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Sharp critical thresholds for a class of nonlocal traffic flow models[☆]

Thomas Hamori, Changhui Tan [*]

Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA

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A B S T R A C T

We study a class of traffic flow models with nonlocal look-ahead interactions. The global regularity of solutions depend on the initial data. We obtain sharp critical threshold conditions that distinguish the initial data into a trichotomy: subcritical initial conditions lead to global smooth solutions, while two types of supercritical initial conditions lead to two kinds of finite time shock formations. The existence of non-trivial subcritical initial data indicates that the nonlocal look-ahead interactions can help avoid shock formations, and hence prevent the creation of traffic jams.

1. Introduction

The history of mathematical theory of traffic flow dates back to the 1920s. Many successful models have been proposed and studied to understand the interactions and the emergent behaviors of vehicles on the road.

One popular class of macroscopic traffic flow models are based on the continuum description of the dynamics of the traffic density

$$\partial_t \rho + \partial_x (f(\rho)) = 0, \quad f(\rho) = \rho u(\rho). \tag{1.1}$$

Here, f is known as the *flux*, which depends on local traffic density $\rho = \rho(t, x)$. The traffic velocity u is modeled through the relation $u = u(\rho)$. A fundamental assumption is that u is a decreasing function in ρ, meaning vehicles slow down as traffic density increases.

A celebrated model under this framework is the Lighthill–Whitham–Richards (LWR) model [1,2], where the velocity $u(\rho) = 1 - \rho$ decays linearly in ρ. The corresponding flux reads

$$f(\rho) = \rho(1 - \rho). \tag{1.2}$$

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* Corresponding author.
E-mail addresses: thamori@email.sc.edu (T. Hamori), tan@math.sc.edu (C. Tan).

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The LWR model successfully captures the phenomenon of shock formation, which is responsible for the creation of traffic jams.

The flux in (1.2) is concave and symmetric (with respect to $\rho = \frac{1}{2}$). However, statistical data from real-world traffic networks suggest that the flux should be neither concave nor symmetric. Rather, observed empirical fluxes are right-skewed and become convex when the density is large, see e.g. [3,4]. In particular, a family of fluxes were introduced in [5] that better fit the data

$$f_J(\rho) = \rho(1-\rho)^J, \quad J > 0. \tag{1.3}$$

For $J > 1$, the flux f_J is right-skewed, and switches from concave to convex at a point $\rho_c = \frac{2}{J+1} \in (0,1)$. Another popular class of models that aim to better fit the data are known as *second order models*, e.g. [6,7]. Instead of imposing the relation $u = u(\rho)$, the velocity u has its own dynamics. These models are not covered in the present work.

In this paper, we consider (1.1) with a general class of fluxes with the following hypotheses

$$f \in C([0,1]) \cap C^\infty([0,1)),\ f(0) = f(1) = 0,\ f'(0) > 0,\ f''(\rho) \begin{cases} < 0 & \rho \in [0, \rho_c), \\ > 0 & \rho \in (\rho_c, 1), \end{cases} \tag{1.4}$$

with a parameter $\rho_c \in (0,1]$. The assumptions in (1.4) cover two scenarios of our concern. First, when $\rho_c = 1$, the flux is concave in $[0,1]$. Examples include the flux (1.2) in the LWR model, as well as fluxes in (1.3) with $J \in (0,1]$. Second, when $\rho_c \in (0,1)$, the convexity of f changes at ρ_c. The fluxes in (1.3) with $J > 1$ lie in this category.

The system (1.1) with flux (1.4) is a scalar conservation law. The behaviors of global solutions have been well-studied, see e.g. the book [8]. In particular, the system develops shock singularity in finite time, for any generic smooth initial data that is not monotone decreasing.

We are interested in the following class of traffic flow models with nonlocal look-ahead interaction

$$\partial_t \rho + \partial_x \big(f(\rho) e^{-\widetilde{\rho}}\big) = 0, \quad \widetilde{\rho}(t,x) = \int_0^\infty K(y)\rho(t, x+y)\, dy. \tag{1.5}$$

Here, the term $e^{-\widetilde{\rho}}$ is known as the Arrhenius-type *slowdown factor*. $\widetilde{\rho}$ represents the heaviness of the traffic ahead, weighted by a kernel K.

The system (1.5) was first introduced by Sopasakis and Katsoulakis [9] where the flux f is taken as in the LWR model (1.2), and the interaction kernel

$$K(x) = 1_{[0,L]}(x), \tag{1.6}$$

where 1_E denotes the indicator function of the set E. They formally derived (1.5) from a microscopic cellular automata (CA) model. In the SK model, the look-ahead distance is L and the weight is a constant. Another class of kernels has been studied numerically in [10] where

$$K(x) = \begin{cases} 1 - \frac{x}{L} & 0 < x < L, \\ 0 & x \geq L. \end{cases} \tag{1.7}$$

Finite time shock formations were observed in both models. The so-called wave breaking phenomenon was studied in [11].

Lee in [12] proposed and studied (1.5) where the flux is taken as (1.3) with $J = 2$. The non-concave-convex flux can lead to different types of shock formations. Later in [13], the system was derived from a class of CA models. An intriguing observation was that the parameter J in (1.3) corresponds to the number of cells a car moved in one step of the microscopic dynamics. See the recent work [14] for generalizations and numerical implementations.

The global wellposedness of (1.5) and related nonlocal traffic flow models have been extensively studied under the framework of nonlocal conservation laws. The theory of entropic weak solutions has been established in [15–24]. These solutions can be discontinuous, allowing the formation of shocks.

One challenging question is *whether* (1.5) *admits global smooth solutions*. In other words, the question asks whether the nonlocal slowdown interaction can help prevent shock formations, and consequently avoid the creation of traffic jams.

A positive answer was given in [25] in a special case when the flux f is (1.2), and the interaction kernel K is (1.6) with look-ahead distance $L = \infty$, namely

$$K(x) = 1_{[0,\infty)}(x), \tag{1.8}$$

and correspondingly

$$\widetilde{\rho}(t,x) = \int_x^\infty \rho(t,y)dy. \tag{1.9}$$

A sharp *critical threshold* on the initial data was established that distinguishes the global behavior of the solutions: subcritical initial data lead to global smooth solutions while supercritical initial data lead to finite-time shock formations. Such critical threshold phenomenon has been studied in the context of Eulerian dynamics, including the Euler–Poisson equations [26–29], the Euler-alignment equations [29–32], and more systems of conservation laws [25,33,34].

In this paper, we study the critical threshold phenomenon for (1.5) with the general class of fluxes in (1.4). Our first result is a generalization of [25], considering concave fluxes.

Theorem 1.1. *Consider Eq.* (1.5) *with smooth initial data* $\rho_0 \in L^1_+ \cap H^k(\mathbb{R})$ *with* $k > 3/2$ *and* $\rho_0(x) \le \rho_M < 1$. *Suppose the flux* f *is concave, satisfying* (1.4) *with* $\rho_c = 1$. *Suppose the nonlocal term* $\widetilde{\rho}$ *satisfies* (1.9). *Then there exists a function* $\sigma : [0,1] \to [0,\infty)$ *such that*

- *If the initial data is subcritical, satisfying*

$$\rho_0'(x) \le \sigma(\rho_0(x)), \quad \forall\, x \in \mathbb{R},$$

then there exists a global smooth solution, namely for any $T > 0$,

$$\rho \in C\big([0,T]; L^1_+ \cap H^k(\mathbb{R})\big). \tag{1.10}$$

- *If the initial data is supercritical, satisfying*

$$\exists\, x_0 \in \mathbb{R} \quad s.t. \quad \rho_0'(x_0) > \sigma(\rho_0(x_0)),$$

then the solution must blow up in finite time. More precisely, there exists a location $x \in \mathbb{R}$ *and a finite time* $T_* > 0$ *such that*

$$\lim_{t\to T_*-} \partial_x \rho(t,x) = +\infty.$$

Remark 1.1. Theorem 1.1 recovers the result in [25] when taking the flux f in (1.2). A similar critical threshold phenomenon is obtained for general concave fluxes. The left graph in Fig. 1 illustrates the shape of the threshold function σ. It can be constructed via the procedure described in Theorem 3.1.

Note that the subcritical region allows $\rho_0'(x)$ to take positive values. Hence, there is a family of non-monotone decreasing initial data that do not lead to shock formations. This provides a strong indication that the nonlocal look-ahead interaction can help preventing the creation of traffic jams, for subcritical initial configurations.

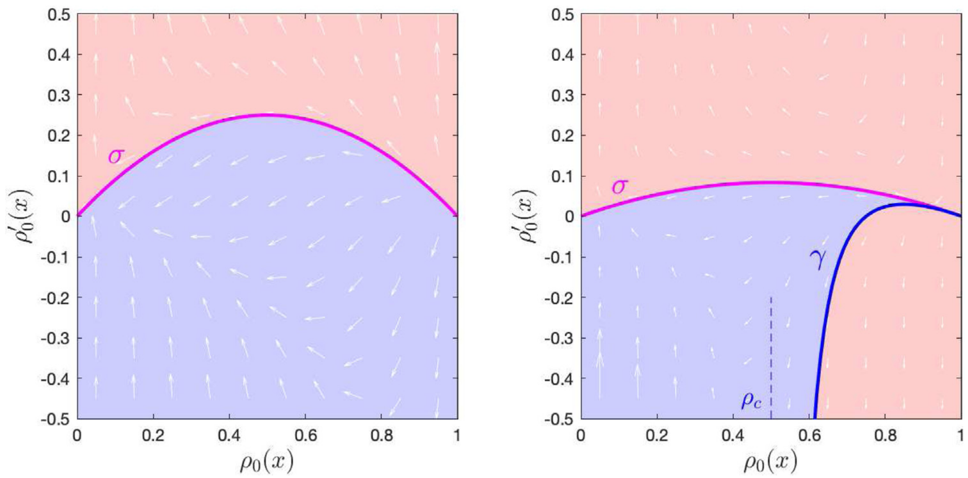

Fig. 1. Illustration of the critical thresholds. Left: when f is concave, the region above σ is supercritical, and the region below σ is subcritical. Right: when f switches from concave to convex at ρ_c, the region above σ is type I supercritical, the region below γ is type II supercritical, and the remaining region is subcritical.

The next main result concerns fluxes that are not concave. The lack of concavity leads to a major difference in the global behaviors of the solutions. In particular, there are two different types of shock formations. There is a trichotomy on initial data that lead to global regularity and two types of finite time blowup. The following theorem provides a sharp characterization on the threshold conditions.

Theorem 1.2. *Consider Eq. (1.5) with smooth initial data $\rho_0 \in L^1_+ \cap H^k(\mathbb{R})$ with $k > 3/2$ and $\rho_0(x) \le \rho_M < 1$. Suppose the flux f satisfies (1.4) with $\rho_c < 1$, that is, f is concave on $[0, \rho_c]$ and convex on $[\rho_c, 1]$. Suppose the nonlocal term $\widetilde{\rho}$ satisfies (1.9). Then there exists two threshold functions σ and γ such that*

- *If the initial data is subcritical, satisfying*

$$\gamma(\rho_0(x)) < \rho_0'(x) \le \sigma(\rho_0(x)), \quad \forall\, x \in \mathbb{R},$$

  then there exists a global smooth solution ρ satisfying (1.10).
- *If the initial data is type I supercritical, satisfying*

$$\exists\, x_0 \in \mathbb{R} \quad s.t. \quad \rho_0'(x_0) > \sigma(\rho_0(x_0)),$$

  then the solution must blow up in finite time. More precisely, there exists a location $x \in \mathbb{R}$ and a finite time $T_* > 0$ such that

$$\lim_{t \to T_*-} \partial_x \rho(t, x) = +\infty,$$

  unless type II blowup occurs earlier than T_*.
- *If the initial data is type II supercritical, satisfying*

$$\exists\, x_0 \in \mathbb{R} \quad s.t. \quad \rho_0'(x_0) \le \gamma(\rho_0(x_0)),$$

  then the solution must blow up in finite time. More precisely, there exists a location $x \in \mathbb{R}$ and a finite time $T_* > 0$ such that

$$\lim_{t \to T_*-} \partial_x \rho(t, x) = -\infty,$$

  unless type I blowup occurs earlier than T_*.

Remark 1.2. The major different phenomenon compared with the models with concave fluxes is the presence of the type II blowup. The description of the new threshold function γ is given in Theorem 3.2. It is a function defined for $\rho > \rho_c$ with a vertical asymptote at $\rho = \rho_c$, namely

$$\lim_{\rho\to\rho_c+} \gamma(\rho) = -\infty.$$

The right graph in Fig. 1 illustrates the shapes of the threshold functions. Our result is sharp: for any $x \in \mathbb{R}$, $(\rho_0(x), \rho_0'(x))$ lies in exactly one of the three regions, which then lead to three types of global behaviors.

Note that the threshold functions σ and γ may only be defined in a subset of $[0, 1]$ and $(\rho_c, 1]$ respectively. See Remark 4.1 for a clarification on the meaning of the threshold conditions if $\sigma(\rho_0(x))$ or $\gamma(\rho_0(x))$ is undefined.

Our final result concerns the class of fluxes in (1.3). Theorems 1.1 and 1.2 can be applied to the system with $f = f_J$ for $J \in (0, 1]$ and $J > 1$, respectively. Remarkably, we find explicit expressions for the corresponding threshold functions.

Theorem 1.3. *Suppose the flux $f = f_J$ satisfies (1.3). Then the threshold functions $\sigma = \sigma_J$ and $\gamma = \gamma_J$ can be explicitly expressed as follows. For any $J > 0$*

$$\sigma_J(\rho) = \frac{\rho(1-\rho)}{J}, \quad \rho \in [0, 1].$$

For any $J > 1$ we have $\rho_c = \frac{2}{J+1}$ and

$$\gamma_J(\rho) = \frac{\rho^2(1-\rho)\left(\rho - \frac{4J}{(J+1)^2}\right)}{J(\rho-\rho_c)^2}, \quad \rho \in (\rho_c, 1].$$

We would like to mention that all our results are based on the particular choice of kernel in (1.8). This allows us to obtain sharp results. The kernel $K = 1_{[0,\infty)}$ features a jump discontinuity at the origin, representing that the interaction is *look-ahead*. Indeed, such jump drives the main phenomenon: global regularity for a class of non-trivial subcritical initial data. We believe the same phenomenon holds for general look-ahead interactions, where the kernel has the same jump structure at the origin, like (1.6) and (1.7). We shall leave the generalization for future investigation.

The rest of the paper is organized as follows. In Section 2, we establish a local wellposedness theory for a general class of nonlocal traffic flow models, including the system (1.5) of our concern. In Section 3, we provide unique constructions of the threshold functions σ and γ. In Section 4, we study the global behaviors of solutions for the three types of initial data, proving Theorems 1.1 and 1.2.

Notations

We denote $L^p(\mathbb{R})$ the Lebesgue spaces in $\mathbb{R}$. The space $L^1_+(\mathbb{R})$ consists non-negative L^1 functions in $\mathbb{R}$. We denote $H^k(\mathbb{R})$ the Sobolev space, endowed with the norm

$$\|g\|_{H^k}^2 = \|g\|_{L^2}^2 + \|\tfrac{d^k}{dx^k} g\|_{L^2}^2,$$

for any non-negative integer k. For non-integer k, the space $H^k(\mathbb{R})$ is defined via Fourier transform

$$\|g\|_{H^k} = \|(I-\Delta)^{k/2} g\|_{L^2} = \left\|\mathcal{F}^{-1}\left[(1+|\xi|^2)^{k/2}\mathcal{F}g\right]\right\|_{L^2},$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the forward and inverse Fourier transforms in $\mathbb{R}$, respectively. We denote $\|\cdot\|_{\dot{H}^k(\mathbb{R})}$ the homogeneous semi-norm with $\|g\|_{\dot{H}^k} = \|\frac{d^k}{dx^k}g\|_{L^2}$, and for general $k > 0$

$$\|g\|_{\dot{H}^k} = \|(-\Delta)^{k/2}g\|_{L^2} = \left\|\mathcal{F}^{-1}\left[|\xi|^k\mathcal{F}g\right]\right\|_{L^2}.$$

The notation $\lceil k\rceil$ refers to the smallest integer greater than or equal to k.

In Section 2, we will repeatedly use the letter C to refer to a constant $C > 0$ whose value may change line by line. The constant might depend on parameters and initial conditions. We write $C(p)$ to represent that the constant depends on the parameter p.

Finally, we denote g' the derivative of g, if g has a single variable; and $\dot{g}$ denotes the material derivative of $g = g(t,x)$ along a characteristic path

$$\dot{g}(t,X(t)) = \frac{d}{dt}g(t,X(t)) = \partial_t g + \left((f'(\rho))e^{-\widetilde{\rho}}\right)\partial_x g,$$

where f is the flux and $X(t)$ is the characteristic path defined in (2.4).

2. Local wellposedness and regularity criteria

In this section, we establish a local wellposedness theory for a general class of nonlocal traffic flow models

$$\partial_t\rho + \partial_x\left(f(\rho)e^{-\widetilde{\rho}}\right) = 0, \quad \widetilde{\rho}(t,x) = \int_{\mathbb{R}} K(y)\rho(t,x+y)\,dy. \tag{2.1}$$

We shall present the theorem with general assumptions on the kernel K:

$$K \in BV(\mathbb{R}), \quad 0 \le K(x) \le \overline{K}. \tag{2.2}$$

Here, we only require K to be bounded, nonnegative, and have bounded total variation. In particular, the interaction does not need to be look-ahead. We shall comment that all look-ahead interactions (1.6), (1.7) and (1.8) satisfy the assumption (2.2), with $\overline{K} = 1$ and $|K|_{BV} \le 2$.

Let us start with the statement of the local wellposedness theory for strong solutions in Sobolev space H^k. We take $k > \frac{3}{2}$ to ensure ρ_0 is Lipschitz.

Theorem 2.1 (*Local Wellposedness*). *Let $k > \frac{3}{2}$. Consider (2.1) with smooth initial condition*

$$\rho_0 \in L^1_+ \cap H^k(\mathbb{R}).$$

Assume the flux f satisfies (1.4), and the kernel K satisfies (2.2). Then there exists a time $T > 0$ such that solution $\rho = \rho(t,x)$ exists and

$$\rho \in C\left([0,T]; L^1_+ \cap H^k(\mathbb{R})\right).$$

Moreover, the solution exists in $[0,T]$ as long as

$$\int_0^T \|\partial_x\rho(t,\cdot)\|_{L^\infty}\,dt < \infty. \tag{2.3}$$

Local wellposedness of (2.1) has been studied in [25] for specific flux (1.2) and interaction kernel (1.8). Here, we extend the result to general fluxes and kernels. We also provide a regularity criterion (2.3). It allows us to study global wellposedness based on the control of $\partial_x\rho$.

In the rest of the section, we present a proof of Theorem 2.1, using *a priori* energy estimates. The focus is on the proper treatment of the nonlinearity in f and the nonlocality in the term $e^{-\widetilde{\rho}}$, where nontrivial commutator and composition estimates are used.

2.1. A priori bounds

First, we state the *conservation of mass*. Integrating (2.1) in x gives

$$\frac{d}{dt}\int_{\mathbb{R}} \rho(t,x)dx = -\int_{\mathbb{R}} \partial_x\big(f(\rho)e^{-\widetilde{\rho}}\big)dx = 0.$$

Let us denote the total mass

$$m := \int_{\mathbb{R}} \rho(t,x)\,dx = \int_{\mathbb{R}} \rho_0(x)\,dx.$$

Next, we consider the characteristic path $X(t,x)$ originated at $x \in \mathbb{R}$

$$\partial_t X(t,x) = f'(\rho(t,X(t,x)))e^{-\widetilde{\rho}(t,X(t,x))}, \quad X(t=0,x) = x. \tag{2.4}$$

We shall suppress the x dependence and write $X(t)$ from now on. Along each characteristic path, we have

$$\frac{d}{dt}\rho(t,X(t)) = -\rho(t,X(t))f(\rho(t,X(t)))e^{-\widetilde{\rho}(t,X(t))}. \tag{2.5}$$

This leads to the following *maximum principle*.

Proposition 2.1 (*Maximum Principle*)**.** *Let $\rho_M \in (0,1]$. Let $\rho = \rho(t,x)$ be a classical solution of (2.1) in $[0,T]\times\mathbb{R}$ with initial condition $\rho_0(x) \in [0,\rho_M]$ for all $x \in \mathbb{R}$. Then, $\rho(t,x) \in [0,\rho_M]$ for all $x \in \mathbb{R}$ and $t \in [0,T]$.*

Proof. Since $f(0) = f(1) = 0$, $\rho = 0$ and $\rho = 1$ are equilibrium states of (2.5). Hence, $\rho_0 \in [0,1]$ implies $\rho(t) \in [0,1]$. Moreover, $-\rho f(\rho)e^{-\widetilde{\rho}} < 0$ for any $\rho \in (0,1)$. Hence, if $\rho_0 \le \rho_M < 1$, we have $\rho(t) < \rho_0 \le \rho_M$ for any $t \ge 0$. □

Finally, we present a priori bounds on the nonlocal term $e^{-\widetilde{\rho}}$. Applying the definition of $\widetilde{\rho}$ in (2.1) and the bounds on the kernel K in (2.2), we obtain the bounds

$$0 \le \widetilde{\rho}(t,x) \le \overline{K}m, \tag{2.6}$$

which then implies

$$e^{-\overline{K}m} < e^{-\widetilde{\rho}} \le 1. \tag{2.7}$$

Furthermore, we have the following bound on $\partial_x(e^{-\widetilde{\rho}})$.

Proposition 2.2. *Under the same assumptions as in Proposition 2.1, we have*

$$\|\partial_x(e^{-\widetilde{\rho}})\|_{L^\infty} \le |K|_{BV}. \tag{2.8}$$

Proof. First, apply (2.7) and get

$$\|\partial_x(e^{-\widetilde{\rho}})\|_{L^\infty} = \|e^{-\widetilde{\rho}}(-\partial_x\widetilde{\rho})\|_{L^\infty} \le \|\partial_x\widetilde{\rho}\|_{L^\infty}.$$

It remains to control $\partial_x\widetilde{\rho}$. We apply maximum principle and compute

$$|\partial_x\widetilde{\rho}(t,x)| = \left|\int_{-\infty}^{\infty} K(y)\partial_x\rho(t,x+y)\,dy\right| \le |K|_{BV}\cdot\rho_M \le |K|_{BV}, \tag{2.9}$$

which directly implies (2.8). □

2.2. L^2 energy estimate

Let us integrate (2.1) against ρ and get

$$\frac{1}{2}\frac{d}{dt}\|\rho(t,\cdot)\|_{L^2}^2 = -\int_{\mathbb{R}} \rho\,\partial_x\big(f(\rho)e^{-\widetilde{\rho}}\big)\,dx = \int_{\mathbb{R}} \partial_x\rho\, f(\rho)e^{-\widetilde{\rho}}\,dx = \int_{\mathbb{R}} \partial_x F(\rho)\,e^{-\widetilde{\rho}}\,dx$$
$$= \int_{\mathbb{R}} -F(\rho)\,\partial_x\big(e^{-\widetilde{\rho}}\big)\,dx \le \|\partial_x\big(e^{-\widetilde{\rho}}\big)\|_{L^\infty}\|F(\rho)\|_{L^1}.$$

Here the function F is the primitive of f with $F(0)=0$. From (1.4), we know that $f(x)\le f'(0)x$ for all $x\in[0,1]$. Therefore, we can estimate

$$|F(x)| = \left|\int_0^x f(y)\,dy\right| \le \frac{f'(0)}{2}x^2.$$

Since $\rho\in[0,1]$, we get

$$\|F(\rho)\|_{L^1} \le \frac{f'(0)}{2}\|\rho\|_{L^2}^2.$$

Apply (2.8) and we conclude with

$$\frac{1}{2}\frac{d}{dt}\|\rho(t,\cdot)\|_{L^2}^2 \le \frac{f'(0)|K|_{BV}}{2}\|\rho(t,\cdot)\|_{L^2}^2. \tag{2.10}$$

2.3. H^k energy estimate

Now, we consider the evolution of the homogeneous $\dot{H}^k$ semi-norm of ρ

$$\|\rho(t,\cdot)\|_{\dot{H}^k} = \|\Lambda^k\rho(t,\cdot)\|_{L^2},$$

where $\Lambda=(-\Delta)^{1/2}$ denotes the fractional Laplacian operator.

Let us first state the following estimates. We refer the proofs to [25] and references therein.

Lemma 2.1 (*Fractional Leibniz Rule*)**.** *Let $k\ge 0$, $g,h\in L^\infty\cap\dot{H}^k(\mathbb{R})$. There exists a constant $C>0$, depending only on k, such that*

$$\|gh\|_{\dot{H}^k} \le C\big(\|g\|_{L^\infty}\|h\|_{\dot{H}^k} + \|g\|_{\dot{H}^k}\|h\|_{L^\infty}\big).$$

Lemma 2.2 (*Commutator Estimate*)**.** *Let $k\ge 1$, $g\in L^\infty\cap\dot{H}^k(\mathbb{R})$, and $h\in L^\infty\cap\dot{H}^{k-1}(\mathbb{R})$. There exists a constant $C>0$, depending only on k, such that*

$$\|[\Lambda^k, g]h\|_{L^2} \le C\big(\|\partial_x g\|_{L^\infty}\|h\|_{\dot{H}^{k-1}} + \|g\|_{\dot{H}^k}\|h\|_{L^\infty}\big),$$

where the commutator is denoted by $[\Lambda^k, f]g = \Lambda^k(fg) - f\Lambda^k g$.

Lemma 2.3 (*Composition Estimate*)**.** *Let $k>0$, $g\in L^\infty\cap\dot{H}^k(\mathbb{R})$, and $h\in C^{\lceil k\rceil}(Range(g))$. Then, the composition $h\circ g\in L^\infty\cap\dot{H}^k(\mathbb{R})$. Moreover, there exists a constant $C>0$, depending on k, $\|h\|_{C^{\lceil k\rceil}(Range(g))}$, and $\|g\|_{L^\infty}$, such that*

$$\|h\circ g\|_{\dot{H}^k} \le C\|g\|_{\dot{H}^k}.$$

To begin with, we act Λ^k on (2.1), integrate against $\Lambda^k\rho$ and get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\rho(t,\cdot)\|_{\dot H^k}^2 &= -\int_{\mathbb{R}} \Lambda^k\rho\cdot\Lambda^k\partial_x\big(f(\rho)e^{-\widetilde{\rho}}\big)\,dx\\ &= -\int_{\mathbb{R}} \Lambda^k\rho\cdot\Lambda^k\Big(f'(\rho)\partial_x\rho\cdot e^{-\widetilde{\rho}} - f(\rho)\cdot e^{-\widetilde{\rho}}\partial_x\widetilde{\rho}\Big)\,dx\\ &= -\int_{\mathbb{R}} \Lambda^k\rho\cdot\Lambda^k\partial_x\rho\cdot f'(\rho)e^{-\widetilde{\rho}}dx - \int_{\mathbb{R}} \Lambda^k\rho\cdot[\Lambda^k, f'(\rho)e^{-\widetilde{\rho}}]\partial_x\rho\,dx\\ &\quad + \int_{\mathbb{R}} \Lambda^k\rho\cdot\Lambda^k\big(f(\rho)e^{-\widetilde{\rho}}\partial_x\widetilde{\rho}\big)\,dx\\ &= \mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

We bound the three terms one by one. For the first term we use integration by parts

$$\mathrm{I} = -\int_{\mathbb{R}} \partial_x\left(\frac{(\Lambda^k\rho)^2}{2}\right)\cdot f'(\rho)e^{-\widetilde{\rho}}\,dx = \frac{1}{2}\int_{\mathbb{R}} (\Lambda^k\rho)^2\cdot\partial_x\big(f'(\rho)e^{-\widetilde{\rho}}\big)\,dx.$$

Applying (2.7) and (2.8), we estimate

$$\left|\partial_x\big(f'(\rho)e^{-\widetilde{\rho}}\big)\right| = \left|f''(\rho)e^{-\widetilde{\rho}}\partial_x\rho + f'(\rho)\partial_x(e^{-\widetilde{\rho}})\right| \le \|f\|_{C^2([0,\rho_M])}(\|\partial_x\rho\|_{L^\infty} + |K|_{BV}).$$

This leads to the bound

$$\mathrm{I} \le \frac{1}{2}\|f\|_{C^2([0,\rho_M])}(\|\partial_x\rho\|_{L^\infty} + |K|_{BV})\|\rho\|_{\dot H^k}^2. \tag{2.11}$$

Moving on to the second term, we apply Lemma 2.2 and get

$$\begin{aligned}\mathrm{II} &\le \|\rho\|_{\dot H^k}\big\|[\Lambda^k, f'(\rho)e^{-\widetilde{\rho}}]\partial_x\rho\big\|_{L^2}\\ &\le C(k)\|\rho\|_{\dot H^k}\Big(\|\partial_x(f'(\rho)e^{-\widetilde{\rho}})\|_{L^\infty}\|\partial_x\rho\|_{\dot H^{k-1}} + \|f'(\rho)e^{-\widetilde{\rho}}\|_{\dot H^k}\|\partial_x\rho\|_{L^\infty}\Big).\end{aligned}$$

For convenience in notations, we shall use C to denote the constants, which can change line by line. We will also keep track of the dependence of the constant with respect to the parameters.

Now we focus on the estimate of $\|f'(\rho)e^{-\widetilde{\rho}}\|_{\dot H^k}$. Apply Lemma 2.1

$$\|f'(\rho)e^{-\widetilde{\rho}}\|_{\dot H^k} \le C(k)\Big(\|f'(\rho)\|_{L^\infty}\|e^{-\widetilde{\rho}}\|_{\dot H^k} + \|f'(\rho)\|_{\dot H^k}\|e^{-\widetilde{\rho}}\|_{L^\infty}\Big). \tag{2.12}$$

The term $\|e^{-\widetilde{\rho}}\|_{\dot H^k}$ can be estimated as follows.

Proposition 2.3. *For $k \ge 1$,*

$$\|e^{-\widetilde{\rho}}\|_{\dot H^k} \le C(k, \overline{K}m, |K|_{BV})\|\rho\|_{\dot H^{k-1}}. \tag{2.13}$$

Proof. We begin by applying Lemma 2.3 with $g(x) = \widetilde{\rho}(t,x)$ and $h(x) = e^{-x}$. From (2.6) we know $\|g\|_{L^\infty} \le \overline{K}m$. Moreover, $\|h\|_{C^\infty([0,\overline{K}m])} \le 1$. Therefore, we have

$$\|e^{-\widetilde{\rho}}\|_{\dot H^k} \le C(k,\overline{K}m)\|\widetilde{\rho}\|_{\dot H^k}.$$

Next, we apply Young's inequality and get

$$\|\widetilde{\rho}\|_{\dot H^k} = \|\partial_x\Lambda^{k-1}\rho\|_{L^2} = \left\|\int_{\mathbb{R}} K(y)\partial_x(\Lambda^{k-1}\rho)\,dy\right\|_{L^2} \le |K|_{BV}\|\Lambda^{k-1}\rho\|_{L^2}. \tag{2.14}$$

Put together and we conclude with (2.13). □

For the term $\|f'(\rho)\|_{\dot{H}^k}$, we again apply Lemma 2.3 with $g(x) = \rho(t,x)$ and $h(x) = f'(x)$. From the maximum principle, $\|g\|_{L^\infty} \le \rho_M < 1$. Moreover,

$$\|h\|_{C^{\lceil k\rceil}(\mathrm{Range}(g))} \le \|f\|_{C^{\lceil k\rceil+1}([0,\rho_M])},$$

which is bounded due to the assumptions on f in (1.4). Hence,

$$\|f'(\rho)\|_{\dot{H}^k} \le C(k, \|f\|_{C^{\lceil k\rceil+1}([0,\rho_M])})\|\rho\|_{\dot{H}^k}. \tag{2.15}$$

Applying (2.8), (2.13) and (2.15) to (2.12) we get

$$\|f'(\rho)e^{-\widetilde{\rho}}\|_{\dot{H}^k} \le C(k, \overline{K}m, |K|_{BV}, \|f\|_{C^{\lceil k\rceil+1}([0,\rho_M])})\|\rho\|_{H^k}.$$

Consequently, we have the bound on the second term

$$\mathrm{II} \le C(k, \overline{K}m, |K|_{BV}, \|f\|_{C^{\lceil k\rceil+1}([0,\rho_M])})(1+\|\partial_x\rho\|_{L^\infty})\|\rho\|_{\dot{H}^k}\|\rho\|_{H^k}. \tag{2.16}$$

Finally, let us estimate the third term using Lemma 2.1

$$\begin{aligned}\mathrm{III} \le& \|\rho\|_{\dot{H}^k}\|f(\rho)e^{-\widetilde{\rho}}\partial_x\widetilde{\rho}\|_{\dot{H}^k}\\ \le& C(k)\|\rho\|_{\dot{H}^k}\Big(\|f(\rho)\|_{\dot{H}^k}\|e^{-\widetilde{\rho}}\partial_x\widetilde{\rho}\|_{L^\infty} + \|e^{-\widetilde{\rho}}\|_{\dot{H}^k}\|f(\rho)\partial_x\widetilde{\rho}\|_{L^\infty}\\ &\qquad + \|\partial_x\widetilde{\rho}\|_{\dot{H}^k}\|f(\rho)e^{-\widetilde{\rho}}\|_{L^\infty}\Big)\\ =& C(k)\|\rho\|_{\dot{H}^k}\big(\mathrm{III}_1+\mathrm{III}_2+\mathrm{III}_3\big).\end{aligned}$$

For III_1, use (2.7), (2.9) and (2.15) (with f' replaced by f)

$$\mathrm{III}_1 \le C(k, \|f\|_{C^{\lceil k\rceil}([0,\rho_M])})|K|_{BV}\|\rho\|_{\dot{H}^k}.$$

For III_2, use (2.9) and (2.13)

$$\mathrm{III}_2 \le C(k, \overline{K}m, |K|_{BV})\|f\|_{C^0([0,\rho_M])}|K|_{BV}\|\rho\|_{\dot{H}^{k-1}}.$$

For III_3, use (2.7) and (2.14)

$$\mathrm{III}_3 \le \|f\|_{C^0([0,\rho_M])}|K|_{BV}\|\rho\|_{\dot{H}^k}.$$

All together, we obtain

$$\mathrm{III} \le C(k, \overline{K}m, |K|_{BV}, \|f\|_{C^{\lceil k\rceil}([0,\rho_M])})\|\rho\|_{\dot{H}^k}\|\rho\|_{H^k}. \tag{2.17}$$

Collecting the estimates (2.11), (2.16), (2.17), we end up with the estimate on H^k energy ($k \ge 1$) as follows.

$$\frac{1}{2}\frac{d}{dt}\|\rho(t,\cdot)\|_{\dot{H}^k}^2 \le C(k, \overline{K}m, |K|_{BV}, \|f\|_{C^{\lceil k\rceil+1}([0,\rho_M])})(1+\|\partial_x\rho\|_{L^\infty})\|\rho\|_{\dot{H}^k}\|\rho\|_{H^k}, \tag{2.18}$$

where the constant C is finite under our assumptions on f and K.

2.4. *Proof of Theorem 2.1*

Define an energy

$$Y(t) = \|\rho(t,\cdot)\|_{L^2}^2 + \|\rho(t,\cdot)\|_{\dot{H}^k}^2.$$

Clearly, $Y(t)$ is equivalent to $\|\rho(t,\cdot)\|_{H^k}^2$. Combining the L^2 and H^k energy estimates (2.10) and (2.18), we have the bound on the evolution of Y as follows

$$Y'(t) \le C(1+\|\partial_x\rho(t,\cdot)\|_{L^\infty})\|\rho(t,\cdot)\|_{H^k}^2. \tag{2.19}$$

Let $k > 3/2$, from the Sobolev embedding theorem, we have $\|\partial_x \rho\|_{L^\infty} \le C(k)\|\rho\|_{H^k}$. This leads to a bound

$$Y'(t) \le C(1 + Y^{1/2})Y(t).$$

Then there exists a time $T_* > 0$, depending on $Y(0)$ and C, such that $Y(t)$ exists and is bounded for $t \in [0, T_*]$. This finishes the local wellposedness proof.

Moreover, we apply Grönwall inequality to (2.19) and obtain

$$Y(T) \le Y(0) \exp\left(\int_0^T C(1 + \|\partial_x \rho(t,\cdot)\|_{L^\infty})\, dt\right).$$

Therefore, $Y(T)$ remains bounded if criterion (2.3) holds.

3. Critical thresholds

In this section, we restrict our attention to our main Eq. (1.5) with the special kernel (1.8). The goal is to construct threshold functions that distinguish the global behaviors of the solutions.

From the regularity criterion (2.3), we know that the solution is globally regular if and only if $\partial_x \rho$ is bounded. Let us denote

$$d = \partial_x \rho.$$

We shall focus on the boundedness of d.

Differentiating (1.5) in x, we can write the dynamics of d as

$$\partial_t d + f'(\rho)e^{-\widetilde{\rho}}\partial_x d = \big(-f''(\rho)d^2 - (f(\rho) + 2\rho f'(\rho))d - \rho^2 f(\rho)\big)e^{-\widetilde{\rho}}.$$

Here we have used the special structure of (1.9). In particular,

$$\partial_x \widetilde{\rho} = -\rho.$$

Let us denote $\dot{d}$ as the time derivative along the characteristic a path $X(t)$, namely

$$\dot{d} = \frac{d}{dt} d(t, X(t)).$$

Then, together with (2.5), we obtain a coupled dynamics of (ρ, d) along each characteristic path

$$\begin{cases} \dot{\rho} = -\rho f(\rho) e^{-\widetilde{\rho}}, \\ \dot{d} = -\big(f''(\rho)d^2 + (f(\rho) + 2\rho f'(\rho))d + \rho^2 f(\rho)\big)e^{-\widetilde{\rho}}. \end{cases} \tag{3.1}$$

Note that the only nonlocality in the coupled dynamics (3.1) appears to be the factor $e^{-\widetilde{\rho}}$. Thus, the trajectories on the phase plane (ρ, d) depend on the local information. Indeed, if we express a trajectory as $d = \mathrm{d}(\rho)$, then it satisfies the following differential equation

$$\mathrm{d}'(\rho) = \frac{f''(\rho)\mathrm{d}^2 + (f(\rho) + 2\rho f'(\rho))\mathrm{d} + \rho^2 f(\rho)}{\rho f(\rho)}. \tag{3.2}$$

We will examine the trajectories in the phase plane and investigate whether the trajectories are bounded or not. The boundedness of d will then lead to global wellposedness of the system (1.5) by Theorem 2.1.

There are two special trajectories that serve as thresholds in the phase plane. They divide the area $\{(\rho, d) : \rho \in [0, 1]\}$ into three regions. Trajectories originated in each region stay inside the region for their life-spans. Trajectories in different regions have different large time behaviors.

We call the two trajectories that separate the regions *critical threshold functions*, and denote them by two functions σ and γ. The trajectories are expressed as $d = \sigma(\rho)$ and $d = \gamma(\rho)$ respectively. Fig. 1 illustrates the shapes of the threshold functions.

In the following, we focus on the wellposedness of the two critical thresholds.

3.1. The threshold function σ

The curve σ represents a trajectory that goes across $(0,0)$ in the phase plane. Since $(0,0)$ is a degenerate equilibrium state of the phase dynamics, there are infinitely many trajectories such that $\mathrm{d}(0)=0$. These trajectories satisfy the following property.

Proposition 3.1. *Let $d=\mathrm{d}(\rho)$ be a trajectory that satisfies (3.2) with $\mathrm{d}(0)=0$, and f satisfies (1.4). Assume $\mathrm{d}'(0)$ exists. Then, we must have*

$$\mathrm{d}'(0)=0 \quad \text{or} \quad \mathrm{d}'(0)=-\frac{2f'(0)}{f''(0)}. \tag{3.3}$$

Proof. We apply (3.2) and take $\rho\to 0$

$$\begin{aligned}\mathrm{d}'(0)&=\lim_{\rho\to 0_+}\frac{f''(\rho)\mathrm{d}(\rho)^2+(f(\rho)+2\rho f'(\rho))\mathrm{d}(\rho)+\rho^2 f(\rho)}{\rho f(\rho)}\\&=\lim_{\rho\to 0+}\frac{\rho f''(\rho)}{f(\rho)}\cdot \mathrm{d}'(0)^2+\lim_{\rho\to 0+}\frac{2\rho f'(\rho)+f(\rho)}{f(\rho)}\cdot \mathrm{d}'(0)=\frac{f''(0)}{f'(0)}\cdot \mathrm{d}'(0)^2+3\mathrm{d}'(0).\end{aligned}$$

This directly leads to (3.3). □

To simplify the notation, we denote

$$\beta=-\frac{2f'(0)}{f''(0)}$$

for the rest of the section. Note that $\beta>0$.

Among these trajectories, there is only one such that $\mathrm{d}'(0)=\beta$. This is the trajectory σ that we seek for. The following theorem ensures a uniquely defined threshold curve σ. The idea of the proof follows from [25, Proposition 3.1].

Theorem 3.1. *Let f satisfy the hypotheses in (1.4). There exists a unique trajectory represented by σ that satisfies the Eq. (3.2), namely*

$$\sigma'(\rho)=\frac{f''(\rho)\sigma(\rho)^2+(f(\rho)+2\rho f'(\rho))\sigma(\rho)+\rho^2 f(\rho)}{\rho f(\rho)}, \tag{3.4a}$$

with initial conditions

$$\sigma(0)=0, \quad \text{and} \quad \sigma'(0)=\beta. \tag{3.4b}$$

Proof. We start with the local existence theory. Fix a small $\epsilon>0$. The classical Cauchy–Peano theorem does not apply directly near $x=0$, as the right hand-side of (3.4a)

$$F(\rho,\sigma):=\frac{f''(\rho)\sigma^2+(f(\rho)+2\rho f'(\rho))\sigma+\rho^2 f(\rho)}{\rho f(\rho)}$$

is not uniformly bounded for $(\rho,\sigma)\in[0,\epsilon]\times[-\epsilon,\epsilon]$. By smallness of ϵ and smoothness of f, we have

$$F(\rho,\sigma)=-\frac{2}{\beta}\left(\frac{\sigma}{\rho}\right)^2+3\left(\frac{\sigma}{\rho}\right)+\mathcal{O}(\epsilon),$$

for any ρ inside the region

$$A=\left\{(\rho,\sigma):0\le\sigma\le\frac{5\beta}{4}\rho,\quad 0\le\rho\le\epsilon\right\}.$$

We can check that if $\frac{\sigma}{\rho} \in [0, \frac{5\beta}{4}]$, then

$$\min\left\{\rho,\ \frac{5\beta}{8} + \mathcal{O}(\epsilon)\right\} \le F(\rho,\sigma) \le \frac{9\beta}{8} + \mathcal{O}(\epsilon).$$

Hence, if we pick ϵ small enough, we would have

$$0 \le F(\rho,\sigma) \le \frac{5\beta}{4}, \quad \forall\ (\rho,\sigma) \in A. \tag{3.5}$$

Now, we can build a sequence of approximate solutions $\{\sigma_n(\rho)\}$ for $\rho \in [0,\epsilon]$. Given $n \in \mathbb{Z}_+$, define equi-distance lattice $\{\rho_k = \frac{k\epsilon}{n}\}_{k=0}^n$.

(i). $\sigma_n(\rho) = \beta\rho, \quad \forall\ \rho \in [0, \rho_1]$.

(ii). $\sigma_n(\rho) = \sigma_n(\rho_k) + F(\rho_k, \sigma_n(\rho_k))(\rho - \rho_k), \quad \forall\ \rho \in [\rho_k, \rho_{k+1}], \quad k = 1, \dots, n-1.$

From (3.5), we know $(\rho, \sigma_n(\rho)) \in A$, for all $\rho \in [0,\epsilon]$. Hence, $\sigma_n(\rho)$ is uniformly bounded and equi-continuous in $\rho \in [0,\epsilon]$. By the Arzela–Ascoli theorem, σ_n converges uniformly to σ, up to an extraction of a subsequence. And by its construction, σ is indeed a solution of (3.4a).

Next, we verify the initial conditions (3.4b). It is clear that $\sigma(0) = 0$ since $\sigma_n(0) = 0$ for every n. To verify $\sigma'(0) = \beta$, we show the following statement: the image of the solution $(\rho, \sigma(\rho))$ lies inside the cone

$$\{(\rho,\sigma) : (1-\delta)\beta\rho \le \sigma \le (1+\delta)\beta\rho, \quad 0 \le \rho \le \epsilon\}.$$

Indeed, we check F at the boundary of the cone

$$\begin{aligned} F\big(\rho, \sigma = (1-\delta)\beta\rho\big) &= \beta(1 + \delta - 2\delta^2) + \mathcal{O}(\epsilon) > \beta, \\ F\big(\rho, \sigma = (1+\delta)\beta\rho\big) &= \beta(1 - \delta - 2\delta^2) + \mathcal{O}(\epsilon) < \beta, \end{aligned} \tag{3.6}$$

where the inequalities can be obtained by choosing $\delta = \sqrt{\epsilon}$ and let ϵ small enough. Therefore, $\sigma'(\rho) \in [(1-\delta)\beta, (1+\delta)\beta]$ for all $\rho \in [0,\epsilon]$. Take $\epsilon \to 0$, we conclude with $\sigma'(0) = \beta$.

Finally, we discuss the local uniqueness. Let $\sigma^{(1)}$ and $\sigma^{(2)}$ be two different solutions of (3.4). Fix a small ϵ. From (3.6) we know that $\sigma^{(i)}(\rho) \ge (1-\delta)\beta\rho$ for $i = 1, 2$. Let $w = \sigma^{(1)} - \sigma^{(2)}$. Note that $\sigma^{(1)}$ and $\sigma^{(2)}$ cannot cross each other for $\rho \in (0,\epsilon]$. Without loss of generality, we may assume $w(\rho) > 0$ for $\rho \in (0,\epsilon]$. (Otherwise, switch $\sigma^{(1)}$ and $\sigma^{(2)}$). Compute

$$\begin{aligned} w'(\rho) &= \frac{f''(\rho)(\sigma^{(1)}(\rho) + \sigma^{(2)}(\rho)) + (f(\rho) + 2\rho f'(\rho))}{\rho f(\rho)} w(\rho) \\ &\le \frac{f''(0)\cdot 2(1-\delta)\beta\rho + 3f'(0)\rho + \mathcal{O}(\rho^2)}{\rho f(\rho)} w(\rho) = \frac{-f'(0)\rho + \mathcal{O}(\delta\rho)}{\rho f(\rho)} w(\rho) < 0, \end{aligned}$$

for any $\rho \in (0,\epsilon]$. Since $w(0) = 0$, it implies $w(\rho) < 0$ for $\rho \in (0,\epsilon]$. This leads to a contradiction.

Once we obtain local wellposedness of σ near $\rho = 0$, global existence and uniqueness for $\rho > 0$ follows from the standard Cauchy–Lipschitz theorem. Indeed, $F(\rho,\sigma)$ is bounded and Lipschitz in σ as long as $\rho \in (0,1)$ and σ is bounded. □

Next, we discuss properties of the threshold function σ.

Proposition 3.2. *Let σ be the solution of (3.4). Then for any $\rho \in (0,1)$ that lies in the domain of σ, $\sigma(\rho) > 0$.*

Proof. Suppose the statement is false. Then there must exist

$$\rho_z = \min\{\rho \in (0,1) : \sigma(\rho) = 0\} > 0,$$

such that $\sigma(\rho_z)$ returns to zero for the first time. Clearly, $\sigma(\rho) > 0$ for all $\rho \in (0, \rho_z)$. This implies $\sigma'(\rho_z) \leq 0$. On the other hand, from the dynamics (3.4a) and $\sigma(\rho_z) = 0$ we have $\sigma'(\rho_z) = \rho_z > 0$. This leads to a contradiction. □

The positivity of σ allows the subcritical regions in Theorems 1.1 and 1.2 to contain initial data ρ_0 that is not monotone decreasing. It is a major indication that the nonlocal slowdown interaction helps to prevent shock formations for a class of non-trivial initial data.

Generally speaking, it is possible that σ can become unbounded. The following Proposition describes the behavior of σ.

Proposition 3.3. *Let σ be the solution of (3.4). Then exactly one of the following statement is true.*

- *σ is well-defined in $[0,1]$.*
- *There exists a $\rho_* \in (0,1]$ such that*

$$\lim_{\rho \to \rho_*-} \sigma(\rho) = +\infty. \tag{3.7}$$

Moreover, we have $\rho_* > \rho_c$.

Proof. Suppose (3.7) does not hold, namely σ is bounded from above in $[0,1]$. Together with Proposition 3.2, we know σ is bounded. Hence, Theorem 3.1 implies the existence and uniqueness of σ in $[0,1]$.

We are left to show that $\rho_* > \rho_c$, namely blowup cannot happen before ρ_c. To this end, we observe that $f''(\rho) \leq 0$ for all $\rho \in [0, \rho_c]$. We can estimate from (3.4a) that

$$\sigma'(\rho) \leq 0 \cdot \sigma(\rho)^2 + M\sigma(\rho) + 1, \quad \text{where} \quad M = \max_{\rho \in [\epsilon, \rho_c]} \frac{f(\rho) + 2\rho f'(\rho)}{\rho f(\rho)} < +\infty,$$

for any $\rho \in [\epsilon, \rho_c]$. This implies the upper bound

$$\sigma(\rho) \leq \left(\sigma(\epsilon) + \frac{1}{M}\right) e^{M\rho}, \quad \forall\ \rho \in [\epsilon, \rho_c].$$

Therefore, the blowup cannot happen when $\rho \leq \rho_c$. □

When the flux f is concave, namely $\rho_c = 1$, the second statement in Proposition 3.3 will not hold. Hence, σ is well-defined in $[0,1]$. When f switches from concave to convex at $\rho_c < 1$, one cannot guarantee that σ will not blow up. However, for the particular f_J in (1.3) of our concern, σ_J is well-defined in $[0,1]$, even if $J > 1$. Moreover, we find the explicit expression of the threshold function σ_J.

Proposition 3.4. *Let $f(\rho) = f_J(\rho) = \rho(1-\rho)^J$ for $J > 0$. Then the trajectory σ_J in (3.4a) can be explicitly expressed by*

$$\sigma_J(\rho) = \frac{\rho(1-\rho)}{J}. \tag{3.8}$$

Proof. We verify that σ_J solves (3.4). For Eq. (3.4a), we plug in f_J and σ_J to the right hand side and get

$$\begin{aligned}&\frac{f_J''(\rho)\sigma_J(\rho)^2 + (f_J(\rho) + 2\rho f_J'(\rho))\sigma_J(\rho) + \rho^2 f_J(\rho)}{\rho f_J(\rho)}\\ &= \frac{J(1-\rho)^{J-2}(-2(1-\rho) + (J-1)\rho) \cdot J^{-2}\rho^2(1-\rho)^2}{\rho^2(1-\rho)^J}\end{aligned}$$

$$+\frac{\rho(1-\rho)^{J-1}(3(1-\rho)-2J\rho)\cdot J^{-1}\rho(1-\rho)}{\rho^2(1-\rho)^J}+\rho$$
$$=\frac{-2(1-\rho)+(J-1)\rho}{J}+\frac{3(1-\rho)-2J\rho}{J}+\rho=\frac{1-2\rho}{J}=\sigma_J'(\rho).$$

For the initial conditions (3.4b), we can verify that $\sigma_J(0)=0$, $\sigma_J'(0)=\frac{1}{J}$ and $\beta_J=-\frac{2f_J'(0)}{f_J''(0)}=\frac{1}{J}$. □

3.2. The threshold function γ

Next, we describe the construction of the other threshold function γ, when the flux switches from concave to convex at $\rho_c<1$. The function γ describes a new type of singularity.

Theorem 3.2. *Let f satisfy the hypotheses in (1.4) with $\rho_c<1$. Then there exists a unique trajectory represented by γ that satisfies the Eq. (3.2), namely*

$$\gamma'(\rho)=\frac{f''(\rho)\gamma(\rho)^2+(f(\rho)+2\rho f'(\rho))\gamma(\rho)+\rho^2f(\rho)}{\rho f(\rho)}, \tag{3.9a}$$

with initial condition

$$\lim_{\rho\to\rho_c+}\gamma(\rho)=-\infty. \tag{3.9b}$$

Proof. Let us first construct γ locally in $(\rho_c,\rho_c+\epsilon)$, for a sufficiently small $\epsilon>0$. γ can be defined via $\eta=\frac{1}{\gamma}$. Indeed, as γ satisfies (3.9a), we must have

$$\eta'(\rho)=-\frac{\gamma'(\rho)}{\gamma(\rho)^2}=\frac{-f''(\rho)-(f(\rho)+2\rho f'(\rho))\eta(\rho)-\rho^2f(\rho)\eta(\rho)^2}{\rho f(\rho)}. \tag{3.10a}$$

Then, Eq. (3.10a) with initial condition

$$\eta(\rho_c)=0 \tag{3.10b}$$

is locally wellposed in $[\rho_c,\rho_c+\epsilon]$. We claim that $\gamma(\rho)=\frac{1}{\eta(\rho)}$ satisfies (3.9) for $\rho\in(\rho_c,\rho_c+\epsilon]$. It suffices to show that $\eta(\rho_c+)<0$. To this end, take Taylor expansion of η around ρ_c

$$\eta(\rho)=\sum_{n=0}^{\infty}\frac{\eta^{(n)}(\rho_c)}{n!}(\rho-\rho_c)^n.$$

The first term of the series is zero due to the initial condition (3.10b). For the second term, observe from the assumption of f in (1.4) that $f''(\rho_c)=0$. Then from (3.10) we get $\eta'(\rho_c)=0$. We continue to calculate the next term

$$\eta''(\rho_c)=-\frac{f'''(\rho_c)}{\rho_cf(\rho_c)}.$$

Since f switches from concave to convex at $\rho=\rho_c$, we have $f'''(\rho_c)\geq 0$. If the strict inequality holds, we have $\eta''(\rho_c)<0$, which yields $\eta(\rho_c+)<0$. If $f'''(\rho_c)=0$, we can continue to the next terms in the Taylor expansion until we have $f^{(n)}(\rho_c)>0$ for some n. Note that such finite n exists, as otherwise f is linear around ρ_c, violating the strict convexity assumption in (1.4). Then

$$\eta(\rho_c)=\eta'(\rho_c)=\cdots=\eta^{(n-2)}(\rho_c)=0,\quad \eta^{(n-1)}(\rho_c)=-\frac{f^{(n)}(\rho_c)}{\rho_cf(\rho_c)}<0,$$

which also leads to $\eta(\rho_c+)<0$. Note that the η defined in (3.10) is unique. Hence, γ can also be uniquely defined in $(\rho_c,\rho_c+\epsilon]$ by $\gamma(\rho)=\frac{1}{\eta(\rho)}$.

Starting from $\rho = \rho_c + \epsilon$, we can consider the dynamics (3.9a) with initial condition $\gamma(\rho_c + \epsilon) = \frac{1}{\eta(\rho_c+\epsilon)}$. From the Cauchy–Lipschitz theorem, γ exists and is unique in $\rho \in [\rho_c + \epsilon, 1)$ as long as γ is bounded.

We now show γ is lower bounded when $\rho \geq \rho_c + \epsilon$. Let us start with an estimate on $f(\rho) + 2\rho f'(\rho)$. Applying convexity of f, we have

$$f(\rho) = f(1) - f'(\rho)(1-\rho) - \frac{f''(\xi)}{2}(1-\rho)^2 \leq -f'(\rho)(1-\rho).$$

Here $\xi \in [\rho, 1]$ and $f''(\xi) \geq 0$. Since $f'(\rho) < 0$ for $\rho \in (\rho_c, 1)$, we obtain

$$f(\rho) + 2\rho f'(\rho) \leq f'(\rho)(3\rho - 1) < 0, \quad \text{if} \quad \rho > \tfrac{1}{3}.$$

Then from (3.9a) it is easy to verify that $\gamma'(\rho) > 0$ as long as $\gamma(\rho) \leq 0$. Therefore, it is not possible that $\gamma(\rho) \to -\infty$ for any $\rho > \max\{\rho_c + \epsilon, \frac{1}{3}\}$. It remains to show that $\gamma(\rho)$ is lower bounded when $\rho \in [\rho_c + \epsilon, \frac{1}{3}]$, in the case $\rho_c + \epsilon < \frac{1}{3}$. From strict convexity in (1.4), we obtain a uniform bound

$$\sup_{\rho \in [\rho_c+\epsilon, \frac{1}{3}]} \frac{f(\rho)}{f''(\rho)} < M,$$

where M depends on ϵ. Now, if $\gamma(\rho) < -M$ we apply (3.9a) and get

$$\gamma'(\rho) = \frac{f''(\rho)\gamma(\rho) + f(\rho)}{\rho f(\rho)}\gamma(\rho) + \frac{2f'(\rho)\gamma(\rho)}{f(\rho)} + \rho > 0,$$

as all three terms above are positive. We conclude with a lower bound of γ

$$\gamma(\rho) \geq \min\{-M, \gamma(\rho_c + \epsilon)\}.$$

For the upper bound, since γ and σ are two trajectories satisfying the same ODE, we have the bound

$$\gamma(\rho) < \sigma(\rho).$$

Therefore, if σ is bounded, so is γ. On the other hand, if σ becomes unbounded as in (3.7), it is possible that γ also becomes unbounded, namely there exists $\rho^* \in (\rho_*, 1)$

$$\lim_{\rho \to \rho^*-} \gamma(\rho) = +\infty. \quad \square \tag{3.11}$$

For the family of fluxes f_J in (1.3), we find an explicit expression to the threshold function γ_J.

Proposition 3.5. *Let $f(\rho) = f_J(\rho) = \rho(1-\rho)^J$ for $J > 1$. Then the trajectory γ_J in (3.4a) can be explicitly expressed by*

$$\gamma_J(\rho) = \frac{\rho^2(1-\rho)\left(\rho - \frac{4J}{(J+1)^2}\right)}{J\left(\rho - \frac{2}{J+1}\right)^2}. \tag{3.12}$$

Proof. First, we calculate

$$f''(\rho) = J\big((J+1)\rho - 2\big)(1-\rho)^{J-2},$$

and hence the inflection point $\rho_c = \frac{2}{J+1}$. Let us denote $\rho_e = \frac{4J}{(J+1)^2}$. Since $\rho_c < \rho_e$, it is easy to verify the condition (3.9b). Now, we verify Eq. (3.9a). Differentiate (3.12) and get

$$\gamma_J'(\rho) = \frac{\rho}{J(\rho - \rho_c)^3} \cdot \Big(-2\rho^3 + (\rho_e + 4\rho_c + 1)\rho^2 - 3(\rho_e + 1)\rho_c\rho + 2\rho_e\rho_c\Big). \tag{3.13}$$

Plug in f_J and γ_J to the right hand side of (3.9a) and get

$$\begin{aligned}
&\frac{f_J''(\rho)\gamma_J(\rho)^2 + (f_J(\rho) + 2\rho f_J'(\rho))\gamma_J(\rho) + \rho^2 f_J(\rho)}{\rho f_J(\rho)} \\
&= J(J+1)(\rho-\rho_c)(1-\rho)^{J-2} \cdot \frac{\rho^4(1-\rho)^2(\rho-\rho_e)^2}{J^2(\rho-\rho_c)^4} \cdot \frac{1}{\rho^2(1-\rho)^J} \\
&\quad + \rho(1-\rho)^{J-1}(3(1-\rho) - 2J\rho) \cdot \frac{\rho^2(1-\rho)(\rho-\rho_e)}{J(\rho-\rho_c)^2} \cdot \frac{1}{\rho^2(1-\rho)^J} + \rho \\
&= \frac{(J+1)\rho^2(\rho-\rho_e)^2 + \rho(3(1-\rho) - 2J\rho)(\rho-\rho_e)(\rho-\rho_c) + J\rho(\rho-\rho_c)^3}{J(\rho-\rho_c)^3} \\
&= \frac{\rho}{J(\rho-\rho_c)^3} \cdot \Big[-2\rho^3 + (\rho_e + (3-J)\rho_c + 3)\rho^2 \\
&\qquad + \big(\rho_e((J+1)\rho_e - 2J\rho_c) - 3(\rho_e\rho_c + \rho_c + \rho_e - J\rho_c^2)\big)\rho + (3\rho_e - J\rho_c^2)\rho_c\Big].
\end{aligned}$$

Using the definitions of ρ_c and ρ_e, we see that the expression matches with (3.13). We conclude that γ_J satisfies (3.9a). □

4. Global behaviors of solutions

In this section, we study the dynamics (3.1) with initial conditions

$$\rho(t=0) = \rho_0 \in [0, \rho_M], \quad d(t=0) = d_0. \tag{4.1}$$

We argue that when (ρ_0, d_0) lie in different regions in the phase plane separated by the threshold functions σ and γ, the global behaviors of the dynamics vary.

Theorem 4.1. *Consider the system (3.1) with initial data (ρ_0, d_0) as in (4.1), and f satisfying the hypotheses in (1.4). Then,*

(a). If (ρ_0, d_0) lies in the type I supercritical region, that is

$$d_0 > \sigma(\rho_0), \tag{4.2}$$

then there exists a finite time t_*, such that

$$\lim_{t\to t_*-} d(t) = +\infty.$$

(b). If (ρ_0, d_0) lies in the type II supercritical region, that is

$$\rho_0 > \rho_c \quad \text{and} \quad d_0 \le \gamma(\rho_0), \tag{4.3}$$

then there exists a finite time t_*, such that

$$\lim_{t\to t_*-} d(t) = -\infty.$$

(c). If (ρ_0, d_0) lies in the subcritical region, meaning neither of the two supercritical regions, that is

$$\rho_0 \le \rho_c \quad \text{and} \quad d_0 \le \sigma(\rho_0), \tag{4.4}$$

or

$$\rho_0 > \rho_c \quad \text{and} \quad \gamma(\rho_0) < d_0 \le \sigma(\rho_0), \tag{4.5}$$

then the solution (ρ, d) exists in all time. Moreover,

$$\lim_{t\to\infty} \rho(t) = 0, \quad \lim_{t\to\infty} d(t) = 0.$$

Remark 4.1. Since σ and γ might not be defined in $[0,1]$ and $(\rho_c,1]$ respectively, in the cases (3.7) and (3.11) when blowups happen, we shall adapt the following conventions in the descriptions of the regions (4.2), (4.3) and (4.5):

$$\sigma(\rho)=+\infty,\quad \forall\ \rho\in[\rho_*,1],\quad \text{and}\quad \gamma(\rho)=+\infty,\quad \forall\ \rho\in[\rho^*,1].$$

For instance, if $\rho_0\geq\rho_*$, (4.2) implies that (d_0,ρ_0) does not lie in the type I supercritical region for any $d_0\in\mathbb{R}$.

Fig. 1 provides an illustration of the three regions. Note that if f is concave, satisfying (1.4) with $\rho_c=1$, the type II supercritical region (4.3) is empty. The region (4.5) is also empty.

Theorems 1.1 and 1.2 follow directly from Theorem 4.1 by collecting all the characteristic paths.

The rest of the section is devoted to the proof of Theorem 4.1. We start with a description of the dynamics of ρ: it decreases in time, and approaches zero as $t\to\infty$.

Lemma 4.1. *Consider (3.1) with initial data (4.1) and f satisfying (1.4). Then $\rho(t)$ is strictly decreasing in time. Moreover, for any $\rho_1\in(0,\rho_0)$, there exists a finite time t_1 such that $\rho(t_1)\leq\rho_1$, unless $d(t)$ blows up before t_1. In particular, if (ρ,d) exists in all time, then*

$$\lim_{t\to\infty}\rho(t)=0.$$

Proof. Apply the bound (2.7) to the ρ-equation in (3.1) and get $\dot\rho\geq-\rho f(\rho)$, which implies $\rho(t)\geq\rho_0e^{-\|f\|_{L^\infty}t}>0$ is strictly positive in all time. On the other hand, we have

$$\dot\rho\leq-e^{-m}\rho f(\rho).$$

Since the right hand side is strictly negative when $\rho\in(0,1)$, we conclude that $\rho(t)$ is strictly decreasing in time.

Moreover, using separation of variables and integrating in $[0,t]$ yield

$$\int_{\rho_0}^{\rho(t)}\frac{d\rho}{\rho f(\rho)}\leq-e^{-m}t. \tag{4.6}$$

Define $G:(0,\rho_0]\to(-\infty,0]$ as follows

$$G(\xi):=\int_{\rho_0}^{\xi}\frac{d\rho}{\rho f(\rho)}.$$

We observe that G is an increasing function.

Now, given any $\rho_1\in(0,\rho_0)$, we can take

$$t_1=-e^mG(\rho_1)=e^m\int_{\rho_1}^{\rho_0}\frac{d\rho}{\rho f(\rho)}<+\infty.$$

Then (4.6) implies $G(\rho(t))\leq G(\rho_1)$, which leads to $\rho(t)\leq\rho_1$. $\square$

Next, we focus on the behaviors of the dynamics of d, which varies in different regions of initial data.

4.1. Blow-up of type I supercritical initial data

For any type I supercritical initial data (4.2), we have the following Lemma on a positive lower bound of d.

Lemma 4.2. *Consider (3.1) with type I supercritical initial data satisfying (4.2) and f satisfying (1.4). Then there exists $c > 0$ such that $d(t) > c$ in the whole lifespan of d.*

Proof. We express the trajectory of the dynamics in the (ρ, d) phase plane by $d = \mathrm{d}(\rho)$, and compare the function d with the threshold function σ. Since $\mathrm{d}(\rho_0) = d_0 > \sigma(\rho_0)$, we have

$$\mathrm{d}(\rho) > \sigma(\rho) > 0, \tag{4.7}$$

for any $\rho \in (0, \rho_0]$ that lies in the domain of d. Here, the second inequality is due to Proposition 3.2. Therefore, we have

$$\inf_t d(t) = \inf_\rho \mathrm{d}(\rho) \geq 0, \tag{4.8}$$

where the equality can only be attained in the case when $\lim_{\rho\to 0} \mathrm{d}(\rho) = 0$, namely that $(0, 0)$ lies on the trajectory. Moreover, from (4.7) we have

$$\mathrm{d}'(0) \geq \sigma'(0) = \beta. \tag{4.9}$$

However, in view of Proposition 3.1, any trajectory with $\mathrm{d}(0) = 0$ must have either $\mathrm{d}'(0) = 0$ or $\mathrm{d}'(0) = \beta$. The former contradicts with (4.9). For the latter case, it has been shown in Theorem 3.1 that σ is the only trajectory that enters $(0, 0)$ with the slope β. Hence, the equality in (4.8) cannot be reached, finishing the proof. □

With the uniform lower bound, we are ready to show that d must blow up in finite time. Let us rewrite the dynamics of d in (3.1) as

$$\dot{d} = -e^{-\widetilde{\rho}}\Big(f''(\rho)d^2 + \big(f(\rho) + 2\rho f'(\rho)\big)d + \rho^2 f(\rho)\Big) = C(\rho)\big(d - \mathrm{d}_-(\rho)\big)\big(d - \mathrm{d}_+(\rho)\big), \tag{4.10}$$

where

$$C(\rho) = -e^{-\widetilde{\rho}} f''(\rho),\ \mathrm{d}_\pm(\rho) = \frac{-\big(f(\rho) + 2\rho f'(\rho)\big) \mp \sqrt{\big(f(\rho) + 2\rho f'(\rho)\big)^2 - 4\rho^2 f(\rho) f''(\rho)}}{2f''(\rho)}.$$

Observe that when $\rho \in (0, \rho_c)$, the coefficient $C(\rho) > 0$. The curves $(\rho, \mathrm{d}_\pm(\rho))$ are the two nullclines of d in the phase plane. We have $\mathrm{d}_-(\rho) < 0 < \mathrm{d}_+(\rho)$, and furthermore

$$\lim_{\rho\to 0+} \mathrm{d}_\pm(\rho) = 0. \tag{4.11}$$

Therefore, when ρ is small enough, the dynamics of d would behave like $\dot{d} \sim C(0)d^2$, which leads to a finite time blowup.

Proof of Theorem 4.1(a). First, from (4.11) we can find a small $\rho_1 > 0$ such that

$$2\mathrm{d}_+(\rho_1) < c,$$

where c is the uniform lower bound of d as in Lemma 4.2.

From Lemma 4.1, there exists a finite time t_1 such that $\rho(t_1) = \rho_1$, unless d already blows up before t_1.

We focus on the dynamics (4.10) starting from t_1. From the hypotheses of f in (1.4), we can obtain a uniform lower bound on $C(\rho)$ as

$$C(\rho) \geq e^{-m} \cdot \min_{\rho\in[0,\rho_1]} \big(-f''(\rho)\big) =: C > 0, \quad \forall\, \rho \in [0, \rho_1].$$

Since $\rho(t) \in (0, \rho_1]$ for any $t \geq t_1$, we deduce from (4.10) that

$$\dot{d} \geq C\big(d - \mathrm{d}_-(\rho)\big)\big(d - \mathrm{d}_+(\rho)\big) \geq C\big(d - \mathrm{d}_+(\rho)\big)^2 \geq C\big(d - \mathrm{d}_+(\rho_1)\big)^2,$$

with initial condition at t_1 satisfying

$$d(t_1) > c > 2\mathrm{d}_+(\rho_1),$$

where we have used Lemma 4.2. Solving the initial value problem would yield

$$d(t) > \mathrm{d}_+(\rho_1) + \frac{\mathrm{d}_+(\rho_1)}{1 - C\mathrm{d}_+(\rho_1)(t - t_1)}.$$

Therefore, we must have

$$\lim_{t \to t_* -} d(t) = +\infty$$

at a finite time

$$t_* \leq t_1 + \frac{1}{C\mathrm{d}_+(\rho_1)}. \quad \square$$

4.2. Blow-up of type II supercritical initial data

For type II supercritical initial data (4.3), the blowup is a direct consequence of a comparison with the threshold function γ.

Proof of Theorem 4.1(b). Compare the trajectory $d = \mathrm{d}(\rho)$ with the threshold function γ. Since $\mathrm{d}(\rho_0) = d_0 \leq \gamma(\rho_0)$, we have

$$\mathrm{d}(\rho) \leq \gamma(\rho),$$

for any $\rho \in (\rho_c, \rho_0]$ that lies in the domain of d. Since $\lim_{\rho \to \rho_c +} \gamma(\rho) = -\infty$, d must blow up to $-\infty$ at some $\rho_1 \geq \rho_c$. Apply Lemma 4.1, $d(t)$ must blow up to at a finite time t_1. $\square$

4.3. Global regularity of subcritical initial data

Let us first consider initial data that lie in the region (4.4). The main idea is that the coefficient $C(\rho)$ in the dynamics (4.10) has a favorable sign that prevents d from going to $-\infty$. On the other hand, d is controlled from above by σ.

Proof of Theorem 4.1(c) for region (4.4). We start with obtaining a lower bound on d. First, suppose $\rho_0 < \rho_c$. In view of the definition of d_-, it has a uniform lower bound on $[0, \rho_0]$

$$\min_{\rho \in [0, \rho_0]} \mathrm{d}_-(\rho) \geq \underline{d} > -\infty,$$

where $\underline{d}$ depends on f and ρ_0. Since $C(\rho) > 0$ for $\rho \in [0, \rho_0]$, (4.10) implies $\dot{d} > 0$ as long as $d < \underline{d}$. This implies the lower bound

$$d \geq \min\{d_0, \underline{d}\}$$

in the whole timespan of d.

For $\rho_0 = \rho_c$, it is easy to verify that the dynamics (4.10) is locally well-posed. Hence, there exists a time $t_1 > 0$ such that solution $d(t_1)$ exists. From the monotonicity of ρ in time, we know $\rho(t_1) < \rho_c$. Hence the dynamics starting from t_1 reduces to the prior case, leading to a lower bound of d.

To obtain an upper bound of d, note that $\mathrm{d}(\rho_0) = d_0 \leq \sigma(\rho_0)$. We compare the trajectory d with σ and get

$$\mathrm{d}(\rho) \leq \sigma(\rho). \tag{4.12}$$

From Proposition 3.3, we know σ is bounded in $[0, \rho_c]$. Since $\rho(t) \le \rho_0$ for all time, we conclude that

$$d(t) \le \max_{\rho\in[0,\rho_0]} \mathrm{d}(\rho) \le \max_{\rho\in[0,\rho_c]} \sigma(\rho) < +\infty. \quad \square$$

For initial data that lie in the region (4.5), note that σ is not necessarily defined for large ρ. Instead, we may bound d from above using the dynamics (4.10). When $\rho \in (\rho_c, 1)$, the coefficient $C(\rho) < 0$. It is possible that the quadratic form in (4.10) has no real roots, and $\mathrm{d}_\pm$ do not exist. In which case we have $\dot{d} < 0$. If $\mathrm{d}_\pm$ exist, we still have $\dot{d} < 0$ when $d > \mathrm{d}_- \ge \mathrm{d}_+$. Hence, d cannot blow up to $+\infty$. The lower bound can be controlled by γ.

Proof of Theorem 4.1(c) for region (4.5). For the upper bound, we apply Proposition 3.3. If σ is well-defined in $[0, \rho_0]$, then d is bounded by σ by comparison (4.12). If σ blows up at $\rho_* \in (\rho_c, \rho_0)$, we observe that d_-, if exists, has a uniform upper bound on $[\rho_*, \rho_0]$

$$\max_{\rho\in[\rho_*,\rho_0]\cap\mathrm{Dom(d)}} \mathrm{d}_-(\rho) \le \overline{d} < +\infty,$$

where $\overline{d}$ depends on f, ρ_* and ρ_0. Since $C(\rho) < 0$ for $\rho \in [\rho_*, \rho_0]$, (4.10) implies $\dot{d} < 0$ as long as $d > \overline{d}$. This leads to the upper bound

$$d \le \max\{d_0, \overline{d}\}.$$

Once $\rho(t)$ drops below ρ_*, it can be controlled by σ.

For the lower bound, can compare the trajectory d with the threshold function γ. Since $\mathrm{d}(\rho_0) = d_0 > \gamma(\rho_0)$, we have

$$\mathrm{d}(\rho) > \gamma(\rho), \quad \forall\ \rho \in [\rho_c, \rho_0].$$

Hence, d has a lower bound as long as $\rho > \rho_c$. Moreover, $\mathrm{d}(\rho_c)$ is bounded. This is because if $\lim_{\rho\to\rho_c+} \mathrm{d}(\rho) = -\infty$, we deduce from Theorem 3.2 that $\mathrm{d} = \gamma$, violating the initial condition (4.5). Therefore, by Lemma 4.1, there exists a time t_1 such that $\rho(t_1) = \rho_c$ and $d(t_1) = \mathrm{d}(\rho_c)$ is bounded. The dynamics enter the region (4.4) at t_1. The global behavior follows from the proof for the region (4.4). $\square$

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