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A NEW APPROACH TO THE GENERALIZED SPRINGER CORRESPONDENCE

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Abstract. The Springer resolution of the nilpotent cone is used to give a geometric construction of the irreducible representations of Weyl groups. Borho and MacPherson obtain the Springer correspondence by applying the decomposition theorem to the Springer resolution, establishing an injective map from the set of irreducible Weyl group representations to simple equivariant perverse sheaves on the nilpotent cone. In this manuscript, we consider a generalization of the Springer resolution using a variety defined by the first author. Our main result shows that in the type A case, applying the decomposition theorem to this map yields all simple perverse sheaves on the nilpotent cone with multiplicity as predicted by Lusztig's generalized Springer correspondence.

1. Introduction

In 1976, Springer introduced a geometric construction of the irreducible representations of the Weyl group of a semisimple algebraic group G on the cohomology of algebraic varieties called Springer fibers [26, 27]. Springer's work was foundational to the field of geometric representation theory. There are many current research directions connected to Springer's classical result, ranging from exploring modular versions of his work [2, 3], to applying geometry, topology, and combinatorics to better understand Springer fibers [14, 20, 29].

The Springer fibers are the fibers of the Springer resolution

$$\mu : \widetilde{\mathcal{N}} \to \mathcal{N}$$

which is a resolution of singularities of the nilpotent cone $\mathcal{N}$ of $\mathfrak{g} = \operatorname{Lie} G$. In this manuscript, we consider an extension of the Springer resolution, which is a map

$$\psi = \widetilde{\eta} \circ \mu : \widetilde{\mathcal{M}} \xrightarrow{\widetilde{\eta}} \widetilde{\mathcal{N}} \xrightarrow{\mu} \mathcal{N},$$

from a variety $\widetilde{\mathcal{M}}$ to the nilpotent cone that factors through the Springer resolution. We refer to this map as the extended Springer resolution. The variety $\widetilde{\mathcal{M}}$ was defined by the first author in [16], and used to construct an analogue of the Springer resolution for $\mathcal{M} = \operatorname{Spec} R(\widetilde{\mathcal{O}}^{pr})$. Here $R(\widetilde{\mathcal{O}}^{pr})$ denotes the ring of regular functions on $\widetilde{\mathcal{O}}^{pr}$, the simply connected cover of the principal nilpotent orbit $\mathcal{O}^{pr} \subset \mathcal{N}$. Although $\widetilde{\mathcal{M}}$ is not smooth, it is locally a quotient of a smooth variety by a finite group. The variety $\widetilde{\mathcal{M}}$ and the map ψ are defined precisely in Section 2.4 below.

When viewed through the lens of the derived category of G-equivariant perverse sheaves, the Springer correspondence is an injective map from the set of

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irreducible representations of the Weyl group to the set of simple G-equivariant perverse sheaves on the nilpotent cone. The latter set is indexed by the pairs $(\mathcal{O}, \mathcal{L})$, where $\mathcal{O}$ is a nilpotent orbit and $\mathcal{L}$ is an irreducible G-equivariant local system on $\mathcal{O}$. Since the Springer correspondence is an injection, we recover only some of the pairs $(\mathcal{O}, \mathcal{L})$ in this manner. Those missing from the Springer correspondence were described by Lusztig. He established a bijection between irreducible representations of what he called relative Weyl groups and all simple G-equivariant perverse sheaves on $\mathcal{N}$ [21], proving that the Springer correspondence is a special case of what is now known as the generalized Springer correspondence. Historically, this result was the beginning of Lusztig's consideration of character sheaves, a key tool in the study of irreducible representations of finite groups of Lie type. A summary of this work can be found in [10].

Unlike the proof of the Springer correspondence given by Borho and MacPherson [8], Lusztig's proof of the generalized Springer correspondence does not proceed by applying the decomposition theorem to the pushforward of the constant sheaf on a single variety. Although Shoji's treatment of the generalized Springer correspondence in [25, Section 11] (based on Lusztig's work) makes use of the decomposition theorem, the generalized Springer correspondence is obtained through pushforwards from multiple varieties. Our main result, stated precisely in Theorem 5.3 below, applies this strategy in the case of $G = SL_n(\mathbb{C})$ to the extended Springer resolution $\widetilde{\mathcal{M}} \to \mathcal{N}$. This theorem shows that the pushforward of the constant sheaf on $\widetilde{\mathcal{M}}$ to the nilpotent cone $\mathcal{N}$ is the direct sum of all the so-called Lusztig sheaves used in the generalized Springer correspondence. This then implies that the pushforward is the direct sum of all the G-equivariant perverse sheaves on $\mathcal{N}$, each occurring with multiplicity as determined by the generalized Springer correspondence. The fact that all G-equivariant perverse sheaves appear here was obtained by the third author in [24], using different methods and without the multiplicity result.

We believe that the geometry of the extended Springer resolution will yield further insights into both Springer fibers and the generalized Springer correspondence. In forthcoming work, the authors will characterize irreducible components of the fibers of the map $\psi : \widetilde{\mathcal{M}} \to \mathcal{N}$, and use the geometric information to analyze the generalized Springer correspondence from this new perspective. The generalized Springer correspondence is related to relative Weyl groups, each of which is a smaller symmetric group, and acts on the cohomology of $\psi^{-1}(x)$ for certain $x \in \mathcal{N}$. Since the map ψ factors through the Springer resolution, our construction yields a new connection between the representation theory of these smaller symmetric groups and the geometry of Springer fibers. This connection sheds light on the Springer fibers themselves and yields previously unknown geometric features of these fibers.

The arguments below rely heavily on the structure theory of the derived category of constructible sheaves. The proof of our main result is comprised of two main steps. In broad terms, the first step requires us to analyze the decomposition theorem when applied to a quotient map $X \to X/Z$ where Z is a finite abelian group acting on a variety X. We then apply this general analysis in the special case of $\widetilde{\eta} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$. The second step of our argument is comprised of studying the simple perverse sheaves that appear as summands of $\widetilde{\eta}_* \underline{\mathbb{C}}_{\widetilde{\mathcal{M}}}[\dim \mathcal{N}]$. In particular, for $G = SL_n(\mathbb{C})$ we prove that each simple perverse sheaf that arises in this way pushes forward to a single Lusztig sheaf under the map μ.

The contents of the paper are as follows. Section 2 covers background information and definitions, including the definition of the partial Springer resolution $\widetilde{\mathcal{N}}^P$. In Section 3, we study the pushforward of the constant sheaf $\underline{\mathbb{C}}_{\widetilde{\mathcal{M}}}$ along $\widetilde{\eta}$. As indicated above, some of our analysis in this section is carried out in a more general setting. Section 4 studies the intersection cohomology complexes appearing in $\widetilde{\eta}_*\underline{\mathbb{C}}_{\widetilde{\mathcal{M}}}[\dim\mathcal{N}]$. We show that each of these pushes forward to an intersection cohomology complex on $\widetilde{\mathcal{N}}^P$ for a particular parabolic subgroup P of $SL_n(\mathbb{C})$. Finally, we realize each of these as an induced complex and prove our main result, Theorem 5.3, in Section 5. We conclude with an explicit example for $G = SL_6(\mathbb{C})$, and a discussion of future work.

2. Preliminaries

2.1. **Group actions and local systems.** We work with schemes over the ground field $\mathbb{C}$. Given a scheme X, $\underline{\mathbb{C}}_X$ denotes the constant sheaf on X with rational coefficients. The Lie algebra of an algebraic group is denoted by the corresponding fraktur letter; if G is an algebraic group, its identity component is denoted G_0, and the component group is G/G_0.

Suppose Z is a finite group acting freely on X on the right, and V is a representation of Z. There is a corresponding local system $\mathcal{L}_V$ on X/Z, defined as the sheaf of locally constant sections of the vector bundle $\mathcal{V} = X \times^Z V \to X/Z$. By abuse of terminology, we may refer to $X \times^Z V$ as a local system on X/Z.

If G is an algebraic group acting on X, G^x denotes the stabilizer in G of $x \in X$. The component group G^x/G_0^x of the stabilizer is sometimes called the equivariant fundamental group of the orbit $G \cdot x$. Suppose V is a representation of G^x/G_0^x, viewed as a representation of G^x via the map $G^x \to G^x/G_0^x$. Then V corresponds to a local system $\mathcal{L}_V$ on $G \cdot x$, defined as the sheaf of locally constant sections of the vector bundle $G \times^{G^x} V \to G/G^x \cong G \cdot x$.

If Z is a subgroup of G, and M is a Z-variety, the mixed space $G \times^Z M$ is the quotient of $G \times M$ by the action of Z defined by $(h, m)z = (hz, z^{-1}m)$ for $z \in Z$. The equivalence class in $G \times^Z M$ of $(g, m) \in G \times M$ is denoted $[g, m]$, or $[g, m]_Z$ if we wish to make the group explicit.

2.2. **Local systems on the principal orbit.** Throughout the rest of the paper, G denotes a simply connected semisimple algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$ and center Z. The group of characters of Z is denoted by $\widehat{Z}$ (with analogous notation for other finite groups). Set $T_{ad} = T/Z$. We denote the set of nilpotent elements in $\mathfrak{g}$, known as the nilpotent cone, by $\mathcal{N}$. The group G has a dense orbit in $\mathcal{N}$, called the principal nilpotent orbit. We denote this orbit by $\mathcal{O}^{pr}$ and refer to its elements as principal nilpotent elements.

Let $\nu \in \mathfrak{g}$ be a principal nilpotent element. Choose a standard triple $\{\nu, s, \nu_-\}$ with nilpositive element ν and semisimple element $s \in \mathfrak{t}$. Since ad_s acts semisimply on $\mathfrak{g}$, we can decompose $\mathfrak{g}$ into its ad_s-eigenspaces:

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j, \text{ where } \mathfrak{g}_j = \{x \in \mathfrak{g} \mid [s, x] = jx\}.$$

For any k, we let $\mathfrak{g}_{\geq k} = \bigoplus_{i \geq k} \mathfrak{g}_i$ and $\mathfrak{g}_{>k} = \bigoplus_{i>k} \mathfrak{g}_i$. Let $\mathfrak{t} = \mathfrak{g}_0$, $\mathfrak{u} = \mathfrak{g}_{>0}$, and $\mathfrak{b} = \mathfrak{t} + \mathfrak{u}$. The corresponding subgroups of G are T, U, and $B = TU$; B is a Borel subgroup of G with maximal torus T.

Let Φ denote the root system of $\mathfrak{g}$, with positive roots Φ^+ chosen so that $\mathfrak{u}$ is the sum of the positive root spaces. The flag variety of G is G/B, and $W = N_G(T)/T$ is the corresponding Weyl group. Much of this paper is focused on the case in which $G = SL_n(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. When $G = SL_n(\mathbb{C})$ we may assume B is the subgroup of upper triangular matrices; the Weyl group in this case is the symmetric group S_n. We have $G^\nu = ZU^\nu$. Since U^ν is connected, being unipotent, the component group G^ν/G_0^ν is identified with Z. Therefore any character $\chi \in \hat{Z}$ corresponds to a local system $\mathcal{L}_\chi$ on $G \cdot \nu = \mathcal{O}^{pr}$.

2.3. **The Springer resolution.** We denote the **Springer resolution** by $\mu : \widetilde{\mathcal{N}} \to \mathcal{N}$ where

$$\widetilde{\mathcal{N}} := \{(gB, x) \in G/B \times \mathcal{N} \mid g^{-1} \cdot x \in \mathfrak{u}\}.$$

Here $g^{-1} \cdot x$ denotes the adjoint action $Ad(g^{-1})(x) = g^{-1}xg$. The map μ is simply projection onto the second factor. Given $x \in \mathcal{N}$, the fiber

$$\mu^{-1}(x) = \{(gB, x) \mid g^{-1} \cdot x \in \mathfrak{u}\}$$

is the **Springer fiber** of x. We can identify the Springer fiber with its image in G/B under the projection to the first factor; under this identification,

$$\mu^{-1}(x) = \{gB \mid g^{-1} \cdot x \in \mathfrak{u}\}.$$

Sometimes G/B is denoted by $\mathcal{B}$ and the Springer fiber in G/B is denoted $\mathcal{B}^x$.

Remark 2.1. There is an isomorphism of varieties

$$G \times^B \mathfrak{u} \to \widetilde{\mathcal{N}}, \quad [g, x] \mapsto (gB, g \cdot x)$$

(cf. [19, pg. 66]). We use this identification frequently below. Under this identification,

$$\mu : G \times^B \mathfrak{u} \to \mathcal{N}, \quad [g, x] \mapsto g \cdot x.$$

Viewed as a subset of $G \times^B \mathfrak{u}$, the Springer fiber over x is

$$\mu^{-1}(x) = \{[g, g^{-1}x] \mid g^{-1} \cdot x \in \mathfrak{u}\}. \tag{2.1}$$

2.4. **The extended Springer resolution.** We now recall from [16] the definition of the variety which is the main geometric focus of this paper.

Let $\Delta \subseteq \Phi^+$ denote the subset of simple roots. Then $\mathfrak{g}_2 = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$. Since $[s, \nu] = 2\nu$, we have $\nu \in \mathfrak{g}_2$, and we can choose root vectors E_α for each $\alpha \in \Delta$ so that $\nu = \sum_{\alpha \in \Delta} E_\alpha$. Since the center Z acts trivially on $\mathfrak{g}$, the action of T on $\mathfrak{g}$ factors through the map $T \to T_{ad} = T/Z$. The map $T_{ad} \to T_{\text{ad}} \cdot \nu$ given by $t \mapsto t \cdot \nu$ embeds T_{ad} in $\mathfrak{g}_2$, so $\mathfrak{g}_2$ is an affine toric variety for T_{ad}, which we denote by

$$\mathcal{V}_{ad} := \mathfrak{g}_2 = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

The composition $\mathcal{V}_{ad} \to \mathfrak{u} \to \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$ is an isomorphism, and using this, we identify $\mathcal{V}_{ad}$ with $\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$. Via this identification, $\mathcal{V}_{ad}$ acquires a B-action (the subgroup U acts trivially), and the projection $p : \mathfrak{u} \to \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}] = \mathcal{V}_{ad}$ is B-equivariant.

An affine toric variety is characterized by the character group of the torus, which can be viewed as a subset of the dual Lie algebra of the torus, together with a cone in the real span of the set of characters. (The character group is a lattice in its real span.) The toric variety $\mathcal{V}_{ad}$ corresponds to the lattice given by the character group of T_{ad}, and the cone equal to the set of $\mathbb{R}_{\geq 0}$-linear combinations of simple roots. The simple roots can be viewed as characters of either T_{ad} or T. We define

$\mathcal{V}$ to be the toric variety for T obtained by changing the lattice for $\mathcal{V}_{ad}$ but keeping the same cone: that is, $\mathcal{V}$ corresponds to the lattice given by the character group of T, and the cone equal to the set of $\mathbb{R}_{\geq 0}$-linear combinations of simple roots. It follows that $\mathcal{V}/Z = \mathcal{V}_{ad}$ (see [15, Section 2.2]).

The composition $T \to B \to B/U$ is an isomorphism, so identifying T with B/U, there is a natural projection $B \to T$. Via this projection, $\mathcal{V}$ acquires a B-action (where U acts trivially), and the projection $\pi : \mathcal{V} \to \mathcal{V}_{ad}$ is B-equivariant.

The variety $\widetilde{\mathcal{M}}$ and map $\psi : \widetilde{\mathcal{M}} \to \mathcal{N}$ discussed in Section 1 are defined as follows. First, consider the maps $p : \mathfrak{u} \to \mathfrak{u}/[\mathfrak{u},\mathfrak{u}] = \mathcal{V}_{ad}$ and $\pi : \mathcal{V} \to \mathcal{V}/Z = \mathcal{V}_{ad}$ defined in the paragraphs above. Set

$$\widetilde{\mathfrak{u}} := \mathcal{V} \times_{\mathcal{V}_{ad}} \mathfrak{u} = \{(v, y) \mid \pi(v) = p(y)\},$$

i.e., we form the following Cartesian diagram.

$$\begin{array}{ccc} \widetilde{\mathfrak{u}} & \longrightarrow & \mathfrak{u} \\ \downarrow & & \downarrow p \\ \mathcal{V} & \xrightarrow{\pi} & \mathcal{V}_{ad} \end{array}$$

Because the maps p and π are both B-equivariant, B acts on $\widetilde{\mathfrak{u}}$. We define $\widetilde{\mathcal{M}} := G \times^B \widetilde{\mathfrak{u}}$. Let $\eta : \widetilde{\mathfrak{u}} \to \mathfrak{u}$ denote projection onto the second factor. We then define $\widetilde{\eta} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ to be the map induced from η, so $\widetilde{\eta}$ maps the element $[g, x] \in \widetilde{\mathcal{M}}$ to $[g, \eta(x)] \in \widetilde{\mathcal{N}}$. The **extended Springer resolution** is the variety $\widetilde{\mathcal{M}}$, together with the map $\psi : \widetilde{\mathcal{M}} \to \mathcal{N}$, where ψ is the composition

$$\psi : \widetilde{\mathcal{M}} \xrightarrow{\widetilde{\eta}} \widetilde{\mathcal{N}} \xrightarrow{\mu} \mathcal{N}$$

of $\widetilde{\eta}$ with the usual Springer resolution μ.

2.5. **Intersection cohomology sheaves and the decomposition theorem.** The decomposition theorem plays an important role in this paper, and we briefly discuss and state it here. The original version of this theorem is due to Beĭlinson, Bernstein, Deligne, and Gabber [4]; we will also need a generalization which can be found in [12] or [11]. Statements and discussions of the theorem relevant for this paper can also be found in [13], [7], and [19]. Before we state the decomposition theorem, we first discuss its essential ingredients, namely intersection cohomology complexes. These are the simple perverse sheaves, and objects in the bounded constructible derived category $D^b(X)$ of a variety X.

A perverse sheaf is a constructible complex of sheaves of $\mathbb{C}$-vector spaces which satisfies certain support and co-support conditions. The intersection cohomology complexes are determined up to canonical isomorphism in $D^b(X)$ by more restrictive conditions. Suppose X is a complex algebraic variety, and let U be a nonsingular, dense open subvariety of X. Given a local system $\mathcal{L}$ on U, and an integer d, we let $\mathcal{L}[d]$ be the complex in $D^b(X)$ with $\mathcal{L}$ shifted to the $-d$th location. The **intersection cohomology complex** $IC(X, \mathcal{L})$ is the object in $D^b(X)$ uniquely determined up to canonical isomorphism by the following properties: the restriction to U is $\mathcal{L}[\dim X]$,

$$\dim\{x \in X \mid \mathcal{H}^i_x(IC(X, \mathcal{L})) \neq 0\} < -i \text{ for all } i > \dim X$$

and

$$\dim\{x \in X \mid \mathcal{H}^i_x(\mathbb{D}IC(X,\mathcal{L})) \neq 0\} < -i \text{ for all } i > -\dim X,$$

where $\mathcal{H}^i_x$ denotes the stalk at x of the cohomology sheaf of the complex, and $\mathbb{D}$ denotes the Verdier dual. If X is smooth and $\mathcal{L}_{triv}$ is the trivial local system, $IC(X,\mathcal{L}_{triv}) = \underline{\mathbb{C}}_X[\dim X]$ (recall that $\underline{\mathbb{C}}_X$ denotes the constant sheaf on X). Spaces with the property that $IC(X,\mathcal{L}_{triv}) = \underline{\mathbb{C}}_X[\dim X]$ are called **rationally smooth**. Examples of rationally smooth varieties include $\widetilde{\mathcal{M}}$ (which is rationally smooth since it is locally a quotient of a smooth variety by a finite group) and $\mathcal{N}$ (see [7, §2.3]).

Suppose $j : Z \hookrightarrow X$ is the inclusion of a closed subvariety. Since we are only working in the derived setting, we use the notation j_* rather than Rj_* to denote the derived direct image $D^b(Z) \to D^b(X)$. The functor j_* is fully faithful and induces an equivalence of categories between $D^b(Z)$ and the set of objects in $D^b(X)$ supported on Z [13, Section 2.3]. Moreover, for any $\mathcal{F} \in D^b(Z)$, there is a natural isomorphism $j^*j_*\mathcal{F} \to \mathcal{F}$ (cf. [18, p. 102]). By convention, if we are working with $D^b(X)$, we omit the symbol j_*, and write $IC(Z,\mathcal{L})$ for $j_*IC(Z,\mathcal{L})$, where $\mathcal{L}$ is a local system on an open subset of Z. More generally, if S is a closed subvariety of Z and $\mathcal{L}$ is a local system on an open subset of X, $IC(S,\mathcal{L})$ may denote an element of $D^b(Z)$ or $D^b(X)$. There is a natural isomorphism $j^*IC(S,\mathcal{L}) \to IC(S,\mathcal{L})$, where $IC(S,\mathcal{L})$ is viewed on the left hand side as an element of $D^b(Z)$ and on the right hand side as an element of $D^b(X)$.

All simple perverse sheaves in $D^b(X)$ are of the form $IC(S,\mathcal{L})$ for some closed subvariety S and local system $\mathcal{L}$. Note that if S has two different open dense subvarieties U_1 and U_2, and local systems $\mathcal{L}_1$ on U_1 and $\mathcal{L}_2$ on U_2 that agree on the intersection $U_1 \cap U_2$, then $IC(S,\mathcal{L}_1)$ and $IC(S,\mathcal{L}_2)$ are canonically isomorphic.

Let $f : X \to Y$ be an algebraic map of irreducible complex varieties. Then f is **semismall** if for all d,

$$\dim\{p \in X \mid \dim f^{-1}(p) \geq d\} \leq \dim X - 2d.$$

If f is semismall, then f is generically finite. The map f is **small** if the inequality above is strict for $d > 0$. A finite map is small.

We now state a version of the decomposition theorem for perverse sheaves. The first statement here can be found in [12] or [11]. The second statement comes from [7] and is a specialization of the original version found in [4].

Theorem 2.2 (The decomposition theorem)**.** *Let $f : X \to Y$ be a proper map of complex algebraic varieties.*

(1) *Let $\mathcal{L}$ be a semisimple local system on X. Then the derived pushforward $f_*IC(X,\mathcal{L})$ is a finite direct sum of shifted simple perverse sheaves on Y.*
(2) *If f is semismall and X is rationally smooth, then $f_*\underline{\mathbb{C}}_X[\dim X]$ is perverse, so no nontrivial shifts occur in* (1)*. That is,*

$$f_*\underline{\mathbb{C}}_X[\dim X] \cong \bigoplus_i IC(\overline{S_i},\mathcal{L}_i) \otimes V_i, \tag{2.2}$$

where each S_i is a locally closed subvariety of Y, $\mathcal{L}_i$ is an irreducible local system on an open set of S_i and V_i is a vector space with dimension equal to the multiplicity of $IC(\overline{S_i},\mathcal{L}_i)$ in the sum.

2.6. **The Springer correspondence and Lusztig's generalized Springer correspondence.** The Springer correspondence is an injective map from the set of isomorphism classes of irreducible W-representations to pairs $(\mathcal{O}, \mathcal{L})$, where $\mathcal{O}$ is a nilpotent orbit and $\mathcal{L}$ is an irreducible local system on $\mathcal{O}$. Note that each pair $(\mathcal{O}, \mathcal{L})$ corresponds to a simple perverse sheaf $IC(\overline{\mathcal{O}}, \mathcal{L})$ on $\mathcal{N}$. In [7], Borho and MacPherson give a proof of the Springer correspondence which relies on the theory of perverse sheaves. They apply Theorem 2.2 to the Springer resolution to obtain

$$\mu_* \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \mathcal{N}] = \bigoplus_{(\mathcal{O},\mathcal{L})} IC(\overline{\mathcal{O}}, \mathcal{L}) \otimes V_{(\mathcal{O},\mathcal{L})}. \tag{2.3}$$

The left side of (2.3) is called the **Springer sheaf**; it is the derived pushforward of the constant sheaf on $\widetilde{\mathcal{N}}$ along the Springer resolution. On the right side of (2.3), the sum is over the pairs $(\mathcal{O}, \mathcal{L})$ that appear in the Springer correspondence. One can show that the Weyl group acts on the left side of (2.3). This implies that each $V_{(\mathcal{O},\mathcal{L})}$ is a W-representation; it is precisely the irreducible representation of W mapping to $(\mathcal{O}, \mathcal{L})$ under the Springer correspondence.

Since the Springer correspondence is an injection, we recover only some of the pairs $(\mathcal{O}, \mathcal{L})$ in this manner. Those missing from the Springer correspondence were described by Lusztig in all cases [21]. Following Lusztig, we consider triples $\mathbf{c} = (L, \mathcal{O}_L, \mathcal{L}^L)$, where L is a standard Levi subgroup of G, $\mathcal{O}_L$ is a nilpotent orbit of L, and $\mathcal{L}^L$ is a local system on $\mathcal{O}_L$. A triple satisfying certain conditions is called a cuspidal datum (see [1, Definition 8.5.1]), and Lusztig has classified all such triples. The **relative Weyl group** corresponding to a cuspidal datum $\mathbf{c}$ is the group $W(L) := N_G(L)/L$. Although $W(L)$ is defined even if $\mathbf{c}$ is not cuspidal, the cuspidal hypothesis is required to ensure that $W(L)$ is a Coxeter group. Details can be found in [21] and [22]. An exposition in language closer to ours can be found in [1, Section 8.5]. The **generalized Springer correspondence** is a bijection between the set of simple perverse sheaves on the nilpotent cone and irreducible representations of relative Weyl groups associated to cuspidal data.

Suppose P is a parabolic subgroup of G with Levi decomposition $P = LU_P$, where L is part of a cuspidal datum $\mathbf{c}$ for G. The corresponding decomposition of Lie algebras is $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}_P$. By means of the isomorphism $\mathfrak{l} \simeq \mathfrak{p}/\mathfrak{u}_P$, the Lie algebra $\mathfrak{l}$ acquires a P-action, where U_P acts trivially. The projection $q : \mathfrak{l} + \mathfrak{u}_P \to \mathfrak{l}$ is P-equivariant, and takes $\mathcal{N}_L + \mathfrak{u}_P$ to $\mathcal{N}_L$. Note that U_P acts trivially on $\mathcal{N}_L$ when $\mathcal{N}_L$ is viewed as a subvariety of $\mathfrak{p}/\mathfrak{u}_P$, but $\mathcal{N}_L$ is not U_P-invariant when $\mathcal{N}_L$ is viewed as a subvariety of $\mathcal{N}_L + \mathfrak{u}_P$.

Let $\widetilde{\mathcal{N}}^P = G \times^P (\mathcal{N}_L + \mathfrak{u}_P)$ denote the variety considered by Borho and MacPherson in [7]; here $\mathcal{N}_L$ is the nilpotent cone of L. We write $[g, x]_P$ for the equivalence class of (g, x) in $\widetilde{\mathcal{N}}^P$, where $g \in G$ and $x \in \mathcal{N}_L + \mathfrak{u}_P$. Borho and MacPherson call the map $\mu_P : \widetilde{\mathcal{N}}^P \to \mathcal{N}$ defined by $\mu_P([g, x]_P) = g \cdot x$ a partial resolution of $\widetilde{\mathcal{N}}$; it is analogous to the Springer resolution. Note that using an isomorphism analogous to the one defined in Remark 2.1, $\widetilde{\mathcal{N}}^P$ can be identified with the variety of pairs $(gP, x) \in G/P \times \mathcal{N}$ such that $g^{-1} \cdot x \in \mathcal{N}_L + \mathfrak{u}_P$.

Let $\pi_P : \widetilde{\mathcal{N}}^P \to G \times^P \mathcal{N}_L$ be the map induced from $q : \mathfrak{l} + \mathfrak{u}_P \to \mathfrak{l}$. We have the following diagram:

$$\mathcal{N} \xleftarrow{\mu_P} \widetilde{\mathcal{N}}^P := G \times^P (\mathcal{N}_L + \mathfrak{u}_P) \xrightarrow{\pi_P} G \times^P \mathcal{N}_L.$$

By the induction of complexes defined in [23] (see, in particular, [23, (1.9.3)] and [6, 2.6.3]), each L-equivariant object in the derived category $D^b(\mathcal{N}_L)$ determines a G-equivariant object in the derived category $D^b(G \times^P \mathcal{N}_L)$. We denote the corresponding functor by Ind_P^G. Consider the simple perverse sheaf $\mathrm{Ind}_P^G IC(\overline{\mathcal{O}}_L, \mathcal{L}^L)$ on $G \times^P \mathcal{N}_L$, where $\mathcal{O}_L$ and $\mathcal{L}^L$ are from the cuspidal datum $\mathbf{c} = (L, \mathcal{O}_L, \mathcal{L}^L)$. We define

$$\mathbb{A}_{\mathbf{c}} := \mu_{P*}\pi_P{}^* \mathrm{Ind}_P^G IC(\overline{\mathcal{O}}_L, \mathcal{L}^L)[\dim \mathfrak{u}_P] \tag{2.4}$$

to be the **Lusztig sheaf** associated to $\mathbf{c}$. In [1, Section 8.5], $\mathbb{A}_{\mathbf{c}}$ is denoted by $\mathrm{ind}_{L\subset P}^G IC(\overline{\mathcal{O}}_L, \mathcal{L}^L)$.

The Lusztig sheaf $\mathbb{A}_{\mathbf{c}}$ decomposes as a sum of simple perverse sheaves, each of which corresponds to an irreducible representation of $W(L)$. The generalized Springer correspondence is obtained by considering the Lusztig sheaves $\mathbb{A}_{\mathbf{c}}$ as $\mathbf{c}$ varies over all cuspidal data. When $L = T$, the only cuspidal datum is $\mathbf{c} = (T, \mathcal{O}_{\{0\}}, \mathcal{L}^T_{triv})$ where $\mathcal{O}_{\{0\}}$ is the zero orbit. In this case, the relative Weyl group is the full Weyl group W and the Lusztig sheaf is the Springer sheaf. We therefore recover the Springer correspondence as a special case of the generalized Springer correspondence.

Remark 2.3. Lusztig's generalized Springer correspondence is proven for connected reductive algebraic groups over any algebraically closed field (of possibly positive characteristic p) and $\overline{\mathbb{Q}}_\ell$-sheaves. Although the discussion in this paper focuses on the characteristic zero setting and $\mathbb{C}$-sheaves, most of the results generalize to the positive characteristic setting. The only potential difficulty is our analysis of varieties with a finite group action (e.g. the Z-action on $\widetilde{\mathcal{M}}$) in Section 3. However, if X is a variety defined over an algebraically closed field of characteristic p and Z is a finite abelian group acting on X, the results of Section 3 hold whenever p is relatively prime to $|Z|$.

2.7. **Cuspidal data and Lusztig sheaves for $G = SL_n(\mathbb{C})$.** When $G = SL_n(\mathbb{C})$, the Springer correspondence and its generalization have a more direct description. In this setting, the only pairs $(\mathcal{O}, \mathcal{L})$ appearing on the right hand side of (2.3) are those corresponding to the trivial local system $\mathcal{L}_{triv}$ on each orbit $\mathcal{O}$, thus the Springer correspondence gives a bijection between irreducible representations of S_n and nilpotent orbits. We adopt the conventions of [19], so the irreducible S_n-representation V_λ indexed by the partition $\lambda \vdash n$ corresponds to the nilpotent orbit $\mathcal{O}_\lambda$ of nilpotent matrices with Jordan type λ. Note that our conventions may differ from some others appearing in the literature up to tensoring V_λ with the sign representation.

Let $\nu \in \mathcal{O}_\lambda$. Each irreducible local system on the orbit $\mathcal{O}_\lambda$ corresponds to a unique irreducible representation of the equivariant fundamental group G^ν/G_0^ν, as described in Section 2.1 above (in fact, this correspondence is an equivalence of categories). If $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$, we have $G^\nu/G_0^\nu \simeq \mathbb{Z}_m$ where $m = \gcd(\lambda_1, \lambda_2, \ldots, \lambda_k)$. The center Z of G is isomorphic to the cyclic group $\mathbb{Z}_n$. Since Z is a subgroup of G^ν, we obtain a map $Z \to G^\nu/G_0^\nu$. This map is always surjective in the type A setting, so we obtain an injective map $\widehat{G^\nu/G_0^\nu} \to \widehat{Z}$ of the character groups. Suppose $\chi \in \widehat{Z}$ is in the image of this map; note that the order of χ divides m. We denote the corresponding G-equivariant local system on $\mathcal{O}_\lambda$ by $\mathcal{L}_\chi$; it depends on the orbit

$\mathcal{O}_\lambda$ as well as on χ, but we omit λ from the notation. Recall that if $\lambda = [n]$, then $\mathcal{O}_\lambda = \mathcal{O}^{pr}$ is the principal orbit and $G^\nu/G_0^\nu \simeq Z$.

From [21] and [22], it follows that there is a character of Z associated to each Lusztig sheaf $\mathbb{A}_{\mathbf{c}}$ (see [21, Section 14.2]). In the type A setting, this association is a bijection (see [22, Section 5.1]). See also [1, Theorem 8.5.8]. Given $\chi \in \widehat{Z}$, we denote by $\mathbf{c}_\chi$ the unique cuspidal datum associated to χ, and $\mathbb{A}_\chi$ the Lusztig sheaf constructed from $\mathbf{c}_\chi$. If χ has order d, then $\mathbf{c}_\chi = (L, \mathcal{O}_L^{pr}, \mathcal{L}_\chi^L)$, where the Levi subgroup L is of the form $S(GL_d(\mathbb{C}) \times \cdots \times GL_d(\mathbb{C}))$ with n/d factors, $\mathcal{O}_L^{pr}$ is the principal orbit in $\mathcal{N}_L$, and $\mathcal{L}_\chi^L$ is a local system defined precisely in Section 5 below (see also [22, §5]). The relative Weyl group in this case is isomorphic to $S_{n/d}$.

Given a character χ of order d, we obtain an induced local system $\mathcal{L}_\chi$ on precisely those orbits $\mathcal{O}_\lambda$ for which $\lambda = [\lambda_1, \ldots, \lambda_k]$ is a partition of n such that d divides λ_i for all i. Let $\overline{\lambda}$ denote the partition of n/d obtained by dividing each part of λ by d. The Lusztig sheaf $\mathbb{A}_\chi$ corresponding to χ decomposes as:

$$\mathbb{A}_\chi = \bigoplus_{\substack{\lambda=[\lambda_1,\ldots,\lambda_k]\vdash n \\ d|\lambda_1,\ldots,d|\lambda_k}} IC(\overline{\mathcal{O}}_\lambda, \mathcal{L}_\chi) \otimes V_{\overline{\lambda}}. \tag{2.5}$$

The generalized Springer correspondence can be stated explicitly in this type A setting as the bijection between simple perverse sheaves on the nilpotent cone and irreducible representations of the relative Weyl groups given by $IC(\overline{\mathcal{O}}_\lambda, \mathcal{L}_\chi) \mapsto V_{\overline{\lambda}}$. Example 2.4 computes the decomposition (2.5) for $G = SL_4(\mathbb{C})$.

Example 2.4. The group $G = SL_4(\mathbb{C})$ has center $Z = \{\omega^k I_4\}$, where $\omega = \exp(\pi i/2)$ and I_4 is the 4×4 identity matrix. Write $\widehat{Z} = \{\iota, \chi_1, \chi_2, \chi_3\}$, where χ_k denotes the central character such that $\chi_k(\omega I_4) = \omega^k$, and ι is the trivial character. The Lusztig sheaf corresponding to the trivial character is the Springer sheaf $\mu_* \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}$. Equation (2.3) becomes

$$\mu_* \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \mathcal{N}] = \bigoplus_{\lambda \vdash 4} IC(\overline{\mathcal{O}}_\lambda, \mathcal{L}_\iota) \otimes V_\lambda.$$

The Lusztig sheaf corresponding to χ_2 is

$$\mathbb{A}_{\chi_2} = \left(IC(\overline{\mathcal{O}}_{[4]}, \mathcal{L}_{\chi_2}) \otimes V_{[2]}\right) \oplus \left(IC(\overline{\mathcal{O}}_{[2,2]}, \mathcal{L}_{\chi_2}) \otimes V_{[1,1]}\right),$$

where the vector spaces V_λ for $\lambda \vdash 2$ are representations of $W(L) \simeq S_2$. Similarly, the Lusztig sheaves corresponding to χ_1 and χ_3 are:

$$\mathbb{A}_{\chi_1} = IC(\overline{\mathcal{O}}_{[4]}, \mathcal{L}_{\chi_1}) \otimes V_{[1]} \quad \text{and} \quad \mathbb{A}_{\chi_3} = IC(\overline{\mathcal{O}}_{[4]}, \mathcal{L}_{\chi_3}) \otimes V_{[1]},$$

where $V_{[1]}$ corresponds to the only representation of the trivial group $W(L) \simeq S_1$ in both these cases. Since $\dim(V_{[1,1]}) = \dim(V_{[2]}) = 1$ and $\dim(V_{[1]}) = 1$, we see that the only simple perverse sheaves with multiplicity more than one in this case are those coming from the Springer sheaf.

In the next sections, we apply the decomposition theorem to the extended Springer resolution when $G = SL_n(\mathbb{C})$. We will show that we recover *all* of the Lusztig sheaves $\mathbb{A}_\chi$ as summands of the pushforward of the constant sheaf. That is, we recover all the simple perverse sheaves on the nilpotent cone. Moreover, each occurs with multiplicity given by the dimension of the corresponding irreducible representation of $S_{n/d}$.

3. Pushing forward the constant sheaf to $\widetilde{\mathcal{N}}$

The main results of the next few sections concern the pushforward of the constant sheaf $\underline{\mathbb{C}}_{\widetilde{\mathcal{M}}}$ via the composition

$$\psi : \widetilde{\mathcal{M}} \xrightarrow{\widetilde{\eta}} \widetilde{\mathcal{N}} \xrightarrow{\mu} \mathcal{N}.$$

In this section, we study the pushforward under the first map $\widetilde{\eta} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$. We prove some of the needed results in a more general setting. The main general result is Proposition 3.6.

3.1. **Pushing forward along a finite quotient map.** Throughout this subsection, X will denote an n-dimensional complex variety with a left action of a finite abelian group Z. It will frequently be convenient to use the right Z-action on X defined by the formula $xz := z^{-1}x$; we write $Y = X/Z$ for the quotient of X by Z (with either action). We will assume that there is an open set $X_{reg} \subset X$ such that Z acts freely on X_{reg}, and let $Y_{reg} = X_{reg}/Z$. We have the following commutative diagram, where the horizontal maps are open embeddings, and the vertical maps are quotient maps.

$$\begin{array}{ccc} X_{reg} & \xrightarrow{i} & X \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \pi} \\ Y_{reg} & \xrightarrow{j} & Y \end{array}$$

The maps π and ρ are finite, so they are trivially both small and semi-small. Moreover, the map ρ is étale. The fibers of π are exactly the orbits of Z on X.

Let $\chi : Z \to \mathbb{C}^*$ denote a character of Z, and let $\mathcal{L}_\chi$ denote the local system defined, as in Section 2.1, by the sheaf of locally constant sections of the line bundle $\mathcal{V}_\chi = X_{reg} \times^Z \mathbb{C}_\chi$ over $X_{reg}/Z = Y_{reg}$. These local systems are studied in [3, Section 3.2]; Lemma 3.1 is essentially a special case of [3, Lemma 3.6].

Lemma 3.1. *With notation as above, we have*

$$\rho_* \underline{\mathbb{C}}_{X_{reg}} = \bigoplus_{\chi \in \widehat{Z}} \mathcal{L}_\chi.$$

For $\chi \in \widehat{Z}$, define

$$X_\chi = \{x \in X \mid Z^x \subseteq \ker(\chi)\}.$$

Since Z is abelian, the set X_χ is Z-stable; let $Y_\chi = X_\chi/Z = \pi(X_\chi)$. Because X_χ is Z-stable, if $y \in Y_\chi$, the entire preimage $\pi^{-1}(y)$ lies in X_χ. In fact, since Z is abelian and $\pi^{-1}(y)$ is a single Z-orbit, the stabilizer groups Z^x are the same for all $x \in \pi^{-1}(y)$.

Given $\chi \in \widehat{Z}$, write $H_\chi = \ker \chi$. If χ is understood, we write $H = H_\chi$. We can view χ either as a character of Z or of Z/H.

Lemma 3.2. *The set X_χ is open in X. Hence Y_χ is open in Y.*

Proof. By definition, $x \in X \setminus X_\chi$ if and only if there exists $g \in Z \setminus H$ such that $x \in X^g = \{x \in X \mid gx = x\}$. Thus,

$$X \setminus X_\chi = \cup_{g \in Z \setminus H} X^g.$$

Since each X^g is closed and Z is finite, $X \setminus X_\chi$ is closed and thus X_χ is open in X. Finally, $\pi : X \to Y$ is proper, so $\pi(X \setminus X_\chi) = Y \setminus Y_\chi$ is closed in Y. Thus, Y_χ is open in Y. □

Lemma 3.3. *The group Z/H acts freely on X_χ/H.*

Proof. Let $\overline{z}, \overline{x}$ denote the images of $x \in Z$ and $x \in X_\chi$ in Z/H and X_χ/H, respectively. Suppose $z \in Z$ and $x \in X_\chi$ such that $\overline{zx} = \overline{x}$. We must show that $\overline{z} = \overline{e}$. By definition, $\overline{zx} = \overline{x}$ implies $zx = hx$ for some $h \in H$. Thus, $zh^{-1}x = x$, implying $zh^{-1} \in Z^x \subset H$. Hence $z \in H$, so $\overline{z} = \overline{e}$, as desired. □

By definition, the open set X_χ contains X_{reg} and hence Y_χ contains Y_{reg}.

Lemma 3.4. *We have*

$$\mathcal{V}_\chi = X_{reg}/H \times^{Z/H} \mathbb{C}_\chi.$$

Proof. By definition, $\mathcal{V}_\chi = X_{reg} \times^Z \mathbb{C}_\chi$ equals $(X_{reg} \times \mathbb{C}_\chi)/Z$, where Z acts by the mixing action $z(u, c) = (uz^{-1}, zc)$. We have

$$(X_{reg} \times \mathbb{C}_\chi)/Z = \big((X_{reg} \times \mathbb{C}_\chi)/H\big)/(Z/H). \tag{3.1}$$

Since H acts trivially on $\mathbb{C}_\chi$, the right hand side of (3.1) is equal to $((X_{reg}/H) \times \mathbb{C}_\chi)/(Z/H)$, which by definition equals $X_{reg}/H \times^{Z/H} \mathbb{C}_\chi$. □

Proposition 3.5. *The local system $\mathcal{L}_\chi$ on $Y_{reg} = X_{reg}/Z$ extends to a local system (also denoted by $\mathcal{L}_\chi$) on Y_χ.*

Proof. It is equivalent to showing that there is a local system on Y_χ whose restriction to Y_{reg} is $\mathcal{L}_\chi$. Since Z/H acts freely on X_χ/H, we have a local system associated to the line bundle $X_\chi/H \times^{Z/H} \mathbb{C}_\chi$ on $(X_\chi/H)/(Z/H) = X_\chi/Z$. By Lemma 3.4, the restriction of this local system to X_{reg}/H is equal to $\mathcal{L}_\chi$. □

Recall that $n = \dim X = \dim Y$. As noted above, given $y \in Y$, the stabilizer group Z^x is independent of the choice of $x \in \pi^{-1}(y)$; a choice of such x gives an identification of $\pi^{-1}(y)$ with Z/Z^x. Given a complex $\mathcal{F}$ of sheaves on Y, the cohomology sheaf $\mathcal{H}^i(\mathcal{F})$ has stalk at $y \in Y$ denoted by $\mathcal{H}^i_y(\mathcal{F})$. We have $\dim \mathcal{H}^i_y(\pi_* \underline{\mathbb{C}}_X) = \dim H^i(\pi^{-1}(y))$. Because π is a finite map, this dimension is 0 if $i > 0$, and is equal to $|\pi^{-1}(y)| = |Z/Z^x|$ if $i = 0$.

The next result describes the pushforward of the constant sheaf on X under the quotient map $\pi : X \to Y$. The strategy of the proof is to first use the decomposition theorem and Lemma 3.1 to show that $\pi_* \underline{\mathbb{C}}_X[n]$ equals the expression in the statement of Proposition 3.6 plus possible extra terms, and then to use dimension arguments to show that these extra terms do not appear.

Proposition 3.6. *Let $\pi : X \to Y$ be the quotient of an n-dimensional variety X by the action of a finite abelian group Z, such that Z acts freely on an open subset X_{reg} of X. Assume that X is rationally smooth. With notation as above, we have*

1. $\pi_* \underline{\mathbb{C}}_X[n] = \bigoplus_{\chi \in \widehat{Z}} IC(Y, \mathcal{L}_\chi)$, *and*
2. $\dim \mathcal{H}^{-n}_y(IC(Y, \mathcal{L}_\chi))$ *is equal to* 1 *if* $y \in Y_\chi$ *and is* 0 *otherwise.*
3. *If* $y \notin Y_\chi$, *then*

$$\mathcal{H}^k_y(IC(Y, \mathcal{L}_\chi)) = 0 \text{ for all integers } k. \tag{3.2}$$

Hence, if $i : Y \setminus Y_\chi \hookrightarrow Y$ is the inclusion, we have $i^* IC(Y, \mathcal{L}_\chi) = 0$ in $D^b(Y \setminus Y_\chi)$.

(4) *If $j_\chi : Y_\chi \to Y$ is the inclusion, then $IC(Y, \mathcal{L}_\chi) = (j_\chi)_! \mathcal{L}_\chi[n]$.*

Proof. Because X is rationally smooth, $\underline{\mathbb{C}}_X[n]$ is a perverse sheaf. Since π is a small map, $\pi_* \underline{\mathbb{C}}_X[n]$ is perverse as well. By the Decomposition Theorem (Theorem 2.2 above),

$$\pi_* \underline{\mathbb{C}}_X[n] = \bigoplus_i IC(Y_i, \mathcal{L}_i), \tag{3.3}$$

where Y_i is a closed subvariety of Y and $\mathcal{L}_i$ is a local system on an open subvariety of Y_i. We claim that for each $\chi \in \widehat{Z}$, the sheaf $IC(Y, \mathcal{L}_\chi)$ occurs in the sum, so the sum takes the form

$$\pi_* \underline{\mathbb{C}}_X[n] = \left(\bigoplus_{\chi \in \widehat{Z}} IC(Y, \mathcal{L}_\chi) \right) \oplus \mathcal{K}, \tag{3.4}$$

where $\mathcal{K}$ is a sum of IC complexes for local systems on subvarieties of Y. To verify the claim, recall that by Lemma 3.1,

$$\rho_* \underline{\mathbb{C}}_{X_{reg}}[n] = \bigoplus_{\chi \in \widehat{Z}} \mathcal{L}_\chi[n].$$

Each term in the decomposition of $\pi_* \underline{\mathbb{C}}_X[n]$ is of the form $IC(Y_i, \mathcal{L}_i)$, where Y_i is a closed subvariety of Y. Recall that $j : Y_{reg} \hookrightarrow Y$. By base change, $j^* \pi_* = \rho_* i^*$. Hence, the restriction of the right side of (3.3) to Y_{reg} equals $\bigoplus_{\chi \in \widehat{Z}} \mathcal{L}_\chi[n]$. Therefore, for each χ, there exists an i such that the restriction of $IC(Y_i, \mathcal{L}_i)$ to Y_{reg} is $\mathcal{L}_\chi[n]$. To finish proving the claim, it suffices to verify that the only intersection cohomology complex on Y whose restriction to Y_{reg} is $\mathcal{L}_\chi[n]$ is $IC(Y, \mathcal{L}_\chi)$. To see this, suppose that the restriction of $IC(Y_i, \mathcal{L}_i)$ to Y_{reg} equals $\mathcal{L}_\chi[n]$. Since the support of $IC(Y_i, \mathcal{L}_i)$ is Y_i, we must have $Y_i = Y$, and then $\mathcal{L}_i$ is a local system on an open subset V_i of Y. This implies that the restrictions of $\mathcal{L}_\chi$ and $\mathcal{L}_i$ to $V_i \cap Y_{reg}$ agree. But if two local systems on open sets agree on the intersection of those open sets, then the corresponding IC-complexes are canonically isomorphic. We conclude that $IC(Y_i, \mathcal{L}_i) = IC(Y, \mathcal{L}_\chi)$, as desired. This proves the claim.

Now let $y \in Y$ and $x \in \pi^{-1}(y)$. By the remarks preceding the proposition,

$$\dim \mathcal{H}_y^{-n}(\pi_* \underline{\mathbb{C}}_X[n]) = \dim \mathcal{H}_y^0(\pi_* \underline{\mathbb{C}}_X) = |\pi^{-1}(y)| = |Z/Z^x|, \tag{3.5}$$

and

$$\dim \mathcal{H}_y^k(\pi_* \underline{\mathbb{C}}_X[n]) = 0 \text{ if } k \neq -n. \tag{3.6}$$

Combining (3.4), (3.5), and (3.6), we see that for any $y \in Y$,

$$|Z/Z^x| = \dim \mathcal{H}_y^{-n}(\pi_* \underline{\mathbb{C}}_X[n]) = \sum_{\chi \in \widehat{Z}} \dim \mathcal{H}_y^{-n}(IC(Y, \mathcal{L}_\chi)) + \dim H_y^{-n}(\mathcal{K}) \tag{3.7}$$

and

$$\dim \mathcal{H}_y^k(IC(Y, \mathcal{L}_\chi)) = \dim \mathcal{H}_y^k(\mathcal{K}) = 0 \text{ if } k \neq -n. \tag{3.8}$$

By Proposition 3.5, $\mathcal{L}_\chi$ extends to a local system (again denoted by $\mathcal{L}_\chi$) on Y_χ so $IC(Y, \mathcal{L}_\chi)|_{Y_\chi} = \mathcal{L}_\chi[n]$. Therefore if $y \in Y_\chi$, then $\dim \mathcal{H}_y^{-n}(IC(Y, \mathcal{L}_\chi)) = 1$. We claim that the number of characters $\chi \in \widehat{Z}$ such that $y \in Y_\chi$ is $|Z/Z^x|$. Indeed, $y \in Y_\chi$ if and only if Z^x is contained in $\ker \chi$. This is equivalent to saying that χ is pulled back from a character $\overline{\chi}$ of Z/Z^x by the quotient map $Z \to Z/Z^x$. Thus,

the number of $\chi \in \widehat{Z}$ such that $y \in Y_\chi$ is equal to the number of distinct characters of Z/Z^x, which is equal to $|Z/Z^x|$ since Z/Z^x is an abelian group. This proves the claim.

The discussion in the previous paragraphs shows that the contribution to the sum on the right hand side of (3.7) from the characters χ such that $y \in Y_\chi$ is equal to $|Z/Z^x|$. Hence, for $k = -n$ and all $y \in Y$, we have $\mathcal{H}^k_y(\mathcal{K}) = 0$. As this equality is also true for $k \neq -n$ by (3.8), it holds for all integers k. Since $\mathcal{K}$ is a direct sum of IC-sheaves, this is only possible if $\mathcal{K} = 0$. Assertion (1) of the proposition follows. Similarly, if $y \notin Y_\chi$, we have $\dim \mathcal{H}^{-n}_y(IC(Y, \mathcal{L}_\chi)) = 0$. Since we have already proved that if $y \in Y_\chi$, then $\dim \mathcal{H}^{-n}_y(IC(Y, \mathcal{L}_\chi)) = 1$, assertion (2) follows. Since $\dim \mathcal{H}^k_y(IC(Y, \mathcal{L}_\chi)) = 0$ for $k \neq -n$ by (3.8), assertion (3) follows as well. Finally, assertion (4) is a consequence of (3) and the fact that the restriction of $IC(Y, \mathcal{L}_\chi)$ to Y_χ is $\mathcal{L}_\chi[n]$. □

3.2. **The subvarieties $\mathcal{V}_\chi$ and $\mathcal{V}_{ad,\chi}$.** In this section, we consider the setup from Section 3.1, in the case that $X = \mathcal{V}$, $Y = \mathcal{V}_{ad}$, and Z is the center of $G = SL_n(\mathbb{C})$. We write $\mathcal{V}_{ad,reg} = (\mathcal{V}_{ad})_{reg}$ and $\mathcal{V}_{ad,\chi} = (\mathcal{V}_{ad})_\chi$. The goal of this section is to give an explicit description of $\mathcal{V}_\chi$ and $\mathcal{V}_{ad,\chi}$ for each $\chi \in \widehat{Z}$ (see Proposition 3.14 and Corollary 3.15).

In this setting $Z = \{\omega^k I_n\} \cong \mathbb{Z}_n$ where $\omega = \exp(2\pi i/n)$ and I_n is the $n \times n$ identity matrix. We use the notation $\underline{\omega}^k = \omega^k I_n$. Let $\chi_k : Z \to \mathbb{C}^*$ be the character defined by $\chi_r(z) = z^k$ for $z \in Z$.

Lemma 3.7.

(1) *Let* $r = n/\gcd(n, c)$. *Then* $\ker \chi_c = \{1, \underline{\omega}^r, \underline{\omega}^{2r}, \ldots\}$.
(2) *Suppose* $\chi \in \widehat{Z}$ *has order* d. *Then* $\ker \chi = \ker \chi_{n/d} = \{1, \underline{\omega}^d, \underline{\omega}^{2d}, \ldots\}$.

Proof. (1) $\ker \chi_c$ is the cyclic group generated by the lowest power of $\underline{\omega}$ in $\ker \chi_c$. Since the smallest positive integer k such that $n|ck$ is $k = r$, the lowest power of $\underline{\omega}$ in $\ker \chi_c$ is $\underline{\omega}^r$.

(2) By definition we have $\chi^k(\underline{\omega}) = \chi(\underline{\omega}^k)$, so $\chi^d = 1$ implies $\{1, \underline{\omega}^d, \underline{\omega}^{2d}, \ldots\} \subset \ker \chi$. Conversely, suppose $\underline{\omega}^k \in \ker \chi$, but k is not a multiple of d. Then $1 \leq m = \gcd(k, d) < d$ and $\underline{\omega}^m \in \ker \chi$, so $\chi^m = 1$. This contradicts the assumption that the order of χ is d. □

Let $\alpha_1, \ldots, \alpha_{n-1}$ be the simple roots associated to the Lie algebra $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, and $\lambda_1, \ldots, \lambda_{n-1}$ denote the fundamental dominant weights (we follow the conventions of [17, Section 13.1]). We let μ_k denote the unique weight in the same coset of λ_i (modulo the root lattice) such that $\mu_k = \sum_{i=1}^{n-1} a_{ki}\alpha_i$, where the $a_{ki} \in \mathbb{Q}$ satisfy $0 \leq a_{ki} < 1$ for all k. We say α_c occurs in μ_k if $a_{kc} \neq 0$.

Recall that λ_k is expressed as a linear combination of simple roots via the formula

$$\lambda_k = \frac{1}{n}\Big((n-k)\alpha_1 + 2(n-k)\alpha_2 + \cdots + k(n-k)\alpha_k + k(n-k-1)\alpha_{k+1} + k(n-k-2)\alpha_{k+2} + \cdots + k\alpha_{n-1}\Big). \tag{3.9}$$

We can use this formula to compute μ_k, as demonstrated in Example 3.8.

Example 3.8. Let $n = 12$. We can view the weights as elements of $\mathbb{R}^{12}$ whose coordinates add up to 0, and each simple root is $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n-1$.

Using equation (3.9) we compute the cosets of the fundamental dominant weights modulo the root lattice. For example,

$$\lambda_1 = \frac{1}{12}(11\alpha_1+10\alpha_2+9\alpha_3+8\alpha_4+7\alpha_5+6\alpha_6+5\alpha_7+4\alpha_8+3\alpha_9+2\alpha_{10}+\alpha_{11}),$$
$$\lambda_2 = \frac{1}{12}(10\alpha_1+20\alpha_2+18\alpha_3+16\alpha_4+14\alpha_5+12\alpha_6+10\alpha_7+8\alpha_8+6\alpha_9+4\alpha_{10}+2\alpha_{11}),$$
$$\lambda_3 = \frac{1}{12}(9\alpha_1+18\alpha_2+27\alpha_3+24\alpha_5+21\alpha_5+18\alpha_6+15\alpha_7+12\alpha_8+9\alpha_9+6\alpha_{10}+3\alpha_{11}),$$
$$\lambda_4 = \frac{1}{12}(8\alpha_1+16\alpha_2+24\alpha_3+32\alpha_4+28\alpha_5+24\alpha_6+20\alpha_7+16\alpha_8+12\alpha_9+8\alpha_{10}+4\alpha_{11}),$$
$$\lambda_5 = \frac{1}{12}(7\alpha_1+14\alpha_2+21\alpha_3+28\alpha_4+35\alpha_5+30\alpha_6+25\alpha_7+20\alpha_8+15\alpha_9+10\alpha_{10}+5\alpha_{11}),$$
$$\lambda_6 = \frac{1}{12}(6\alpha_1+12\alpha_2+18\alpha_3+24\alpha_4+30\alpha_5+36\alpha_6+30\alpha_7+24\alpha_8+18\alpha_9+12\alpha_{10}+6\alpha_{11});$$

the fundamental weights $\lambda_7, \ldots, \lambda_{11}$ are given by similar formulas. Each μ_i is obtained from λ_i by taking the fractional portion of the coefficients a_{ik}:

$$\mu_1 = \lambda_1,$$
$$\mu_2 = \frac{1}{6}(5\alpha_1+4\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+5\alpha_7+4\alpha_8+3\alpha_9+2\alpha_{10}+\alpha_{11}),$$
$$\mu_3 = \frac{1}{4}(3\alpha_1+2\alpha_2+\alpha_3+3\alpha_5+2\alpha_6+\alpha_7+3\alpha_9+2\alpha_{10}+\alpha_{11}),$$
$$\mu_4 = \frac{1}{3}(2\alpha_1+\alpha_2+2\alpha_4+\alpha_5+2\alpha_7+\alpha_8+2\alpha_{10}+\alpha_{11}),$$
$$\mu_5 = \frac{1}{12}(7\alpha_1+2\alpha_2+9\alpha_3+4\alpha_4+11\alpha_5+6\alpha_6+\alpha_7+8\alpha_8+3\alpha_9+10\alpha_{10}+5\alpha_{11}),$$
$$\mu_6 = \frac{1}{2}(\alpha_1+\alpha_3+\alpha_5+\alpha_7+\alpha_9+\alpha_{11}).$$

We can compute μ_i for $7 \leq i \leq 11$ using the same methods. In this case we find:

- α_1 and α_{11} occur in all μ_i
- α_2 and α_{10} occur in all μ_i except μ_6
- α_3 and α_9 occur in all μ_i except μ_4, μ_8
- α_4 and α_8 occur in all μ_i except μ_3, μ_6, μ_9
- α_5 and α_7 occur in all μ_i,
- and α_6 occurs in all μ_i for odd i.

Given a weight λ of T, we write e^λ for the corresponding function on T. If λ is in the root lattice, e^λ can also be viewed as a function on T_{ad}. Set $x_i = e^{\alpha_i}$ and $v_i = e^{\mu_i}$ for all $1 \leq i \leq n-1$. By definition, $\mathcal{V} = \operatorname{Spec} A$, where $A = \mathbb{C}[v_1, \ldots, v_{n-1}, x_1, \ldots, x_{n-1}]/I$ for some ideal I, and $\mathcal{V}_{ad} = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_{n-1}] \cong \mathbb{C}^{n-1}$. The ideal I contains all elements of the form

$$v_1^{a_1} v_2^{a_2} \cdots v_{n-1}^{a_{n-1}} - x_1^{b_1} x_2^{b_2} \cdots x_{n-1}^{b_{n-1}}$$

for all $a_1, a_2, \ldots, a_{n-1}, b_1, b_2, \ldots, b_{n-1} \in \mathbb{Z}$ such that

$$a_1\mu_1 + a_2\mu_2 + \cdots + a_{n-1}\mu_{n-1} = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_{n-1}\alpha_{n-1}.$$

We view $v_1, \ldots, v_{n-1}, x_1, \ldots, x_{n-1}$ as coordinates on $\mathbb{C}^{2n-2}$, and $\mathcal{V}$ as the subscheme of $\mathbb{C}^{2n-2}$ defined by the ideal I.

Given a point x in $\mathcal{V}$, we write $v_i(x)$ and $x_i(x)$ for the values of the coordinate functions v_i and x_i at x. Similarly, if $y \in \mathcal{V}_{ad}$, $x_i(y)$ is the value of the coordinate

x_i at y. Recall that $\pi : \mathcal{V} \to \mathcal{V}_{ad}$ is the quotient by Z. We have $\pi(x) = y$ if and only if $x_i(x) = x_i(y)$ for all i. If $z \in Z$, we have $\pi(zx) = \pi(x)$, so $x_i(zx) = x_i(x)$ for all i.

The center Z acts on $x \in \mathbb{C}^{2n-2}$ by

$$z(v_1(x), v_2(x), \dots, v_{n-1}(x), x_1(x), \dots, x_{n-1}(x)) \\ = (zv_1(x), z^2 v_2(x), \dots, z^{n-1} v_{n-1}(x), x_1(x), \dots, x_{n-1}(x)).$$

The action of Z on the coordinate functions v_i and x_i is given by $z \cdot v_i = z^{-i} v_i$ and $z \cdot x_i = x_i$, respectively. The Z-action on $\mathbb{C}^{2n-2}$ preserves the subscheme $\mathcal{V}$ since the ideal I is T-invariant.

Example 3.9. We continue Example 3.8 for $n = 12$. Rewriting μ_i for $i = 1, 2, 4$ as a function on T and expanding in terms of v_i and x_i, we find that

$$v_1^{12} = x_1^{11} x_2^{10} x_3^9 x_4^8 x_5^7 x_6^6 x_7^5 x_8^4 x_9^3 x_{10}^2 x_{11},$$
$$v_2^6 = x_1^5 x_2^4 x_3^3 x_4^2 x_5 x_7^5 x_8^4 x_9^3 x_{10}^2 x_{11},$$
$$v_4^3 = x_1^2 x_2 x_4^2 x_5 x_7^2 x_8 x_{10}^2 x_{11}.$$

Using similar methods, we obtain

$$v_3^4 = x_1^3 x_2^2 x_3 x_5^3 x_6^2 x_7 x_9^3 x_{10}^2 x_{11},$$
$$v_5^{12} = x_1^7 x_2^2 x_3^9 x_4^4 x_5^{11} x_6^6 x_7 x_8^8 x_9^3 x_{10}^{10} x_{11}^5,$$
$$v_6^2 = x_1 x_3 x_5 x_7 x_9 x_{11}.$$

Lemma 3.10 tells us when α_c occurs in μ_k; the reader can verify this result when $n = 12$ using the data provided in Example 3.8.

Lemma 3.10. *Let $c \in \{1, \dots, n-1\}$ and let $r = n/\gcd(n, c)$. Then α_c occurs in μ_k (for $1 \le k \le n-1$) if and only if k is not a multiple of r.*

Proof. We prove the equivalent statement that α_c does not occur in μ_k if and only if r divides k. The root α_c does not occur in μ_k if and only if the coefficient of α_c in μ_k is an integer. Equation (3.9) shows that if $c \le k$ then this coefficient is equal to $c(n-k)/n$; if $c > k$ then this coefficient is equal to $[k(n-k-(c-k))]/n = k(n-c)/n$. In either case, the coefficient is an integer if and only if n divides ck. Writing $s = \gcd(n, c)$, we have $n = rs$ and $c = ms$, where r and m are relatively prime. Then $n = rs$ divides $ck = kms$ if and only if r divides km if and only if r divides k. □

Note that if $r = n/\gcd(n, c)$, the lemma implies that α_c occurs in each μ_i. Our first step toward computing $\mathcal{V}_\chi$ is to compute the stabilizer groups Z^x for $x \in \mathcal{V}$.

Lemma 3.11. *For $x \in \mathcal{V}$, $Z^x = \cap \ker \chi_k$ where the intersection is over all $k \in \{1, \dots, n-1\}$ such that $v_k(x) \neq 0$.*

Proof. Given $x, x' \in \mathcal{V}$, we have $x = x'$ if and only if $x_k(x) = x_k(x')$ and $v_k(x) = v_k(x')$ for all k. Hence, for $z \in Z$, we have $z \in Z^x$ if and only if $x_k(zx) = x_k(x)$ and $v_k(zx) = v_k(x)$ for all k. The first equality always holds, by the discussion above. We have $v_k(zx) = \chi_k(z) v_k(x)$, and this equals $v_k(x)$ if and only if $\chi_k(z) = 1$ whenever $v_k(x) \neq 0$, which is equivalent to the assertion of the lemma. □

Motivated by Lemma 3.11, we now examine the condition $v_k(x) = 0$.

Lemma 3.12. *Suppose $x \in \mathcal{V}$. We have $v_k(x) = 0$ if and only if $x_i(x) = 0$ for some i such that α_i occurs in μ_k.*

Proof. We have $\mu_k = \sum_{i=1}^{n-1} a_{ki}\alpha_i$ for $a_{ki} \in \mathbb{Q}$ such that $0 \leq a_{ki} < 1$. Choose a positive integer d such that $da_{ki} \in \mathbb{Z}$ for all i. Then $d\mu_k = \sum_{i=1}^{n-1} da_{ki}\alpha_i$, so in the ring A, we have

$$v_k^d = \prod_{i=1}^{n-1} x_i^{da_{ki}}. \tag{3.10}$$

The root α_i occurs in μ_k if and only if $a_{ki} > 0$. So evaluating both sides of (3.10) at x, we see that if α_i occurs in μ_k and $x_i(x) = 0$, then $v_k(x)^d = 0$, and hence $v_k(x) = 0$. On the other hand, if $x_i(x) \neq 0$ for all i such that α_i occurs in μ_k, then $(v_k(x))^d \neq 0$, so $v_k(x) \neq 0$. □

Proposition 3.13 describes the stabilizer groups Z^x for $x \in \mathcal{V}$.

Proposition 3.13. *Let $x \in \mathcal{V}$, and $\{c_1, c_2, \ldots, c_\ell\}$ be the set of integers k such that $x_k(x) = 0$. Let $r_i = n/\gcd(n, c_i)$ and $r = \operatorname{lcm}(r_1, r_2, \ldots, r_\ell)$. (Note that r divides n.) Then $Z^x = \ker \chi_r$.*

Proof. By Lemma 3.12, $v_k(x) = 0$ if and only if $x_i(x) = 0$ for some i such that α_i occurs in μ_k. Equivalently, $v_k(x) = 0$ if and only if α_{c_i} occurs in μ_k for some i. Finally, by Lemma 3.10 α_{c_i} occurs in μ_k if and only if k is not a multiple of r_i. We therefore conclude that $v_k(x) \neq 0$ if and only if k is a multiple of r. The desired statement now follows from Lemma 3.11. □

Let $\chi : Z \to \mathbb{C}^*$ be a character of Z. Recall that $\mathcal{V}_\chi = \{x \in \mathcal{V} \mid Z^x \subset \ker \chi\}$. If χ has order n then $\ker \chi = \{1\}$, so $\mathcal{V}_\chi = \mathcal{V}_{reg}$. The main result of this section is Proposition 3.14, which describes $\mathcal{V}_\chi$.

Proposition 3.14. *Suppose $\chi \in \widehat{Z}$ has order d and let $x \in \mathcal{V}$. Then $x \in \mathcal{V}_\chi$ if and only if for each k in $\{1, \ldots, n-1\}$, $x_k(x) \neq 0$ if d does not divide k.*

Proof. Let $x \in \mathcal{V}$, and let $c_1, \ldots, c_\ell$ be the integers k with $x_k(x) = 0$. The proposition is equivalent to the assertion that $x \in \mathcal{V}_\chi$ if and only if d divides c_i for all i.

Proposition 3.13 says that $Z^x = \ker \chi_r$, where $r = \operatorname{lcm}(r_1, r_2, \ldots, r_\ell)$ for $r_i = n/\gcd(n, c_i)$. Applying this fact together with Lemma 3.7 we now have:

$$x \in \mathcal{V}_\chi \Leftrightarrow Z^x \subseteq \ker \chi = \ker \chi_{n/d} \Leftrightarrow r \text{ divides } \frac{n}{d} \Leftrightarrow r_i \text{ divides } \frac{n}{d} \text{ for all } i.$$

Finally, we have that r_i divides $\frac{n}{d}$ for all i if and only if d divides $\gcd(n, c_i)$ for all i. The last condition is equivalent to requiring that d divide c_i for all i, as desired. □

Corollary 3.15. *Suppose $x \in \mathcal{V}$. Then $x \in \mathcal{V}_{reg}$ if and only if $x_k(x) \neq 0$ for each $k \in \{1, \ldots, n-1\}$.*

Proof. By definition, $x \in \mathcal{V}_{reg}$ if and only if $Z^x = \{1\}$. Since $\{1\} = \ker \chi$ when χ has order n, Proposition 3.14 implies that $\mathcal{V}_{reg}$ consists of the set of $x \in \mathcal{V}$ such that $x_k(x) \neq 0$ for all $k \in \{1, \ldots, n-1\}$. □

Corollary 3.16. *Suppose $\chi \in \widehat{Z}$ has order d. Then $y \in \mathcal{V}_{ad,\chi}$ if and only if for each k in $\{1, \ldots, n-1\}$, $x_k(y) \neq 0$ if d does not divide k. In particular, $\mathcal{V}_{ad,reg}$ consists of the set of $y \in \mathcal{V}_{ad}$ such that $x_k(y) \neq 0$ for all $k \in \{1, \ldots, n-1\}$.*

Proof. By definition, y is in $\mathcal{V}_{ad,\chi}$ if $y = \pi(x)$ for $x \in \mathcal{V}_\chi$. In this case, $x_k(y) = x_k(x)$, so the result follows from Proposition 3.14 and Corollary 3.15. □

3.3. **The subvarieties $\widetilde{\mathcal{M}}_\chi$ and $\widetilde{\mathcal{N}}_\chi$.** Recall that we have $\eta : \widetilde{\mathfrak{u}} \to \mathfrak{u}$, defined by

$$\eta = \pi \times 1 : \widetilde{\mathfrak{u}} = \mathcal{V} \times_{\mathcal{V}_{ad}} \mathfrak{u} \to \mathfrak{u} = \mathcal{V}_{ad} \times_{\mathcal{V}_{ad}} \mathfrak{u}.$$

The Z-action on $\widetilde{\mathfrak{u}}$ comes from the action on the first factor, and we have $\widetilde{\mathfrak{u}}/Z = \mathfrak{u}$. As varieties, $\widetilde{\mathfrak{u}} = \mathcal{V} \times \mathfrak{g}_{\geq 4}$ and $\mathfrak{u} = \mathcal{V}_{ad} \times \mathfrak{g}_{\geq 4}$. Both $\widetilde{\mathfrak{u}}$ and $\mathfrak{u}$ have B-actions and the map η is B-equivariant. From this we obtain

$$\widetilde{\eta} : \widetilde{\mathcal{M}} = G \times^B \widetilde{\mathfrak{u}} \to \widetilde{\mathcal{N}} = G \times^B \mathfrak{u}.$$

The Z-action on $\widetilde{\mathcal{M}}$ is the restriction of the left G-action to the subgroup Z; note that for $z \in Z$, $g \in G$, and $x \in \mathcal{V}$, we have

$$z[g, x] = [zg, x] = [gz, x] = [g, zx],$$

where the second equality is because z is central, and the third holds since $Z \subset B$. The equality $\widetilde{\mathfrak{u}}/Z = \mathfrak{u}$ implies that $\widetilde{\mathcal{M}}/Z = \widetilde{\mathcal{N}}$ (see [16]).

Proposition 3.17. *Let $\chi \in \widehat{Z}$.*

(1) *We have $\widetilde{\mathfrak{u}}_\chi = \mathcal{V}_\chi \times_{\mathcal{V}_{ad}} \mathfrak{u}$ and $\mathfrak{u}_\chi = \mathcal{V}_{ad,\chi} \times_{\mathcal{V}_{ad}} \mathfrak{u}$. As varieties, $\widetilde{\mathfrak{u}}_\chi = \mathcal{V}_\chi \times \mathfrak{g}_{\geq 4}$ and $\mathfrak{u}_\chi = \mathcal{V}_{ad,\chi} \times \mathfrak{g}_{\geq 4}$.*

(2) *We have $\widetilde{\mathcal{M}}_\chi = G \times^B \widetilde{\mathfrak{u}}_\chi$ and $\widetilde{\mathcal{N}}_\chi = G \times^B \mathfrak{u}_\chi$.*

Proof. (1) Suppose $\widetilde{u} = (x, u) \in \widetilde{\mathfrak{u}}$, where $x \in \mathcal{V}$ and $u \in \mathfrak{u}$. Then $Z^{(x,u)} = Z^x$, so $\widetilde{u} \in \widetilde{\mathfrak{u}}_\chi$ if and only if $x \in \mathcal{V}_\chi$. This proves that $\widetilde{\mathfrak{u}}_\chi = \mathcal{V}_\chi \times_{\mathcal{V}_{ad}} \mathfrak{u}$. Since $\mathfrak{u}_\chi$ is the image of $\widetilde{\mathfrak{u}}_\chi$ under the map η, it follows that $\mathfrak{u}_\chi = \mathcal{V}_{ad,\chi} \times_{\mathcal{V}_{ad}} \mathfrak{u}$. The assertions about the structure of $\widetilde{\mathfrak{u}}_\chi$ and $\mathfrak{u}_\chi$ as varieties follow from writing $\mathfrak{u} = \mathcal{V}_{ad} \times \mathfrak{g}_{\geq 4}$.

(2) If $[g, \widetilde{u}] \in \widetilde{\mathcal{M}}$, then $Z^{[g,\widetilde{u}]} = Z^{\widetilde{u}}$, so $[g, \widetilde{u}] \in \widetilde{\mathcal{M}}_\chi$ if and only if $u \in \widetilde{\mathfrak{u}}_\chi$. Hence $\widetilde{\mathcal{M}}_\chi = G \times^B \widetilde{\mathfrak{u}}_\chi$. The assertion $\widetilde{\mathcal{N}}_\chi = G \times^B \mathfrak{u}_\chi$ follows because $\widetilde{\mathcal{N}}_\chi$ is the image of $\widetilde{\mathcal{M}}_\chi$ under $\widetilde{\eta}$. □

Example 3.18. Let $n = 6$ and $\chi = \chi_3 \in \widehat{Z}$; note that χ is a character of order $d = 2$. In this example, we consider the intersections $\widetilde{\mathcal{N}}_\chi \cap \mu^{-1}(\nu)$ for various nilpotent elements $\nu \in \mathcal{N}$. Here $\mu^{-1}(\nu)$ is the Springer fiber over ν; recall from (2.1) that $\mu^{-1}(\nu) = \{[g, g^{-1}\nu] \mid g^{-1} \cdot \nu \in \mathfrak{u}\}$. Proposition 3.17 shows that $\widetilde{\mathcal{N}}_\chi$ is determined by $\mathfrak{u}_\chi$: we have $[g, g^{-1}\nu] \in \mathcal{N}_\chi$ if and only if $g^{-1}\nu \in \mathfrak{u}_\chi$. We have $\mathfrak{u}_\chi = p^{-1}(\mathcal{V}_{ad,\chi})$, where $p : \mathfrak{u} \to \mathcal{V}_{ad} = \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$ is the natural projection. Write

$$p(g^{-1} \cdot \nu) = c_1 E_{\alpha_1} + c_2 E_{\alpha_2} + c_3 E_{\alpha_3} + c_4 E_{\alpha_4} + c_5 E_{\alpha_5}. \tag{3.11}$$

We deduce from Proposition 3.14 that $(g, g^{-1} \cdot \nu) \in \widetilde{\mathcal{N}}_\chi$ if and only if $c_k \neq 0$ for all odd k.

It is well-known that the irreducible components of $\mu^{-1}(\nu)$ for $\nu \in \mathcal{O}_\lambda$ are indexed by standard tableaux of shape λ. Indeed, if we construct these components as Steinberg does in [28, §2], each standard tableau $S \in \mathrm{ST}(\lambda)$ corresponds to an open subset $C_S \subset \mu^{-1}(\nu)$ of maximal dimension. The decomposition of $\mu^{-1}(\nu)$ into irreducible components is $\mu^{-1}(\nu) = \cup_{S \in \mathrm{ST}(\lambda)} \overline{C}_S$. In this example, it is straightforward

to show that $C_S \subset \widetilde{\mathcal{N}}_\chi$ whenever S is one of the following standard tableaux.

(3.12)

1	2
3	4
5	6

1	2	3	4
5	6		

1	2	5	6
3	4		

1	2	3	4	5	6

This computation is discussed in greater detail for the first standard tableau above in Example 5.4.

Corollary 3.16 implies that $\mathfrak{u}_{reg}$ consists of the elements of $\mathfrak{u}$ such that the coefficient of each root vector corresponding to a simple root is nonzero. This set is exactly the set of principal nilpotent elements in $\mathfrak{u}$. Therefore, $\widetilde{\mathcal{N}}_{reg}$ is the set of $[g,\nu] \in G\times^B \mathfrak{u}$ where ν is principal nilpotent in $\mathfrak{u}$. The map $\mu : [g,\nu] \to g\cdot\nu$ identifies $\widetilde{\mathcal{N}}_{reg}$ with the principal nilpotent orbit $\mathcal{O}^{pr}$ in $\mathcal{N}$. As discussed in Section 2.2, each character χ of Z induces a local system $\mathcal{L}_\chi$ on $\mathcal{O}^{pr}$ and consequently on $\widetilde{\mathcal{N}}_{reg}$. We are now in the setting of Section 3.1, with $X = \widetilde{\mathcal{M}}$ and $Y = \widetilde{\mathcal{N}}$. Applying Proposition 3.6 to $\widetilde{\eta} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ yields the following result.

Proposition 3.19. *Let $\widetilde{\eta} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ denote the map induced by $\eta : \widetilde{\mathfrak{u}} \to \mathfrak{u}$. Then*

$$\widetilde{\eta}_* \underline{\mathbb{C}}_{\widetilde{\mathcal{M}}}[\dim \mathcal{N}] = \bigoplus_{\chi\in\widehat{Z}} IC(\widetilde{\mathcal{N}}, \mathcal{L}_\chi).$$

Moreover, if i is the inclusion of $\widetilde{\mathcal{N}} \setminus \widetilde{\mathcal{N}}_\chi$ into $\widetilde{\mathcal{N}}$, then $i^* IC(\widetilde{\mathcal{N}}, \mathcal{L}_\chi) = 0$.

4. IC sheaves on $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{N}}^P$

In this section we continue our study of the IC-complexes $IC(\widetilde{N}, \mathcal{L}_\chi)$ for $\chi \in \widehat{Z}$. The main result of the section is Theorem 4.6, which shows that the pushforward of this complex to the variety $\widetilde{\mathcal{N}}^P$ studied in [7] is equal to the IC-complex $IC(\widetilde{\mathcal{N}}^P, \mathcal{L}_\chi)$.

We continue to assume $G = SL_n(\mathbb{C})$. Throughout this section, we fix positive integers d and r such that $n = dr$. Let L_d be the Levi subgroup of G containing T whose simple roots are the $\alpha_k \in \Delta$ such that k is not divisible by d. Then L_d is a block diagonal matrix with $d \times d$ blocks; that is,

$$L_d \cong S(GL_d(\mathbb{C}) \times GL_d(\mathbb{C}) \times \cdots \times GL_d(\mathbb{C})),$$

with r factors on the right hand side. We let P_d denote the standard parabolic subgroup of G with Levi factor L_d and unipotent radical U_d. Since B is upper triangular, P_d is block upper triangular.

Let I_d denote the identity matrix in $GL_d(\mathbb{C})$. The center $Z(L_d)$ of L_d consists of the block diagonal matrices where the i-th block is $a_i I_d$ for $a_i \in \mathbb{C}$, subject to the condition $(a_1 a_2 \cdots a_r)^d = 1$. For simplicity, we denote such a matrix by $z_d(a_1, \ldots, a_r)$. The identity component $Z(L_d)_0$ is the subset of elements of $Z(L_d)$ satisfying $a_1 a_2 \cdots a_r = 1$. Therefore, the component group $C_d = Z(L_d)/Z(L_d)_0$ is isomorphic to $\mathbb{Z}_d$. The character group $\widehat{C_d}$ is generated by the character

$$\phi : z_d(a_1, \ldots, a_r) \mapsto a_1 a_2 \cdots a_r.$$

The inclusion map $Z \to Z(L_d)$ induces a surjection $Z \to C_d$, with kernel $\{1, \underline{\omega}^d, \underline{\omega}^{2d}, \ldots\}$. Pullback via this surjection yields an injective map $\widehat{C_d} \to \widehat{Z}$.

Lemma 4.1.

(1) *The map* $\widehat{C_d} \to \widehat{Z}$ *takes* ϕ *to* χ_r.
(2) *If* $\chi \in \widehat{Z}$ *is a character of order* d, *then* χ *is the image of an element of* $\widehat{C_d}$.

Proof. (1) holds by direct calculation. (2) follows since any element χ of $\widehat{Z}$ of order d is a power of χ_r, which is the image of an element of $\widehat{C_d}$. □

Remark 4.2. By abuse of notation, we will frequently use the same letter χ for an element of $\widehat{C_d}$ and its image in $\widehat{Z}$.

For the remainder of this section, since d is fixed, we simplify the notation by writing $P = LU_P$ for $P_d = L_dU_d$. We write $U_L = U \cap L$. We let $\chi \in \widehat{Z}$ denote a character of order d. We may repeat these assumptions for emphasis.

Let ν_d be the matrix which has entries equal to 1 above the diagonal in each $d \times d$ block, and zeroes elsewhere. In other words, ν_d is the sum of simple root vectors corresponding to $\alpha_k \in \Delta$ such that k is not divisible by d. Thus, ν_d is a principal nilpotent element of $\mathfrak{l}$. The next proposition describes the stabilizer groups P^{ν_d} and L^{ν_d} and the corresponding component groups.

Lemma 4.3.

(1) $L^{\nu_d} = Z(L)U_L^{\nu_d}$.
(2) $P^{\nu_d} = L^{\nu_d}U_P^{\nu_d}$.
(3) *The inclusions* $Z(L) \subset L^{\nu_d} \subset P^{\nu_d}$ *induce identifications of component groups*

$$Z(L)/Z(L)_0 = L^{\nu_d}/(L^{\nu_d})_0 = P^{\nu_d}/(P^{\nu_d})_0.$$

Proof. Statement (1) follows from a straightforward computation of the stabilizer of a principal nilpotent element in $GL_d(\mathbb{C})$, which we omit. We prove (2). Suppose $p = u\ell \in P^{\nu_d}$ with $u \in U_P$ and $\ell \in L$. For any $x \in \mathfrak{l}$, the image of $u\ell x$ under the projection $\mathfrak{p} \to \mathfrak{l}$ equals ℓx. Since $u\ell\nu_d = \nu_d \in \mathfrak{l}$, we see that $\ell\nu_d = \nu_d$, so $\ell \in L^{\nu_d}$. This implies that $u \in U_P^{\nu_d}$, proving (2). Finally, the equalities in (3) follow from the fact that the groups $U_P^{\nu_d}$ and $U_L^{\nu_d}$ are unipotent, and therefore connected. □

We have a Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}_P$. Let $\mathcal{N}_L$ denote the nilpotent cone in $\mathfrak{l}$. Recall from Section 2.6 that

$$\widetilde{\mathcal{N}}^P = G \times^P (\mathcal{N}_L + \mathfrak{u}_P);$$

we write $[g, x]_P$ for the element of $\widetilde{\mathcal{N}}^P$ corresponding to (g, x) for $g \in G$, $x \in \mathcal{N}_L + \mathfrak{u}_P$. Borho and MacPherson defined a stratification of $\widetilde{\mathcal{N}}^P$ indexed by L-orbits on the nilpotent cone $\mathcal{N}_L$; the stratum corresponding to the orbit $\mathcal{O}_L \subset \mathcal{N}_L$ is $G \times^P (\mathcal{O}_L + \mathfrak{u}_P)$ (see [7, Section 2.10]; note that $\mathcal{O}_L + \mathfrak{u}_P$ is P-stable). The stratum corresponding to the principal nilpotent orbit $\mathcal{O}_L^{pr}$ in $\mathcal{N}_L$ is denoted $\widetilde{\mathcal{N}}_{reg}^P$.

Lemma 4.4. *Suppose* $\chi \in \widehat{Z}$ *has order* d, *and let* $P = P_d$, $L = L_d$. *Then* $\mathfrak{u} \cap (\mathcal{O}_L^{pr} + \mathfrak{u}_P) = \mathfrak{u}_\chi$.

Proof. Write $x \in \mathfrak{u}$ as a sum of root vectors:

$$x = \sum_{i=1}^{n-1} a_{\alpha_i} X_{\alpha_i} + \sum_{\beta \in \Phi^+ - \Delta} a_\beta X_\beta,$$

where the α_i are the simple roots and $a_\gamma \in \mathbb{C}$ for all $\gamma \in \Phi^+$. We have $x \in \mathcal{O}_L^{pr} + \mathfrak{u}_P$ if and only if for each simple root α_i of L, $a_{\alpha_i} \neq 0$. The simple roots of L are the α_i where i is not divisible by d. By Propositions 3.14 and 3.17, x is in $\mathfrak{u}_\chi$ if and only if a_{α_i} is nonzero when i is not divisible by d. Hence, x is in $\mathfrak{u}_\chi$ if and only if x is in $\mathfrak{u} \cap (\mathcal{O}_L^{pr} + \mathfrak{u}_P)$. □

There is a semismall map $\rho_P : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}^P$ [7, Lemma 2.10(e)], defined as follows. Given $g \in G$ and $x \in \mathfrak{u}$ (resp. $x \in \mathcal{N}_L + \mathfrak{u}_P$), temporarily write $[g,x]_B$ (resp. $[g,x]_P$) for the equivalence class of (g,x) in $\widetilde{\mathcal{N}} = G \times^B \mathfrak{u}$ (resp. in $\widetilde{\mathcal{N}}^P = G \times^P (\mathcal{N}_L + \mathfrak{u}_P)$). By definition, $\rho_P([g,x]_B) = [g,x]_P$. We have $\mu = \mu_P \circ \rho_P$, where as in Section 2.6, $\mu_P([g,x]_P) = g \cdot x$ (see [7, Section 2.10]).

Proposition 4.5. *Suppose $\chi \in \widehat{Z}$ has order d, and let $P = P_d$. The map $\rho_P : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}^P$ satisfies $\rho_P^{-1}(\widetilde{\mathcal{N}}^P_{reg}) = \widetilde{\mathcal{N}}_\chi$. Moreover, ρ_P takes $\widetilde{\mathcal{N}}_\chi$ isomorphically onto $\widetilde{\mathcal{N}}^P_{reg}$.*

Proof. Since $\mathcal{O}_L^{pr} + \mathfrak{u}_P$ is B-invariant (as it is P-invariant), the intersection $(\mathcal{O}_L^{pr} + \mathfrak{u}_P) \cap \mathfrak{u}$ is B-invariant. The definition of ρ_P implies that

$$\rho_P^{-1}(\widetilde{\mathcal{N}}^P_{reg}) = G \times^B ((\mathcal{O}_L^{pr} + \mathfrak{u}_P) \cap \mathfrak{u}).$$

By Lemma 4.4, $(\mathcal{O}_L^{pr} + \mathfrak{u}_P) \cap \mathfrak{u} = \mathfrak{u}_\chi$, so

$$\rho_P^{-1}(\widetilde{\mathcal{N}}^P_{reg}) = G \times^B \mathfrak{u}_\chi = \widetilde{\mathcal{N}}_\chi,$$

where the last equality is by Proposition 3.17. This proves the first assertion of the proposition.

We now show that ρ_P takes $\widetilde{\mathcal{N}}_\chi$ isomorphically onto $\widetilde{\mathcal{N}}^P_{reg}$. Borho and MacPherson [7, Lemma 2.10(b)] show that the fiber of ρ_P over the stratum corresponding to $\mathcal{O}_L$ is isomorphic to the Springer fiber $\mathcal{B}_L^x$; here $\mathcal{B}_L$ is the flag variety for L, and x is an element of $\mathcal{O}_L$. For the stratum corresponding to the principal orbit $\mathcal{O}_L^{pr}$, the Springer fiber $\mathcal{B}_L^x$ is a single point. Hence ρ_P induces a bijection

$$\widetilde{\mathcal{N}}_\chi = \rho_P^{-1}(\widetilde{\mathcal{N}}^P_{reg}) \to \widetilde{\mathcal{N}}^P_{reg}. \tag{4.1}$$

Since $\widetilde{\mathcal{N}}^P_{reg}$ is smooth, Zariski's Main Theorem implies that (4.1) is an isomorphism of schemes, completing the proof. □

Observe that $\mathcal{O}^{pr}$ can be viewed as an open dense subset of any of $\mathcal{N}$, $\widetilde{\mathcal{N}}$, or $\widetilde{\mathcal{N}}^P$. Indeed, the Springer resolution $\mu : \widetilde{\mathcal{N}} \to \mathcal{N}$ is an isomorphism over $\mathcal{O}^{pr}$; the map μ factors through $\rho_P : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}^P$ (see the remarks before Proposition 4.5), and ρ_P takes the open set $\widetilde{\mathcal{N}}_\chi$ (which contains $\mathcal{O}^{pr}$) isomorphically onto its image. A character χ of Z corresponds to a local system $\mathcal{L}_\chi$ on $\mathcal{O}^{pr}$, and we obtain IC complexes $IC(\widetilde{\mathcal{N}}, \mathcal{L}_\chi)$ and $IC(\widetilde{\mathcal{N}}^P, \mathcal{L}_\chi)$.

Theorem 4.6. *Suppose $\chi \in \widehat{Z}$ has order d, and let $P = P_d$. Then $(\rho_P)_* IC(\widetilde{\mathcal{N}}, \mathcal{L}_\chi) = IC(\widetilde{\mathcal{N}}^P, \mathcal{L}_\chi)$.*

Proof. By the Decomposition Theorem (see Theorem 2.2), $(\rho_P)_* IC(\widetilde{\mathcal{N}}, \mathcal{L}_\chi)$ is a direct sum of shifted IC complexes on $\widetilde{\mathcal{N}}^P$. We claim that the shifts are all trivial. To prove the claim, it suffices to show that $(\rho_P)_* IC(\widetilde{\mathcal{N}}, \mathcal{L}_\chi)$ is perverse, since no nontrivial shift of an intersection cohomology complex remains perverse (see [7, p. 37]). Observe that the composition $\rho_P \circ \widetilde{\eta} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}^P$ is semismall, since ρ_P is semismall and $\widetilde{\eta}$ is finite. Moreover, $\widetilde{\mathcal{M}}$ is rationally smooth, since it is locally

a quotient of a smooth variety by a finite group. Therefore, by [7, Section 1.7], $(\rho_P \circ \widetilde{\eta})_* \underline{\mathbb{C}}_{\widetilde{\mathcal{M}}}$ is perverse. By Proposition 3.19,

$$(\rho_P \circ \widetilde{\eta})_* \underline{\mathbb{C}}_{\widetilde{\mathcal{M}}} = (\rho_P)_* (\oplus_{\chi \in \widehat{Z}} IC(\widetilde{\mathcal{N}}, \mathcal{L}_\chi)).$$

Since the right hand side satisfies the support condition to be a perverse sheaf, each summand also satisfies the same support condition and therefore each summand is also perverse. The claim follows.

The proper base change theorem implies that the pushforward $(\rho_P)_*$ commutes with the restriction to the open set $\widetilde{\mathcal{N}}_\chi \cong \widetilde{\mathcal{N}}^P_{reg}$. Since ρ_P is an isomorphism over this open set, and the local system $\mathcal{L}_\chi$ extends to this open set, the perverse sheaf $(\rho_P)_* IC(\widetilde{\mathcal{N}}, \mathcal{L}_\chi)$ restricts to $\mathcal{L}_\chi[n]$ on $\widetilde{\mathcal{N}}^P_{reg}$ (where $n = \dim \mathcal{N}$). Arguing as in the proof of Proposition 3.6, we see that the only intersection cohomology complex on $\widetilde{\mathcal{N}}^P$ whose restriction to $\widetilde{\mathcal{N}}^P_{reg}$ is $\mathcal{L}_\chi[n]$ is $IC(\widetilde{\mathcal{N}}_P, \mathcal{L}_\chi)$. Therefore, $IC(\widetilde{\mathcal{N}}^P, \mathcal{L}_\chi)$ must occur as a summand in $(\rho_P)_* IC(\widetilde{\mathcal{N}}, \mathcal{L}_\chi)$. Hence we can write

$$(\rho_P)_* IC(\widetilde{\mathcal{N}}, \mathcal{L}_\chi) = IC(\widetilde{\mathcal{N}}^P, \mathcal{L}_\chi) \oplus \mathcal{K},$$

where $\mathcal{K}$ is a direct sum of intersection cohomology complexes on subvarieties of $\widetilde{\mathcal{N}}^P$. To complete the proof, we must show that $\mathcal{K} = 0$.

As a step toward this, we claim that if $IC(Z, \mathcal{F})$ occurs in $\mathcal{K}$, then Z must be contained in $\widetilde{\mathcal{N}}^P \setminus \widetilde{\mathcal{N}}^P_{reg}$. Indeed, if not, then $Z \cap \widetilde{\mathcal{N}}^P_{reg}$ would be open and dense in Z, so it would contain a point z of the smooth open set where the local system $\mathcal{F}$ is defined. Therefore $\mathcal{H}^*_z((\rho_P)_* IC(\widetilde{\mathcal{N}}, \mathcal{L}_\chi))$ would be at least 2-dimensional, since it would have contributions from $IC(\widetilde{\mathcal{N}}^P, \mathcal{L}_\chi)$ and $IC(Z, \mathcal{F})$. This contradicts the fact that the restriction of $(\rho_P)_* IC(\widetilde{\mathcal{N}}, \mathcal{L}_\chi)$ to $\widetilde{\mathcal{N}}^P_{reg}$ is $\mathcal{L}_\chi[n]$. Therefore $Z \subset \widetilde{\mathcal{N}}^P \setminus \widetilde{\mathcal{N}}^P_{reg}$, proving the claim.

Consider the Cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathcal{N}} \setminus \widetilde{\mathcal{N}}_\chi & \xrightarrow{i} & \widetilde{\mathcal{N}} \\ \downarrow{\scriptstyle \rho'_P} & & \downarrow{\scriptstyle \rho_P} \\ \widetilde{\mathcal{N}}^P \setminus \widetilde{\mathcal{N}}^P_{reg} & \xrightarrow{j} & \widetilde{\mathcal{N}}^P \end{array}$$

where ρ'_P is the restriction of ρ_P. By the proper base change theorem,

$$j^*(\rho_P)_* IC(\widetilde{\mathcal{N}}, \mathcal{L}_\chi) = (\rho'_P)_* i^* IC(\widetilde{\mathcal{N}}, \mathcal{L}_\chi) = 0,$$

where the second equality holds since $i^* IC(\widetilde{\mathcal{N}}, \mathcal{L}_\chi) = 0$ by Proposition 3.19. Since $\mathcal{K}$ is a direct summand in $(\rho_P)_* IC(\widetilde{\mathcal{N}}, \mathcal{L}_\chi)$, we deduce that $j^*\mathcal{K} = 0$.

We know that $\mathcal{K}$, if nonzero, is a direct sum of terms of the form $IC(Z, \mathcal{F})$, where Z is a subvariety of $\widetilde{\mathcal{N}}^P \setminus \widetilde{\mathcal{N}}^P_{reg}$. Observe that for such a Z, following the conventions of Section 2.5, we have $j^* IC(Z, \mathcal{F}) = IC(Z, \mathcal{F})$. Such a complex has a nonzero stalk at any point z in the open set of Z where the local system $\mathcal{F}$ is defined. Since $j^*\mathcal{K} = 0$, we see that no term of the form $IC(Z, \mathcal{F})$ can appear in $\mathcal{K}$, and therefore $\mathcal{K} = 0$, as desired. □

5. The generalized Springer correspondence and Lusztig Sheaves

Recall that our goal is to prove that $\psi_* \underline{\mathbb{C}}_{\widetilde{\mathcal{M}}}$ is a direct sum of Lusztig sheaves, which were defined in Section 2.6 above. In order to proceed we need to realize

each $IC(\widetilde{\mathcal{N}}^P, \mathcal{L}_\chi)$ as an induced complex. After some preliminary discussion, we prove our main theorem, which is Theorem 5.3 below.

We keep the notation of Section 4. Thus, $G = SL_n(\mathbb{C})$, $\chi \in \widehat{Z}$ has order d, and $\nu_d \in \mathcal{N}$, $P_d = L_d U_d$ are as in Section 4. As in that section, we write $P = LU_P$ for $P_d = L_d U_d$.

We recall some notation from Section 2.6. The projection $q : \mathfrak{p} = \mathfrak{l} + \mathfrak{u}_P \to \mathfrak{l}$ is P-equivariant, where the P-action on the target is the extension of the adjoint L-action to P by requiring that U_P acts trivially. We use the same notation q for the induced maps $\mathcal{N}_L + \mathfrak{u}_P \to \mathcal{N}_L$ and $\mathcal{O}_L^{pr} + \mathfrak{u}_P \to \mathcal{O}_L^{pr}$, which are also P-equivariant. The map q induces a map $\pi_P : \widetilde{\mathcal{N}}^P \to G \times^P \mathcal{N}_L$.

Note that if $x \in \mathfrak{l}$, even though $q(x) \in \mathfrak{l}$ is the same element as x, the notation indicates that the P-action is different. Indeed, if $u \in U_P$, then $u \cdot x = x + y$ for some $y \in \mathfrak{u}_P$, but $u \cdot q(x) = q(x)$. Given $x \in \mathfrak{l}$, we will write $\overline{x} = q(x)$ for the same element, but with trivial action of U_P. Recall that ν denotes a principal nilpotent element.

Lemma 5.1. *$P \cdot \nu$ is open in $\mathcal{N}_L + \mathfrak{u}_P$.*

Proof. Observe that $P \cdot \nu$ has the same dimension as $\mathcal{N}_L + \mathfrak{u}_P$. Indeed, $P^\nu = G^\nu$ has dimension $n-1$, so $\dim P \cdot \nu = \dim P - (n-1)$; on the other hand, $\dim \mathcal{N}_L = \dim L - (n-1)$, so $\dim \mathcal{N}_L + \dim \mathfrak{u}_P = \dim L + \dim U_P - (n-1) = \dim P - (n-1)$. This verifies the assertion about dimensions. Since $\mathcal{N}_L + \mathfrak{u}_P$ is irreducible, the closure of $P \cdot \nu$ must equal $\mathcal{N}_L + \mathfrak{u}_P$. Since any orbit is open in its closure (see [19, Section 2.1]), $P \cdot \nu$ is open in $\mathcal{N}_L + \mathfrak{u}_P$. □

Remark 5.2. An alternative proof of Lemma 5.1 is as follows. We know that $\mathcal{O}^{pr} = G \cdot \nu$ is open and dense in $\mathcal{N}$. Since $\widetilde{\mathcal{N}}^P \to \mathcal{N}$ is an isomorphism over $\mathcal{O}^{pr}$, we can identify $G \cdot \nu$ with its inverse image in $\widetilde{\mathcal{N}}^P$, and that inverse image is therefore open and dense in $\widetilde{\mathcal{N}}^P$. We write $\mathcal{O}_L^{pr}$ for either the orbit $L \cdot \nu_d$ or $L \cdot \overline{\nu_d}$. The fiber over eP of the projection $\widetilde{\mathcal{N}}^P \to G/P$ is identified with $\mathcal{O}_L^{pr} + \mathfrak{u}_P$. The intersection of $G \cdot \nu$ with this fiber is open and dense in the fiber, and is identified with $P \cdot \nu$, completing the proof.

Lemma 4.1 implies that the character χ of Z is the pullback of a character of C_d, which we again denote by χ (cf. Remark 4.2). By Lemma 4.3, the component group L^ν / L_0^ν of L^ν is equal to C_d. Therefore, χ induces a local system $\mathcal{L}_\chi^L$ on the orbit $\mathcal{O}_L^{pr}$. The Lusztig sheaf $\mathbb{A}_\chi$ is defined by

$$\mathbb{A}_\chi = \mu_{P*} \pi_P^* \operatorname{Ind}_P^G IC(\mathcal{N}_L, \mathcal{L}_\chi^L)[d_P],$$

where $d_P = \dim \mathfrak{u}_P$. Recall the map $\psi : \widetilde{\mathcal{M}} \to \mathcal{N}$, which factors as

$$\widetilde{\mathcal{M}} \xrightarrow{\widetilde{\eta}} \widetilde{\mathcal{N}} \xrightarrow{\rho_P} \widetilde{\mathcal{N}}^P \xrightarrow{\mu_P} \mathcal{N}.$$

Our main theorem is the following.

Theorem 5.3. *Let $\psi : \widetilde{\mathcal{M}} \to \mathcal{N}$ be the extended Springer resolution. Then*

$$\psi_* \underline{\mathbb{C}}_{\widetilde{\mathcal{M}}}[\dim \mathcal{N}] = \bigoplus_{\chi \in \widehat{Z}} \mathbb{A}_\chi.$$

Proof. By Proposition 3.19 and Theorem 4.6, we have

$$(\rho_P \circ \widetilde{\eta})_* \underline{\mathbb{C}}_{\widetilde{\mathcal{M}}}[\dim \mathcal{N}] = \bigoplus_{\chi \in \widehat{Z}} IC(\widetilde{\mathcal{N}}, \mathcal{L}_\chi).$$

Comparing with the definition of the Lusztig sheaf, we see that it suffices to prove that as G-equivariant perverse sheaves on $\widetilde{\mathcal{N}}^P$, we have

$$\pi_P^* \operatorname{Ind}_P^G(IC(\mathcal{N}_L, \mathcal{L}_\chi^L))[d_P] = IC(\widetilde{\mathcal{N}}^P, \mathcal{L}_\chi). \tag{5.1}$$

From the map $q : \mathcal{O}_L^{pr} + \mathfrak{u}_P \to \mathcal{O}_L^{pr}$ we obtain a local system $q^*\mathcal{L}_\chi^L$ on $\mathcal{O}_L^{pr} + \mathfrak{u}_P$. We have

$$\begin{aligned} \pi_P^* \operatorname{Ind}_P^G IC(\mathcal{N}_L, \mathcal{L}_\chi^L)[d_P] &= \operatorname{Ind}_P^G(q^* IC(\mathcal{N}_L, \mathcal{L}_\chi^L))[d_P] \\ &= \operatorname{Ind}_P^G IC(\mathcal{N}_L + \mathfrak{u}_P, q^*\mathcal{L}_\chi^L)[d_P] \\ &= IC(\widetilde{\mathcal{N}}^P, \operatorname{Ind}_P^G(q^*\mathcal{L}_\chi^L)). \end{aligned}$$

Here the first equality is because $\pi_P = \operatorname{Ind}_P^G(q)$ and Ind_P^G is a functor; the second equality is because of the compatibility of IC complexes with smooth pullback (see [19, Lemma 2.15]); and the third equality is because induction equivalence is compatible with the construction of IC-complexes (see [6, §5.2]). To complete the proof, it suffices to show that there is some G-stable open set of $\widetilde{\mathcal{N}}^P$ on which the restrictions of the local systems $\operatorname{Ind}_P^G(q^*\mathcal{L}_\chi^L)$ and $\mathcal{L}_\chi$ are isomorphic. This is equivalent to showing that there is some P-stable open set of $\mathcal{O}_L^{pr} + \mathfrak{u}_P$ on which the restrictions of $q^*\mathcal{L}_\chi^L$ and $\mathcal{L}_\chi$ are isomorphic. We will verify this for the open set $P \cdot \nu$ of $\mathcal{O}_L^{pr} + \mathfrak{u}_P$.

The map q induces (by restriction) a map of orbits

$$q : P \cdot \nu \to P \cdot \overline{\nu}; \tag{5.2}$$

here the notation $\overline{\nu}$ is as discussed at the beginning of the section. We need to check that

$$q^*(\mathcal{L}_\chi^L|_{P\cdot\overline{\nu}}) \cong \mathcal{L}_\chi|_{P\cdot\nu}. \tag{5.3}$$

The map (5.2) corresponds to the map of component groups

$$P^\nu/P_0^\nu = G^\nu/G_0^\nu \cong Z \to P^{\overline{\nu}}/P_0^{\overline{\nu}} = L^\nu/L_0^\nu = Z(L)/Z(L)_0.$$

By construction, under this map of component groups the character χ of $C_d = Z(L)/Z(L)_0$ pulls back to the character of Z which we have also denoted by χ. This implies (5.3), and the result follows. □

Theorem 5.3 implies that we can use the geometry of $\widetilde{\mathcal{M}}$ to study the generalized Springer correspondence. For example, Z must permute the irreducible components of the "generalized Springer fibers" $\psi^{-1}(\nu)$ for each $\nu \in \mathcal{N}$. This is referred to as a monodromy representation. From the discussion in [7, §1.2], it follows that the multiplicity of $IC(\overline{\mathcal{O}}_\lambda, \mathcal{L}_\chi)$ in $\psi_* \underline{\mathbb{C}}_{\widetilde{\mathcal{M}}}[\dim \mathcal{N}]$ is exactly the multiplicity of the irreducible Z-representation with character χ in the monodromy representation on the irreducible components of $\psi^{-1}(\nu)$ for $\nu \in \mathcal{O}_\lambda$. Example 5.4 is a continuation of Example 3.18, and computes this monodromy representation in a few cases.

Example 5.4. Let $n = 6$ and $\chi \in Z$ be the character of order $d = 2$. Consider the following standard tableau of shape $\lambda = [2, 2, 2]$.

$$S = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array}$$

Example 3.18 shows that $C_S \subseteq \widetilde{\mathcal{N}}_\chi$ where C_S is the open subset of the Springer fiber $\mu^{-1}(\nu_2)$ of maximal dimension constructed as by Steinberg in [28, §2]. Let $g_0 B \in \mu^{-1}(\nu_2)$ denote a generic element. Then equation (3.11) becomes

$$p(g_0^{-1} \cdot \nu_2) = E_{\alpha_1} + c_2 E_{\alpha_2} + E_{\alpha_3} + E_{\alpha_5},$$

and $p(g_0^{-1} \cdot \nu_2)$ corresponds to the coordinate values $(1, c_2, 1, 0, 1) \in \mathcal{V}_{ad}$ (we have $\mathcal{V}_{ad} = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_{n-1}] \cong \mathbb{C}^{n-1}$ as in Section 3.2). This confirms $C_S \subset \widetilde{\mathcal{N}}_\chi$ since $c_k \neq 0$ for all odd k.

Recall from Section 3.2 that we identify $\mathcal{V}$ with a subscheme of

$$\mathbb{C}^{2n-2} \simeq \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_{n-1}; v_1, \ldots, v_{n-1}].$$

Note that $x \in \mathcal{V}$ satisfies $\pi(x) = p(g_0^{-1} \cdot \nu_2)$ if and only if x has coordinates

$$(1, c_2, 1, 0, 1; 0, 0, v_3, 0, 0) \text{ where } v_3^2 = 1$$

since $v_3^2 = x_1 x_3 x_5$ and α_4 occurs in μ_k for all $k \neq 3$ (so $v_k(x) = 0$ for all $k \neq 3$). In other words, there are exactly two points in $\widetilde{\mathcal{M}}$ over each point of $C_S \subset \widetilde{\mathcal{N}}$. It is straightforward to show that $\widetilde{\eta}^{-1}(\overline{C}_S)$ consists of precisely two irreducible components (which are also irreducible components of $\psi^{-1}(\nu_2)$), each corresponding to the two possible values for v_3, namely $v_3 \in \{\pm 1\}$. These irreducible components are permuted by the Z-action (given by $v_3 \mapsto \omega^3 v_3 = -v_3$); yielding a Z-representation with character $\iota + \chi$, where ι is the trivial character. The Z-action on all other irreducible components of $\psi^{-1}(\nu_2)$ is trivial.

Continuing in this way, we can compute $\widetilde{\eta}^{-1}(\overline{C}_S)$ for the other standard tableaux appearing in (3.12). Our computations show that when S is either of

1	2	3	4
5	6		

1	2	5	6
3	4		

then $\widetilde{\eta}^{-1}(\overline{C}_S)$ consists of precisely two irreducible components, permuted by the Z-action as above. We also show that Z acts trivially on all other irreducible components of $\psi^{-1}(\nu)$ for $\nu \in \mathcal{O}_{[4,2]}$.

When S is the standard tableau with a single row, then the corresponding nilpotent element ν of $\mathcal{N}$ is regular so $\mu^{-1}(\nu)$ is a single point. The fiber in $\widetilde{\mathcal{M}}$ over $\mu^{-1}(\nu)$ consists of exactly 6 points and Z acts by the regular representation on these components of $\psi^{-1}(\nu)$.

These monodromy computations show that we expect the simple G-equivariant sheaves corresponding to the local system $\mathcal{L}_\chi$ to appear with a total multiplicity of 4, since χ only appears in the Z-action on $\widetilde{\eta}^{-1}(\overline{C}_S)$ for S from (3.12). In particular:

- $IC(\overline{\mathcal{O}}_{[2,2,2]}, \mathcal{L}_\chi)$ has multiplicity 1,
- $IC(\overline{\mathcal{O}}_{[4,2]}, \mathcal{L}_\chi)$ has multiplicity 2, and
- $IC(\overline{\mathcal{O}}_{[6]}, \mathcal{L}_\chi)$ has multiplicity 1.

The table in Table 1 gives the generalized Springer correspondence for $SL_6(\mathbb{C})$. The reader may confirm the information of the table in Figure 1 by computing the generalized Springer correspondence for $SL_6(\mathbb{C})$ as discussed in Section 2.7. Column 6 of the table shows that the multiplicities computed above using the monodromy action match those of the simple perverse sheaves appearing in the generalized Springer correspondence since $\dim(V_{[1,1,1]}) = 1$, $\dim(V_{[2,1]}) = 2$, and $\dim(V_{[3]}) = 1$.

TABLE 1. The generalized Springer correspondence for $SL_6(\mathbb{C})$. Local systems on each orbit are identified by the order of the corresponding central character. Each simple perverse sheaf has multiplicity given by dimension of an irreducible $S_{n/d}$ representation (where d denotes the order of the character).

Orbit	Local System	$L_d = T$ $S_{n/d} = S_6$	$L_d \cong S(GL_3 \times GL_3)$ $S_{n/d} = S_2$	$L_d \cong S(GL_3 \times GL_3)$ $S_{n/d} = S_2$	$L_d \cong S(GL_2 \times GL_2 \times GL_2)$ $S_{n/d} = S_3$	$L_d = G$ (Cuspidal) $S_{n/d}$ trivial
$[1,1,1,1,1,1]$	Triv	$\dim V = 1$				
$[2,1,1,1,1]$	Triv	$\dim V = 5$				
$[2,2,1,1]$	Triv	$\dim V = 9$				
$[2,2,2]$	Triv Order 2	$\dim V = 5$			$\dim V = 1;\ [1,1,1]$	
$[3,1,1,1]$	Triv	$\dim V = 10$				
$[3,2,1]$	Triv	$\dim V = 16$				
$[3,3]$	Triv Order 3 Order 3	$\dim V = 5$	$\dim V = 1;\ [1,1]$	$\dim V = 1;\ [1,1]$		
$[4,1,1]$	Triv	$\dim V = 10$				
$[4,2]$	Triv Order 2	$\dim V = 9$			$\dim V = 2;\ [2,1]$	
$[5,1]$	Triv	$\dim V = 5$				
$[6]$	Triv Order 6 Order 3 Order 2 Order 3 Order 6	$\dim V = 1$	$\dim V = 1;\ [2]$	$\dim V = 1;\ [2]$	$\dim V = 1;\ [3]$	$\dim V = 1$ $\dim V = 1$

The reader may note that there is a combinatorial connection between the standard tableaux appearing in (3.12) and the partitions of 3 indexing irreducible representations of the relative Weyl group S_3 appearing in column 6 of Figure 1. This pattern generalizes. In a forthcoming paper, the authors give an explicit description of the irreducible components of $\psi^{-1}(\nu)$ as $\nu \in \mathcal{N}$ varies and compute the monodromy representation.

The discussion in Section 2.6 tells us that, in the type A case, only the simple perverse sheaves corresponding to G_{ad}-equivariant local systems (that is, the trivial local systems) appear in the Springer correspondence. In other types, there are G_{ad}-equivariant local systems which do not appear in the Springer correspondence. For example, if the center Z is trivial, then $G = G_{ad}$, so any G-equivariant local system is G_{ad}-equivariant, and therefore any local systems missing from the Springer correspondence are G_{ad}-equivariant. This phenomenon occurs in type G_2, where exactly one local system is missing from the Springer correspondence (see [9, Section 13.3]). Since the center is trivial in type G_2, the construction of this paper does not yield all the simple perverse sheaves on the nilpotent cone. Even if the center is nontrivial, there can be missing G_{ad}-equivariant local systems. An example in classical types occurs in type C_6, for the orbit indexed by the partition $[6, 4, 2]$. The relevant data can be found in [5]. In this example, the G_{ad}-equivariant fundamental group is $\mathbb{Z}_2 \times \mathbb{Z}_2$, so there are four G_{ad}-equivariant local systems on this orbit, but only three of these occur in the Springer correspondence. Even though the results of Theorem 5.3 may not hold in full generality, many of the arguments above do hold in the general setting, and it is likely $\psi_* \underline{\mathbb{C}}_{\widetilde{\mathcal{M}}}[\dim \mathcal{N}]$ is a sum of Lusztig sheaves in this case also. The authors plan to study the geometric constructions of this manuscript for arbitrary reductive algebraic groups in future work.

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