Upper-Triangular Linear Relations on Multiplicities and the Stanley–Stembridge Conjecture

Megumi Harada [*]

Department of Mathematics and Statistics
McMaster University
1280 Main Street West
Hamilton, Ontario L8S4K1
Canada

Megumi.Harada@math.mcmaster.ca

Martha Precup[†]

Department of Mathematics and Statistics
Washington University in St. Louis
One Brookings Drive
St. Louis, Missouri 63130
U.S.A.

martha.precup@wustl.edu

Submitted: Jun 11, 2021; Accepted: Jun 15, 2022; Published: Jul 15, 2022

Abstract

In 2015, Brosnan and Chow, and independently Guay-Paquet, proved the Shareshian–Wachs conjecture, which links the Stanley–Stembridge conjecture in combinatorics to the geometry of Hessenberg varieties. This link is made precise through Tymoczko's permutation group action on the cohomology ring of regular semisimple Hessenberg varieties. In previous work, the authors exploited this connection to prove a graded version of the Stanley–Stembridge conjecture for a special case in which only irreducible representations of the permutation group indexed by partitions with at most two parts can appear. In this manuscript, we derive a new set of linear relations satisfied by the multiplicities of certain permutation representations in Tymoczko's representation. We also show that these relations are upper-triangular in an appropriate sense and that they uniquely determine the multiplicities. As an application of these results, we prove an inductive formula for the multiplicity coefficients corresponding to partitions with a maximal number of parts.

Mathematics Subject Classifications: 14M17, 05E05

1 Introduction

Recent results have forged exciting new connections between algebraic combinatorics and the geometry and topology of certain subvarieties of the flag variety called *Hessenberg*

---

[*]Partially supported by a Canada Research Chair (Tier 2) Award and an NSERC Discovery Grant.

[†]Partially supported by NSF grant DMS-1954001.

varieties. In particular, the Shareshian–Wachs conjecture [12], proven in 2015 by Brosnan and Chow [5] and independently by Guay-Paquet [8], established a new connection between Hessenberg varieties and the long-standing *Stanley–Stembridge conjecture* in combinatorics. This conjecture states that the chromatic symmetric function of the incomparability graph of a (3+1)-free poset is e-positive, i.e., it is a non-negative linear combination of elementary symmetric functions. The Stanley–Stembridge conjecture is well-known in the field of combinatorics and related to various other deep conjectures about immanants.

The results mentioned above establish the following research problem: *use the properties of Hessenberg varieties to prove the Stanley–Stembridge conjecture.* The problem can in fact be made more specific, as follows. The results of Brosnan–Chow and Guay-Paquet connect the *dot action representation*, defined by Tymoczko in [15] on the cohomology groups of regular semisimple Hessenberg varieties, to the Stanley–Stembridge conjecture. From this it follows that if Tymoczko's dot action representation is a permutation representation in which each point stabilizer is a Young subgroup, then the Stanley–Stembridge conjecture is true. We refer the reader to [9, Introduction and Section 2] for a more leisurely account of the historical background and motivation for this circle of ideas.

There are already partial results to the problem stated above. For instance, we used Hessenberg varieties to prove a graded refinement of the Stanley–Stembridge conjecture in the so-called *abelian case* by giving an inductive description of the nontrivial permutation representations that appear in that case [9]. Moreover, in that manuscript we additionally stated a conjecture which gives, in the general case, an inductive description of the multiplicities of certain nontrivial permutation representations [9, Conjecture 8.1]. One motivation for the present manuscript was to prove this conjecture using the geometry and combinatorics of Hessenberg varieties. In doing so, we discovered new properties obeyed by the multiplicities of the so-called tabloid representations in Tymoczko's representation.

We now describe the results of this manuscript in more detail. Hessenberg varieties in type A are subvarieties of the full flag variety $\mathcal{F}\ell ags(\mathbb{C}^n)$ of nested sequences of linear subspaces in $\mathbb{C}^n$. These varieties are parameterized by a choice of linear operator $\mathsf{X} \in \mathfrak{gl}(n, \mathbb{C})$ and Hessenberg function $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. (For details see Section 2.) For the purpose of this discussion it suffices to consider only the case when the operator is a regular semisimple operator S in $\mathfrak{gl}(n, \mathbb{C})$; we denote the corresponding Hessenberg variety by $\mathcal{H}ess(\mathsf{S}, h)$. As mentioned above, Tymoczko defined [15] an action of the symmetric group $\mathfrak{S}_n$ on $H^{2i}(\mathcal{H}ess(\mathsf{S}, h))$ for each $i \geqslant 0$. From the work of Shareshian–Wachs, Brosnan–Chow, and Guay-Paquet it follows that in order to prove the (graded) Stanley–Stembridge conjecture, it suffices to prove that the cohomology $H^{2i}(\mathcal{H}ess(\mathsf{S}, h))$ for each i is a non-negative combination of the tabloid representations M^μ [6, Part II, Section 7.2] of $\mathfrak{S}_n$ for μ a partition of n. In other words, given the decomposition

$$H^{2i}(\mathcal{H}ess(\mathsf{S}, h)) = \sum_{\mu \vdash n} c_{\mu,i} M^\mu \tag{1}$$

in the representation ring $\mathcal{R}ep(\mathfrak{S}_n)$ of $\mathfrak{S}_n$, it suffices to show that the coefficients $c_{\mu,i}$ are non-negative.

We take a moment to mention here that the coefficients $c_{\mu,i}$ appearing in (1) were previously known to satisfy a matrix equation

$$\sum_{\mu \vdash n} N_{\lambda\mu} c_{\mu,i} = y_{\lambda,i}$$

where the $y_{\lambda,i}$ are derived from the Betti numbers, i.e. the dimensions of ordinary cohomology groups, of certain regular Hessenberg varieties. Here, $N_{\lambda\mu} = \sum_{\nu \vdash n} K_{\nu,\lambda} K_{\nu,\mu}$ where the $K_{\nu,\lambda}, K_{\nu,\mu}$ are Kostka numbers [9, Section 2]. However, the Kostka numbers and the matrix N are computationally unwieldy, and it was not clear (to us) how to exploit the above matrix equation to prove the non-negativity of the $c_{\mu,i}$. Another motivation for this manuscript was to find other relations satisfied by these coefficients which are more computationally tractable.

The main results of this manuscript are as follows. Let n be a positive integer and $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ a Hessenberg function. Let $i \geqslant 0$ be a fixed non-negative integer and $X_i = (c_{\mu,i})$ denote the (column) vector whose entries are the coefficients appearing in (1) above.

- In Corollary 23, we derive a family of (new) matrix equations $AX_i = W_i$ satisfied by the column vectors X_i for $i \geqslant 0$. The matrix $A = (A(\lambda, \mu))_{\lambda,\mu \vdash n}$ is obtained by counting certain subsets of the permutation group $\mathfrak{S}_n$ using the data of a pair of partitions $\lambda, \mu \vdash n$, and is independent of both the choice of Hessenberg function h and the integer $i \geqslant 0$. The column vectors W_i are obtained by counting certain subsets of the permutation group $\mathfrak{S}_n$ using the data of a partition λ, the Hessenberg function h, and the integer $i \geqslant 0$.

- In Theorem 29, we prove that the above matrix $A = (A(\lambda, \mu))$ is *upper-triangular, with* 1*'s along the diagonal*, with respect to an appropriately chosen linear order on the set $\mathrm{Par}(n)$ of partitions of n. We additionally prove an inductive formula for its matrix entries (Proposition 24, cf. also Corollary 40).

- Generalizing results of [9, Section 4], we obtain a sink set decomposition of the subsets of $\mathfrak{S}_n$ defining the column vector W_i above (Proposition 50). As a consequence we obtain an inductive formula for the entries of W_i for the special case in which λ has the maximal possible number of parts (Theorem 65).

- As an application of the above results, we prove [9, Conjecture 8.1]; more precisely, we obtain an inductive formula for the coefficients $c_{\mu,i}$ in (1) for the special case in which μ has the maximal possible number of parts (Theorem 66), thus providing further evidence for the Stanley–Stembridge conjecture.

Some remarks are in order. Firstly, the main contribution of this manuscript are the new linear relations in Corollary 23; most particularly, the upper-triangularity of the matrix A gives substantial reason to expect that these matrix equations will play a significant role in the solution to the full Stanley–Stembridge conjecture. Secondly, we are aware that there exist other proofs of our conjecture as stated in [9, Conjecture 8.1], using the coproduct structure on the ring of symmetric functions [10]. Thirdly, in his original paper on the subject, Stanley derives a different set of linear relations obeyed by the coefficients c_λ [13, 14, Theorem 3.4, cf. also the erratum posted on Stanley's personal webpage], in which he uses a notion of *sink sequences*. It should be noted that Stanley's definition of sink sequences uses the cardinality of the sinks in an inductively defined set of graphs, whereas the sink set decompositions which are used in our arguments are a decomposition based on the sink *subsets* (i.e. not just their cardinalities, but the subsets themselves). As of this writing, we are not aware of a precise relationship between our linear relations and those of Stanley's.

We now give a brief overview of the contents of the manuscript. Section 2 is devoted to the setup and definitions of appropriate notation and terminology. In Section 3 we derive the new matrix equations $AX_i = W_i$, and in Section 4 we prove that A is upper-triangular, with 1's along the diagonal. We also derive the inductive formula for the numbers $A(\lambda, \mu)$. In Section 5 we derive a separate inductive formula for the entries of the "constant vector" W_i. Finally, in Section 6 we prove Conjecture 8.1 from [9].

Finally, we take a moment to report on a recent development in this line of inquiry, which was made public after the initial announcement of our results in the present manuscript. Specifically, Abreu and Nigro [2] have shown that the coefficients $c_{\mu,i}$ are uniquely determined by a set of linear relations known as the modular law, first obtained by Guay-Paquet using entirely different methods from ours [7]. It should be noted that the modular law relates coefficients $c_{\mu,i}$ associated to *different* Hessenberg functions h, whereas our linear relations are between the $c_{\mu,i}$ for a *fixed* Hessenberg function. As of this writing, we do not know whether some combination of these relations can solve the conjecture. We leave this open for future work.

2 Background and Terminology

In this section we briefly recall the setting of our paper. For a more leisurely account we refer to [9]. Let n be a positive integer and set $[n] := \{1, 2, \ldots, n\}$. Hessenberg varieties in Lie type A are subvarieties of the (full) flag variety $\mathcal{F}\ell ags(\mathbb{C}^n)$, which is the collection of sequences of nested linear subspaces of $\mathbb{C}^n$:

$$\mathcal{F}\ell ags(\mathbb{C}^n) := \{V_\bullet = (\{0\} \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n) \mid \dim_{\mathbb{C}}(V_i) = i, \forall\, i \in [n]\}.$$

A Hessenberg variety in $\mathcal{F}\ell ags(\mathbb{C}^n)$ is specified by two pieces of data: a **Hessenberg function**, that is, a nondecreasing function $h : [n] \to [n]$ such that $h(i) \geqslant i$ for all i, and a choice of an element X in $\mathfrak{gl}(n, \mathbb{C})$. We frequently write a Hessenberg function by listing its values in sequence, i.e., $h = (h(1), h(2), \ldots, h(n))$. The **Hessenberg variety** associated to the linear operator X and Hessenberg function h is defined as

$$\mathcal{H}ess(\mathsf{X}, h) = \{V_\bullet \in \mathcal{F}\ell ags(\mathbb{C}^n) \mid \mathsf{X}V_i \subseteq V_{h(i)} \text{ for all } i\}. \tag{2}$$

When the linear operator X is chosen to be a regular semisimple operator S (i.e., diagonalizable with distinct eigenvalues), we refer to the corresponding Hessenberg variety

$\mathcal{H}ess(\mathsf{S}, h)$ as a **regular semisimple Hessenberg variety**. Tymoczko defined an action of the symmetric group $\mathfrak{S}_n$ on the cohomology of a regular semisimple Hessenberg variety $H^*(\mathcal{H}ess(\mathsf{S}, h))$ which is called the **dot action** [15]. She defines the dot action by first defining a $\mathfrak{S}_n$-action on the T-equivariant cohomology ring $H_T^*(\mathcal{H}ess(\mathsf{S}, h))$ in terms of its Goresky-Kottwitz-MacPherson description, which is a purely combinatorial characterization of this ring using certain labelled graphs. She then shows that this $\mathfrak{S}_n$-action descends to an action on the ordinary cohomology $H^*(\mathcal{H}ess(\mathsf{S}, h))$. (The reader can find a more detailed synopsis of this story in [1, Section 8].) Tymoczko's dot action on $H^*(\mathcal{H}ess(\mathsf{S}, h))$ preserves the grading on cohomology, so in fact $\mathfrak{S}_n$ acts on each $H^{2i}(\mathcal{H}ess(\mathsf{S}, h))$ for $i \geqslant 0$ (the cohomology is concentrated in even degrees). For μ a partition of n, we denote by M^μ the complex vector space with basis given by the set of tabloids of shape μ. Since $\mathfrak{S}_n$ acts on the set of tabloids, M^μ is a $\mathfrak{S}_n$-representation, and is called the tabloid representation (corresponding to μ) [6, Part II, Section 7.2]. It is well-known that the set of these tabloid representations form a $\mathbb{Z}$-basis for the representation ring $\mathcal{R}ep(\mathfrak{S}_n)$ of $\mathfrak{S}_n$, so we can decompose $H^*(\mathcal{H}ess(\mathsf{S}, h))$ with respect to Tymoczko's dot action as follows:

$$H^*(\mathcal{H}ess(\mathsf{S}, h)) = \sum_{\mu \vdash n} c_\mu M^\mu \quad \text{ and } \quad H^{2i}(\mathcal{H}ess(\mathsf{S}, h)) = \sum_{\mu \vdash n} c_{\mu,i} M^\mu \tag{3}$$

where $c_\mu, c_{\mu,i} \in \mathbb{Z}$.

As explained in the Introduction, the motivation of this manuscript is to prove the graded Stanley–Stembridge conjecture. We refer the reader to [9] for more history; for the present manuscript we take the 'graded Stanley–Stembridge conjecture' to mean the following.

Conjecture 1. Let n be a positive integer, $h : [n] \to [n]$ be a Hessenberg function, and S be a regular semisimple linear operator. Then the integers $c_{\mu,i}$ appearing in (3) are non-negative.

2.1 Hessenberg data

For later use, we introduce some Lie-theoretic and combinatorial notation associated to Hessenberg varieties. We fix a Hessenberg function $h : [n] \to [n]$.

Let $\mathfrak{t} \subseteq \mathfrak{gl}(n, \mathbb{C})$ denote the Cartan subalgebra of diagonal matrices and let t_i denote the coordinate on $\mathfrak{t}$ reading off the (i, i)-th matrix entry along the diagonal. Denote the root system of $\mathfrak{gl}(n, \mathbb{C})$ by Φ. Then the positive roots of $\mathfrak{gl}(n, \mathbb{C})$ are $\Phi^+ = \{t_i - t_j \mid 1 \leqslant i < j \leqslant n\}$ where $t_i - t_j \in \Phi^+$ corresponds to the root space spanned by the elementary matrix E_{ij}, denoted $\mathfrak{g}_{t_i - t_j}$. Similarly, the negative roots of $\mathfrak{gl}(n, \mathbb{C})$ are $\Phi^- = \{t_i - t_j \mid 1 \leqslant j < i \leqslant n\}$. We denote the simple positive roots in Φ^+ by $\Delta = \{\alpha_i := t_i - t_{i+1} \mid 1 \leqslant i \leqslant n-1\}$. Finally, it is clear that each root $t_i - t_j \in \Phi$ can be uniquely identified with an ordered pair (i, j), with $i \neq j$. We will make this identification below whenever it is notationally convenient.

For each permutation $w \in \mathfrak{S}_n$, let

$$\operatorname{inv}(w) := \{(i, j) \mid i > j \text{ and } w(i) < w(j)\}$$

denote the set of inversions of w. Note that we adopt the nonstandard notation of listing the larger number in the pair $(i, j) \in \mathrm{inv}(w)$ first. This is because we identify $\mathrm{inv}(w)$ with a subset of negative roots in Section 5 below. Under the correspondence between ordered pairs and roots discussed in the last paragraph, this set indexes the negative roots which become positive under the action of w. This action can be expressed concretely as $w(t_i - t_j) = t_{w(i)} - t_{w(j)}$.

The Hessenberg function $h : [n] \to [n]$ uniquely determines two subsets of roots as follows:

$$\Phi_h^- := \{t_i - t_j \mid i > j \text{ and } i \leqslant h(j)\} \quad \text{and} \quad \Phi_h := \Phi_h^- \sqcup \Phi^+ = \{t_i - t_j \mid i \leqslant h(j)\}.$$

Let $\mathrm{inv}_h(w) := \mathrm{inv}(w) \cap \Phi_h^-$; this set of inversions is used later to compute the Betti numbers of certain Hessenberg varieties.

Recall that an ideal I of Φ^- is defined to be a collection of negative roots such that if $\alpha \in I$, $\beta \in \Phi^-$, and $\alpha + \beta \in \Phi^-$, then $\alpha + \beta \in I$. The relation defining Φ_h^- immediately implies that

$$I_h := \Phi^- \setminus \Phi_h^- = \{t_i - t_j \mid i > h(j)\}$$

is an ideal in Φ^-. We call it the **ideal corresponding to $\boldsymbol{h}$**.

Given an ideal $I \subseteq \Phi^-$, its **lower central series** is the sequence of ideals defined inductively by

$$I_1 = I \quad \text{and} \quad I_j = \{\gamma + \beta \mid \gamma, \beta \in I_{j-1} \text{ and } \gamma + \beta \in \Phi^-\} \quad \text{for all } j \geqslant 2.$$

The **height of an ideal $\boldsymbol{I}$** is the length of its lower central series and we denote it by $ht(I)$.

Example 2. Let $h = (2, 4, 4, 5, 5)$. Then

$$\Phi_h^- = \{t_2 - t_1, t_3 - t_2, t_4 - t_2, t_4 - t_3, t_5 - t_4\}$$

and

$$I_h = \{t_3 - t_1, t_4 - t_1, t_5 - t_1, t_5 - t_2, t_5 - t_3\}$$

and $ht(I_h) = 2$ since

$$(I_h)_2 = \{t_5 - t_1\} \quad \text{and} \quad (I_h)_3 = \varnothing.$$

The data of a Hessenberg function can also be encoded by way of a graph. Given a Hessenberg function $h : [n] \to [n]$, the **incomparability graph** associated to h is the graph $\Gamma_h = (V_h, E_h)$ with vertex set $V_h = [n]$ and edge set $E_h = \{\{i, j\} \mid i < j \text{ and } h(i) \geqslant j\}$. Notice that the edges of Γ_h correspond bijectively to the roots in Φ_h^-.

Example 3. The graph corresponding to the Hessenberg function $h = (2, 4, 4, 5, 5)$ from Example 2 is

$$\begin{tikzcd} 1 \arrow[r, dash] & 2 \arrow[r, dash] \arrow[rr, dash, bend left] & 3 \arrow[r, dash] & 4 \arrow[r, dash] & 5 \end{tikzcd}$$

In many ways, the combinatorial structure of the graph Γ_h and the ideal I_h mirror one another. For example, [9, Proposition 5.8] shows that $m(\Gamma_h) = ht(I_h) + 1$, where $m(\Gamma_h)$ denotes the maximum cardinality of an independent subset of vertices (that is, vertices which are pairwise nonadjacent) in Γ_h. The reader can confirm this equation for the Hessenberg function $h = (2, 4, 4, 5, 5)$ appearing in Example 2 and Example 3. This correspondence is essential for the arguments of Section 5 below. Furthermore, the structure of the ideal I_h, and that of the graph Γ_h, is closely connected to the dot action representation. The following theorem relates the multiplicities of the tabloid representations appearing in (3) with the height of I_h. This is a restatement of [9, Corollary 5.12].

Theorem 4. *Let c_μ and $c_{\mu,i}$ be the coefficients appearing in* (3)*. Then $c_\mu = c_{\mu,i} = 0$ for all $\mu \vdash n$ with more than $m(\Gamma_h) = ht(I_h) + 1$ parts.*

2.2 Partitions and subsets

In this section we establish some combinatorial terminology and notation which we use below. Let n be a positive integer.

Definition 5. Let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$. We define J_λ to be the subset of $[n-1]$ defined by

$$J_\lambda := [n-1] \setminus \{\lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_1 + \cdots + \lambda_{k-1}\}$$

Remark 6. We frequently identify the set $[n-1] := \{1, 2, \ldots, n-1\}$ with the set of simple positive roots Δ by the association $\alpha_i \mapsto i$. Under this identification, we may view J_λ as the subset of simple roots

$$J_\lambda := \Delta \setminus \{\alpha_{\lambda_1}, \alpha_{\lambda_1+\lambda_2}, \ldots, \alpha_{\lambda_1+\cdots+\lambda_{k-1}}\} \subseteq \Delta.$$

We illustrate in Example 7 how Definition 5 can be visualized. Note that any partition of n corresponds to a Young diagram with n boxes, and by slight abuse of notation we denote both the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and the corresponding Young diagram as λ.

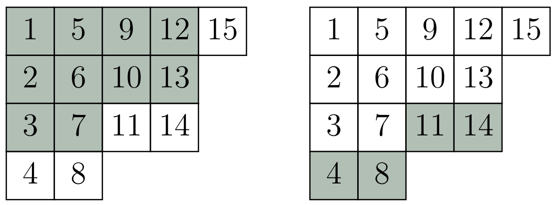
Example 7. Let $\lambda = (5, 4, 4, 2) \vdash 15$. Using the Young tableau of this diagram which fills the boxes of λ with the integers $\{1, 2, \ldots, n\}$ in order starting from the top left and reading across rows from left to right, starting from the top row to the bottom row, as indicated below, the set $J_\lambda = [14] \setminus \{5, 9, 13\}$ corresponds to those boxes which are not at the rightmost end of a row. In the figure below, these boxes are shaded in grey.

1	2	3	4	5
6	7	8	9	
10	11	12	13	
14	15			

Recall that the **dual (or transpose) partition** of λ is the partition λ' obtained by swapping the rows and the columns of the Young diagram of λ. We will also be interested in the set $J_{\lambda'}$ corresponding to λ'. In fact it will be useful to introduce notation for the complement of $J_{\lambda'}$. We let

$$\mathbb{J}_\lambda := [n-1] \setminus J_{\lambda'}. \tag{4}$$

Example 8. Continuing Example 7, let $\lambda = (5,4,4,2) \vdash 15$. Then it is straightforward to see that $\lambda' = (4,4,3,3,1)$ and $J_{\lambda'} = [14] \setminus \{4,8,11,14\}$ and thus $\mathbb{J}_\lambda := [14] \setminus J_{\lambda'} = \{4,8,11,14\}$. Below, the shaded boxes in the figure on the left correspond to the elements of $J_{\lambda'}$, while the shaded boxes in the figure on the right correspond to those in $\mathbb{J}_\lambda := \Delta \setminus J_{\lambda'}$. Note that the diagram for λ is drawn, but the labelling of the boxes corresponds to the Young tableau of the dual partition λ' with filling as in Example 7. The box labelled 15 in the diagram is contained in neither $J_{\lambda'}$ nor $\mathbb{J}_\lambda$ since both sets are contained in $[n-1] = [14]$, not $[n] = [15]$.



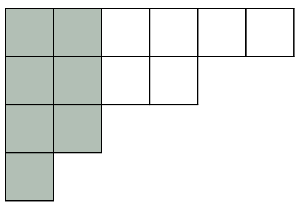
We will also be interested in certain subdiagrams of a Young diagram λ. First recall that for $\lambda = (\lambda_1, \ldots, \lambda_k)$ a partition with $\lambda_k > 0$, the integer k is often called the number of parts of λ (also known as the *length* of λ). By definition, the number of parts of λ is equal to λ'_1, the first entry of the dual partition λ'. Thus we will sometimes use the notation λ'_1 for the number of parts.

We will also need to refer to the number of boxes in the bottom row of λ, which is equal to $\lambda_{\lambda'_1}$; however, to avoid cumbersome notation we denote this as $r(\lambda)$ and call it the **bottom length** of λ. (Thus, if λ has k parts, then $r(\lambda) = \lambda_k$.) It follows from the definitions that the maximum number of boxes in a column of λ is exactly λ'_1, and there are precisely $r(\lambda)$ many such columns in λ.

In the inductive arguments given in the later sections, we will need to remove columns from λ as follows.

Definition 9. Let λ be a partition of n and let ℓ be a positive integer. Then we denote by $\lambda[\ell]$ the partition obtained by removing the leftmost ℓ columns from the Young diagram associated to λ.

Example 10. Let $\lambda = (6,4,2,1)$ and let $\ell = 2$. Then $\lambda[2]$ is the partition $\lambda = (4,2)$ obtained by removing the leftmost 2 columns of λ. In the figure below, the boxes that are removed are shaded, and the white boxes correspond to the smaller partition $\lambda[2]$.



Remark 11. Using the terminology and notation introduced above, we note that if λ is a partition of n with exactly k parts and $r = r(\lambda)$, and $\ell \in \mathbb{Z}$ with $1 \leqslant \ell \leqslant r-1$, then the partition $\lambda[\ell]$ still has k parts, while $\lambda[r]$ is a partition of $n - rk$ which has strictly fewer than k parts.

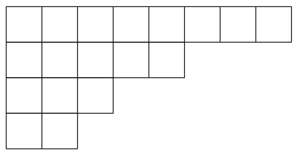
Definition 12. Let λ be a partition. We say a consecutive sequence $\{s, s+1, \ldots, s+t\} \subseteq [\lambda_1]$ is a **step of $\boldsymbol{\lambda}$** if

$$\lambda'_s = \lambda'_{s+1} = \cdots = \lambda'_{s+t}$$

and if this sequence is maximal with respect to this property, i.e., assuming the quantities are defined, both $\lambda'_{s-1} \neq \lambda'_s$ and $\lambda'_{s+t+1} \neq \lambda'_{s+t}$ (with the convention that $\lambda'_0 = 0$).

The terminology above is motivated by viewing the Young diagram of λ as an (upside-down) staircase.

Example 13. If $\lambda = (8, 5, 3, 2)$ so that $\lambda' = (4, 4, 3, 2, 2, 1, 1, 1)$ as in the diagram below



then there are four steps of λ, namely $A_1 = \{1, 2\}, A_2 = \{3\}, A_3 = \{4, 5\}, A_4 = \{6, 7, 8\}$. Each step gives the labels of a set of columns (starting from the left) of λ with the same length.

It is clear that every column in λ belongs to exactly one step of λ, giving us the following decomposition.

Definition 14. The **step decomposition** of $\lambda \vdash n$ is the decomposition

$$[\lambda_1] = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_{\mathrm{step}(\lambda)}$$

where each A_i is a step of λ and $\mathrm{step}(\lambda)$ is a positive integer which we call the **number of steps** (or **step number**) of λ. We will always assume that the A_i are listed in increasing order, i.e. $A_1 = \{1, 2, \ldots, a_1\}$, $A_2 = \{a_1 + 1, \ldots, a_2\}$, and so on, for some sequence of integers $1 \leqslant a_1 < a_2 < \cdots < a_{\mathrm{step}(\lambda)} = \lambda_1$.

Example 15. Continuing with Example 13 above, the step decomposition of the partition $\lambda = (8, 5, 3, 2)$ is

$$A_1 \sqcup A_2 \sqcup A_3 \sqcup A_4 = \{1, 2\} \sqcup \{3\} \sqcup \{4, 5\} \sqcup \{6, 7, 8\}.$$

Since there are 4 steps, we have $\mathrm{step}(\lambda) = 4$.

3 Linear equations satisfied by representation multiplicities

The main result of this section, Theorem 18, gives a set of linear equations satisfied by the multiplicity coefficients c_μ and $c_{\mu,i}$ of equation (3). In Corollary 23 below, we also reformulate our main result into a family of matrix equations by applying Theorem 18 to the special cases when the set J below is chosen to be $\mathbb{J}_\lambda$ for a partition λ of n.

The following sets of permutations play a key role in the analysis below.

Definition 16. Let $J \subseteq [n-1]$ and $i \in \mathbb{Z}$, $i \geqslant 0$. We define

$$\mathcal{W}(J,h) := \left\{ w \in \mathfrak{S}_n \mid \begin{array}{c} w^{-1}(j) \leqslant h(w^{-1}(j+1)) \text{ for all } j \in J \text{ and} \\ w^{-1}(j) > h(w^{-1}(j+1)) \text{ for all } j \in [n-1] \setminus J \end{array} \right\} \subseteq \mathfrak{S}_n.$$

We also define

$$\mathcal{W}_i(J,h) := \mathcal{W}(J,h) \cap \{w \in \mathfrak{S}_n \mid |\mathrm{inv}_h(w)| = i\}.$$

Note that

$$\mathcal{W}(J,h) := \bigsqcup_i \mathcal{W}_i(J,h)$$

where the union is taken over all i such that $\mathcal{W}_i(J,h) \neq \varnothing$.

Remark 17. Identifying J as a subset of Δ as in Remark 6 we may also identify the sets $\mathcal{W}_i(J,h)$ as

$$\mathcal{W}_i(J,h) := \{w \in \mathfrak{S}_n \mid w^{-1}(J) \subseteq \Phi_h \text{ and } w^{-1}(\Delta \setminus J) \subseteq I_h \text{ and } |\mathrm{inv}_h(w)| = i\} \subseteq \mathfrak{S}_n.$$

We use the above interpretation of the $\mathcal{W}(J,h)$-sets, below, when we connect them with the Betti numbers of certain Hessenberg varieties.

Next, let $w \in \mathfrak{S}_n$ be a permutation. Then we let

$$\mathrm{Des}_R(w) := \{i \in [n-1] \mid w(i) > w(i+1)\} \tag{5}$$

denote the set of **right descents of w** (also called the **descents of w**) and

$$\mathrm{Des}_L(w) := \{i \in [n-1] \mid w^{-1}(i) > w^{-1}(i+1)\} \tag{6}$$

denote the set of **left descents of w** (also called the **inverse descents of w**, i.e. descents of w^{-1}).[1] Both of these sets have a natural interpretation in terms of the one-line notation for w. The set of left descents corresponds to the set of ordered pairs $(i, i+1)$ such that $i+1$ appears before i in the one-line notation for w. Similarly, the set of right descents corresponds to the pairs $(i, i+1)$ such that, in the one-line notation of w, the $(i+1)$-st entry is less than the i-th entry.

For two subsets J and K of $[n-1]$ we also define

$$\mathcal{D}(J,K) := \{w \in \mathfrak{S}_n \mid \mathrm{Des}_L(w) = [n-1] \setminus J \text{ and } \mathrm{Des}_R(w) \subseteq [n-1] \setminus K\}. \tag{7}$$

The goal of this section is to prove the following.

[1] We note that both sets of terminology are used in the literature. For instance, in [4, p.17] we see the terms 'left and right descents', whereas in other research manuscripts such as [3, 16], the terms 'descent and inverse descents' are used.

Theorem 18. *Let $J \subseteq [n-1]$ and $i \in \mathbb{Z}$, $i \geqslant 0$. Then*

$$|\mathcal{W}_i(J,h)| = \sum_{\mu \vdash n} c_{\mu,i}\,|\mathcal{D}(J,J_\mu)| \tag{8}$$

and

$$|\mathcal{W}(J,h)| = \sum_{\mu \vdash n} c_{\mu}\,|\mathcal{D}(J,J_\mu)|. \tag{9}$$

We organize this section as follows. In Section 3.1 we prove Theorem 18 modulo some elementary lemmas and a previous result of Brosnan-Chow, and in Section 3.2 we record the proofs of the lemmas. Put together, this section therefore paves the way for Section 4, in which we re-organize a certain subset of these linear relations obtained in Theorem 18 (namely, those for which $J = \mathbb{J}_\lambda$) into a set of matrix equations, one for each $i \geqslant 0$. The analysis of this matrix equation will occupy much of the remainder of the paper.

3.1 Proof of Theorem 18 modulo some lemmas

The proof of Theorem 18 relies on three results which we list below. The first is a result of Brosnan–Chow [5] which relates the representation multiplicities in (3) to the Betti numbers of regular Hessenberg varieties. The last two are straightforward inclusion-exclusion arguments.

For a given subset $J \subseteq [n-1]$, let $\mathsf{X}_J \in \mathfrak{gl}(n,\mathbb{C})$ be the regular element such that $\mathsf{X}_J = \mathsf{N}_J + \mathsf{S}_J$ where

$$\mathsf{N}_J = \sum_{j \in J} E_{j,j+1}$$

and S_J is a semisimple linear operator such that N_J is a regular nilpotent element in the Levi subalgebra $\mathfrak{z}_{\mathfrak{g}}(\mathsf{S}_J)$. A Hessenberg variety associated to such a regular operator X_J as above is called a **regular Hessenberg variety**. Moreover, let $\mathfrak{S}_J := \langle s_i \mid i \in J\rangle$ be the subgroup of the symmetric group generated by the simple reflections $s_i := s_{\alpha_i}$ for $i \in J$. The theorem of Brosnan and Chow, which we recall below, identifies the dimension of the subspaces $H^{2i}(\mathcal{H}ess(\mathsf{S},h))^{\mathfrak{S}_J}$ with the dimension of the degree-$2i$-cohomology (i.e., the $2i$-th Betti number) of a certain regular Hessenberg variety.

Theorem 19. (Brosnan–Chow, [5, Theorem 127]) *Let n be a positive integer and $h : [n] \to [n]$ a Hessenberg function. Let X_J and $\mathfrak{S}_J$ for $J \subseteq [n-1]$ be as above, and S be a regular semisimple operator. Then for each non-negative integer i, we have*

$$\dim(H^{2i}(\mathcal{H}ess(\mathsf{S},h)))^{\mathfrak{S}_J} = \dim H^{2i}(\mathcal{H}ess(\mathsf{X}_J,h)).$$

The next two results are straightforward inclusion-exclusion arguments which are based on a combinatorial formula for the Betti numbers of regular Hessenberg varieties obtained by the second author [11].

Lemma 20. *Let $J \subseteq [n-1]$, h any Hessenberg function, and $i \in \mathbb{Z}$ with $i \geqslant 0$. Then*

$$|\mathcal{W}_i(J,h)| = \sum_{I\,:\,J\subseteq I} (-1)^{|I|-|J|} \dim(H^{2i}(\mathcal{H}ess(\mathsf{X}_I, h))). \tag{10}$$

Lemma 21. *Let μ be a partition of n and $J \subseteq [n-1]$. Then*

$$|\mathcal{D}(J, J_\mu)| = \sum_{I\,:\,J\subseteq I} (-1)^{|I|-|J|} \dim(M^\mu)^{\mathfrak{S}_I}. \tag{11}$$

We now give a proof of Theorem 18, assuming Lemma 20 and Lemma 21 and using Theorem 19.

Proof of Theorem 18. We have:

$$\begin{aligned}
|\mathcal{W}_i(J,h)| &= \sum_{I:J\subseteq I} (-1)^{|I|-|J|} \dim(H^{2i}(\mathcal{H}ess(\mathsf{X}_I, h))) \qquad \text{by Lemma 20} \\
&= \sum_{I:J\subseteq I} (-1)^{|I|-|J|} \sum_{\mu\vdash n} c_{\mu,i} \dim(M^\mu)^{\mathfrak{S}_I} \qquad \text{by Theorem 19} \\
&= \sum_{\mu\vdash n} c_{\mu,i} \left(\sum_{I:J\subseteq I} (-1)^{|I|-|J|} \dim(M^\mu)^{\mathfrak{S}_I} \right) \\
&= \sum_{\mu\vdash n} c_{\mu,i}\, |\mathcal{D}(J, J_\mu)| \quad \text{by Lemma 21}
\end{aligned}$$

which proves equation (8). Equation (9) follows directly from (8) by summing over i. □

3.2 Möbius inversion on the Boolean lattice

We now give proofs of the elementary lemmas used in the previous section. Both follow from an application of the well-known Möbius inversion formula on the Boolean lattice, which is a version of the principle of inclusion-exclusion. We will need the following Betti number formula [11, Lemma 1].

Theorem 22. *Let $J \subseteq [n-1]$ and h be any Hessenberg function. Then for each non-negative integer i, we have*

$$\dim(H^{2i}(\mathcal{H}ess(\mathsf{X}_J, h))) = |\{w \in \mathfrak{S}_n \mid w^{-1}(j) \leqslant h(w^{-1}(j+1))\,\forall j \in J \text{ and } |\mathrm{inv}_h(w)| = i\}|.$$

Using the above, we first prove Lemma 20.

Proof of Lemma 20. Let $\mathcal{W}_i := \{w \in \mathfrak{S}_n \mid |\mathrm{inv}_h(w)| = i\}$ and for each $I \subseteq \Delta$ define $f_I : \mathcal{W}_i \to \{0,1\}$ as follows:

$$f_I(w) = \begin{cases} 1 & \text{if } w \in \mathcal{W}_i(I,h) \\ 0 & \text{else.} \end{cases}$$

For each $I \subseteq \Delta$, let us also define a function $g_I : \mathcal{W}_i \to \{0,1\}$ by

$$g_I(w) = \begin{cases} 1 & \text{if } w^{-1}(j) \leqslant h(w^{-1}(j+1)) \text{ for all } j \in I \\ 0 & \text{else.} \end{cases}$$

Then it is clear that, by definition of f_J,

$$|\mathcal{W}_i(J,h)| = \sum_{w \in \mathcal{W}_i} f_J(w).$$

Next we examine the RHS of (10). By Theorem 22 the RHS is equal to

$$\sum_{I:J\subseteq I} (-1)^{|I|-|J|} |\{w \in \mathcal{W}_i \mid w^{-1}(j) \leqslant h(w^{-1}(j+1)) \text{ for all } j \in I\}|.$$

On the other hand, from the definition of g_I, this is in turn equal to

$$\sum_{I:J\subseteq I} (-1)^{|I|-|J|} \sum_{w \in \mathcal{W}_i} g_I(w) = \sum_{w \in \mathcal{W}_i} \sum_{I:J\subseteq I} (-1)^{|I|-|J|} g_I(w).$$

Therefore, to prove the proposition it would suffice to show that

$$f_J = \sum_{I:J\subseteq I} (-1)^{|I|-|J|} g_I. \tag{12}$$

By definition of g_J and f_I we have $g_J = \sum_{I:J\subseteq I} f_I$ so (12) follows immediately from the Möbius inversion formula on the Boolean lattice. This completes the proof. $\square$

To prove Lemma 21 we first recall the following well-known description of the numbers $\dim(M^\mu)^{\mathfrak{S}_I}$, which can be easily seen from the fact that the integer $\dim(M^\mu)^{\mathfrak{S}_I}$ counts the number of double cosets $\mathfrak{S}_\mu \backslash \mathfrak{S}_n / \mathfrak{S}_I$ [4, Section 2.4]:

$$\dim(M^\mu)^{\mathfrak{S}_I} = |\{w \in \mathfrak{S}_n \mid \mathrm{Des}_L(w) \subseteq [n-1] \setminus I \text{ and } \mathrm{Des}_R(w) \subseteq [n-1] \setminus J_\mu\}|. \tag{13}$$

Proof of Lemma 21. Consider $\mathcal{A}_\mu := \{w \in \mathfrak{S}_n \mid \mathrm{Des}_R(w) \subseteq [n-1] \setminus J_\mu\}$. On $\mathcal{A}_\mu$ define for each $I \subseteq [n-1]$ a function $f_I : \mathcal{A}_\mu \to \{0,1\}$ by

$$f_I(w) = \begin{cases} 1 & \text{if } \mathrm{Des}_L(w) = [n-1] \setminus I \\ 0 & \text{else.} \end{cases}$$

On $\mathcal{A}_\mu$ also define for each $I \subseteq [n-1]$ a function g_I as follows:

$$g_I(w) = \begin{cases} 1 & \text{if } \mathrm{Des}_L(w) \subseteq [n-1] \setminus I \\ 0 & \text{else.} \end{cases}$$

Then it is clear that $|\mathcal{D}(J, J_\mu)| = \sum_{w\in\mathcal{A}_\mu} f_J(w)$ by definition of f.

We now examine the RHS of (11). We have

$$\begin{aligned}
\text{RHS} &= \sum_{I:J\subseteq I} (-1)^{|I|-|J|} \dim(M^\mu)^{\mathfrak{S}_I} \\
&= \sum_{I:J\subseteq I} (-1)^{|I|-|J|} \left|\{w \in \mathfrak{S}_n \mid \mathrm{Des}_L(w) \subseteq [n-1] \setminus I \text{ and } \mathrm{Des}_R(w) \subseteq [n-1] \setminus J_\mu\}\right| \quad \text{by (13)} \\
&= \sum_{I:J\subseteq I} (-1)^{|I|-|J|} \sum_{w\in\mathcal{A}_\mu} g_I(w) \\
&= \sum_{w\in\mathcal{A}_\mu} \sum_{I:J\subseteq I} (-1)^{|I|-|J|} g_I(w).
\end{aligned}$$

Thus it suffices to show that

$$f_J(w) = \sum_{I:J\subseteq I} (-1)^{|I|-|J|} g_I.$$

As in the proof of the previous lemma, this follows immediately from the Möbius inversion formula on the Boolean lattice. This completes the proof. □

4 The matrix equation $AX = W$

We now introduce the matrix equation that is the subject of this section. We will be particularly interested in the sets $\mathcal{W}_i(J, h)$ in the case that $J = \mathbb{J}_\lambda$. Thus, we introduce notation for the cardinality of the sets in (7) for the case $J = \mathbb{J}_\lambda$ and $K = J_\mu$ for two partitions $\lambda, \mu \vdash n$. Specifically, we define

$$A(\lambda, \mu) := |\mathcal{D}(\mathbb{J}_\lambda, J_\mu)|. \tag{14}$$

Using the above notation, Theorem 18 has an immediate corollary, as follows. Let $\mathrm{Par}(n)$ denote the set of partitions of n.

Corollary 23. *Let $A = (A(\lambda, \mu))_{\lambda,\mu\in\mathrm{Par}(n)}$ be the matrix whose coefficients are the integers* (14) *and let $i \in \mathbb{Z}$, $i \geqslant 0$. Let X_i be the (column) vector whose entries are the $c_{\mu,i} \in \mathbb{Z}$ specified in* (3)*. Let W_i be the (column) vector whose entries are the integers $|\mathcal{W}_i(\mathbb{J}_\lambda, h)|$. Then $AX_i = W_i$.*

The main results of this section show that the matrix A has computationally convenient properties with respect to an appropriate choice of total order on $\mathrm{Par}(n)$, in a sense we now explain. The previous section showed that the multiplicity coefficients $c_{\mu,i}$ in (3) obey a set of linear equations, where there is one such linear equation for each partition $\lambda \vdash n$, and by putting these together, Corollary 23 interprets this set of linear equations as a single matrix equation $AX_i = W_i$. Here, each row corresponds to a single linear equation associated to a partition λ. Since the coefficients $c_{\mu,i}$ are also indexed by the set of partitions of n, we see that the matrix $A = (A(\lambda, \mu))$ is in fact a square matrix. With this in mind, we can state the main results of this section. Proposition 24 states that certain matrix entries of A have an inductive description or are equal to 0. Next, Theorem 29 states that – with respect to an appropriately defined total order on the set of partitions of n – the matrix A is upper-triangular with 1's along the diagonal.

We begin with a precise statement of the first main result. Recall that $\lambda[\ell]$ denotes the partition obtained by deleting the first ℓ columns from λ as in Definition 9.

Proposition 24. *Let $\lambda \vdash n$ be a partition with exactly k parts. Let $\mu \vdash n$ be a partition with at most k parts. Then*

1. *if $\mu_k < \lambda_k$, then*
$$\mathcal{D}(\mathbb{J}_\lambda, J_\mu) = \varnothing \text{ and therefore } A(\lambda, \mu) = 0$$
and

2. *if $\mu_k \geqslant \lambda_k$, then for any $\ell \in \mathbb{Z}$ with $0 \leqslant \ell \leqslant \lambda_k$, there exists a natural bijection between the sets*
$$\mathcal{D}(\mathbb{J}_\lambda, J_\mu) \text{ and } \mathcal{D}(\mathbb{J}_{\lambda[\ell]}, J_{\mu[\ell]})$$
and in particular we have
$$A(\lambda, \mu) = A(\lambda[\ell], \mu[\ell]).$$

Before proving Proposition 24, we state the second main result of this section – an upper-triangularity property of the matrix A with respect to an appropriate total order on the set of partitions of n. We have the following.

Definition 25. Let n be a positive integer and let $\mathrm{Par}(n)$ denote the set of partitions of n. We define a total ordering $\preceq$ on $\mathrm{Par}(n)$ as follows:
$$\mu \preceq \lambda \Leftrightarrow \mu' \leqslant_{\text{lex}} \lambda'. \tag{15}$$

Remark 26. We are not aware of where, or whether, the above total order has been studied or used elsewhere in the literature, particularly in the area related to chromatic symmetric functions and the Stanley–Stembridge conjecture.

Example 27. Let $n = 6$ and consider $\lambda = (3, 3)$ and $\mu = (4, 1, 1)$. Note that λ and μ are incomparable in the dominance order, but $\lambda' = (2, 2, 2)$ and $\mu' = (3, 1, 1, 1)$ so $\lambda' <_{\text{lex}} \mu'$ and therefore, according to our definition (15), we have $\lambda \prec \mu$.

Remark 28. It is straightforward to see that lexicographical order of $\mathrm{Par}(n)$, which is a total order, respects the dominance (partial) ordering on $\mathrm{Par}(n)$, in the sense that $\mu \trianglelefteq \lambda$ implies $\mu \leqslant_{\text{lex}} \lambda$. It is also well known that $\mu \trianglelefteq \lambda$ if and only if their dual partitions satisfy the reverse relation, i.e. $\lambda' \trianglelefteq \mu'$. It follows that the total order $\preceq$ of Definition 25 on $\mathrm{Par}(n)$ respects the reversed dominance order.

We now state our upper-triangularity theorem.

Theorem 29. *The matrix* $(A(\lambda,\mu))_{\lambda,\mu\in\mathrm{Par}(n)}$*, written with respect to the total order* (15) *on the indexing set* $\mathrm{Par}(n)$*, is upper-triangular with* 1*'s along the diagonal. Equivalently, for* $\lambda,\mu\in\mathrm{Par}(n)$*, we have the following:*

1. *If* $\mu\prec\lambda$ *with respect to the total order* (15) *then* $\mathcal{D}(\mathbb{J}_\lambda, J_\mu)=\varnothing$*, so in particular,* $A(\lambda,\mu)=0$*.*

2. *The set* $\mathcal{D}(\mathbb{J}_\lambda, J_\lambda)$ *contains a unique element, so in particular,* $A(\lambda,\lambda)=1$*.*

Example 30. When $n=2$ we get the matrix:

$$\begin{aligned}(A(\lambda,\mu))_{\lambda,\mu\in\mathrm{Par}(2)} &= \begin{bmatrix} A((2),(2)) & A((2),(1,1)) \\ A((1,1),(2)) & A((1,1),(1,1)) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\end{aligned}$$

and similarly for $n=3$ we have $\mathrm{Par}(3)=\{(3)\prec(2,1)\prec(1,1,1)\}$ and it can be checked directly that we get the matrix

$$\begin{aligned}(A(\lambda,\mu))_{\lambda,\mu\in\mathrm{Par}(3)} &= \begin{bmatrix} A((3),(3)) & A((3),(2,1)) & A((3),(1,1,1)) \\ A((2,1),(3)) & A((2,1),(2,1)) & A((2,1),(1,1,1)) \\ A((1,1,1),(3)) & A((1,1,1),(2,1)) & A((1,1,1),(1,1,1)) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.\end{aligned}$$

Given that A is upper-triangular, it is natural to ask whether *every* entry above the diagonal is non-zero. The next example shows that the answer is no, i.e., it can happen that $\lambda\prec\mu$ (so $A(\lambda,\mu)$ lies strictly above the diagonal) but that $A(\lambda,\mu)=0$.

Example 31. Let $n=6$ and $\lambda=(3,3)$. In this case $[5]\setminus\mathbb{J}_\lambda=\{1,3,5\}$, so if $w\in\mathcal{D}(\mathbb{J}_\lambda,J_\mu)$ then (in the one-line notation of w) 2 appears to the left of 1, 4 is to the left of 3, and 6 is to the left of 5. When $\mu=\lambda=(3,3)$, the permutation w has $\mathrm{Des}_R(w)\subseteq[5]\setminus J_{(3,3)}=\{3\}$, and thus $w=[2,4,6,1,3,5]$ is the unique permutation in $\mathcal{D}(\mathbb{J}_\lambda,J_\mu)$. This shows that $A((3,3),(3,3))=1$, as expected. Consider now the case $\mu=(4,1,1)$. Then $[5]\setminus J_{(4,1,1)}=\{4,5\}$. Any $w\in\mathcal{D}(\mathbb{J}_\lambda,J_\mu)$ must satisfy $\mathrm{Des}_R(w)\subseteq\{4,5\}$. However, given the constraints on the left descents of w, no such permutation can exist. Thus, although $\lambda\prec\mu$, we still have $A(\lambda,\mu)=0$.

The previous example shows that determining when $A(\lambda,\mu)$ is non-zero (for $\lambda\prec\mu$) is not immediate. On the other hand, we observe in the above example that, although $(3,3)\prec(4,1,1)$ for our total order $\prec$, these two partitions are incomparable in the *dominance* order (c.f. Example 27 and Remark 28). This motivates the following question, which (as far as we know) remains open. We leave further investigation of this question to future work.

Question 32. If $A(\lambda, \mu) \neq 0$, then is it true that $\mu \trianglelefteq \lambda$?

Returning to the previous discussion, the remainder of this section is devoted to the proofs of Proposition 24 and Theorem 29. We need several preliminaries. Let $J = \{i_1 < i_2 < \cdots < i_\ell\} \subseteq [n-1]$. The **staircase decomposition (of $[n]$) corresponding to J** is the decomposition

$$[n] = \{i_0 = 1, 2, \ldots, i_1\} \sqcup \{i_1+1, i_1+2, \ldots, i_2\} \sqcup \cdots \sqcup \{i_{\ell-1}+1, \ldots, i_\ell\} \sqcup \{i_\ell+1, \ldots, n = i_{\ell+1}\}.$$

where by convention we set $i_0 := 1$ and $i_{\ell+1} := n$. Each subset appearing in the above decomposition is called a **staircase**, and we denote by $\mathbb{F}(J) := \ell + 1$ the **number of staircases** in the associated staircase decomposition. The motivation for the "staircase" terminology comes from studying the set of right descents of a permutation $w \in \mathfrak{S}_n$. It follows directly from the definition of $\mathrm{Des}_R(w)$ in (5) that if $\mathrm{Des}_R(w) \subseteq J = \{i_1 < i_2 < \cdots < i_\ell\} \subseteq [n-1]$ then for all $0 \leqslant s \leqslant \ell$ we have

$$w(i_s + 1) < w(i_s + 2) < \cdots < w(i_{s+1}) \tag{16}$$

on each staircase $\{i_s + 1, i_s + 2, \ldots, i_{s+1}\}$ of J.

We also find it convenient to introduce analogous terminology for the permutations themselves. Let $w \in \mathfrak{S}_n$ and $\{i_s + 1, i_s + 2, \ldots, i_{s+1}\} \subseteq [n-1]$ for $i_{s+1} > i_s$ be a sequence of consecutive integers, possibly of length 1 (when $i_{s+1} = i_s + 1$). We say w **is a staircase on the interval $\{i_s + 1, i_s + 2, \ldots, i_{s+1}\}$** if (16) holds. We also say that $\{i_s + 1, i_s + 2, \ldots, i_{s+1}\}$ **is a staircase of w**. A staircase $\{i_s + 1, i_s + 2, \ldots, i_{s+1}\}$ of w is **maximal** if neither $\{i_s, i_s + 1, i_s + 2, \ldots, i_{s+1}\}$ nor $\{i_s + 1, i_s + 2, \ldots, i_{s+1}, i_{s+1} + 1\}$ is a staircase of w. The following is immediate from the definition of the right descent set given in (5) and we omit the proof.

Lemma 33. *Let $w \in \mathfrak{S}_n$. Suppose $J = \{i_1 < i_2 < \cdots < i_\ell\}$. Let $i_0 := 1$ and $i_{\ell+1} := n$. If $\mathrm{Des}_R(w) \subseteq J$, then w is a staircase on each interval $\{i_s+1, i_s+2, \ldots, i_{s+1}\}$ for $0 \leqslant s \leqslant \ell$, and there are at most $\ell + 1$ maximal staircases in the staircase decomposition of w. In particular, suppose $\mu = (\mu_1, \ldots, \mu_k)$ is a partition of n with k parts, and $\mathrm{Des}_R(w) \subseteq [n-1] \setminus J_\mu = \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \cdots + \mu_{k-1}\}$. Then there are at most $\mathbb{F}([n-1] \setminus J_\mu) = k$ maximal staircases of w.*

Example 34. Let $w = [1, 4, 7, 8, 2, 5, 6, 3] \in \mathfrak{S}_8$. Then $\mathrm{Des}_R(w) = \{4, 7\}$ since it is between the 4th and 5th entries, as well as the 7th and 8th entries, that there is a decrease in the one-line notation of w. The maximal staircases of w are $\{1, 2, 3, 4\}$, $\{5, 6, 7\}$ and $\{8\}$. Note that $\mathrm{Des}_R(w) \subseteq [7] \setminus J_\mu$ where $\mu = (4, 3, 1)$. In this case, $\mathbb{F}([7] \setminus J_\mu) = \mathbb{F}(\{4, 7\}) = 3$ is the number of maximal staircases of w, in agreement with the lemma above.

We now turn our attention to left descents. As already noted, for a permutation $w \in \mathfrak{S}_n$, if $i \in \mathrm{Des}_L(w)$ then the pair $(i, i+1)$ has the property that $i + 1$ appears before i in the one-line notation of w. Let $w \in \mathfrak{S}_n$ and $i \in [n]$. For a given staircase of w, we say i **occurs in that staircase** if i appears in the segment of the one-line notation of w corresponding to that staircase.

Example 35. Continuing with Example 34, let $w = [1,4,7,8,2,5,6,3] \in \mathfrak{S}_8$. Then $\{1,2,3,4\}$ is a staircase, and we say that 7 appears in that staircase since 7 occurs as one of the entries in positions 1, 2, 3, or 4 in the one-line notation of w.

Note that any $j \in [n]$ occurs in exactly one maximal staircase of w for any $w \in \mathfrak{S}_n$.

Lemma 36. *Let $w \in \mathfrak{S}_n$. Suppose that $\{j, j+1, \ldots, j+\ell-1\} \subseteq \mathrm{Des}_L(w)$ is a sequence of ℓ consecutive integers contained in $\mathrm{Des}_L(w)$. Then the $\ell+1$ many integers $j+\ell > j+\ell-1 > \cdots > j+1 > j$ must appear in distinct maximal staircases of w, each strictly to the right of the previous one. In particular, the number of maximal staircases of w must be greater than or equal to $\ell+1$.*

Proof. Within each staircase, the entries in the one-line notation of w must be increasing, so any pair of consecutive integers which must appear in inverted order cannot appear in the same staircase. Moreover, if they must be inverted, then the smaller integer must appear to the right of the greater integer i.e., must appear in a staircase strictly to the right of the greater integer. □

The next statement follows from Lemmas 33 and 36.

Corollary 37. *Suppose $K \subseteq [n-1]$ is a subset of $[n-1]$ containing a consecutive sequence of length ℓ. Let $\mu = (\mu_1, \ldots, \mu_k)$ be a partition of n with k parts. Then the set*

$$\mathcal{D}([n-1] \setminus K, J_\mu) = \{w \in \mathfrak{S}_n \mid \mathrm{Des}_L(w) = K \text{ and } \mathrm{Des}_R(w) \subseteq [n-1] \setminus J_\mu\} \tag{17}$$

is empty if $\ell + 1 > k$.

Proof. Suppose $w \in \mathfrak{S}_n$ and that $\mathrm{Des}_L(w) = K$. Since K contains a consecutive sequence of length ℓ, from Lemma 36 it follows that the number of maximal staircases of w is at least $\ell+1$. On the other hand, if $\mathrm{Des}_R(w) \subseteq [n-1] \setminus J_\mu$ then by Lemma 33, we have $\mathbb{F}([n-1] \setminus J_\mu) = k$, and w has at most k maximal staircases. Since $\ell+1 > k$, this cannot occur. Hence (17) is empty as desired. □

In fact, we can say more. The following statement is straightforward and we omit the proof.

Lemma 38. *Let μ be a partition of n with k many parts. Let $w \in \mathfrak{S}_n$ and suppose $\mathrm{Des}_L(w)$ contains a sequence $\{a, a+1, \ldots, a+k-2\} \subseteq [n-1]$ of maximal cardinality $k-1$ and $\mathrm{Des}_R(w) \subseteq [n-1] \setminus J_\mu$. Then:*

1. *$\mathrm{Des}_R(w) = [n-1] \setminus J_\mu$, so the one-line notation of w contains precisely k maximal staircases, and*

2. *for each i such that $0 \leqslant i \leqslant k-1$, the element $a+i$ in the sequence $\{a, a+1, \ldots, a+k-1\}$ must appear in the $(i+1)$st staircase of the one-line notation of w (counting from the left).*

In particular, the staircases in which each $a+i$ must occur is fixed, and exactly one element in the sequence $\{a, a+1, \ldots, a+k-1\}$ occurs in each of the k maximal staircases.

In the course of the argument below it will be useful to have the following terminology. Suppose $w \in \mathfrak{S}_n$ and suppose $m \in \mathbb{Z}$, $1 \leqslant m < n$. There is a map (which is not a group homomorphism)

$$d_{n,m} : \mathfrak{S}_n \to \mathfrak{S}_{n-m}$$

obtained by deleting the entries $\{1, 2, \ldots, m\} = [m]$ in the one-line notation of w, and interpreting what remains as a permutation of $n-m$, under the identification $\{m+1, m+2, \ldots, n\} \cong \{1, 2, \ldots, n-m\}$ given by $j \mapsto j-m$. We will refer to this procedure of applying $d_{n,m}$ as **ignoring the $[\boldsymbol{m}]$ entries** (of the one-line notation of w).

Example 39. Let $m = 2$ and $n = 5$. Let $w = [4, 3, 2, 5, 1]$. Then $d_{5,2}(w) = [2, 1, 3]$ because we first ignore the entries 1 and 2 in $w = [4, 3, \mathbf{2}, 5, \mathbf{1}]$ to obtain $[4, 3, 5]$ and then use the identification $j \mapsto j-2$ to obtain $[2, 1, 3]$.

We are now ready to prove Proposition 24.

Proof of Proposition 24. The proof can be separated into two parts according to the cases given in the statement of the proposition. For simplicity denote $r := \lambda_k$.

We begin with the case $\mu_k < r = \lambda_k$, which itself can be separated into two subcases, namely, $\mu_k = 0$ and $0 < \mu_k < \lambda_k$. First suppose $\mu_k = 0$, i.e., μ has strictly fewer than k parts. From the definition of the set $\mathbb{J}_\lambda$, it follows that there are r many distinct sequences in $[n-1] \setminus \mathbb{J}_\lambda = J_{\lambda'}$, of the form

$$\{1, 2, \ldots k-1\}, \{k+1, k+2, \ldots, 2k-1\}, \ldots, \{(r-1)k+1, \ldots, kr-1\}.$$

This means in particular that the set $[n-1] \setminus \mathbb{J}_\lambda$ contains at least one consecutive sequence of length $k-1$. Applying Corollary 37, we immediately obtain that $\mathcal{D}(\mathbb{J}_\lambda, J_\mu) = \varnothing$ if μ has strictly fewer than k parts.

Next suppose that μ has k parts (i.e. $\mu_k > 0$) but $\mu_k < r = \lambda_k$. Seeking a contradiction, suppose that $w \in \mathcal{D}(\mathbb{J}_\lambda, J_\mu)$, so $\mathrm{Des}_L(w) = [n-1] \setminus \mathbb{J}_\lambda$ and $\mathrm{Des}_R(w) \subseteq [n-1] \setminus J_\mu$. Then w satisfies the hypotheses of Lemma 38 and it follows that the given conditions completely determine the staircases in which the integers $\{1, 2, \ldots, kr\}$ must occur in the one-line notation of w. In fact, since these are the smallest kr integers in $[n]$ and since each staircase must have increasing entries, the hypotheses determine the precise *location* (not just the staircase) in which these entries must occur. In particular, the r many integers $\{1, k+1, 2k+1, \ldots, (r-1)k+1\}$ must appear in the rightmost staircase of w, which contains μ_k many entries. This implies that $\mu_k \geqslant r$, contradicting the assumptions of this case. This concludes the proof of statement (1) of the proposition.

Now we consider the case of $\mu_k \geqslant r = \lambda_k$. By similar reasoning as in the previous paragraph, it follows that if a permutation $w \in \mathfrak{S}_n$ satisfies $\mathrm{Des}_L(w) = [n-1] \setminus \mathbb{J}_\lambda = J_{\lambda'}$ and $\mathrm{Des}_R(w) \subseteq [n-1] \setminus J_\mu$, then w is determined by the location (in the one-line notation) of the integers $\{kr+1, kr+2, \ldots, n\} \cong [n-kr]$, i.e., the image of w under the map $d_{n,kr}$ described above. It is straightforward to see that w is also determined by its image under the map $d_{n,k\ell}$ for any $1 \leqslant \ell \leqslant r$. In what follows, for concreteness we make the argument in detail for the special case $\ell = r$; the argument for ℓ such that $1 \leqslant \ell < r$ is addressed below. Consider the image in $\mathfrak{S}_{n-kr}$ of the set

$$\mathcal{D}(\mathbb{J}_\lambda, J_\mu) = \{w \in \mathfrak{S}_n \mid \mathrm{Des}_L(w) = [n-1] \setminus \mathbb{J}_\lambda \text{ and } \mathrm{Des}_R(w) \subseteq [n-1] \setminus J_\mu\} \tag{18}$$

under the map $d_{n,kr}$ which ignores the $[kr]$ entries. By the above argument, $d_{n,kr}$ is injective on (18). To prove the desired claim, it suffices to show that

$$d_{n,kr}(\mathcal{D}(\mathbb{J}_\lambda, J_\mu)) = \mathcal{D}(\mathbb{J}_{\lambda[r]}, J_{\mu[r]}),$$

that is, that the image of (18) under $d_{n,kr}$ is precisely

$$\mathcal{D}(\mathbb{J}_{\lambda[r]}, J_{\mu[r]}) = \{w' \in \mathfrak{S}_{n-kr} \mid \mathrm{Des}_L(w') = [n-kr-1] \setminus \mathbb{J}_{\lambda[r]} \text{ and } \mathrm{Des}_R(w') \subseteq [n-kr-1] \setminus J_{\mu[r]}\}. \tag{19}$$

To see this, we first show that any $w' = d_{n,kr}(w)$ for w in (18) must lie in (19). Since $\mathrm{Des}_L(w) = [n-1] \setminus \mathbb{J}_\lambda = J_{\lambda'}$, we already know that the $j \in [n-1]$ such that $j+1$ occurs before j in the one-line notation for w with $j > kr$ are precisely the ones of the form

$$\{kr+1, kr+2, \ldots, n\} \setminus \{kr + \lambda'_{r+1}, kr + \lambda'_{r+1} + \lambda'_{r+2}, \ldots, kr + \lambda'_{r+1} + \cdots + \lambda'_t\}$$

where $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_t)$ has t parts and $\lambda'_1 = \cdots = \lambda'_r = k$ by assumption. Notice that $\lambda[r]' = (\lambda'_{r+1}, \lambda'_{r+2}, \ldots, \lambda'_t)$. Under the identification of $\{kr+1, kr+2, \ldots, n\}$ with $[n-kr]$ given by $j \mapsto j - kr$, this means that w' has left descent set $[n-kr-1] \setminus \mathbb{J}_{\lambda[r]}$.

Next we need to show that $\mathrm{Des}_R(w') \subseteq [n-kr-1] \setminus J_{\mu[r]}$. It follows from the above that the entries $\{1, 2, \ldots, kr\}$ distribute themselves in the k staircases of the one-line notation of w in such a way that each staircase contains precisely r many of the entries within $\{1, 2, \ldots, kr\}$. Therefore, when ignoring the $[kr]$ entries in w to obtain w', the locations where the right descents can possibly occur are precisely at

$$\{\mu_1 - r, \mu_1 + \mu_2 - 2r, \ldots, \mu_1 + \cdots + \mu_{k-1} - (k-1)r\}$$

which is exactly the set $[n-kr-1] \setminus J_{\mu[r]}$ for the partition $\mu[r] = (\mu_1 - r, \mu_2 - r, \ldots, \mu_k - r)$. In particular we conclude $\mathrm{Des}_R(w') \subseteq [n-kr-1] \setminus J_{\mu[r]}$ as desired.

Thus $d_{n,kr}$ sends the set (18) into the set (19). In fact, the argument given above is reversible, i.e., any $w' \in \mathfrak{S}_{n-kr}$ lying in (19) can be extended to an element in $\mathfrak{S}_n$ by reversing the correspondence to $j \mapsto j + kr$ and adding the entries $\{1, 2, \ldots, kr\}$ in exactly the locations specified by the hypotheses in (18), and it is clear that this extension then lies in (18). This proves the claim in the special case $\ell = r$. For any $1 \leqslant \ell < r$, by arguments similar to those above it follows that the entries of $d_{n,\ell k}(w)$ corresponding to the integers $\{1, 2, \ldots, (r-\ell)k\}$ are already determined, and so an argument essentially identical to the one above proves the desired claim. This concludes the proof of the proposition. $\square$

From Proposition 24 we readily obtain the following.

Corollary 40. *Let λ, μ be partitions of n and suppose that there exists $\ell \in \mathbb{Z}, \ell \geqslant 1$, such that the dual partitions λ' and μ' agree up to the ℓ-th entry, i.e. $\lambda'_s = \mu'_s$ for all $1 \leqslant s \leqslant \ell$. Then*

$$A(\lambda, \mu) = A(\lambda[\ell], \mu[\ell]).$$

Proof. The argument is a simple induction on the number of steps (in the sense of Definition 12) in the partitions λ and μ on which they agree. More precisely, suppose

$$[\lambda_1] = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_{\mathrm{step}(\lambda)}$$

is the step decomposition of λ and define u to be the index of the step in which ℓ occurs, i.e., suppose $\ell \in A_u$.

We first consider the base case. Suppose $u = 1$. Let k denote the number of parts of λ. Then the Young diagrams of λ and μ both contain as their leftmost ℓ columns a rectangular $k \times \ell$ box. Proposition 24 then implies $A(\lambda, \mu) = A(\lambda[\ell], \mu[\ell])$ as desired. This proves the base case.

Now suppose $u > 1$. Also suppose by induction that the claim is proved for $u - 1$. Since $u > 1$ we know λ and μ both contain a rectangular $k \times r$ box where k is the number of parts of both λ and μ and $r = \lambda_k$ is the bottom length of both λ and μ. Another application of Proposition 24 implies that $A(\lambda, \mu) = A(\lambda[r], \mu[r])$. By assumption, the dual partitions of $\lambda[r]$ and $\mu[r]$ agree up to entry $\ell - r$, and in the step decomposition of $\lambda[r]$, the number $\ell - r$ occurs in step A_{u-1} since we have deleted a full step from λ to obtain $\lambda[r]$. Hence by induction we know $A((\lambda[r])[\ell - r], (\mu[r])[\ell - r]) = A(\lambda[r], \mu[r])$, but from Definition 9 it is clear that $\nu[s][t] = \nu[s + t]$ for any partition ν and s, t for which the statement makes sense, so the result follows. $\square$

We are finally in a position to prove the upper-triangularity property.

Proof of Theorem 29. Since $\lambda' >_{\mathrm{lex}} \mu'$, there exists some $\ell \in \mathbb{Z}_{\geqslant 1}$ such that $\lambda'_s = \lambda'_s$ for all $0 \leqslant s \leqslant \ell$ and $\lambda'_{\ell+1} > \mu'_{\ell+1}$. If no such ℓ exists, then $\lambda_1 > \mu_1$ and we may apply Proposition 24 directly. By Corollary 40, we know $A(\lambda, \mu) = A(\lambda[\ell], \mu[\ell])$. By construction, $\lambda[\ell]$ and $\mu[\ell]$ have the property that $(\lambda[\ell]')_1 > (\mu[\ell]')_1$. Hence by Proposition 24, we have $A(\lambda[\ell], \mu[\ell]) = 0$, as desired.

We also need to show that for any λ, we have $A(\lambda, \lambda) = 1$. Indeed, applying Corollary 40 to $\ell = \lambda_1 - 1$ we obtain that $A(\lambda, \lambda) = A(\lambda[\lambda_1 - 1], \lambda[\lambda_1 - 1])$. By construction, $\lambda[\lambda_1 - 1]$ is a partition with only one column. Therefore we are now reduced to showing that if a partition ν is of the form $\nu = (1, 1, \ldots, 1)$ then $A(\nu, \nu) = 1$. Let ν be such a partition of m for some positive integer $m \leqslant n$. By definition, $\mathbb{J}_\nu = \varnothing = J_\nu$ so we have $[m-1] \setminus \mathbb{J}_\nu = [m-1] = \Delta \setminus J_\nu$. This means $\mathcal{D}(\mathbb{J}_\nu, J_\nu)$ consists of permutations w in $\mathfrak{S}_m$ with the property that every pair $(i, i+1)$ for all $1 \leqslant i \leqslant m-1$ appears inverted in the one-line notation of w, and that for all i such that $1 \leqslant i \leqslant m-1$, we have $w(i) > w(i+1)$. The only such permutation is the longest element $[m, m-1, \ldots, 2, 1] \in \mathfrak{S}_m$, so $\mathcal{D}(\mathbb{J}_\nu, J_\nu)$ is a singleton set and $A(\nu, \nu) = 1$ as desired. This concludes the proof. $\square$

5 An inductive formula for the W-vector

We saw in Corollary 23 that the coefficients $c_{\mu,i}$ of (3), when written as a column vector $X_i = (c_{\mu,i})$, satisfy a matrix equation $AX_i = W_i$. In order to solve this matrix equation, we need to analyze the "constant vector" W_i for each i. This is the purpose of this section.

Recall that the vector W_i is defined to have entries $|\mathcal{W}_i(\mathbb{J}_\lambda, h)|$, where the sets $\mathcal{W}_i(\mathbb{J}_\lambda, h)$ are introduced in Definition 16, and λ varies over the partitions of n. The main result (Theorem 65) of this section is an inductive description of the set $\mathcal{W}_i(\mathbb{J}_\lambda, h)$ in the case that λ has $k = ht(I_h) + 1$ parts. However, we also emphasize that this assumption – namely, that λ has exactly $k = ht(I_h) + 1$ parts – is *not* required for many of the results in this section which lead up to Theorem 65.

For simplicity, in this section we identify the subsets $J \subseteq [n-1]$ with subsets of the simple roots $\{\alpha_i \mid i \in J\} \subseteq \Delta$, as explained in Remark 6. Recall that, as explained in Remark 17, this identification yields a corresponding root-theoretic description of the sets $\mathcal{W}_i(\mathbb{J}_\lambda, h)$. This Lie-theoretic language is more convenient for our purposes here and below, so henceforth we use these root-theoretic identifications.

5.1 Sink sets and subsets of height k

In order to obtain our inductive formula, we exploit the structural relationship between the ideal I_h and graph Γ_h alluded to in Section 2. Recall the following notation from [9].

- We let $\mathcal{A}(\Gamma_h)$ denote the set of all acyclic orientations of Γ_h and $\mathcal{A}_k(\Gamma_h)$ denote the set of all acyclic orientations with exactly k sinks.
- Given $\omega \in \mathcal{A}(\Gamma_h)$ we denote the subset of vertices that occur as sinks of ω by $\mathrm{sk}(\omega)$. Note that each independent set of vertices in Γ_h occurs as the sink set of some acyclic orientation and $\mathrm{sk}(\omega)$ is independent for each $\omega \in \mathcal{A}(\Gamma_h)$.
- Let $\mathrm{SK}_k(\Gamma_h)$ be the set of all possible sink sets (or, independent sets) of Γ of cardinality k.
- The maximum sink set size $m(\Gamma_h)$ is the maximum of the cardinalities of the sink sets $\mathrm{sk}(\omega)$ associated to all possible acyclic orientations of Γ_h, i.e.,

$$m(\Gamma_h) := \max\{|\mathrm{sk}(\omega)| \mid \omega \in \mathcal{A}(\Gamma_h)\}.$$

The **sink set decomposition** is

$$\mathcal{A}_k(\Gamma_h) = \bigsqcup_{T \in \mathrm{SK}_k(\Gamma_h)} \{\omega \in \mathcal{A}_k(\Gamma_h) \mid \mathrm{sk}(\omega) = T\}. \tag{20}$$

With this terminology in place, our goal is to extend the sink set decomposition of $\mathcal{A}_k(\Gamma_h)$ to a sink set decomposition of the set $\mathcal{W}(\mathbb{J}_\lambda, h)$.

For $T \in \mathrm{SK}(\Gamma_h)$ let $\Gamma_h[T] := \Gamma_h - T$ be the graph obtained from Γ_h be deleting the vertices in T and all incident edges. Then $\Gamma_h[T]$ is the incomparability graph for a Hessenberg function $h[T] : [n-k] \to [n-k]$ as shown in [9, Lemma 4.3].

Remark 41. It is not difficult to see from the definitions of $ht(I_h)$ and $m(\Gamma_h)$ that $m(\Gamma_{h[T]}) \leqslant m(\Gamma_h)$, or equivalently, that $ht(I_{h[T]}) \leqslant ht(I_h)$ (cf. also [9, Proposition 5.8, Corollary 5.12, Lemma 5.13]).

Note that any acyclic orientation of Γ_h induces an acyclic orientation of $\Gamma_h[T]$, as demonstrated in the example below.

Example 42. Let $h = (2, 3, 5, 6, 7, 8, 8, 8)$, and consider the following acyclic orientation ω of Γ_h displayed below.

$$\begin{tikzcd} 1 & 2 \arrow[l] \arrow[r] & 3 & 4 \arrow[l] \arrow[r] \arrow[rr, bend left] & 5 \arrow[r] \arrow[ll, bend right] & 6 & 7 \arrow[l] \arrow[r] \arrow[ll, bend right] & 8 \arrow[ll, bend right] \end{tikzcd}$$

This acyclic orientation has $T = \mathrm{sk}(\omega) = \{1, 3, 6\}$, where the vertices in $\mathrm{sk}(\omega)$ and all incident edges are highlighted in red for emphasis. For this graph, we have $m(\Gamma_h) = 3$. The graph below shows $\Gamma[T]$ with the acyclic orientation induced from Γ_h.

$$\begin{tikzcd} 2 & 4 \arrow[r] & 5 & 7 \arrow[l] \arrow[r] & 8 \end{tikzcd}$$

which corresponds to the Hessenberg function $h[T] = (1, 3, 4, 5, 5)$. Note that we could also re-index the vertices of $\Gamma[T]$ to obtain the following acyclic graph.

$$\begin{tikzcd} 1 & 2 \arrow[r] & 3 & 4 \arrow[l] \arrow[r] & 5 \end{tikzcd}$$

An orientation $\omega \in \mathcal{A}(\Gamma_h)$ assigns each edge e a source and a target; we notate the source (respectively target) of e according to the orientation ω by $\mathrm{src}_\omega(e)$ (respectively $\mathrm{tgt}_\omega(e)$). Given an orientation ω of Γ_h we let

$$\mathrm{asc}(\omega) := \{e = \{a, b\} \mid \mathrm{src}_\omega(e) = a, \mathrm{tgt}_\omega(e) = b, \text{ and } a < b\}.$$

In other words, if Γ_h is drawn as in Example 42 with the labels of the vertices increasing from left to right, then $\mathrm{asc}(\omega)$ counts the number of edges which point to the right.

Given a sink set $T \in \mathrm{SK}(\Gamma_h)$ the **degree of $\boldsymbol{T}$** is

$$\deg_h(T) := \min\{\mathrm{asc}(\omega) \mid \omega \in \mathcal{A}(\Gamma_h) \text{ and } \mathrm{sk}(\omega) = T\}.$$

For example, $\deg_h(T) = 3$ for the h and T as appearing in Example 42. The next lemma is [9, Lemma 4.8], and shows that in practice it is easy to compute $\deg_h(T)$ for any $T \in \mathrm{SK}(\Gamma_h)$.

Lemma 43. *([9, Lemma 4.8]) Let $T \in \mathrm{SK}(\Gamma_h)$. Then*

$$\deg_h(T) = |\{e = \{a, b\} \in E(\Gamma_h) \mid a < b,\, b \in T\}|.$$

We will see that sink sets in Γ_h correspond bijectively to certain subsets of roots in I_h. In particular, we need the following definition.

Definition 44. Let $R \subseteq \Phi^-$. We say R is a **subset of height k** if there exist integers $q_1, q_2, \ldots, q_k, q_{k+1} \in [n]$ such that $q_1 < q_2 < \ldots < q_k < q_{k+1}$ and $R = \{t_{q_2} - t_{q_1}, t_{q_3} - t_{q_2}, \ldots, t_{q_{k+1}} - t_{q_k}\}$. We let $\mathcal{R}_k(I)$ denote the set of all subsets of height k in an ideal I, and define $\mathcal{R}(I) := \bigsqcup_{k \geqslant 0} \mathcal{R}_k(I)$.

It is easy to show that $R \subseteq \Phi^-$ is a subset of height k if and only if there exists $w \in \mathfrak{S}_n$ such that $w(R)$ is a subset of simple roots corresponding to k consecutive vertices in the Dynkin diagram for $\mathfrak{gl}(n, \mathbb{C})$. The set $\mathcal{R}(I)$ can also be used to compute the height of the ideal. The following is [9, Lemma 5.5].

Lemma 45. *([9, Lemma 5.5]) Let I be a nonempty ideal in Φ^-. Then*

$$ht(I) = \max\{|R| \mid R \in \mathcal{R}(I)\}.$$

Recall that [9, Section 5] defines a bijection:

$$\mathrm{SK}_k(\Gamma_h) \to \mathcal{R}_{k-1}(I_h); \;\; T \mapsto R_T := \{\beta_i = t_{\ell_{i+1}} - t_{\ell_i} \mid 1 \leqslant i \leqslant k-1\} \tag{21}$$

where $T = \{\ell_1 < \ell_2 < \cdots < \ell_k\}$. By Lemma 45, this bijection shows that the maximum size of any sink set in Γ_h is precisely $ht(I_h) + 1$, as noted in Section 2.

Example 46. Let h and T be an in Example 42. The bijection defined in (21) above associates $T = \{1, 3, 6\} \in \mathrm{SK}_3(\Gamma_h)$ to the subset

$$\{t_3 - t_1, t_6 - t_3\} \in \mathcal{R}_2(\Gamma_h).$$

Since $3 = m(\Gamma_h) = ht(\Gamma_h) + 1$, we know that I_h cannot contain any subsets of height $k \geqslant 3$. This line of reasoning is essential for proving the inductive formulas later in this section.

5.2 Another sink-set decomposition

Throughout this section, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition of n with k parts. In this section we will show that the sets $\mathcal{W}(\mathbb{J}_\lambda, h)$ have a sink set decomposition. First we define a subset of $\mathcal{W}(\mathbb{J}_\lambda, h)$ associated to each sink set.

Definition 47. Given $T = \{\ell_1 < \ell_2 < \cdots < \ell_k\} \in \mathrm{SK}_k(\Gamma_h)$ we define

$$\mathcal{W}_i(\mathbb{J}_\lambda, h, T) := \{w \in \mathcal{W}_i(\mathbb{J}_\lambda, h) \mid w(\ell_j) = k - j + 1,\, 1 \leqslant j \leqslant k\}.$$

and let $\mathcal{W}(\mathbb{J}_\lambda, h, T) = \sqcup_i \mathcal{W}_i(\mathbb{J}_\lambda, h, T)$ where the union is taken over all i such that $\mathcal{W}_i(\mathbb{J}_\lambda, h, T) \neq \varnothing$.

The conditions defining $\mathcal{W}(\mathbb{J}_\lambda, h, T)$ tell us that if $w \in \mathcal{W}(\mathbb{J}_\lambda, h, T)$ then:

$$k, k-1, \ldots, 2, 1 \text{ appear in positions } \ell_1, \ell_2, \ldots, \ell_{k-1}, \ell_k \text{ in the one-line notation for } w. \tag{22}$$

In particular, $(k, k-1, \ldots, 2, 1)$ is a subsequence of the one-line notation for w.

Example 48. Let $h = (2,3,5,6,7,8,8,8)$ and $T = \{1,3,6\}$ as in Example 42. Consider $\lambda = (3,3,2)$; in this case $\mathbb{J}_\lambda = \{\alpha_3, \alpha_6\}$. We have, for example, that $w \in \mathcal{W}(\mathbb{J}_\lambda, h, T)$ where

$$w = [\mathbf{3}, 6, \mathbf{2}, 8, 5, \mathbf{1}, 7, 4].$$

Note that in the example above, $w^{-1}(\{\alpha_1, \alpha_2\}) = \{t_3 - t_1, t_6 - t_3\} = R_T$, where R_T was computed in Example 46. The next lemma shows that this property characterizes the elements of $\mathcal{W}_i(\mathbb{J}_\lambda, h, T)$.

Lemma 49. *Let $T \in \mathrm{SK}_k(\Gamma_h)$. Then $w \in \mathcal{W}_i(\mathbb{J}_\lambda, h, T)$ if and only if $w \in \mathcal{W}_i(\mathbb{J}_\lambda, h)$ and $R_T = w^{-1}(\{\alpha_1, \ldots, \alpha_{k-1}\})$.*

Proof. If $w \in \mathcal{W}_i(\mathbb{J}_\lambda, h, T)$ for $T = \{\ell_1, \ell_2, \ldots, \ell_k\}$, then $w \in \mathcal{W}_i(\mathbb{J}_\lambda, h)$ and

$$w^{-1}(\alpha_{k-j}) = w^{-1}(t_{k-j} - t_{k-j+1}) = t_{\ell_{j+1}} - t_{\ell_j} \text{ for all } j = 1, \ldots, k-1$$

by definition of $\mathcal{W}_i(\mathbb{J}_\lambda, h, T)$. Now the definition of R_T given in (21) implies

$$w^{-1}(\{\alpha_1, \ldots, \alpha_{k-1}\}) = R_T \in \mathcal{R}_{k-1}(I_h)$$

as desired.

To show the converse, suppose $w \in \mathcal{W}_i(\mathbb{J}_\lambda, h)$ and $w^{-1}(\{\alpha_1, \ldots, \alpha_{k-1}\}) = R_T$ where $T = \{\ell_1, \ell_2, \ldots, \ell_k\} \in \mathrm{SK}_k(\Gamma_h)$. Then

$$w^{-1}(\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}\}) = R_T := \{t_{\ell_2} - t_{\ell_1}, t_{\ell_3} - t_{\ell_2}, \ldots, t_{\ell_k} - t_{\ell_{k-1}}\}.$$

All that remains to show is that $w(\ell_j) = k - j + 1$ for all $1 \leqslant j \leqslant k$. The equation above implies $w(\ell_j) \in \{1, 2, \ldots, k\}$. Observe that $w^{-1}(\{\alpha_1, \ldots, \alpha_{k-1}\}) = R_T$ implies $w(R_T) = \{\alpha_1, \ldots, \alpha_{k-1}\}$. Thus we also know $w(\ell_j) = w(\ell_{j+1}) + 1$ since

$$w(t_{\ell_{j+1}} - t_{\ell_j}) = t_{w(\ell_{j+1})} - t_{w(\ell_j)} \in \{\alpha_1, \ldots, \alpha_{k-1}\}.$$

This can only be the case if $\ell_1 = k$, $\ell_2 = k-1$, and so on. We conclude $w(\ell_j) = k - j + 1$ for each k as desired. $\square$

The next proposition generalizes the sink set decomposition given in (20) and gives a sink set decomposition of the set $\mathcal{W}_i(\mathbb{J}_\lambda, h)$ for each i.

Proposition 50. *Let n be a positive integer and $h : [n] \to [n]$ a Hessenberg function. Let $i \in \mathbb{Z}$, $i \geqslant 0$ and λ be a partition of n with k parts. Then*

$$\mathcal{W}_i(\mathbb{J}_\lambda, h) = \bigsqcup_{T \in \mathrm{SK}_k(\Gamma_h)} \mathcal{W}_i(\mathbb{J}_\lambda, h, T). \tag{23}$$

We call the decomposition (23) the **sink set decomposition of $\boldsymbol{\mathcal{W}(\mathbb{J}_\lambda, h)}$.**

Proof. It is straightforward from the definition of the sets $\mathcal{W}_i(\mathbb{J}_\lambda, h, T)$ that the RHS of (23) is contained in the LHS. Thus we have only to prove the opposite inclusion. Let $w \in \mathcal{W}_i(\mathbb{J}_\lambda, h)$. By definition, $w^{-1}(\Delta \setminus \mathbb{J}_\lambda) \subseteq I_h$. Since $\{\alpha_1, \ldots, \alpha_{k-1}\} \subseteq \Delta \setminus \mathbb{J}_\lambda = J_{\lambda'}$ it follows immediately that

$$t_{w^{-1}(1)} - t_{w^{-1}(2)}, t_{w^{-1}(2)} - t_{w^{-1}(3)}, \ldots, t_{w^{-1}(k-1)} - t_{w^{-1}(k)} \in I_h.$$

In particular, $R = w^{-1}(\{\alpha_1, \ldots, \alpha_{k-1}\})$ is a subset of I_h of height $k-1$. Since (21) is a bijection, there exists a unique sink set $T \in \mathrm{SK}_k(\Gamma_h)$ such that $R = R_T$ and therefore $w \in \mathcal{W}_i(\mathbb{J}_\lambda, h, T)$ by Lemma 49. □

5.3 Inductive Formulas

Our next goal is to identify each set $\mathcal{W}_i(\mathbb{J}, h, T)$ with a subset of permutations in $\mathfrak{S}_{n-k}$. The following notation generalizes [9, Definition 7.3].

Definition 51. Suppose $T \in \mathrm{SK}_k(\Gamma_h)$ with $T = \{\ell_1 < \ell_2 < \cdots < \ell_k\}$ and $\lambda \vdash n$ with k parts. Define a permutation in $\mathfrak{S}_n$, denoted w_T, by:

1. $w_T(\ell_j) = k - j + 1$, $1 \leqslant j \leqslant k$, i.e. w_T satisfies (22), and
2. the remaining entries in the one-line notation of w_T list the integers $[n] \setminus T$ in increasing order from left to right.

Example 52. Let $h = (2, 3, 5, 6, 7, 8, 8, 8)$ and $T = \{1, 3, 6\}$ as in Example 42. Then

$$w_T = [\mathbf{3}, 4, \mathbf{2}, 5, 6, \mathbf{1}, 7, 8]$$

where the entries in positions $\ell_1 = 1$, $\ell_2 = 3$ and $\ell_3 = 6$ are bolded for emphasis. Note that w_T need not be an element of $\mathcal{W}(\mathbb{J}_\lambda, h, T)$. For example $w_T \notin \mathcal{W}(\mathbb{J}_\lambda, h, T)$ when $\lambda = (3, 3, 2)$ is the same partition considered in Example 48 since

$$w_T^{-1}(\alpha_4) = w_T^{-1}(t_4 - t_5) = t_2 - t_4 \in \Phi_h$$

so w_T does not satisfy the condition that $w_T^{-1}(\Delta \setminus \mathbb{J}_\lambda) \subseteq I_h$.

For each sink set $T = \{\ell_1 < \ell_2 < \cdots < \ell_k\}$ let $\mathsf{f}_T : ([n] \setminus T) \to [n-k]$ be the bijection such that $\phi_T(j) = j - j'$ where j' denotes the number of elements $i \in T$ such that $i \leqslant j$. This bijection can be used to give explicit formulas for w_T, as noted in the following remark.

Remark 53. The conditions defining w_T can be written explicitly in formulas involving f_T as follows.

- If $j > k$ then $w_T^{-1}(j)$, the position of j in the one-line notation for w_T, is the unique element of $[n]$ such that $\mathsf{f}_T(w_T^{-1}(j)) = j - k$, and
- if $j \in [n] \setminus T$ we have $w_T(j) = \mathsf{f}_T(j) + k$.

In fact, the above formulas uniquely determine the bijection f_T, which must then be as defined in the preceding paragraph.

Example 54. We continue Example 52 from above. Here $k = 3$ and $w_T = [\mathbf{3}, 4, \mathbf{2}, 5, 6, \mathbf{1}, 7, 8]$. In particular, we have $T = \{1, 3, 6\}$ and

$$\mathsf{f}_T(2) = 1,\ \mathsf{f}_T(4) = 2,\ \mathsf{f}_T(5) = 3,\ \ \mathsf{f}_T(7) = 4,\ \ \mathsf{f}_T(8) = 5.$$

Notice that f_T is the natural bijection we used to relabel the vertices of $\Gamma_h[T]$ in Example 42. The reader can easily verify the formulas given in Remark 53 in this case. For example,

$$w_T(2) = \mathsf{f}_T(2) + 3 = 4 \ \text{ and } \ \mathsf{f}_T(w_T^{-1}(6)) = \mathsf{f}_T(5) = 3 = 6 - 3.$$

The following is a generalization of [9, Lemma 7.6].

Lemma 55. *Let $T = \{\ell_1 < \ell_2 < \cdots < \ell_k\}$ be a sink set of cardinality k. Each element $w \in \mathfrak{S}_n$ satisfying condition (1) of Definition 51 can be written uniquely as $w = w_T\sigma$ where $\sigma \in \operatorname{Stab}(\ell_1, \ell_2, \ldots, \ell_k)$.*

Proof. The hypotheses on w determine the entries in positions $\ell_1, \ell_2, \ldots, \ell_k$ in one-line notation. The other entries must be a permutation of the set $[n] \setminus \{\ell_1, \ell_2, \ldots, \ell_k\}$, and the hypotheses on w place no conditions on this permutation. Recall that for w_T and any permutation $\sigma \in \mathfrak{S}_n$, right-composition with σ acts on the positions, i.e. if w_T sends i to $w_T(i)$, then $w_T\sigma$ sends i to $w_T(\sigma(i))$. Thus, if σ stabilizes $\ell_1, \ell_2, \ldots, \ell_k$, then $w = w_T\sigma$ satisfies $w(\ell_j) = w_T(\ell_j) = k - j + 1$ for all $j = 1, \ldots, k$. Moreover, it is straightforward to see that such a σ is unique. □

Corollary 56. *Each $w \in \mathcal{W}_i(\mathbb{J}_\lambda, h, T)$ can be written uniquely as $w = w_T\sigma$ where $\sigma \in \operatorname{Stab}(\ell_1, \ell_2, \ldots, \ell_k)$.*

Proof. By definition, each element of $\mathcal{W}_i(\mathbb{J}_\lambda, h, T)$ satisfies condition (1) of Definition 51. □

Example 57. Let $w = [\mathbf{3}, 6, \mathbf{2}, 8, 5, \mathbf{1}, 7, 4] \in \mathcal{W}(\mathbb{J}_{(3,3,2)}, h, T)$ for $h = (2, 3, 5, 6, 7, 8, 8, 8)$, as shown in Example 48. In this case, the factorization $w = w_T\sigma$ gives us

$$\sigma = [\mathbf{1}, 5, \mathbf{3}, 8, 4, \mathbf{6}, 7, 2] \in \operatorname{Stab}(1, 3, 6).$$

The bijection f_T defined above induces a natural isomorphism:

$$\operatorname{Stab}(\ell_1, \ell_2, \ldots, \ell_k) \to \mathfrak{S}_{n-k}; \quad \sigma \mapsto x_\sigma$$

defined as follows. Given $\sigma \in \operatorname{Stab}(\ell_1, \ell_2, \ldots, \ell_{rk})$, delete positions $\ell_1, \ell_2, \ldots, \ell_k$ from the one-line notation for σ and then apply f_T to the remaining entries to obtain x_σ. The result is clearly an element in $\mathfrak{S}_{n-k}$ and each element of $\mathfrak{S}_{n-k}$ arises in this way.

Example 58. The element $\sigma = [\mathbf{1}, 5, \mathbf{3}, 8, 4, \mathbf{6}, 7, 2] \in \operatorname{Stab}(1, 3, 6)$ obtained in Example 57 above maps to $x_\sigma = [3, 5, 2, 4, 1] \in \mathfrak{S}_5$.

By Lemma 55, for each $T \in \mathrm{SK}_k(\Gamma_h)$ we get a well defined bijection

$$\Psi_T : \{w \in \mathfrak{S}_n : w \text{ satisfies condition (1) of Definition 51 }\} \to \mathfrak{S}_{n-rk}$$

defined by $\Psi_T(w_{\lambda,T}\sigma) = x_\sigma$. Note that Ψ_T is very similar to the map $d_{n,m} : \mathfrak{S}_n \to \mathfrak{S}_{n-m}$ defined in Section 4 and used in the proof of Proposition 24. Indeed, using the language of that section, applying Ψ_T can be described as ignoring the $[k]$ entries in the one-line notation of w.

Recall that there is a natural Lie subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ obtained by "setting the variables in row/columns $\{\ell_1, \ell_2, \dots, \ell_k\}$ equal to zero." More precisely, there is a natural Lie algebra isomorphism

$$\{X \in \mathfrak{gl}(n, \mathbb{C}) \,|\, X_{ij} = 0 \text{ if } \{i, j\} \cap T \neq \varnothing\} \cong \mathfrak{gl}(n-k, \mathbb{C}). \tag{24}$$

defined explicitly on the basis $\{E_{ij} \,|\, \{i, j\} \cap T = \varnothing\}$ of the LHS by $E_{ij} \mapsto E_{\mathfrak{f}_T(i)\mathfrak{f}_T(j)}$.

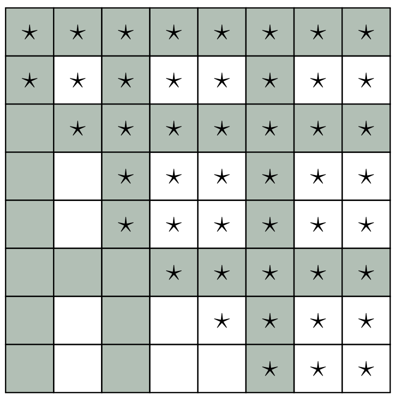
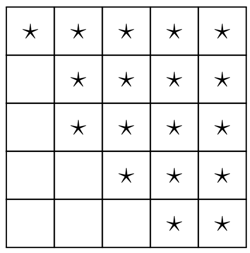
Recall that for each $T \in \mathrm{SK}_k(\Gamma_h)$ we have an associated Hessenberg function $h[T] : [n-k] \to [n-k]$ whose incomparability graph is obtained by deleting the vertices in T and any incident edges from Γ_h. In fact, this Hessenberg function corresponds to the Hessenberg space $H \cap \mathfrak{gl}(n-k, \mathbb{C})$ under the identification in (24). (See [9, Section 4] for more details on this perspective.) We identify the set of roots

$$\Phi[T] := \{t_i - t_j \in \Phi \,|\, \{i, j\} \cap T = \varnothing\} \subseteq \Phi$$

with the root system of $\mathfrak{gl}(n-k, \mathbb{C})$ via

$$t_i - t_j \mapsto t_{\mathfrak{f}_T(i)} - t_{\mathfrak{f}_T(j)}. \tag{25}$$

Example 59. We demonstrate the identifications from (24) and (25) in the running example started in Example 42, with $h = (2, 3, 5, 6, 7, 8, 8, 8)$. To visualize what is going on, we represent $\mathfrak{gl}(8, \mathbb{C})$ as an 8×8 grid with a star placed in the (i, j)-box precisely when the root (i, j) is contained in Φ_h. The boxes highlighted in grey correspond the roots in $\Phi \setminus \Phi[T]$ so the white boxes containing a star correspond to the roots in $\Phi_h[T] := \Phi[T] \cap \Phi_h$, to be discussed further below.

$\mathfrak{gl}(8, \mathbb{C})$:  $\mathfrak{gl}(5, \mathbb{C})$: 

Note that the map in (25) is an isomorphism of root systems, where $\Phi[T]$ is viewed as a subroot system of Φ (since $\Phi[T]$ is closed under addition in Φ). Moreover, the subsets $\Phi_h[T] := \Phi_h \cap \Phi[T]$ and $\Phi_h^-[T] := \Phi_h^- \cap \Phi[T]$ correspond to $\Phi_{h[T]}$ and $\Phi_{h[T]}^-$ respectively, via (25).

Remark 60. The root system isomorphism given in (25) is compatible with the corresponding identification $\mathrm{Stab}(\ell_1, \ldots, \ell_k)$ given in (24) in the following sense. If $\sigma \in \mathrm{Stab}(\ell_1, \ldots, \ell_k)$ and $t_i - t_j \in \Phi[T]$ then $\sigma(t_i - t_j) \in \Phi[T]$ and

$$t_k - t_\ell = \sigma(t_i - t_j) \Leftrightarrow t_{\mathsf{f}_T(k)} - t_{\mathsf{f}_T(\ell)} = x_\sigma(t_{\mathsf{f}_T(i)} - t_{\mathsf{f}_T(j)}).$$

Recall that for a permutation $w \in \mathfrak{S}_n$ we define

$$\mathrm{inv}(w) := \{(i,j) \mid i > j \text{ and } w(i) < w(j)\}.$$

We identify $\mathrm{inv}(w)$ with the subset of negative roots $\Phi^- \cap w^{-1}(\Phi^+) = \{t_i - t_j \mid (i,j) \in \mathrm{inv}(w)\}$ throughout this section. Then (25) gives a bijection between $\mathrm{inv}(\sigma) \cap \Phi[T]$ and $\mathrm{inv}(x_\sigma)$ and a bijection between $\mathrm{inv}_h(\sigma) \cap \Phi[T]$ and $\mathrm{inv}_{h[T]}(x_\sigma)$.

Lemma 61. *Let $T \in \mathrm{SK}_k(\Gamma_h)$. Then*

1. $\mathrm{inv}(w_T) = \{(i,j) \,|\, i > j \text{ and } i \in T\}$, *and*
2. *if $w = w_T\sigma$ for $\sigma \in \mathrm{Stab}(\ell_1, \ldots, \ell_k)$ then*

$$\mathrm{inv}(w) = \mathrm{inv}(w_T) \sqcup (\mathrm{inv}(\sigma) \cap \Phi[T]). \tag{26}$$

Proof. We begin by proving statement (1). If $(i,j) \in \mathrm{inv}(w_T)$ then $i > j$ and $w_T(i) < w_T(j)$. If $i \notin T$, then $w_T(j) > w_T(i) > k$ so from the construction of w_T we conclude $j \notin T$. But the entries in the one-line notation of w_T for $i, j \notin T$ cannot be inverted, by Definition 51(2). Hence $w_T(j) > w_T(i)$, yielding a contradiction. Therefore $i \in T$ as desired. On the other hand, consider (i,j) with $i > j$ and $i \in T$. Since $i \in T$, we may write $i = \ell_{i_0}$ for some i_0 with $1 \leqslant i_0 \leqslant k$. If $j \in T$, then $j = \ell_{j_0}$ for some j_0 with $1 \leqslant j_0 \leqslant k$ such that $j_0 < i_0$ (since $j < i$) and we have

$$w_T(i) = w_T(\ell_{i_0}) = k - i_0 + 1 < k - j_0 + 1 = w_T(\ell_{j_0}) = w_T(j)$$

so $(i,j) \in \mathrm{inv}(w_T)$. If $j \notin T$, then $w_T(j) > k$ and therefore

$$w_T(i) \leqslant k < w_T(j)$$

so $(i,j) \in \mathrm{inv}(w_T)$ also. This proves (1).

Next we prove (2). Let w be as given. Note that since $\sigma \in \mathrm{Stab}(\ell_1, \ldots, \ell_k)$, we have $w(T) = w_T(T) = \{1, 2, \ldots, k\}$. Our proof relies on this fact, as well as the formulas given in Remark 53. We first show the inclusion $\mathrm{inv}(w) \subseteq \mathrm{inv}(w_T) \sqcup (\mathrm{inv}(\sigma) \cap \Phi[T])$. Let $(i,j) \in \mathrm{inv}(w)$. If $i \in T$ then $(i,j) \in \mathrm{inv}(w_T)$ by (1). If $i \notin T$, then $k < w(i) < w(j)$ so $j \notin T$ as above and we conclude $(i,j) \in \Phi[T]$. Since $\sigma \in \mathrm{Stab}(\ell_1, \ldots, \ell_k)$ and σ is a permutation σ also preserves the complement $[n] \setminus \{\ell_1, \dots, \ell_k\} = [n] - T$. Hence if $i \notin T$ then $\sigma(i) \notin T$ also. Using this fact and the formulas from Remark 53 we now have

$$\mathsf{f}_T(\sigma(i)) + k = w_T\sigma(i) < w_T\sigma(j) = \mathsf{f}_T(\sigma(j)) + k \Rightarrow \mathsf{f}_T(\sigma(i)) < \mathsf{f}_T(\sigma(j)) \Rightarrow \sigma(i) < \sigma(j)$$

since f_T^{-1} is an increasing function. Therefore $(i, j) \in \mathrm{inv}(\sigma) \cap \Phi[T]$.

To prove the opposite inclusion, suppose $(i, j) \in \mathrm{inv}(w_T)$. By (1), we know $i \in T$. If $j \in T$ then

$$w(i) = w_T\sigma(i) = w_T(i) < w_T(j) = w_T\sigma(j) = w(j)$$

so $(i, j) \in \mathrm{inv}(w)$. If $j \notin T$ then $w(j) = w_T\sigma(j) > k$ and

$$w(i) = w_T\sigma(i) = w_T(i) \leqslant k < w(j)$$

so $(i, j) \in \mathrm{inv}(w)$ in this case also. Hence $\mathrm{inv}(w_T) \subseteq \mathrm{inv}(w)$. Next suppose $(i, j) \in \mathrm{inv}(\sigma) \cap \Phi[T]$. This means $i, j \notin T$ and thus we know, as above, that $\sigma(i), \sigma(j) \notin T$ also. Hence

$$w(i) = w_T\sigma(i) = f_T(\sigma(i)) + k < f_T(\sigma(j)) + k = w_T\sigma(j) = w(j)$$

since f_T is increasing and $\sigma(i) < \sigma(j)$ by assumption. Therefore $\mathrm{inv}(\sigma) \cap \Phi[T] \subseteq \mathrm{inv}(w)$ also. This completes the proof. $\square$

Example 62. Continuing the running example, we have

$$w_T\sigma = w = [\mathbf{3}, 6, \mathbf{2}, 8, 5, \mathbf{1}, 7, 4] \in \mathcal{W}(\mathbb{J}_{(3,3,2)}, h, T)$$

where $w_T = [\mathbf{3}, 4, \mathbf{2}, 5, 6, \mathbf{1}, 7, 8]$ and $\sigma = [\mathbf{1}, 5, \mathbf{3}, 8, 4, \mathbf{6}, 7, 2] \in \mathrm{Stab}(1, 3, 6)$. In this case it can be checked that

$$\begin{aligned}\mathrm{inv}(w) = \{&(6,1), (6,2), (6,3), (6,4), (6,5), (3,1), (3,2),\\ &(8,2), (8,4), (8,5), (8,7), (5,2), (5,4), (7,4)\}\end{aligned}$$

where

$$\mathrm{inv}(w_T) = \{(6,1), (6,2), (6,3), (6,4), (6,5), (3,1), (3,2)\}$$

and

$$\mathrm{inv}(\sigma) \cap \Phi[T] = \{(8,2), (8,4), (8,5), (8,7), (5,2), (5,4), (7,4)\}.$$

It is, in general, not the case that $\ell(w) = \ell(w_T) + \ell(\sigma)$ (where $\ell(w)$ denotes the Bruhat length of $w \in \mathfrak{S}_n$); indeed, this is not true for the example above. Therefore the decomposition of the inversions given in Lemma 61 above is not a simple application of known formulas for the inversion set of a given permutation (see [4, Sections 2.4-2.5]).

Lemma 63. *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition of n with exactly k parts and $T \in \mathrm{SK}_k(\Gamma_h)$. Then:*

1. *$w_T^{-1}(\mathbb{J}_\lambda) \cap \Phi[T]$ is mapped to $\mathbb{J}_{\lambda[1]}$ under the identification in* (25) *and*

2. $w_T^{-1}(\Delta \setminus \mathbb{J}_\lambda) \cap \Phi[T]$ *is mapped to* $\{\alpha_1, \ldots, \alpha_{n-k-1}\} \setminus \mathbb{J}_{\lambda[1]}$ *under the identification in* (25)*,*

where $\lambda[1] = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_k - 1)$.

Proof. By definition, $w_T(T) = \{1, 2, \ldots, k\}$. Therefore

$$\begin{aligned} w_T^{-1}(\alpha_j) = t_{w_T^{-1}(j)} - t_{w_T^{-1}(j+1)} \in \Phi[T] \quad &\Leftrightarrow \quad \{w_T^{-1}(j), w_T^{-1}(j+1)\} \cap T = \varnothing \\ &\Leftrightarrow \quad \{j, j+1\} \cap \{1, 2, \ldots, k\} = \varnothing \end{aligned}$$

and we conclude that $w_T^{-1}(\alpha_j) \in \Phi[T]$ if and only if $k+1 \leqslant j \leqslant n-1$. This shows that, for the remainder of the argument, and since we are only interested in simple roots α_j such that $w_T^{-1}(\alpha_j)$ lands in $\Phi[T]$, we may assume that $j > k$. To simply notation we define $\mathbb{J}_{\lambda[1]+k} := \{\alpha_{i+k} : \alpha_i \in \mathbb{J}_{\lambda[1]}\}$ and $\mathbb{J}^c_{\lambda[1]+k} := \{\alpha_{k+1}, \ldots, \alpha_{n-1}\} \setminus \mathbb{J}_{\lambda[1]+k}$. Then by definition

$$\mathbb{J}_\lambda = \{\alpha_k\} \sqcup \mathbb{J}_{\lambda[1]+k} \text{ and } \Delta \setminus \mathbb{J}_\lambda = \{\alpha_1, \ldots, \alpha_k\} \sqcup \mathbb{J}^c_{\lambda[1]+k}.$$

Thus, $w_T^{-1}(\mathbb{J}_\lambda) \cap \Phi[T] = w_T^{-1}(\mathbb{J}_{\lambda[1]+k})$ and $w_T^{-1}(\Delta \setminus \mathbb{J}_\lambda) \cap \Phi[T] = w_T^{-1}(\mathbb{J}^c_{\lambda[1]+k})$. From the formula given in Remark 53 we have

$$w_T^{-1}(\alpha_j) = t_{w_T^{-1}(j)} - t_{w_T^{-1}(j+1)} \mapsto t_{\mathsf{f}_T(w_T^{-1}(j))} - t_{\mathsf{f}_T(w_T^{-1}(j+1))} = t_{j-k} - t_{j+1-k}$$

under the identification in (25). Therefore (25) maps $w_T^{-1}(\mathbb{J}_{\lambda[1]+k})$ to $\mathbb{J}_{\lambda[1]}$ and $w_T^{-1}(\mathbb{J}^c_{\lambda[1]+k})$ to $\{\alpha_1, \ldots, \alpha_{n-k-1}\} \setminus \mathbb{J}_{\lambda[1]}$ as desired. □

The next lemma is the technical heart of our argument. Notice that this is the first time we require the assumption that $k = ht(I_h) + 1$.

Lemma 64. *Let* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ *be a partition of* n *with* k *parts, where* $k = ht(I_h) + 1$ *and* $T \in \mathrm{SK}_k(\Gamma_h)$. *Then* $w = w_T\sigma \in \mathcal{W}(\mathbb{J}_\lambda, h, T)$ *if and only if* $\Psi_T(w) = x_\sigma \in \mathcal{W}(\mathbb{J}_{\lambda[1]}, h[T])$.

Proof. By Corollary 56, each $w \in \mathcal{W}(\mathbb{J}_\lambda, h, T)$ is of the form $w = w_T\sigma$ for a unique $\sigma \in \mathrm{Stab}(\ell_1, \ell_2, \ldots, \ell_k)$ and

$$w^{-1}(\mathbb{J}_\lambda) \subseteq \Phi_h \text{ and } w^{-1}(\Delta \setminus \mathbb{J}_\lambda) \subseteq I_h.$$

Since $\Phi[T]$ is invariant under σ and $\Phi_h[T] = \Phi[T] \cap \Phi_h$, intersecting the sets appearing in the equations above with $\Phi[T]$ yields

$$\sigma^{-1}(w_T^{-1}(\mathbb{J}_\lambda) \cap \Phi[T]) \subseteq \Phi_h[T] \text{ and } \sigma^{-1}(w_T^{-1}(\Delta \setminus \mathbb{J}_\lambda) \cap \Phi[T]) \subseteq I_h[T] \tag{27}$$

where $I_h[T] := \Phi[T] \cap I_h$. The forward direction of the statement now follows directly from Lemma 63 and Remark 60.

On the other hand, if $x_\sigma \in \mathcal{W}(\mathbb{J}_{[1]}, h[T])$ then Lemma 63 and Remark 60 together imply that equation (27) still holds. In order to show $w = w_T\sigma \in \mathcal{W}(\mathbb{J}_\lambda, h, T)$ we must prove $w^{-1}(\alpha_k) \in \Phi_h$ and $w^{-1}(\{\alpha_1, \ldots, \alpha_{k-1}\}) \subseteq I_h$. The latter fact is straightforward, since from the definition of w_T we have

$$w^{-1}(\{\alpha_1, \ldots, \alpha_{k-1}\}) = R_T \subseteq I_h.$$

Thus, we have only to show that $w^{-1}(\alpha_k) \in \Phi_h$. If not, then $w^{-1}(\alpha_k) \in I_h$ and

$$R = w^{-1}(\{\alpha_1, \ldots, \alpha_{k-1}, \alpha_k\}) \subseteq I_h$$

is a subset of height k in I_h. Lemma 45 now implies $ht(I_h) > k - 1$, a contradiction. We conclude that $w \in \mathcal{W}(\mathbb{J}_\lambda, h, T)$ as desired. □

We are now ready to prove the main result of this section.

Theorem 65. *Let λ be a partition of n with k parts, where $k = ht(I_h) + 1$ and $T \in \mathrm{SK}_k(\Gamma_h)$. Then Ψ_T maps $\mathcal{W}_i(\mathbb{J}_\lambda, h, T)$ bijectively onto $\mathcal{W}_{i-\deg_h(T)}(\mathbb{J}_{\lambda[1]}, h[T])$.*

Proof. Let $w \in \mathcal{W}_i(\mathbb{J}_\lambda, h, T)$ and $T = \{\ell_1 < \ell_2 < \cdots < \ell_k\}$. By Corollary 56, $w = w_T\sigma$ for a unique $\sigma \in \mathrm{Stab}(\ell_1, \ldots, \ell_k)$ and $\Psi_T(w) = x_\sigma$ by definition. Lemma 64 implies $\Psi_T : \mathcal{W}(\mathbb{J}_\lambda, h, T) \to \mathcal{W}(\mathbb{J}_{\lambda[1]}, h[T])$ is a bijection, so we have only to show that this bijection respects the grading as indicated. But this follows from Lemma 61 by intersecting both sides of (26) with Φ_h. We obtain

$$\mathrm{inv}_h(w) = \mathrm{inv}_h(w_T) \sqcup (\mathrm{inv}(\sigma) \cap \Phi_h[T])$$

so

$$i = |\mathrm{inv}_h(w)| = |\mathrm{inv}_h(w_T)| + |\mathrm{inv}_{h[T]}(x_\sigma)| = \deg_h(T) + |\mathrm{inv}_{h[T]}(x_\sigma)|$$

where the equation above follows directly from Lemma 43 and Remark 60. From this it follows that $\Psi_T(w) \in \mathcal{W}_{i-\deg_h[T]}(\mathbb{J}_{\lambda[1]}, h[T])$ as desired. □

6 Inductive formulas for the multiplicities associated to maximal sink sets

The main result of this section is a first application of the results obtained in the previous sections. Specifically, we derive an inductive formula for the multiplicities $c_{\mu,i}$ of the tabloid representations in the decomposition of the dot action representation on $H^{2i}(\mathcal{H}ess(\mathsf{S}, h))$, for partitions μ with the maximal number of parts. This result proves [9, Conjecture 8.1].

In the following we use the notation and terminology of Section 5. Let n be a positive integer, $h : [n] \to [n]$ a Hessenberg function, Γ_h its associated incomparability graph. Let $k = ht(I_h) + 1$. Let $\omega \in \mathcal{A}_k(\Gamma_h)$ be an acyclic orientation of Γ_h and let $T = \mathrm{sk}(\omega)$ be the sink set of ω of maximal size k. We can delete the vertices of T and all incident edges from Γ_h to obtain a strictly smaller graph $\Gamma_{h[T]}$ associated to a smaller Hessenberg function $h[T] : [n-k] \to [n-k]$ (see [9, Section 4] for more details).

Let $\mathsf{S}_{n-k} \in \mathfrak{gl}(n-k,\mathbb{C})$ be a regular semisimple operator. The cohomology of the Hessenberg variety $\mathcal{H}ess(\mathsf{S}_{n-k}, h[T]) \subseteq \mathcal{F}\ell ags(\mathbb{C}^{n-k})$ has a dot action of the permutation group $\mathfrak{S}_{n-k}$ and therefore has a corresponding decomposition analogous to (3). We denote the coefficients for this decomposition by $c^T_{\mu',i}$ as follows:

$$H^{2i}(\mathcal{H}ess(\mathsf{S}_{n-k}, h[T])) = \sum_{\mu' \vdash (n-k)} c^T_{\mu',i} M^{\mu'}. \tag{28}$$

With the notation in place we can state our inductive formula, which was first stated as Conjecture 8.1 in [9].

Theorem 66. *Let n be a positive integer and $h : [n] \to [n]$ a Hessenberg function. Let $k = ht(I_h)+1$. Suppose $\mu \vdash n$ is a partition of n with exactly $k = ht(I_h)+1$ parts. Then for all $i \geqslant 0$ we have*

$$c_{\mu,i} = \sum_{T \in \mathrm{SK}_k(\Gamma_h)} c^T_{\mu[1],i-\deg_h(T)}. \tag{29}$$

Proof. Let $\mathrm{Par}_{\geqslant k}(n)$ denote the set of all partitions of n with at least k parts and $\mathrm{Par}_k(n)$ denote the set of all partitions of n with exactly k parts. Let $\overline{A} = (A(\lambda,\mu))_{\lambda,\mu\in\mathrm{Par}_{\geqslant k}(n)}$. By definition, if $\lambda \in \mathrm{Par}_{\geqslant k}(n)$ and $\lambda \preceq \mu$, then μ has at least k parts so $\overline{A}$ is the lower right-hand $|\mathrm{Par}_{\geqslant k}(n)| \times |\mathrm{Par}_{\geqslant k}(n)|$ submatrix of A. In particular, $\overline{A}$ is upper-triangular since A is by Theorem 29. We consider the matrix equation

$$\overline{A}\,\overline{X}_i = \overline{W}_i \quad \text{where} \quad \overline{X}_i = (c_{\mu,i})_{\mu\in\mathrm{Par}_{\geqslant k}(n)} \quad \text{and} \quad \overline{W}_i = (|\mathcal{W}_i(\mathbb{J}_\lambda, h)|)_{\lambda\in\mathrm{Par}_{\geqslant k}(n)}. \tag{30}$$

The matrix equation appearing in (30) is consistent since we already know a priori that there exists a solution, given by the coefficients $c_{\mu,i}$ of (3). Moreover, since $\overline{A}$ is upper-triangular, this solution is unique. Furthermore, $c_{\mu,i} = 0$ for all partitions μ with more than k parts by Theorem 4. We may therefore rewrite the matrix equation $\overline{A}\,\overline{X}_i = \overline{W}_i$ as the following system of linear equations, one equation for each partition $\lambda \vdash n$ with exactly k parts:

$$|\mathcal{W}_i(\mathbb{J}_\lambda, h)| = \sum_{\mu\in\mathrm{Par}_k(n)} c_{\mu,i} A(\lambda,\mu). \tag{31}$$

In order to proved the desired result, it suffices to show that the RHS of (29) satisfies, as μ varies among all partitions of n with exactly k parts, the linear relations obtained in (31). From the sink set decomposition of $\mathcal{W}_i(\mathbb{J}_\lambda, h)$ given in Proposition 50 and the bijection between $\mathcal{W}_i(\mathbb{J}_\lambda, h, T)$ and $\mathcal{W}_{i-\deg_h(T)}(\mathbb{J}_{\lambda[1]}, h[T])$ given in Theorem 65 we obtain

$$\begin{aligned} |\mathcal{W}_i(\mathbb{J}_\lambda, h)| &= \sum_{T\in\mathrm{SK}_k(\Gamma_h)} |\mathcal{W}_{i-\deg_h(T)}(\mathbb{J}_{\lambda[1]}, h[T])| \\ &= \sum_{T\in\mathrm{SK}_k(\Gamma_h)} \sum_{\mu'\vdash(n-k)} c^T_{\mu',i-\deg_h(T)}\, |\mathcal{D}(\mathbb{J}_{\lambda[1]}, J_{\mu'})| \end{aligned}$$

where the second equality follows from Theorem 18, applied to $J = \mathbb{J}_{\lambda[1]}, h[T]$ and $n-k$. Notice that $|\mathcal{D}(\mathbb{J}_{\lambda[1]}, J_{\mu'})| = A(\lambda[1], \mu')$ by (14).

From Remark 41 it follows that for any $T \in \mathrm{SK}_k(\Gamma_h)$, the height of the ideal $I_{h[T]}$ is at most $k-1 = ht(I_h)$ and hence the coefficient $c^T_{\mu', i-\deg_h(T)}$ appearing in the last expression above is zero if μ' has more than k parts. Therefore we may rewrite the above expression and exchange the summation operations as follows:

$$\sum_{T \in \mathrm{SK}_k(\Gamma_h)} \sum_{\mu' \vdash (n-k)} c^T_{\mu', i-\deg_h(T)} A(\lambda[1], \mu') = \sum_{T \in \mathrm{SK}_k(\Gamma_h)} \sum_{\substack{\mu' \vdash (n-k) \\ \mu' \text{ has } \leqslant k \text{ parts}}} c^T_{\mu', i-\deg_h(T)} A(\lambda[1], \mu')$$
$$= \sum_{\substack{\mu' \vdash (n-k) \\ \mu' \text{ has } \leqslant k \text{ parts}}} \left(\sum_{T \in \mathrm{SK}_k(\Gamma_h)} c^T_{\mu', i-\deg_h(T)} \right) A(\lambda[1], \mu').$$

Next we observe that any partition μ' of $n-k$ which has at most k parts is equal to $\mu[1]$ for a unique partition μ of n with the properties that μ has exactly k parts. Indeed, it is not hard to see that $\mu := (\mu'_1 + 1, \mu'_2 + 1, \ldots, \mu'_k + 1)$ is precisely this (unique) μ.

Using this correspondence $\mu \leftrightarrow \mu[1] = \mu'$, we may therefore conclude that the last expression in the displayed equations above is equal to

$$\sum_{\mu \in \mathrm{Par}_k(n)} \left(\sum_{T \in \mathrm{SK}_k(\Gamma_h)} c^T_{\mu[1], i-\deg_h(T)} \right) A(\lambda[1], \mu[1])$$

which is in turn equal to

$$\sum_{\mu \in \mathrm{Par}_k(n)} \left(\sum_{T \in \mathrm{SK}_k(\Gamma_h)} c^T_{\mu[1], i-\deg_h(T)} \right) A(\lambda, \mu)$$

by Corollary 40. Putting the above together we have obtained

$$|\mathcal{W}_i(\mathbb{J}_\lambda, h)| = \sum_{\mu \in \mathrm{Par}_k(n)} \left(\sum_{T \in \mathrm{SK}_k(\Gamma_h)} c^T_{\mu[1], i-\deg_h(T)} \right) A(\lambda, \mu). \tag{32}$$

This proves the desired result. □

Acknowledgements

We are grateful for the hospitality and financial support of the Fields Institute for Research in the Mathematical Sciences in Toronto, Canada. The Fields Research Fellowship allowed us to spend a fruitful month together at the Fields Institute in August 2018, during which we had many of the ideas in this manuscript. Finally, we thank the anonymous referees for extremely careful and thorough readings of the manuscript and for the many helpful comments which improved the paper.

References

[1] H. Abe and M. Harada and T. Horiguchi and M. Masuda. The cohomology rings of regular nilpotent Hessenberg varieties in Lie type A. *Int. Math. Res. Not.*, **17** (2019), 5316–5388.

[2] A. Abreu and A. Nigro. Chromatic symmetric functions from the modular law. *J. Combin. Theory Ser. A*, **180** (2021), Paper No. 105407, 30.

[3] R. Adin and E. Bagno and E. Eisenberg and S. Reches and M. Sigron. On two-sided gamma-positivity for simple permutations. *Electron. J. Combin.*, **25** (2018), no. 2, #P2.38.

[4] A. Björner and F. Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, New York, 2005.

[5] P. Brosnan and T. Chow. Unit interval orders and the dot action on the co-homology of regular semisimple Hessenberg varieties. *Adv. Math.*, **329** (2018), 955-1001.

[6] W. Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997.

[7] M. Guay-Paquet. A modular relation for the chromatic symmetric functions of (3+1)-free posets, arXiv:1306.2400.

[8] M. Guay-Paquet. A second proof of the Shareshian-Wachs conjecture, by way of a new Hopf algebra, arXiv:1601.05498.

[9] M. Harada and M. Precup. The cohomology of abelian Hessenberg varieties and the Stanley-Stembridge conjecture. *Algebraic Combinatorics*, **2** (2019), no. 6, 1059–1108.

[10] S. J. Lee. Personal communication.

[11] M. Precup. The Betti numbers of regular Hessenberg varieties are palindromic. *Transform. Groups*, **23** (2018), 491-499.

[12] J. Shareshian and M. L. Wachs. Chromatic quasisymmetric functions. *Adv. Math.*, **295** (2016), 497-551.

[13] R. P. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math.*, **111** (1995), 166-194.

[14] R. P. Stanley. Erratum to "A symmetric function generalization of the chromatic polynomial of a graph", Theorem 3.4. http://math.mit.edu/~rstan/pubs/pubfiles/xg_erratum.pdf

[15] J. S. Tymoczko. Permutation actions on equivariant cohomology of flag varieties. In *Toric topology*, volume 460 of *Contemp. Math.*, pages 365–384. Amer. Math. Soc., Providence, RI, 2008.

[16] M. Visontai. Some remarks on the joint distribution of descents and inverse descents. Electron. J. Combin., **20** (2013), no. 1, #P52.