https://doi.org/10.1038/s41567-022-01691-z

nature physics

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Physics of highly multimode nonlinear optical systems

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Linear multimode optical systems have enabled clean experimental observations and the applications of numerous phenomena that continually extend the boundaries of wave physics. The infrastructure that has enabled these studies facilitates the study of an even richer world of nonlinear multimode optical systems. Multimode nonlinear optical physics is full of emergent phenomena, including robust spatial attractors, multimode wave instabilities, and conservative and dissipative multimode solitons. Many of these effects push the limits of existing theoretical techniques, demanding new insights and approaches that could emerge from other fields, such as statistical mechanics, physics-informed machine learning, network science and beyond. Here we provide an overview of recent investigations of wave propagation in highly multimode nonlinear systems, principally multimode fibre waveguides and laser cavities. These systems, with their multifaceted control, low cost, scalability and ultra-high bandwidth, are ideal physical platforms for exploring—and ultimately applying—high-dimensional nonlinear physics, from orderly but elusive objects like spatiotemporal solitons to dynamical complexity itself, both near and far from equilibrium.

Optical systems offer several attractive features for experimental studies of exotic physical phenomena. First, they can be precisely and diversely engineered, and their characteristic scale (approximately micrometre) makes it possible for even complex optical systems to be manufactured. Second, the separation of characteristic frequencies allows optical phenomena to be isolated, so experiments can be accurately described by theoretical models that neglect lower-frequency thermal, electromagnetic or mechanical excitations. A partial list of phenomena for which linear optical systems have uniquely enabled clean observations include localization and coherent transport in disordered media[1], PT-symmetric physics[2], as well as a variety of topological processes[3]. These endeavours are enabled by the design of multimode structures and linear mode-coupling, which is analogous to designing potentials and perturbations for the Schrödinger equation (Box 1), but including both conservative (Hermitian) and dissipative (non-Hermitian) effects.

Techniques to control the modes, dispersion and mode-coupling in fabricated structures are available both in the form of design paradigms (such as photonic crystals and graded-index structuring) and platforms (such as free-space cavities, fibre optics and integrated nanophotonics), which usually benefit from the low-cost, high-quality components available for imaging or telecommunications. Modes and mode-coupling properties may also be controlled in a reconfigurable manner, for example, by the use of spatial light modulators (SLMs) and electro- and acousto-optic modulators, such that experiments can today easily achieve millions of adjustable degrees of freedom. In short, experiments with multimode linear optical systems are limited mainly by imagination, and have therefore been intensively, if not yet exhaustively, explored.

In contrast, the physics of multimode nonlinear optical (MMNLO) systems remains comparatively unexplored, even though experiments can be constructed from the same scaffold as linear optical systems, and therefore enjoy most of the same advantages. Notable exceptions to this statement include soliton mode-locking in lasers[4] and microresonators[5], lasing in multimode resonators (both ordered and disordered) and many forms of quantum optical-state generation. Considering the substantial and far-reaching impacts these excursions into highly MMNLO physics have had, it is natural to ask questions such as 'What has limited the study of MMNLO systems to date?' and 'What is known, and what remains unresolved?'.

What has limited the broader study of MMNLO systems? Perhaps the most important factor is dimensionality. Techniques suitable for theoretical analysis—analytical, numerical and even conceptual reasoning frameworks—that apply to linear or to low-dimensional nonlinear systems generally do not scale well to highly MMNLO systems. Experimentally, a similar challenge exists. MMNLO systems feature so many meaningful dimensions to control and observe that traditional experimental tools often prove insufficient. Low-dimensional measurements frequently fail to directly resolve the signatures of multimode nonlinear phenomena, for example, by averaging over too many dimensions. For the semiclassical wave physics that is the focus of this Review, the weakness of optical nonlinearities is rarely a limitation. By combining the intensity of modern pulsed lasers with low-loss propagation in optical resonators or fibre waveguides, and/or by using laser gain to compensate losses, highly multimode nonlinear wave propagation can easily be observed, with essentially no limit to evolution time or distance.

What is known about the physics of MMNLO systems, and what remains stubbornly unresolved? Early studies of lasing in the 1960s necessarily considered the multimode nonlinear physics of interacting lasing modes, a pursuit that led to the understanding of mode competition and the discovery of mode-locking[6]. Early work involving optical fibres in the 1970s and 1980s followed a similar trajectory[7–9]. However, in both domains, research soon focused on single-mode systems, in which disorder could be neglected, and for which vastly simpler, one-dimensional (1D) models could be applied.

MMNLO wave propagation re-emerged as a prominent research topic through studies of light propagation in disordered media, mainly between 2003 and 2013. In complete analogy with electrons, depending on the degree of randomness in the medium, light transport can range from ballistic to diffusive[10], or even undergo Anderson localization[1,11–13]. Unlike diffusion, which is possible even under dynamic scattering conditions, because it is an interference effect Anderson localization requires the random potential itself to remain invariant during propagation. Disordered multimode optical structures provide ideal testbeds[14–18] to explore wave localization

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Box 1 | Introduction to spatiotemporal nonlinear wave propagation in multimode optical systems

Light propagation in a multimode waveguide or resonator (see the figure in this Box) can be understood through a formalism mathematically analogous to the Schrödinger wave formulation of quantum mechanics, considering the classical envelope of the electric field A rather than the wavefunction. In this formalism, pulse propagation is described by a generalized nonlinear Schrödinger equation:

$$\partial_z A = \hat{D}(k_x, k_y, \omega)A + \hat{W}(x, y, \omega)A + \hat{N}(x, y, t)A. \quad (1)$$

In equation (1), each term affects the electric-field envelope A in the indicated domains (space/wavevector, time/frequency) of the operator. The linear dispersion/diffraction operator $\hat{D}$, which acts on A in frequency and wavevector space, is

$$\hat{D}(k_x, k_x, \omega) = i\left[\sqrt{\beta_{\mathrm{eff}}^2(\omega) - k_x^2 - k_y^2} - \beta_{\mathrm{eff}}(\omega_0) - (\omega - \omega_0)/v_{\mathrm{ref}}\right], \quad (2)$$

where β_{eff} is the wavenumber and defines a phase velocity reference, $k_{x,y}$ are transverse wavevector components, and ω_0 is the centre radial frequency. Time coordinate of equation (1) is in a moving reference frame, with a group velocity v_{ref} chosen for convenience. The waveguide operator $\hat{W}$ defines the space- and frequency-dependent linear potential:

$$\hat{W}(x, y, \omega, z) = i\frac{\beta_{\mathrm{eff}}(\omega)}{2}\left[\left(\frac{n(x, y, \omega, z)}{n_{\mathrm{eff}}(\omega)}\right)^2 - 1\right], \quad (3)$$

where n is the refractive index and n_{eff} is the refractive index at the centre of the waveguide. Finally, the nonlinear operator $\hat{N}$, where

$$\hat{N}(x, y, t) = i\gamma|A(x, y, t, z)|^2, \quad (4)$$

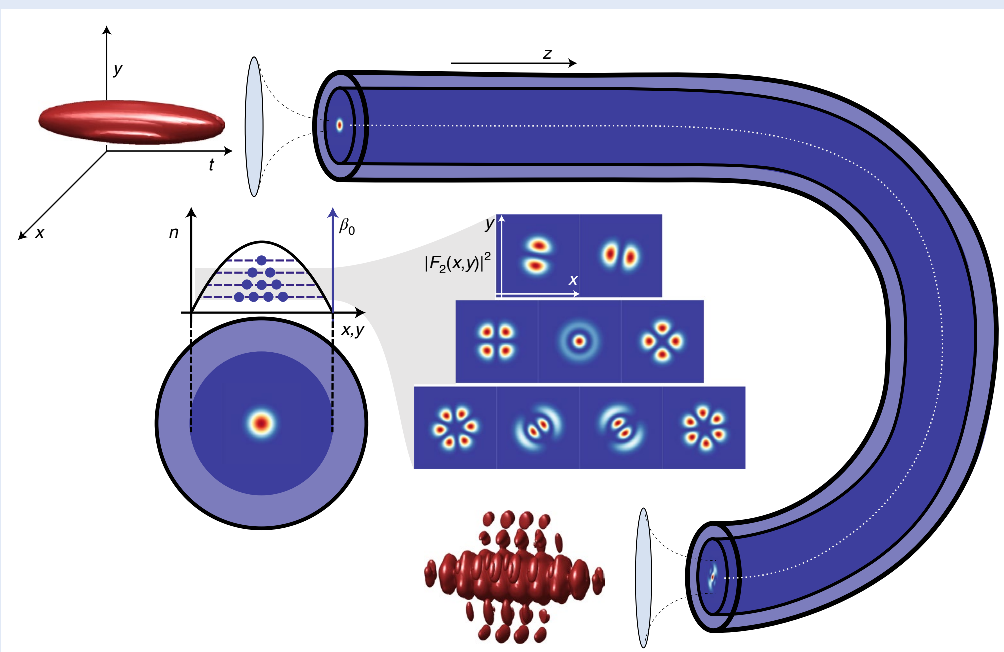
acts as a self-induced spatiotemporal potential, which may, similar to $\hat{W}$, confine and steer the propagating wave(s).

Equation (1) is qualitatively similar to the Schrödinger equation, $\partial_t\psi = i\nabla^2\psi - iV\psi$, and specifically to the Gross–Pitaevskii equation used to describe trapped Bose–Einstein condensates. In suitable units, equation (1)'s first term can be approximated as $i\nabla^2 A$, and the second and third terms can be approximated together as a nonlinear potential term, $iV(x, y, z, A)A$. A key difference is that equation (1)'s evolution coordinate is not time t but rather z, the distance along the propagation axis of the waveguide or cavity.

In the optical systems considered here, light is confined more strongly in the transverse dimensions (x, y) than in other dimensions. When this is true, we can perform a change-of-basis operation on equation (1), expressing $A(x, y, t, z)$ at any point in space–time by a linear complex combination of transverse modes[32,33,37]:

$$A(x, y, t, z) = \sum_{m=1}^{M} a_m(t, z)F_m(x, y). \quad (5)$$

In equation (5), the transverse modes $F_m(x,y)$ are eigenfunctions of equation (1) without the final nonlinear term, for a monochromatic field at ω_0 and assuming longitudinally invariant $n(x, y, \omega_0)$. Changing the basis from Cartesian coordinates



A pulsed beam described by an envelope $A(x, y, t, z = 0)$ is coupled into a multimode waveguide. Subsequent evolution along z of this waveform is described by equation (1), in (x, y, t) coordinates, or equation (6), in mode space. The structure of the multimode waveguide influences the physics by modifying all terms in equation (6), but the effects on $\beta_0^{(m)}$ and $\beta_1^{(m)}$ are most pronounced. In parabolic graded-index (GRIN) fibres, the refractive index profile is a cylindrically symmetric parabolic potential. This potential's modes form degenerate groups whose eigenvalues are equally spaced. $\beta_1^{(m)}$ are also closely clustered in GRIN fibres; in other words, the modal dispersion is small. By contrast, other designs generally exhibit less symmetric modal structures, with more varied $\beta_1^{(m)}$. The figure shows the intensity distribution $|F_m(x, y)|^2$ of modes 1 to 10.

Box 1 | Introduction to spatiotemporal nonlinear wave propagation in multimode optical systems (Continued)

(that is, real space) into mode space by applying equation (2) to equation (1), one obtains the multimode nonlinear Schrödinger equations (MMNLSE):

$$\partial_z a_m(t,z) = i\beta_0^{(m)} a_m - \beta_1^{(m)}\partial_t a_m - \frac{i\beta_2^{(m)}}{2}\partial_{tt} a_m + \sum_{j=1}^{M} L_{mj}(z) a_j + i\gamma \sum_{j=1}^{M}\sum_{k=1}^{M}\sum_{l=1}^{M} \Gamma_{mjkl} a_j a_k^* a_l \quad (6)$$

Equation (6) is a system of coupled 1D equations for waves within each of the waveguide's transverse modes. The first and second terms on the right side describe the phase and group velocity, and the third describes group velocity dispersion. The second-last term can be used to describe linear coupling between modes due to weak, possibly z-dependent, perturbations to the fixed linear potential, $n(x, y, \omega_0)$, such as disordered manufacturing imperfections. In multimode nonlinear optical systems, the last term, which describes the tensorial coupling between modes by the Kerr nonlinearity, is typically the most important. The terms in this sum include self-phase modulation terms, $\sim i|a_m(t,z)|^2 a_m(t,z)$, which act as self-induced potentials for pulses in each mode, cross-phase modulation terms, $\sim i|a_j(t,z)|^2 a_m(t,z)$, which allow pulses in different modes to steer or confine light in other modes, and four-wave mixing terms, $\sim i a_j a_k^* a_l$, which allow energy to be exchanged between modes. The description here is simplified, but representative. Other effects, such as Raman scattering or saturable gain or loss, can be incorporated by similar techniques.

phenomena in both the classical and quantum domains[19–21]. Many studies have focused on how nonlinear processes affect Anderson localization in random systems. In general, for strong linear disorder, the extended Bloch modes of a lattice morph into highly localized states[22]. Meanwhile, nonlinearity itself (for example, the Kerr nonlinearity) can also underlie confined optical entities known as discrete solitons or breathers[23]. The question naturally arises as to whether these two different processes can be synergistic or antagonistic. To address this, experiments have been carried out in disordered photonic lattices, where Anderson localization effects appear to be enhanced by self-focusing nonlinearities[19,20,24]. However, at this point, it is not clear if asymptotically (in infinite lattices) the interplay between nonlinearity and disorder can eventually spoil or enhance Anderson localization[25], nor if a concise, general answer is even possible.

Today, the study of MMNLO wave propagation has been revitalized. This new research phase has been enabled and shaped by developments that have matured since 2010. New theoretical concepts and experimental methods, driven by applications in imaging and telecommunications, have enabled advances in understanding and controlling multimode waves. These include principal modes[26,27] and related concepts[28,29], wavefront shaping with SLMs, and mode-multiplexing techniques, to mention just a few. These are treated in other articles in this issue[30,31]. Meanwhile, relatively efficient techniques for simulating the propagation of intense pulses in multimode waveguides have emerged[32,33]. Applying these and other techniques has led to a research trend that has—so far—emphasized the role of nonlinearity relative to disorder.

In this Review we will overview progress mainly within this recent wave of activity and highlight open questions and challenges that may finally be resolved with the benefit of nascent technology and techniques. Our aim here is a high-level overview; readers interested in exhaustive reviews of recent work may consult refs. [34–36], whereas readers who desire more detailed tutorials may refer to refs. [34,37–39]. Recent work has primarily utilized multimode optical fibres and fibre resonators, so we will focus on these settings before discussing progress and prospects in other platforms.

Spatial self-organization

When light propagates linearly in a highly multimode waveguide, the complex interference of many modes generally leads to a disordered pattern (Fig. 1a); one of the most unexpected observations of recent experiments has been that, with nonlinearity present, a completely different outcome occurs: the field 'self-cleans' into a bell-shaped beam (Fig. 1b)[40]. Experiments by several groups soon verified that, across a range of conditions[41,42], the optical power in a GRIN multimode fibre irreversibly migrates toward lower-order modes even at modest levels (Fig. 1). These observations incited a debate as to what is behind the self-cleaning mechanism. Clearly, the power was not enough to induce nonlinear self-focusing effects, and beam self-cleaning manifested itself well before any Raman lines appeared in the spectrum. Instead, it seems that this effect stems from the conservative component of the Kerr nonlinearity that allows the waveguide modes to exchange power via four-wave mixing[40,43]. Although nonlinear attractors that result in 'cleaned' beams with reduced disorder are known, such as in Raman beam cleaning[44] or self-cleaning in fibres with heterogeneous dissipation[45], dissipation is a fundamental part of these processes. The experiments by Krupa et al.[40] and others instead presented a perplexing mystery—robust, attractor-like behaviour, including the possibility of disorder reduction, but without dissipation.

Subsequent investigations resolved this mystery. Systematic experiments showed that, although energy flows on average towards

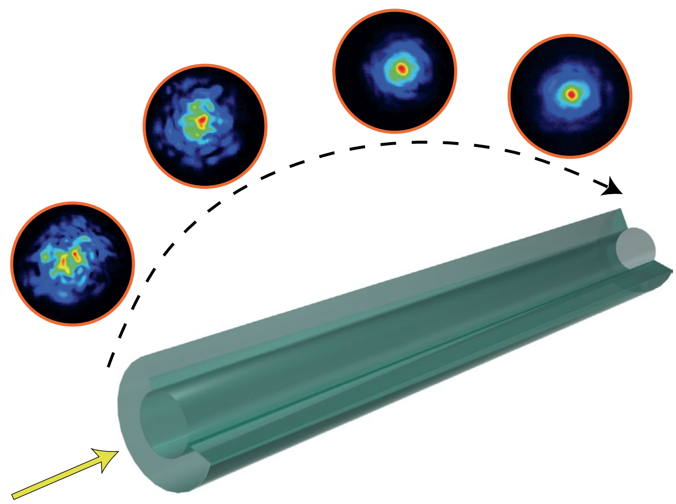


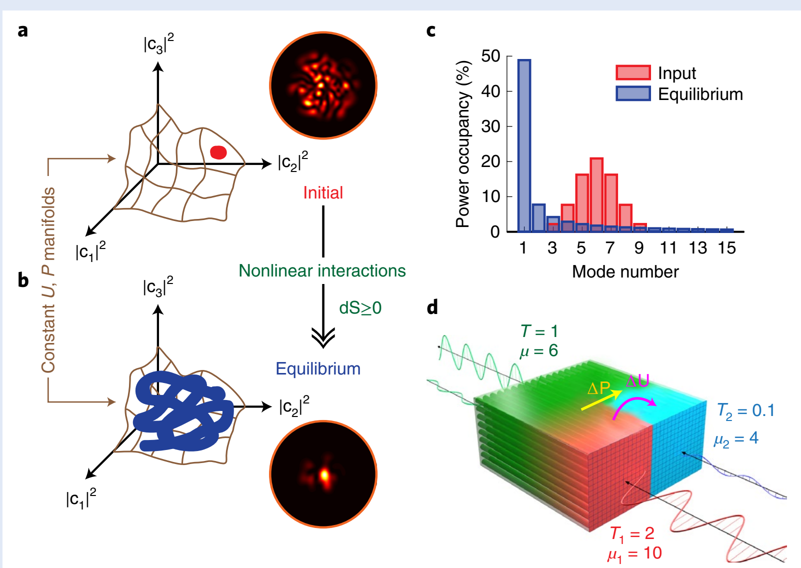
Fig. 1 | Beam self-cleaning in GRIN multimode fibre. Low-intensity light that propagates linearly in a multimode waveguide typically displays a speckled, multi-peaked beam due to the complex interference of many modes. However, recently, many experiments have observed that, when intense light is launched into a GRIN multimode waveguide, the beam profile (circles) gradually evolves towards a 'clean' bell-shaped beam[40–42,45–48,54,56,167]. Figure adapted with permission from ref. [40], Springer Nature Limited.

Box 2 | Introduction to optical thermodynamics

Following the tenets of statistical mechanics, in building an optical thermodynamic theory, one has to first identify the invariants associated with these particular systems. In highly transparent (lossless) optical waveguides, two conservation laws can be ascertained. These are (1) the total optical power P propagating in the guide channel and (2) the total electrodynamic Minkowski momentum density U (the internal energy of the system) flowing in the waveguide[168]. Against the backdrop of these two constants, we next invoke the ergodic hypothesis, which asserts that the nonlinear multimode system will explore, in a fair manner (with the same probability), all its accessible microstates in its phase space that lie on the constant energy (U) and power (P) manifolds[169]. In this regard, if a nonlinear multimode optical system has been initially positioned somewhere in its phase space (see panel a in the figure in this Box), it is expected to evolve along a chaotic path (see panel b in the figure in this Box), and in doing so it will increase its entropy. At this point, the question is 'Where will the system macroscopically relax?' In particular, how will the optical power be redistributed within the various modes (say M modes) in such a nonlinear multimode optical configuration? To address this question, we invoke the second law of thermodynamics, which implies that, at equilibrium, the entropy S of the system must be maximized under the constraints imposed by the two conservation laws, $P=\sum_{i=1}^{M}|c_i|^2$ and $U=-\sum_{i=1}^{M}\beta_0^{(i)}|c_i|^2$, where the modal occupancy $|c_i|^2$ represents the optical power conveyed in the ith mode, and $\beta_0^{(i)}$ denotes the associated propagation constant (eigenvalue). With these assumptions in mind, one finds that the mode power distribution is governed by Bose–Einstein statistics. Given that, and if the number of 'photons' is much larger than the number of modes M, one can then show that, at equilibrium, the power allocation $|c_i|^2$ obeys the celebrated Rayleigh–Jeans law[51,55,170,171]. The entropy S associated with this distribution is also given as follows:

$$|c_i|^2=\frac{-T}{\beta_i+\mu};\ S=\sum_{i=1}^{M}\ln|c_i|^2, \tag{7}$$

where the quantities T and μ appearing in equation (4) represent, respectively, the temperature and chemical potential of this optical system at equilibrium. In a way analogous to the standard thermodynamics of an ideal classical gas, we can also derive the universal equation of state $U-\mu P=MT$ (ref. [51]). From this latter equation, one can uniquely determine the final T and μ, once the initial conditions are known[172]. In other words, the collective dynamics of a complex nonlinear multimode optical system are fully predictable (in a statistical manner after an ensemble average[52]) by using the formalism of optical thermodynamics (see panel c in the figure in this Box). More importantly, one can formally show that T and μ act as actual thermodynamic forces that govern the energy (momentum) and power exchange among subsystems (ΔU, ΔP), in full accord with the second law of thermodynamics[51,52] (see panel d in the figure in this Box). It is worth emphasizing that the optical temperature T is not measured in kelvins and has nothing to do with the actual temperature of the environment in which the optical system is embedded; rather, it is a measure of the disorder of the optical field. To better illustrate these concepts, panel e in the figure in this Box provides a correspondence between variables and relations for classical (standard) statistical mechanics and the optical thermodynamic theory outlined here.



e

Optical thermodynamics	Standard thermodynamics
Power P	Number of particles N
EM momentum U	Energy E
Number of modes M	Volume V
Equation of state $U-\mu P=MT$	Equation of state $PV=Nk_BT$
Interactions: optical nonlinearity	Interactions: molecular forces

a,**b**, Phase space of a nonlinear multimode optical system in the intensity domain with only the first three axes shown here for illustration purposes. Due to nonlinear modal interactions, starting from the point indicated (**a**, red spot), the system evolves chaotically (**b**, blue path) on the constant U, P manifolds (brown surface), to maximize its entropy S. **c**, In equilibrium, the modal occupancies settle into a Rayleigh-Jeans distribution that can be uniquely predicted using only the U and P invariants, regardless of the specific modes that are initially excited. **d**, When two optical systems interact through nonlinearity, energy always flows from a hot subsystem (with higher optical temperature T) to the colder one. Meanwhile, the subsystem with the higher chemical potential always supplies power to the other, until their temperatures and chemical potentials equalize. **e**, Fundamental variables and relations in optical thermodynamics and their counterparts in standard statistical mechanics. Panel **d** adapted with permission from ref. [51], Springer Nature Limited.

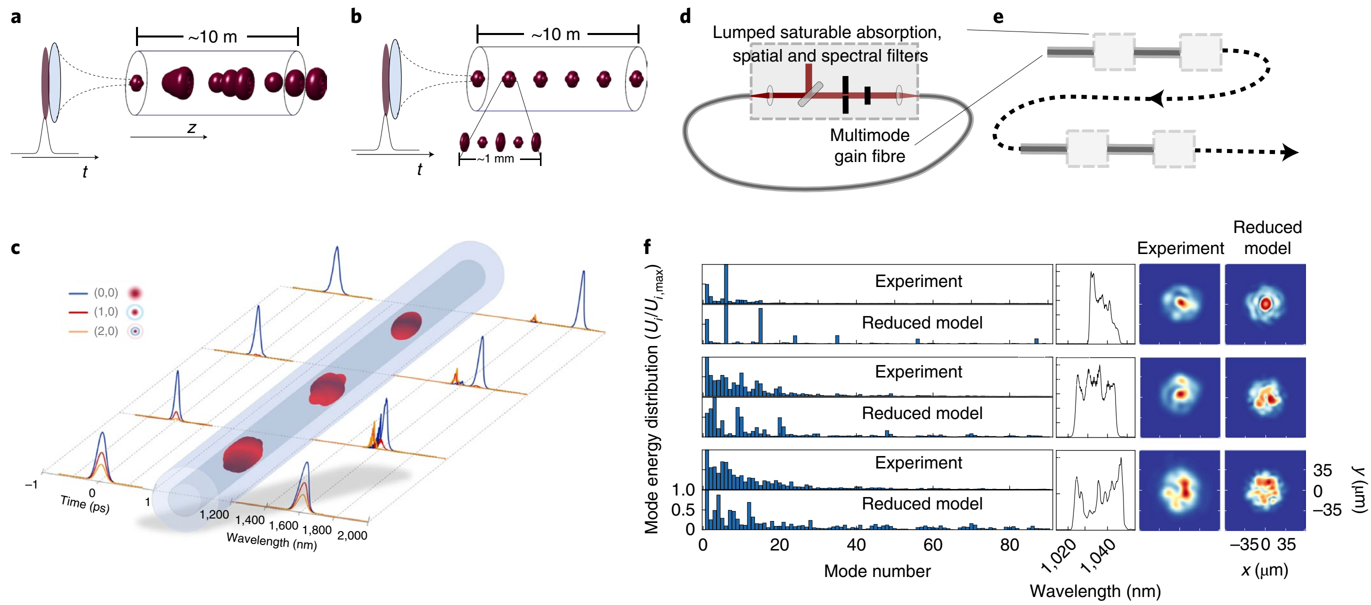


Fig. 2 | Multimode solitons and spatiotemporal dynamics in multimode waveguides and cavities. a, With low optical intensity, pulses launched into a multimode waveguide disperse due to group velocity dispersion within each mode, and pulses within each mode walk off from one another due to modal dispersion. **b**, In a multimode soliton, pulses within multiple transverse modes synchronize in time to form a non-dispersing wavepacket[62,63,66,67,69,72]. In the multimode solitons observed so far, periodic breathing due to modal interference is still present. **c**, Over long distances, multimode solitons in passive fibre decay, with energy becoming localized in the fundamental mode[69,72]. **d**,**e**, A multimode laser cavity consists of a gain medium (here, a multimode fibre doped with rare-earth media and optically pumped by a diode laser), as well as other linear and nonlinear elements used to shape the periodic evolution of light within the cavity (**d**), which can be thought of as an infinite periodic medium (**e**). **f**, In highly multimode laser cavities, a variety of dissipative multimode solitons can be observed[79–83], a few of which are shown. Left, modal composition, with U_i the energy in mode number i; middle, optical spectrum; right, output beam profiles. Panel **d** adapted with permission from ref. [80], Springer Nature Limited. Panels reproduced with permission from: **c**, ref. [69], under a Creative Commons licence CC BY 4.0; **f**, ref. [80], Springer Nature Limited.

the fundamental mode and, as a result, the beam looks cleaner, the output beam quality does not actually improve as the power increases[46,47]. Other studies also demonstrated that a correlation exists between dissipation or disorder effects and the rate at which beam self-cleaning can occur[45,48,49].

Theoretical understanding of this phenomenon has meanwhile required the development of new strategies that can globally and universally describe the dynamics that unfold in dissipationless highly multimode nonlinear systems. Even though global or multimode nonlinear wave solvers[32,50] (Box 1) can be used to simulate these phenomena, these calculations are time-consuming and yield limited physical understanding.

To this end, an optical thermodynamic theory was put forward that can self-consistently describe, by means of statistical mechanics, the complex processes of energy/power exchange in multimode systems under weak nonlinear conditions[51,52] (Box 2). These results are universal to any conservative multimode optical configuration. In this theory, the multimode wave is a kind of dilute photon gas whose temperature has nothing to do with the actual thermal environment in which the optical multimode arrangement is embedded. Instead, it is inherent to the optical system itself, given that photons are nonlinearly reallocated among modes in a probabilistic fashion without involving phonons as in traditional thermodynamics. Intuitively, one may say that the optical temperature characterizes the randomness of an optical beam, with higher temperatures corresponding to more speckled intensity patterns.

This approach reveals that optical thermalization plays an important role in understanding beam self-cleaning effects. Recent experiments suggest that in such a scenario the power among modes is redistributed based on the Rayleigh–Jeans law that favours the lower-order modes under positive temperature. This outcome is distinct from earlier predictions that viewed beam cleaning as a condensation phenomenon[53]. These two perspectives have recently been reconciled by a combination of statistical mechanics and systematic experiments, revealing the relationship between descriptions of beam cleaning as either a thermalization or condensation process[54].

Beyond beam cleaning, the optical thermodynamic theory should support a variety of devices designed to manipulate highly multimode waves. As initial examples, the laws that govern optical isentropic processes were obtained and the prospect of Carnot cycles was also proposed[51].

A central concept behind statistical mechanics is ergodicity, by means of which a system explores its phase space in a fair manner[55]. In general, ergodic behaviour in nonlinear multimode systems is enabled by chaotic nonlinear power exchanges among the various modes. Whereas the unequally spaced eigenvalues in step-index fibres promote chaotic and ergodic dynamics, this is more than offset by the resulting poor phase matching of four-wave mixing compared to that in GRIN fibres. Clearly, in situations where ergodicity is suppressed, such as in strongly nonlinear configurations where solitons form[56], the applicability of such statistical formulations will be limited. Although the theory's predictions are accurate even in real experiments with weak loss and finite propagation length, the extent to which non-equilibrium statistical methods can be employed in arrangements where dissipation or gain is substantial[57,58] is currently a point of debate.

Spatiotemporal self-organization

Solitons. *Solitons in passive fibres.* Solitons are particle-like localized waves that do not disperse because of a balance between linear and nonlinear processes. Temporal solitons in single-mode optical

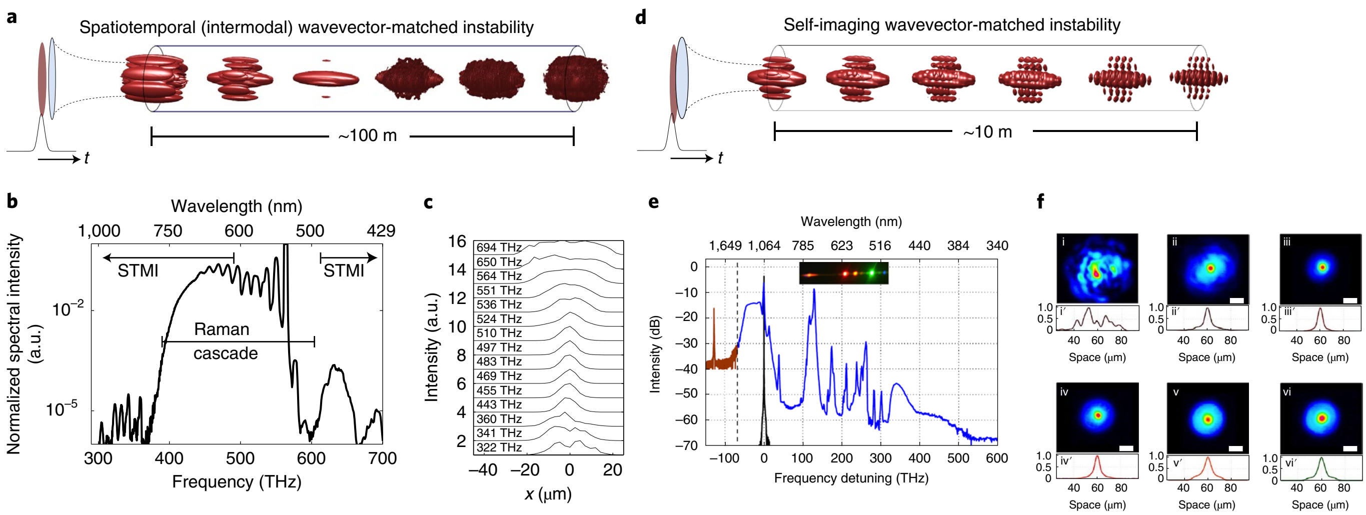


Fig. 3 | Spatiotemporal instabilities of nonlinear waves in multimode waveguides. a–**c**, Spatiotemporal, or intermodal, wavevector matching[7,43,45,84–86,89–91,95,96] can give rise to sidebands with distinct spatial profiles from the pump (**a**). The figure[45] shows a multimode beam at 532 nm, which is first attracted to the fundamental mode, and then experiences STMI, resulting in spectral sidebands, shown in **b** and **c**. The large, asymmetric sideband in **b** from ~400 to 600 THz originates in part from cascaded Raman scattering, rather than only STMI. **d**, Coupled with the Kerr nonlinearity, periodic breathing in parabolic GRIN fibre (Fig. 2a) facilitates energy transfer between sideband waves of different frequencies but with similar modal composition[87,93]. **e**, When intense pulses are launched into a GRIN multimode fibre, this leads to a series of distinct spectral peaks[93]. **f**, The beam profile of the beam exiting the fibre (i, i′) at low power, (ii, ii′) at high intensity at the pump wavelength of 1,064 nm, and (iii–vi, iii′–vi′) at the first four anti-Stokes sidebands 750, 650, 600 and 550 nm, which are similar in shape to the pump (ii, ii′)[93]. Figure adapted with permission from: **b**,**c**, ref. [45], Springer Nature Limited; **e**,**f**, ref. [93], APS.

fibres (SMFs) are remarkable for their ability to propagate robustly over a million kilometres[59]. Just as important, solitons often act like eigenmodes in nonlinear systems; complex fields can be decomposed into soliton and radiation components[60]. Many complex phenomena in SMF or 1D systems, such as continuum generation, rogue waves, and mode-locking in lasers, can be understood in terms of soliton dynamics. In optics, stable 2D spatial solitons can form only in materials with particular nonlinear properties[61], and 3D or spatiotemporal solitons have not been observed in homogeneous media. Higher-dimensional soliton dynamics is thus a largely unexplored frontier.

Solitons in multimode waveguides combine the multidimensional freedom of higher-dimensional spatial and spatiotemporal solitons with the rich, long-lived dynamics of 1D fibre solitons. The first systematic experimental studies of solitons in multimode fibre[62–67] observed a surprisingly wide variety of distinct soliton-like waves. In these multimode solitons (here, we use the 'soliton' terminology loosely), the self-focusing nonlinear potential due to each pulse causes pulses in different modes to mutually trap one another in time, forming a kind of bound state (Fig. 2b). This mutual self-focusing results in frequency shifts of each mode such that at steady state the multimode wavepacket propagates without dispersing. Because the different spatial modes within the wavepacket have different propagation constants, a multimode soliton undergoes rapid spatiotemporal oscillations as its constituent modes interfere (Fig. 2b).

A high-intensity pulse launched into a multimode fibre undergoes fission, a spatiotemporal process through which the pulse decomposes into a multitude of distinct multimode solitons[66,67] and multimode dispersive radiation[64,68]. The most intense solitons formed from fission are sensitive to stimulated Raman scattering (SRS), through which the soliton dissipates energy into the medium, resulting in a redshifting centre frequency. As SRS results in a spectrally isolated multimode soliton, it has proven valuable in isolating single multimode solitons by simple bandpass-filtering. Study of these isolated multimode Raman solitons revealed an intriguing relationship between the soliton's energy and spatiotemporal volume, which reflects the internal balance of the processes[63].

Although initial studies of spatiotemporal soliton fission considered only the transient, short-lived multimode soliton products, which approximate the intuitive picture in Fig. 2a,b, multimode fibres permit nonlinear wave propagation over kilometres, corresponding to thousands of characteristic lengths. Whereas stable single-mode solitons comprise a continuous family of arbitrary-energy wavepackets, the constraints imposed by internally balancing modal dispersion limits multimode solitons to much more limited configurations[69]; indeed, formation of a multimode soliton is only possible if the pulses in the constituent modes have adequate bandwidth and energy[70,71]. Over longer distances, even these metastable multimode solitons appear to collapse into the fundamental mode[72] (Fig. 2c). Although the mechanisms of this process are still not understood, observations of long-term energy localization are—though seemingly paradoxical—a remarkably common feature of multimode nonlinear wave propagation even beyond the regime of beam cleaning described previously, arising not only from purely conservative[40–42] but also from dissipative[44,45] nonlinear intermodal interactions.

Solitons in resonators. Optical solitons in the driven-dissipative setting of a laser or coherently driven cavity[73–77] display an even richer scope of behaviours. Synchronized (that is, phase-locked) collective states of the cavity's longitudinal modes underlie mode-locked lasers[4] and microresonator frequency combs[5]. In addition to solitons formed in cavities with anomalous dispersion, which resemble solitons of the 1D nonlinear Schrödinger equation, normal-dispersion soliton pulses, solutions of the cubic-quintic complex Ginzburg–Landau equation, are also possible in the dissipative setting[78].

Investigations into mode-locking in fibre lasers with multiple transverse modes resulted in the discovery of spatiotemporal mode-locking (STML)—the formation of stable, multimode dissipative solitons[79,80].

The conceptual prototype is a laser constructed with multimode gain fibre, spatial and spectral filters, and components that act as a spatiotemporal saturable absorber (Fig. 2d). Considering the number of independent controls in such a laser, it may not seem surprising that certain special configurations could admit stable spatiotemporal pulses. However, decades of effort have shown that high-dimensional solitons are typically fragile, prone to disintegration at the slightest symmetry breaking. The discovery of STML was surprising not only because the observed multimode dissipative solitons persist stably indefinitely, but because these new solitons display remarkable diversity and complexity. Many such solitons robustly maintain a complex pattern of internal coherence, even in the presence of considerable intracavity disorder and perturbations, and across a wide range of experimental parameters[79–83]. Although there are many distinct mechanisms through which multimode dissipative solitons form[80], in general they result from a dynamic balance between saturable absorption and spectral and spatial filtering (which tend to shorten the multimode pulse) and nonlinearity, gain and dispersions (which tend to broaden the pulse).

Spatiotemporal instabilities

Instabilities are perhaps the most fundamental processes of nonlinear wave physics. They influence phenomena that range from short-term dynamics to steady-state pattern formation. Exchange of energy between waves by four-wave mixing underlies many instabilities. One wave grows at the expense of another—the first wave experiences gain whereas the second decays and is therefore unstable. That gain is a strong function of the wavevector and frequency. In experiments, components of unavoidable noise with wavevector $\mathbf{k}$ and frequency ω at the peak of the instability gain function typically grow dramatically and determine the spatiotemporal characteristics of the resulting field. This energy exchange requires momentum and energy conservation, with the former implying the coupling requires wavevector or phase matching, $\sum_i \mathbf{k}_i(\omega_i) = 0$, where $\mathbf{k}_i(\omega_i)$ are the propagation constants of the participating modes. Alternatively, waves can be coupled if some other process, acting as an additional virtual wave with wave vector $\mathbf{k}_p$, makes up any difference between the wavevectors of the coupled waves, that is, $\sum_i \mathbf{k}_i(\omega_i) = \mathbf{k}_p$.

Spatiotemporal wavevector matching. Spatial (that is, modal) dispersion can compensate for chromatic dispersion, in a similar manner to what occurs in spatiotemporal modulation instability (STMI) in homogeneous media[84]. Nanosecond-duration pulses launched into multiple modes of a GRIN fibre can undergo beam self-cleaning owing to the Kerr nonlinearity as mentioned above, and the cleaning is enhanced by stimulated Raman scattering[44]. This state is a spatial attractor (the system is dissipative) for a wide range of multimode input fields. However, the attractor is unstable against STMI (Fig. 3a); four-wave mixing subsequently mediates the transfer of power from the fundamental to higher-order modes in new frequency bands[7,45,85–88] (Fig. 3b,c). In other multimode fibres, the manifestation of STMI depends on the details of the fibre's structure[89–92]. However, STMI always produces a steady-state field in which different frequency components have different spatial profiles.

Self-imaging wavevector matching. In a GRIN fibre, superpositions of modes undergo periodic spatial compression and expansion (Fig. 2b). Through the Kerr nonlinearity, this results in an effective longitudinal refractive index grating that includes virtual wavevectors with magnitudes that are multiples of $2\pi/Z_p$, where Z_p is the grating period. The self-induced grating thus allows wavevector matching of new processes via a route that is fundamentally distinct from STMI[36,64,87,93,94]. So-called parametric instabilities arise in many physical settings when a parameter of the medium is modulated in time (equivalently, along the longitudinal direction in passive waveguides).

If the modulation arises from collective oscillations rather than from a periodic drive or perturbation, the instability is referred to as geometric parametric instability (GPI). The strong self-imaging evolution in GRIN fibres is an example of such a periodic self-modulation, which allows coupling between waves with different frequencies and the same transverse mode. GPI based on this coupling was observed by launching 900-ps pulses at 1,064 nm, where the dispersion is normal, into GRIN fibre[93]. With a peak input power of 50 kW, the output consists of a series of non-uniformly spaced spectral peaks between 450 and 730 nm (Fig. 3e). The output beams look 'cleaned', with a bell-shaped profile consisting of the fundamental mode (together with a background of higher-order modes) that maintains the self-imaging evolution (Fig. 3f). The similarity of the spatial profiles of each frequency component is a signature of self-imaging-based wavevector matching.

Soliton instabilities. Qualitatively similar processes to those that destabilize quasi-continuous waves can also cause a soliton to adjust its shape, which can include undergoing fission, and dissipate energy into one or more dispersive waves. Through spatiotemporal wavevector matching, the soliton can induce formation of the dispersive wave by intermodal four-wave mixing[95] or cross-phase modulation[96]. In GRIN fibres, self-imaging wavevector matching enables resonant energy transfer from femtosecond multimode fields to dispersive waves that extend over hundreds of nanometres of wavelength[64,65,68].

Wavefront-shaping for control of multimode nonlinear waves

It was observed, early on, that, by manually controlling the modes excited in a multimode fibre, the ensuing nonlinear processes could be dramatically affected[7,85]. A similar strategy was taken to explore the dynamics of multimode solitons and supercontinuum[65,97]. The millions of degrees of spatial control offered by modern SLMs offer an enticing opportunity for extending this exploratory strategy with fully automated, computer-driven optimization of MMNLO wave propagation. Mirroring the typical configuration of wavefront-shaping experiments in disordered media and linear multimode waveguides, Tzang and colleagues launched intense nanosecond pulses into highly multimode, normal-dispersion GRIN fibre after reflection from an SLM[98]. They used a genetic algorithm to adjust the phase applied to the SLM to optimize objective functions defined by measurements of the light exiting the fibre, such as maximizing the energy produced in a particular frequency band.

Similar strategies have been employed to control light propagation in multimode fibre amplifiers[99], as well as to explore different regimes of beam self-cleaning[100] and multimode fibre lasers[101]. Aside from controlling features of nonlinear wave interactions in passive or driven-dissipative settings, high-dimensional wavefront shaping also offers a route to utilizing complex nonlinear dynamics in multimode waveguides for information processing[102], extending prior work using linear random media[103]. So far, all these studies have relied on translating an SLM-pixel basis to a multimode fibre, usually in a way that prevents easy interpretation of experiments. To move beyond these blackbox perspectives, fully mode-resolved excitation and measurement will be helpful[104].

Outlook

Towards spatiotemporal solitons. Although a variety of spatiotemporal soliton physics have been observed in multimode fibres, no observations constitute what we would describe as spatiotemporal

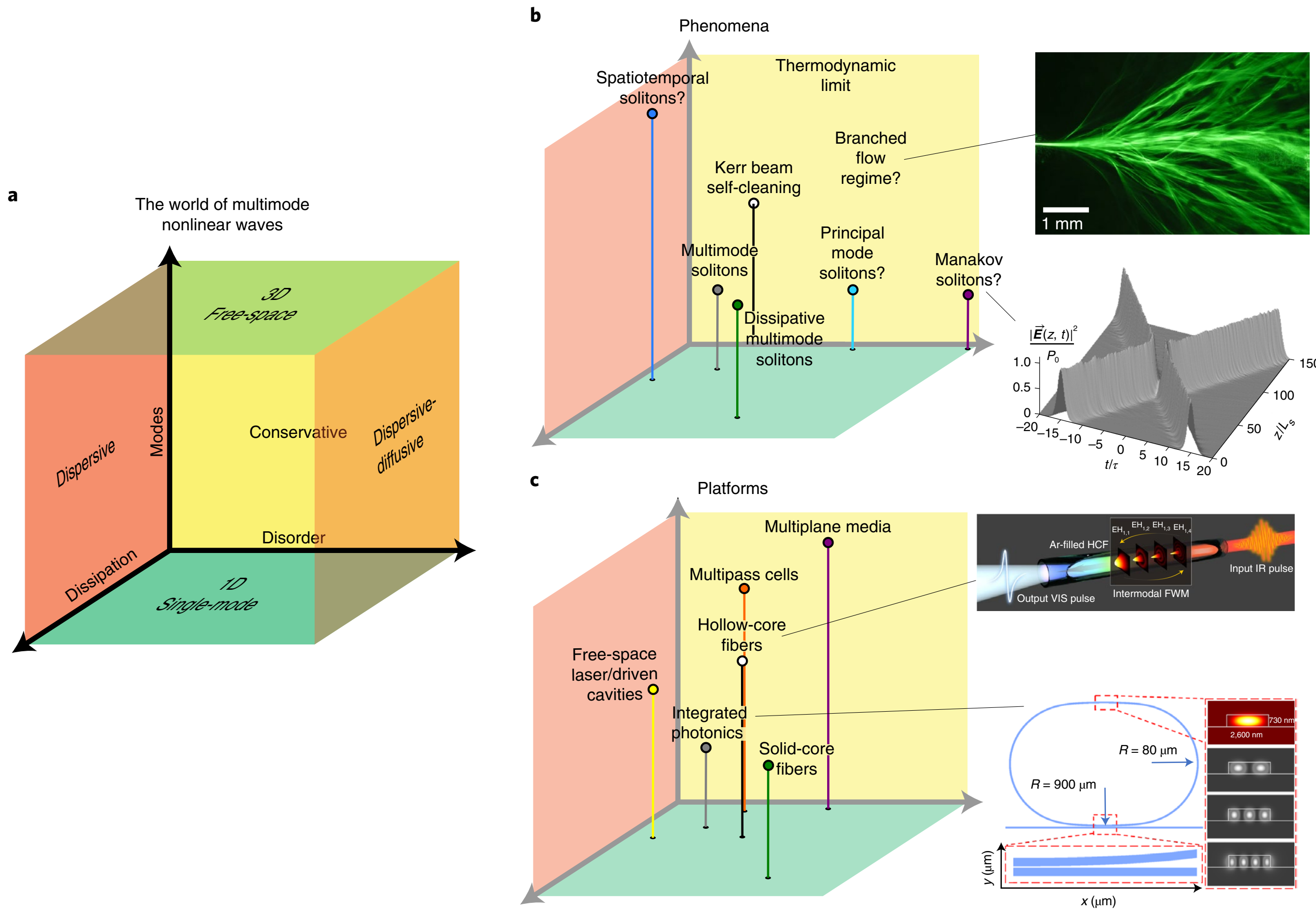


Fig. 4 | Future directions for MMNLO systems. a, The physics of MMNLO systems depends on numerous factors, but one insightful parameter space to visualize different regimes involves (1) the number of involved modes, as well as the prevalence of (2) dissipative and (3) disordered processes. In waveguides, the number of modes is scaled by increasing the transverse dimensions or refractive potential depth (numerical aperture). Disorder can be introduced by deliberately distorting the waveguide or cavity, or by increasing the length scale so that inherent manufacturing disorder becomes important. Dissipation can be controlled by introducing optical filters and compensating with optical gain. The high gain possible with fibre amplifiers (>20 dB per pass) in particular enables the study of strongly dissipative persistent phenomena, such as multimode dissipative solitons. **b**, Different phenomena have been observed, or are hypothesized to exist, in different regimes. **c**, To explore new phenomena, different physical platforms provide a means of accessing other parts of parameter space that are challenging for solid-core fibres. In **b** and **c**, the indicated points are intended only as representative centre coordinates—in all cases the phenomena and platform extend over a larger area of nearby parameter space. τ is the soliton duration, L_s is the soliton propagation length, P_0 is the soliton power, and R is radius of curvature. Images in right-hand column adapted with permission from (top to bottom): ref. [120], Springer Nature Limited; ref. [116], The Optical Society; ref. [126], Springer Nature Limited; ref. [135], Wiley.

solitons (STSs). By STSs, we mean solitons whose propagation is characterized by a simultaneous balance of both spatial diffraction and temporal dispersion by nonlinear self-focusing, waves first envisioned as 'light bullets' by Silberberg[105]. Instead, the solitons observed so far are metastable complexes of mutually trapped pulses, more multimodal soliton molecules or soliton-conical wave[106] hybrids than STSs. Indeed, after formation, the spatial confinement for observed multimode solitons is almost completely provided by the waveguide potential. Pulses resembling STSs in multimode waveguides have, however, been theoretically predicted in both passive[107–111] and active[112,113] settings, although not all theoretical predictions consider solitons whose self-confinement is predominantly due to nonlinearity. To observe such unusual solitons, researchers will need to consider wave propagation with relatively weak confinement, as in multipass cells[114]. Although multimode STS could perhaps exist in conservative settings, we expect that saturable dissipation[115] will play an important role in stability and excitation, as in most work on multidimensional cavity solitons[73–77].

Role of disorder in nonlinear multimode systems. Recent work on MMNLO physics has emphasized nonlinearity relative to disorder. However, the coherent wave propagation in beam self-cleaning and spatiotemporal mode-locking occurs despite disorder and, indeed, these self-organization processes are evidently enhanced by disorder[40,45,48,49,79,80]. Studies of nonlinear multimode systems with disorder ranging from negligible to dominant should offer insight—and perhaps even resolution—on the relationship between disorder and nonlinearity that has long fascinated the field.

For this, multimode solitons offer one promising platform. Multimode soliton formation and propagation as described above do not involve disorder at all. Remarkably however, in the so-called Manakov limit of strong disordered mode-coupling, temporal multimode solitons with the particle-like properties of solitons of the 1D nonlinear Schrödinger equation re-emerge[34,116–118]. Disorder causes fast intermodal energy exchange within the envelope of these Manakov solitons, which minimizes modal dispersion and four-wave mixing, similar to the rapid phase variations that exist within the envelope of an incoherent soliton[61]. However, despite this pivotal role of disorder, the temporal multimode Manakov soliton is a completely coherent wave structure. Realization of the Manakov limit remains a challenge, but the experimental observation of multimode Manakov solitons would be an important milestone.

Between the regime of perturbative disorder and the Manakov limit should lie a regime of smooth but non-negligible disorder[119], analogous to the branched-flow regime[120]. If processes like dispersion and nonlinearity are comparable to this smooth disorder, we expect physics—and perhaps soliton-like structures—that is fundamentally a product of all factors. One hypothetical possibility, which would generalize the concept of a soliton in disordered multimode systems, would be principal nonlinear eigenmodes or principal mode solitons, a fusion of the soliton and linear principal mode[26] concepts. Like multimode STS, the dissipative selection of a multimode laser provides one enticing route to clean observations of these nonlinear waves[80].

New experimental platforms. Most studies of MMNLO waves have opportunistically relied on commercially available fibres. Although the low cost of these fibres has benefited early development, they have also biased and limited research. Thus, critical goals for the field ought to be to observe new phenomena and enable applications by utilizing or exploring MMNLO concepts in existing photonic platforms besides commercial solid-core fibres, and by developing new photonic structures designed specifically to access new regimes and enable practical applications of MMNLO wave propagation (Fig. 4c).

So far, the most immediately clear application of MMNLO is generating very-high-brightness supercontinuum; for this, a seemingly straightforward innovation comprises multimode fibres that are suitable for efficient and broader supercontinuum[121]. To achieve this with glasses beyond fused silica can, however, require completely novel fibre designs, because conventional methods for graded-index doping are not possible[122]. More work, involving, for example, tapered fibres[123,124] or chalcogenide glasses[125], could extend these results, perhaps even into the long-wave infrared or soft UV regimes. For this and other aims, hollow-core fibres offer another compelling platform. As hollow-core fibres guide light primarily in a pressurized gas, extreme, ultrabroadband pulse propagation, where ionized plasma and self-focusing play pivotal roles, can be safely studied[126–130]. Nonlinear wave propagation in cavities or multipass cells[114,131] may also explore highly multimode regimes. By controlling the dispersion of the mirrors or by changing the optical media placed within the cavity (for example, adding gain)—or both—a range of regimes could be explored. Intracavity or intracell SLMs could permit dynamic control of intracavity coupling and mode characteristics across 10^7 degrees of freedom[132].

Although few-mode nonlinear optical physics in integrated platforms, such as silicon nitride-on-insulator, have been studied[133,134], the potential for highly multimode nonlinear wave physics in these devices remains unexamined. Traditionally, single-mode structures have been employed for strong transverse confinement, but GRIN or GRIN-like structures could allow for confined modes with lower sidewall overlap, leading to lower propagation losses[135]. Designs that reliably control multimode nonlinear physics are challenging, but emerging strategies such as inverse design[136] could allow for integrated multimode nonlinear photonics structures that leverage multimode phenomena to outperform traditional devices.

New platforms may also offer solutions to control the effect of disorder on MMNLO phenomena. Some phenomena—like beam self-cleaning or STML—appear to be resilient to disorder, but disorder can still pose practical challenges. Methods used in the broader field of complex photonics (discussed elsewhere in this issue) to coherently control light in disordered media are worth pursuing, in part because of the rich questions inherent in generalizing them to nonlinear systems[137]. However, a more pragmatic approach—especially for developing tools to be used by non-specialists—is to design multimode structures in which disorder is intrinsically suppressed by choice of the mode structure or topology[138,139], or by use of inflexible photonic structures, including rod-like waveguides, integrated chips and bulk optical systems such as multipass cells.

To study dissipative MMNLO dynamics, waveguides providing high optical gain are necessary. Initial works on multimode nonlinear waves in fibre lasers and amplifiers used custom Yb-doped fibres, which are not commercially available[79,80,99,140] and whose rarity has hindered research progress. However, these fibres are not necessary, or even optimal, for studying dissipative MMNLO or for developing useful multimode mode-locked laser systems. For example, high-quality pulse formation with diffraction-limited beam quality and energy far beyond single-mode limits has been predicted in widely available step-index multimode gain fibres, although this requires a strong saturable absorber[80]. Initial steps to observe this regime have been made[81]. Fibres with optimized inhomogeneous loss in addition to refractive index may also facilitate novel forms of STML[112]. Bulk laser systems, based on high-gain laser or parametric[141] processes, are also appealing for studying and applying dissipative MMNLO. For fibre-based STML meanwhile, amplifiers being developed for mode-division multiplexed communication[142–144] may soon offer a convenient solution.

Experimental techniques for multimode nonlinear physics. The multidimensional complexity of MMNLO underlies many aspects of its fundamental interest, but also some of its experimental challenges. In the context of MMNLO, the measurement challenge is how to efficiently obtain all the information needed to adequately describe broadband, highly multimode states of light. Efficient measurements must minimize measurement time as well as experimental cost and complexity, but should still facilitate unambiguous comparisons with theory, both for decisive science and to quantitatively inform design optimization and control.

Techniques to measure complex spatiotemporal optical fields are too plentiful, and feature trade-offs that are too numerous, to exhaustively discuss here—readers may consult refs. [145,146] and others. A variety of measurements used in MMNLO also involve scanning or sampling the multimode field using a single-mode fibre (for example, refs. [147–149]), which can be convenient and useful qualitatively, but do not always allow direct comparison with theory.

For MMNLO, comparison of experiments and theory benefits from measurement of the complex field amplitude in each relevant mode. Moreover, a well-chosen or calibrated modal basis can maximize both the interpretability and efficiency of measurements. Modal bases are effectively exploited by physical mode-demultiplexing measurements[104,150], in which a field is physically separated into modal components, for example by phase plates, and then measured by single-mode measurement devices. However, this approach is currently complex to implement, especially for highly multimode fields. An alternative is to measure the entire spatiotemporal electric field and then perform mode decomposition directly at each wavelength (for example, ref. [151]). This is often experimentally simpler, but will usually be less time-efficient.

Looking forward, we expect that broadband modal decomposition should be possible with simple optical set-ups by extending monochromatic numerical mode decomposition techniques (for example, refs. [57,152–155]). Sufficient information for reconstruction might be obtained from images of the near-field and far-field beams after diffraction from gratings, or by using a tunable spectral filter to acquire wavelength-dependent beam profiles. For MMNLO, we believe that physical mode-resolved measurements are the ideal choice. Although the equipment requirements for these measurements are not intense (a single SLM is sufficient), regular application of these techniques will require efforts to improve their accessibility, such as open-source instrumentation and self-calibration procedures.

Theoretical techniques for multimode nonlinear physics. Many regimes of MMNLO physics remain challenging to model due to the number of interacting modes, the heterogeneous, nonlinear or nonlocal quality of interactions, and the frequent lack of ergodicity and conservative equilibrium. These challenges are ubiquitous throughout physics, limiting our understanding from network physics to many-body quantum systems, and even intergalactic and biological phenomena. Compared to these settings, the accessibility of multimode nonlinear optics experiments is both remarkable and exciting. In the past, multimode optical experiments have provided the means to experimentally isolate complex phenomena, such as replica symmetry breaking in nonlinear wave propagation[156] or localization in passive or active disordered optical media. Thus, looking to the future, MMNLO systems may provide a testbed to develop theories and techniques to understand complex physical phenomena well beyond optical waves[157–159].

To this end, MMNLO systems are ideal subjects for techniques that combine traditional physics models, such as the coupled-mode (Box 1) or thermodynamic (Box 2) theories, with machine learning models. Like the many-body quantum systems to which neural network techniques have been effectively applied[160], MMNLO systems contain emergent high-dimensional phenomena, which—though not always intuitive—should form the basis for understandable compressed models. The relative ease of acquiring enormous quantities of experimental data and of controlling millions of degrees of freedom further supports the promise of these methods. Initial work applying machine learning techniques to linear multimode propagation is encouraging[161–163], but extensions to nonlinear multimode optical systems have been inspiring but less effective than anticipated[98,164].

We expect that major improvements will be possible by applying physics-informed models[165], by better controlling or learning experimental noise and drift, by controlling input fields in both space and time[166], and by considering richer dynamical settings such as the anomalous dispersion regime[65]. Besides providing an ideal testbed for data-driven techniques that achieve both physical insight and accurate predictions, data-assisted methods will probably prove essential in achieving systematic control over highly multimode nonlinear dynamics.

Received: 13 December 2021; Accepted: 24 June 2022;
Published online: 8 September 2022

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Acknowledgements
This effort was sponsored, in part, by the Department of the Navy, Office of Naval Research under ONR awards nos. N00014-20-1-2789 and N00014-18-1-2347. Portions of the work were sponsored by the National Science Foundation (ECCS-1912742 and EECS-1711230), the Army Research Office (award no. W911NF1710481), the Simons Foundation (733682) and the BSF (2016381).

Author contributions
All authors contributed to the writing and editing of the manuscript.

Competing interests
L.G.W. and F.W.W. hold patent number US 10,965,092 B2 on spatiotemporal mode-locking. The other authors declare no competing interests.

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Peer review information *Nature Physics* thanks Stefan Wabnitz and the other, anonymous, reviewer(s) for their contribution to the peer review of this work.