Journal of Algebra 619 (2023) 323–346

Contents lists available at ScienceDirect

Journal of Algebra

journal homepage: www.elsevier.com/locate/jalgebra

Classification of pivotal tensor categories with fusion rules related to $SO(4)$

Daniel Copeland [a], Cain Edie-Michell [b,*]

[a] The University of California, San Diego, United States
[b] University of New Hampshire, United States

ARTICLE INFO

Article history:
Received 4 January 2022
Available online 15 December 2022
Communicated by Nicolás Andruskiewitsch

Keywords:
Tensor categories
Classification
Lie groups

ABSTRACT

In this paper we classify all semisimple pivotal tensor categories with the same fusion rules as Rep($SO(4)$), or one of the associated truncations. We show that such categories are explicitly classified by two non-zero complex numbers. Furthermore we show these tensor categories are always braided, and aside from a small number of degenerate cases, there exist exactly 8 braidings.

1. Introduction

In this note we continue the program to classify tensor categories with fusion rules the same as Rep(G) for G a semisimple Lie group (or of the associated fusion categories). The classification is currently known for the majority of the classical Lie groups. The known results are for: $SU(2)$ [10], $SU(N)$ [12], $O(N)$ and $Sp(N)$ [16], and $SO(N)$ ($N \neq 4$) [6]. The latter three results apply to ribbon categories, while the first two do not require

* Corresponding author.
E-mail addresses: daniel.copeland@gmail.com (D. Copeland), cain.edie-michell@unh.edu (C. Edie-Michell).

https://doi.org/10.1016/j.jalgebra.2022.12.003
0021-8693/

any assumption of braiding and provide a classification for pivotal tensor categories. Our technique for $SO(4)$-type categories also does not require a braiding assumption.

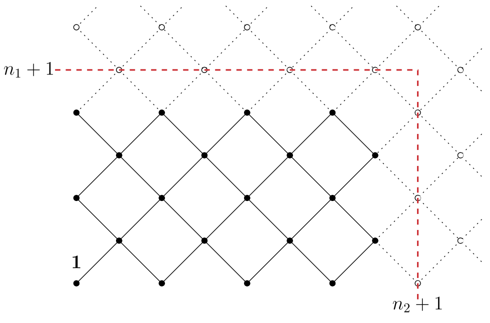
The standard technique for these classification problems is to identify the endomorphism algebras of tensor powers of the "vector representation" in an arbitrary tensor category with the same fusion rules of $\mathrm{Rep}(G)$, and to show that this algebra must agree with the known examples coming from quantum groups. In the case of $SU(N)$ this gives well-known quotients of the Hecke algebras [18], and in the $O(N)$ and $SO(N)$ cases we find quotients of BMW algebras [5]. For $SO(N)$ with $N \neq 4$ the endomorphism algebras also afford representations of the BMW algebra, but the image of the BMW algebra does not generate the endomorphism algebra for $SO(2n)$ for $n > 2$.

The gap at $SO(4)$ is due to the fact that the tensor square of the vector representation splits into four simples, rather than three (as is the case for every other $SO(N)$ with $N \geq 3$). This means that a braid element on $X^{\otimes 2}$ need not satisfy the cubic BMW skein relation, which was required for the method of [6].

There is another important distinction between $SO(4)$ and $SO(2n)$ with $n > 2$, which is that the root system for $SO(4)$ is not irreducible (its root system is the product $A_1 \times A_1$). As we shall see, this manifests in categorifications of $SO(4)$ fusion rules being described by two independent parameters q_1, q_2, rather than a single parameter q.

In this paper we close this gap by studying a known $SO(4)$-type category and identifying the monoidal subcategory whose objects are tensor powers of the vector representation. This subcategory is essentially a planar algebra, and we describe it by generators and relations in a planar algebraic way, although we do not use that language. The planar algebras we describe can be seen as natural extensions of the Fuss-Catalan planar algebras [2]. We then show that the corresponding subcategory of any category with $SO(4)$-type fusion rules must have the same presentation. We then obtain the classification of tensor categories with $SO(4)$ fusion rules from standard reconstruction arguments.

We say a tensor category has $SO(4)$ fusion rules if its Grothiendieck ring is isomorphic to $K(\mathrm{Rep}(SO(4)))$, or isomorphic to the Grothendieck ring of one of the associated fusion categories. We label these fusion rings by K_{n_1,n_2} where $n_i \in \mathbb{N} \cup \{\infty\}$ (see Definition 2.2 for a precise definition). The fusion graph of K_{n_1,n_2} for the vector representation is given by (shown here with $n_1 = 5$ and $n_2 = 8$):



For any non-zero complex numbers q_1 and q_2, there exists a category $\mathcal{C}_{q_1,q_2}$, defined in Definition 2.1. For any n_1 and n_2 there exist q_1 and q_2 so that $\mathcal{C}_{q_1,q_2}$ has fusion rules K_{n_1,n_2}. The classification of all categories with these fusion rules is given in our main theorem.

Theorem 1.1. *Let $\mathcal{C}$ be a semisimple pivotal tensor category with $K(\mathcal{C}) = K_{n_1,n_2}$ where $n_1, n_2 \in \mathbb{N}_{\geq 2} \cup \infty$. We have the following:*

1. *If $n_1 = n_2 = 3$, then $\mathcal{C}$ is a Tambara-Yamagami fusion category with $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. There are exactly four of these categories up to monoidal equivalence [17, Theorem 4.1]. Two of these categories are equivalent to $\mathcal{C}_{\zeta_8,\zeta_8}$ and $\mathcal{C}_{\zeta_8^5,\zeta_8}$, the other two are non-equivalent to any $\mathcal{C}_{q_1,q_2}$. There are exactly 8 braidings on each of these categories [14, Theorem 1.2].*
2. *If either n_1 or n_2 is not equal to 3, then the category $\mathcal{C}$ is monoidally equivalent to $\mathcal{C}_{q_1,q_2}$ where $q_1, q_2 \in \mathbb{C}^\times$, with the order of q_i^2 equal to $n_i + 1$ (or possibly $q_i^2 = 1$ if $n_i = \infty$). Further we have the monoidal equivalences*

$$\mathcal{C}_{q_1,q_2} \simeq \mathcal{C}_{q_2,q_1} \simeq \mathcal{C}_{q_1,q_2^{-1}} \simeq \mathcal{C}_{q_1^{-1},q_2} \simeq \mathcal{C}_{-q_1,-q_2}.$$

3. *The category $\mathcal{C}_{q_1,q_2}$ is braided, and the possible braidings on these categories are parameterized by the set*

$$\{(s_1, s_2) : s_1^2 = -q_1^{\pm 1} \quad \text{and} \quad s_2^2 = -q_2^{\pm 1}\}/\{(s_1, s_2) = (-s_1, -s_2)\}.$$

When both $n_1, n_2 > 2$, these eight braidings are all distinct. If either n_1 or n_2 are equal to 2, then four of these braidings are distinct. If both n_1 and n_2 are equal to 2, then two of these braidings are distinct.

Constructions of these categories are given in Definition 2.1.

Remark 1.2. The above classification is up to equivalences which preserve the distinguished object X corresponding to the vector representation of $SO(4)$ in the categories $\mathcal{C}_{q_1,q_2}$. The equivalences given in Theorem 1.1 are all the possible equivalences which preserve X. There can exist additional equivalences between the categories $\mathcal{C}_{q_1,q_2}$ which don't preserve X.

An illustrating example is seen in the case when q_2^2 is a root of unity of even order $n_2 + 1$ such that $[n_2]_{q_2} = -1$. For these parameters, we have that $\mathcal{C}_{q_1,q_2}$ is monoidally equivalent to $\mathcal{C}_{q_1,-q_2}$ but the equivalence does not fix the distinguished object X.

This paper is outlined as follows.

In Section 2 we define the categories $\mathcal{C}_{q_1,q_2}$ which are examples of categories with $SO(4)$ fusion rules. We define what it means to give a based semisimple presentation of a pivotal tensor category, and give such a presentation for the categories $\mathcal{C}_{q_1,q_2}$.

In Section 3 we use techniques inspired by the theory of planar algebras [3,13] to classify arbitrary pivotal tensor categories with $SO(4)$ fusion rules. The presentation we describe is exactly the same as the category $\mathcal{C}_{q_1,q_2}$, hence reconstruction techniques allow us to deduce that the arbitrary pivotal tensor category must be $\mathcal{C}_{q_1,q_2}$. Our methods to describe the arbitrary presentation rely heavily on the $SO(4)$ fusion rules for objects appearing in the tensor square, and the tensor cube, of the "vector representation". By working in the idempotent basis, we are able to use these fusion rules to pin down a large number of relations in our arbitrary category. The hard part of the argument is determining the Fourier transformation of our generators. By playing off the standard algebra multiplication in $\mathrm{End}(X \otimes X)$ against the special convolution algebra structure, we are able to fully pin down the Fourier transform, and finish our presentation.

We conclude the paper with Section 4, where we classify all the braidings on the monoidal categories $\mathcal{C}_{q_1,q_2}$. The key idea to classify these braidings is to consider the adjoint subcategory $\mathcal{C}^{\mathrm{ad}}_{q_1,q_2}$, which we know is equivalent to a product of $SO(3)$ type categories. The braidings on the $SO(3)$ type categories are fully classified [16], and we can leverage this information up via some technical computations to classify all braidings on the full category $\mathcal{C}_{q_1,q_2}$.

Acknowledgments

The second author was supported by NSF grant DMS 2245935 and a AMS-Simons Travel Grant. Both authors would like to thank the referee for many useful suggestions, and Hans Wenzl for useful conversations.

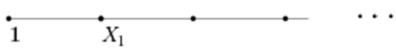
2. Preliminaries

We refer the reader to [8] for the basics on tensor categories. For us a tensor category is a $\mathbb{C}$-linear, abelian, monoidal, and rigid category with simple unit.

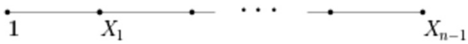
2.1. Tensor categories with $SO(4)$ fusion rules

In this subsection we present a family of pivotal tensor categories with $SO(4)$ fusion rules. We build these categories using Deligne products of $SU(2)$ categories.

Categories with $SU(2)$ fusion rules (and their truncations) are known as type A categories. In the generic case there are infinitely many simples up to isomorphism, labeled $\mathbf{1} = X_0, X_1, X_2, \ldots$ The fusion graph for multiplication by X_1 is



In the fusion case there are finitely many isomorphism types of simples $\mathbf{1}, X_1, X_2, \ldots, X_{n-1}$ and the fusion graph for multiplication by X_1 is the truncated graph



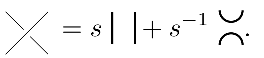
Fusion categories with these fusion rules are known as A_n categories.

Type A and A_n categories are classified up to monoidal equivalence [10] by the dimension of the object X_1, which can be expressed as

$$\dim(X_1) = [2]_q = q + q^{-1} \tag{1}$$

where q is a non-zero complex number which is either ± 1 or not a root of unity in the generic case, and is a root of unity in the fusion case. These categories are spherical (see [8, Section 4.7]) and there is a unique choice of spherical structure such that X_1 is symmetrically self-dual. We denote a type A or A_n category with parameter q by $\mathcal{A}_q$. Note that $\mathcal{A}_q = \mathcal{A}_{q^{-1}}$.

The categories $\mathcal{A}_q$ are all braided. Type A and A_n categories are classified up to braided equivalence (which fixes the distinguished object X_1), by the two eigenvalues of the braiding σ_{X_1,X_1}. These eigenvalues are s and $-s^{-3}$ where s is a solution to either $s^2 = -q$ or $s^2 = -q^{-1}$. Hence there are four distinct braidings on each of the monoidal categories $\mathcal{A}_q$, which are defined by



With the categories $\mathcal{A}_q$ in hand, we can define the categories $\mathcal{C}_{q_1,q_2}$ which appear in our main theorem.

Definition 2.1. Let $\mathcal{C}_{q_1,q_2}$ denote the sub-tensor category of $\mathcal{A}_{q_1} \boxtimes \mathcal{A}_{q_2}$ generated by $X := X_1 \boxtimes Y_1$ (we use X_1, resp. Y_1, to denote the generating object of $\mathcal{A}_{q_1}$, resp. $\mathcal{A}_{q_2}$).

Note that we can only refer to the object $X_1 \boxtimes Y_1$ when both $\mathcal{A}_{q_1}$ and $\mathcal{A}_{q_2}$ are non-trivial.

The categories $\mathcal{C}_{q_1,q_2}$ inherit 16 braidings from the four braidings on each of $\mathcal{A}_{q_1}$ and $\mathcal{A}_{q_2}$. These are parameterized by solutions to $s_1^2 = -q_1^{\pm 1}$ and $s_2^2 = -q_2^{\pm 1}$. The braided categories corresponding to the solutions (s_1, s_2) and $(-s_1, -s_2)$ are braided equivalent. Hence we get 8 distinct braidings on the categories $\mathcal{C}_{q_1,q_2}$.

Definition 2.2. For $n_1, n_2 \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ we define the fusion ring K_{n_1,n_2} by

$$K_{n_1,n_2} := K(\mathcal{C}_{q_1,q_2})$$

where each q_i is a non-zero complex number such that q^2 has order $n_i + 1$.

We say a category has $SO(4)$ type fusion rules if its Grothendieck ring is isomorphic to K_{n_1,n_2} for some $n_1, n_2 \in \mathbb{N}_{\geq 2} \cup \{\infty\}$.

For convenience let us label the simple elements of these fusion rings. The simple elements of K_{n_1,n_2} are those $X_i \boxtimes Y_j$ with $0 \leq i \leq n_1 - 1, 0 \leq j \leq n_2 - 1$ and $i + j \in 2\,\mathbb{Z}$. In this notation, the distinguished object X is $X_1 \boxtimes Y_1$.

From the fusion graphs we see that all the fusion rings are $\mathbb{Z}_2$-graded (since $\mathbf{1}$ only appears in even powers of X). The adjoint subcategories (see [8, Section 4.14]) have fusion rules of $SO(3) \times SO(3)$ type, an important fact we will use later.

2.2. Presentations for semisimple tensor categories

We recall some basic facts regarding presentations of semisimple spherical tensor categories, before providing a presentation of the categories $\mathcal{C}_{q_1,q_2}$.

Definition 2.3. A *based tensor category* will be a pair $(\mathcal{C}, X)$ where X is a chosen tensor generator of a spherical tensor category $\mathcal{C}$.

The A_n categories are conventionally based by picking a simple object corresponding to the vector representation of $SU(2)$. Likewise, we consider any $SO(4)$-category based by a simple object X corresponding to the vector rep of $SO(4)$.

A based presentation of a (small) spherical based tensor category $(\mathcal{C}, X)$ is a set of morphisms F between tensor powers of X, and a set of relations R satisfied in $\mathcal{C}$ such that

$$\mathcal{C} \cong \overline{\mathcal{C}(F)/\mathcal{R}}$$

where $\mathcal{C}(F)$ is the free (based, strictly pivotal and strict monoidal) spherical $\mathbb{C}$-linear monoidal category (possibly not abelian and with non-simple unit) generated by one object and the morphisms F, $\mathcal{R}$ is the smallest tensor ideal of $\mathcal{C}(F)$ containing R, and the notation $\overline{\mathcal{C}}$ denotes the *Cauchy completion* (additive and idempotent completion [1, Theorem 1]) of a category $\mathcal{C}$.

For instance, an A_n category has a based presentation with no generators and the relations

$$\bigcirc = [2]_q \qquad \text{and} \qquad f^{(n)} = 0$$

where q^2 is a primitive $n+1$-st root of 1 and $f^{(n)}$ denotes the n-th Jones-Wenzl projection. Note that here we have chosen a spherical structure which makes the generating object symmetrically self-dual (this differs from the standard quantum group convention, where the other spherical structure is chosen. For the quantum group convention, we have that the closed loop has value $-[2]_q$). This allows us to draw unorientated strands.

Given a spherical monoidal category $\mathcal{C}$, let $\mathcal{N}(\mathcal{C})$ denote the monoidal ideal of negligible morphisms in $\mathcal{C}$ [9, Section 2]. Under various assumptions on the category $\mathcal{C}$, the quotient $\mathcal{C}\,/\,\mathcal{N}(\mathcal{C})$ is semisimple (however in our set-up we only require the result of Lemma 2.5 below).

Definition 2.4. A *based semisimple presentation* of a based semisimple spherical tensor category $(\mathcal{C}, X)$ is a set of morphisms F between tensor powers of X and a set of relations R satisfied in $\mathcal{C}$ such that

$$\mathcal{C}' = \overline{\mathcal{C}(F)/\mathcal{R}}.$$

has simple unit, and

$$\mathcal{C} \cong \mathcal{C}' \,/\, \mathcal{N}(\mathcal{C}').$$

A based semisimple presentation generally contains less information than a presentation (since we do not need to provide relations for the negligible ideal). For example, an A_n category has a based semisimple presentation with no generators and the single relation

$$\bigcirc = [2]_q$$

where q^2 is a primitive $(n+1)$-st root of 1. This is a based semisimple presentation. Indeed, the unit is simple, as every closed diagram of cups and caps can be evaluated to a scalar by popping closed loops, and the quotient by negligibles is shown to be equivalent to an A_n category in [15, Chapter 12: Sections 6-8]. The relation $f^{(n)} = 0$ is not necessary since the element $f^{(n)}$ gets sent to 0 when we quotient by negligibles.

The condition that $\overline{\mathcal{C}(F)/\mathcal{R}}$ (or equivalently $\mathcal{C}(F)/\mathcal{R}$) has a simple tensor unit is often summarized as "having enough relations to evaluate closed diagrams". The following well-known fact states that having enough relations to evaluate closed diagrams is a sufficient condition to produce a based semisimple presentation.

Lemma 2.5. *[4, Proposition 3.5] Suppose a based semisimple spherical tensor category $(\mathcal{C}, X)$ is generated by morphisms F between tensor powers of X and satisfies relations R such that $\mathcal{C}(F)/\mathcal{R}$ has a simple tensor unit. Then (F, R) is a based semisimple presentation for $\mathcal{C}$.*

Classification outline

We can outline our argument for classifying $SO(4)$-type categories:

Step 1. Provide a based semisimple presentation for the categories $\mathcal{C}_{q_1,q_2}$ (the presentation depends on q_1, q_2).

Step 2. Given a semisimple pivotal tensor category $\mathcal{D}$ with $SO(4)$-type fusion rules, find parameters q_1, q_2 and morphisms in $\mathcal{D}$ which satisfy the relations for $\mathcal{C}_{q_1,q_2}$ from Step 1.

Step 3. Conclude that $\mathcal{D} \cong \mathcal{C}_{q_1,q_2}$, as follows. Let $\mathcal{C}' = \overline{\mathcal{C}(F)/\mathcal{R}}$ where (F, R) is the based semisimple presentation of $\mathcal{C}_{q_1,q_2}$ from Step 1. Observe that Step 2 provides a tensor functor

$$\Phi : \mathcal{C}' \to \mathcal{D}.$$

The kernel of Φ is a tensor ideal of $\mathcal{C}'$, which must be contained in $\mathcal{N}(\mathcal{C}')$ since $\mathcal{N}(\mathcal{C}')$ is the unique maximal tensor ideal of $\mathcal{C}'$. Let $\mathrm{Im}(\Phi)$ denote the image of Φ, a $\mathbb{C}$-linear monoidal subcategory of $\mathcal{D}$. If $X^{\otimes i}$ and $X^{\otimes j}$ are any objects of $\mathcal{C}'$, then we may also consider them objects of $\mathcal{C}_{q_1,q_2}$ and $\mathcal{D}$ (through mild abuse of notation), and the previous two sentences give inequalities

$$\dim \mathrm{Hom}_{\mathcal{C}_{q_1,q_2}}(X^{\otimes i}, X^{\otimes j}) \leq \dim \mathrm{Hom}_{\mathrm{Im}(\Phi)}(X^{\otimes i}, X^{\otimes j}) \leq \dim \mathrm{Hom}_{\mathcal{D}}(X^{\otimes i}, X^{\otimes j}).$$

On the other hand,

$$\dim \mathrm{Hom}_{\mathcal{D}}(X^{\otimes i}, X^{\otimes j}) = \dim \mathrm{Hom}_{\mathcal{C}_{q_1,q_2}}(X^{\otimes i}, X^{\otimes j})$$

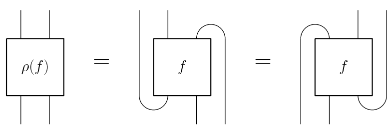
since $\mathcal{D}$ and $\mathcal{C}_{q_1,q_2}$ have the same fusion rules. Hence both inequalities above are equalities and in particular $\mathrm{Im}(\Phi) \simeq \mathcal{D}$. Since $\mathcal{D}$ is semisimple, all negligible morphisms are zero [7, Proposition 5.7] so the kernel of Φ must be equal to $\mathcal{N}(\mathcal{C}')$. In conclusion, this shows $\mathcal{D} \cong \mathcal{C}' / \mathcal{N}(\mathcal{C}') \cong \mathcal{C}_{q_1,q_2}$.

Based semisimple presentation for $\mathcal{C}_{q_1,q_2}$

With the above ansatz in mind, let's give a based semisimple presentation for the categories $\mathcal{C}_{q_1,q_2}$. To reduce clutter, we abbreviate the quantum numbers

$$[n]_{q_1} \text{ by } [n]_1, \text{ and } [n]_{q_2} \text{ by } [n]_2.$$

Given a morphism $f \in \mathrm{End}(X^{\otimes 2})$, we let $\rho(f)$ denote the *Fourier transform*, or one-click rotation of f:



The second equality (which is equivalent to $\rho^2(f) = f$) follows from the facts that we assume our categories are strictly pivotal, every object is self-dual and X^2 is multiplicity-free. Our presentation for $\mathcal{C}_{q_1,q_2}$ will use two generators P and Q in $\mathrm{End}(X^{\otimes 2})$. They are defined by

$$P = \frac{1}{[2]_2} f^{(2)} \boxtimes \underset{\frown}{\smile} \text{ and } Q = \frac{1}{[2]_1} \underset{\frown}{\smile} \boxtimes f^{(2)}, \tag{2}$$

where $f^{(2)} := \big|\,\big| - \frac{1}{[2]_i}\asymp$ denotes the second Jones-Wenzl projection in the respective factors. With these definitions, P is the projection with image $X_2 \boxtimes \mathbf{1} \subset X^{\otimes 2}$ and Q is the projection with image $\mathbf{1} \boxtimes Y_2 \subset X^{\otimes 2}$.

Lemma 2.6. *The morphisms P and Q generate $\mathcal{C}_{q_1,q_2}$ as a spherical tensor category.*

Proof. This has been proved in greater generality using planar algebra language by Liu [13, Corollary 3.2]. We provide a proof in our case for the reader's convenience. We will show that the simpler morphisms $g = \big|\,\big| \boxtimes \asymp$ and $h = \asymp \boxtimes \big|\,\big|$ generate $\mathcal{C}_{q_1,q_2}$. Since P and Q are related to g and h by the equations

$$P = \frac{1}{[2]_2}\left(g - \frac{1}{[2]_1}\asymp \boxtimes \asymp\right) \text{ and } Q = \frac{1}{[2]_1}\left(h - \frac{1}{[2]_2}\asymp \boxtimes \asymp\right),$$

the result will follow.

To show that g and h generate, it suffices to check they generate all the morphisms in the full tensor subcategory of $\mathcal{C}_{q_1,q_2}$ with objects $\mathbf{1}, X, X^{\otimes 2}, X^{\otimes 3}, \ldots$ (since X tensor generates $\mathcal{C}_{q_1,q_2}$). Furthermore, $\mathcal{C}_{q_1,q_2}$ is $\mathbb{Z}_2$-graded, so by Frobenius reciprocity it's enough to show that g and h generate the endomorphism algebras $\mathrm{End}(X^{\otimes k})$. We have

$$\mathrm{End}(X^{\otimes k}) \cong \mathrm{End}_{\mathcal{A}_{q_1}}(X_1^{\otimes k}) \otimes_{\mathbb{C}} \mathrm{End}_{\mathcal{A}_{q_2}}(Y_1^{\otimes k}).$$

The subalgebra $\mathrm{End}_{\mathcal{A}_{q_1}}(X_1^{\otimes k}) \boxtimes \mathrm{id}_k$ is generated (as an algebra) by the cup/cap elements $g_1, g_2, \ldots, g_{k-1}$ where

$$g_i = \mathrm{id}_{i-1} \otimes g \otimes \mathrm{id}_{k-i-1}\,.$$

Similarly, $\mathrm{id}_k \boxtimes \mathrm{End}_{\mathcal{A}_{q_2}}(Y_1^{\otimes k})$ is generated (as an algebra) by the corresponding h_i's. Hence g and h generate $\mathrm{End}(X^{\otimes k})$ (as a Hom space in a spherical tensor category). $\square$

Now that we know P and Q generate $\mathcal{C}_{q_1,q_2}$, we can give a based semisimple presentation with two generators. By choosing spherical structures on the categories $\mathcal{A}_{q_1}$ and $\mathcal{A}_{q_2}$, we can ensure that $\mathcal{C}_{q_1,q_2}$ is generated by a symmetrically self-dual object.

Proposition 2.7. *For q_1, q_2 non-zero complex numbers, the pivotal category $\mathcal{C}_{q_1,q_2}$ is tensor generated by the symmetrically self-dual object $X = X_1 \boxtimes Y_1$, and has a based semisimple presentation with two generators $P, Q \in End(X^{\otimes 2})$ and the following relations:*

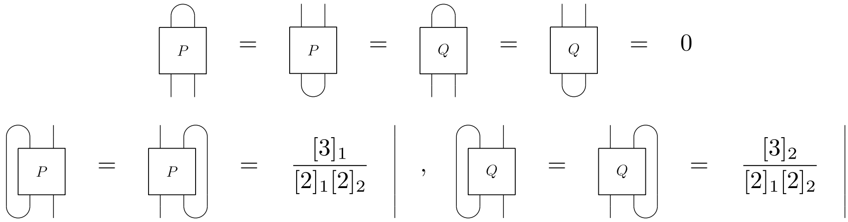
(a) $\bigcirc = [2]_1[2]_2$

(b) $P^2 = P, Q^2 = Q$ *and* $PQ = QP = 0$

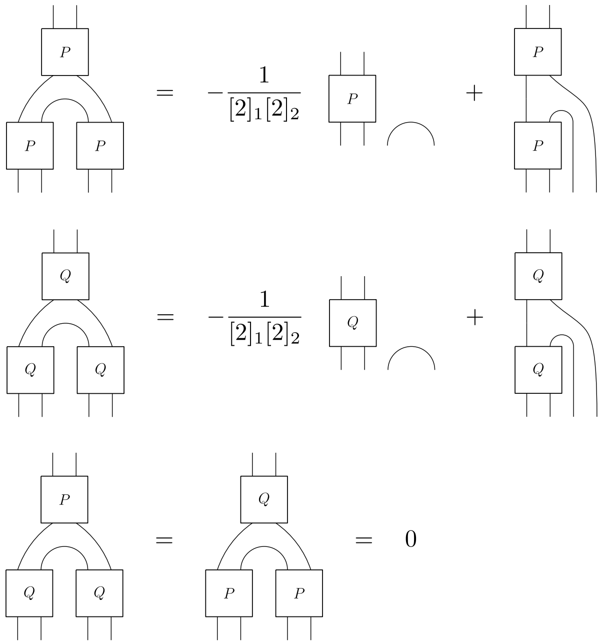
(c) Fourier equation:

$$\rho(P) = \frac{-1}{[2]_1[2]_2}\big|\,\big| + \frac{1}{[2]_2^2}\asymp + \frac{[2]_1}{[2]_2}Q.$$

(d) Bubble popping:



(e) Triangle popping:



Furthermore, for (s_1, s_2) solutions to $s_1^2 = -q_1^{\pm 1}$ and $s_2^2 = -q_2^{\pm 1}$, we have a braiding on $\mathcal{C}_{q_1,q_2}$ defined by

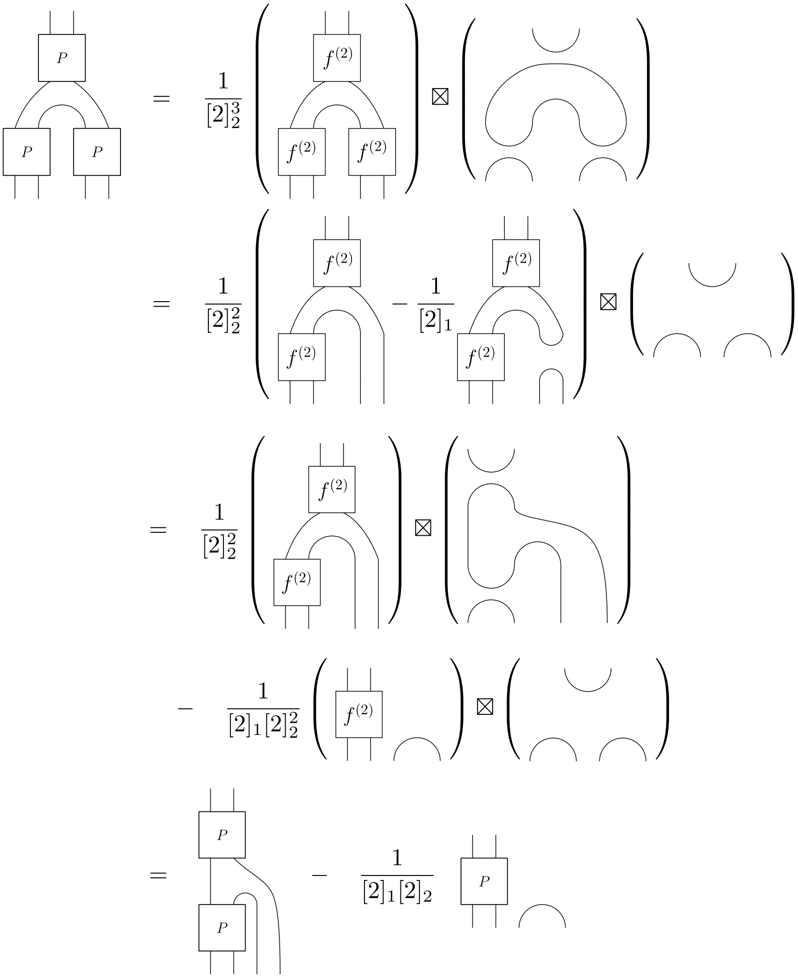
$$\diagup\!\!\!\!\diagdown = s_1 s_2 \mid\,\mid + \frac{\frac{q_1 s_1^2}{q_1^2+1} + \frac{q_2 s_2^2}{q_2^2+1} + 1}{s_1 s_2} \asymp + \frac{\left(q_2^2+1\right) s_1}{q_2 s_2} P + \frac{\left(q_1^2+1\right) s_2}{q_1 s_1} Q.$$

Remark 2.8. This presentation is closely related to the Fuss-Catalan algebras of [2].

Remark 2.9. Note that the Fourier equation (c) implies

$$\rho(Q) = \frac{-1}{[2]_1[2]_2} \mid\,\mid + \frac{1}{[2]_1^2} \asymp + \frac{[2]_2}{[2]_1} P.$$

Proof. By Lemma 2.6, the morphisms P and Q generate the category. Checking that they satisfy the given relations we leave as an exercise in type A skein theory. We provide an example for a triangle popping relation, using type A skein theory in the respective factors:



We must check that we have enough relations to describe the category $\mathcal{C}_{q_1,q_2}$. By Lemma 2.5, it suffices to show that we can use the provided relations to evaluate any closed planar diagram made from P's and Q's to a scalar. We can represent such a diagram as a planar 4-valent graph with vertices labeled by P, Q, $\rho(P)$ or $\rho(Q)$.

The following standard argument implies any planar 4-valent graph must contain either a loop, bigon, triangle. Suppose not, then as the graph is planar, 4-valent, and does not contain a loop, bigon, or triangle, then we respectively get

$$V - E + F = 1, \quad 4V = 2E, \quad \text{and} \quad 4F \leq 2E.$$

These three equations are incompatible, so the graph must contain a loop, bigon, or triangle.

We prove by induction on the number of vertices that the diagram can be reduced to a scalar using the relations. If there are no vertices, then relation (a) reduces any diagram (made of cups/caps) to a scalar. For the inductive step, note that if the graph contains any self-loops then the bubble popping relations allow one to reduce the number of vertices. If there are no self-loops, then the graph must contain a bigon or a triangle. The relations (b) and (c) imply that any diagram with a bigon can be reduced to a sum of diagrams with fewer vertices. Finally, any triangle can be reduced in a similar way using the triangle popping relations and relation (c) (possibly after applying a 2 or 4-click rotation to the triangle).

The braidings described in the final statement come from the known braidings on $\mathcal{A}_{q_1} \boxtimes \mathcal{A}_{q_2}$, which are inherited from the well-known braidings on the factors. Checking our formula for the braiding is another skein theory exercise. $\square$

3. Monoidal classification

In this section we classify pivotal categories $\mathcal{C}$ with $K(\mathcal{C}) \cong K_{n_1,n_2}$. We may identify the Grothendieck ring of $\mathcal{C}$ with that of $\mathcal{C}_{q_1,q_2}$ thus use the symbols $X_a \boxtimes Y_b$ to denote simple objects in $\mathcal{C}$.

The subcategories tensor generated by $X_2 \boxtimes 1$ and $1 \boxtimes Y_2$ have $SO(3)$-type fusion rules. A result of Etingof and Ostrik ([9], Thms. A.1, A.3 and Remark A.4) states that (apart from the case where the fusion rules are $K(\mathrm{Vec}(\mathbb{Z}_2))$ any pivotal category with $SO(3)$ type fusion rules is monoidally equivalent to $\mathrm{Rep}(SO(3)_q) \cong \mathcal{A}_q^{\mathrm{ad}}$ where q is not a root of unity or $q^2 = 1$ (if there are infinitely many simples) or q is an appropriate root of unity in the fusion case. This allows us to prove the following.

Lemma 3.1. *We have that*

$$\langle X_2 \boxtimes \mathbf{1} \rangle \cong \mathcal{A}_{q_1}^{ad} \text{ and } \langle \mathbf{1} \boxtimes Y_2 \rangle \cong \mathcal{A}_{q_2}^{ad}$$

where

- *In the $K_{\infty,\infty}$ case, q_1 is either not a root of unity or $q_1^2 = 1$, and similarly q_2 is either not a root of unity or $q_2^2 = 1$.*
- *In the K_{n_1,n_2} case, q_1^2 is a primitive (n_1+1)-st root of 1 and q_2^2 is a primitive (n_2+1)-st root of 1.*
- *In the $K_{n_1,\infty}$ case, q_1^2 a primitive (n_1+1)-st root of unity, and q_2 is not a root of 1 (or $q_2^2 = 1$).*

In particular we have

$$\dim(X_1 \boxtimes \mathbf{1}) = [3]_1 \text{ and } \dim(\mathbf{1} \boxtimes Y_1) = [3]_2 \tag{3}$$

Proof. From the fusion rules of $\mathcal{C}$, we have that $\langle X_2 \boxtimes \mathbf{1}\rangle$ has $SO(3)$-type fusion rules. This gives the result immediately, apart from the special case where $K(\langle X_2 \boxtimes \mathbf{1}\rangle) \cong K(\mathrm{Vec}(\mathbb{Z}_2))$. In this case we can either have that $\langle X_2 \boxtimes \mathbf{1}\rangle \simeq \mathrm{Vec}(\mathbb{Z}_2) \simeq \mathcal{A}_{q_1}$ where q_1^2 is a primitive fourth root of unity, or that $\langle X_2 \boxtimes \mathbf{1}\rangle \simeq \mathrm{Vec}^\omega(\mathbb{Z}_2)$ where ω is the non-trivial element of $H^3(\mathbb{Z}_2, \mathbb{C}^\times)$. The latter case is non-equivalent to any $\mathcal{A}_{q_1}$, hence we have to show that it can't occur in our setting. To do this, note that $(X_1 \boxtimes Y_1) \otimes (X_1 \boxtimes Y_1)$ is an associative algebra object in $\mathcal{C}$ (as it is of the form $X \otimes X^*$). The restriction of this algebra object to $\langle X_2 \boxtimes \mathbf{1}\rangle \simeq \mathrm{Vec}^\omega(\mathbb{Z}_2)$ gives the algebra object $\mathbf{1}\boxtimes\mathbf{1} \oplus \mathbf{X_2}\boxtimes\mathbf{1}$. However the category $\mathrm{Vec}^\omega(\mathbb{Z}_2)$ has no algebra objects of this form [8, Example 7.8.3. (4)]. Thus $\langle X_2 \boxtimes \mathbf{1}\rangle \simeq \mathrm{Vec}(\mathbb{Z}_2) \simeq \mathcal{A}_{q_1}$. The argument for the subcategory $\langle \mathbf{1} \boxtimes Y_2\rangle$ is identical. □

Since $\mathcal{A}_{q_i}^{\mathrm{ad}} \simeq \mathcal{A}_{-q_i}^{\mathrm{ad}}$, q_1 and q_2 are only determined up to sign. The following lemma fixes our choice of q_1.

Lemma 3.2. *By possibly replacing q_1 with $-q_1$ and/or modifying the spherical structure, we may assume $X = X_1 \boxtimes Y_1$ is symmetrically self-dual and*

$$\dim(X) = [2]_1[2]_2.$$

Proof. By changing the pivotal structure by an element of $\mathrm{Hom}(\mathbb{Z}_2 \to \mathbb{C}^\times)$ we can assume that X is symmetrically self-dual. Indeed, X has odd degree with respect to the $\mathbb{Z}_2$-grading, and so changing the spherical structure negates the second Frobenius-Schur indicator of X.

The fusion rules for $\mathcal{C}$ dictate

$$X^{\otimes 2} \cong \mathbf{1} \oplus X_2 \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes Y_2 \oplus X_2 \boxtimes Y_2. \tag{4}$$

Taking dimensions we find

$$\dim(X)^2 = 1 + [3]_1 + [3]_2 + [3]_1[3]_2.$$

Hence

$$\dim(X) = \pm[2]_1[2]_2.$$

By possibly replacing q_1 with $-q_1$ we can ensure that $\dim(X) = [2]_1[2]_2$. □

Remark 3.3. We note some small degenerate cases, which will allows us to restrict q_1 and q_2 (and hence n_1 and n_2). If either of n_1 or n_2 is equal to 2 (corresponding to q_1^2 or q_2^2 having order 3), then K_{n_1,n_2} has either type A or type A_n fusion rules. The classification is already known in these cases [10] and the results of Theorem 1.1 part 2 hold. Hence we can assume both q_1^2 and q_2^2 have orders larger than three. In practical terms, this means we can assume that $[3]_i \neq 0$.

If both n_1 and n_2 are equal to 3 (corresponding to q_1^2 and q_2^2 having order 4), then K_{n_1,n_2} is a Tambara-Yamagami fusion ring with group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, which is another case where a classification is known [17] and the results of Theorem 1.1 part 1 also hold. Hence, for convenience, we may assume that one of q_i^2 has order strictly greater than four. However, the techniques of the section work without this assumption, see Remark 3.16 below. Without loss of generality we assume the order of q_2^2 is strictly greater than four. Hence we can assume $[3]_2 \neq 1$.

3.1. Planar calculations

We now wish to obtain a based semisimple presentation for the category $\mathcal{C}$. To do this we first need to find generators. Using the fusion rule Eq. (4) we can define morphisms P and Q:

Definition 3.4. Let P and Q in $\mathrm{End}_{\mathcal{C}}(X^{\otimes 2})$ denote the minimal idempotents with images isomorphic to $X_2 \boxtimes \mathbf{1}$ and $\mathbf{1} \boxtimes Y_2$, respectively. Let R denote the minimal idempotent whose image is isomorphic to $X_2 \boxtimes Y_2$.

Lemma 3.5. *The set* $\{P, Q, \mid\,\mid, \substack{\smile\\ \frown}\}$ *forms a basis for* $\mathit{End}_{\mathcal{C}}(X^{\otimes 2})$.

Proof. The set $\{P, Q, R, \frac{1}{[2]_1[2]_2}\substack{\smile\\ \frown}\}$ is a complete set of minimal idempotents for $\mathrm{End}(X^{\otimes 2})$, which has dimension 4. So this is a basis for $\mathrm{End}(X^{\otimes 2})$. Since

$$P + Q + R + \frac{1}{[2]_1[2]_2}\substack{\smile\\ \frown} = \mid\,\mid, \tag{5}$$

we see that $\{P, Q, \mid\,\mid, \substack{\smile\\ \frown}\}$ is also a basis. $\square$

Our goal will be to show that P and Q generate a category with the same based semisimple presentation as $\mathcal{C}_{q_1,q_2}$. As discussed in Section 2, this will show that $\mathcal{C}$ is monoidally equivalent to $\mathcal{C}_{q_1,q_2}$.

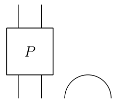
Note that relation (a) is true from our choice of normalization and (b) follows from the fact P and Q are orthogonal idempotents. We show the rest of the relations hold in a series of lemmas.

Lemma 3.6. *The bubble popping relations are satisfied in* $\mathcal{C}$.

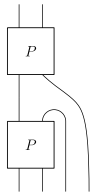
Proof. If we cap off P or Q on the top or bottom, we must get 0 since P and Q are projections onto nontrivial objects of $\mathcal{C}$. Capping the sides of P or Q must result in a scalar times the identity of X, and taking traces yields the result. $\square$

Lemma 3.7. *The triangle popping relations are satisfied in* $\mathcal{C}$.

Proof. The relations that include both P and Q follow from the fusion rules. For instance, $\mathrm{Hom}_{\mathcal{C}}(P \otimes P, Q) = 0$, so the triangle with two $P's$ and a Q is 0. Let us prove the triangle relation involving three P's. By the fusion rules, $\mathrm{Hom}_{\mathcal{C}}(P \otimes X^{\otimes 2}, P)$ is 2-dimensional if $n_1 > 3$, and 1-dimensional if $n_1 = 3$. In either case we claim the space is spanned by the following diagrams:



and



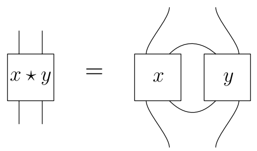
Indeed, by turning the lower right strand upwards to obtain corresponding morphisms in $\mathrm{End}_{\mathcal{C}}(P \otimes X)$, it is seen that the first diagram corresponds to an idempotent whose image is isomorphic to $P \otimes X$, and the second diagram corresponds to a scalar multiple of an idempotent whose image is isomorphic to X. When $n_1 > 3$, we have $P \otimes X \ncong X$, so the diagrams are linearly independent.

Therefore the triangle with 3 P's is a linear combination of these two diagrams. By precomposing with $\mathrm{id}_{X^{\otimes 2}} \otimes P$ and $\mathrm{id}_{X^{\otimes 2}} \otimes \asymp$, the coefficients are determined and give the triangle popping relation.

The case of a triangle with three Q's is very similar. □

The trickiest relation to prove is the Fourier transform equation (c). In order to do this we need to study the convolution algebra of $\mathrm{End}_{\mathcal{C}}(X^{\otimes 2})$.

Definition 3.8. The *convolution algebra* is the vector space $\mathrm{End}_{\mathcal{C}}(X^{\otimes 2})$ with multiplication $x \star y$ defined as follows:



Note that we have $x \star y = \rho(\rho(x)\rho(y))$ (this uses the assumption that $\mathcal{C}$ is strictly pivotal, which we make without loss of generality).

Before deriving the Fourier relation (c), we compute the structure coefficients of the convolution algebra of $\mathrm{End}_{\mathcal{C}}(X^{\otimes 2})$ in the $\{P, Q, \asymp, |\,|\}$ basis. The convolution of anything with $\asymp$ or $|\,|$ is easy to figure out, so it suffices to compute $P \star P$, $P \star Q$ and $Q \star Q$. Recall that the minimal idempotent R has the expression

$$R = |\,| - \frac{1}{[2]_1[2]_2} \asymp - P - Q,$$

by Equation (5).

Lemma 3.9. *We have the following relations in $\mathcal{C}$:*

$$\begin{aligned} P \star P &= \frac{[3]_1}{[2]_1^2[2]_2^2} \genfrac{}{}{0pt}{}{\smile}{\frown} + \frac{[3]_1 - 1}{[2]_1[2]_2} P \\ Q \star Q &= \frac{[3]_2}{[2]_1^2[2]_2^2} \genfrac{}{}{0pt}{}{\smile}{\frown} + \frac{[3]_2 - 1}{[2]_1[2]_2} Q \\ P \star Q &= \frac{1}{[2]_1[2]_2} R = \frac{1}{[2]_1[2]_2} \left(|\,| - \frac{1}{[2]_1[2]_2} \genfrac{}{}{0pt}{}{\smile}{\frown} - P - Q \right). \end{aligned}$$

Proof. Consider the diagram for $P \star P$. This morphism factors through $P \otimes P$, which is an idempotent whose image does not contain $1 \boxtimes Y_2$ or $X_2 \boxtimes Y_2$ (by the fusion rules). Thus $P \star P$ is annihilated by Q and R, so $P \star P$ is contained in the span of $\genfrac{}{}{0pt}{}{\smile}{\frown}$ and P. The coefficients are determined by applying caps to the bottom and side (and using $\dim(P) = [3]_1$ and $\dim(X) = [2]_1[2]_2$).

The equation for $Q \star Q$ is verified similarly. To derive the equation for $P \star Q$, note that $P \otimes Q$ is a minimal idempotent of $\mathrm{End}_{\mathcal{C}}(X^{\otimes 4})$ whose image is a simple object isomorphic to $X_2 \boxtimes Y_2$. Since $X_2 \boxtimes Y_2$ appears with multiplicity 1 in $X \otimes X$ and $P \star Q$ factors through $P \otimes Q$, we see that $P \star Q$ is a scalar multiple of R. The scalar is computed by taking the trace and using a bubble popping relation. □

Remark 3.10. The above lemma shows that the structure constants of the convolution algebra in the $P, Q, |\,|, \genfrac{}{}{0pt}{}{\smile}{\frown}$ basis depend only on q_1 and q_2.

Now that we know the multiplication structure on the convolution algebra, it is routine to compute the minimal idempotents.

Lemma 3.11. *A complete set of minimal idempotents for the convolution algebra $(End_{\mathcal{C}}(X \otimes X), \star)$ is given by*

$$\begin{aligned} \Bigg\{ &\frac{1}{[2]_1[2]_2} |\,|, \\ &\frac{-1}{[2]_1[2]_2} |\,| + \frac{1}{[2]_2^2} \genfrac{}{}{0pt}{}{\smile}{\frown} + \frac{[2]_1}{[2]_2} Q, \\ &\frac{-1}{[2]_1[2]_2} |\,| + \frac{1}{[2]_1^2} \genfrac{}{}{0pt}{}{\smile}{\frown} + \frac{[2]_2}{[2]_1} P, \\ &\frac{1}{[2]_1[2]_2} |\,| + \left(1 - \frac{1}{[2]_1^2} - \frac{1}{[2]_2^2}\right) \genfrac{}{}{0pt}{}{\smile}{\frown} - \frac{[2]_2}{[2]_1} P - \frac{[2]_1}{[2]_2} Q \Bigg\}. \end{aligned}$$

Proof. Using the structure constants given in the previous lemma, we check directly that these elements are mutually orthogonal idempotents. Since $\mathrm{End}_{\mathcal{C}}(X \otimes X)$ is 4-dimensional, they form a complete set of minimal idempotents. □

In the following lemma, we observe that the Fourier transform sends minimal idempotents (with respect to composition) to minimal idempotents (with respect to convolution), and use this to pin down the Fourier transform of P (and hence Q) to one of two possibilities.

Lemma 3.12. *We have two possibilities for the Fourier transform of P. Either*

$$\rho(P) = \frac{-1}{[2]_1[2]_2}\Big|\ \Big| + \frac{1}{[2]_2^2}\genfrac{}{}{0pt}{}{\smile}{\frown} + \frac{[2]_1}{[2]_2}Q$$

or

$$\rho(P) = \frac{-1}{[2]_1[2]_2}\Big|\ \Big| + \frac{1}{[2]_1^2}\genfrac{}{}{0pt}{}{\smile}{\frown} + \frac{[2]_2}{[2]_1}P$$

with the latter case only occurring when $q_1 = \pm q_2^{\pm 1}$.

Proof. The Fourier transform ρ intertwines the standard product and convolution product in $\mathrm{End}_{\mathcal{C}}(X \otimes X)$, so $\rho(P)$ must be a minimal idempotent with respect to the convolution product. Hence it must belong to the set listed in the previous lemma. A simple computation shows that

$$\rho\left(\frac{1}{[2]_1[2]_2}\genfrac{}{}{0pt}{}{\smile}{\frown}\right) = \frac{1}{[2]_1[2]_2}\Big|\ \Big|$$

in the space $\mathrm{End}_{\mathcal{C}}(X \otimes X)$. Since ρ is an involution and P is distinct from $\frac{1}{[2]_1[2]_2}\genfrac{}{}{0pt}{}{\smile}{\frown}$, this rules out one of the possibilities for $\rho(P)$. Thus

$$\begin{aligned}\rho(P) \in \{&\frac{-1}{[2]_1[2]_2}\Big|\ \Big| + \frac{1}{[2]_2^2}\genfrac{}{}{0pt}{}{\smile}{\frown} + \frac{[2]_1}{[2]_2}Q,\\ &\frac{-1}{[2]_1[2]_2}\Big|\ \Big| + \frac{1}{[2]_1^2}\genfrac{}{}{0pt}{}{\smile}{\frown} + \frac{[2]_2}{[2]_1}P,\\ &\frac{1}{[2]_1[2]_2}\Big|\ \Big| + \left(1 - \frac{1}{[2]_1^2} - \frac{1}{[2]_2^2}\right)\genfrac{}{}{0pt}{}{\smile}{\frown} - \frac{[2]_2}{[2]_1}P - \frac{[2]_1}{[2]_2}Q\}.\end{aligned}$$

We want to rule out the third listed solution. Indeed, if $\rho(P)$ was equal to that solution then taking traces gives

$$[3]_1 = [3]_1[3]_2,$$

which implies $[3]_2 = 1$ or $[3]_1 = 0$, a contradiction to Remark 3.3.

In a similar fashion, if $\rho(P)$ was equal to the second solution, then taking traces shows $[3]_1 = [3]_2$. This can only happen if $q_1 = \pm q_2^{\pm 1}$. $\square$

Remark 3.13. When $n_1 = n_2 = 3$, the third listed solution is possible, but does not produce a new category. More precisely, if we rewrite the presentation for $\mathcal{C}_{q_1,q_2}$ in terms of generators $P' = Q$ and $Q' = R$, then P' and Q' satisfy the same relations as P and Q, except $\rho(P')$ is given by the third listed solution above.

Finally, by considering fusion of depth three objects, we can deduce the Fourier transform equation (c):

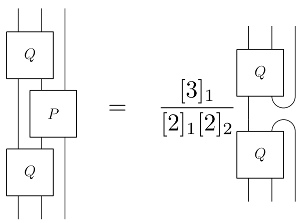
Lemma 3.14. *In $\mathcal{C}$ we have the equation*

$$\rho(P) = \frac{-1}{[2]_1[2]_2} \Big| \; \Big| + \frac{1}{[2]_2^2} \substack{\smile \\ \frown} + \frac{[2]_1}{[2]_2} Q.$$

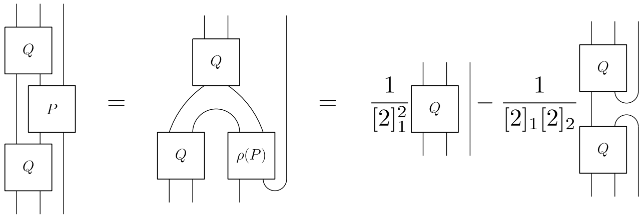
Proof. It suffices to prove that the second solution for $\rho(P)$ and $\rho(Q)$ in the previous lemma is not possible. So assume for contradiction that

$$\rho(P) = \frac{-1}{[2]_1[2]_2} \Big| \; \Big| + \frac{1}{[2]_1^2} \substack{\smile \\ \frown} + \frac{[2]_2}{[2]_1} P$$

To find a contradiction, consider $(Q \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes P)(Q \otimes \mathrm{id}_X)$. Note that $Q \otimes \mathrm{id}_X$ is a sum of two minimal idempotents, one a projection onto a simple isomorphic to X and the other a projection onto a simple isomorphic to $X_1 \boxtimes Y_3$. Since $X_1 \boxtimes Y_3$ does not occur in the image of $\mathrm{id}_X \boxtimes P$, we have that $(Q \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes P)(Q \otimes \mathrm{id}_X)$ must be a scalar times the projection onto X. Taking traces, this proves that



On the other hand, we have:



In the second equality we used our assumption about $\rho(P)$ and also the triangle popping relation to remove a triangle with two Q's and a P. As in the proof of Lemma 3.7, the two diagrams on the right side of the last equation are linearly independent unless $n_2 = 3$. Hence the two expressions for $(Q \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes P)(Q \otimes \mathrm{id}_X)$ can only be equal if $n_2 = 3$. However by Remark 3.3 we assume $n_2 > 3$. $\square$

Remark 3.15. We remark that there do exist categories satisfying the relations of Proposition 2.7, except with the different Fourier transformation

$$(c') \qquad \rho(P) = \frac{-1}{[2]_1[2]_2}\Big|\;\Big| + \frac{1}{[2]_1^2}\genfrac{}{}{0pt}{}{\smile}{\frown} + \frac{[2]_2}{[2]_1}P, \qquad \rho(Q) = \frac{-1}{[2]_1[2]_2}\Big|\;\Big| + \frac{1}{[2]_2^2}\genfrac{}{}{0pt}{}{\smile}{\frown} + \frac{[2]_1}{[2]_2}Q$$

This category is constructed as follows.

If $q_1 = \pm q_2^{\pm 1}$, then the category $\mathcal{C}_{q_1,q_2}$ has an order two monoidal auto-equivalence, which is the restriction of the swap auto-equivalence on $\mathcal{A}_{q_1} \boxtimes \mathcal{A}_{q_2}$. This auto-equivalence exchanges the minimal idempotents P and Q. We claim that the subcategory of the crossed product [11, Section 3.3] $\mathcal{C}_{q_1,q_2} \rtimes \mathbb{Z}_2$ generated by the object X in the non-trivial grading gives the desired category. We leave the proof of this fact to an interested reader.

Note that this subcategory of $\mathcal{C}_{q_1,q_2} \rtimes \mathbb{Z}_2$ does not have $SO(4)$-type fusion rules (except when $n_1 = n_2 = 3$, see the remark below). This differing of fusion rules can first be seen in the third tensor power of X, which explains why we have to consider 3 box relations in order to prove Lemma 3.14.

Remark 3.16. We remark on the case $n_1 = n_2 = 3$. In this case the fusion rules were previously categorified by [17]. There are four categories, parametrized by a choice of bicharacter of $\mathbb{Z}_2 \times \mathbb{Z}_2$, and a choice of sign. The quantum group construction uses q_1 and q_2 primitive eighth roots of unity, and different choices for q_1 and q_2 yield only two inequivalent categories, which are $\mathcal{C}_{\zeta_8,\zeta_8}$ and $\mathcal{C}_{\zeta_8^5,\zeta_8}$ where $\zeta_8 = e^{2\pi i/8}$. By comparing the braidings with those of the TY-categories [14], we see these two categories account for the two TY-categories with bicharacter χ_c (in the notation of [17], Section 4).

The other two TY-categories (with bicharacter χ_1 in the notation of [17], Section 4), are obtained by applying the construction in the previous remark to $\mathcal{C}_{\zeta_8,\zeta_8}$ and $\mathcal{C}_{\zeta_8^5,\zeta_8}$. They have a based semisimple presentation obtained by replacing the Fourier equation (c) with the different equation (c') above.

The only places where the arguments in this section break down for $n_2 = n_1 = 3$ are Lemmas 3.12 and 3.14, where two possibilities for $\rho(P)$ were excluded. When $n_1 = n_2 = 3$, neither of these possibilities cannot be excluded. As mentioned above, the possibility excluded in Lemma 3.14 corresponds to the TY categories not obtained as $\mathcal{C}_{q_1,q_2}$. On the other hand, it can be checked that the possibility excluded in Lemma 3.12 is also possible, and provides an alternate presentation for $\mathcal{C}_{q_1,q_2}$ (see Remark 3.13). Thus the classification technique of this section works for $n_1 = n_2 = 3$ as well.[1]

[1] We thank the referee for suggesting this.

Putting everything together, we have found morphisms P and Q in $\mathcal{C}$ which satisfy the relations of the based semisimple presentation for $\mathcal{C}_{q_1,q_2}$. As explained in the preliminaries, the fact that $\mathcal{C}$ and $\mathcal{C}_{q_1,q_2}$ have the same fusion rules implies $\mathcal{C}$ is equivalent to $\mathcal{C}_{q_1,q_2}$ as a pivotal tensor category.

4. Classification of braidings

In this section we classify all braidings on the fixed monoidal category $\mathcal{C}_{q_1,q_2}$. We will show that the eight braidings given in Definition 2.1 and described in Proposition 2.7 are the only braidings on $\mathcal{C}_{q_1,q_2}$.

We begin by considering the two distinguished subcategories $\mathcal{A}^{\mathrm{ad}}_{q_1}$ and $\mathcal{A}^{\mathrm{ad}}_{q_2}$. As these subcategories are equivalent to $SO(3)$ type categories, we know that if the order of q_i^2 is greater than 4 and not equal to 6, then their braidings are classified by a choice of $q_1^{\pm 1}$ and $q_2^{\pm 1}$ [16].[2]

The next lemma shows that the braidings on these subcategories determine the braiding on their product (which as explained at the start of Section 3 is the adjoint subcategory of $\mathcal{C}_{q_1,q_2}$).

Lemma 4.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be semisimple tensor categories, with universal grading groups [8, Definition 4.14.2] $U(\mathcal{C})$ and $U(\mathcal{D})$. Then braidings on $\mathcal{C} \boxtimes \mathcal{D}$ are determined by braidings on $\mathcal{C}$ and $\mathcal{D}$, together with a bicharacter*

$$a : U(\mathcal{C}) \times U(\mathcal{D}) \to \mathbb{C}\,.$$

Proof. First we show how a braiding on $\mathcal{C} \boxtimes \mathcal{D}$ gives rise to braidings on $\mathcal{C}$ and $\mathcal{D}$ and a bicharacter. Clearly the braiding on the product gives braidings on the factors. Now suppose X is an object of $\mathcal{C}$ and Y an object of $\mathcal{D}$. Then the braiding

$$c_{\mathbf{1}\boxtimes Y, X\boxtimes \mathbf{1},} : \mathbf{1} \boxtimes Y \otimes X \boxtimes \mathbf{1} \to X \boxtimes \mathbf{1} \otimes \mathbf{1} \boxtimes Y$$

describes a morphism $a_{X,Y} \in \mathrm{End}_{\mathcal{C}\boxtimes\mathcal{D}}(X \boxtimes Y)$. The naturality of the braiding on $\mathcal{C} \boxtimes \mathcal{D}$ implies $a_{X,Y}$ is an automorphism of the identity functor of $\mathcal{C} \boxtimes \mathcal{D}$. If we fix one of the factors (say fix an object X in $\mathcal{C}$) then the hexagon identity for the braiding implies $a_{X,-}$ is identified with a monoidal isomorphism of the identity functor of $\mathcal{D}$. In other words, the morphisms $a_{X,Y}$ for X fixed are described by a character of $U(\mathcal{D})$. The same

[2] In the case that the order of q_i^2 is either equal to 6, or less than or equal to 4, we have that $n_i \in \{3, 5\}$. The results of [16] do not apply, and there exist additional Tannakian braidings on the categories $\mathcal{A}^{ad}_{q_i}$. These Tannakian braidings come from the categories $\mathrm{Rep}(\mathbb{Z}_2)$ and $\mathrm{Rep}(S_3)$ respectively. We can repeat the analysis of this section for these special cases. We find that these Tannakian braidings cannot lift to braidings of the categories $\mathcal{C}_{q_1,q_2}$. Furthermore, in the case of $n_1 = 3$, we have that only two of the braidings on the subcategory $\mathcal{A}^{ad}_{q_1} \boxtimes \mathcal{A}^{ad}_{q_2}$ lift to the category $\mathcal{C}_{q_1,q_2}$. However in this case each of these two braidings on $\mathcal{A}^{ad}_{q_1} \boxtimes \mathcal{A}^{ad}_{q_2}$ has four extensions to $\mathcal{C}_{q_1,q_2}$. Hence these special cases are still covered by Theorem 1.1 (albeit via a non-natural bijection). We leave the details to a motivated reader.

considerations hold when fixing an object Y of $\mathcal{D}$, and the conclusion is that $a_{X,Y}$ may be identified with a bicharacter of $U(\mathcal{C}) \times U(\mathcal{D})$.

Now we show that braidings c_{X_1,X_2} on $\mathcal{C}$ and d_{Y_1,Y_2} on $\mathcal{D}$ together with a bicharacter a uniquely determine a braiding on $\mathcal{C} \boxtimes \mathcal{D}$. Suppose X_1, X_2 are in $\mathcal{C}$ and Y_1, Y_2 are in $\mathcal{D}$. Then the braiding in $\mathcal{C} \boxtimes \mathcal{D}$ on $(X_1 \boxtimes Y_1) \otimes (X_2 \boxtimes Y_2)$ factors as

$$(c_{X_1,X_2} \boxtimes d_{Y_1,Y_2}) \circ (1 \otimes a_{X_2,Y_1} \otimes 1)$$

which shows how the braiding on the product is completely determined by c, d and a. $\square$

Corollary 4.2. *There exist four distinct braidings on the subcategory*

$$\mathcal{C}^{ad}_{q_1,q_2} = \mathcal{A}^{ad}_{q_1} \boxtimes \mathcal{A}^{ad}_{q_2}.$$

These are parameterized by the four choices of $q_1^{\pm 1}$ and $q_2^{\pm 1}$.

Proof. The universal grading group of $\mathcal{A}^{\mathrm{ad}}_q$ is trivial, so by the previous lemma the braiding on $\mathcal{C}^{\mathrm{ad}}_{q_1,q_2}$ is determined by the braidings on the factors. By the classification of braidings on $SO(3)$ type categories by Tuba and Wenzl [16] there are exactly two braidings on $\mathcal{A}^{\mathrm{ad}}_q$, parametrized by the choice of q or q^{-1}. $\square$

Let us fix one of these four possible braidings. As the monoidal category $\mathcal{C}_{q_1,q_2}$ is determined up to $q_1 \to q_1^{-1}$ and $q_2 \to q_2^{-1}$, we can freely choose q_1 and q_2 so that this braiding corresponds to the choice q_1^{+1} and q_2^{+1} in the above lemma. In particular using [16, Lemma 8.4] we see the twists of P and Q are q_1^4 and q_2^4 respectively. As $R \cong P \otimes Q$ we can use Corollary 4.2 to see the twist of R is $(q_1 q_2)^4$. Summarizing, we have the following twists in $\mathcal{C}_{q_1,q_2}$:

$$\theta_{\mathbf{1}} = 1, \qquad \theta_P = q_1^4, \qquad \theta_Q = q_2^4, \quad \text{and} \quad \theta_R = (q_1 q_2)^4.$$

With these twists in hand, it is straightforward to determine all possible braidings on $\mathcal{C}_{q_1,q_2}$ compatible with the fixed braiding on $\mathcal{C}^{\mathrm{ad}}_{q_1,q_2}$.

Lemma 4.3. *There exist two braidings on $\mathcal{C}_{q_1,q_2}$ which restrict to a fixed braiding on $\mathcal{C}^{ad}_{q_1,q_2}$.*

Proof. For this proof it is more convenient to work in the idempotent basis of $\mathrm{End}_{\mathcal{C}_{q_1,q_2}}(X \otimes X)$. The braiding on $\mathcal{C}_{q_1,q_2}$ is determined by

 $= \alpha_{\mathbf{1}} \dfrac{1}{[2]_1 [2]_2}$  $+ \alpha_P P + \alpha_Q Q + \alpha_R R,$

where $\alpha_{\mathbf{1}}, \alpha_P, \alpha_Q, \alpha_R \in \mathbb{C}$. As we know the twists on $\mathbf{1}, P, Q$, and R we can use the balancing equation [8, Equation 8.32] to find

$$1 = \theta_X^2 \alpha_{\mathbf{1}}^2, \qquad q_1^4 = \theta_X^2 \alpha_P^2, \qquad q_2^4 = \theta_X^2 \alpha_Q^2, \quad \text{and} \quad (q_1 q_2)^4 = \theta_X^2 \alpha_R^2.$$

This allows us to determine α_P, α_Q and α_R in terms of $\alpha_{\mathbf{1}}$, up to sign. For some $\epsilon_P, \epsilon_Q, \epsilon_R \in \{-1, 1\}$ we have

$$\alpha_P = \epsilon_P q_1^2 \alpha_{\mathbf{1}}, \qquad \alpha_Q = \epsilon_Q q_2^2 \alpha_{\mathbf{1}}, \quad \text{and} \quad \alpha_R = \epsilon_R (q_1 q_2)^2 \alpha_{\mathbf{1}}.$$

To determine $\alpha_{\mathbf{1}}$ and the three signs, we use that the inverse of the braiding is equal to its Fourier transform [15, Theorem 2.5], as follows. Using Eq. (5), we have

$$\times = \alpha_R \,|\;|\, + \left(\frac{\alpha_{\mathbf{1}} - \alpha_R}{[2]_1[2]_2}\right) \asymp + (\alpha_Q - \alpha_R)P + (\alpha_Q - \alpha_R)Q.$$

Applying the Fourier transform and using the Fourier relations for P and Q we find

$$\rho\left(\times\right) = \left(\frac{\alpha_{\mathbf{1}} - \alpha_P - \alpha_Q + \alpha_R}{[2]_1[2]_2}\right) |\;|\, + \left(\alpha_R + \frac{\alpha_P - \alpha_R}{[2]_2^2} + \frac{\alpha_Q - \alpha_R}{[2]_1^2}\right) \asymp$$
$$+ \frac{(\alpha_Q - \alpha_R)[2]_2}{[2]_1} P + \frac{(\alpha_P - \alpha_R)[2]_1}{[2]_2} Q.$$

On the other hand,

$$\left(\times\right)^{-1} = \alpha_{\mathbf{1}}^{-1} \frac{1}{[2]_1[2]_2} \asymp + \alpha_P^{-1} P + \alpha_Q^{-1} Q + \alpha_R^{-1} R$$
$$= \alpha_R^{-1} \,|\;|\, + \frac{\alpha_{\mathbf{1}}^{-1} - \alpha_R^{-1}}{[2]_1[2]_2} \asymp + (\alpha_P^{-1} - \alpha_R^{-1})P + (\alpha_Q^{-1} - \alpha_R^{-1})Q.$$

Now the equality $\rho\left(\times\right) = \left(\times\right)^{-1}$ produces 4 equations:

$$\alpha_R^{-1} = \frac{1}{[2]_1[2]_2}(\alpha_{\mathbf{1}} - \alpha_P - \alpha_Q + \alpha_R)$$
$$\frac{\alpha_{\mathbf{1}}^{-1} - \alpha_R^{-1}}{[2]_1[2]_2} = \frac{\alpha_P}{[2]_2^2} + \frac{\alpha_Q}{[2]_1^2} + \alpha_R\left(1 - \frac{1}{[2]_1^2} - \frac{1}{[2]_2^2}\right)$$
$$\alpha_P^{-1} - \alpha_R^{-1} = \frac{[2]_2}{[2]_1}(\alpha_Q - \alpha_R)$$
$$\alpha_Q^{-1} - \alpha_R^{-1} = \frac{[2]_1}{[2]_2}(\alpha_P - \alpha_R).$$

The last two equations yield

$$[2]_1^2(2 - \epsilon_P \epsilon_R (q_2^2 + q_2^{-2})) = [2]_2^2(2 - \epsilon_Q \epsilon_R (q_1^2 + q_1^{-2})).$$

Solving this equation shows four cases:

$$\begin{aligned}
\epsilon_P = \quad & \epsilon_Q = \quad & -\epsilon_R \qquad & \text{for all } q_1 \text{ and } q_2, \\
\epsilon_P = \quad & \epsilon_Q = \quad & \epsilon_R \qquad & \text{for } q_1 = \pm q_2^{\pm 1}, \\
\epsilon_P = \quad & -\epsilon_Q = \quad & \epsilon_R \qquad & \text{for } q_1^2 = -1, \text{ or } q_2^4 = -1, \\
\epsilon_P = \quad & -\epsilon_Q = \quad & -\epsilon_R \qquad & \text{for } q_1^4 = -1, \text{ or } q_2^2 = -1.
\end{aligned}$$

Immediately we can disregard the latter two cases, due to Remark 3.3. In the second case we can use the third equation to find

$$\alpha_{\mathbf{1}}^2 = \begin{cases} \pm 1 & \text{if } q_2 = \pm q_1^{-1} \\ \mp q_1^{-6} & \text{if } q_2 = \pm q_1. \end{cases}$$

However we can now consider the first equation which tells us that either $q_2^4 = -1$ or q_2 is a primitive 6-th root of unity, both of which have already been dealt with in Remark 3.3.

Finally we have the first case. Again we use the third equation to find

$$\alpha_{\mathbf{1}}^2 = \frac{1}{q_1^3 q_2^3}.$$

Comparing this to the first equation shows that $\epsilon_P = -1$. Hence we have two possible solutions for the braiding, corresponding to the two square roots of $\alpha_{\mathbf{1}}^2 = \frac{1}{q_1^3 q_2^3}$. These two braidings exist as they are realized in Proposition 2.7. □

Putting everything together, we have classified all braidings on the categories $\mathcal{C}_{q_1,q_2}$. This completes the proof of part 2 of Theorem 1.1.

Data availability

No data was used for the research described in the article.

References

[1] Francis Borceux, Dominique Dejean, Cauchy completion in category theory, Cah. Topol. Géom. Différ. Catég. 27 (2) (1986) 133–146.
[2] Dietmar Bisch, Vaughan Jones, Algebras associated to intermediate subfactors, Invent. Math. 128 (1) (1997) 89–157.
[3] Dietmar Bisch, Vaughan Jones, Singly generated planar algebras of small dimension, Duke Math. J. 101 (1) (2000) 41–75.
[4] Stephen Bigelow, Emily Peters, Scott Morrison, Noah Snyder, Constructing the extended Haagerup planar algebra, Acta Math. 209 (1) (2012) 29–82.
[5] Joan S. Birman, Hans Wenzl, Braids, link polynomials and a new algebra, Trans. Am. Math. Soc. 313 (1) (1989) 249–273.
[6] Daniel Copeland, Classification of ribbon categories with the fusion rules of $SO(N)$, PhD thesis, 2020.

[7] P. Deligne, La catégorie des représentations du groupe symétrique S_t, lorsque t n'est pas un entier naturel, in: Algebraic Groups and Homogeneous Spaces, in: Tata Inst. Fund. Res. Stud. Math., vol. 19, Tata Inst. Fund. Res, Mumbai, 2007, pp. 209–273.
[8] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, Victor Ostrik, Tensor Categories, Mathematical Surveys and Monographs, vol. 205, American Mathematical Society, Providence, RI, 2015.
[9] P. Etingof, V. Ostrik, On semisimplification of tensor categories, https://arxiv.org/abs/1801.04409, January 2018.
[10] Jürg Fröhlich, Thomas Kerler, Quantum Groups, Quantum Categories and Quantum Field Theory, Lecture Notes in Mathematics, vol. 1542, Springer-Verlag, Berlin, 1993.
[11] César Galindo, Crossed product tensor categories, J. Algebra 337 (2011) 233–252.
[12] D. Kazhdan, H. Wenzl, Reconstructing monoidal categories, Adv. Sov. Math. 16 (1993) 111–136.
[13] Zhengwei Liu, Exchange relation planar algebras of small rank, Trans. Am. Math. Soc. 368 (2016).
[14] Jacob Siehler, Near-group categories, https://arxiv.org/abs/math/0011037, 2000.
[15] V.G. Turaev, Quantum Invariants of Knots and 3-Manifolds, 3rd corr. ed. edition, De Gruyter, 2016.
[16] I. Tuba, H. Wenzl, On braided tensor categories of type BCD, J. Reine Angew. Math. 581 (2005) 31–69.
[17] Daisuke Tambara, Shigeru Yamagami, Tensor categories with fusion rules of self-duality for finite abelian groups, J. Algebra 209 (2) (1998) 692–707.
[18] Hans Wenzl, Hecke algebras of type A_n and subfactors, Invent. Math. 92 (2) (1988) 349–383.