A SEPARATION THEOREM FOR SIMPLE THEORIES

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Abstract. This paper builds model-theoretic tools to detect changes in complexity among the simple theories. We develop a generalization of dividing, called shearing, which depends on a so-called context $\mathbf{c}$. This leads to defining $\mathbf{c}$-superstability, a syntactical notion, which includes supersimplicity as a special case. The main result is a separation theorem showing that for any countable context $\mathbf{c}$ and any two theories T_1, T_2 such that T_1 is $\mathbf{c}$-superstable and T_2 is $\mathbf{c}$-unsuperstable, and for arbitrarily large μ, it is possible to build models of any theory interpreting both T_1 and T_2 whose restriction to $\tau(T_1)$ is μ-saturated and whose restriction to $\tau(T_2)$ is not $\aleph_1$-saturated. (This suggests "$\mathbf{c}$-superstable" is really a dividing line.) The proof uses generalized Ehrenfeucht-Mostowski models, and along the way, we clarify the use of these techniques to realize certain types while omitting others. In some sense, shearing allows us to study the interaction of complexity coming from the usual notion of dividing in simple theories and the more combinatorial complexity detected by the general definition. This work is inspired by our recent progress on Keisler's order, but does not use ultrafilters, rather aiming to build up the internal model theory of these classes.

1. Introduction and motivation

This paper aims to develop internal model-theoretic tools to detect significant changes in complexity among the simple theories.

Motivating examples of simple theories [17] include the random graph and random k-uniform hypergraphs for arbitrary finite k. It was subsequently shown that pseudofinite fields, certain higher-order analogues of the triangle-free random graph, and the theory ACFA are also simple, see [5], [5], [1]. The 90s saw a great deal of work on simple theories, as recorded in the 2002 survey [3]. Still, basic questions about simple theories, such as 1.1 below, remain open. The tools we have to detect structural changes in stable theories, such as dividing, still work well in simple theories but the extent to which they explain the whole picture is less clear.

In the course of our recent work on Keisler's order, a large-scale classification program in model theory which compares theories roughly according to the likelihood of saturation in their regular ultrapowers, we made a surprising discovery. Although the union of the first two classes in Keisler's order is precisely the stable theories [18], it turns out that this order has infinitely many classes, already within the simple unstable theories with no nontrivial dividing, those 'near' the random graph [10]. A key role was played by what were essentially disjoint unions of the higher analogues of the triangle-free random graph, studied by Hrushovski [5].

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The thesis that differences seen by ultrafilters should be significant (as ultrafilters are, in some sense, very forgiving) suggests that if a stratification of levels of randomness is appearing in this presumably simple part of the map, one should look for an internal explanation.

It is useful to remember what Keisler's order tells us about the stable theories. When the second author proved that the union of the first two classes in Keisler's order is precisely the stable theories, his proof used a characterization of the saturated models of stable theories: a model of a stable theory is λ^+-saturated iff it is $\kappa(T)$-saturated and every maximal indiscernible set has size at least λ^+ [18, III.3]. This required developing forking (dividing) and uniqueness of nonforking extensions in stable theories.

The analogous characterization of saturated models of simple theories seems to be a real challenge to our understanding:

Question 1.1. *Give a characterization of the saturated models of simple theories analogous to the theorem that a model of a stable theory is λ^+-saturated iff it is $\kappa(T)$-saturated and every maximal indiscernible set has size at least λ^+.*

Although Question 1.1 remains for the time being open, in what follows, we will be guided by and will further develop this core idea of the relation between understanding dividing and understanding saturation.

It is also useful to recall some particulars of the higher analogues of triangle-free graphs from [5]. Let $T_{n,k}$ denote the $(n+1)$-free $(k+1)$-hypergraph, i.e. the model completion of the theory of a uniform $(k+1)$-ary hypergraph in which there are no $(n+1)$ vertices of which every $(k+1)$ form a hyperedge. The triangle-free random graph is not simple, however Hrushovski showed that for $n > k \geq 2$, $T_{n,k}$ is simple with only trivial dividing, see 5.11 and 5.12 below. So where does the complexity of the $T_{n,k}$s come from? 'Amalgamation' is a natural answer, and was key to [5] and to the property in [10, 1.5]. Moreover, these amalgamation problems appeared orthogonal to forking.

However, the methods of the present paper open up a different answer.

We introduce a natural extension of dividing, which we call *shearing*, and which includes dividing as a special case. This definition is developed by looking at dividing in a certain canonical context, that of Ehrenfeucht-Mostowski models, and studying realization of types there. In the first part of the paper, extending an idea from [11], we develop the relation of weak definability of types in generalized Ehrenfeucht-Mostowski models to realizing those types in larger templates. There are many parallels to stable phenomena, and various definitions which specialize to the familiar ones in the stable case, but they have their own flavor.

In the second part of the paper, we isolate the main mechanism of this correspondence as the definition of *shearing*, which a priori makes no reference to GEM models or to realizing types. Dividing involves inconsistency of a formula instantiated along an indiscernible sequence; shearing involves inconsistency of a formula instantiated along a generalized indiscernible sequence. The definition of shearing involves choosing an element I from a class $\mathcal{K}$ of index models, extending the class of linear orders and satisfying certain basic requirements. A *countable context* $\mathbf{c} = (I, \mathcal{K})$ is essentially a choice of some nontrivial countable I in some allowed $\mathcal{K}$. We introduce a notion of a theory being $\mathbf{c}$-*superstable*, essentially the analogue of superstability (or supersimplicity) for the corresponding shearing.

Theorem 7.1 below, the "separation theorem," then explains the connection between shearing and saturation: it says essentially that given two theories T_1, T_2 and a countable context $\mathbf{c}$ such that T_1 is $\mathbf{c}$-superstable and T_2 is $\mathbf{c}$-unsuperstable, it is possible to build a model (of any theory interpreting both T_1 and T_2, without loss of generality in disjoint signatures) whose reduct to $\tau(T_1)$ is arbitrarily saturated while the reduct to $\tau(T_2)$ is not even $\aleph_1$-saturated. In some sense, we may add weak definitions for all relevant types from T_1 while types from T_2 remain in this sense undefinable. (Alternately, either half of the theorem can be taken as a recipe for building very saturated or very unsaturated models of a given theory according to its $\mathbf{c}$-superstability for a given context.)

Some consequences for $\trianglelefteq^*$ are given in §8. In §9, we prove, in some sense, that the focus of shearing is within simplicity. §10 outlines natural extensions and some open problems. A companion paper [12] in progress gives a full analysis of the case of the random graph, characterizing the contexts for which it is $\mathbf{c}$-superstable, and proving that the theories $T_{n,k}$ are strictly more complex in the sense of shearing.

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Contents

2. Basic notation and definitions

Convention 2.1. *All theories are complete and first order unless otherwise stated.*

Convention 2.2. *Given a universal class of models $\mathcal{K}$, we will write "$J \in \mathcal{K}$ is $\aleph_0$-saturated" to mean "J is countably homogeneous and countably universal for elements of $\mathcal{K}$," which makes sense even if $\mathcal{K}$ is not elementary.*

We work in the setup of generalized Ehrenfeucht-Mostowski (GEM) models. These methods begin with the EM models of Ehrenfeucht and Mostowski 1965 [2] and were further developed in e.g. Shelah 1978, chapters VII-VIII [18] and Shelah [19]. A self-contained introduction may be found in our recent paper [11], §3, which takes up the development of these techniques and adds the "G" for "generalized" to stress that we may vary the index model I, see below. Here we review some basic definitions motivated there, and clarify our assumptions for the present paper.

For Ehrenfeucht and Mostowski, index models were linear orders; we will use expansions of linear orders, which need not come from an elementary class. (An

example from [11] is the class $\mathcal{K}_\mu$ of linear orders expanded by μ unary predicates which partition the domain; note the "partition" requirement implies the class is not elementary.) The following general definition will suffice for this paper.[1]

Definition 2.3 (Index model class)**.** *Call $\mathcal{K}$ an index model class, abbreviated* imc*, when for some signature $\tau = \tau_{\mathcal{K}} \supseteq \{<\}$,*

(1) $\mathcal{K}$ is a class of τ-models, closed under isomorphism, but not necessarily an elementary class.
(2) For each $I \in \mathcal{K}$, $<^I$ linearly orders I.
(3) $\mathcal{K}$ is universal,[2] so $I \in \mathcal{K}$ iff every finitely generated submodel of I is in $\mathcal{K}$.
(4) We could allow partial functions, so for every function symbol $F \in \tau$, there is a predicate P_F which is always interpreted as its domain.
(5) For every $I \in \mathcal{K}$ there is an $\aleph_0$-saturated $J \in \mathcal{K}$ with $I \subseteq J$.
(6) $\mathcal{K}$ is Ramsey, see 2.9 below.

Definition 2.4 (GEM models and proper templates, [19] Definition 1.8)**.** *We say $N = \mathrm{GEM}(I, \Phi) = \mathrm{GEM}(I, \Phi, \mathbf{a})$ is a generalized Ehrenfeucht-Mostowski model with skeleton $\mathbf{a}$ when for some vocabulary $\tau = \tau_\Phi$ we have the following.*

(1) I is a model, called the index model.
(2) N is a τ_Φ-structure and $\mathbf{a} = \{\bar{a}_t : t \in I\}$ generates N.
(3) $\langle \bar{a}_t : t \in I \rangle$ is quantifier free indiscernible in N.
(4) Φ is a template, taking (for each $n < \omega$) the quantifier free type of $\bar{t} = \langle t_0, \dots, t_{n-1} \rangle$ in I to the quantifier free type of $\bar{a}_{\bar{t}}$ in N. (So Φ determines τ_Φ uniquely, and also a theory T_Φ, the maximal τ_Φ-theory which holds in every such N.)

(*Note that* $\mathrm{GEM}(I, \Phi)$ *is not uniquely determined as we have to choose the elements of e.g.* $N \setminus \mathrm{Range}(\mathbf{a})$*, but as usually no confusion arises we may omit the additional information. So really, "$N = \mathrm{GEM}(I, \Phi)$" is a relation.*)

When it is useful to specify the skeleton $\mathbf{a}$ generating a given GEM model we may display it. Templates are simply possible instructions, which may not be 'coherent' or give rise to a model; properness says they do.

Definition 2.5. *The template Φ is called proper for I if there is M such that $M = \mathrm{GEM}(I, \Phi)$. We say Φ is proper for a class $\mathcal{K}$ if Φ is proper for all $I \in \mathcal{K}$.*

Definition 2.6. *Given a class $\mathcal{K}$, write $\Upsilon_{\mathcal{K}}$ for the class of templates proper for $\mathcal{K}$, and write Υ when $\mathcal{K}$ is clear from context.*

Convention 2.7. *All templates we consider are assumed to satisfy:*

(a) nontriviality, i.e. we may add in the GEM *definition the condition that $\lg(\bar{a}_t) \geq 1$ and $\langle \bar{a}_t : t \in I \rangle$ is without repetition,*
(b) T_Φ is well defined and has Skolem functions, where well defined means:
  (1) T_Φ is complete.
  (2) for every $I \in \mathcal{K}$, $\langle \bar{a}_t : t \in I \rangle$ is indiscernible, not just quantifier-free indiscernible, in $\mathrm{GEM}(I, \Phi)$*. (this really follows)*

[1] Item 2.3(5) is more than is needed but simplifies our proofs here; asking that $\mathbf{D}(\mathcal{K})$, the set of quantifier-free types, has amalgamation would suffice.

[2] This implies $\mathcal{K}$ is an $EC(\emptyset, \Gamma)$-class, that is, the set of models of a first order theory which omit some (possibly empty) set of types. Inversely, if T is universal in $\mathcal{L}(T)$, Γ a set of q.f. types then $EC(T, \Gamma)$ is a universal class. As mentioned, $\mathcal{K}$ need not be an elementary class.

(3) $\mathrm{GEM}(I,\Phi)$ *is unique in the sense that it depends, up to isomorphism, on* Φ *and the isomorphism type of* I*. More fully: if* $N = \mathrm{GEM}(I,\Phi)$ *then for some* $\mathbf{a}$*,* $N = \mathrm{GEM}(I,\Phi,\mathbf{a})$ *and if* $N' = \mathrm{GEM}(I,\Phi,\mathbf{a}')$ *then there is a unique isomorphism from* N *onto* N' *mapping* $\mathbf{a}$ *to* $\mathbf{a}'$*, i.e.* $\bar{a}_s$ *to* $\bar{a}'_s$ *for* $s \in I$*.*

(4) *for every* $J \supseteq I$ *from* $\mathcal{K}$ *we have* $\mathrm{GEM}(I,\Phi) \preceq \mathrm{GEM}(J,\Phi)$*.*
More fully, considering the parenthesis in 2.4 *we should say: for every* $J \supseteq I$ *and* $N_2 = \mathrm{GEM}(J,\Phi)$ *there is* $N_1 = \mathrm{GEM}(I,\Phi)$ *such that* $N_1 \subseteq N_2$ *hence* $N_1 \preceq N_2$*. Moreover if* $N_2 = \mathrm{GEM}(J,\Phi,\mathbf{a})$ *then* $N_1 = \mathrm{GEM}(I,\Phi,\mathbf{a} \restriction I)$*. (As in* (3)*, if* $N' = \mathrm{GEM}(I,\Phi,\mathbf{a}')$ *then there is a unique isomorphism from* N' *onto* N_1 *mapping* $\mathbf{a}'$ *onto* $\mathbf{a} \restriction I$*).*

Definition 2.8. *Given a class of templates* Υ*, let* $\leq_\Upsilon$ *be the natural partial order on* Υ*, that is,* $\Phi \leq_\Upsilon \Psi$ *means that* $\tau(\Phi) \subseteq \tau(\Psi)$ *and* $\mathrm{GEM}(I,\Phi) \subseteq \mathrm{GEM}(I,\Psi)$ *and* $\mathrm{GEM}_{\tau(\Phi)}(I,\Phi) \preceq \mathrm{GEM}_{\tau(\Phi)}(I,\Psi)$*. We may use* $\leq$ *when* Υ *is clear from context.*

Definition 2.9. *We say the class* $\mathcal{K}$ *is Ramsey when: given any*

a) $J \in \mathcal{K}$ *which is* $\aleph_0$*-saturated,*
b) model M*, and*
c) sequence $\mathbf{b} = \langle \bar{b}_t : t \in J \rangle$ *of finite sequences from* M *with the length of* $\bar{b}_t$ *determined by* $\mathrm{tp}_{\mathrm{qf}}(t,\emptyset,J)$*,*

there exists a template Ψ *which is proper for* $\mathcal{K}$ *such that:*

i) $\tau(M) \subseteq \tau(\Psi)$
ii) Ψ *reflects* $\mathbf{b}$ *in the following sense:*
for any $s_0,\dots,s_{n-1}$ *from* J*,*
any $\varphi = \varphi(x_0,\dots,x_{m-1}) \in \mathcal{L}(\tau(M))$*,*
and any $\tau(M)$*-terms* $\sigma_\ell(\bar{y}_0,\dots,\bar{y}_{n-1})$ *for* $\ell = 0,\dots,m-1$*,*

$$\underline{\text{if}}\ M \models \varphi[\sigma_0(\bar{b}_{t_0},\dots,\bar{b}_{t_{n-1}}),\dots,\sigma_{m-1}(\bar{b}_{t_0},\dots,\bar{b}_{t_{n-1}})]$$
$$\text{for every } t_0,\dots,t_{n-1} \text{ realizing } \mathrm{tp}_{\mathrm{qf}}(s_0 \frown \dots \frown s_{n-1},\emptyset,J) \text{ in } J,$$
$$\underline{\text{then}}\ \mathrm{GEM}(J,\Psi) \models \varphi[\sigma_0(\bar{a}_{s_0},\dots,\bar{a}_{s_{n-1}}),\dots,\sigma_{m-1}(\bar{a}_{s_0},\dots,\bar{a}_{s_{n-1}})]$$

where $\langle \bar{a}_s : s \in J \rangle$ *denotes the skeleton of* $\mathrm{GEM}(J,\Psi)$*.*

We will generally use this definition in the form of Corollary 2.10.

Corollary 2.10. *If* $\mathcal{K}$ *is Ramsey, whenever we are given:*

a) $J \in \mathcal{K}$ *is* $\aleph_0$*-saturated*
b) Φ *a template proper for* $\mathcal{K}$
c) $M = \mathrm{GEM}(J,\Phi)$ *with skeleton* $\mathbf{a}$
d) N^+*, an elementary extension or expansion of* M*, or both*

then there is a template Ψ *proper for* $\mathcal{K}$ *with* $\tau(\Psi) \supseteq \tau(N^+)$ *and* $\Psi \geq \Phi$*. Moreover,* Ψ *reflects* $\mathbf{a}$ *in the sense described in* 2.9 *ii), with* $\mathbf{a}$ *here replacing* $\mathbf{b}$ *there.*

Remark 2.11. *The term "Ramsey property" for an index model class is also justified by Scow's result that (in our language) this corresponds naturally to the set of finite substructures of elements of the class being a Ramsey class in the sense of Nešetřil* [13] *and of Kechris-Pestov-Todorčević* [7]*. See Scow* [16] *Theorem 4.31.*

The last definition of this section will be crucial for the rest of the paper. Recall the definition of "index model class," 2.3, which had various mild restrictions on which classes of index models we may consider. For many of our arguments we will

fix not only some index model class $\mathcal{K}$ but some particular $I \in \mathcal{K}$, and the following conditions ensure in various ways that our I is not trivial.

Definition 2.12 (Context)**.** *A context* $\mathbf{c}$ *is a tuple* $(I, \mathcal{K}) = (I_{\mathbf{c}}, \mathcal{K}_{\mathbf{c}})$ *such that* $\mathcal{K}$ *is an index model class and* $I \in \mathcal{K}$*, and in addition:*

(1) *If* $\tau(\mathcal{K})$ *includes function symbols, then in addition we require that* $I = \mathrm{cl}(I)$.
(2) $I_{\mathbf{c}}$ *is* nontrivial*, meaning that* $I \neq \mathrm{cl}(\bar{t}, I)$ *for every finite* $\bar{t} \subseteq I$.
(3) I *is* reasonable*, meaning that whenever* $I \subseteq J$ *where* $J \in \mathcal{K}$ *is* $\aleph_0$*-saturated, if* $\bar{t} \in {}^{\omega >}I$*,* $s \in J$ *and*

$$(\text{for all } r \in J)\ (\mathrm{tp}_{\mathrm{qf}}(r, \bar{t}, J) = \mathrm{tp}_{\mathrm{qf}}(s, \bar{t}, J) \text{ implies } r = s)$$

then $s \in \mathrm{cl}_I(\bar{t})$.
(4) I *is* non-1-trivial*, meaning that whenever* $I \subseteq J$ *where* $J \in \mathcal{K}$ *is* $\aleph_0$*-saturated,* $\bar{t} \in {}^{\omega >}I$*,* $s \in J$ *and* $s \notin \mathrm{cl}(\bar{t})$ *then*

$$\{r \in J \ : \ \mathrm{tp}_{\mathrm{qf}}(r, \bar{t}, J) = \mathrm{tp}_{\mathrm{qf}}(s, \bar{t}, J)\} \text{ is infinite.}$$

Notation 2.13. *Given a context* $\mathbf{c}$*, which fixes* $\mathcal{K} = \mathcal{K}_{\mathbf{c}}$ *and* $\Upsilon = \Upsilon_{\mathcal{K}}$*, and given a theory* T*,*

(a) *Let* $\Upsilon[T]$ *be the class of* $\Phi \in \Upsilon$ *such that* $\tau_\Phi \supseteq \tau(T)$ *and* $T_\Phi \supseteq T$ *and* T_Φ *has Skolem functions for* T.
(b) *Let* $\Upsilon[\lambda, T]$ *be the class of* $\Phi \in \Upsilon[T]$ *such that* $\tau(\Phi)$ *has size* $\leq \lambda$.

Notation 2.14. *Given any linearly ordered set* I*, let* $\mathrm{inc}_n(I)$ *denote the set of strictly increasing* n*-element sequences from* I*, and let* $\mathrm{inc}(I) = \bigcup_n \mathrm{inc}_n(I)$.

3. $\mathcal{K}$-indiscernible sequences

This section discusses $\mathcal{K}$-indiscernible sequences, for a given index model class $\mathcal{K}$, Definition 2.3 above. These were introduced in [18] and have an interesting and varied history in the model theoretic literature, both in works of the second author and many others. Notably, the idea that generalized indiscernibles could give insight into model-theoretic dividing lines has been developed in a different direction by Scow [16] and Guingona-Hill-Scow [4].

Readers familiar with some such definition are nonetheless encouraged to read the remark after Definition 3.2.

To start, for the purposes of discussion, the familiar definition of an indiscernible sequence may be written as follows.

Definition 3.1. *Suppose we are given an ordered set* $(I, <)$*, a model* N *of* T*,* $A \subseteq N$*, and a map* $f : I \to {}^{\omega >}N$*. For each* $t \in I = \mathrm{Dom}(f)$*, write* $\bar{b}_t$ *for* $f(t)$*, so the image of* f *is the sequence* $\mathbf{b} = \langle \bar{b}_t : t \in I \rangle$*. We say* $\mathbf{b}$ *is an* indiscernible sequence over A *when it satisfies: for all* $k < \omega$*, all* $t_0, \ldots, t_{k-1}$ *and* $t'_0, \ldots, t'_{k-1}$ *from* I*, if*

$$\mathrm{tp}_{\mathrm{qf}}(t_0 \frown t_1 \frown \cdots \frown t_{k-1}, \emptyset, I) = \mathrm{tp}_{\mathrm{qf}}(t'_0 \frown t'_1 \frown \cdots \frown t'_{k-1}, \emptyset, I) \tag{1}$$

then in N*, or equivalently in the monster model* $\mathfrak{C} = \mathfrak{C}_T$*,*

$$\mathrm{tp}_{\tau(T)}(\bar{b}_{t_0} \frown \bar{b}_{t_1} \frown \cdots \frown \bar{b}_{t_{k-1}}, A, \mathfrak{C}) = \mathrm{tp}_{\tau(T)}(\bar{b}_{t'_0} \frown \bar{b}_{t'_1} \frown \cdots \frown \bar{b}_{t'_{k-1}}, A, \mathfrak{C}). \tag{2}$$

In the following key definition, we choose an I which may be an expansion of a linear order, the domain of f changes from I to ${}^{\omega>}I$, and 3.1(1) is updated in the natural way (note the inset line beginning "$\ell < k$" in 3.2 is trivially satisfied when the t's are singletons).

Definition 3.2 ($\mathcal{K}$-indiscernible sequence)**.** *Suppose we are given a context* $\mathbf{c}$*, thus* $I = I_{\mathbf{c}}$ *and* $\mathcal{K} = \mathcal{K}_{\mathbf{c}}$*. Suppose we are given a model* N *of* T*,* $A \subseteq N$*, and a map* $f : {}^{\omega>}I \to {}^{\omega>}N$*. For each* $\bar{t} \in \mathrm{Dom}(f)$*, write* $\bar{b}_{\bar{t}}$ *for* $f(\bar{t})$*, so the image of* f *is the sequence* $\mathbf{b} = \langle \bar{b}_{\bar{t}} : \bar{t} \in {}^{\omega>}I \rangle$*. We say* $\mathbf{b}$ *is a* $\mathcal{K}$-indiscernible sequence over A *when it satisfies: for all* $k < \omega$*, all* $\bar{t}_0, \dots, \bar{t}_{k-1}$ *and all* $\bar{t}'_0, \dots, \bar{t}'_{k-1}$ *from* ${}^{\omega>}I$*, if*

$$\ell < k \implies \lg(\bar{t}_\ell) = \lg(\bar{t}'_\ell) \quad \text{and}$$

$$\mathrm{tp}_{\mathrm{qf}}(\bar{t}_0 \frown \bar{t}_1 \frown \cdots \frown \bar{t}_{k-1}, \emptyset, I) = \mathrm{tp}_{\mathrm{qf}}(\bar{t}'_0 \frown \bar{t}'_1 \frown \cdots \frown \bar{t}'_{k-1}, \emptyset, I)$$

then in N*, or equivalently in the monster model* $\mathfrak{C} = \mathfrak{C}_T$*,*

$$\mathrm{tp}_{\tau(T)}(\bar{b}_{\bar{t}_0} \frown \bar{b}_{\bar{t}_1} \frown \cdots \frown \bar{b}_{\bar{t}_{k-1}}, A, \mathfrak{C}) = \mathrm{tp}_{\tau(T)}(\bar{b}_{\bar{t}'_0} \frown \bar{b}_{\bar{t}'_1} \frown \cdots \frown \bar{b}_{\bar{t}'_{k-1}}, A, \mathfrak{C}).$$

Definition 3.2 improves the range of 3.1 substantially. A very useful and less obvious way it does so may be observed as follows. If $\mathcal{K}$ is a class of linear orders and $I \in \mathcal{K}$, and if we are given a function f and a sequence $\mathbf{b}$ satisfying Definition 3.1, we may extend the domain of f naturally to ${}^{\omega>}I$ by setting

$$\bar{b}_{\bar{t}} = \bar{b}_{t_0} \frown \cdots \frown \bar{b}_{t_{\ell-1}} \text{ when } \bar{t} = \langle t_0, \dots, t_{\ell-1} \rangle \tag{3}$$

to generate a sequence $\mathbf{b}$ satisfying 3.2. However, Definition 3.2 doesn't ask that something like (3) be true. A priori, in 3.2,

$$\bar{b}_{t \frown s} \text{ may not be equal to } \bar{b}_t \frown \bar{b}_s. \tag{4}$$

Consider the following family of examples (3.5), which will require a few definitions.

Notation 3.3. *Given a context* $\mathbf{c}$*,* $\mathbf{D}(I)$ *is the set of quantifier-free types*[3] *of strictly increasing finite sequences of elements of* I*.*

Let us name the set of tuples in I sharing a quantifier-free type (inc_n: 2.14).

Definition 3.4. *For* $I \in \mathcal{K}$ *and* $\mathfrak{r} \in \mathbf{D}(I)$*, and implicitly* $n = n(\mathfrak{r})$*, let*

$$Q^I_{\mathfrak{r}} = Q^I_{\mathfrak{r},n} = \{\bar{t} : \bar{t} \in \mathrm{inc}_n(I), \mathrm{tp}_{\mathrm{qf}}(\bar{t}, \emptyset, I) = \mathfrak{r}\}$$

be the set of realizations of $\mathfrak{r}$ *in* I*.*

Definition 3.5 ($\mathfrak{r}$-indiscernible sequence)**.** *Suppose we are given a context* $\mathbf{c}$*, thus* $\mathcal{K} = \mathcal{K}_{\mathbf{c}}$ *and* $I = I_{\mathbf{c}}$ *. Suppose we are given a theory* T*, a model* $N \models T$*,* $A \subseteq N$*, a type* $\mathfrak{r} \in \mathbf{D}(I)$*, and a map* $f : Q^I_{\mathfrak{r}} \to {}^{\omega>}N$*. For each* $\bar{t} \in Q^I_{\mathfrak{r}}$ *write* $\bar{b}_{\bar{t}}$ *for* $f(t)$*. We say*

$$\mathbf{b} = \langle \bar{b}_{\bar{t}} : \bar{t} \in Q^I_{\mathfrak{r}} \rangle$$

is a $\mathfrak{r}$-indiscernible sequence over A *when:*

(a) *for all* $\bar{t} \in Q^I_{\mathfrak{r}}$*,* $\lg(\bar{b}_{\bar{t}})$ *is finite and constant.*

(b) *for all finite* k*, if* $\bar{t}_0, \dots, \bar{t}_{k-1}, \bar{t}'_0, \dots, \bar{t}'_{k-1} \in Q^I_{\mathfrak{r}}$ *and*

$$\mathrm{tp}_{\mathrm{qf}}(\bar{t}_0 \frown \bar{t}_1 \frown \cdots \frown \bar{t}_{k-1}, \emptyset, I) = \mathrm{tp}_{\mathrm{qf}}(\bar{t}'_0 \frown \bar{t}'_1 \frown \cdots \frown \bar{t}'_{k-1}, \emptyset, I)$$

then in N*, or equivalently in the monster model* $\mathfrak{C} = \mathfrak{C}_T$*,*

$$\mathrm{tp}_{\tau(T)}(\bar{b}_{\bar{t}_0} \frown \bar{b}_{\bar{t}_1} \frown \cdots \frown \bar{b}_{\bar{t}_{k-1}}, A, \mathfrak{C}) = \mathrm{tp}_{\tau(T)}(\bar{b}_{\bar{t}'_0} \frown \bar{b}_{\bar{t}'_1} \frown \cdots \frown \bar{b}_{\bar{t}'_{k-1}}, A, \mathfrak{C}).$$

[3] It would be more consistent with standard notation, if a little less readable, to write $\mathbf{D}_{\mathrm{qf}}(I)$.

Observation 3.6. *Definition 3.5 can naturally be considered as a special case of Definition 3.2.*

Proof. Extend f in 3.5 to ${}^{\omega>}I$ by setting $f(u) = \emptyset$ for all $u \in {}^{\omega>}I \setminus Q^I_{\mathfrak{r}}$. □

As another example, $\mathcal{K}$-indiscernible sequences arise naturally in GEM models.

Example 3.7. *For any context $\mathbf{c} = (I, \mathcal{K})$ and any $M = \mathrm{GEM}(I, \Phi, \mathbf{a})$, the template Φ determines an f showing that the skeleton $\mathbf{a}$ is a $\mathcal{K}$-indiscernible sequence.*

Remark 3.8. *In the example of a skeleton, of course, equation (3) above does hold; see also convention 4.1.*

So far we have been careful to write $\bar{t}$ for finite tuples from I of length possibly > 1, as distinguished from singletons $t \in I$, in order to clearly make the point in equation (4), p. 7 above. However, for the remainder of the paper, it will greatly simplify readability to also allow s, t to range over elements of $\mathrm{inc}(I)$.

Convention 3.9 (Dropping some overlines)**.** *Beginning in §4 and to the end of the paper, unless otherwise stated, we allow s, t to range over elements of ${}^{\omega>}I$, not just I. For example, referring to sequences as in 3.5, we may write*

$$\mathbf{b} = \langle \bar{b}_t : t \in Q^I_{\mathfrak{r}} \rangle$$

when $n = n(\mathfrak{r})$ is not necessarily 1. (This convention doesn't mean we won't continue to use overlines; it just means that the lack of an overline doesn't mean the length is 1.)

Classically in model theory, a main use of indiscernible sequences is in the definition of dividing, and so we may expect that the more robust notion of indiscernible sequence would give us a more finely calibrated notion of dividing. This will be developed in §5, after a section which may justify some particulars of that definition.

4. Weak definability and saturation

Developing an idea from [11] §9,[4] this section shows that GEM-models reveal a useful weakening of the phenomenon of definability of types from stable theories. Moreover, we will see that existence of these weak definitions may be characterized in terms of realization and omission of types in extensions of the given GEM model, and so is tightly connected to the problem of building saturated models in this setup.

Convention 4.1. *When $M = \mathrm{GEM}(I, \Phi)$ with skeleton $\mathbf{a} = \langle \bar{a}_t : t \in I \rangle$, then whenever $\bar{t} = \langle t_0, \dots, t_{k-1} \rangle \in \mathrm{inc}(I)$,*

$$\bar{a}_{\bar{t}} \text{ abbreviates } \bar{a}_{t_0} \frown \bar{a}_{t_1} \frown \cdots \frown \bar{a}_{t_{k-1}}.$$

Discussion 4.2. To motivate the first main definition of the section, Definition 4.9, suppose we are given a context $\mathbf{c} = (I, \mathcal{K})$, a complete theory T, and $M = \mathrm{GEM}(I, \Phi) \models T$ with skeleton $\mathbf{a} = \langle \bar{a}_t : t \in I \rangle$. Suppose $p \in \mathbf{S}_{\tau(T)}(M)$ is a type or a partial type, so we may enumerate it as

$$\langle \varphi_\alpha(x, \bar{b}_\alpha) : \alpha < \kappa \rangle \tag{5}$$

[4] the reader does not need to have seen that paper to follow the present section.

for some κ depending on p. Since we are in a GEM model, we may write a more informative version of (5),

$$\langle \varphi_\alpha(x, \bar{\sigma}_\alpha(\bar{a}_{\bar{t}_\alpha})) \ : \ \alpha < \kappa \rangle \tag{6}$$

where each $\bar{\sigma}_\alpha$ abbreviates some finite sequence of $\tau(\Phi)$-terms $\langle \sigma_\ell(\bar{y}_0, \dots, \bar{y}_{n-1}) : \ell < m(\alpha)\rangle$, and $\bar{t}_\alpha = \langle t_0, \dots, t_{n-1}\rangle \in \mathrm{inc}_n(I)$, so $\bar{a}_{\bar{t}_\alpha}$ is a sequence from the skeleton; thus, $\bar{\sigma}_\alpha(\bar{a}_{\bar{t}_\alpha})$ abbreviates $\langle \sigma_\ell(\bar{a}_{\bar{t}_\alpha}) : \ell < m(\alpha)\rangle$. [In order to evaluate this expression, it should of course be the case that for each $i < n$, $\lg(\bar{a}_{t_i}) = \lg(\bar{y}_i)$.] The choice of $\bar{\sigma}_\alpha$, $\bar{a}_{\bar{t}_\alpha}$ need not be unique; any choice with the property that $\langle \sigma_\ell(\bar{a}_{\bar{t}_\alpha}) : \ell < m(\alpha)\rangle$ evaluates correctly in M to $\bar{b}_\alpha$, will do.

Definition 4.3. *Given a context* $\mathbf{c} = (I, \mathcal{K})$, $M = \mathrm{GEM}(I, \Phi) = \mathrm{GEM}(I, \Phi, \mathbf{a})$ *and a type* $p \in \mathbf{S}(M)$, *call any enumeration of* p *satisfying* (6) *of Discussion* 4.2 *a* detailed enumeration.

Discussion 4.4. Continuing 4.2, each item in the sequence (6) has three natural ingredients: the formula φ_α, the sequence of $\tau(\Phi)$-terms $\bar{\sigma}_\alpha$, and $\mathfrak{r}_\alpha = \mathrm{tp}_{\mathrm{qf}}(\bar{t}_\alpha, \emptyset, I)$. The move from $\bar{t}_\alpha$ to its quantifier-free type $\mathfrak{r}_\alpha$ potentially loses information. Our question is whether this is serious, i.e. whether there is a partial function

$$F : (\ \tau(T)\text{-formulas}\ ) \times (\ \text{finite sequences of } \tau(\Phi)\text{-terms}\ ) \times \mathbf{D}(I) \longrightarrow \{0, 1\} \tag{7}$$

such that given any J with $I \subseteq J \in \mathcal{K}$, the set of formulas

$$\{\varphi(x, \bar{\sigma}(\bar{a}_{\bar{t}}))^{\mathbf{i}} : (\varphi, \bar{\sigma}, \mathfrak{r}) \in \mathrm{Dom}(F),\ \bar{t} \in \mathrm{inc}(J), \mathrm{tp}_{\mathrm{qf}}(\bar{t}, \emptyset, J) = \mathfrak{r},\ F(\varphi, \bar{\sigma}, \mathfrak{r}) = \mathbf{i}\} \tag{8}$$

when evaluated in $N = \mathrm{GEM}(J, \Phi)$, is consistent and extends p.

We will formally define such functions F in 4.9 below after a few additional remarks and adjustments.

First, why do we consider all larger J's? The deeper answer will be that, just as the usual definability of types is most useful in controlling extensions of the given type to larger models, here we will use F in applications of 2.10, which will require J to be sufficiently saturated. The simpler, initial answer is that for many natural I, restricting to $I = J$ gives F trivially, as the next example explains.

Definition 4.5. *Let* $\mathbf{c} = (I, \mathcal{K})$ *be a context. We say* I *is* separated *when* $s \neq t \in I$ *implies* $\mathrm{tp}_{\mathrm{qf}}(s, \emptyset, I) \neq \mathrm{tp}_{\mathrm{qf}}(t, \emptyset, I)$.

An example of 4.5 which played a key role in [11] §5: for a given infinite μ, $\mathcal{K}_\mu$ is the class of linear orders expanded by μ unary predicates which partition the domain, which is known to be an index model class. A separated $I \in \mathcal{K}_\mu$ is one in which each element of I has its own color.

Remark 4.6. *When* I *is separated, each* $\bar{t} \in \mathrm{inc}(I)$ *is the unique realization of its quantifier-free type* $\mathfrak{r} = \mathrm{tp}_{\mathrm{qf}}(\bar{t}, \emptyset, I)$, *so for the case* $I = J$, *a function* F *following* (7) *exists trivially, and the more interesting question concerns* $J \supsetneq I$.

One more example will explain the appearance of the finite $\bar{t}_*$ in Definition 4.9.

Example 4.7. Let $\mathcal{K}$ be the class of infinite linear orders, and $I = (\mathbb{Q}, <)$. Let T be the theory of an equivalence relation with infinitely many infinite classes. Choose $M = \mathrm{GEM}(I, \Phi, \mathbf{a})$ to be a countable model with $\mathbf{a} = \langle a_t : t \in I\rangle$ a sequence of elements from distinct equivalence classes. By our assumption 2.7, there are Skolem functions for T, say, $\langle f_i : i < \omega\rangle$ in $\tau(\Phi)$ interpreted so that $\langle f_i^M(a_t) : i < \omega\rangle$

enumerates the equivalence class of a_t. Let b be any element of M and let p be the partial type $\{E(x,b)\}$. Then we may choose a detailed enumeration of p, say,

$$p = \langle\ E(x, f_i(a_{t_*}))\ \rangle$$

for some $i = i_p < \omega$ and some $t_* = t_p \in I$. But since any two t, t' in I have the same quantifier-free type, no function F satisfying (7)-(8) above exists. This is easily solved by allowing F to depend on some finite sequence from I, here t_*.

Definition 4.8. *For $\bar{t}_* \in \mathrm{inc}(I)$, let $\mathbf{D}(I, \bar{t}_*)$ denote the set of quantifier-free types over $\bar{t}_*$ of strictly increasing finite sequences of elements of I, i.e.*

$$\mathbf{D}(I, \bar{t}_*) = \{\mathrm{tp}_{\mathrm{qf}}(\bar{t}, \bar{t}_*, I) : \bar{t} \in \mathrm{inc}(I)\}.$$

We arrive at the main definition of the section.

Definition 4.9 (Weakly definable type)**.** *Suppose we are given a context $\mathbf{c} = (I, \mathcal{K})$, a complete theory T, $M = \mathrm{GEM}(I, \Phi) = \mathrm{GEM}(I, \Phi, \mathbf{a}) \models T$ and a partial type or type $p \in \mathbf{S}_{\tau(T)}(M)$, of $\mathcal{L}(\tau_T)$. Say p is* weakly definable *when there exist*

(a) a detailed enumeration $\langle \varphi_\alpha(x, \bar{\sigma}_\alpha(\bar{a}_{\bar{t}_\alpha}))\ :\ \alpha < \kappa\rangle$ of p, where $\kappa = |p|$,
(b) a finite sequence $\bar{t}_* \in \mathrm{inc}(I)$,
(c) a partial function F depending on $\bar{t}_*$, such that

$$F : \{\bar{t}_*\} \times (\ \tau(T)\text{-formulas}\ ) \times (\ \text{finite sequences of } \tau(\Phi)\text{-terms}\ ) \times \mathbf{D}(I, \bar{t}_*) \longrightarrow \{0,1\}$$

and for some $\aleph_0$-saturated J with $I \subseteq J \in \mathcal{K}$, the set of formulas

$$q = \{\varphi(x, \bar{\sigma}(\bar{a}_{\bar{t}}))^{F(\bar{t}_*, \varphi, \bar{\sigma}, \mathfrak{r})} : (\bar{t}_*, \varphi, \bar{\sigma}, \mathfrak{r}) \in \mathrm{Dom}(F),\ \bar{t} \in \mathrm{inc}(J), \mathrm{tp}_{\mathrm{qf}}(\bar{t}, \bar{t}_*, J) = \mathfrak{r}\}$$

when evaluated in $N = \mathrm{GEM}(J, \Phi)$, is consistent and extends p.

Notation 4.10. *In the context of* 4.9*, we may also write "p is weakly definable over $\bar{t}_*$" or "p has a weak definition over $\bar{t}_*$" to emphasize the choice of the finite $\bar{t}_*$.*

Observation 4.11. *If p is a partial type of $\mathrm{GEM}(I, \Phi)$ and is weakly definable, then p remains weakly definable in $\mathrm{GEM}(I, \Psi)$ for any Ψ with $\Phi \leq \Psi \in \Upsilon[T]$, as witnessed by the same $\bar{t}_*$ and F.*

Observation 4.12. In the context of 4.9, note that it follows from the definition of GEM-model that if p has a weak definition over some $\bar{t}_* \in \mathrm{inc}(I)$, and if $\bar{s}^*$ is any other sequence from J with $\mathrm{tp}_{\mathrm{qf}}(\bar{s}^*, \emptyset, J) = \mathrm{tp}_{\mathrm{qf}}(\bar{t}_*, \emptyset, J)$, then the set of formulas

$$\{\varphi(x, \bar{\sigma}(\bar{a}_{\bar{t}}))^{F(\bar{s}^*, \varphi, \bar{\sigma}, \mathfrak{r})} : (\bar{s}^*, \varphi, \bar{\sigma}, \mathfrak{r}) \in \mathrm{Dom}(F),\ \bar{t} \in \mathrm{inc}(J), \mathrm{tp}_{\mathrm{qf}}(\bar{t}, \bar{s}^*, J) = \mathfrak{r}\}$$

when evaluated in $N = \mathrm{GEM}(J, \Phi)$, is consistent. Moreover, this consistent set of formulas extends a natural analogue of p, namely, the type obtained by replacing every occurrence of $\bar{t}_*$ in the given detailed enumeration of p by $\bar{s}^*$.

Remark 4.13. *In Definition* 4.9 *the particular choice of J will not matter, only that it is $\aleph_0$-saturated and extends I. We could have stated the definition for some, equivalently every, $\aleph_0$-saturated $J \supseteq I$ from $\mathcal{K}$.*

Discussion 4.14. In Definition 4.9, existence of a weak definition depends on $I, \mathcal{K}, \Phi$, not only on the type. We might also say it is the extension q of p which has the weak definition. When such a weak definition exists, then for each J the extension $q \supseteq p$ we obtain is unique. (We aren't asserting this is independent of the choice of the detailed enumeration, and we have also left open the possibility

of varying the domain of F to include e.g. formulas not used in p – but once F is given, for each larger J there is no ambiguity.) It may not be a complete type over N, since in J there may be many $\bar{t}$'s which do not realize any type in $\mathbf{D}(I)$ and so are never used, for example, if I is separated and J is $\aleph_0$-saturated, J will contain many finite sequences in which distinct elements have the same quantifier-free 1-types, and these have no analogue in I. However, if I is $\aleph_0$-saturated, then q will be a complete type.

Claim 4.15 (Definable implies weakly definable, for formulas)**.** *Suppose* $\mathbf{c} = (I, \mathcal{K})$ *is a context,* T *a complete theory,* $M = \mathrm{GEM}(I, \Phi) = \mathrm{GEM}(I, \Phi, \mathbf{a}) \models T$.

(a) Suppose $\Delta = \{\varphi, \neg\varphi\}$ *for some stable formula* φ *of* T*. Any type* $p \in \mathbf{S}_\Delta(M)$ *has a weak definition over some finite* $\bar{t}_* \in \mathrm{inc}(I)$.

(b) In the previous item, Δ *may be of any finite size as long as it contains only formulas which are stable in* T.

Proof. Since definitions operate formula-by-formula and the concatenation of finitely many finite $\bar{t}_*$'s is still finite, it will suffice to prove case (a). So let us assume $\Delta = \{\varphi(x, \bar{y}), \neg\varphi(x, \bar{y})\}$ where φ is a stable formula, and $\ell(x)$ need not be 1.

As φ is stable and M is a model, there is a formula $\theta = \theta(\bar{y}, \bar{z})$ and a sequence of elements $\bar{c} \in {}^{\mathrm{lg}(\bar{z})}M$ such that for all $\bar{b} \in {}^{\mathrm{lg}(\bar{y})}M$,

$$\varphi(x, \bar{b}) \in p \text{ if and only if } M \models \theta(\bar{b}, \bar{c}).$$

Fix some sequence $\bar{\sigma}_*$ of $\tau(\Phi)$-terms and some $\bar{t}_* \in \mathrm{inc}(I)$ so that evaluated in M,

$$\bar{c} = \bar{\sigma}_*(\bar{a}_{\bar{t}_*}).$$

Fix any detailed enumeration of p :

$$\langle \varphi(x, \bar{\sigma}_\alpha(\bar{a}_{\bar{t}_\alpha}))^{\mathbf{i}_\alpha} \ : \ \alpha < \kappa \rangle.$$

Consider the function F given by

$$(\bar{t}_*, \varphi, \bar{\sigma}_\alpha, \mathrm{tp}_{\mathrm{qf}}(\bar{t}_\alpha, \bar{t}_*, I)) \ \mapsto \ \mathbf{i}_\alpha.$$

Fix any $\aleph_0$-saturated $J \supseteq I$ from $\mathcal{K}$ and we would like to show the application of F defines a consistent $q \supseteq p$. Recall from 2.7(3) that $\mathrm{GEM}_{\tau(T)}(I, \Phi) \preceq \mathrm{GEM}_{\tau(T)}(J, \Phi)$. So if $\varphi(x, \bar{\sigma}_\alpha(\bar{a}_{\bar{t}_\alpha}))^{\mathbf{i}_\alpha} \in p$, then for any other $\bar{t}' \in \mathrm{inc}(J)$ such that

$$\mathrm{tp}_{\mathrm{qf}}(\bar{t}', \bar{t}_*, J) = \mathrm{tp}_{\mathrm{qf}}(\bar{t}_\alpha, \bar{t}_*, I)$$

we have that in $N = \mathrm{GEM}_{\tau(T)}(J, \Phi)$, again recalling 2.7(2),

$$\mathrm{tp}_{\tau(T)}(\bar{a}_{\bar{t}'}, \bar{a}_{\bar{t}_*}, N) = \mathrm{tp}_{\tau(T)}(\bar{a}_{\bar{t}_\alpha}, \bar{a}_{\bar{t}_*}, N).$$

In particular,

$$\psi(\bar{a}_{\bar{t}'}, \bar{a}_{\bar{t}_*}) := \theta(\bar{\sigma}_\alpha(\bar{a}_{\bar{t}'}), \bar{\sigma}_*(\bar{a}_{\bar{t}_*})) = \theta(\bar{\sigma}_\alpha(\bar{a}_{\bar{t}'}), \bar{c})$$

will hold in N if and only if

$$\psi(\bar{a}_{\bar{t}_\alpha}, \bar{a}_{\bar{t}_*}) := \theta(\bar{\sigma}_\alpha(\bar{a}_{\bar{t}_\alpha}), \bar{\sigma}_*(\bar{a}_{\bar{t}_*})) = \theta(\bar{\sigma}_\alpha(\bar{a}_{\bar{t}_\alpha}), \bar{c})$$

holds in N, so if and only if $\mathbf{i}_\alpha = 1$. So F agrees with the definition given by $\theta(\bar{y}, \bar{c})$, thus its output will be consistent. □

Discussion 4.16. In the precursor to this paper [11] §9 we summarized the main results proved there by suggesting a definition corresponding to (in the present notation) weak definitions over the empty set. The proof of [11], Claim 5.10 there established that for $T = T_{\mathbf{rg}}$ the theory of the random graph, $\mathcal{K} = \mathcal{K}_\mu$ the class of linear orders expanded by μ unary predicates which partition the domain, and a context $\mathbf{c} = (I, \mathcal{K})$ where I is separated, we have that in any $M = \mathrm{GEM}(I, \Phi) \models T$, any partial Δ-type over M for $\Delta = \{R(x,y), \neg R(x,y)\}$ has a weak definition over the empty set.

Conclusion 4.17. *Weak definability of φ-types is strictly weaker than definability of φ-types, since a φ-type over a model is definable if and only if φ is stable.*

Discussion 4.18. The extension of stability in 4.17 requires looking locally. Notice we have *not* called a type weakly definable when each of its formulas is. Rather, we require a single finite sequence $\bar{t}_*$ which works for the entire type. (What if each formula is weakly definable but the type is not? Then there is no problem in realizing each φ-type in some larger Ψ, but we won't be able to realize the entire type at the same time.) This is justified by Claim 4.22 below, and indeed, the careful reader may guess that non-superstability, suitably extended, will have an important role to play in what follows.

We now connect weak definability to the construction of saturated models. For the remainder of the section, let the following be arbitrary but fixed.

Hypothesis 4.19. For the rest of the section,

(1) $\mathbf{c}$ *a context, so* $I = I_{\mathbf{c}}$ *and* $\mathcal{K} = \mathcal{K}_{\mathbf{c}}$ *are given.*
(2) T *a complete first-order theory.*
(3) $\Phi \in \Upsilon = \Upsilon_{\mathcal{K}}[T]$*, recalling notation* 2.13*.*
(4) $M = \mathrm{GEM}_{\tau(T)}(I, \Phi)$*.*
(5) "there exists $\Psi \geq \Phi$*" always means* $\Psi \in \Upsilon[T]$*.*

Claim 4.20. *Let $p = p(\bar{x})$ be a partial type in M. Suppose p has a weak definition over some finite $\bar{t}_* \subseteq I$. Then there exists $\Psi \in \Upsilon$, $\Psi \geq \Phi$ such that p is realized in* $\mathrm{GEM}_{\tau(T)}(I, \Psi)$*.*

Remark 4.21. *In the special case when $\bar{t}_*$ is empty, this was noted in* [11]*,* 9.6*.*

Proof of 4.20. Let $J \in \mathcal{K}$ be an $\aleph_0$-saturated extension of I. Let $N = \mathrm{GEM}(J, \Phi)$. By hypothesis, there is a finite $\bar{t}_* \in \mathrm{inc}(I)$ and a function $F = F_{\bar{t}_*}$ giving a weak definition of p over $\bar{t}_*$. Applying F in the larger setting of J, let q be the type $q = q_{\bar{t}_*}(\bar{x})$ from Definition 4.9. Let

$$\mathcal{S} = \{\bar{s} \subseteq J : \mathrm{tp}_{\mathrm{qf}}(\bar{s}, \emptyset, J) = \mathrm{tp}_{\mathrm{qf}}(\bar{t}_*, \emptyset, I)\}.$$

Recalling Observation 4.12, for each $\bar{s} \in \mathcal{S}$, let $F_{\bar{s}}$ denote the result of replacing $\bar{t}_*$ by $\bar{s}$ in the definition of F, and let $q_{\bar{s}}(\bar{x})$ denote the corresponding set of formulas. As N is a GEM-model, for each $\bar{s} \in \mathcal{S}$, $q_{\bar{s}}(\bar{x})$ is also a partial type. Let N_1 be a large elementary extension of N in which each of the partial types in the set

$$\{q_{\bar{s}}(\bar{x}) : \bar{s} \in \mathcal{S}\}$$

is realized, noting that $\bar{t}_* \in \mathcal{S}$ and therefore $q = q_{\bar{t}_*}$ belongs to this set. Let $\bar{c}_{\bar{s}}$ denote a realization of $q_{\bar{s}}$ in N_1. Let $\mathbf{a} = \langle \bar{a}_t : t \in J \rangle$ denote the skeleton of the

GEM-model $N = \mathrm{GEM}(J, \Phi)$. Let $k = \lg(\bar{x})$. Let $G_0, \ldots, G_{k-1}$ be new $\lg(\bar{a}_{\bar{t}_*})$-place function symbols. As $N \preceq N_1$, we may expand N_1 by interpreting the G_i's so that

$$\langle G_i^{N_1}(\bar{a}_{\bar{s}}) : i < k \rangle = \bar{c}_{\bar{s}}$$

for each $\bar{s} \in \mathcal{S}$. Finally, we may further expand N_1 by adding Skolem functions. Let N_1^+ denote this expanded model. Apply the Ramsey property, Corollary 2.10, with $\mathbf{a}$, J, N_1^+, Φ to obtain $\Psi \in \Upsilon[T]$, $\Psi \geq \Phi$.

Why is this enough? By the reflection property mentioned in 2.10, the template Ψ will record from N_1^+ the information that for each $\bar{s} \in \mathcal{S}$, and each $(\bar{s}, \varphi, \bar{\sigma}, \mathfrak{r}) \in \mathrm{Dom}\, F_{\bar{s}}$ [where recall that $\mathfrak{r} \in \mathbf{D}(I, \bar{s})$],

$$N_1^+ \models \varphi(\, G_0^{N_1}(\bar{a}_{\bar{s}}), \ldots, G_{k-1}^{N_1}(\bar{a}_{\bar{s}})\, ,\ \bar{s}(\bar{a}_{\bar{t}})\,)^{F_{\bar{s}}(\bar{s}, \varphi, \bar{\sigma}, \mathfrak{r})} \text{ for every } \bar{t} \in \mathrm{inc}(J) \text{ realizing } \mathfrak{r}.$$

That is, Ψ records the truth or falsity of this formula as a property of $\mathrm{tp}_{\mathrm{qf}}(\bar{t} \frown \bar{s}, \emptyset, J)$. This will ensure that in $\mathrm{GEM}(J, \Psi)$, for every $\bar{s} \in \mathcal{S}$, $\bar{G}(\bar{a}_{\bar{s}})$ will realize every formula of $q_{\bar{s}}$. This holds a fortiori in $\mathrm{GEM}(I, \Psi)$, which completes the proof. □

Claim 4.22. *Let p be a partial type of $M = \mathrm{GEM}_{\tau(T)}(I, \Phi)$. Suppose there is some $\Psi \geq \Phi$ such that p is realized in $\mathrm{GEM}(I, \Psi)$. Then p has a weak definition over some finite $\bar{t}_* \in \mathrm{inc}(I)$ in $\mathrm{GEM}(I, \Psi)$.*

Proof. Suppose $\bar{c}$ realizes p in $GEM(I, \Psi)$. Let $\bar{t}_*$ be a finite subset of I such that $\bar{c} \in \mathrm{GEM}(\bar{t}_*, \Psi)$, noting that if $\bar{c}$ is named by constants or is otherwise in the algebraic closure of the empty set, we may choose $\bar{t}_*$ to be empty. Let $J \supseteq I$ be $\aleph_0$-saturated, and assume the skeleton $\mathbf{a}$ of $\mathrm{GEM}(J, \Psi)$ extends that of $\mathrm{GEM}(I, \Psi)$.

For any[5] tuple $(\bar{t}_*, \psi, \bar{\sigma}, \mathfrak{r})$, let F be given by $F(\bar{t}_*, \psi, \bar{\sigma}, \mathfrak{r}) = 1$ when for some, equivalently every, finite $\bar{s} \subseteq I$ with $\mathrm{tp}_{\mathrm{qf}}(\bar{s}, \bar{t}_*, I) = \mathfrak{r}$, we have that $\mathrm{GEM}(I, \Psi) \models \psi[\bar{c}, \bar{\sigma}(\bar{a}_{\bar{s}})]$; and $F(\bar{t}_*, \psi, \bar{\sigma}, \mathfrak{r}) = 0$ otherwise. Since Ψ is a template, and since $\bar{c}$ realizes p, this function is well defined and has the required properties. □

Discussion 4.23. If p is a partial type of $M = \mathrm{GEM}_{\tau(T)}(I, \Phi)$, and has a weak definition in some $\mathrm{GEM}(I, \Psi)$ for $\Psi \geq \Phi$, must there be a weak definition already in $\mathrm{GEM}(I, \Phi)$? After all, I has not changed. This question has to do with the choice of detailed enumeration. If we fix a detailed enumeration of p in $\mathrm{GEM}(I, \Phi)$, then whether or not this specific detailed enumeration gives rise to a weak definition is determined by Φ; a later, larger $\Psi \geq \Phi$ won't be able to change the situation. However, our definition 4.9 starts by choosing in the given model, some detailed enumeration, and certainly with richer templates, the available detailed enumerations may increase. This is why a priori, a weak definition may become available later in some $\mathrm{GEM}(I, \Psi)$.

Corollary 4.24. *Let p be a partial type of $M = \mathrm{GEM}_{\tau(T)}(I, \Phi)$.*
Then we have (a) if and only if (b):

- *(a) there exists some $\Psi_1 \geq \Phi$ such that p has a weak definition in $\mathrm{GEM}(I, \Psi_1)$ (and therefore has a weak definition in $\mathrm{GEM}(I, \Psi)$ for all $\Psi \geq \Psi_1$).*
- *(b) there exists some $\Psi_2 \geq \Phi$ such that p is realized in $\mathrm{GEM}(I, \Psi_2)$ (and therefore realized in $\mathrm{GEM}(I, \Psi)$ for all $\Psi \geq \Psi_2$).*

[5] assuming the given lengths of the variables, types, sequences are compatible.

Proof. If p has a weak definition in $\mathrm{GEM}(I, \Psi_1)$ then Claim 4.20 gives $\Psi_2 \geq \Psi_1$ such that p is realized in $\mathrm{GEM}(I, \Psi_2)$. If p is realized in $\mathrm{GEM}(I, \Psi_2)$ then letting $\Psi_1 = \Psi_2$, Claim 4.22 shows that p has a weak definition [that is, over some finite $\bar{t}_*$] in $\mathrm{GEM}(I, \Psi_1)$. For the parentheticals, being realized is clearly preserved under increasing the template by definition of the order on templates, and Observation 4.11 records that being weakly definable is too. □

Corollary 4.25. *Let p be a partial type of $M = \mathrm{GEM}_{\tau(T)}(I, \Phi)$.*
Then we have (a) if and only if (b):

- *(a) for no $\Psi_1 \geq \Phi$ does p have a weak definition in $\mathrm{GEM}(I, \Psi_1)$.*
- *(b) for no $\Psi_2 \geq \Phi$ is p is realized in $\mathrm{GEM}(I, \Psi_2)$.*

What is the core mechanism underlying the appearance of weak definability? Suppose we look locally: this suggests:

Question 4.26. *Let p be a φ-type or partial φ-type in $M = \mathrm{GEM}_{\tau(T)}(I, \Phi)$. Does there exist a finite sequence $\bar{t}_* \in \mathrm{inc}(I)$, a formula $\psi(x, \bar{b}) \in p$ (so $\psi = \varphi$ or $\neg\varphi$), a finite sequence $\bar{\sigma}$ of $\tau(\Phi)$-terms, and $\bar{t} \in \mathrm{inc}(I)$ such that*

$$\psi(x, \bar{b}) = \psi(x, \bar{\sigma}^{\mathrm{GEM}(I,\Phi)}(\bar{a}_{\bar{t}})) \tag{9}$$

and such that for some $\aleph_0$-saturated J, with $I \subseteq J \in \mathcal{K}$, in $N = \mathrm{GEM}(J, \Phi)$, the set of formulas

$$\{\psi(\bar{x}, \sigma^N(\bar{a}_{\bar{s}})) : \mathrm{tp}_{\mathrm{qf}}(\bar{s}, \bar{t}_*, J) = \mathrm{tp}_{\mathrm{qf}}(\bar{t}, \bar{t}_*, I)\} \tag{10}$$

extends p and *is consistent?*

Note that as $\Phi \leq \Psi$, and $\mathrm{GEM}(I, \Phi) \preceq \mathrm{GEM}(J, \Psi)$ by 2.7, $\sigma(\bar{a}_{\bar{t}})$ evaluates identically in both the larger and smaller model. Recall that the notation $J[\bar{t}_*]$ means J expanded by constants for elements of $\bar{t}_*$.

Discussion 4.27. Equations (9) and (10) tell us that in a possibly larger model there is a sequence

$$\langle \sigma^N(\bar{a}_{\bar{s}}) : \bar{s} \in Q_{\mathfrak{r}}^{J[\bar{t}_*]} \rangle$$

which is $\mathcal{K}$-indiscernible over $\mathrm{GEM}(\bar{t}_*, \Psi)$ and which includes $\bar{b} = \sigma^N(\bar{a}_{\bar{t}})$, and the question essentially asks whether ψ instantiated along this sequence is inconsistent.

The clarity brought by the larger J is important, recalling e.g. 4.5. Still, the instructions as to whether or not to realize p in $\mathrm{GEM}(I, \Phi)$ must come from the template Φ. If inconsistency appears in $\mathrm{GEM}(J, \Phi)$, this template cannot produce a realization even for I.

Discussion 4.28. In the next sections, we will see that this definition has a special explanatory power considered alone, a priori free of connection to GEM-models.

5. Shearing

In this section we develop a definition that will be central to the rest of the paper. Informally, it is the right extension of dividing (in the usual sense of model theory) to the case where we allow $\mathcal{K}$-indiscernible sequences, for $\mathcal{K}$ any index model class, not necessarily only linear orders.

Notation 5.1. *When $I_0 \subseteq J \in \mathcal{K}$ is a set, writing $J[I_0]$ means J expanded by constants for the elements of I_0, and likewise for $J[\bar{s}]$ when $\bar{s} \subseteq J$ is a sequence.*

Definition 5.2 (Shearing)**.** *Suppose we are given a context $\mathbf{c}$, a theory T, $M \models T$, $A \subseteq M$, and a formula $\varphi(\bar{x}, \bar{c})$ of the language of T with $\bar{c} \in {}^{\omega >}M$.*

We say that

$$\text{the formula } \varphi(\bar{x}, \bar{c}) \text{ shears over } A \text{ in } M \text{ for } (I_0, I_1, \mathbf{c})$$

when there exist a model N, a sequence $\mathbf{b}$ in N, enumerations $\bar{s}_0$ of I_0 and $\bar{t}$ of I_1, and an $\aleph_0$-saturated $J \supseteq I$ such that:

1. *$I_0 \subseteq I_1$ are finite subsets of I*
2. *$M \preceq N$*
3. *$\mathbf{b} = \langle \bar{b}_{\bar{s}} : \bar{s} \in {}^{\omega >}(J[I_0]) \rangle$ is $\mathcal{K}$-indiscernible in N over A*
4. *$\bar{c} = \bar{b}_{\bar{t}}$, and*
5. *the set of formulas*

$$\{\varphi(\bar{x}, \bar{b}_{\bar{t}'}) : \bar{t}' \in {}^{\lg(\bar{t})}(J), \mathrm{tp}_{\mathrm{qf}}(\bar{t}', \bar{s}_0, J) = \mathrm{tp}_{\mathrm{qf}}(\bar{t}, \bar{s}_0, I)\}$$

is contradictory.

Convention 5.3. Some conventions for Definition 5.2:

1. If $\mathbf{c}$ is clear, we may write "...for (I_0, I_1)" instead of "...for $(I_0, I_1, \mathbf{c})$."
2. We may write "the formula $\varphi(\bar{x}, \bar{c})$ shears over A in M" to mean that there is some (I_0, I_1) for which this holds.

Observation 5.4. *Changing J in the definition 5.2 does not matter as long as $I_1 \subseteq J \in \mathcal{K}$ and J is $\aleph_0$-saturated.*

Discussion 5.5. Definition 5.2 is parallel to the usual notion of dividing in that M is not assumed to be a GEM-model. Even if it is, $\mathbf{b}$ need have no connection to the skeleton. Only the parameter $\bar{c}$ of $\varphi(\bar{x}, \bar{c})$ and the set A are required to be in M (though by the first of these $\mathbf{b}$ has nonempty intersection with M). A priori, the sequence $\mathbf{b}$ belongs to N.

Claim 5.6. *Suppose $\varphi(\bar{x}, \bar{c})$ shears over A in M for $(I_0, I_1, \mathbf{c})$. Suppose $I_1' \supseteq I_0' \subseteq I_0$ and $I_1 \subseteq I_1'$. Then $\varphi(\bar{x}, \bar{c})$ shears over A in M for $(I_0', I_1', \mathbf{c})$.*

Discussion 5.7. Suppose $\varphi(\bar{x}, \bar{c})$ shears over A in M for $(I_0, I_1, \mathbf{c})$, as witnessed by $\bar{s}_0, \bar{t}$, N, and $\mathbf{b}$. The same data work to show that $\varphi(\bar{x}, \bar{c})$ shears in elementary extensions of N, and also if we take the reduct of N to a language which still contains $\tau(\varphi)$. However, shearing does not necessarily persist under expansions (consider what happens if we name $\bar{c}$ by a constant.)

Claim 5.8 (Dividing implies shearing)**.** *Let T be any complete theory and suppose $\varphi(\bar{x}, \bar{a})$ divides over some set A in the monster model of T. Then $\varphi(\bar{x}, \bar{a})$ $\mathbf{c}$-shears over A for any context $\mathbf{c}$.*

Remark 5.9. *The proof will show more, namely that we can choose any finite $I_0 \subseteq I$ and any $I_1 \supseteq I_0$ such that $I_1 = I_0 \cup \{t\}$ where $t \notin \mathrm{cl}(I_0)$, and φ will $(I_0, I_1, \mathbf{c})$-shear over A. (In fact there is nothing in the proof that prevents t from having length longer than* 1.)

Proof of 5.8. The idea of the proof is simple: use the Ramsey property to upgrade a dividing sequence to a sequence witnessing shearing. However, we check all the details.

By our assumption, there are $1 < k < \omega$ and a formula

$$\varphi(\bar{x}, \bar{a}) \tag{11}$$

which k-divides over A in the monster model of T. Let $\mathbf{c} = (I, \mathcal{K}) = (I_{\mathbf{c}}, \mathcal{K}_{\mathbf{c}})$ be given, recalling that this means satisfying 2.3 and 2.12. Fix any finite $I_0 \subseteq I$ and choose $t \in I_{\mathbf{c}} \setminus I_0$ with t not in the definable closure of I_0 in $I_{\mathbf{c}}$. Fix an enumeration $\bar{s}_0$ of I_0. Let J be $\aleph_0$-saturated, and without loss of generality, $I_0 \subseteq J$. Let

$$\mathcal{S} = \{s \in J : \mathrm{tp}_{\mathrm{qf}}(t, \bar{s}_0, J) = \mathrm{tp}_{\mathrm{qf}}(s, \bar{s}_0, I)\}$$

which is infinite by our assumption 2.12, and inherits a linear ordering from J. Let $I_1 = I_0 \cup \{t\}$. We will show that φ shears for $(I_0, I_1, \mathbf{c})$. Let

$$\mathbf{b} = \langle \bar{b}_s : s \in \mathcal{S} \rangle, \text{ i.e. } \langle \bar{b}_s : s \in (\mathcal{S}, <) \rangle$$

[where the intention is: indexed by $\mathcal{S}$ considered as a linearly ordered set] be an indiscernible sequence over A in the monster model of T witnessing the k-dividing of φ. To belabor the point, $\mathbf{b}$ is only indiscernible in the usual sense, as by compactness we can choose such a sequence indexed by any infinite linear order. In particular, $\bar{a}$ from (11) belongs to $\mathbf{b}$, and

$$\{\ \varphi(\bar{x}, \bar{b}_s) : s \in \mathcal{S}\ \}$$

is 1-consistent but k-contradictory.

We now appeal to a GEM-model. Choose some template Ψ proper for $\mathcal{K}$ so that $M \restriction \tau(T) \models T$, where $M = \mathrm{GEM}(J, \Phi)$ (this is always possible, see e.g. 6.8). Let $\mathbf{a} = \langle \bar{a}_s : s \in J \rangle$ be the skeleton of M. Let M_0 be an elementary extension of M which contains $\mathbf{b}$ and A. Let M_0^+ be the expansion of M_0 in which every element of A is named by a constant. Let $m_0 = |I_0 \cup \{t\}|$ and $m_1 = \mathrm{lg}(\bar{a}_s)$ where $\bar{a}_s$ is any member of the skeleton, and let $\ell = \mathrm{lg}(\bar{b}_s)$ for any $\bar{b}_s \in \mathbf{b}$. Let $m = m_0 \cdot m_1$. [Even though we are looking to build an $\mathcal{S}$-indexed sequence, recall we are working over I_0, so we need to carry out the next part of the construction uniformly over all copies of I_0 for the Ramsey property to work as desired.] Let M_0^{++} be the following further expansion of M_0^+: add to the language a new sequence of ℓ-many m-ary function symbols. Interpret these functions in M_0^{++} so that for each $\bar{s}'_0 \in {}^{\omega >}J$ with $\mathrm{tp}_{\mathrm{qf}}(\bar{s}'_0, \emptyset, J) = \mathrm{tp}_{\mathrm{qf}}(\bar{s}_0, \emptyset, I)$, and for each $s \in \mathcal{S}$,

$$F_i^{M_0^{++}}(\bar{a}_{\bar{s}'_0 \frown \langle s \rangle}) : i < \ell\rangle = \bar{b}_s.$$

Apply the Ramsey property to $M, \mathbf{a}, M_0^{++}, \Phi$ and let $\Psi \geq \Phi$ be the template returned.

Let $N = \mathrm{GEM}(J, \Psi)$. Note that in $N \restriction \tau(T)$ there will be an automorphic image A' of A, which is named by constants in the GEM-model N. For each $s \in \mathcal{S}$, let

$$\bar{b}'_s = \langle F_i^N(\bar{a}_{\bar{s}_0 \frown \langle s \rangle}) : i < \ell\rangle.$$

To match the notation of Definition 5.2: for some, equivalently every, $s \in \mathcal{S}$, let $\mathfrak{r} = \mathrm{tp}_{\mathrm{qf}}(s, \emptyset, J[\bar{s}_0]) = \mathrm{tp}_{\mathrm{qf}}(t, \emptyset, J[\bar{s}_0])$. Recall that $\{s \in Q_{\mathfrak{r}}^{J[\bar{s}_0]}\}$ denotes $\{s \in J, \mathrm{tp}_{\mathrm{qf}}(s, \emptyset, J[\bar{s}_0]) = \mathfrak{r}\}$. Then by the Ramsey property,

$$\mathbf{b}' = \langle \bar{b}'_s : s \in Q_{\mathfrak{r}}^{J[\bar{s}_0]} \rangle$$

is an $\mathfrak{r}$-indiscernible sequence, and [because A' is named by constants in N] it is $\mathfrak{r}$-indiscernible over A', the copy of A in N. Moreover, by the Ramsey property and the choice of our original sequence $\mathbf{b}$,

$$\{\varphi(x, \bar{b}'_s) : s \in Q_{\mathfrak{r}}^{J[\bar{s}_0]}\}$$

will be 1-consistent but k-inconsistent. Finally, observe that we may without loss of generality assume $\bar{a}$ from (11) belongs to the sequence $\mathbf{b}'$, as follows. By the Ramsey property, the type of $A, \bar{a}$ in the monster model for T will be the same as that of $A', \bar{b}'_t$. So we may move A' to A by an automorphism G, and then move $G(\bar{b}'_t)$ to $\bar{a}$ by an automorphism F which fixes A pointwise. Then the sequence $F(G(\mathbf{b}'))$ will witness the $(I_0, I_1, \mathbf{c})$-shearing of φ as desired. □

For later quotation, we single out the special case of linear orders.

Corollary 5.10. *Let M be any model, $A \subseteq M$ and let $\varphi(\bar{x}, \bar{a})$ be any formula of M. Suppose $\varphi(\bar{x}, \bar{a})$ divides over A in the usual sense. Let $\mathcal{K}$ be the class of linear orders, I any infinite member of $\mathcal{K}$, and $\mathbf{c} = (I, \mathcal{K})$. Then for any finite $I_0 \subseteq I$ and any $t \in I \setminus I_0$, writing $I_1 = I_0 \cup \{t\}$, we have that $\varphi(\bar{x}, \bar{a})$ $(I_0, I_1, \mathbf{c})$-shears.*

Discussion and examples. We outline here some additional results (originally part of the present paper, but moved to a separate manuscript for reasons of space) to give the reader some idea of the landscape.

The definition of "$\mathbf{c}$-unsuperstable" is given in the next section, and for now can be understood as a strong version of "shearing occurs".

Recall that $T_{3,2}$ is the theory of the generic tetrahedron-free three-hypergraph. (We have kept the notation consistent with our earlier papers. In his work Ulrich has suggested a reasonable notational change, adding one to the subscripts.)

Fact 5.11. *For each $n \geq 2$, let $T_{n,1}$ be the theory of the generic K_{n+1}-free graph. Then $T_{n,1}$ is not simple.*

Fact 5.12 (Hrushovski c. 2002, see [5])**.** *For $n > k \geq 2$, $T_{n,k}$ is simple unstable with only trivial dividing (i.e. only dividing coming from equality).*

The following is worked out in [12]:

Example 5.13 ([12])**.** *There is a countable context $\mathbf{c}$ such that the random graph is $\mathbf{c}$-superstable, and $T_{3,2}$ is $\mathbf{c}$-unsuperstable; in $T_{3,2}$, this shearing arises from a formula which is a Boolean combination of positive instances of the edge relation.*

The following general class of examples are constructed in [12]:

Example 5.14 ([12])**.** *Given any $n > k \geq 2$, there is a countable context $\mathbf{c}$ such that $T_{\mathbf{rg}}$ is $\mathbf{c}$-superstable and $T_{n,k}$ is not $\mathbf{c}$-superstable.*

Conclusion 5.15. *Shearing is strictly weaker than dividing.*

Discussion 5.16. Since shearing is not the same as dividing in simple theories, it necessarily fails some of the usual properties of independence relations.

6. Unsuperstability

Hypothesis 6.1.

1) $\mathbf{c}$ *is a context, so* I *and* $\mathcal{K}$ *are fixed.*
2) In this section I *is countable. We may say:* $\mathbf{c}$ *is a countable context.*
3) J *is* $\aleph_0$*-saturated,* $I \subseteq J \in \mathcal{K}$.
4) T *will vary, but will always be a complete first order theory.*
5) $\Upsilon_{\mathbf{c}}$ *denotes the templates proper for* $\mathcal{K}$. *We will assume the templates* Φ *in question satisfy* $T_\Phi \supseteq T$ *and have Skolem functions for* T, *i.e. belong to* $\Upsilon_{\mathbf{c}}[T]$.
6) Note: when we write $\mathrm{tp}_{\mathrm{qf}}(\bar{s}, ..., J) = \mathrm{tp}_{\mathrm{qf}}(\bar{t}, ..., J)$ *or something of the sort, it's understood that* $\lg(\bar{s}) = \lg(\bar{t})$.

In this section we define "T is (un)superstable for the countable context $\mathbf{c}$" and prove Theorem 6.14.[6] To do so we step back from our assumption that I must be the index set for the skeleton of a given GEM model, to simply using I (or a saturated J extending it) as the index set for *some* $\mathcal{K}$-indiscernible sequence which will witness e.g. inconsistency or dividing. Notice that in the next definition, I is not a priori an input to a GEM model, and the B_n are just sets in the monster model, a priori not related to the I_n's beyond what is written there.

Definition 6.2. *Let* $\mathbf{c}$ *be a countable context. We say* T *is* unsuperstable *for* $\mathbf{c}$ *when there are:*

(a) an increasing sequence of nonempty finite sets $\langle I_n : n < \omega \rangle$ *with* $I_m \subseteq I_n \subseteq I$ *for* $m < n < \omega$ *and* $\bigcup_n I_n = I$, *which are given along with a choice of enumeration* $\bar{s}_n$ *for each* I_n *where* $\bar{s}_n \trianglelefteq \bar{s}_{n+1}$ *for each* n
(b) an increasing sequence of nonempty, possibly infinite, sets $B_n \subseteq B_{n+1} \subseteq \mathfrak{C}_T$ *in the monster model for* T, *with* $B := \bigcup_n B_n$
(c) and a partial type p *over* B, *such that*

$$p \restriction B_{n+1} \text{ } (I_n, I_{n+1})\text{-shears over } B_n.$$

Remark 6.3. *To extend this definition to* I *each of whose strict subsets is finitely generated, add that* $\mathrm{cl}(I_n) \subsetneq \mathrm{cl}(I_{n+1})$ *and make the parallel changes to the proofs so that the* I_n*'s list finite sequences of generators rather than their closures.*

Definition 6.4. *When* T *is not unsuperstable for* $\mathbf{c}$, *we say* T is superstable for $\mathbf{c}$, *or just* $\mathbf{c}$-superstable.

Remark 6.5. Definition 6.2 uses countability of I in an essential way, as it is the union of an increasing chain of finite sets.

For later reference, we state the local version separately. Comparing to 6.2, note "T is superstable for the countable context $\mathbf{c}$" is just the case where (T, Δ) is $\mathbf{c}$-superstable and Δ is the set of all formulas of the language.

[6] We might have said "unsupersimple."

Definition 6.6. *Let* $\mathbf{c}$ *be a countable context and* Δ *a set of formulas of* T*. We say*

$$(T, \Delta) \text{ is unsuperstable for } \mathbf{c}$$

when 6.2 *holds in the case that we replace "*$\mathbf{S}(B)$*" in* 6.2(c) *by "*$\mathbf{S}_\Delta(B)$*," i.e., the type* p *in* 6.2(c) *may be taken to be a* Δ*-type.*

Claim 6.7. *Suppose* (T, Δ) *is not supersimple in the usual sense and* $\mathbf{c} = (I, \mathcal{K})$ *is any countable context. Then* (T, Δ) *is unsuperstable for* $\mathbf{c}$*.*

Proof. Immediate from 5.8; the countability of the context is used only in the definition of $\mathbf{c}$-superstable. □

For the complementary claim, see 6.18 below.

Observation 6.8. *For any theory* T *and context* $\mathbf{c}$*, for any* $\Phi \in \Upsilon_{\mathbf{c}}$*, there is* Ψ *with* $\Phi \leq \Psi \in \Upsilon_{\mathbf{c}}[T]$*.*

Proof. Let $J \supseteq I$ be $\aleph_0$-saturated. Let $M = \mathrm{GEM}(J, \Phi)$ with skeleton $\mathbf{a}$. If Φ is not already in $\Upsilon_{\mathbf{c}}[T]$, then without loss of generality, we may suppose the signature of T and of T_Φ are disjoint. Let N be an elementary extension of M which may also be expanded to a model of T with Skolem functions for T (of course this expansion need not have anything to do with the structure on N). Let N^+ be this expansion. Let Ψ be the template returned by applying the Ramsey property to N^+, $\mathbf{a}$, Φ. Then $\Psi \geq \Phi$ and Ψ will be in $\Upsilon_{\mathbf{c}}[T]$. □

First we consider the case where T is superstable for a given context, i.e., not unsuperstable. The larger role of I mentioned above plays little role in this proof, since superstability ensures good behavior for *all* relevant I_n's and B_n's, including those which have natural meaning in a GEM model.

Claim 6.9. *Assume* T *is superstable for the countable context* $\mathbf{c}$*. Suppose* $\Phi \in \Upsilon_{\mathbf{c}}[T]$ *and* $M = \mathrm{GEM}_{\tau(T)}(I, \Phi)$*. Let* $p \in \mathbf{S}(M)$ *be any type. Then there is* $\Psi \geq \Phi$ *such that* p *is realized in* $\mathrm{GEM}(I, \Psi)$*.*

Proof. Let $\langle i_n : n < \omega \rangle$ list I. Let $I_n = \{i_k : k < n\}$. Let B_n name the model $\mathrm{GEM}_{\tau(T)}(I_n, \Phi)$. Note that $\bigcup_n I_n = I$, $\bigcup_n B_n = M$, and $p \in \mathbf{S}(M)$. We ask: is there $m < \omega$ such that for no $n > m$ does $p \restriction B_n$ (I_m, I_n)-shears over M_m?

If there is no such m, so we contradict superstability. More precisely, choose $n(i)$ increasing with i such that $i = j + 1$ implies $p \restriction B_{n(i)}$ $(I_{n(j)}, I_{n(i)})$-shears over $B_{n(j)}$. Now the sequences $\langle I_{n(i)} : i < \omega \rangle$, $\langle B_{n(i)} : i < \omega \rangle$, and the type $p \in \mathbf{S}_{\tau(T)}(B) = \mathbf{S}_{\tau(T)}(\bigcup_i B_{n(i)})$ witness that T is $\mathbf{c}$-unsuperstable.

So there must be one such, call it m_*. Now we proceed similarly to the case where we have a weak definition. Let $J \supseteq I$ be $\aleph_0$-saturated. Since M is a GEM-model, we may choose a detailed enumeration (recalling 4.3)

$$p(\bar{x}) = \{\varphi_\alpha(\bar{x}, \bar{\sigma}_\alpha(\bar{a}_{\bar{t}_\alpha})) : \alpha < \alpha_*\}$$

where each $\bar{a}_{\bar{t}}$ is from the skeleton and each $\bar{\sigma}_\alpha$ is sequence of $\tau(\Phi)$-terms. Let $\bar{s}_{m_*}$ be the enumeration of I_{m_*}. Consider the larger set of formulas

$$(12) \quad q(\bar{x}) = \{\varphi_\alpha(\bar{x}, \bar{\sigma}_\alpha(\bar{a}_{\bar{t}})) : \alpha < \alpha_*,\ \bar{t} \in {}^{\omega >}J,\ \mathrm{tp}_{\mathrm{qf}}(\bar{t}, \bar{s}_{m_*} J) = \mathrm{tp}_{\mathrm{qf}}(\bar{t}_\alpha, \bar{s}_{m_*}, I)\}.$$

Suppose $q(\bar{x})$ were not a partial type. There would be $\alpha_1, \ldots, \alpha_k$ such that

$$\{\varphi_\alpha(\bar{x}, \bar{\sigma}_\alpha(\bar{a}_{\bar{t}})) : \alpha \in \{\alpha_1, \ldots, \alpha_k\}, \bar{t} \in {}^{\omega >}J, \mathrm{tp}_{\mathrm{qf}}(\bar{t}, \bar{s}_{m_*} J) = \mathrm{tp}_{\mathrm{qf}}(\bar{t}_\alpha, \bar{s}_{m_*}, I)\}$$

is inconsistent. Assuming the model M is infinite (if not it would already be saturated), without loss of generality[7] there is some single α such that

$$\{\varphi_\alpha(\bar{x}, \bar{\sigma}_\alpha(\bar{a}_{\bar{t}})) : \bar{t} \in {}^{\omega>}J, \mathrm{tp}_{\mathrm{qf}}(\bar{t}, \bar{s}_{m_*} J) = \mathrm{tp}_{\mathrm{qf}}(\bar{t}_\alpha, \bar{s}_{m_*}, I)\}$$

is contradictory. Now the sequence

$$\langle \bar{\sigma}_\alpha(\bar{a}_{\bar{t}})) : \bar{t} \in {}^{\omega>}J[\bar{s}_{m_*}]\rangle$$

is $\mathcal{K}$-indiscernible (the intended interpretation is that when $\bar{a}_{\bar{t}}$ has the wrong length to input to $\bar{\sigma}_\alpha$, the expression evaluates to $\emptyset$) over $\bar{a}_{\bar{s}_{m_*}}$. In other words, it is $\mathcal{K}$-indiscernible over B_{m_*}. Let n be such that $\bar{t}_\alpha \subseteq I_n$. Then we've shown that the formula $\varphi_\alpha(\bar{x}, \bar{\sigma}_\alpha(\bar{a}_{\bar{t}_\alpha}))$ here (I_{m_*}, I_n)-shears over B_{m_*}. This contradicts the choice of m_* from the beginning of the proof. We conclude that $q(\bar{x})$ is indeed a partial type, and of course $q(\bar{x}) \supseteq p(\bar{x})$.

Now, for any other $\bar{s} \in {}^{\omega>}J$ such that $\mathrm{tp}_{\mathrm{qf}}(\bar{s}, \emptyset, J) = \mathrm{tp}_{\mathrm{qf}}(\bar{s}_{m_*}, \emptyset, I)$, let

$$q_{\bar{s}}(\bar{x})$$

denote the result of replacing $\bar{s}_{m_*}$ by $\bar{s}$ in (12) above. This takes place in $\mathrm{GEM}(J, \Phi)$, and $q(\bar{x}) = q_{\bar{s}_{m_*}}(\bar{x})$ is a partial type, so each $q_{\bar{s}}(\bar{x})$ must also be a partial type.

Thus, in some larger elementary extension $N_\star$ of $\mathrm{GEM}(J, \Phi)$, we may realize all of these types $q_{\bar{s}}(\bar{x})$. Let $\bar{d}_{\bar{s}}$ denote the realization in $N_\star$ of $q_{\bar{s}}(\bar{x})$. Expand $N_\star$ to $N_\star^+$ by new functions F_ℓ, $\ell < \mathrm{lg}(\bar{x})$, interpreted so that for each $\bar{s} \subseteq J$ realizing $\mathrm{tp}_{\mathrm{qf}}(\bar{s}_{m_*}, \emptyset, J)$, we have

$$\langle F_\ell(\bar{a}_{\bar{s}}) : \ell\rangle = \bar{d}_{\bar{s}}.$$

Finally, let $N_\star^{++}$ be the expansion of $N_\star^+$ to a model with Skolem functions. Applying the Ramsey property with $\mathrm{GEM}(J, \Phi)$, $\mathbf{a}$, and $N_\star^{++}$, let Ψ be the template returned. Then Ψ will be nice, proper for $\mathcal{K}$, and in $\mathrm{GEM}(I, \Psi)$ the type p will be realized, as will be many of its copies. □

Corollary 6.10. *Assume (T, Δ) is superstable for the countable context $\mathbf{c}$. Suppose $\Phi \in \Upsilon_{\mathbf{c}}[T]$ and $M = \mathrm{GEM}_{\tau(T)}(I, \Phi)$. Let $p \in \mathbf{S}_\Delta(M)$ be any type. Then there is $\Phi \geq \Psi$ such that p is realized in $\mathrm{GEM}(I, \Psi)$.*

Proof. The same proof works at a slight notational cost; simply replace $\mathbf{S}$ by $\mathbf{S}_\Delta$, and add Δ to $\mathbf{S}_{\tau(T)}(B)$. □

Claim 6.11. *Let Δ be any set of formulas of T, in our main case all formulas. Assume (T, Δ) is superstable for the countable context $\mathbf{c}$. Let μ and λ be such that $\mu > |T|$, $\lambda = \lambda^{<\mu}$. Then for a dense set of $\Psi \in \Upsilon_{\mathbf{c}}[\lambda, T]$ the model $\mathrm{GEM}_{\tau(T)}(I, \Psi)$ is μ-saturated for Δ-types.*

Proof. Choose $\Phi_0 \in \Upsilon_{\mathbf{c}}[\lambda, T]$, recalling this denotes the templates Ψ proper for $\mathcal{K}_{\mathbf{c}}$ with $|\tau(\Psi)| \leq \lambda$ and $T_\Psi \supseteq T$. We need to show that for any such Φ_0 there is $\Psi \geq \Phi_0$ as required.

By induction on $\alpha \leq \lambda$ we will construct an increasing continuous chain of templates $\Phi_\alpha \in \Upsilon_{\mathbf{c}}[\lambda, T]$ so that Φ_λ will have the desired property. It suffices to describe the successor stage. Let $M_\alpha = \mathrm{GEM}_{\tau(T)}(I, \Phi_\alpha)$. Since $\tau(\Phi_\alpha)$ has $\leq \lambda$

[7] More precisely, there is some β: let φ_β be the conjunction of the formulas φ_α. Then φ_β belongs to the type so is consistent, but if we allow the relevant $\bar{t}$ to vary in J, we get inconsistency by definition of β. This is the only point in the proof where we use that p is a type, that is, that p is a Δ-type where Δ is a set of formulas closed under conjuction. However, notice that if p is a φ-type, we can always choose φ_β to be (an instance of φ) $\wedge$ (an instance of $\neg\varphi$).

symbols, this will be a model of size $\leq \lambda$. Counting types, there will be $\lambda = \lambda^{<\mu}$ choices of a parameter set A of size $< \mu$, and over each such A, up to $2^{<\mu} \leq \lambda$ types, for a total of $\leq \lambda$ types. Applying Claim 6.11 (either applying that Claim λ times in succession, or better, simply modifying that proof by adding λ-many different functions F and realizing the types all at once), we find $\Phi_{\alpha+1} \geq \Phi_\alpha$ with $|\tau(\Phi)_{\alpha+1}| \leq |\tau(\Phi_\alpha)| + \lambda$ so that in $\mathrm{GEM}_{\tau(T)}(I, \Phi_{\alpha+1})$ the types we had just counted are all realized.

By the end of the induction, $M_\lambda = \mathrm{GEM}_{\tau(T)}(I, \Phi_\lambda)$ will be μ-saturated. □

A comment on the operation of Claim 6.11. At first it may seem strange that saturated models are built up around a single unchanging I, but what one should notice is the change and expansion in the template as Φ_0 becomes Φ_λ. In some sense the induction of 6.11 is simply adding a growing list of precise construction instructions to the 'scaffolding' of the model (the saturation will be for $\tau(T)$ once the 'scaffolding is taken off'). The inclusion of both μ and λ in the statement of the claim points out how we may increase saturation even further as we allow an increased distance between the size of $\tau(\Phi)$ and the "constant" size of T. If we hope to build a μ-saturated model for some large μ, the statement of Claim 6.11 tells us what kind of λ we will need.

Next we consider **c**-unsuperstability. In this direction, the potential difference between the I- or J-indexed sequence witnessing shearing and the I- or J-indexed skeleton of the GEM models in the picture will be noted.

Discussion 6.12. *Our theorems will continue to be true locally as will be obvious from the proofs (the type ultimately omitted is a progressive automorphic image of the type realizing un-superstability), but we emphasize the global versions as there is marginally less notation, and state the local versions after for later reference.*

As a warm-up for Theorem 6.14, we explain how to copy a single instance of shearing into a GEM-model. (Claim 6.13 is illustrated in Figure 1.)

Claim 6.13 (Folding an instance of shearing into a GEM-model)**.** *Fix a background theory T and suppose we are given a countable context $\mathbf{c} = (I, \mathcal{K})$, along with:*

(1) (an instance of shearing) *Let $I_0 = \emptyset$ and $I_1 \subseteq I$ be finite. Let $J \supseteq I$, $J \in \mathcal{K}$ be $\aleph_0$-saturated. Let $\bar{s}_1$ enumerate I_1. Let $\mathfrak{r}_1 = \mathrm{tp}_{\mathrm{qf}}(\bar{s}_1, \emptyset, I)$. Suppose the formula $\varphi(x, \bar{c}_1)$ shears for $(\emptyset, I_1)$, witnessed by the $\mathcal{K}$-indiscernible sequence $\mathbf{b} = \langle \bar{b}_{\bar{s}} : \bar{s} \in Q^J_{\mathfrak{r}_1} \rangle$, with $\bar{b}_{\bar{s}_1} = \bar{c}_1$. Let $m = \lg(\bar{c}_1)$.*

(2) (a GEM-model) *Let $M = \mathrm{GEM}(J, \Phi)$ where $\Phi \in \Upsilon[T]$. Let $\langle \bar{a}_j : j \in J \rangle$ be the skeleton of M, and let $k = \lg(\bar{a}_{\bar{s}_1})$.*

Then there exists $\Psi \geq \Phi$ and a sequence of k-place function symbols $\langle \bar{F}_\ell : \ell < m \rangle$ of $\tau(\Psi) \setminus \tau(\Phi)$ such that in the model $\mathrm{GEM}(J, \Psi)$ which has skeleton $\langle \bar{a}_j : j \in J \rangle$, we have that the formula $\varphi(x, \bar{F}(\bar{a}_{\bar{s}_1}))$ itself $(\emptyset, I_0)$-shears, witnessed by the sequence $\langle \bar{F}(\bar{a}_{\bar{s}}) : \bar{s} \in Q^J_{\mathfrak{r}_1} \rangle$. Moreover the type of this sequence is the same as the type of $\mathbf{b}$.

Proof. Consider $M \restriction \tau(T)$ and $\mathbf{b}$ inside the monster model, having a priori nothing to do with each other. (We are initially in $\mathfrak{C}_T$, though when we expand before applying the Ramsey property, it is really $\mathfrak{C}_{T_\Phi}$. So the reader should understand that $\mathfrak{C}_T$ denotes "$\mathfrak{C}_{T_\Phi} \restriction \tau(T)$.") Choose an elementary extension N of M which includes $\langle \bar{b}_{\bar{s}} : \bar{s} \in Q^J_{\mathfrak{r}_1} \rangle$. Let $\langle F_\ell : \ell < m \rangle$ be new k-place function symbols, where

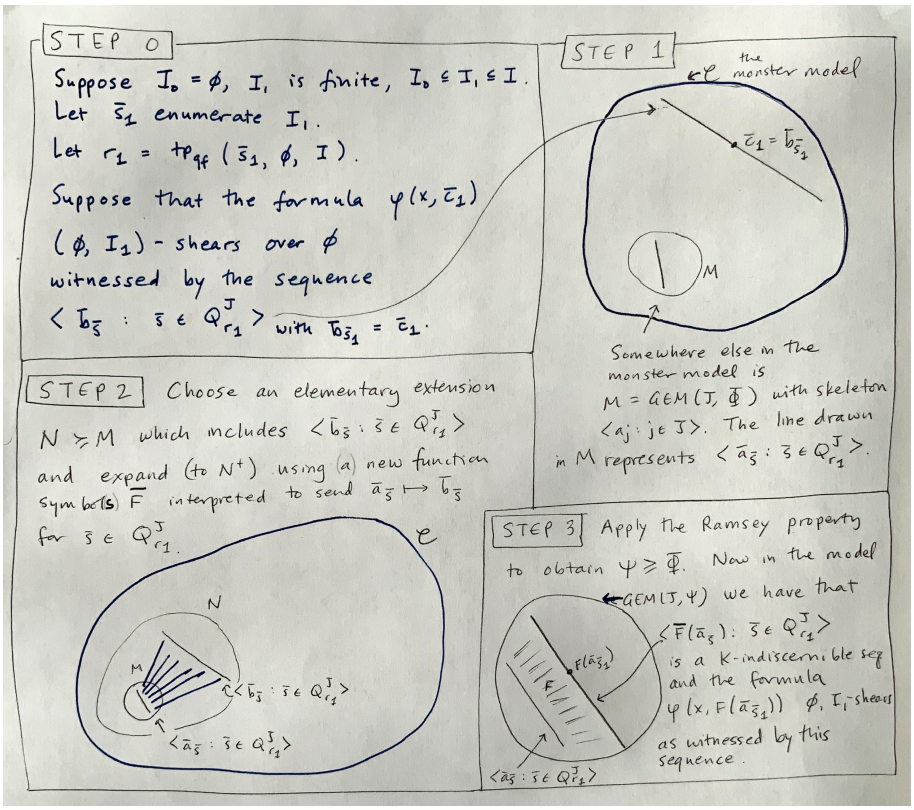

FIGURE 1. Folding an instance of shearing into a GEM-model (6.13).

new means not in $\tau(\Phi)$. Expand N to N^+ by interpreting the functions F so that

$$\bar{F}(\bar{a}_{\bar{s}}) = \bar{b}_{\bar{s}} \text{ for each } \bar{s} \in Q^J_{\mathfrak{r}_1}.$$

Recall our notation: if $\bar{F} = \langle F_i : i < k \rangle$ then $\bar{F}(\bar{a})$ is the sequence $\langle F_i(\bar{a}) : i < k \rangle$. Expand also to add Skolem functions. Apply the Ramsey property to N^+ to obtain $\Psi \geq \Phi$. Now in GEM(J, Ψ), with skeleton $\langle \bar{a}_j : j \in J \rangle$, we have that

$$\{\bar{F}(\bar{a}_{\bar{s}}) : \bar{s} \in Q^J_{\mathfrak{r}_1}\}$$

is a $\mathcal{K}$-indiscernible sequence, necessarily contained in GEM(J, Ψ), and it witnesses the $(\emptyset, I_1)$-shearing of the formula $\varphi(x, \bar{F}(\bar{a}_{\bar{s}_1}))$, as desired. The last line of the claim follows by the Ramsey property, *because* $\mathbf{b}$ was a $\mathcal{K}$-indiscernible sequence. □

Theorem 6.14. *Suppose* $\mathbf{c}$ *is a countable context and assume* T *is* $\mathbf{c}$*-unsuperstable. For every* $\Phi \in \Upsilon_{\mathbf{c}}[T]$ *there is* $\Phi_* \geq \Phi$ *with* $|\tau(\Phi_*)| \leq |T| + |\tau_\Phi| + \aleph_0$ *such that for every* $\Psi \geq \Phi_*$ *we have* $\mathrm{GEM}_{\tau(T)}(I, \Psi)$ *is not* $\aleph_1$*-saturated.*

Proof. We will write $I = I_{\mathbf{c}}$ and $\mathcal{K} = \mathcal{K}_{\mathbf{c}}$ for the duration of the proof.[8] Let $M = \mathrm{GEM}(I, \Phi)$, and let $J \supseteq I$ be some $\aleph_0$-saturated member of $\mathcal{K}$.

Let $\langle I_n : n < \omega \rangle$, $\langle \bar{s}_n : n < \omega \rangle$, $\langle B_n : n < \omega \rangle$, and a partial type p over B be given from Definition 6.2 to witness the $\mathbf{c}$-unsuperstability of T. To review and fix notation, this means we start with the data of:

(i) $I_0 \subseteq \cdots \subseteq I_n \cdots$ are an increasing sequence of finite subsets of I, with $\bigcup_n I_n = I$. Without loss of generality (see 5.6) $I_0 = \emptyset$.
For each n, $\bar{s}_n$ is an enumeration of I_n, and let us assume $\bar{s}_n \trianglelefteq \bar{s}_{n+1}$.

(ii) $B_0 \subseteq \cdots \subseteq B_n \cdots$ are an increasing sequence of nonempty, possibly infinite, sets of parameters in the monster model for T, with $B := \bigcup_n B_n$. For each n, there is a formula $\varphi_n(\bar{x}, \bar{c}_n)$ witnessing that $p \restriction B_n$ is a type which (I_{n-1}, I_n)-shears over B_{n-1}.

For simplicity in the present proof, without loss of generality:

(iii) Since p is allowed to be a partial type, we may assume that:

$$p = \{\varphi_n(\bar{x}, \bar{c}_n) : 1 \leq n < \omega\}.$$

(iv) Moreover, since a sequence which is $\mathcal{K}$-indiscernible over some C remains $\mathcal{K}$-indiscernible over $C' \subseteq C$, we may assume[9]: that $B_0 = \{\bar{c}_{0,\ell} : \ell < \lg(\bar{c}_0)\}$ and for each $n \geq 1$, $B_n = B_{n-1} \cup \{\bar{c}_{n,\ell} : \ell < \lg(\bar{c}_n)\}$.

(v) Finally, for each n, to fix notation, say the shearing of φ_n is witnessed by $\mathbf{b}^n = \langle \bar{b}^n_{\bar{s}} : \bar{s} \in {}^{\omega >}(J[I_{n-1}]) \rangle$ in the monster model, which is $\mathcal{K}$-indiscernible over B_{n-1}. Note in particular that $\bar{b}^n_{\bar{s}_n} = \bar{c}_n$. (We could also have asked that $\bar{c}_{n-1} \trianglelefteq \bar{c}_n$.)

As our notation suggests, without loss of generality the $\aleph_0$-saturated sequence $J \supseteq I$ indexing each $\mathbf{b}^n$ is the same J mentioned in the second line of the proof. As before, we may consider $M \restriction \tau(T)$ and B as subsets of the monster model, and a priori they may have nothing to do with each other.[10]

By induction on $n < \omega$ we will define Φ_n and q_n (and auxiliary objects $\mathfrak{r}_n$, $\bar{F}_{\mathfrak{r}_n}$, G_n), which will be objects of the following kind.

for each $n \geq 0$,

(a) $\Phi_n \in \Upsilon_{\mathbf{c}}[T]$, and $m < n$ implies $\Phi_m \leq \Phi_n$, and $\Phi_0 = \Phi$.

(b) $|\tau(\Phi_n)| \leq |\tau(\Phi_0)| + |T|$

for each $n \geq 1$,

(b) q_n is a partial type of $\mathrm{GEM}_{\tau(T)}(I, \Phi_n)$, which (I_{n-1}, I_n)-shears over the submodel $\mathrm{GEM}_{\tau(T)}(I_{n-1}, \Phi_{n-1})$, witnessed by the sequence

$$\langle \bar{F}_{\mathfrak{r}_n}(\bar{a}_{\bar{s}}) : \bar{s} \text{ is from } J \text{ and realizes } \mathrm{tp}_{\mathrm{qf}}(\bar{s}_n, \bar{s}_{n-1}, I) \rangle$$

[8] Informal proof summary: suppose $p = \{\varphi_n(x, \bar{c}_n) : n < \omega\}$ witnesses $\mathbf{c}$-unsuperstability. At stage n, copy the nth instance of shearing into the model $\mathrm{GEM}(J, \Phi_{n-1})$, over what came before. The 'copy' in $\mathrm{GEM}(J, \Phi_n)$ of the $\mathcal{K}$-indiscernible sequence witnessing this instance of shearing has the same type as the 'original', over what came before. Take a partial automorphism of the monster model fixing the parameters so far, sending the original sequence onto the copy. Replace p by its image under this map; note the first $n-1$ formulas don't change. Keep going.

[9] recall notation: if $\lg(\bar{c}) = k$, then $\{\bar{c}_\ell : \ell < k\}$ is simply the set of its elements.

[10] Most of the time we will work in $\mathfrak{C}_T$, though when we are expanding before applying the Ramsey property, it is really $\mathfrak{C}_{T_\Phi}$. Perhaps it is best to consider that $\mathfrak{C}_T$ denotes "$\mathfrak{C}_{T_\Phi} \restriction \tau(T)$." Note also that there there is no connection asserted between the various J-indexed sequences witnessing dividing, or between these and possible skeletons of GEM models. Moreover, the types of the various J-indexed sequences $\mathbf{b}^n$ could certainly be different.

where $\bar{F}_{\mathfrak{r}_n}$ is a sequence of $\ell(\bar{b}^n_{\bar{s}_n})$-many function symbols of $\tau(\Phi_n)$, each of arity $\ell(\bar{a}_{\bar{s}_n})$, where here $\langle a_j : j \in J\rangle$ denotes the skeleton.

(c) each q_n is contained in the image of p under some partial automorphism of $\mathfrak{C}$, and $m < n$ implies $q_m \subseteq q_n$. In slight abuse of notation,

$$q_n = G_n(G_{n-1}(\cdots G_1(\ \{\varphi_m(\bar{x}, \bar{c}_m) : m \leq n\}\ )\cdots))$$

where the G's are from item (f).

for each $n \geq 1$, auxiliary objects used in the construction:

(d) $\mathfrak{r}_n \in \mathbf{D}_{\rm qf}(J)$.

(e) $\bar{F}_{\mathfrak{r}_n}$ are new function symbols of $\tau(\Phi_n)$ with domain $Q^J_{\mathfrak{r}_n}$, see (b).

(f) G_n is a partial automorphism of $\mathfrak{C}_T$, see (c). The domain of G_1 includes each of the shearing sequences $\mathbf{b}^i$ for $i < \omega$, *thus* it also includes the domain B of p. The domain of G_n includes the range of G_{n-1}. Finally, for $m < n$, G_n leaves $\bar{c}_m$ fixed, in slight abuse of notation $G_n \restriction \bar{c}_m = G_m \restriction \bar{c}_m$.

Note that it will follow from (b) that q_n shears "up to stage n" in $\mathrm{GEM}(J, \Phi_n)$, it will follow from (c) that q_n extends to a partial type which continues to witness $\mathbf{c}$-unsuperstability, and as we shall verify, it will follow from both that $\bigcup_n q_n$ is a partial type of $\mathrm{GEM}(J, \bigcup_n \Phi_n)$, or indeed of $\mathrm{GEM}(I, \bigcup_n \Phi_n)$, which witnesses $\mathbf{c}$-unsuperstability, in a precise sense verified at the end of the proof. When the skeleton is indexed by I instead of J, the shearing sequences may not be contained in the model, but the parameters for the type are, which will be sufficient.

Stage $n = 0$. Let $\Phi_0 = \Phi$.

Stage $n = 1$. Let $N_1 = \mathrm{GEM}(J, \Phi_0)$ with skeleton $\mathbf{a}$. The formula

$$\varphi_1(\bar{x}, \bar{c}_1) \tag{13}$$

$(\emptyset, \bar{s}_1)$-shears over B_0. The shearing is witnessed by $\mathbf{b}^1$, and in particular, $\bar{b}^1_{\bar{s}_1} = \bar{c}_1$. Let $\mathfrak{r}_1 = \mathrm{tp}_{\rm qf}(\bar{s}_1, \emptyset, I)$. Let $\bar{F}_{\mathfrak{r}_1}$ be a sequence of $\ell(\bar{b}^1_{\bar{s}_1})$-many new $\ell(\bar{a}_{\bar{s}_1})$-ary function symbols. Considering N_1 as an elementary submodel of $\mathfrak{C}_{T_\Phi}$, interpret $\bar{F}_{\mathfrak{r}_1}$ so that

$$\bar{F}_{\mathfrak{r}_1}(\bar{a}_{\bar{s}}) = \bar{b}^1_{\bar{s}} \ \text{ for each } \bar{s} \in Q^J_{\mathfrak{r}_1}.$$

(So in particular $\bar{F}_{\mathfrak{r}_1}(\bar{a}_{\bar{s}_1}) = \bar{b}^1_{\bar{s}_1} = \bar{c}_1$.) Let $N_{1,1}$ be an elementary extension of N_1 in this larger language, which is closed under the functions $\bar{F}_{\mathfrak{r}_1}$ and which is then also expanded to have Skolem functions.

Apply the Ramsey property to $N_{1,1}$, $\mathbf{a}$, Φ_0. We obtain $\Phi_1 \geq \Phi_0$, and the functions from $\bar{F}_{\mathfrak{r}_1}$ are in $\tau(\Phi_1)$. Because of the Skolem functions, Φ_1 is nice. In $\mathrm{GEM}(J, \Phi_1)$, the sequence

$$\langle F_{\mathfrak{r}_1}(\bar{a}_{\bar{s}}) : \bar{s} \in Q^J_{\mathfrak{r}_1}\rangle \tag{14}$$

need no longer be identical to $\mathbf{b}^1$, but because of the reflection clause in the Ramsey property and the fact that $\mathbf{b}^1$ is $\mathcal{K}$-indiscernible, this new sequence is also $\mathcal{K}$-indiscernible and will have the same type as $\mathbf{b}^1$. Let G_1 be a partial automorphism of $\mathfrak{C}_T$ whose domain includes B, which sends

$$\langle \bar{b}^1_{\bar{s}} : \bar{s} \in Q^J_{\mathfrak{r}_1}\rangle$$

to the sequence from (14) in the natural way, i.e.

$$\bar{b}^1_{\bar{s}} \mapsto \bar{F}^{\mathrm{GEM}(J,\Phi_1)}_{\mathfrak{r}_1}(\bar{a}_{\bar{s}}) \qquad \text{for } \bar{s} \in Q^J_{\mathfrak{r}_1}.$$

In slight abuse of notation, write $G_1(p)$ for the image of the partial type p after applying G_1. Then $G_1(p)$ is a type of $\mathbf{S}(G_1(B))$ such that for every $n > 0$, we have that $G_1(p) \restriction G_1(B_n)$ is a type which (I_{n-1}, I_n)-shears over $G_1(B_{n-1})$, and moreover for $n = 1$ this shearing is witnessed by the sequence (14) and the formula

$$\varphi_1(\bar{x}, G_1(\bar{b}^1_{\bar{s}_1})) \equiv \varphi_1(\bar{x}, \bar{F}_{\mathfrak{r}_1}(\bar{a}_{\bar{s}_1})) \equiv \varphi_1(\bar{x}, G_1(\bar{c}_1)).$$

We define

$$q_1 = \{\varphi_1(\bar{x}, G_1(\bar{c}_1))\}$$

[i.e. the G_1-image of (13)] so as $\bar{s}_1$ is a sequence of elements of I, by the equivalence just given, q_1 is a partial type of $\mathrm{GEM}_{\tau(T)}(I, \Phi_1)$ which $(\emptyset, I_1)$-shears over $\emptyset$, and is contained in an automorphic image of p. This completes the base stage.

Stage $n = k + 1$**.** As the stage begins, we have a template Φ_{n-1}, partial automorphisms $G_1, \ldots, G_{n-1}$, and a partial type[11] q_{n-1} of $\mathrm{GEM}_{\tau(T)}(I, \Phi_{n-1})$, such that

$$\begin{aligned} q_{n-1} = &\{\ \varphi_1(\bar{x}, G_1(\bar{c}_1))\ \} \cup \{\ \varphi_2(\bar{x}, G_2(G_1((\bar{c}_2)))\ \} \cup \cdots \\ &\cdots \cup \{\ \varphi_{n-1}(\bar{x}, G_{n-1}(G_{n-2}(\cdots G_1(\bar{c}_{n-1})))\ )\ \}. \end{aligned}$$

By inductive hypothesis,

- for each $1 \leq j < n$, q_j is a partial type with parameters from the submodel $\mathrm{GEM}_{\tau(T)}(I_j, \Phi_j)$;
- q_j is a partial type which (I_{j-1}, I_j)-shears over $\mathrm{GEM}_{\tau(T)}(I_{j-1}, \Phi_{j-1})$.
- if we consider q_j as a partial type in $\mathrm{GEM}_{\tau(T)}(J, \Phi_j)$, there are a sequence of function symbols $\bar{F}_{\mathfrak{r}_j}$ of $\tau(\Phi_j)$ such that in this model,

  $$\langle \bar{F}_{\mathfrak{r}_j}(\bar{a}_{\bar{s}}) \ :\ \bar{s}_{j-1} \frown \bar{s} \in Q^J_{\mathfrak{r}_j} \rangle$$

  witnesses this shearing.

To simplify notation, locally in this stage, write G to abbreviate the composition $G_{n-1} \circ G_{n-2} \circ \cdots \circ G_1$. Let $M_n = \mathrm{GEM}(I, \Phi_{n-1})$. Let $N_n = \mathrm{GEM}(J, \Phi_{n-1})$. As G is a partial automorphism, $\varphi_n(\bar{x}, G(\bar{c}_n))$ is a formula which is consistent with q_k and which (I_{n-1}, I_n)-shears over $G(B_{n-1})$,[12] as witnessed by $G(\mathbf{b}^n)$. Let

$$\mathfrak{r}_n = \mathrm{tp}_{\mathrm{qf}}(\bar{s}_n, \emptyset, I).$$

Let $\bar{F}_{\mathfrak{r}_n}$ be a sequence of $\ell(\bar{b}^n_{\bar{s}_n})$-many new $\ell(\bar{a}_{\bar{s}_n})$-ary function symbol(s). Considering N_n as an elementary submodel of $\mathfrak{C}_{T_\Phi}$, and recalling that $\bar{s}_{n-1}$ is the distinguished enumeration of I_{n-1} (and an initial segment of $\bar{s}_n$), interpret them as follows. First, we consider elements coming from the skeleton of the form $\bar{a}_{\bar{s}}$ where $\bar{s} \in Q^J_{\mathfrak{r}}$ and $\bar{s}_{n-1} \trianglelefteq \bar{s}$. In this case, interpret so that

$$\bar{F}_{\mathfrak{r}_n}(\bar{a}_{\bar{s}}) = G(\bar{b}^n_{\bar{s}}) \ \text{ for each } \bar{s} \in Q^J_{\mathfrak{r}_n} \text{ such that } \bar{s}_{n-1} \trianglelefteq \bar{s}.$$

Next, we consider elements coming from the skeleton of the form $\bar{a}_{\bar{t}}$ where $\bar{t} \in Q^J_{\mathfrak{r}_n}$ and $\bar{t} \restriction \ell(\bar{s}_{n-1}) = \bar{s}'$ for some $\bar{s}' \in Q^J_{\mathfrak{r}_{n-1}}$ possibly different from $\bar{s}_{n-1}$. For each such $\bar{s}'$, fix some partial automorphism of the monster model $H_{\bar{s}_{n-1} \mapsto \bar{s}'}$ whose domain

[11] i.e. in the slight abuse of notation from above, q_k is $G_1(\{\varphi_1\}) \cup G_2(G_1(\{\varphi_2\})) \cup \cdots \cup G_{n-1}(\cdots G_1(\{\varphi_{n-1}\})\ )$. Or recalling item (f) of the list at the beginning of the proof, q_k is just $G_{n-1}(G_{n-2}(\cdots(\{\varphi_1, \ldots, \varphi_{n-1}\})\cdots))$.

[12] Note that the set $G(B_{n-1})$ includes the domain of q_k. However, the set $G(B_{n-1})$ certainly need not include the algebraic closure of the domain of q_k, such as $\mathrm{GEM}(I_k, \Phi_k)$. Indiscernibility over this possibly larger or possibly different set will be guaranteed only after we let the Ramsey property make a better choice of $\mathcal{K}$-indiscernible sequence for us.

includes $G(B) \cup N_n \cup G(\mathbf{b}^n)$ and which extends the automorphism of N_n induced by sending $\bar{s}_{n-1}$ to $\bar{s}'$ in the index model. Then, for this fixed $\bar{s}'$, interpret $F_{\mathfrak{r}_n}$ so that

$$\bar{F}_{\mathfrak{r}_n}(\bar{a}_{\bar{s}}) = H_{\bar{s}_{n-1} \mapsto \bar{s}'}(G(\bar{b}^n_{\bar{s}})) \text{ for each } \bar{s} \in Q^J_{\mathfrak{r}_n} \text{ with } \bar{s}' \trianglelefteq \bar{s}.$$

Note that the reason to do the parallel expansion for all $\bar{s}' \in Q^J_{\mathfrak{r}_n}$ is so that the Ramsey property will record the type of the $\mathcal{K}$-indiscernible sequence correctly, over each $\bar{a}_{\bar{s}'}$. Let $N_{n,1}$ be an elementary extension of N_n in this larger language, which is closed under $F_{\mathfrak{r}_n}$ and which is then also expanded to have Skolem functions. Apply the Ramsey property to $N_{n,1}$, $\mathbf{a}$, Φ_{n-1}. We obtain $\Phi_n \geq \Phi_{n-1}$. Note that $F_{\mathfrak{r}_n} \in \tau(\Phi_n)$. Again because of the Skolem functions, we are assured Φ_n is nice.

Just as in the base case, in the model $\mathrm{GEM}(J, \Phi_n)$, the sequence

$$\langle F_{\mathfrak{r}_n}(\bar{a}_{\bar{s}}) : \bar{s} \in Q^J_{\mathfrak{r}_n},\ \bar{s}_{n-1} \trianglelefteq \bar{s} \rangle \tag{15}$$

need no longer be identical to the sequence

$$\langle G(\bar{b}^n_{\bar{s}}) \ :\ \mathrm{tp}_{\mathrm{qf}}(\bar{s}, \bar{s}_{n-1}, J) = \mathrm{tp}_{\mathrm{qf}}(\bar{s}_n, \bar{s}_{n-1}, I) \rangle \tag{16}$$

but because of the reflection clause in the Ramsey property and the fact that (16) is $\mathcal{K}$-indiscernible, this new sequence (15) is also $\mathcal{K}$-indiscernible and will have the same type as (16). [We really use the $J[I_{n-1}]$ in the definition of $\mathbf{b}_n$ from Definition 5.2 here: the $\mathcal{K}$-indiscernible sequence we use is indiscernible over $\bar{a}_{\bar{s}_{n-1}}$, i.e. "over I_{n-1}".] Let G_n be a partial automorphism of $\mathfrak{C}_T$ which is the identity on the domain of q_{n-1}, whose domain includes B, and which sends (16) to (15) in the natural way:

$$G(\bar{b}^n_{\bar{s}}) \mapsto F_{\mathfrak{r}_n}(\bar{a}_{\bar{s}}).$$

Then $q_n = G(p_{n-1} \cup \{\varphi_n\})$ is a partial type of $\mathrm{GEM}(I, \Phi_n)$, indeed of $\mathrm{GEM}(I_n, \Phi_n)$, which satisfies the inductive hypothesis.

This completes the inductive step, and so the induction.

Verification. Let $\Psi \geq \Psi_* = \bigcup_n \Phi_n$ and let $q = \bigcup_n q_n$. Let $M = \mathrm{GEM}(I, \Psi)$ and let $N = \mathrm{GEM}(J, \Psi)$. Then q is a partial type of $M \restriction \tau(T)$. Let us show it is not realized in M. Assume for a contradiction that it were realized, say by $\bar{d}$. Then for some $k < \omega$, $\bar{d} \subseteq \mathrm{GEM}(I_k, \Phi_k)$. We know that in $\mathrm{GEM}(J, \Phi_{k+1})$ there is a formula of q which (I_k, I_{k+1})-shears over $\mathrm{GEM}(I_k, \Phi_k)$. Since the sequence $\bar{d}$ cannot realize the type in this larger model N, a fortiori it cannot realize the type in the smaller model M. $\square$

Remark 6.15. *The proof of Theorem* 6.14 *builds a type p which does not have a weak $\bar{t}_*$-definition for any finite $\bar{t}_*$ in I, and moreover cannot have one in any* $\mathrm{GEM}(I, \Psi)$ *for $\Psi \geq \Phi_*$. The failure of $\aleph_1$-saturation in* $\mathrm{GEM}_{\tau(T)} I, \Psi)$ *for any $\Psi \geq \Phi_*$ will always be due to this p (of course other types may be omitted as well).*

Corollary 6.16 (Local unsuperstability)**.** *Let Δ be a set of formulas of T. Suppose $\mathbf{c}$ is a countable context, and assume (T, Δ) is $\mathbf{c}$-unsuperstable. For every $\Phi \in \Upsilon_{\mathbf{c}}[T]$ there is $\Phi_* \geq \Phi$ with $|\tau(\Phi_*)| \leq |T| + |\tau_\Phi| + \aleph_0$ such that for every $\Psi \geq \Phi_*$ we have* $\mathrm{GEM}_{\tau(T)}(I, \Psi)$ *is not $\aleph_1$-saturated, in particular, it will omit a Δ-type over a countable set.*

Discussion 6.17. In 6.14 and 6.16 we make no assumptions on the size of the language. We can require $|\tau(\Phi_*)| \leq |\tau(\Phi)| + |T|$, with no requirement on $\tau(\Psi)$.

Claim 6.18. *There exists a countable context* $\mathbf{c}$ *such that if* T *is any theory which is supersimple in the usual sense, then* T *is* $\mathbf{c}$*-superstable, and if* T' *is any theory which is not supersimple in the usual sense, then* T *is* $\mathbf{c}$*-unsuperstable.*

Proof. The existence of such a $\mathbf{c}$ is given by Theorem 9.8 below. □

Discussion and examples. Several examples may shed light on the interaction between ambient shearing and its appearance in GEM-models. We continue under Hypothesis 6.1.

From the proofs of this section, we see:

Conclusion 6.19. *In determining whether a theory* T *is* $\mathbf{c}$*-superstable, for a countable context* $\mathbf{c}$*, it suffices to consider* m*-types for some fixed* $m < \omega$*, e.g.* $m = 1$*.*

Proof. We have established an equivalence in terms of saturation, and for saturation this is true. □

Example 6.20. *It may be the case that for every nonalgebraic formula* $\varphi(x, c)$ *of some* $\mathrm{GEM}_{\tau(T)}(I, \Phi)$*, if we write* φ *as* $\varphi(x, \bar{\sigma}(\bar{a}_{\bar{t}}))$ *for some sequence* $\bar{\sigma}$ *of* $\tau(\Phi)$*-terms and* $\bar{a}_{\bar{t}}$ *of the appropriate length, then*

$$\{\varphi(x, \bar{\sigma}(\bar{a}_{\bar{s}})) : \mathrm{tp}_{\mathrm{qf}}(\bar{s}, \emptyset, J) = \mathrm{tp}_{\mathrm{qf}}(\bar{t}, \emptyset, I)\}$$

is consistent in $\mathrm{GEM}(J, \Phi)$ *for every* $J \supseteq I$*,* even though *some of these formulas divide, thus shear.*

Proof. Let T be the theory of an equivalence relation with infinitely many infinite classes. By quantifier elimination it suffices to consider $\varphi(x, c) = E(x, c)$. Suppose we have set up the GEM-model M so that the skeleton $\mathbf{a}$ is an infinite sequence within a single class, with Skolem functions $\langle F_n : n < \omega \rangle$ interpreted so that F_n^M copies $\mathbf{a}$ over to the n-th class. Then φ clearly divides (so a fortiori shears), and even does so along an indiscernible sequence in $\mathrm{GEM}_{\tau(T)}(I, \Phi)$. Still, the set of formulas in the statement will remain consistent since, in M, no two s, t which share a quantifier-free type satisfy $\neg E(a_s, a_t)$. However, the proof of Theorem 6.14 shows that we may find $\Psi \geq \Phi$ in which an analogous instance of dividing does occur "along the skeleton". Informally, first choose an elementary extension of M in which there are many nonstandard classes, interpret a new function symbol G which maps each a_t from $\mathbf{a}$ to an element in a distinct nonstandard E-class, and apply the Ramsey property to obtain Ψ. Then letting t be any nonalgebraic element of any $\aleph_0$-saturated $J \supseteq I$ and $\varphi(x, c)$ be any formula of $\mathrm{GEM}_{\tau(T)}(I, \Phi)$ such that $c = G(a_t)$ for some $t \in I$, the sequence $\langle G(a_s) : \mathrm{tp}_{\mathrm{qf}}(s, \emptyset, J) = \mathrm{tp}_{\mathrm{qf}}(t, \emptyset, I)$ will be in distinct E-equivalence classes, and so will witness the dividing of $\varphi(x, c) = E(x, c)$. □

This example may be easily modified with finitely many equivalence relations to produce examples where the given GEM-model does or does not "witness" the natural "superstability rank" (for dividing or shearing), and even more, showing the importance of the template Ψ in "witnessing" shearing:

Example 6.21. *Let* $\mathbf{c}$ *be a countable context, so* $I = I_{\mathbf{c}}$*. It may be the case that for every nonalgebraic formula* $\varphi(x, c)$ *of some* $\mathrm{GEM}_{\tau(T)}(I, \Phi)$*, if we write* φ *as* $\varphi(x, \bar{\sigma}(\bar{a}_{\bar{t}}))$ *for some sequence* $\bar{\sigma}$ *of* $\tau(\Phi)$*-terms and* $\bar{a}_{\bar{t}}$ *of the appropriate length, then*

$$\{\varphi(x, \bar{\sigma}(\bar{a}_{\bar{s}})) : \mathrm{tp}_{\mathrm{qf}}(\bar{s}, \emptyset, J) = \mathrm{tp}_{\mathrm{qf}}(\bar{t}, \emptyset, I)\}$$

is consistent in $\mathrm{GEM}(J, \Phi)$ *for every* $J \supseteq I$, even though T *is not superstable for the context* $\mathbf{c}$.

Proof. Let T be the theory of infinitely many equivalence relations, $\{E_i : i < \omega\}$, where each E_i has infinitely many infinite classes and for each $i < \omega$, each E_{i+1}-class is the union of infinitely many E_i-classes. This theory is stable but not superstable. The previous example extends naturally to this case, provided we have set up the GEM-model M so that the skeleton $\mathbf{a}$ is an infinite sequence within a single class for each E_i, with Skolem functions $\langle F_{n,i} : n < \omega\rangle$ interpreted so that $F^M_{n,i}$ copies $\mathbf{a}$ over to the n-th class of the i-th equivalence relation. □

7. The separation theorem

Theorem 7.1 (Separation Theorem)**.** *Let* T_0, T_1 *be any two theories, without loss of generality in disjoint signatures, and of any size. Suppose there exists a countable context* $\mathbf{c}$ *such that* T_0 *is* $\mathbf{c}$*-superstable and* T_1 *is* $\mathbf{c}$*-unsuperstable. Then for any* $\Phi \in \Upsilon_{\mathbf{c}}$*, for arbitrarily large* μ*, there exists* $\Psi \geq \Phi$ *with* $\Psi \in \Upsilon_{\mathbf{c}}[T_0] \cap \Upsilon_{\mathbf{c}}[T_1]$ *such that writing* $M = \mathrm{GEM}(I, \Psi)$*, we have that* $M \restriction \tau(T_0)$ *is* μ*-saturated but* $M \restriction \tau(T_1)$ *is not* $\aleph_1$*-saturated.*

Proof. Let κ be any infinite cardinal. Let $\Phi \in \Upsilon_{\mathbf{c}}$ be given. By applying Observation 6.8 twice, if necessary, we may assume $\Phi \in \Upsilon_{\mathbf{c}}[T_0] \cap \Upsilon_{\mathbf{c}}[T_1]$. By Theorem 6.14, we may find $\Phi_1 \geq \Phi$ such that $\mathrm{GEM}_{\tau(T_1)}(I, \Phi')$ will not be $\aleph_1$-saturated for any $\Phi' \geq \Phi_1$. Next, choose μ and λ so that $\mu \geq \kappa$ and μ, λ satisfy the hypotheses of Claim 6.11. Apply Claim 6.11 to find $\Phi_2 \geq \Phi_1$ so that $\mathrm{GEM}_{\tau(T_0)}(I, \Phi_2)$ is μ-saturated. Then $\Psi = \Phi_2$ is as desired. □

Corollary 7.2. *Let* $\mathbf{c} = (I, \mathcal{K})$ *be a countable context. If* T_0 *is superstable for* $\mathbf{c}$ *and* T_1 *is not superstable for* $\mathbf{c}$*, for* every *theory* T_* *interpreting both of them (without loss of generality in disjoint signatures), and for arbitrarily large* μ*, there is a model* $M_* \models T_*$ *such that* $M_* \restriction \tau(T_0)$ *is* μ*-saturated but* $M_* \restriction \tau(T_1)$ *is not* $\aleph_1$*-saturated.*

Note that one genre of corollary of the Separation Theorem is to point out various constraints on models arising as GEM models.

Conclusion 7.3. *If* $\mathbf{c}$ *is a countable context and* T *is* $\mathbf{c}$*-unsuperstable, then for some* Φ*, for every* $\Psi \geq \Phi$*, no* GEM *model* $\mathrm{GEM}(I_{\mathbf{c}}, \Psi) \models T$ *is* $\aleph_1$*-saturated, already when restricted to* $\tau(T)$*.*

8. Consequences for $\trianglelefteq^*$

In this section we apply 7.2 to obtain a series of results about the structure of the interpretability order $\trianglelefteq^*_1$. The setup suggests a method for obtaining various further results. We emphasize that all results are in ZFC.

Convention 8.1. *All theories in this section are complete.*

Recall that $\trianglelefteq^*_1$ means (for readers used to all three subscripts): $\trianglelefteq^*_{\lambda,\chi,\kappa}$, i.e. for all sufficiently large λ, for $\chi = |T_0| + |T_1|$, for $\kappa = 1$ (so "for every 1-saturated

model" abbreviates "for every model"). For a complete definition and motivation, see sections 1-2 of [11].

First we recall a fact which spells out the sense in which $\trianglelefteq_1^*$ naturally refines Keisler's order $\trianglelefteq$. In the context of Keisler's order, writing $\trianglelefteq_\lambda$ means that we restrict to regular ultrafilters on λ.

Fact 8.2 ([11] Corollary 2.11)**.** *If for arbitrarily large λ we have $T_0 \trianglelefteq_\lambda T_1$ in Keisler's order, then $\neg(T_1 \trianglelefteq_1^* T_0)$.*

Claim 8.3. *Let T_0, T_1 be complete countable theories and let $\mathbf{c}$ be a countable context. Suppose T_0 is $\mathbf{c}$-superstable and T_1 is $\mathbf{c}$-unsuperstable for $\mathbf{c}$. Then*

$$\neg(T_1 \trianglelefteq_1^* T_0).$$

Proof. This just applies Corollary 7.2 to the definition of $\trianglelefteq_1^*$. □

Corollary 8.4. *Let T_0, T_1 be complete countable theories.. Suppose T_0 is supersimple and T_1 is not supersimple. Then*

$$\neg(T_1 \trianglelefteq_1^* T_0).$$

Proof. Claim 8.3 and Claim 6.18. □

For the next few results, Keisler's order is invoked in the proofs so we restrict to countable theories (for which Keisler's order is defined).

Lemma 8.5. *Let T_i be superstable with the fcp and let T_j be strictly stable nfcp. Then T_i, T_j are $\trianglelefteq_1^*$-incomparable.*

Proof. $T_j \trianglelefteq_\lambda T_i$ for arbitrarily large λ in Keisler's order, so by 8.2 $\neg(T_i \trianglelefteq_1^* T_j)$. For the other direction, let $\mathbf{c}$ be from 6.18. Then T_i is $\mathbf{c}$-superstable and T_j is $\mathbf{c}$-unsuperstable, so by 8.3, $\neg(T_j \trianglelefteq_1^* T_i)$. □

Conclusion 8.6. *The interpretability order $\trianglelefteq_1^*$ is not linear even on the stable theories.*

Theorem 8.7. *Let T_i, T_j be complete countable theories. Suppose T_i is strictly stable. Suppose T_j is supersimple unstable. Then T_i and T_j are $\trianglelefteq_1^*$-incomparable.*

Proof. We know $T_i \triangleleft_\lambda T_j$ in Keisler's order for arbitrarily large λ, so $\neg(T_j \trianglelefteq_1^* T_i)$. For the other direction, let $\mathbf{c}$ be from 6.18. T_i is not $\mathbf{c}$-superstable, because it is strictly stable, but T_j is $\mathbf{c}$-superstable, because it is supersimple. So by 8.3, $\neg(T_j \trianglelefteq_1^* T_i)$. □

Theorem 8.7 has various immediate, but perhaps more quotable, corollaries. Recall from [11] that $T_{\mathbf{rg}}$ (theory of the random graph) is $\trianglelefteq_1^*$-minimum among unstable theories.

Conclusion 8.8. *It is not the case that all stable theories are below all unstable theories in $\trianglelefteq_1^*$.*

Corollary 8.9. *Let T be countable and strictly stable. Then T and $T_{\mathbf{rg}}$ are $\trianglelefteq_1^*$-incomparable.*

Corollary 8.10. *Let T be Hrushovski's strictly stable $\aleph_0$-categorical pseudoplane. Then T and $T_{\mathbf{rg}}$ are $\trianglelefteq_1^*$-incomparable.*

Conclusion 8.11. *The order $\trianglelefteq_1^*$ is not linear even on the countable $\aleph_0$-categorical theories. Moreover, it is not linear even on countable $\aleph_0$-categorical graphs.*

Discussion 8.12. Of course, a priori we do not know that $\neg(T \trianglelefteq_1^* T')$, or even just $\trianglelefteq$, means that there is a way to see the difference via superstability for some context. One could naturally define a new triangle ordering saying that T_1 below T_2 means that if T_2 is $\mathbf{c}$-superstable for some countable context $\mathbf{c}$ then so is T_1.

9. Simple and supersimple

In this section we characterize the pairs (T, Δ) which are $\mathbf{c}$-superstable for *some* countable context $\mathbf{c}$.

Discussion 9.1. In [11] Lemma 7.10, we proved that that for any simple theory T with $\kappa(T) \leq \kappa$, for arbitrarily large μ, for a certain context (which took as a parameter κ), for every Φ, there was $\Psi \geq \Phi$ such that $M = \mathrm{GEM}_{\tau(T)}(I, \Psi)$ is μ-saturated. Notice that this does not contradict the results of §6, since that section used countability of the context in an essential way (informally, this is the case "$\kappa(T) = \aleph_0$"). Some further remarks in this line are given in §10.

We will use the following index model class, which is Ramsey, [11] Fact 3.20. Note that we demand $\mathcal{K}_{\mathbf{c}}$ have a linear order.

Definition 9.2.

(1) *$\mathcal{K}^{tr}_\kappa$ is the class of trees with κ levels and lexicographic order which are normal, meaning that a member η at a limit level is determined by*

$$\{\nu : \nu \trianglelefteq \eta\}.$$

(So the tree has the function $\cap(\eta, \nu) = \min\{\rho : \rho \trianglelefteq \nu, \rho \trianglelefteq \eta\}$.)

(2) *We call $I \in \mathcal{K}^{tr}_\kappa$ standard when the ith level, P^I_i, of I consists of sequences of length i and $n \in P_i$, $j < i$, $\eta \restriction j \in P_j$ and $\eta \restriction j \trianglelefteq_I \eta$, so every $I \in \mathcal{K}^{tr}$ is isomorphic to a standard one (this is justified by the assumption of normality).*

Fact 9.3 ([11] 7.12, update)**.** *Let $I \in \mathcal{K}^{tr}_\kappa$ be standard with universe ${}^{\omega>}\{0\}$. Suppose Δ is a set of formulas of T such that every Δ-type in every model of T does not fork over some finite set. Then for every $\Phi \in \Upsilon$, there is $\Psi \in \Upsilon$ with $\Phi \leq \Psi$ such that $M = \mathrm{GEM}_{\tau(T)}(I, \Psi)$ is μ-saturated for Δ-types.*

We will need the following local update of [11] Lemma 7.12.

Definition 9.4. *Recall that $\kappa_{\mathrm{loc}}(T) = \aleph_0$ means that for every formula φ, every φ-type does not fork over a finite set, while $\kappa(\varphi, T) = \aleph_0$ means that every φ-type does not fork over a finite set.*

Lemma 9.5. *Let T be any complete theory and let φ be a formula of T which is simple and $\kappa(\varphi, T) = \aleph_0$. Then $(T, \{\varphi, \neg\varphi\})$ is $\mathbf{c}$-superstable for the countable context $({}^{\omega>}\{0\}, \mathcal{K}^{tr}_\kappa)$.*

Proof. Let T, φ be given and let $\Delta = \{\varphi, \neg\varphi\}$. Let $\mathbf{c} = (I, \mathcal{K})$ where $\mathcal{K} = \mathcal{K}^{tr}_\kappa$ and I is standard with universe ${}^{\omega>}\{0\}$, so is a single branch. Suppose for a contradiction that (T, Δ) is not $\mathbf{c}$-superstable. Then Fact 9.3 gives us a Φ_* such that for every $\Psi \geq \Phi_*$, $\mathrm{GEM}_{\tau(T)}(I, \Psi)$ omits some φ-type over a countable set. For $\Psi \geq \Phi_*$ given by Fact 9.3, we get a contradiction. $\square$

Theorem 9.6. *Let T be first order complete and φ a formula of T. Then the following are equivalent:*

(a) φ is simple and $\kappa(\varphi, T) = \aleph_0$.
(b) $\mathbf{c}$-supersimple where $\mathcal{K}_{\mathbf{c}} = \mathcal{K}^{tr}_{\kappa}$ and $I_{\mathbf{c}} = {}^{\omega>}\{0\}$.
(c) for some countable context $\mathbf{c}$, $(T, \{\varphi, \neg\varphi\})$ is $\mathbf{c}$-superstable.

Proof of Theorem 9.6. (a) implies (b): apply Lemma 9.5.

(b) implies (c): immediate.

(c) implies (a): we prove the contrapositive. If φ is not simple or $\kappa(\varphi, T) > \aleph_0$, this amounts to saying that saying that (T, Δ) is not superstable for $\Delta = \{\varphi, \neg\varphi\}$ so we may apply 6.7. □

Corollary 9.7. *Let T be first order complete. Then the following are equivalent:*

(a) T is simple and $\kappa_{\rm loc}(T) = \aleph_0$.
(b) for some countable context $\mathbf{c}$, for every φ, $(T, \{\varphi, \neg\varphi\})$ is $\mathbf{c}$-superstable.

Theorem 9.8. *Suppose that T is a complete first order theory. Then the following are equivalent:*

(a) T is supersimple, i.e. every $p \in \mathbf{S}(M)$ does not fork over a finite set for $M \in {\rm Mod}_T$.
(b) T is $\mathbf{c}$-supersimple where $\mathcal{K}_{\mathbf{c}} = \mathcal{K}^{tr}_{\kappa}$ and $I_{\mathbf{c}} = {}^{\omega>}\{0\}$.
(c) There is some countable context $\mathbf{c}$ for which T is $\mathbf{c}$-supersimple.

Proof. (a) implies (b): Assume T is supersimple and let $\mathbf{c} = ({}^{\omega>}\{0\}, \mathcal{K}^{tr}_{\kappa})$. Assume for a contradiction that T were not $\mathbf{c}$-superstable. By the Separation Theorem, there would be a template Φ such that for every $\Psi \geq \Psi$, ${\rm GEM}(I, \Psi)$ is not $\aleph_1$-saturated. This contradicts Fact 9.3.

(b) implies (c): immediate.

(c) implies (a): for the contrapositive, assume T is not supersimple. Then by Claim 6.7, T is not $\mathbf{c}$-superstable for any countable context $\mathbf{c}$. □

We end this section with a motivating example for the work above and for [12]. *Reminder: all contexts are countable.* Fix for awhile $\mathbf{c} = (I, \mathcal{K}) = (I_{\mathbf{c}}, \mathcal{K}_{\mathbf{c}})$ some countable context and we shall investigate how $\mathbf{c}$-dividing may arise for $T_{\mathbf{rg}}$ inside a GEM-model.[13] Consider $M = {\rm GEM}(I, \Phi)$, where $\Phi \in \Upsilon[T_{\mathbf{rg}}]$ thus $(M, R^M) \models T_{\mathbf{rg}}$, and let $p \in \mathbf{S}(M \restriction \tau(T_{\mathbf{rg}}))$ be a nonalgebraic type. Fix J such that $I \subseteq J \in \mathcal{K}$ and J is $\aleph_0$-saturated. Let $N = {\rm GEM}(J, \Phi)$. By our assumption that all contexts are nice, see 2.7, $M \preceq N$, so we will identify the sequence $\langle \bar{a}_t : t \in I \rangle$ which generates M with a subsequence of $\langle \bar{a}_t : t \in J \rangle$.

By quantifier elimination, p is equivalent to $\{R(x, b_\alpha)^{\mathbf{i}_\alpha} \wedge x \neq b_\alpha \ : \ \alpha < \kappa\}$ for some κ, where each $\mathbf{i}_\alpha \in \{0, 1\}$. As M is generated by $\{\bar{a}_t : t \in I\}$, each b_α may be written as $\sigma^M_\alpha(\bar{a}_{\bar{t}_\alpha})$ for some $\tau(\Phi)$-term σ_α and some $\bar{t}_\alpha \in {\rm inc}(I)$. This representation may, of course, not be unique. Since we are ultimately looking for a criterion that will prevent $\mathbf{c}$-dividing, there is a priori no harm in choosing our enumeration to include all such representations. That is, without loss of generality, for some $\kappa = \kappa + |\tau(\Phi)|$,

$$p(x) \equiv \{R(x, \sigma^M_\alpha(\bar{a}_{\bar{t}_\alpha}))^{\mathbf{i}_\alpha} \wedge x \neq \sigma^M_\alpha(\bar{a}_{\bar{t}_\alpha}) : \alpha < \kappa\}. \tag{17}$$

[13] This is similar to [11] Claim 5.10, though slightly more general.

where for each $\alpha < \kappa$ and $\bar{t} \in {}^{\omega >}I$ if $\sigma^M(\bar{a}_{\bar{t}}) = \sigma^M_\alpha(\bar{a}_{\bar{t}_\alpha})$, then

$$\text{for some } \beta < \kappa,\ \sigma_\beta = \sigma \text{ and } \bar{t}_\beta = \bar{t}. \tag{18}$$

Why? This simply says we may choose to include all ways of generating the mentioned elements from the skeleton. Note this may increase the length of the enumeration, but will not change the size of the type in $\tau(T_{\mathbf{rg}})$. [From the point of view of $M \restriction \tau(T_{\mathbf{rg}})$, it may appear that we have repeated various formulas of the form $R(x, b_\alpha)^{\mathbf{i}_\alpha} \wedge x \neq b_\alpha$ many times, because we have listed an instance for each way of writing b_α in M in terms of the skeleton; whereas from the point of view of our enumeration which has access to $\tau(\Phi)$, for each b_α there are potentially $|I| + |\tau(\Phi)|$ such representations.]

Recalling our fixed $\aleph_0$-saturated J extending I and its associated $N = \mathrm{GEM}(J, \Phi)$, we ask about potential $\mathbf{c}$-dividing. Working in N, consider the set of formulas

$$\begin{aligned} q(x) = q_{I_0}(x) = \{&R(x, \sigma^N_\alpha(\bar{a}_{\bar{u}}))^{\mathbf{i}_\alpha} \wedge x \neq \sigma^M_\alpha(\bar{a}_{\bar{u}}) : \alpha < \kappa, \\ &\mathrm{tp}_{\mathrm{qf}}(\bar{u}, I_0, J) = \mathrm{tp}_{\mathrm{qf}}(\bar{t}_\alpha, I_0, I)\}. \end{aligned}$$

To show $q(x)$ is consistent, it would suffice to check that whenever

$$R(x, \sigma^N_\alpha(\bar{a}_{\bar{v}}))^{\mathbf{i}_\alpha} \in q \text{ and } R(x, \sigma^N_\beta(\bar{a}_{\bar{w}}))^{\mathbf{i}_\beta} \in q \tag{19}$$

we have that if $\sigma^N_\alpha(\bar{a}_{\bar{v}}) = \sigma^N_\beta(\bar{a}_{\bar{w}})$ then $\mathbf{i}_\alpha = \mathbf{i}_\beta$. Suppose this fails. That is, suppose for some suitable $\alpha, \beta, \bar{v}_*, \bar{w}_*$ which we fix for awhile, q contains the contradictory formulas

$$R(x, \sigma^N_\alpha(\bar{a}_{\bar{v}_*})) \text{ and } \neg R(x, \sigma^N_\beta(\bar{a}_{\bar{w}_*})). \tag{20}$$

In other words,

$$\sigma^N_\alpha(\bar{a}_{\bar{v}_*}) = \sigma^N_\beta(\bar{a}_{\bar{w}_*}) = b \text{ but } \mathbf{i}_\alpha \neq \mathbf{i}_\beta \text{ (here w.l.o.g. } \mathbf{i}_\alpha = 1,\ \mathbf{i}_\beta = 0). \tag{21}$$

We may visualize what has happened as follows. In N, there is a "positive line" consisting of the elements

$$\mathrm{Pos} = \{\sigma^N_\alpha(\bar{a}_{\bar{v}}) : \mathrm{tp}_{\mathrm{qf}}(\bar{v}, I_0, J) = \mathrm{tp}_{\mathrm{qf}}(\bar{t}_\alpha, I_0, I)\} \subseteq \mathrm{Dom}(N)$$

and a "negative line" consisting of the elements

$$\mathrm{Neg} = \{\sigma^N_\beta(\bar{a}_{\bar{w}}) : \mathrm{tp}_{\mathrm{qf}}(\bar{w}, I_0, J) = \mathrm{tp}_{\mathrm{qf}}(\bar{t}_\beta, I_0, I)\} \subseteq \mathrm{Dom}(N)$$

and we have that

$$\mathrm{Pos} \cap \mathrm{Neg} \neq \emptyset, \text{ witnessed by } b = \sigma^N_\alpha(\bar{a}_{\bar{v}_*}) = \sigma^N_\beta(\bar{a}_{\bar{w}_*}). \tag{22}$$

However, it is also important to notice that both "lines" have "points from I", and that these are not the point(s) of intersection:[14]

$$\sigma^N_\alpha(\bar{a}_{\bar{t}_\alpha}) \neq \sigma^N_\beta(\bar{a}_{\bar{t}_\beta}) \tag{23}$$

else our original p would be inconsistent. (So both Pos and Neg have size ≥ 2.)

To see the key property of $\mathbf{c}$ hidden in this picture, let us pull back the picture from N to a picture on I. Definition 9.9 may be suggested by two observations. First, writing $\mathfrak{r}_\alpha = \mathrm{tp}_{\mathrm{qf}}(\bar{t}_\alpha, I_0, I)$, we have that "$\sigma^N_\alpha(\bar{a}_{\bar{v}}) = \sigma^N_\alpha(\bar{a}_{\bar{u}})$" is an equivalence relation on $Q^J_{\mathfrak{r}_\alpha}$ (asserting that $\bar{v}, \bar{u}$ are equivalent), and similarly for σ^N_β and $\mathfrak{r}_\beta$. Second, the fact that $\bar{t}_\alpha$ and $\bar{t}_\beta$ may have different types is a red herring (as

[14] If p is a complete type in M, then we will have the stronger statement that "the restrictions of Pos and Neg to I have no intersection," i.e. $\{\sigma^N_\alpha(\bar{a}_{\bar{v}}) : \mathrm{tp}_{\mathrm{qf}}(\bar{v}, I_0, I) = \mathrm{tp}_{\mathrm{qf}}(\bar{t}_\alpha, I_0, I)\} \cap \{\sigma^N_\beta(\bar{a}_{\bar{w}}) : \mathrm{tp}_{\mathrm{qf}}(\bar{w}, I_0, I) = \mathrm{tp}_{\mathrm{qf}}(\bar{t}_\beta, I_0, I)\} = \emptyset$, but we do not need this here.

will be explained). The third formula, F, will give the analogue of (21) "points of intersection."

Definition 9.9. *The context $\mathbf{c} = (I, \mathcal{K})$ has $\mathbf{c}$ has property $\odot$ when:*

For every finite $I_0 \subseteq I$ there is a finite I_1 with $I_0 \subseteq I_1 \subseteq I$, letting $\bar{s}$, $\bar{t}$ list I_0, I_1 respectively, such that for any $\aleph_0$-saturated $J \supseteq I$ there exist quantifier-free (possibly infinitary) formulas of $\tau(\mathcal{K})$ called $F(\bar{x}_1, \bar{x}_2; \bar{y})$, $E_1(\bar{x}_1, \bar{x}_2; \bar{y})$, $E_2(\bar{x}_1, \bar{x}_2; \bar{y})$, such that $\ell(\bar{x}_1) = \ell(\bar{x}_2) = \lg(\bar{t})$, $\lg(\bar{y}) = \lg(\bar{s})$, and:

- (i) *for $\mathbf{i} = 1, 2$ $E_{\mathbf{i}}(\bar{x}_1, \bar{x}_2; \bar{s})$ defines an equivalence relation on $Y_{\bar{s}} = \{\bar{t}' \in {}^{\lg(\bar{t})}J : \mathrm{tp}_{\mathrm{qf}}(\bar{t}', \bar{s}, J) = \mathrm{tp}_{\mathrm{qf}}(\bar{t}, \bar{s}, J)\}$.*
- (ii) *$F(\bar{x}_1, \bar{x}_2; \bar{y})$ defines a nonempty one to one partial function from $Y_{\bar{s}}/E_1(-, -; \bar{s})$ to $Y_{\bar{s}}/E_2(-, -; \bar{s})$, and*
- (iii) *F has no fixed points, in other words for no $\bar{t} \in Y_{\bar{s}}$ is it the case that $F(\bar{t}, \bar{t}; \bar{s})$.*

In a manuscript in preparation, we use property $\odot$ to characterize countable contexts for which the random graph is $\mathbf{c}$-unsuperstable [12].

10. Further remarks and open questions

As noted above, the results in the present paper dealing with countable contexts may be seen as the case of "$\kappa = \aleph_0$." In this section we record the natural extensions of the definitions and theorems to the case of arbitrary κ without proofs, noting the main work of the proofs is already done in the case $\kappa = \aleph_0$ (so even though we plan to give some details in future work, it is worth stating these versions here for the interested reader). Complementary to the above comments:

Discussion 10.1. *Beginning with §6, we use repeatedly that I is a countable context. "Countably generated" is likely enough, but one would have to check carefully.*

Definition 10.2.

- *(1) We say $\mathbf{c} = (I, \mathcal{K})$ is a κ-context when it is a context and in addition if κ is regular, I is generated by κ elements but not by $< \kappa$ elements.*
- *(2) We say $(\bar{I}, \mathfrak{t})$ κ-represents $I \in \mathcal{K}$ when $\bar{I} = \langle I_\zeta : \zeta < \kappa \rangle$ is increasing, $\mathfrak{t} = \{\bar{t}_\zeta : \zeta < \kappa\rangle$, $\bar{t}_\zeta \subseteq I_\zeta$ is finite, $I_\zeta = \mathrm{cl}_I(\bar{t}_\zeta)$, $\bigcup_\zeta I_\zeta = I$.*
- *(3) $I \in \mathcal{K}$ has a κ-representation iff $(I, \mathcal{K})$ is a κ-context for $\mathcal{K}$ as usual.*

Definition 10.3. *Let $\mathbf{c} = (I, \mathcal{K})$ be a κ-context, $(\bar{I}, \mathfrak{t})$ a κ-representation. We say a complete first-order theory T is $\mathbf{c}$-superstable when there is no $\preceq$-increasing $\langle M_\zeta : \zeta \leq \kappa \rangle$ sequence of models of T and $p \in \mathbf{S}(\bigcup_{\zeta<\kappa} M_\zeta)$ such that $p \restriction (M_{\zeta+1})$ $\mathbf{c}$-shears over M_ζ.*

Extension 10.4. *Let $\mathbf{c}$ be a κ-context, $\mathbf{c} = (I, \mathcal{K})$, T first order complete, and $\lambda > \kappa$. T is $\mathbf{c}$-superstable iff for a dense set of $\Phi \in \Upsilon_{\mathcal{K}}[T]$, $\mathrm{GEM}_{\tau(T)}(I, \Phi)$ is λ-saturated.*

Extension 10.5. *Let T be complete, κ regular. The following are equivalent:*

- *(1) T is simple, $\kappa(T) \leq \kappa$.*
- *(2) For some κ-context $\mathbf{c}$, T is $\mathbf{c}$-superstable.*

Some questions. Next we turn to some natural questions.

Question 10.6. *Recalling* 5.16*, shearing is strictly weaker than dividing, therefore (non-)shearing cannot satisfy all the usual properties of independence relations in simple theories. Which such properties hold for shearing, and which fail? Does this depend on the context?*

Question 10.7. *Does the analogue of "forking = dividing" hold for shearing in simple theories?*

For the next question, note that we may characterize the class of theories T such that "for any countable context $\mathbf{c}$, T is $\mathbf{c}$-superstable if and only if the random graph is $\mathbf{c}$-superstable" indirectly by property $\odot$. Namely, this is the class of T which are $\mathbf{c}$-superstable for a countable context $\mathbf{c}$ if and only if $\neg\odot_{\mathbf{c}}$.

Question 10.8. *Consider the class of theories T such that "for any countable context $\mathbf{c}$, T is $\mathbf{c}$-superstable if and only if the random graph is $\mathbf{c}$-superstable". Give an internal model-theoretic characterization of this class.*

Question 10.9. *We may define an ordering on theories: $T_1 \leq T_2$ if for every countable context $\mathbf{c}$, if T_2 is $\mathbf{c}$-superstable then T_1 is $\mathbf{c}$-superstable. What is the structure of this ordering?*

Question 10.9 requires both understanding theories and building contexts. The results of the present paper give some partial information:

Lemma 10.10. *In the ordering of* 10.9*,*

- *The superstable theories are minimal, since every type in a superstable theory is definable (thus weakly definable) over a finite set, so it follows that such theories are $\mathbf{c}$-superstable for every countable context $\mathbf{c}$.*
- *The random graph is minimal among the unstable theories. (If T has the independence property, it will be susceptible to $\odot$, and if T is not simple, see next item.)*
- *The non-supersimple theories are precisely those which are not $\mathbf{c}$-superstable for any countable context $\mathbf{c}$, see Theorem 9.8.*

Question 10.11. *Is there a maximal class among the supersimple theories?*

One could eventually consider the analogue of 10.9 for contexts which are not necessarily countable, but for this it would make sense to first extend the results of this paper along the lines of §10 above.

References

[1] Z. Chatzidakis and E. Hrushovski, "Algebraically closed fields with an automorphism." Trans. AMS. (1999) Volume 351, Number 8, Pages 2997–3071.

[2] A. Ehrenfeucht and A. Mostowski. "Models of axiomatic theories admitting automorphisms." Fund. Math. 43 (1956), 50–68.

[3] R. Grossberg, J. Iovino, O. Lessmann, "A primer of simple theories." Arch. Math. Logic 41 (2002), no. 6, 541–580.

[4] V. Guingona, C. D. Hill, and L. Scow. "Characterizing model-theoretic dividing lines via collapse of generalized indiscernibles." Ann. Pure Appl. Logic 168 (2017), no. 5, 1091–1111.

[5] E. Hrushovski, "Pseudofinite fields and related structures." Model Theory and Applications, Quaderni di Matematica, vol. 11, Aracne, Rome, 2002, pp. 151–212.

[6] J. Hubička and J. Nešetřil. "All those Ramsey classes (Ramsey classes with closures and forbidden homomorphisms)." Advances in Math 356C (2019), 106791.

[7] A. S. Kechris, V. G. Pestov and S. Todorčević, "Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups." Geom. Funct. Anal. 15 (2005), no. 1, 106–189.

[8] M. Malliaris, "Hypergraph sequences as a tool for saturation of ultrapowers." Journal of Symbolic Logic, 77, 1 (2012) 195–223.
[9] M. Malliaris and S. Shelah, "Existence of optimal ultrafilters and the fundamental complexity of simple theories." Advances in Math. 290 (2016) 614–681.
[10] M. Malliaris and S. Shelah, "Keisler's order has infinitely many classes." Israel J. Math. 224 (2018), no. 1, 189–230.
[11] M. Malliaris and S. Shelah, "A new look at interpretability and saturation." Ann Pure Appl Logic 170, 5 (2019) 642–671.
[12] M. Malliaris and S. Shelah. Manuscript F2061, in progress.
[13] J. Nešetřil. "Ramsey classes and homogeneous structures." Combin. Probab. Comput. 14 (2005), no. 1-2, 171–189.
[14] J. Nešetřil and V. Rödl. "Ramsey classes of set systems." J. Comb. Th. A 34 (1983), 183–201.
[15] M. O. Rabin, "Universal groups of automorphisms of models." In *Symposium on the Theory of Models*, proceedings of the 1963 International Symposium at Berkeley, J. W. Addison, L. Henkin, A. Tarski, eds. North-Holland Publ. Co., Amsterdam (1965).
[16] L. Scow. "Characterization of NIP theories by ordered graph-indiscernibles," Annals of Pure and Applied Logic 163 (2012), pp. 1624–1641.
[17] S. Shelah, "Simple unstable theories." Ann. Math. Logic, 19, 177–203 (1980).
[18] S. Shelah, *Classification Theory and the number of non-isomorphic models*, North-Holland, rev. ed. 1990 (first ed., *Classification Theory*, 1978). Results cited here are from 1978.
[19] Shelah, "General non structure theory." E59 (2016). http://shelah.logic.at/files/E59.pdf .
[20] S. Shelah, "Toward classifying unstable theories." Ann Pure Appl Logic 80 (1996) 229–255.

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