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Feasibility Guaranteed Traffic Merging Control Using Control Barrier Functions

Kaiyuan Xu Wei Xiao Christos G. Cassandras

***Abstract*— We consider the merging control problem for Connected and Automated Vehicles (CAVs) aiming to jointly minimize travel time and energy consumption while providing speed-dependent safety guarantees and satisfying velocity and acceleration constraints. Applying the joint optimal control and control barrier function (OCBF) method, a controller that optimally tracks the unconstrained optimal control solution while guaranteeing the satisfaction of all constraints is efficiently obtained by transforming the optimal tracking problem into a sequence of quadratic programs (QPs). However, these QPs can become infeasible, especially under tight control bounds, thus failing to guarantee safety constraints. We solve this problem by deriving a control-dependent feasibility constraint corresponding to each CBF constraint, add it to each QP and show that such modified QPs are guaranteed to be feasible. Extensive simulations of the merging control problem illustrate the effectiveness of this feasibility guaranteed controller.**

I. INTRODUCTION

The performance of transportation networks critically depends on the management of traffic at conflict areas such as intersections, roundabouts and merging roadways [1]. Coordinating and controlling vehicles in such conflict areas is a challenging problem in terms of reducing congestion and energy consumption while also ensuring passenger comfort and guaranteeing safety [2], [3]. The emergence of Connected and Automated Vehicles (CAVs) [1] and the development of new traffic infrastructure technologies [4] provide a promising solution to this problem.

Both centralized and decentralized methods have been studied to deal with the control and coordination of CAVs at conflict areas. Centralized mechanisms are often used in forming platoons in merging problems [5] and determining passing sequences at intersections [6]. These approaches tend to work better when the safety constraints are independent of speed and they generally require significant computation resources, especially when traffic is heavy. They are also not easily amenable to disturbances.

Decentralized mechanisms restrict all computation to be done on board each CAV with information sharing limited to a small number of neighbor vehicles [7]–[10]. Optimal control problem formulations are often used, with Model Predictive Control (MPC) techniques employed as an alternative to account for additional constraints and to compensate for disturbances by re-evaluating optimal actions [11]–[13]. The objectives typically target the minimization of acceleration or the maximization of passenger comfort (measured as the acceleration derivative or jerk). An alternative to MPC has recently been proposed through the use of Control Barrier Functions (CBFs) [14], [15] which provide provable guarantees that safety constraints are always satisfied.

For the merging control problem under CAV traffic, a decentralized optimal merging control framework with a complete solution is given in [16]. The objective jointly minimizes (i) the travel time of each CAV over a given road segment from a point entering a Control Zone (CZ) to the eventual merging point and (ii) a measure of its energy consumption. However, the computational complexity in deriving this solution, even under simple vehicle dynamics, becomes prohibitive for real-time applications when safety constraints (e.g., preventing rear-end collisions) become active. This limitation can be overcome by the joint Optimal Control with Control Barrier Functions (OCBF) approach in [14]. In this approach, we first derive the solution of the optimal merging control problem when no constraints become active. Then, we solve another problem to optimally track this solution while also guaranteeing the satisfaction of all constraints using CBFs [15]. As shown in [14], this also allows the use of more accurate vehicle dynamics, possibly including noise, and the presence of more complicated objective functions. The OCBF controller is derived through a sequence of Quadratic Programs (QPs) over time which are simple to solve. However, they may become infeasible when the control bounds conflict with the CBF constraints, in which case the safety constraints can no longer be guaranteed. Thus, a basic question in CBF-based methods is: how can we guarantee the feasibility of all QP problems that need to be solved in deriving explicit solutions?

This paper resolves the QP feasibility problem above when an OCBF controller is used in decentralized merging control, thus ensuring that feasible trajectories are always possible. The merging control problem is formulated as in [16] to jointly minimize the travel time and energy consumption subject to speed-dependent safety constraints as well as vehicle limitations. We adopt the OCBF approach and guar-

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K. Xu and C. G. Cassandras are with the Division of Systems Engineering and Center for Information and Systems Engineering, Boston University, Brookline, MA, 02446, USA {xky, cgc}@bu.edu

W. Xiao is with the Computer Science and Artificial Intelligence Lab, Massachusetts Institute of Technology. weixy@mit.edu

antee the feasibility of each QP problem by adding a single feasibility constraint to it following the strategy developed in [17] for general optimal control problems. While the feasibility constraints constructed in [17] are limited to be independent of the control, here we exploit the structure of the safety constraints in the merging problem to derive control-dependent feasibility constraints and prove that all QPs needed to fully solve the merging problem are feasible.

The paper is organized as follows. In Section II, the formulation of the merging control problem is reviewed. In Section III, we explain how to transition from an optimal control solution to the OCBF controller using a sequence of QPs. In Section IV, we derive the new constraints added to the QPs and prove that this ensures their feasibility. Simulation results in Section V illustrate how to prevent CAV trajectories from becoming eventually infeasible by using the new feasibility constraints included in the OCBF controller.

II. PROBLEM FORMULATION AND APPROACH

The merging problem arises when traffic must be joined from two different roads, usually associated with a main lane and a merging lane as shown in Fig.1. We consider the case where all traffic consists of CAVs randomly arriving at the two roads joined at the Merging Point (MP) M where a collision may occur. A coordinator is associated with the MP whose function is to maintain a First-In-First-Out (FIFO) queue of CAVs based on their arrival time at the CZ and enable real-time Vehicle-to Infrastructure (V2I) communication with the CAVs that are in the CZ, as well as the last one leaving the CZ. The segment from the origin O or O' to the MP M has a length L for both roads, where L is selected to be as large as possible subject to the coordinator's communication range and the road network's configuration and it defines the CZ. Since we consider single-lane roads in this merging problem, CAVs may not overtake each other in the CZ (extensions to multi-lane merging are given in [18]). The FIFO assumption imposed so that CAVs cross the MP in their order of arrival is made for simplicity (and often to ensure fairness), but can be relaxed through dynamic resequencing schemes as in [19] where optimal solutions are similarly derived but for different selected CAV sequences.

Let $S(t)$ be the set of FIFO-ordered indices of all CAVs located in the CZ at time t along with the CAV (whose index is 0 as shown in Fig.1) that has just left the CZ. Let $N(t)$ be the cardinality of $S(t)$. Thus, if a CAV arrives at time t it is assigned the index $N(t)$. All CAV indices in $S(t)$ decrease by one when a CAV passes over the MP and the vehicle whose index is -1 is dropped.

The vehicle dynamics for each CAV $i \in S(t)$ along the lane to which it belongs take the form

$$\begin{bmatrix} \dot{x}_i(t) \\ \dot{v}_i(t) \end{bmatrix} = \begin{bmatrix} v_i(t) \\ u_i(t) \end{bmatrix} \tag{1}$$

where $x_i(t)$ denotes the distance to the origin O (O') along the main (merging) lane if the vehicle i is located in the main (merging) lane, $v_i(t)$ denotes the velocity, and $u_i(t)$ denotes the control input (acceleration). We consider two objectives for each CAV subject to three constraints, as detailed next.

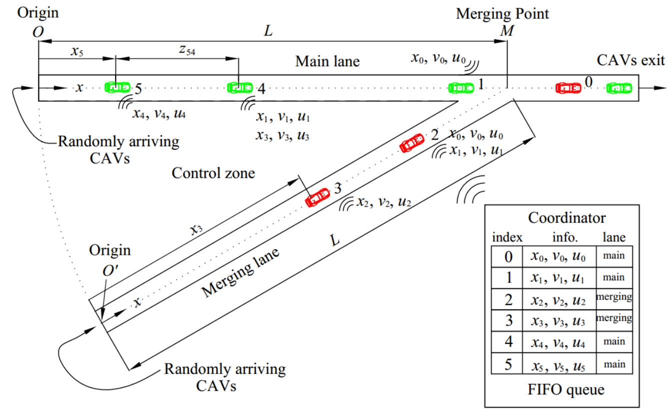

Figure 1. The merging problem. CAVs randomly arrive at the origins of the main and merging roads. Collisons may occur at the MP. A coordinator is associated with the MP to maintain the FIFO queue and share information among CAVs as needed.

Objective 1 (Minimizing travel time): Let t_i^0 and t_i^m denote the time that CAV $i \in S(t)$ arrives at the origin O or O' and the MP M, respectively. We wish to minimize the travel time $t_i^m - t_i^0$ for CAV i.

Objective 2 (Minimizing energy consumption): We also wish to minimize energy consumption for each CAV $i \in S(t)$ expressed as

$$J_i(u_i(t)) = \int_{t_i^0}^{t_i^m} C(u_i(t))dt, \tag{2}$$

where $C(\cdot)$ is a strictly increasing function of its argument.

Constraint 1 (Safety constraints between i and i_p): Let i_p denote the index of the CAV which physically immediately precedes i in the CZ (if one is present). We require that the distance $z_{i,i_p}(t) := x_{i_p}(t) - x_i(t)$ be constrained so that

$$z_{i,i_p}(t) \geq \varphi v_i(t) + \delta, \quad \forall t \in [t_i^0, t_i^m], \tag{3}$$

where $v_i(t)$ is the speed of CAV $i \in S(t)$ and φ denotes the reaction time (as a rule, $\varphi = 1.8$s is used, e.g., [20]). If we define z_{i,i_p} to be the distance from the center of CAV i to the center of CAV i_p, then δ is a constant determined by the length of these two CAVs (generally dependent on i and i_p but taken to be a constant for simplicity).

Constraint 2 (Safe merging (terminal constraint) between i and $i-1$): When $i - 1 = i_p$, this constraint is redundant since (3) is enforced, but when $i - 1 \neq i_p$ there should be enough safe space at the MP M for a merging CAV to cut in, i.e.,

$$z_{i,i-1}(t_i^m) \geq \varphi v_i(t_i^m) + \delta. \tag{4}$$

Constraint 3 (Vehicle limitations): Finally, there are constraints on the speed and acceleration for each $i \in S(t)$, i.e.,

$$v_{\min} \leq v_i(t) \leq v_{\max}, \forall t \in [t_i^0, t_i^m], \tag{5}$$

$$u_{i,\min} \leq u_i(t) \leq u_{i,\max}, \forall t \in [t_i^0, t_i^m], \tag{6}$$

where $v_{\max} > 0$ and $v_{\min} \geq 0$ denote the maximum and minimum speed allowed in the CZ, while $u_{i,\min} < 0$ and

$u_{i,\max} > 0$ denote the minimum and maximum control input, respectively.

Optimization Problem Formulation. Our goal is to determine a control law (as well as optimal merging time t_i^m) to achieve objectives 1-2 subject to constraints 1-3 for each $i \in S(t)$ governed by the dynamics (1). The common way to minimize energy consumption is by minimizing the control input effort $u_i^2(t)$, noting that the OCBF method allows for more elaborate fuel consumption models, e.g., as in [21]. By normalizing travel time and $u_i^2(t)$, and using $\alpha \in [0, 1)$, we construct a convex combination as follows:

$$J_i(u_i(t), t_i^m) = \int_{t_i^0}^{t_i^m} \left(\alpha + \frac{(1-\alpha)\frac{1}{2}u_i^2(t)}{\frac{1}{2}\max\{u_{i,\max}^2, u_{i,\min}^2\}} \right) dt. \quad (7)$$

If $\alpha = 1$, then we solve (7) as a minimum time problem. Otherwise, by defining $\beta := \frac{\alpha \max\{u_{i,\max}^2, u_{i,\min}^2\}}{2(1-\alpha)}$ and multiplying (7) by $\frac{\beta}{\alpha}$, we have:

$$J_i(u_i(t), t_i^m) := \beta(t_i^m - t_i^0) + \int_{t_i^0}^{t_i^m} \frac{1}{2}u_i^2(t)dt, \quad (8)$$

where $\beta \geq 0$ is a weight factor that can be adjusted to penalize travel time relative to the energy cost, subject to (1), (3)-(6) and the terminal constraint $x_i(t_i^m) = L$, given $t_i^0, x_i(t_i^0), v_i(t_i^0)$.

III. Optimal Control and Control Barrier Function Controller

The merging control problem can be analytically solved, however, as pointed out in [14], it becomes computationally intensive when one of more constraints become active. To obtain a solution in real time while guaranteeing that no safety constraint is violated, the OCBF approach [14] is adopted through the following steps: (i) an optimal control solution for the *unconstrained* optimal control problem is first obtained as a reference control, (ii) the resulting reference trajectory is optimally tracked subject to the control bounds (6) as well as a set of CBF constraints enforcing (3), (4). Using the forward invariance property of CBFs [15], these constraints are guaranteed to be satisfied at all times if they are initially satisfied. The importance of CBFs is that they impose linear constraints on the control which, if satisfied, guarantee the satisfaction of the associated original constraints that involve the state and/or control. This comes at the expense of potential conservativeness in the control since the CBF constraint is a sufficient condition for ensuring its associated original problem constraint. This whole process leads to a sequence of QPs solved over discrete time steps, since the objective function is quadratic and the CBF constraints are linear in the control.

Unconstrained optimal control solution: With all constraints inactive (including at t_i^0), the solution of **Problem 1** is achieved by standard Hamiltonian analysis [22] so that, as shown in [16], the optimal control, velocity and position trajectories of CAV i have the form:

$$u_i^*(t) = a_i t + b_i \quad (9)$$

$$v_i^*(t) = 1/2 \cdot a_i t^2 + b_i t + c_i \quad (10)$$

$$x_i^*(t) = 1/6 \cdot a_i t^3 + 1/2 \cdot b_i t^2 + c_i t + d_i \quad (11)$$

where the parameters a_i, b_i, c_i, d_i and t_i^m are obtained by solving a set of nonlinear algebraic equations ((36) in [16]). This set of equations only needs to be solved when CAV i enters the CZ and this can be done very efficiently.

Optimal tracking controller with CBFs: Once we obtain the unconstrained optimal control solutions (9)-(11), we use a function $h(u_i^*(t), x_i^*(t), x_i(t))$ as a control reference $u_{ref}(t) = h(u_i^*(t), x_i^*(t), x_i(t))$, where $x_i(t)$ provides feedback from the actual observed CAV trajectory. We then design a controller that minimizes $\int_{t_i^0}^{t_i^m} \frac{1}{2}(u_i(t) - u_{\text{ref}}(t))^2 dt$ subject to all constraints (3), (4) and (6). This is accomplished as reviewed next (see also [14]).

First, let $\boldsymbol{x}_i(t) \equiv (x_i(t), v_i(t))$, $\boldsymbol{x}(t) = (\boldsymbol{x}_1(t), ..., \boldsymbol{x}_N(t))$. Due to the vehicle dynamics (1), define $f(\boldsymbol{x}_i(t)) = [v_i(t), 0]^T$ and $g(\boldsymbol{x}_i(t)) = [0, 1]^T$. The constraints (3) and (4) are expressed in the form $b_q(\boldsymbol{x}(t)) \geq 0, q \in \{1, ..., B_i\}$ where B_i is the number of constraints CAV i needs to satisfy. The CBF method maps $b_q(\boldsymbol{x}(t)) \geq 0$ to a new constraint which directly involves the control $u_i(t)$ and takes the (linear in the control) form

$$L_f b_q(\boldsymbol{x}(t)) + L_g b_q(\boldsymbol{x}(t)) u_i(t) + \gamma(b_q(\boldsymbol{x}(t))) \geq 0, \quad (12)$$

where L_f, L_g denote the Lie derivatives of $b_q(\boldsymbol{x}(t))$ along f and g respectively and $\gamma(\cdot)$ denotes some class-$\mathcal{K}$ function [15]. The forward invariance property of CBFs guarantees that a control input that keeps (12) satisfied will also enforce $b_q(\boldsymbol{x}(t)) \geq 0$, i.e., the constraints (3), (4) are never violated.

To optimally track the reference speed trajectory, a CLF function $V(\boldsymbol{x}_i(t))$ is used. A CLF function is not a necessity in OCBF but often helps tracking the velocity trajectory. Letting $V(\boldsymbol{x}_i(t)) = (v_i(t) - v_{\text{ref}}(t))^2$, the CLF constraint takes the form

$$L_f V(\boldsymbol{x}_i(t)) + L_g V(\boldsymbol{x}_i(t)) u_i(t) + \epsilon V(\boldsymbol{x}_i(t)) \leq e_i(t), \quad (13)$$

where $\epsilon > 0$, and $e_i(t)$ is a relaxation variable which makes the constraint soft. Then, the OCBF controller optimally tracks the reference trajectory by solving the optimization problem:

$$\min_{u_i(t), e_i(t)} \int_{t_i^0}^{t_i^m} \left(\beta e_i^2(t) + \frac{1}{2}(u_i(t) - u_{\text{ref}(t)})^2 \right) \quad (14)$$

subject to the CBF constraints (12), the CLF constraints (13) and the control bounds (6). There are several possible choices for $u_{\text{ref}}(t)$ and $v_{\text{ref}}(t)$. In the sequel, we choose the following referenced trajectory with feedback included to reduce the tracking position error:

$$v_{\text{ref}}(t) = \frac{x_i^*(t)}{x_i(t)} v_i^*(t), \quad u_{\text{ref}}(t) = \frac{x_i^*(t)}{x_i(t)} u_i^*(t) \quad (15)$$

where $u_i^*(t)$, $v_i^*(t)$, $x_i^*(t)$ are obtained from (9)-(11).

We can solve problem (14) by discretizing $[t_i^0, t_i^m]$ into intervals of equal length Δt and solving (14) over each time interval. The decision variables $u_i(t)$ and $e_i(t)$ are assumed to be constant on each such time interval $[t_i^0 + k\Delta t, t_i^0 + (k+1)\Delta t]$ and can be easily obtained by solving a Quadratic Program (QP) problem (16) since all CBF constraints are linear in the decision variables $u_i(t)$ and $e_i(t)$ (fixed over each interval $[t_i^k, t_i^k + \Delta t]$).

$$\begin{aligned}\min_{u_i(t),e_i(t)} \; & \beta e_i(t)^2 + \frac{1}{2}(u_i(t) - u_{\mathrm{ref}}(t))^2 \\ s.t. \;\; & (12), (13), (6), t = t_i^0 + k\Delta t\end{aligned} \tag{16}$$

By repeating this process until CAV i exits the CZ, the solution to (14) is obtained *if* (16) *is feasible for each time interval*. This approach is simple and computationally efficient. However, it is also myopic since each QP is solved over a single time step, which may lead to infeasible QPs at future time steps, especially when (6) is tight.

IV. Feasibility Guaranteed OCBF

To avoid the infeasibility caused by the myopic QP solving approach in the CBF method, an additional "feasibility constraint" $b_F(\boldsymbol{x}(t), u_i(t)) \geq 0$ is introduced in [17]. A feasibility constraint is defined as a constraint that makes the QP corresponding to the next time interval feasible, thus, in the case of (16), a feasibility constraint has the following properties: (i) it guarantees that (12) and (6) do not conflict, (ii) the feasibility constraint itself conflicts with neither (12) nor (6). In general, we call any two state and/or control constraints (e.g., any CBF constraint) *conflict-free* if their intersection is non-empty in terms of the control.

In [17], a general sufficient condition for feasibility is provided based on the assumption that the feasibility constraint $b_F(\boldsymbol{x}(t), u(t))$ is independent of the control $u(t)$. This assumption is hard to meet when it comes to the merging control problem. In what follows, we show that it is possible to find a feasibility constraint $b_F(\boldsymbol{x}(t), u(t)) \geq 0$ for each CAV i without this assumption and explicitly derive this constraint which can provably guarantee feasibility.

In the merging control problem, the form of the safety constraint depends on whether CAV i and CAV $i-1$ are in the same road. If so, $i - 1 = i_p$ and the rear-end safety constraint needs to be considered. Otherwise, the safe merging constraint (4) must be included.

A. Rear-end Safety Constraint

When CAV i and $i - 1 = i_p$ are in the same road, only the rear-end safety constraint needs to be considered:

$$b_1(\boldsymbol{x}(t)) = z_{i,i_p}(t) - \varphi v_i(t) + \delta \geq 0 \tag{17}$$

Note that in (17) only CAV i_p's position $x_{i_p}(t)$ is needed in addition to CAV i's state $\boldsymbol{x}_i(t)$, which can be easily implemented in a decentralized way. As $b_1(\boldsymbol{x}(t))$ is differentiable, we can calculate the Lie derivatives $L_f(b_1(\boldsymbol{x}(t))) = v_{i_p} - v_i(t)$, $L_g(b_1(\boldsymbol{x}(t))) = -\varphi$. Applying (12) and choosing a linear function $\gamma(x) = k_1 x$ as the class-$\mathcal{K}$ function, the rear-end safety constraint (17) can be directly transformed into the CBF constraint:

$$b_{\mathrm{cbf}_1}(\boldsymbol{x}, u_i) = v_{i_p} - v_i - \varphi u_i + k_1 b_1(\boldsymbol{x}) \geq 0 \tag{18}$$

As $-L_g(b_1(\boldsymbol{x}(t))) = \varphi > 0$, multiplying both sides of the control bound (6) by $-L_g(b_1(\boldsymbol{x}(t)))$ will not change the direction of the inequalities. Hence, we have

$$\varphi u_{i,\min} \leq \varphi u_i(t) \leq \varphi u_{i,\max} \tag{19}$$

Note that (18) can be rewritten as

$$\varphi u_i(t) \leq v_{i_p}(t) - v_i(t) + k_1 b_1(\boldsymbol{x}(t)) \tag{20}$$

where $\varphi u_i(t) \leq u_{i,\max}$ never conflicts with (20) as they have the same inequality direction. Thus, we can guarantee that (18) and (19) are conflict-free by adding

$$b_F(\boldsymbol{x}(t)) = v_{i_p}(t) - v_i(t) + k_1 b_1(\boldsymbol{x}(t)) - \varphi u_{i,\min} \geq 0 \tag{21}$$

We can now consider this feasibility constraint as a new CBF and apply (12) to transform it into a CBF constraint. Choosing a linear function $\gamma(x) = k_F x$ as the class $\mathcal{K}$ function, the corresponding CBF constraint is

$$\begin{aligned}& u_{i_p} - u_i + k_1(v_{i_p} - v_i - \varphi u_i) + k_F(v_{i_p} - v_i) \\ & + k_F k_1(z_{i,i_p} - \varphi v_i + \delta) - k_F \varphi u_{i,\min} \geq 0\end{aligned} \tag{22}$$

where the argument t of the functions above is omitted.

Next, we determine a feasible constraint to be added to every QP so that it guarantees the QP of the next time interval is feasible. Choosing $k_F = k_1$, (22) becomes

$$\begin{aligned}& u_{i_p} - u_i + k_1(v_{i_p} - v_i - \varphi u_{i,\min}) \\ & + k_1(v_{i_p} - v_i - \varphi u_i + k_1(z_{i,i_p} - \varphi v_i + \delta)) \geq 0\end{aligned} \tag{23}$$

Define a "candidate function" $\eta(\boldsymbol{x}, u_i)$ [17] as

$$\eta_1(\boldsymbol{x}, u_i) = u_{i_p} - u_i + k_1(v_{i_p} - v_i - \varphi u_{i,\min}) \tag{24}$$

Then, replacing the first three terms of the feasibility CBF constraint (23) with $\eta_1(\boldsymbol{x}, u_i)$ and noting that the remaining terms are given by $b_{\mathrm{cbf}_1}(\boldsymbol{x}, u_i)$ defined in (18), to substitute the second row with $b_{\mathrm{cbf}_1}(\boldsymbol{x}, u_i)$, (23) becomes

$$\eta_1(\boldsymbol{x}, u_i) + k_1 b_{\mathrm{cbf}_1}(\boldsymbol{x}, u_i) \geq 0 \tag{25}$$

Since $b_{\mathrm{cbf}_1}(\boldsymbol{x}, u_i) \geq 0$ is required in (18), it follows that (25) will be satisfied if

$$\eta_1(\boldsymbol{x}, u_i) \geq 0 \tag{26}$$

Setting

$$b_{\eta_1}(\boldsymbol{x}) = v_{i_p} - v_i - \varphi u_{i,\min} \tag{27}$$

in (24), we can view $b_{\eta_1}(\boldsymbol{x})$ as a CBF and apply (12) to observe that the corresponding CBF constraint coincides with (26). Adding the CBF constraint (26) to the QP (16), we will show next that (26) guarantees the feasibility of the QP corresponding to the next time interval. Before establishing this result, we make the following two assumptions.

Assumption 1: All CAVs have the same minimum acceleration, i.e. $u_{i,\min} = u_{\min}$.

This is a weak assumption guaranteeing that (26) and (6) are conflict-free. It can be easily enforced since all CAVs are operating within the same CZ, i.e., they can reach agreement on a common $u_{\min} = \min_i\{u_{i,\min}\}$.

Assumption 2: Δt is adequately small such that the forward invariance property of CBFs remains in force.

This assumption is made to utilize the forward invariance property of CBFs to guarantee safety. It can be met by decreasing the time interval or by using the recently proposed event-driven technique [23].

Theorem 1: If $b_{\eta_1}(\boldsymbol{x}(t)) \geq 0$ and the QP (16) subject to (18), (6) and (26) is feasible at time t, then the QP corresponding to time $t + \Delta t$ is also feasible.

The proof of this and all other theorems is omitted due to page limitations but is included in [24].

Assumption 3: The following initial conditions are satisfied: $b_1(\boldsymbol{x}(t_i^0)) \geq 0, b_F(\boldsymbol{x}(t_i^0)) \geq 0, b_{\eta_1}(\boldsymbol{x}(t_i^0)) \geq 0$

The constraint $b_1(\boldsymbol{x}(t_i^0)) \geq 0$ requires CAV i to meet the rear-end safety with the immediately preceding CAV (if one exists) when entering the CZ. In addition, $b_F(\boldsymbol{x}(t_i^0)) \geq 0$ requires that the CBF constraint is initially conflict-free with the control bounds and $b_{\eta_1}(\boldsymbol{x}(t_i^0)) \geq 0$ indicates that CAV i should not be too faster than the preceding CAV. These constraints are reasonable and can be met using a Feasibility Enforcement Zone (FEZ) [25] that precedes the CZ.

Theorem 2: Under *Assumptions 1,2,3*, the QP (16) subject to (18), (6) and (26) corresponding to any time interval $[t_i^0 + k\Delta t, t_i^0 + (k+1)\Delta t] \subset [t_i^0, t_i^m]$ is feasible.

B. Safe Merging Constraint

When CAVs i and $i-1$ are in different roads, they should also satisfy the merging safety constraint

$$z_{i,i-1}(t_i^m) - \varphi v_i(t_i^m) - \delta \geq 0 \tag{28}$$

This differs from the rear-end safety constraint in that it only applies to specific time instants t_i^m. This poses a technical complication as a CBF must always be in a continuously differentiable form. We can convert (28) to such a form using a technique similar to the one in [14] to define

$$b_2(\boldsymbol{x}(t)) = z_{i,i-1}(t) - \Phi(\boldsymbol{x}(t))v_i(t) - \delta \geq 0 \tag{29}$$

where $\Phi(\boldsymbol{x}(t)) = \frac{\varphi}{x_i(t_i^m)} x_i(t)$. Note that $\Phi(\boldsymbol{x}(t_i^m)) = \varphi$ consistent with (28). Setting $\varphi_2 = \frac{\varphi}{x_i(t_i^m)}$ and omitting the argument t, we get $L_f(b_2(\boldsymbol{x})) = v_{i-1} - v_i - \varphi_2 v_i^2$ and $L_g(b_2(\boldsymbol{x})) = -\varphi_2 x_i$. Thus, using (12) and choosing $\gamma(x) = k_2 x$, (28) is mapped onto the CBF constraint

$$b_{\text{cbf}_2}(\boldsymbol{x}, u_i) = v_{i-1} - v_i - \varphi_2 v_i^2 - \varphi_2 x_i u_i + k_2 b_2(\boldsymbol{x}) \geq 0 \tag{30}$$

Proceeding as in Sec. IV-A, we define $\eta_2(\boldsymbol{x}, u_i) = u_{i-1} - u_i - 2\varphi_2 v_i u_i - \varphi_2 v_i u_{\min} + k_2(v_{i-1} - v_i - \varphi_2 v_i^2 - \varphi_2 x_i u_{\min})$ and derive two conditions corresponding to (26) and (27):

$$\eta_2(\boldsymbol{x}, u_i) \geq 0 \tag{31}$$

$$b_{\eta_2}(\boldsymbol{x}) = v_{i-1} - v_i - \varphi_2 v_i^2 - \varphi_2 x_i u_{\min} \tag{32}$$

We can then obtain similar theorems as before:

Theorem 3: If $b_{\eta_2}(\boldsymbol{x}(t)) \geq 0$, $v_i \geq 0, u_{\min} \leq 0$ and the QP (16) subject to (30), (6) and (31) is feasible at time t, then the QP corresponding to time $t + \Delta t$ is also feasible.

Assumption 4: The following initial conditions are satisfied: $b_2(\boldsymbol{x}(t_i^0)) \geq 0, b_F(\boldsymbol{x}(t_i^0)) \geq 0, b_{\eta_2}(\boldsymbol{x}(t_i^0)) \geq 0$

Theorem 4: Under *Assumptions 1,2,4*, if $v_i \geq 0, u_{\min} \leq 0$, the QP (16) subject to (30), (6) and (31) corresponding to any time interval $[t_i^0 + k\Delta t, t_i^0 + (k+1)\Delta t] \subset [t_i^0, t_i^m]$ is feasible.

Theorem 5: If $b_{\eta_1}(\boldsymbol{x}(t)) \geq 0$, $b_{\eta_2}(\boldsymbol{x}(t)) \geq 0$, $v_i \geq 0, u_{\min} \leq 0$, the QP (16) subject to (18), (30), (6), (26) and (31) is feasible at time t, then the QP corresponding to time $t + \Delta t$ is also feasible.

V. Simulation Results

All simulations are performed in MATLAB using quadprog as the solver for the QPs. We first build the model shown in Fig. 1 with parameters $L = 400, u_{\min} = -2m/s^s, u_{\max} = 3m/s^2$ and adopt the OCBF controller without feasibility guarantee for each CAV. When a QP for optimally tracking the unconstrained optimal control of a CAV i becomes infeasible, we record its index and consider two different cases corresponding to the rear-end safety constraint and the safe merging constraint separately. We re-run the simulations of the two cases with feasibility constraints added to the ego CAV, keeping all other conditions same.

Rear-end Safety Constraint: We limit ourselves to simulation results for the rear-end safety constraint, with additional results included in [24]. A particular CAV, labeled "CAV 25", is chosen as the first case study. As CAV 25 and CAV 24 are in the same road, the possibly active constraint of interest is the rear-end safety constraint. We adopt the OCBF controller and run the simulation twice to derive the two trajectories of CAV 25, one with the feasibility guarantee and the other without. Note that in the merging problem, the control $u_i(t)$ is 1-dimensional, thus the feasible set of the QP is an interval. To illustrate the performance of the feasibility constraint, the evolution of the feasible set of the QPs over time are plotted in Fig. 2.

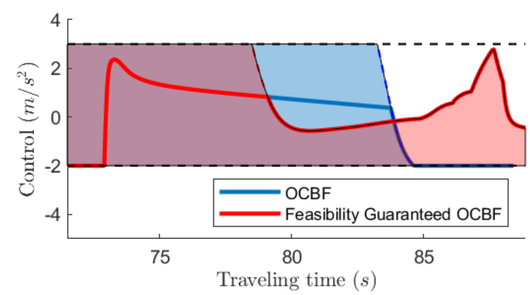

Figure 2. Control History Comparison

In Fig. 2, the solid blue curve shows the control history $u(t)$ generated by the OCBF controller. The shaded blue area shows the feasible set of the QP. For each time t, the shaded blue area marks the maximum and minimum acceleration allowed by the QP. Note that when $t = 85s$, the QP becomes infeasible and the control is set to be $-2m/s^2$ to continue the program execution. The solid red line is the control history generated by the OCBF controller with the feasibility constraint added to the QP. The shaded red area shows the feasible set corresponding to the revised QP. The dashed black lines shows the control bounds of the CAV.

The figure shows that the OCBF and feasibility guaranteed OCBF have the same feasible set before 79s, which generates the same control history. However, after 79s, the feasible set of the feasibility guaranteed OCBF shrinks due to the feasibility constraint while the myopic OCBF approach keeps the same feasible set. This leads to a large difference after 84s. The feasible set of the myopic OCBF approach rapidly shrinks and even becomes empty, indicating that the QP is infeasible. The feasibility guaranteed OCBF, however, remains feasible with the help of the advance action introduced by the feasibility constraint.

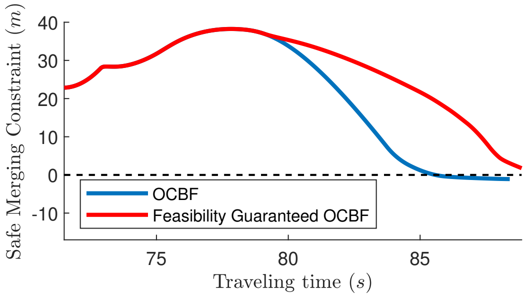

Figure 3. Rear-end Safety Constraint Satisfaction Comparison

The infeasible QP makes the safety constraints unguaranteed as we no longer benefit from the forward invariance property of the CBF. The rear-end safety constraints of the two cases are plotted in Fig. 3. We can see that the rear-end safety constraint is violated after 85s using the OCBF controllers. This corresponds to the infeasible QP shown in Fig. 2 after 85s. With the feasibility constraint, the red curve remains positive, indicating that the rear-end safety constraint is always satisfied.

VI. Conclusion

We have presented a feasibility guaranteed decentralized control framework combining optimal control and CBFs leading to OCBF controllers for merging CAVs jointly minimizing both the travel time and the energy consumption subject to speed-dependent safety constraints and vehicle limitations. We resolve the QP feasibility problem which arises by adding a single control-dependent feasibility constraint corresponding to each CBF constraint. The explicit control-dependent feasibility constraints we have derived rely on the velocity-dependent safety constraint structure. Thus, future research will explore extending this method to feasibility constraints for arbitrary optimal control problems, as well as integrating recently developed event-driven QPs which relaxes the assumption of an adequately small time interval in *Assumption 2*.

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