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Master equation for Cournot mean field games of control with absorption

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Abstract

We establish the existence and uniqueness of a solution to the master equation for a mean field game of controls with absorption. The mean field game arises as a continuum limit of a dynamic game of exhaustible resources modeling Cournot competition between producers. The proof relies on an analysis of a forward-backward system of nonlocal Hamilton-Jacobi/Fokker-Planck equations with Dirichlet boundary conditions. In particular, we establish new a priori estimates to prove that solutions are differentiable with respect to the initial measure.

Keywords: Mean field games; Exhaustible resources

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1. Introduction

In [26], the authors introduced a dynamic game of exhaustible resource production modeling Cournot competition between producers of a good in finite supply, for instance oil, whose Markov perfect (Nash) equilibrium was characterized there by a system of coupled nonlinear PDEs. This built on the influential continuous-time study of the monopoly (single-player) version of the problem by Hotelling from 1931 [27]. By Cournot competition, we mean that the decision or control variable of the players is their quantity (or rate) of production, the market price or prices of the goods being determined by a decreasing function of the aggregate (or average) production.

When the goods each player produces are homogeneous, there is a single price p of the good which depends, in the Cournot framework, on the average $\frac{1}{N}\sum_j^N q_j$, where $q_j \geq 0$ is player j's quantity, and there are $N < \infty$ players. When the goods are substitutable, for instance oil of different grades from different suppliers, or consumer goods such as televisions, a typical model has that the price p_i that producer i receives for its good depends in a decreasing manner on $q_i + \frac{\epsilon}{N-1}\sum_{j\neq i} q_j$. That is, its price is influenced by the average of the other players' quantities (thereby viewing them as exchangeable), where $\epsilon > 0$ measures the degree of interaction. A dynamic exhaustible resources problem in this case is analyzed in [31].

Mean field games, in which there is a continuum of players, have been much-studied in the past 15 years. We refer, for instance, to [1] and [8], for surveys from PDE and probabilistic perspectives respectively. In the context of the Cournot model, the homogeneous goods case leads to a continuum approximation model whose optimal strategies are of (unrealistic) bang-bang type: the players either produce nothing or as quickly as possible. The substitutable goods case has a more reasonable mean field game model, as studied in [10] and [11]. As mean field games of control, and because the state variable is absorbed at zero (exhaustion of the resource), they differ from the vast majority of problems studied in the literature where interaction is through the mean of the state variable, which lives on the full space. Rigorous existence results are thus more recent and under various restrictions, for instance [19,22,23,20]. We refer the reader to [7,17,18,28] for benchmark results on mean field games of controls.

There has been much recent interest in describing mean field games through a Master Equation [4,2,9]. The study of such equations now has a large body of literature, going back to such works as [15,12]. Again the existing results in the literature concern mean field interaction through the state. See the recent results found in [13,14,32,33]. As for boundary conditions, most references contain results only for master equations on the whole space or with periodic boundary conditions. See, however, the recent work by Ricciardi for Neumann boundary conditions [35]. Here we introduce and analyze the Master Equation of Cournot mean field games of control with absorption. Our main result is the existence and uniqueness of a classical solution.

Once one has a unique classical solution to the master equation, a natural application is to the convergence problem for N-player games corresponding to a mean field game. Using the arguments of [4, Chapter 6], one can hope to obtain estimates that prove the closed-loop Nash equilibrium strategies for N-player games converge to the mean field equilibrium strategy. In our case, the infinite time horizon, the dependence of the dynamics on the distribution of controls, and especially the absorbing boundary conditions add technical obstacles to a straightforward application of the arguments found in [4]. We leave this application to future research.

In the rest of this section, we introduce the main notation needed and give our main results. In Section 1.1, we give the precise description of the Cournot model as a mean field game and write the corresponding master equation. In Section 1.3, we define a metric on the space of measures and introduce a notion of derivative for functions defined on this space. In Section 1.4, we give the definition of a solution to the master equation and present Theorem 1.5, which gives precise conditions under which a unique solution exists. Finally, in Section 1.5 we present the outline of the rest of the paper, which is devoted to the proof of Theorem 1.5.

1.1. Description of the model

Let $P : [0, \infty) \to \mathbb{R}$ be a given price function, satisfying the following:

Assumption 1.1. P is continuous on $[0, \infty)$ with $P(0) > 0$. For some $n \geq 4$, P is n times continuously differentiable on $(0, \infty)$, $P^{(n)}$ is locally Lipschitz, and $P' < 0$. In addition, $\limsup_{q \to 0+} P'(q)$ is strictly less than zero (it could be $-\infty$), and there exists a finite *saturation point* $\eta > 0$ such that $P(\eta) = 0$.

The *profit function* $\pi : [0, \infty)^4 \to \mathbb{R}$ for an individual producer is given by

$$\pi(\epsilon, q, Q, a) = \begin{cases} q\left(P(\epsilon Q + q) - a\right) & \text{if } q > 0, \\ 0 & \text{if } q = 0. \end{cases}$$

Here q is the rate of production chosen by the producer, Q is the market's aggregate rate of production, a is the marginal cost of production, and $\epsilon \geq 0$ is a fixed parameter that determines the substitutability of goods.

It will be convenient to define the *relative prudence*

$$\rho(Q) := -\frac{Q P''(Q)}{P'(Q)}$$

Notice that by Assumption 1.1, ρ is continuously differentiable on $(0, \infty)$. If, for example, we take $P'(q) = -q^{-\rho}$ for some fixed $\rho \in \mathbb{R}$ (cf. [26]), then $\rho(Q) = \rho$ (constant relative prudence).

Assumption 1.2 *(Relative prudence).* We assume

$$\bar{\rho} := \sup_{Q \in (0,\infty)} \rho(Q) < \frac{2 + \epsilon}{1 + \epsilon} \leq 2.$$

Assumptions 1.1 and 1.2 guarantee a Hamiltonian of the following continuous time game is well-defined.

In the finite N-player differential game introduced in [26], each player i has remaining stock (or reserves) $x_i(t)$ at time $t \geq 0$ and we denote by $\bar{q}_i(t) \geq 0$ their chosen rate of production, so $x_i(t)$ satisfies the stochastic differential equation

$$\mathrm{d}x_i(t) = \left(-\bar{q}_i(t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_i(t)\right)\mathbb{I}_{\{x_i(t)>0\}},$$

where each $W_i(t)$ is an independent standard Brownian motion representing, for instance, uncertainty in the extraction process. The producers start with initial ($t = 0$) reserves $\mathbf{x} \in \mathbb{R}_+^N$ and each maximizes expected discounted lifetime profit. The value function $u_i : \mathbb{R}_+^N \to \mathbb{R}$ of player i is given by

$$u_i(\mathbf{x}) = \sup_{\bar{q}_i} \mathbb{E} \int_0^{\tau_i} e^{-rt} \pi\left(\epsilon, \bar{q}_i(t), \bar{Q}_{-i}(t), 0\right) \mathrm{d}t, \tag{1.1}$$

where τ_i is the first time x_i hits (and is absorbed at) zero, $r \geq 0$ is the common discount rate on future profits, $\bar{Q}_{-i}(t)$ is the mean production rate of the other producers:

$$\bar{Q}_{-i}(t) = \frac{1}{N-1}\sum_{j \neq i} \bar{q}_j(t),$$

and we assume for simplicity that marginal costs of production are zero.

The Hamilton-Jacobi-Bellman equation corresponding to each player's optimal control problem in (1.1) is as follows. Define

$$H(\epsilon, Q, a) := \sup_{q \geq 0} \pi(\epsilon, q, Q, a) \quad \text{from which it follows}$$

$$\operatorname{argmax}_{q \geq 0} \pi(\epsilon, q, Q, a) = -\frac{\partial H}{\partial a}(\epsilon, Q, a).$$

In a Markov perfect (Nash) equilibrium of the N-player differential game the associated system of Hamilton-Jacobi-Bellman (HJB) partial differential equations (PDEs) for the value functions is

$$H\left(\epsilon, \bar{Q}^*_{-i}(\mathbf{x}), \frac{\partial u_i}{\partial x_i}\right) + \sum_{j \neq i} \frac{\partial H}{\partial a}\left(\epsilon, \bar{Q}^*_{-j}(\mathbf{x}), \frac{\partial u_j}{\partial x_j}\right)\frac{\partial u_i}{\partial x_j} - r u_i + \frac{\sigma^2}{2}\sum_{j=1}^{N} \frac{\partial^2 u_i}{\partial x_j^2} = 0, \qquad (1.2)$$

coupled with

$$\bar{Q}^*_{-i}(\mathbf{x}) = -\frac{1}{N-1}\sum_{j \neq i} \frac{\partial H}{\partial a}\left(\epsilon, \bar{Q}^*_{-j}(\mathbf{x}), \frac{\partial u_j}{\partial x_j}\right).$$

See [26, Equation (3.4)]; here we have additional diffusion terms due to the Brownian noise in the dynamics.

The mean field game (MFG) version of this problem, corresponding to a continuum of players with density of initial reserves m was introduced in [10] and further studied in [11], where it is characterized by two PDEs and a fixed point condition (which are given here in Section 1.5). An explicit solution of the deterministic MFG ($\sigma = 0$) when the price function P is linear is given in [24].

We next introduce the master equation formulation of this MFG.

1.2. Master equation heuristics

Let m be a measure representing the initial distribution of stock over all producers. Let $U(x, m)$ be the maximum discounted lifetime profit for an individual producer that starts with a stock of x. If we assume that U is smooth with respect to both variables (see Definition 1.3 below for derivatives in the space of measures), then U will satisfy

$$H\left(\epsilon, Q^*, \frac{\partial U}{\partial x}(m,x)\right) + \int_{\mathcal{D}} \frac{\partial H}{\partial a}\left(\epsilon, Q^*, \frac{\partial U}{\partial x}(m,y)\right) \frac{\partial}{\partial y}\frac{\delta U}{\delta m}(m,x,y)\,\mathrm{d}m(y) - rU(m,x) + \frac{\sigma^2}{2}\left(\frac{\partial^2 U}{\partial x^2}(m,x) + \int_{\mathcal{D}} \frac{\partial^2}{\partial y^2}\frac{\delta U}{\delta m}(m,x,y)\,\mathrm{d}m(y)\right) = 0, \tag{1.3}$$

where Q^* is defined as the unique fixed solution of the equation

$$Q^* = -\int_{\mathcal{D}} \frac{\partial H}{\partial a}\left(\epsilon, Q^*, \frac{\partial U}{\partial x}(m,y)\right)\mathrm{d}m(y). \tag{1.4}$$

Equation (1.3) is called the *master equation*.

Formally, the master equation can be derived from the system of Hamilton-Jacobi-Bellman (HJB) equations (1.2) for the N-player game. Letting $N \to \infty$, we formally interpret each sum as an integral with respect to the population distribution. See [4,9] for a detailed interpretation of the master equation.

1.3. Metric and derivative on a space of measures

Before we can state our main result, we will need to define a notion of derivative with respect to a measure. Let $\mathcal{M} = \mathcal{M}(\mathcal{D})$ be the space of all finite signed Radon measures μ on $\mathcal{D}$. We denote by $\mathcal{M}_+$ the subset of $\mathcal{M}$ consisting only of positive measures. The topology on $\mathcal{M}$ is that of narrow convergence. We say that a sequence $\{\mu_n\}$ in $\mathcal{M}$ converges narrowly if for every bounded continuous function ϕ on $\mathcal{D}$, we have

$$\int_{\mathcal{D}} \phi(x)\,\mathrm{d}\mu_n(x) \to \int_{\mathcal{D}} \phi(x)\,\mathrm{d}\mu(x).$$

We now introduce the derivative on $\mathcal{M}(\mathcal{D})$.

Definition 1.3 *(Differentiability with respect to measures).* Let $\mathscr{M}$ be any dense subset of $\mathcal{M}_+$. Given a function $F : \mathscr{M} \to \mathbb{R}$, we say that F is continuously differentiable if there exists a continuous function $f : \mathscr{M} \times \mathcal{D} \to \mathbb{R}$, satisfying

$$\left|f(m,x)\right| \le C(m)\ \forall x \in \mathcal{D}$$

for some constant $C(m)$, such that

$$\lim_{t\to 0+} \frac{F\left(m + t(\hat{m} - m)\right) - F(m)}{t} = \int_{\mathcal{D}} f(m,x)\,\mathrm{d}(\hat{m} - m)(x) \quad \forall m, \hat{m} \in \mathscr{M}. \tag{1.5}$$

The function $f(m,x)$ is unique, and we denote it $f(m,x) = \dfrac{\delta F}{\delta m}(m,x)$.

Definition 1.3 is essentially the classical Gâteaux derivative, though we only take $m, \hat{m}$ from the convex subset $\mathscr{M}$ of the vector space $\mathcal{M}$. Uniqueness follows from the fact that the measure $\hat{m} - m$ in (1.5) can be taken to be an essentially arbitrary positive measure (by density of $\mathscr{M}$ in $\mathcal{M}_+$); contrast with the situation in which $m, \hat{m}$ must be probability measures (cf. [4]).

1.4. Statement of the main result

To state our main result, we will first define a set of measures on which the master equation (1.3) is supposed to hold. Fix $\alpha \in (0, 1)$ and let $\mathcal{M}^{2+\alpha}$ denote the set of all positive measures m on $\mathcal{D} = (0, \infty)$ satisfying the condition

$$\int_{\mathcal{D}} x^{-2-\alpha}\, \mathrm{d}m(x) < \infty.$$

Definition 1.4. We say that a function $U : \mathcal{D} \times \mathcal{M}^{2+\alpha} \to \mathbb{R}$ is a (classical) solution of the master equation (1.3)-(1.4) with absorbing boundary conditions provided it satisfies the following:

(1) $U(0, m) = 0$ for every $m \in \mathcal{M}^{2+\alpha}$;
(2) U and $\dfrac{\delta U}{\delta m}$ are twice continuously differentiable with respect to x;
(3) for every $m \in \mathcal{M}^{2+\alpha}$ and $x > 0$, Equation (1.3) is satisfied.

The Dirichlet boundary condition $U(0, m) = 0$ is an absorbing type boundary condition, representing the fact that players exit the game as they run out of resources (cf. [10,25]). Theorem 1.5 is the first result, as far as we know, on the Master Equation with boundary conditions of this type.

Our main result in this paper is as follows.

Theorem 1.5. *Under Assumptions 1.1 and 1.2, the following assertions hold.*

(1) *There exist constants $r^* > 0$ (large) and $\epsilon^* > 0$ (small) such that whenever $r \geq r^*$ and $0 < \epsilon \leq \epsilon^*$, the master equation (1.3) has a solution, which is unique under the condition (6.7) (cf. Section 5.5).*
(2) *If P is linear, and in particular if (without loss of generality) $P(q) = 1 - q$, then there exists a constant r^* such that for every $r \geq r^*$ and $\epsilon < 2$, the master equation (1.3) has a solution, which is unique under the condition (6.7) (cf. Section 5.5).*

Remark 1.6. The precise conditions on r^* and ϵ^* in Theorem 1.5 are contained in Assumptions 5.26 and 5.27. Although these two conditions are essentially in dichotomy, nevertheless in this paper we make an attempt to utilize as much as possible a unified method of proof for both cases. See Remark 4.17 for more details.

1.5. Structure of the proof

In a generalized sense, we use the method of characteristics to solve the master equation (1.3)-(1.4). Consider the HJB/Fokker-Planck system

$$\begin{cases} (i) & \dfrac{\partial u}{\partial t} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 u}{\partial x^2} + H\left(\epsilon, Q^*(t), \dfrac{\partial u}{\partial x}\right) - ru = 0, \\ (ii) & \dfrac{\partial m}{\partial t} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 m}{\partial x^2} + \dfrac{\partial}{\partial x}\left(\dfrac{\partial H}{\partial a}\left(\epsilon, Q^*(t), \dfrac{\partial u}{\partial x}\right) m\right) = 0, \\ (iii) & Q^*(t) = -\displaystyle\int_{\mathcal{D}} \dfrac{\partial H}{\partial a}\left(\epsilon, Q^*(t), \dfrac{\partial u}{\partial x}\right) \mathrm{d}m(t), \\ (iv) & m|_{x=0} = u|_{x=0} = 0, \ m|_{t=0} = m_0 \in \mathcal{M}_+(\mathcal{D}) \end{cases} \tag{1.6}$$

where $\mathcal{D} := (0, \infty)$. We can think of System (1.6) as the characteristics of Equation (1.3). Indeed, suppose U is a smooth solution to (1.3) and (u, m) is a smooth solution to (1.6). Then formally the two are related by the equation $u(x, t) = U(x, m(t))$, and in particular $U(x, m_0) = u(x, 0)$. In the proof of our main result, our strategy will be to *define* a function U in this way, then prove that it satisfies (1.3). To do this, we follow these steps:

(1) Prove that (1.6) has a unique solution (u, m) for any $m_0 \in \mathcal{M}^\alpha$. Define $U(x, m_0) = u(x, 0)$.
(2) Prove that U is differentiable with respect to the measure variable m_0:
   (a) Formally differentiate (1.6) with respect to the measure variable to obtain a linearized system.
   (b) Prove that the linearized system has a unique solution.
   (c) Prove that the unique solution thus obtained is indeed the derivative of U with respect to the measure.
(3) Use the smoothness of U to establish that System (1.3)-(1.4) is satisfied.

The remainder of this paper is structured as follows. In Section 2 we establish notation and define function spaces as needed. In Section 3 we study the Fokker-Planck equation with absorbing boundary conditions and establish some results that allow us to prove existence of solutions to System (1.6); they may also have independent interest. In Section 4 we present existence, uniqueness, and regularity results on System (1.6). Section 5 is the core this paper, in which we derive all of the a priori estimates on linearized systems that will allow us to prove differentiability of the master field $U(x, m)$. Here the reader will find some parallels with a recent work by Graber and Laurel that also deals with linearized systems in order to analyze sensitivity of solutions to the parameter ϵ [21]. In the present work, the analysis is considerably more sophisticated because we are taking derivatives with respect to a *measure* and not a scalar parameter; this requires estimates on a linearized system in appropriate norms, in particular dual spaces that introduce a great deal of technicalities. The main result is proved in Section 6, essentially as a corollary of Section 5. Proofs of some technical results are left in the appendix.

2. Preliminaries

2.1. Function spaces

Let $\mathcal{D} = (0, \infty)$. For $n \in \mathbb{N}$, we denote by $\mathcal{C}^n = \mathcal{C}^n(\overline{\mathcal{D}})$ the space of all n times continuously differentiable functions on $\overline{\mathcal{D}}$ such that the norm

$$\|f\|_{\mathcal{C}^n(\overline{\mathcal{D}})} = \sum_{k=0}^{n} \sup_{x \in \overline{\mathcal{D}}} \left| \frac{\mathrm{d}^k f}{\mathrm{d}x^k}(x) \right|$$

is finite; $\mathcal{C}^n(\overline{\mathcal{D}})$ is a Banach space endowed with this norm. In particular, $\mathcal{C}^0(\overline{\mathcal{D}})$ is simply the space of all continuous functions, endowed with the supremum norm. We denote by $\mathcal{C}_c^n = \mathcal{C}_c^n(\mathcal{D})$ the space of all n times continuously differentiable functions which have compact support contained in $\mathcal{D}$; this is a subspace of $\mathcal{C}^n(\overline{\mathcal{D}})$, and $\mathcal{C}_0^n(\mathcal{D})$ denotes its closure. We also denote $\mathcal{C}_c^\infty(\mathcal{D}) = \cap_{n=1}^\infty \mathcal{C}_c^n(\mathcal{D})$.

For any $\alpha \in (0,1)$, define the Hölder seminorm

$$[f]_\alpha := \sup_{x,y\in\overline{\mathcal{D}},x\neq y} \frac{\left|f(x)-f(y)\right|}{\left|x-y\right|^\alpha}.$$

Define $\mathcal{C}^{n+\alpha} = \mathcal{C}^{n+\alpha}(\overline{\mathcal{D}})$ to be the space of all n times continuously differentiable functions f whose nth derivative is Hölder continuous, such that the norm

$$\|f\|_{\mathcal{C}^{n+\alpha}(\overline{\mathcal{D}})} = \|f\|_{\mathcal{C}^n(\overline{\mathcal{D}})} + \left[\frac{\mathrm{d}^n f}{\mathrm{d}x^n}\right]_\alpha$$

is finite. In particular, when $n=0$ the space $\mathcal{C}^\alpha(\overline{\mathcal{D}})$ is simply the space of all α-Hölder continuous functions with standard norm. We define $\mathcal{C}_\diamond^\alpha = \mathcal{C}_\diamond^\alpha(\mathcal{D})$ to be the space of all $f \in \mathcal{C}^\alpha(\overline{\mathcal{D}})$ such that $f(0)=0$.

When $\alpha = 1$, the quantity $[f]_\alpha$ defined above is referred to as the Lipschitz constant of f, denoted $\mathrm{Lip}(f)$ instead of $[f]_1$. We define $\mathrm{Lip}(\overline{\mathcal{D}})$ to be the space of all Lipschitz continuous functions on $\overline{\mathcal{D}}$, with norm

$$\|f\|_{\mathrm{Lip}(\overline{\mathcal{D}})} = \|f\|_{\mathcal{C}^0} + \mathrm{Lip}(f),$$

and the subspace $\mathrm{Lip}_\diamond(\mathcal{D})$ the set of all $f \in \mathrm{Lip}(\overline{\mathcal{D}})$ such that $f(0)=0$.

We now define Hölder spaces of functions on space-time. Let $I=[0,T]$ or $I=\left[0,\infty\right)$. For any number $\beta \geq 0$ we define the space $\mathcal{C}^{\beta,0}(\overline{\mathcal{D}}\times I)$ to be the set of all functions $u : \overline{\mathcal{D}} \times I \to \mathbb{R}$ such that the following norm is finite:

$$\|u\|_{\mathcal{C}^{\beta,0}} = \|u\|_{\mathcal{C}^{\beta,0}(\overline{\mathcal{D}}\times I)} := \sup_{t\in I}\left\|u(\cdot,t)\right\|_{\mathcal{C}^\beta(\overline{\mathcal{D}})}.$$

For any $\alpha \in (0,1)$ define

$$[u]_{\alpha,\alpha/2} := \sup_{x,y\in\overline{\mathcal{D}},t,s\in I,x\neq y,t\neq s} \frac{\left|u(x,t)-u(y,s)\right|}{\left|x-y\right|^\alpha + |t-s|^{\alpha/2}}.$$

We denote by $\mathcal{C}^{\alpha,\alpha/2}(\overline{\mathcal{D}}\times I)$ the subspace of $\mathcal{C}^{0,0}(\overline{\mathcal{D}}\times I)$ such that the norm

$$\|u\|_{\mathcal{C}^{\alpha,\alpha/2}(\overline{\mathcal{D}}\times I)} := \|u\|_{\mathcal{C}^{0,0}(\overline{\mathcal{D}}\times I)} + [u]_{\alpha,\alpha/2}$$

is finite. The space $\mathcal{C}^{2,1}(\overline{\mathcal{D}}\times I)$ consists of functions such that

$$\|u\|_{\mathcal{C}^{2,1}(\overline{\mathcal{D}}\times I)} := \|u\|_{\mathcal{C}^{0,0}(\overline{\mathcal{D}}\times I)} + \left\|\frac{\partial u}{\partial x}\right\|_{\mathcal{C}^{0,0}(\overline{\mathcal{D}}\times I)} + \left\|\frac{\partial^2 u}{\partial x^2}\right\|_{\mathcal{C}^{0,0}(\overline{\mathcal{D}}\times I)} + \left\|\frac{\partial u}{\partial t}\right\|_{\mathcal{C}^{0,0}(\overline{\mathcal{D}}\times I)}$$

is finite, and the subspace $\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathcal{D}} \times I)$ such that

$$\|u\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathcal{D}}\times I)} := \|u\|_{\mathcal{C}^{2,1}(\overline{\mathcal{D}}\times I)} + \left[\frac{\partial^2 u}{\partial x^2}\right]_{\alpha,\alpha/2} + \left[\frac{\partial u}{\partial t}\right]_{\alpha,\alpha/2}$$

is finite. Cf. [29, Section 1.1]. Note that there exist constants C_α such that

$$\|u\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathcal{D}}\times I)} \le C_\alpha \left(\|u\|_{\mathcal{C}^{0,0}(\overline{\mathcal{D}}\times I)} + \left[\frac{\partial^2 u}{\partial x^2}\right]_{\alpha,\alpha/2} + \left[\frac{\partial u}{\partial t}\right]_{\alpha,\alpha/2}\right).$$

We define the Lebesgue spaces L^p in the usual way, and we write the norms $\|f\|_p = \|f\|_{L^p}$ interchangeably.

2.2. Norms on the space of measures

We define the total variation norm $\|\mu\|_{TV} = |\mu|(\mathcal{D})$, which can also be expressed as

$$\|\mu\|_{TV} = \sup\left\{\int_{\mathcal{D}} \phi(x)\,\mathrm{d}\mu(x) : \phi \in \mathcal{C}^0(\mathcal{D}),\ \|\phi\|_{\mathcal{C}^0} \le 1\right\}.$$

Under this norm, $\mathcal{M}$ becomes a Banach space. On the other hand, it is not necessary to converge in this norm in order to converge narrowly. For this it suffices to consider $\mathcal{M}$ as a subspace of the dual of $\mathcal{C}^\alpha_\diamond$, with norm

$$\|\mu\|_{(\mathcal{C}^\alpha_\diamond)^*} = \sup\left\{\int_{\mathcal{D}} \phi(x)\,\mathrm{d}\mu(x) : \phi \in \mathcal{C}^\alpha_\diamond(\mathcal{D}),\ \|\phi\|_{\mathcal{C}^\alpha} \le 1\right\}.$$

We may also replace $\mathcal{C}^\alpha_\diamond$ with $\mathrm{Lip}_\diamond$.

Lemma 2.1. *Let $\{\mu_n\}$ be a sequence in $\mathcal{M}$. If $\|\mu_n\|_{TV}$ is bounded, if $\|\mu_n - \mu\|_{(\mathcal{C}^\alpha_\diamond)^*} \to 0$, and if $\mu_n(\mathcal{D}) \to \mu(\mathcal{D})$, then μ_n converges narrowly to μ.*

Proof. Let ϕ be a bounded, continuous function on $\mathcal{D}$, and let $\varepsilon > 0$. Choose $\psi \in \mathcal{C}^\alpha_\diamond$ such that $\|\phi - \phi(0) - \psi\|_{\mathcal{C}^0} < \varepsilon$. Then

$$\left|\int_{\mathcal{D}} \phi\,\mathrm{d}(\mu_n - \mu)\right| \le \varepsilon\left(\|\mu_n\|_{TV} + \|\mu\|_{TV}\right) + |\phi(0)||\mu_n(\mathcal{D}) - \mu(\mathcal{D})| + \left|\int_{\mathcal{D}} \psi\,\mathrm{d}(\mu_n - \mu)\right|.$$

Using the fact that $\|\mu_n\|_{TV}$ is bounded, we let $n \to \infty$ and then $\varepsilon \to 0$ to conclude. □

2.3. Remark on constants

Throughout this manuscript, C will denote a generic positive constant, whose precise value may change from line to line. When C depends on the data from the problem, will attempt to specify all the parameters on which C depends. In particular, we may write $C(a_1, \dots, a_n)$ to denote a positive number which depends on given parameters $a_1, \dots, a_n$. When no parameters are specified, this means C depends only on the number of steps in the proof (and is generally an increasing function thereof).

3. Fokker-Planck equation with absorbing boundary conditions

Recall $\mathcal{D} := (0, \infty)$. In this section we study weak solutions to a Fokker-Planck equation with Dirichlet boundary conditions:

$$\begin{cases} \dfrac{\partial m}{\partial t} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 m}{\partial x^2} - \dfrac{\partial}{\partial x}(bm) = 0, \\ m|_{x=0} = 0,\ m|_{t=0} = m_0 \end{cases} \tag{3.1}$$

for a given velocity function $b = b(x,t)$. We want an interpretation of (3.1) that makes sense for any $m_0 \in \mathcal{M}(\mathcal{D})$. Thus we say that $m \in \mathcal{C}^0\left([0,T]; \mathcal{M}(\mathcal{D})\right)$ is a *weak solution* of (3.1) provided that, for all $\phi \in \mathcal{C}_c^\infty(\mathcal{D} \times \left[0,T\right))$, we have

$$\int_0^T \int_{\mathcal{D}} \left(-\frac{\partial \phi}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 \phi}{\partial x^2} + b\frac{\partial \phi}{\partial x}\right) m(\mathrm{d}x, t)\,\mathrm{d}t = \int_{\mathcal{D}} \phi(x,0) m_0(\mathrm{d}x). \tag{3.2}$$

Our main existence/uniqueness result is contained in the following lemma. Its proof is fairly standard and is found in Appendix A.

Lemma 3.1. *Let b be a bounded continuous function on $\mathcal{D} \times [0,T]$, and let $m_0 \in \mathcal{M}_{1,+}(\mathcal{D})$. Then there exists a unique weak solution m of (3.1). It satisfies*

$$\left\|m(t)\right\|_{TV} \le \|m_0\|_{TV} \quad \forall t \ge 0. \tag{3.3}$$

It is also Hölder continuous with respect to the $\mathcal{C}^\alpha_\diamond(\mathcal{D})^*$ and $\mathrm{Lip}_\diamond(\mathcal{D})^*$ metrics, and in particular

$$\begin{gathered} \left\|m(t)\right\|_{\mathcal{C}^\alpha_\diamond(\mathcal{D})^*} \le \|m_0\|_{TV}\left(\int_{\mathcal{D}} x^\alpha m_0(\mathrm{d}x) + 2\left(\|b\|_\infty^\alpha + \sigma^\alpha\right)\max\left\{t^\alpha, t^{\alpha/2}\right\}\right) \\ \left\|m(t_1) - m(t_2)\right\|_{\mathcal{C}^\alpha_\diamond(\mathcal{D})^*} \le 2\|m_0\|_{TV}\left(\|b\|_\infty^\alpha + \sigma^\alpha\right)|t_1 - t_2|^{\alpha/2}\ \forall t_1, t_2 \ge 0 \text{ s.t. } |t_1 - t_2| \le 1, \end{gathered} \tag{3.4}$$

where for $\alpha = 1$ we replace $\mathcal{C}^1_\diamond(\mathcal{D})^*$ with $\mathrm{Lip}_\diamond(\mathcal{D})^*$. Its total mass function $\eta(t)$ is continuous and decreasing on $[0,T]$.

Lemma 3.1 has the following straightforward corollary, whose proof we omit.

Corollary 3.2. *Let b be a bounded continuous function on $\mathcal{D} \times [0, T]$, let $m_0 \in \mathcal{M}_1(\mathcal{D})$, and let m_0^+ and m_0^- denote the positive and negative parts, respectively, of m_0. Then there exists a unique weak solution m of (3.1), whose positive part m^+ is precisely the solution of (3.1) with m_0 replaced by m_0^+, and whose negative part m^- is the solution of (3.1) with m_0 replaced by m_0^-. The estimates (3.4) still hold, with m_0 replaced by $|m_0|$.*

3.1. The mass function

Let m be a weak solution to (3.1). We define the *total mass function* $\eta : [0, T] \to \mathbb{R}$ by

$$\eta(t) := \int_{\mathcal{D}} m(\mathrm{d}x, t).$$

Notice that η is in general not constant. Since the equations in System (1.6) depend on η, we are motivated to study the regularity of η as a function of time, and in particular we would like to know when it is Hölder continuous in order to establish the existence of classical solutions to the system. Note that it is insufficient to know how regular it is only for t away from zero, because the behavior of the population mass as $t \to 0$ influences the regularity of solutions to the backward-in-time Hamilton-Jacobi equation.

As a first step, we analyze the case where $b = 0$, so that (3.1) reduces to the heat equation with absorbing boundary conditions. Our goal is to determine whether the heat semigroup itself produces a Hölder continuous flow of total population mass. Recall that the heat kernel is given by

$$S(x, t) = (2\sigma^2 \pi t)^{-1/2} \exp\left\{-\frac{x^2}{2\sigma^2 t}\right\} \tag{3.5}$$

and that the solution of the heat equation with absorbing boundary condition at $x = 0$

$$\frac{\partial m}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 m}{\partial x^2}, \quad m|_{t=0} = m_0, \quad m|_{x=0} = 0 \tag{3.6}$$

is given by

$$m(x, t) = \int_{\mathcal{D}} \big(S(x - y, t) - S(x + y, t)\big)\, m_0(\mathrm{d}y). \tag{3.7}$$

For a measure $m_0 \in \mathcal{M}(\mathcal{D})$ the corresponding mass function generated by the heat equation is

$$\eta^h[m_0](t) := \int_{\mathcal{D}} \int_{\mathcal{D}} \big(S(x - y, t) - S(x + y, t)\big)\, m_0(\mathrm{d}y)\, \mathrm{d}x. \tag{3.8}$$

By Fubini's theorem, one can reverse the order of integration in (3.8) and then write $\eta^h[m](t)$ explicitly in terms of the cdf of m:

$$\eta^h[m_0](t) = \frac{2}{\sqrt{2\sigma^2\pi}} \int_0^\infty \exp\left\{-\frac{x^2}{2\sigma^2}\right\} m_0\left(\left(t^{1/2}x, \infty\right)\right) \mathrm{d}x.$$

To the question, "Is $\eta^h[m_0](\cdot)$ Hölder continuous on $[0, T]$ for every measure $m_0 \in \mathcal{M}(\mathcal{D})$?" the answer is a straightforward "no," as the following example shows.

Example 3.3. Define m as a density

$$m(x) = \frac{1}{x(\ln x)^2} \mathbb{I}_{(0,e^{-1})}(x).$$

Note that m is a probability density on $\mathcal{D}$ with cdf

$$F(x) = \int_0^x m(s)\,\mathrm{d}s = -\frac{1}{\ln x} \mathbb{I}_{(0,e^{-1})}(x) + \mathbb{I}_{[e^{-1},\infty)}(x).$$

Assume that $\eta^h[m](\cdot)$ is α-Hölder continuous on $[0, T]$ for some $\alpha \in (0, 1)$. Then there exists a constant C such that

$$1 - \eta^h[m](s) = \frac{2}{\sqrt{2\sigma^2\pi}} \int_0^\infty F(\sqrt{s}x) e^{-\frac{x^2}{2\sigma^2}} \,\mathrm{d}x \le C s^\alpha \;\forall s > 0,$$

and so, by Fatou's Lemma,

$$\frac{2}{\sqrt{2\sigma^2\pi}} \int_0^\infty \liminf_{s\to 0^+} s^{-\alpha} F(\sqrt{s}x)^{-\frac{x^2}{2\sigma^2}} \,\mathrm{d}x \le C.$$

But for any $x > 0$, we have

$$\lim_{s\to 0+} s^{-\alpha} F(\sqrt{s}x) = \lim_{s\to 0+} \frac{-1}{s^\alpha \ln(\sqrt{s}x)} = +\infty.$$

This is a contradiction.

For $0 < \alpha < 1$ we define $\mathscr{M}_\alpha(\mathcal{D})$ to be the space of all $m \in \mathcal{M}(\mathcal{D})$ on $\mathcal{D}$ such that $\eta^h[m] \in \mathcal{C}^\alpha\left([0,\infty)\right)$, with norm

$$\|m\|_{\mathscr{M}_\alpha} = \left\|\eta^h[m]\right\|_{\mathcal{C}^\alpha\left([0,\infty)\right)} + \|m\|_{TV}.$$

It is straightforward to see that $\mathscr{M}_\alpha$ is a Banach space. The heat equation (3.6) generates a semigroup of contractions on $\mathscr{M}_\alpha$. Indeed, let $m(t)$ denote the (measure-valued) solution at time t. First we deduce $\left\|m(t)\right\|_{TV} \le \|m_0\|_{TV}$ by integrating (3.7). Moreover, by the semigroup property (i.e. by uniqueness of solutions to the heat equation) we have $\eta^h[m(t)](s) = \eta^h[m_0](t+s)$, so that

$$\left\|\eta^h[m(t)]\right\|_{\mathcal{C}^\alpha\left([0,\infty)\right)} = \left\|\eta^h[m_0](t+\cdot)\right\|_{\mathcal{C}^\alpha\left([0,\infty)\right)} \leq \left\|\eta^h[m_0]\right\|_{\mathcal{C}^\alpha\left([0,\infty)\right)} \quad \forall t \geq 0.$$

Example 3.3 shows that measures which have a steep concentration of mass near 0 will fail to be in $\mathscr{M}_\alpha$. We now show prove that the converse is true, i.e. an estimate on the concentration of mass near zero will guarantee inclusion in $\mathscr{M}_\alpha$. For any $\alpha > 0$, denote by $\mathcal{M}^\alpha$ the set of all $m \in \mathcal{M}$ satisfying

$$\int_{\mathcal{D}} |x|^{-\alpha}\, \mathrm{d}|m|(x) < \infty. \tag{3.9}$$

For instance, $\mathcal{M}^\alpha$ contains all finite measures with support in $[z,\infty)$ for some $z > 0$. In particular, $\mathcal{M}^\alpha$ is dense in $\mathcal{M}$. If we endow $\mathcal{M}^\alpha$ with the norm

$$\|m\|_{\mathcal{M}^\alpha} = \|m\|_{TV} + \int_{\mathcal{D}} |x|^{-\alpha}\, \mathrm{d}|m|(x) = \int_{\mathcal{D}} \left(1 + |x|^{-\alpha}\right) \mathrm{d}|m|(x),$$

then it is straightforward to see that $\mathcal{M}^\alpha$ is a Banach space. We will also denote $\mathcal{M}^\alpha_+ = \mathcal{M}^\alpha \cap \mathcal{M}_+$, i.e. the set of all *positive* measures such that (3.9) holds.

Proposition 3.4. *Let $\alpha \in (0,2)$. Then $\mathcal{M}^\alpha \subset \mathscr{M}_{\alpha/2}$, and there exists a constant $C(\alpha)$ such that*

$$\|m\|_{\mathscr{M}_{\alpha/2}} \leq C(\alpha)\|m\|_{\mathcal{M}^\alpha} \quad \forall m \in \mathcal{M}^\alpha.$$

In particular, $\mathscr{M}_{\alpha/2}$ is dense in $\mathcal{M}$.

Proof. We can write

$$\eta^h[m](t) = \int_{\mathcal{D}} f(y,t) m_0(\mathrm{d}y),$$

where

$$f(y,t) = \int_0^\infty \big(S(x-y,t) - S(x+y,t)\big)\, \mathrm{d}x.$$

We observe that

$$\frac{\partial f}{\partial t}(y,t) = \int_0^\infty \left(\frac{\partial S}{\partial t}(x-y,t) - \frac{\partial S}{\partial t}(x+y,t)\right) \mathrm{d}x$$

$$= \frac{\sigma^2}{2} \int_0^\infty \left(\frac{\partial^2 S}{\partial x^2}(x-y,t) - \frac{\partial^2 S}{\partial x^2}(x+y,t) \right) \mathrm{d}x$$

$$= -\sigma^2 \frac{\partial S}{\partial x}(y,t) = \frac{y}{\sqrt{2\sigma^2\pi} t^{3/2}} e^{-\frac{y^2}{2\sigma^2 t}}.$$

Let $p \geq 1$, $y > 1$. By a change of variables $s = y^2/t$, we deduce

$$\left(\int_0^\infty \left| f_t(y,s) \right|^p \mathrm{d}s \right)^{1/p} = C(p) y^{-2/p'}, \; p' := p/(p-1).$$

Therefore

$$\left| f(y,t_1) - f(y,t_2) \right| \leq C(p) y^{-2/p'} |t_1 - t_2|^{1/p'}$$

We choose $p = 2/(2-\alpha)$, or equivalently $p' = 2/\alpha$. Then we have

$$\left| \eta^h[m](t_1) - \eta^h[m](t_2) \right| \leq \int_0^\infty \left| f(y,t_1) - f(y,t_2) \right| m(y)\,\mathrm{d}y \leq C(\alpha) |t_1 - t_2|^{\alpha/2} \int_0^\infty y^{-\alpha} m(y)\,\mathrm{d}y.$$

The claim follows. □

Recall that the heat semigroup is a semigroup of contractions on $\mathscr{M}_\alpha$. It turns out that the heat semigroup is also bounded on $\mathcal{M}^\alpha$ for arbitrary $\alpha > 0$, as the following lemma implies.

Lemma 3.5. *Let m_0 be a positive measure satisfying (3.9) for some $\alpha > 0$. There exists a constant $C(\alpha)$ such that if m is the solution of the heat equation (3.6), then*

$$\int_{\mathcal{D}} |x|^{-\alpha} m(\mathrm{d}x, t) \leq C(\alpha) \int_{\mathcal{D}} |x|^{-\alpha} m_0(\mathrm{d}x). \tag{3.10}$$

The proof of Lemma 3.5, which is found in Appendix A, relies on the following result, which will be useful for other estimates on parabolic equations.

Lemma 3.6. *Let $S(x,t)$ be the heat kernel, defined in (3.5). For all $n = 0, 1, 2, \ldots$, there exists a (Hermite) polynomial P_n of degree n such that*

$$\frac{\partial^n S}{\partial x^n}(x,t) = \left(\sigma^2 t\right)^{-n/2} P_n\left(\frac{|x|}{\sqrt{\sigma^2 t}} \right) S(x,t). \tag{3.11}$$

As a corollary, for all $n = 0, 1, 2, \ldots$, and $k = 1, 2, 3, \ldots$ the constants

$$m_n := \sup_{x,t} |x|^{n+1} \left| \frac{\partial^n S}{\partial x^n}(x,t) \right|, \quad m_{n,k} := \sup_{x,t} |x|^{n+1-k} \left(\sigma^2 t\right)^{k/2} \left| \frac{\partial^n S}{\partial x^n}(x,t) \right|$$

are finite and depend only on n and k.

Proof. The proof of (3.11) is elementary using induction. The second claim follows from the fact that $\sup_{x\geq 0} x^\alpha e^{-x}$ is finite for any $\alpha \geq 0$. □

We conclude this section by generalizing our results to the Fokker-Planck equation for an arbitrary bounded continuous drift term $b(x,t)$. The proofs are found in Appendix A.

Lemma 3.7. *Let b be a bounded continuous function on $\mathcal{D} \times [0,T]$, let $m_0 \in \mathcal{M}_+(\mathcal{D}) \cap \mathscr{M}_\alpha(\mathcal{D})$, and let m be the unique weak solution m of (3.1), given by Lemma 3.1. Then the total mass function $\eta(t) := \int_{\mathcal{D}} m(\mathrm{d}x,t)$ is β-Hölder continuous for $\beta = \min\{\alpha, 1/2\}$, with*

$$\|\eta\|_{C^\beta([0,T])} \leq C(\sigma)\left(\|m_0\|_{\mathscr{M}_\alpha} + \|b\|_\infty\right). \tag{3.12}$$

Lemma 3.8. *Let b be a bounded continuous function on $\mathcal{D} \times [0,T]$, let $m_0 \in \mathcal{M}_+^\alpha(\mathcal{D})$ for some $\alpha > 0$, and let m be the unique weak solution m of (3.1), given by Lemma 3.1. Then there exists some constants $C(\alpha)$ and $C(\alpha,\sigma)$ such that*

$$\int_{\mathcal{D}} |x|^{-\alpha}\, m(\mathrm{d}x,t) \leq C(\alpha) e^{C(\alpha,\sigma)\|b\|_\infty t} \int_{\mathcal{D}} |x|^{-\alpha}\, m_0(\mathrm{d}x). \tag{3.13}$$

4. Forward-backward system

In this section we prove existence and uniqueness of solutions to *infinite time horizon forward-backward system* (1.6). Many of the ideas in this section can already be found in [20]. Our result is novel in that (i) the time horizon is infinite and (ii) the initial measure m_0 need not be smooth nor even a density. The proof is based on a priori estimates followed by an application of the Leray-Schauder fixed point theorem (see e.g. [16, Theorem 11.3]). Most of the proofs in this section involve either standard computations or ideas that can be found in the previous works [20,23,19], and so we relegate them to Appendix B. However, in the sequel we will make frequent reference to the *estimates* found in this section.

4.1. The Hamiltonian

In this subsection we deduce a number of structural features of the Hamiltonian, using only Assumptions 1.1 and 1.2. The proofs can be found in Appendix B.1.

Lemma 4.1 *(Unique optimal quantity). The function $q^* : [0,\infty)^3 \to [0,\infty)$ given by $q^*(\epsilon, Q, a) = \operatorname{argmax}_{q\geq 0} \pi(\epsilon, q, Q, a)$ is well-defined and locally Lipschitz continuous. It is non-increasing in the variable a. With respect to ϵ and Q, it satisfies*

$$-\epsilon \leq \frac{\partial q^*}{\partial Q} \leq \epsilon \frac{\bar{\rho}-1}{2-\bar{\rho}}, \quad -Q \leq \frac{\partial q^*}{\partial \epsilon} \leq Q\frac{\bar{\rho}-1}{2-\bar{\rho}}. \tag{4.1}$$

Define $H(\epsilon, Q, a) = \pi\left(\epsilon, q^*(Q,a), Q, a\right) \geq 0$. Then H is locally Lipschitz, decreasing in all variables, and convex in a; its derivative $\dfrac{\partial H}{\partial a} = -q^*$ is also locally Lipschitz.

Corollary 4.2 *(Smoothness and uniform convexity). Let $\bar{\epsilon} \geq 0$, $\bar{Q} \geq 0$, and $\bar{a} > 0$ be constants such that $\bar{a} < P(\bar{\epsilon}\bar{Q})$. Consider the restriction of $H = H(\epsilon, Q, a)$ to the domain $[0, \bar{\epsilon}] \times [0, \bar{Q}] \times [0, \bar{a}]$. Then H is n times continuously differentiable with Lipschitz continuous derivatives, where n is the same as in Assumption 1.1. It is also uniformly convex in the a variable, and in particular there exists a constant $C_H = C(\bar{\epsilon}, \bar{Q}, \underline{a}, \bar{a}) \geq 1$ such that*

$$C_H^{-1} \leq \frac{\partial^2 H}{\partial a^2}(\epsilon, Q, a) \leq C_H \quad \forall(\epsilon, Q, a) \in [0, \bar{\epsilon}] \times [0, \bar{Q}] \times [\underline{a}, \bar{a}]. \tag{4.2}$$

Corollary 4.3 *(Q dependence). We have the following estimates in the region where $P(\epsilon Q) > a$:*

$$\left|\frac{\partial H}{\partial Q}\right| \leq \epsilon(P(0) - a), \quad \left|\frac{\partial^2 H}{\partial Q \partial a}\right| \leq \epsilon \max\left\{\left|\frac{\bar{\rho} - 1}{\bar{\rho} - 2}\right|, 1\right\} =: \bar{P}\epsilon. \tag{4.3}$$

Lemma 4.4 *(Unique aggregate quantity). Let $\epsilon \geq 0, \phi \in L^\infty(\mathcal{D})$ and $m \in \mathcal{M}_+(\mathcal{D})$ with $\int_{\mathcal{D}} \mathrm{d}m(x) \leq 1$ and $\phi \geq 0$ (a.e.). Then there exists a unique $Q^* = Q^*(\epsilon, \phi, m) \geq 0$ such that*

$$Q^* = \int_{\mathcal{D}} q^*(\epsilon, Q^*, \phi(x))\, \mathrm{d}m(x) = -\int_{\mathcal{D}} \frac{\partial H}{\partial a}\left(\epsilon, Q^*, \phi(x)\right) \mathrm{d}m(x). \tag{4.4}$$

Moreover, Q^* satisfies the a priori estimate

$$Q^* \leq c(\bar{\rho}, \epsilon) q^*(0, 0, 0), \;\; c(\bar{\rho}, \epsilon) := \max\left\{\frac{2 - \bar{\rho}}{2 + \epsilon - (1 + \epsilon)\bar{\rho}}, 1\right\}. \tag{4.5}$$

Finally, Q^* is locally Lipschitz in the following sense. If $\epsilon_1, \epsilon_2 \in [0, \epsilon]$, ϕ_1, ϕ_2 Lipschitz functions with $\|\phi_i\|_\infty \leq M$, and $m_1, m_2 \in \mathcal{M}_{1,+}(\mathcal{D})$ with $\int_{\mathcal{D}} \mathrm{d}m_i(x) \leq 1$, set $Q_i^* = Q^*(\epsilon_i, \phi_i, m_i)$ to be the solution of (4.4) corresponding to ϵ_i, ϕ_i, m_i for $i = 1, 2$. Then there exists a constant $C = C(\epsilon, \bar{\rho}, M)$ such that

$$\begin{aligned} &\left|Q_1^* - Q_2^*\right| \\ &\leq C\left(|\epsilon_1 - \epsilon_2| + \int_{\mathcal{D}} \left|\phi_1(x) - \phi_2(x)\right| \mathrm{d}m_1(x) + \max_{i=1,2}\left\|\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\right\|_\infty \mathbf{d}_1(m_1, m_2) + \left|\int_{\mathcal{D}} \mathrm{d}(m_1 - m_2)(x)\right|\right) \\ &\leq C\left(|\epsilon_1 - \epsilon_2| + \|\phi_1 - \phi_2\|_\infty + \max_{i=1,2}\left\|\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\right\|_\infty \mathbf{d}_1(m_1, m_2) + \left|\int_{\mathcal{D}} \mathrm{d}(m_1 - m_2)(x)\right|\right). \end{aligned} \tag{4.6}$$

Remark 4.5. The function $c(\bar{\rho}, \epsilon)$ in equation (4.5) is an increasing function of ϵ.

Corollary 4.6. *Let ϵ, ϕ, m, and $Q^* = Q^*(\epsilon, \phi, m)$ be as in Lemma 4.4. Then*

$$q^*\left(\epsilon, Q^*, \phi(x)\right) \leq c(\bar{\rho}, \epsilon) q^*(0, 0, 0) \tag{4.7}$$

for a.e. $x \in \mathcal{D}$.

4.2. Finite time horizon problem

In this section we fix a final time $T > 0$ and consider the forward-backward system only on this time horizon. For technical reasons, we will need to replace the constant ϵ with a function $\epsilon(t)$ such that $\epsilon(T) = 0$. System (1.6) becomes

$$\begin{cases} (i) & \dfrac{\partial u}{\partial t} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 u}{\partial x^2} + H\left(\epsilon(t), Q^*(t), \dfrac{\partial u}{\partial x}\right) - ru = 0, \\ (ii) & \dfrac{\partial m}{\partial t} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 m}{\partial x^2} + \dfrac{\partial}{\partial x}\left(\dfrac{\partial H}{\partial a}\left(\epsilon(t), Q^*(t), \dfrac{\partial u}{\partial x}\right) m\right) = 0, \\ (iii) & Q^*(t) = -\displaystyle\int_{\mathcal{D}} \dfrac{\partial H}{\partial a}\left(\epsilon(t), Q^*(t), \dfrac{\partial u}{\partial x}\right) \mathrm{d}m(t), \\ (iv) & m|_{x=0} = u|_{x=0} = 0,\ m|_{t=0} = m_0 \in \mathcal{P}(\mathcal{D}),\ u|_{t=T} = u_T \in C^{2+\alpha}. \end{cases} \tag{4.8}$$

We define (u, m) to be a solution to (4.8) provided that u is a smooth function on $\overline{\mathcal{D}} \times [0, T]$ (twice continuously differentiable with respect to x, continuously differentiable with respect to t), $m \in \mathcal{C}([0, T]; \mathcal{P}(\mathcal{D}))$, Equations (i) and (iii) are satisfied pointwise, the boundary conditions for u in (iv) are satisfied pointwise, and Equation (ii) with the boundary conditions for m from (iv) holds in the sense of distributions (see Section 3). Note that a solution (u, m) must satisfy $\dfrac{\partial u}{\partial x} \geq 0$, because the domain of H is $\left[0, \infty\right)^3$. It is possible to relax this somewhat by extending the domain of $H(\epsilon, Q, a)$ to include all $a > \lim_{q \to \infty} P(q)$, but we need not do so here.

Assumption 4.7 *(Structure of $\epsilon(t)$).* We assume ϵ is a smooth, non-negative, non-increasing function on $\left[0, T\right]$ such that $\epsilon(T) = 0$ and $\left\|\epsilon'\right\|_\infty \leq 1$.

Assumption 4.8 *(Structure of u_T).* For each $T > 0$, the function u_T is an element of $\mathcal{C}^{2+\alpha}\left(\overline{\mathcal{D}}\right)$ that satisfies the following conditions:

(1) $u_T(0) = 0$;
(2) $\frac{\sigma^2}{2} u_T''(0) + H\left(0, 0, u_T'(0)\right) = 0$;
(3) $0 \leq u_T(x) \leq c_1$ for all $x \in \mathcal{D}$, where $c_1 > 0$ is some constant;
(4) there exists a constant $c_3 > 0$, independent of T, such that $0 \leq u_T'(x) \leq c_3$ for all $x \in \mathcal{D}$ and all $T > 0$;
(5) there exists a constant C_α, independent of T, such that $\|u_T\|_{\mathcal{C}^{2+\alpha}\left(\overline{\mathcal{D}}\right)} \leq C_\alpha$ for all $T > 0$.

Remark 4.9. It is always possible to satisfy Assumption 4.8 for an arbitrary constant $c_3 > 0$. Here we give one possible construction. Set $h = \frac{2}{\sigma^2} H(0, 0, c_3)$, so that condition (2) becomes $u_T''(0) = -h$. If $h > 0$, then Assumption 4.8 is satisfied by the function

$$u_T(x) = \frac{2(c_3)^2}{3h} + \frac{h^2}{12 c_3}\left(x - \frac{2c_3}{h}\right)_-^3,$$

where $x_- := \min\{x, 0\}$. In the case where $h = 0$, Assumption 4.8 is satisfied by the function

$$u_T(x) = \begin{cases} c_3 x - \frac{x^3}{6} & \text{if } x \le (c_3)^{1/2}, \\ \frac{1}{2}(c_3)^{3/2} + \frac{1}{3}\left(x - 2(c_3)^{1/2}\right)^3, & \text{if } (c_3)^{1/2} \le x \le 2(c_3)^{1/2}, \\ \frac{1}{2}(c_3)^{3/2}, & \text{if } 2(c_3)^{1/2} \le x. \end{cases}$$

Note also that these examples can be slightly modified to produce globally $\mathcal{C}^\infty$ functions satisfying Assumption 4.8.

4.3. Estimates on the Hamilton-Jacobi equation

Lemma 4.10 *(A priori estimates for HJ equation). Let $Q^*(t)$ be any bounded, non-negative function. Let u be a solution of the Hamilton-Jacobi equation*

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + H\left(\epsilon(t), Q^*(t), \frac{\partial u}{\partial x}\right) - ru = 0, \ x \in \mathcal{D}, \ t \in \left[0, T\right) \tag{4.9}$$

with Dirichlet boundary conditions $u(0,t) = 0$ and final condition $u(x,T) = u_T(x)$, which satisfies Assumption 4.8. Then for all $x \in \mathcal{D}$ and $t \in [0,T]$, we have

$$0 \le u(x,t) \le \frac{1}{r}H(0,0,0) + c_1, \quad 0 \le u_x(x,t) \le M, \tag{4.10}$$

where

$$M = M(\sigma, r, c_1, c_3) := \begin{cases} 2\sqrt{\frac{2H(0,0,0)\left(H(0,0,0)+rc_1\right)}{\sigma^2 r}} & \text{if } c_3 \le \sqrt{\frac{2}{\sigma^2 r}}H(0,0,0) \\ c_3 + \frac{2}{\sigma^2 r c_3}H(0,0,0)^2 + \frac{2c_1}{\sigma^2 c_3}H(0,0,0) & \text{if } c_3 \ge \sqrt{\frac{2}{\sigma^2 r}}H(0,0,0) \end{cases}. \tag{4.11}$$

Proof. See Appendix B.2. Cf. [20, Section 4]. □

4.4. Estimates on the coupling

Lemma 4.11. *Let (u, m) be a solution of (4.8). Then Q^*, given by (4.8)(iii), satisfies the following bounds:*

$$0 \le Q^*(t) \le c(\bar\rho, \epsilon(0))q^*(0,0,0) = -c(\bar\rho, \epsilon(0))\frac{\partial H}{\partial a}(0,0,0), \tag{4.12}$$

where $c(\bar\rho, \epsilon)$ is defined in (4.5).

Suppose, moreover, that $m_0 \in \mathscr{M}_{\alpha/2}$ for some $\alpha \in (0,1]$. Then $Q^*(t)$ is Hölder continuous on $\left[0, T\right]$ with

$$\|Q^*\|_{\mathcal{C}^{\alpha/2}} \le C\left(\left\|\frac{\partial u}{\partial x}\right\|_{\mathcal{C}^{\alpha,\alpha/2}} + \left\|\frac{\partial^2 u}{\partial x^2}\right\|_\infty + 1\right), \tag{4.13}$$

for some $C = C\left(\bar{\rho}, \epsilon(0), \sigma, M, \|m_0\|_{\mathscr{M}_{\alpha/2}}\right)$*, where* M *is the constant from Lemma 4.10 that gives an upper bound on* $\left\|\frac{\partial u}{\partial x}\right\|_\infty$.

Proof. See Appendix B.3. □

4.5. Parabolic estimates

Before stating our result on the existence of smooth solutions to the system, we present some estimates on solutions to parabolic problems that *do not depend on the time horizon*. These estimates will be useful in study of the linearized system (Section 5).

Lemma 4.12. *Let* $T > 0, r > 0$ *be given. For any* $f \in \mathcal{C}^{\alpha,\alpha/2}(\overline{\mathcal{D}} \times [0,T])$*, and* $u_0 \in \mathcal{C}^{2+\alpha}(\overline{\mathcal{D}})$*, there exists a unique solution* $u \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathcal{D}} \times [0,T])$ *of*

$$\frac{\partial u}{\partial t} + ru - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} = f, \ \forall x \in \mathcal{D}, t > 0; u(0,t) = 0 \ \forall t > 0; \ u(x,0) = u_0(x) \ \forall x \in \mathcal{D} \tag{4.14}$$

satisfying

$$\|u\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathcal{D}}\times[0,T])} \leq C(\sigma,r,\alpha)\left(\|f\|_{\mathcal{C}^{\alpha,\alpha/2}(\overline{\mathcal{D}}\times[0,T])} + \|u_0\|_{\mathcal{C}^{2+\alpha}(\overline{\mathcal{D}})}\right). \tag{4.15}$$

The constant $C(\sigma,r,\alpha)$ *in* (4.15) *does* not *depend on* T*. More specifically, we can say that if* $r \geq 1$,

$$\|u\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathcal{D}}\times[0,T])} \leq C(\sigma,\alpha)\left([f]_{\alpha,\alpha/2} + r^{\frac{\alpha}{2}}\|f\|_0 + [u_0]_{2+\alpha} + r^{1+\frac{\alpha}{2}}\|u_0\|_0\right). \tag{4.16}$$

Proof. The result follows from potential estimates found in [29, Chapter IV]. See Appendix B.4. □

4.6. Existence of solutions

Lemma 4.13. *Let* $m_0 \in \mathscr{M}_{\alpha/2}$ *and* $0 < \alpha \leq 1$*. Then there exists a constant*

$$C = C(\bar{\rho}, \epsilon(0), \sigma, M, c_1, \|m_0\|_{\mathscr{M}_{\alpha/2}}, \alpha)$$

such that for any solution (u,m) *of* (4.8),

$$\|u\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}} \leq C\left(1 + r^{\frac{\alpha}{2}} + C_\alpha + r^{1+\frac{\alpha}{2}}c_1\right), \tag{4.17}$$

where M *is the constant from Lemma 4.10 and* c_1, c_3, C_α *are the constants from Assumption 4.8.*

Remark 4.14. The constant on the right-hand side of (4.17) does not depend on T.

For the proof of Lemma 4.13, see Appendix B.5.

Theorem 4.15 *(Existence of classical solutions for (4.8)). Let $m_0 \in \mathscr{M}_{\alpha/2}$ and $0 < \alpha \le 1$. Then there exists a solution (u, m) satisfying the finite time horizon problem (4.8) and having the following regularity: $u \in \mathcal{C}^{2+\alpha,1+\alpha/2}\left(\overline{\mathcal{D}} \times [0, T]\right)$, $m \in \mathcal{C}^{1/2}\left([0, T]; \mathcal{M}_{1,+}(\mathcal{D})\right)$. Thus, Equation (4.8)(i) is satisfied in a classical sense, while Equation (4.8)(ii) is satisfied in the weak sense defined in (3.2), and Equation (4.8)(iii) holds pointwise.*

Proof. We use the Leray-Schauder fixed point theorem in a more or less standard way, cf. [20, 23,19]. The details are given in Appendix B.5. □

Theorem 4.16 *(Existence of solutions to the infinite horizon problem (1.6)). Let $m_0 \in \mathscr{M}_{\alpha/2}$ and $0 < \alpha \le 1$. Then there exists a solution $(u, m) \in \mathcal{C}^{2+\alpha,1+\alpha/2}\left(\overline{\mathcal{D}} \times \left[0, \infty\right)\right) \times \in \mathcal{C}^{1/2}\Big(\left[0, \infty\right); \mathcal{M}_{1,+}(\mathcal{D})\Big)$ solving the infinite time horizon problem (1.6) and satisfying the following estimates:*

$$
\begin{aligned}
&\|u\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}} \le C(\bar{\rho}, \epsilon, \sigma, M, \|m_0\|_{\mathscr{M}_{\alpha/2}}, \alpha)\left(1 + r^{\frac{\alpha}{2}} + C_\alpha\right),\\
&\mathbf{d}_1\left(m(t_1), m(t_2)\right) \le 2\,(M + \sigma)|t_1 - t_2|^{1/2}\ \forall |t_1 - t_2| \le 1,\\
&0 \le u(x,t) \le \frac{1}{r} H(0,0,0), \quad 0 \le \frac{\partial u(x,t)}{\partial x} \le M\ \forall x \in \overline{\mathcal{D}}, t \ge 0,\\
&0 \le Q^*(t) \le \bar{Q},\\
&0 \le -\frac{\partial H}{\partial a}\left(\epsilon, Q^*(t), \frac{\partial u}{\partial x}(x,t)\right) \le \bar{Q} \quad \forall (x,t) \in \mathcal{D} \times \left[0, \infty\right)
\end{aligned}
\tag{4.18}
$$

where M and $\bar{Q}$ are defined by

$$
M := 2\sqrt{\frac{2}{\sigma^2 r}} H(0,0,0), \quad \bar{Q} := -c\left(\bar{\rho}, \epsilon\right) \frac{\partial H}{\partial a}(0,0,0) \tag{4.19}
$$

Proof. For each $T > 0$, we will let $\epsilon(t)$ be a function satisfying Assumption 4.7 as well as $\epsilon(0) = \epsilon$, and we let u_T be a function satisfying Assumption 4.8. By Theorem 4.15 there exists a solution of (4.8), which we denote (u^T, m^T). Fix an arbitrary $T_0 > 0$. By Lemmas 4.13 and 3.1, (u^T, m^T) is uniformly bounded in $\mathcal{C}^{2+\alpha,1+\alpha/2}\left(\overline{\mathcal{D}} \times \left[0, T_0\right]\right) \times \mathcal{C}^{1/2}\left(\left[0, T_0\right]; \mathcal{M}_{1,+}(\mathcal{D})\right)$ for all $T \ge T_0$, with norms bounded by a constant that does not depend on T_0. Thus, by standard diagonalization, we may pass to a subsequence, still denoted (u^T, m^T), that converges to some fixed (u, m), where the convergence is in $\mathcal{C}^{2,1}\left(\overline{\mathcal{D}} \times \left[0, T_0\right]\right) \times \mathcal{C}^0\left(\left[0, T_0\right]; \mathcal{M}_{1,+}(\mathcal{D})\right)$ for *every* T_0. By the uniform estimates on (u^T, m^T) it also follows that $(u, m) \in \mathcal{C}^{2+\alpha,1+\alpha/2}\left(\overline{\mathcal{D}} \times \left[0, \infty\right)\right) \times \mathcal{C}^{1/2}\left(\left[0, \infty\right); \mathcal{M}_{1,+}(\mathcal{D})\right)$. To see that (u, m) is indeed a solution to (1.6), it suffices to pass to the limit in the equations satisfied by (u^T, m^T) on arbitrary time horizons. Finally, note that the following estimates hold:

$$
\begin{aligned}
&\|u\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}} \le C(\bar{\rho}, \epsilon(0), \sigma, M, c_1, c_3, \|m_0\|_{\mathscr{M}_{\alpha/2}}, \alpha)\left(1 + r^{\frac{\alpha}{2}} + C_\alpha + r^{1+\frac{\alpha}{2}} c_1\right),\\
&\mathbf{d}_1\left(m(t_1), m(t_2)\right) \le 2\,(M + \sigma)|t_1 - t_2|^{1/2}\ \forall |t_1 - t_2| \le 1,
\end{aligned}
$$

$$0 \le u(x,t) \le \frac{1}{r} H(0,0,0) + c_1, \quad 0 \le \frac{\partial u(x,t)}{\partial x} \le M(\sigma, r, c_1, c_3)\ \forall x \in \overline{\mathcal{D}}, t \ge 0,$$
$$0 \le Q^*(t) \le -c\left(\bar{\rho}, \epsilon\right) \frac{\partial H}{\partial a}(0,0,0),$$
$$0 \le -\frac{\partial H}{\partial a}\left(\epsilon, Q^*(t), \frac{\partial u}{\partial x}(x,t)\right) \le -c\left(\bar{\rho}, \epsilon\right) \frac{\partial H}{\partial a}(0,0,0) \quad \forall (x,t) \in \mathcal{D} \times \left[0, \infty\right)$$

where $M = M(\sigma, r, c_1, c_3)$ is defined in (4.11). This follows because they hold for (u^T, m^T) uniformly in T (Lemmas 3.1, 4.10, and 4.11, also Corollary 4.6). Now by Remark 4.9, c_1, c_2 and c_3 can be made arbitrarily close to zero. Letting $c_1, c_3 \to 0$ and using the continuity of H and $\frac{\partial H}{\partial a}$, we deduce the estimates (4.18). □

4.7. Uniqueness and smoothness of the Hamiltonian

When the demand schedule is linear, uniqueness of solutions to (1.6) follows with no further conditions on the data, cf. [23]. In the case of a general, nonlinear demand schedule satisfying Assumptions 1.1 and 1.2, we can prove uniqueness of solutions for small enough parameter ϵ. Cf. [20]. The smallness of ϵ makes two contributions. First, it ensures that the Hamiltonian H is a smooth, uniformly convex function on the domain where solutions exist. Second, it ensures that certain "energy estimates" à la Lasry-Lions (see [30]) hold, which prove uniqueness. The case where ϵ is small has independent interest, aside from being a technical condition that yields uniqueness. (Cf. Remark 4.17.)

Remark 4.17. The inspiration for taking $\epsilon > 0$ small is taken from the basic idea that Chan and Sircar use to compute solutions [10,11] Namely, it is natural to try take a formal Taylor expansion of the solution with respect to ϵ around zero, since at $\epsilon = 0$ the system of equations is decoupled. (See [21] for a justification of this technique.) Now when $\epsilon > 0$ is small enough, one might think to simplify our approach by devising a contraction mapping argument. In the present work, we do not take this approach, but instead seek to unify as much as possible with the case where the demand schedule is linear. For in this latter case, it is essentially from the structure of the Hamiltonian that one obtains the "propagation of monotonicity" (cf. [14]) that is needed to prove uniqueness. We show that the same is true when ϵ is small, and we do so by proving the same type of estimates as we do for the linear demand schedule. One could, in principle, generalize this idea to other "smallness" conditions; for example, if the demand schedule is "close enough to linear" in a suitable sense, then our arguments for uniqueness will go through for a wide range of parameters ϵ. In the present work, however, we do not pursue this direction, so as to avoid a multiplication of technicalities.

In this section we consider both the smoothness of the Hamiltonian and uniqueness of solutions separately. The former can at first be viewed as a tool for proving the latter, in the case of a nonlinear demand schedule. However, when we prove the regularity of the master field in Sections 5 and 6, the smoothness of the Hamiltonian will be required even when the demand schedule is linear. Therefore we address it in a separate subsection.

4.7.1. Assumptions ensuring that the Hamiltonian is smooth

The following assumption ensures in general that H can be treated as a smooth, uniformly convex function in System (1.6).

Assumption 4.18. We assume that $M < P\left(\epsilon\bar{Q}\right)$, where M and $\bar{Q}$ are defined in (4.19).

Remark 4.19. [Sufficient conditions to give Assumption 4.18] There necessarily exists r^* large enough so that

$$2\sqrt{\frac{2}{\sigma^2 r}}H(0,0,0) < P(0) \quad \forall r \geq r^*.$$

Then, since $\epsilon \mapsto P\left(\epsilon\bar{Q}\right)$ is a continuous, decreasing function of ϵ, there exists $\epsilon^* > 0$ such that Assumption 4.18 holds for all $0 < \epsilon \leq \epsilon^*$ and all $r \geq r^*$.

Under Assumption 4.18, it follows from Corollary 4.2 and the a priori estimates (4.18) from Theorem 4.16 that in System (1.6) (or (4.8), provided c_2 from Assumption 4.8 is chosen small enough), H can be treated as n times continuously differentiable with Lipschitz continuous derivatives, and moreover it is uniformly convex. In particular, from (4.2) there exists a constant $C_H \geq 1$ such that

$$C_H^{-1} \leq \frac{\partial^2 H}{\partial a^2}\left(\epsilon, Q(t), \frac{\partial u}{\partial x}(x,t)\right) \leq C_H \quad \forall (x,t) \in \mathcal{D} \times (0,\infty) \tag{4.20}$$

whenever u is a solution of (1.6).

An interesting special case is when the demand schedule is linear; without loss of generality we take $P(q) = 1 - q$. In this case (and in general when $\bar{\rho} \leq 1$) we have $c(\bar{\rho},\epsilon) = 1$, and a simple computation shows $\bar{Q} = 1/2$ and $M = (2\sigma^2 r)^{-1/2}$. For any $\epsilon^* < 2$, it is possible to take r^* sufficiently large so that Assumption 4.18 holds for any $r \geq r^*$ and any $\epsilon \leq \epsilon^*$. In this case, the smoothness of H on the domain where solutions lie implies that the solution to (1.6) is the same as the solution to

$$\begin{cases} (i) & \dfrac{\partial u}{\partial t} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{1}{4}\left(1 - \epsilon Q^*(t) - \dfrac{\partial u}{\partial x}\right)^2 - ru = 0, \\ (ii) & \dfrac{\partial m}{\partial t} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 m}{\partial x^2} - \dfrac{\partial}{\partial x}\left(\dfrac{1}{2}\left(1 - \epsilon Q^*(t) - \dfrac{\partial u}{\partial x}\right)m\right) = 0, \\ (iii) & Q^*(t) = -\displaystyle\int_{\mathcal{D}} \dfrac{1}{2}\left(1 - \epsilon Q^*(t) - \dfrac{\partial u}{\partial x}(\cdot,t)\right)\mathrm{d}m(t), \\ (iv) & m|_{x=0} = u|_{x=0} = 0, \; m|_{t=0} = m_0 \end{cases} \tag{4.21}$$

4.7.2. *Uniqueness*

Theorem 4.20. *In addition to Assumption 4.18, suppose that*

$$r \geq 1000 \max\left\{1 + c(\bar{\rho},\epsilon)\bar{P}\epsilon, 1 + c(\bar{\rho},\epsilon)\bar{Q}, \bar{Q} + \epsilon P(0) + 1\right\}^2 \quad \textit{and} \tag{4.22}$$

$$\epsilon \leq \left(4C_H c(\bar{\rho},\epsilon)\left(1+\bar{Q}\right)\left(C_H\left(P(0)+1\right)+\bar{P}\right)\right)^{-1}, \tag{4.23}$$

where C_H is the constant from (4.20). Then there is at most one solution (u, m, Q^*) of (4.8), and likewise there is at most one solution (u, m, Q^*) of (1.6) such that u and $\frac{\partial u}{\partial x}$ are bounded.

Proof. Suppose that (u, m, Q^*) and $(\hat{u}, \hat{m}, \hat{Q}^*)$ are both solutions of (4.8), or of (1.6) with u, $\frac{\partial u}{\partial x}$, $\hat{u}$, and $\frac{\partial \hat{u}}{\partial x}$ bounded. We will employ the results of Sections 5.3 and 5.4, which are proved independently. Equation (4.22) (which is surely an overestimate, see Remark 5.13) implies that Assumption 5.12 holds. Then Equation (4.23) implies that Lemma 5.15 holds. Since the initial conditions are the same, i.e. $\hat{m}_0 = m_0$, we have

$$\int_0^T \int_{\mathcal{D}} e^{-rt} \left| \frac{\partial u}{\partial x} - \frac{\partial \hat{u}}{\partial x} \right|^2 \big(m(\mathrm{d}x, t) + \hat{m}(\mathrm{d}x, t) \big) \, \mathrm{d}t = 0,$$

where T is the (finite or infinite) time horizon. It follows that $\frac{\partial u}{\partial x} = \frac{\partial \hat{u}}{\partial x}$ on the support of m and $\hat{m}$, and so by Lemma 4.4 we deduce that $Q^* = \hat{Q}^*$. Then by standard uniqueness for parabolic equations, it follows that $m = \hat{m}$; we also get $u = \hat{u}$ in a straightforward way if $T < \infty$.

For the infinite time horizon case, let $w(x, t) = e^{-rt} \big(u(x, t) - \hat{u}(x, t) \big)$ and note that it satisfies

$$-\frac{\partial w}{\partial t} - \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial x^2} = e^{-rt} \left(H\left(\epsilon, Q^*(t), \frac{\partial u}{\partial x}\right) - H\left(\epsilon, Q^*(t), \frac{\partial \hat{u}}{\partial x}\right) \right) \le C \left| \frac{\partial w}{\partial x} \right|,$$

since $\frac{\partial u}{\partial x}$ and $\frac{\partial \hat{u}}{\partial x}$ are bounded. Let $c > 0$. Multiply by $(w - c)_+ := \max\{w - c, 0\}$ and integrate to get

$$\int_0^\infty (w - c)_+(x, t)^2 \, \mathrm{d}x + \frac{\sigma^2}{2} \int_t^T \int_0^\infty \left| \frac{\partial (w - c)_+}{\partial x} \right|^2 \mathrm{d}x \, \mathrm{d}\tau$$
$$\le \int_0^\infty (w - c)_+(x, T)^2 \, \mathrm{d}x + C \int_t^T \int_0^\infty \left| \frac{\partial (w - c)_+}{\partial x} \right| (w - c)_+ \, \mathrm{d}x \, \mathrm{d}\tau,$$

from which we deduce

$$\int_0^\infty (w - c)_+(x, t)^2 \, \mathrm{d}x \le \int_0^\infty (w - c)_+(x, T)^2 \, \mathrm{d}x + C \int_t^T \int_0^\infty (w - c)_+^2 \, \mathrm{d}x \, \mathrm{d}\tau.$$

By Gronwall's inequality (applied backward in time), we obtain

$$\int_0^\infty (w - c)_+(x, t)^2 \, \mathrm{d}x \le e^{C(T - t)} \int_0^\infty (w - c)_+(x, T)^2 \, \mathrm{d}x.$$

Since $u, \hat{u}$ are bounded, taking T large enough we deduce $w(x,T) \le c$, and thus the right-hand side is zero. We deduce that $w \le c$ everywhere. Since c is arbitrary, it follows that $w \le 0$, i.e. $u \le \hat{u}$. By reversing the roles of u and $\hat{u}$ we see that $u = \hat{u}$. □

The following result does not require any of the assumptions made in this section, but simply imposes a linear demand schedule.

Theorem 4.21. *Under the assumption $P(q) = 1 - q$ (but no additional assumptions), there is at most one solution to the finite horizon problem (4.8), and likewise at most one solution to the infinite time horizon problem (1.6) such that u and $\frac{\partial u}{\partial x}$ are bounded.*

Proof. Let (u, m, Q^*) and $(\hat{u}, \hat{m}, \hat{Q}^*)$ be two solutions to the PDE system (1.6), then set $q^* := q^*(\epsilon, Q^*, \frac{\partial u}{\partial x})$ and $\hat{q}^* = q^*(\epsilon, \hat{Q}^*, \frac{\partial \hat{u}}{\partial x})$. Following the calculations in [23], we derive

$$\int_0^T \int_0^\infty e^{-rt} \left(\hat{q}^* - q^*\right)^2 (m + \hat{m}) \, \mathrm{d}x \, \mathrm{d}t + \epsilon \int_0^T e^{-rt} \left(Q^*(t) - \hat{Q}^*(t)\right)^2 \mathrm{d}t$$
$$\le \int_0^\infty \left(e^{-rT} \left(u - \hat{u}\right)(x,T) \left(m - \hat{m}\right)(x,T) - \left(u - \hat{u}\right)(x,0) \left(m - \hat{m}\right)(x,0)\right) \mathrm{d}x.$$

Because the initial/final data are the same, the right-hand side is zero, and we conclude using the same arguments as in the proof of Theorem 4.20. □

5. A priori estimates on the linearized system

In this section our goal is to prove a priori estimates and existence of solutions for a system of the form

$$\begin{cases} (i) \ \dfrac{\partial w}{\partial t} + \dfrac{\sigma^2}{2} \dfrac{\partial^2 w}{\partial x^2} + V_1(x,t) \dfrac{\partial w}{\partial x} + V_2(x,t) \mathcal{Q}(t) - rw = f, \\ (ii) \ \dfrac{\partial \mu}{\partial t} - \dfrac{\sigma^2}{2} \dfrac{\partial^2 \mu}{\partial x^2} + \dfrac{\partial}{\partial x}\left(V_3(x,t)\mu\right) + \dfrac{\partial}{\partial x}\left(\left(V_4(x,t)\dfrac{\partial w}{\partial x} + V_5(x,t)\mathcal{Q}(t)\right) m + \nu\right) = 0, \\ (iii) \ \mathcal{Q}(t) = \left(1 + \displaystyle\int_{\mathcal{D}} V_5(\cdot,t) \, \mathrm{d}m(t)\right)^{-1} \\ \qquad \times \left(-\displaystyle\int_{\mathcal{D}} \mathrm{d}\nu(t) - \int_{\mathcal{D}} V_3(\cdot,t) \, \mathrm{d}\mu(t) - \int_{\mathcal{D}} V_4(\cdot,t) \dfrac{\partial w}{\partial x}(\cdot,t) \, \mathrm{d}m(t)\right), \\ (iv) \ \mu|_{x=0} = w|_{x=0} = 0, \ \ \mu|_{t=0} = \mu_0. \end{cases} \tag{5.1}$$

It is useful to study System (5.1) at a sufficiently high level of abstraction because our estimates will serve three purposes:

(1) proving that U is Lipschitz with respect to the measure variable,
(2) proving the existence of a candidate for $\frac{\delta U}{\delta m}$, and
(3) proving that the candidate is indeed a derivative in the sense of Definition 1.3.

To see this, let (u, m, Q^*) and $(\hat{u}, \hat{m}, \hat{Q}^*)$ be the solutions of (1.6) corresponding to initial conditions m_0 and $\hat{m}_0$, respectively. For $s \in [0, 1]$ define

$$u_s = s\hat{u} + (1-s)u, \quad Q_s^* = s\hat{Q}^* + (1-s)Q^*.$$

If $w = \hat{u} - u$, $\mu = \hat{m} - m$, and $\mathcal{Q} = \hat{Q}^* - Q^*$, then (5.1) is satisfied with

$$\begin{aligned}
V_1(x,t) &= \int_0^1 \frac{\partial H}{\partial a}\left(\epsilon, Q_s^*(t), \frac{\partial u_s}{\partial x}\right) \mathrm{d}s,\\
V_2(x,t) &= \int_0^1 \frac{\partial H}{\partial Q}\left(\epsilon, Q_s^*(t), \frac{\partial u_s}{\partial x}\right) \mathrm{d}s,\\
V_3(x,t) &= \frac{\partial H}{\partial a}\left(\epsilon, \hat{Q}^*(t), \frac{\partial \hat{u}}{\partial x}\right),\\
V_4(x,t) &= \int_0^1 \frac{\partial^2 H}{\partial a^2}\left(\epsilon, Q_s^*(t), \frac{\partial u_s}{\partial x}\right) \mathrm{d}s,\\
V_5(x,t) &= \int_0^1 \frac{\partial^2 H}{\partial Q \partial a}\left(\epsilon, Q_s^*(t), \frac{\partial u_s}{\partial x}\right) \mathrm{d}s,
\end{aligned} \tag{5.2}$$

with $f = 0$ and $\nu = 0$.

Next, we formally take the derivative of System (1.6) with respect to the measure. The result is System (5.1) if we define

$$\begin{cases}
V_1(x,t) = \dfrac{\partial H}{\partial a}\left(\epsilon, Q^*(t), \dfrac{\partial u}{\partial x}\right),\\
V_2(x,t) = \dfrac{\partial H}{\partial Q}\left(\epsilon, Q^*(t), \dfrac{\partial u}{\partial x}\right),\\
V_3(x,t) = \dfrac{\partial H}{\partial a}\left(\epsilon, Q^*(t), \dfrac{\partial u}{\partial x}\right),\\
V_4(x,t) = \dfrac{\partial^2 H}{\partial a^2}\left(\epsilon, Q^*(t), \dfrac{\partial u}{\partial x}\right),\\
V_5(x,t) = \dfrac{\partial^2 H}{\partial Q \partial a}\left(\epsilon, Q^*(t), \dfrac{\partial u}{\partial x}\right),
\end{cases} \tag{5.3}$$

with $f = 0$ and $\nu = 0$. If (w, μ) is the solution to System (5.1) assuming (5.3) and initial conditions $\mu_0 = \delta_y$, then $w(x, 0) = \frac{\delta U}{\delta m}(m, x, y)$ is the candidate derivative of the master field $U(m_0, x)$ with respect to m_0, where m_0 is a given initial condition in System (1.6).

Finally, let $\tilde{w} = \hat{u} - u - w$, $\tilde{\mu} = \hat{m} - m - \mu$, $\widetilde{\mathcal{Q}} = \hat{Q}^* - Q^* - \mathcal{Q}$. Then $(\tilde{w}, \tilde{\mu}, \widetilde{\mathcal{Q}})$ satisfies (5.1) with $V_1, \dots, V_5$ defined as in (5.3) and with

$$f(x,t) = -\int_0^1 \left(\frac{\partial H}{\partial a}\left(\epsilon, Q_s^*(t), \frac{\partial u_s}{\partial x}\right) - \frac{\partial H}{\partial a}\left(\epsilon, Q^*(t), \frac{\partial u}{\partial x}\right)\right)\left(\frac{\partial \hat{u}}{\partial x} - \frac{\partial u}{\partial x}\right) \mathrm{d}s$$
$$- \int_0^1 \left(\frac{\partial H}{\partial Q}\left(\epsilon, Q_s^*(t), \frac{\partial u_s}{\partial x}\right) - \frac{\partial H}{\partial Q}\left(\epsilon, Q^*(t), \frac{\partial u}{\partial x}\right)\right)\left(\hat{Q}^*(t) - Q^*(t)\right) \mathrm{d}s,$$
$$\nu(t) = \frac{\partial^2 H}{\partial Q \partial a}\left(\epsilon, Q^*(t), \frac{\partial u}{\partial x}\right)(\hat{Q} - Q)(\hat{m} - m) + \frac{\partial^2 H}{\partial a^2}\left(\epsilon, Q^*(t), \frac{\partial u}{\partial x}\right)\left(\frac{\partial \hat{u}}{\partial x} - \frac{\partial u}{\partial x}\right)(\hat{m} - m)$$
$$+ \hat{m} \int_0^1 \left(\frac{\partial^2 H}{\partial Q \partial a}\left(\epsilon, Q_s^*(t), \frac{\partial u_s}{\partial x}\right) - \frac{\partial^2 H}{\partial Q \partial a}\left(\epsilon, Q^*(t), \frac{\partial u}{\partial x}\right)\right)\left(\hat{Q}^*(t) - Q^*(t)\right) \mathrm{d}s$$
$$+ \hat{m} \int_0^1 \left(\frac{\partial^2 H}{\partial a^2}\left(\epsilon, Q_s^*(t), \frac{\partial u_s}{\partial x}\right) - \frac{\partial^2 H}{\partial a^2}\left(\epsilon, Q^*(t), \frac{\partial u}{\partial x}\right)\right)\left(\frac{\partial \hat{u}}{\partial x} - \frac{\partial u}{\partial x}\right) \mathrm{d}s. \quad (5.4)$$

Our a priori estimates on $(\tilde{w}, \tilde{\mu}, \widetilde{\mathcal{Q}})$ will allow us to conclude that our candidate satisfies the definition of derivative given in Definition 1.3.

Conceptually, the a priori estimates are organized in the following progression. A crucial point is to obtain *energy estimates*, which are derived by developing $\frac{\mathrm{d}}{\mathrm{d}t}\langle w, \mu\rangle$ using the equations and isolating positive terms. However, it was already noticed in [21] that the integral terms appearing in system such as (5.1) interfere with the energy estimates. Because of this, we first introduce a set of technical estimates on the Fokker-Planck equation, which require substantial preliminary results on parabolic equations. Once this major step is accomplished, we are then to proceed to the energy estimates, followed by Hölder regularity in time, and concluded by full Schauder type estimates. Combining the a priori estimates with the Leray-Schauder fixed point theorem, we also deduce an existence result for System (5.1).

5.1. Preliminaries: global-in-time interior estimates

In the context of our study of System (5.1), the main purpose of this section is to introduce some function spaces which, together with their *dual* spaces, will be useful for technical reasons in the sequel. There is a more general motivation, however, which is to find higher-order estimates on parabolic equations with Dirichlet boundary conditions, while bypassing the compatibility conditions on the boundary. So as not to distract the reader from the main purpose of this section, we have moved all the proofs to the appendix.

5.1.1. Interior estimates on the heat equation

Define $d(x) := \min\{x, 1\}$. Let n be a non-negative integer and let $k \geq 0$. For a function $\phi : [0, \infty) \to \mathbb{R}$, we define the seminorm

$$[\phi]_{n,k} := \left\| d^{n+k}\phi^{(n)} \right\|_0 = \sup_{x \geq 0} d(x)^{n+k} \left| \phi^{(n)}(x) \right|$$

and the norm

$$\|\phi\|_{n,k} := \max_{0 \leq j \leq n} [\phi]_{j,k} .$$

When $k = 0$ we will simply write $[\phi]_{n,0} = [\phi]_n$ and $\|\phi\|_{n,0} = \|\phi\|_n$. We will define $X_{n,k}$ to be the space of all function $\phi : [0, \infty) \to \mathbb{R}$ such that $\|\phi\|_{n,k}$ is finite, and $X_n := X_{n,0}$.

We will also make use of the following norm:

$$\|\phi\|^*_{n,1} := \sup_{0 \leq x \leq 1} \left| \int_0^x \phi(\xi)\, \mathrm{d}\xi \right| + \|\phi\|_{n,1} .$$

Consider now the following potentials:

$$\begin{aligned} u(x,t) &= \int_0^\infty S(x-y,t) u_0(y)\, \mathrm{d}y, \\ v(x,t) &= \int_0^t \int_0^\infty S(x-y,t-s) f(y,s)\, \mathrm{d}y\, \mathrm{d}s, \\ w(x,t) &= -2\int_0^t \frac{\partial S}{\partial x}(x,t-s)\psi(s)\, \mathrm{d}s. \end{aligned} \tag{5.5}$$

Proposition 5.1. *Let $u_0 \in X_n$, $f \in \mathcal{C}\left([0,T]; X_{n-1,1}\right)$, and $\psi \in \mathcal{C}([0,T])$. Then there exists a constant M_n, depending only on n, such that for u, v, w defined as in (5.5), we have*

$$\begin{aligned} \|u(\cdot,t)\|_n &\leq M_n \|u_0\|_n , \\ \|v(\cdot,t)\|_n &\leq M_n \int_0^t (t-s)^{-1/2} \|f(\cdot,s)\|^*_{n-1,1}\, \mathrm{d}s, \\ \|w(\cdot,t)\|_n &\leq M_n \sup_{0 \leq s \leq t} |\psi(s)| . \end{aligned} \tag{5.6}$$

Proof. See Appendix C.1.1. □

A corollary of Proposition 5.1 is an estimate of solutions to the Dirichlet problem:

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + f(x,t), \quad u(0,t) = \psi(t), \quad u(x,0) = u_0(x). \tag{5.7}$$

Theorem 5.2. *Let $u_0 \in X_n$, $f \in \mathcal{C}\left([0,T]; X_{n-1,1}\right)$, and $\psi \in \mathcal{C}([0,T])$. Let u be the solution of (5.7). Then there exists a constant M_n, depending only on n, such that*

$$\left\|u(\cdot,t)\right\|_n \le M_n\left(\|u_0\|_n + t^{1/2}\sup_{0\le s\le t}\left\|f(\cdot,s)\right\|^*_{n-1,1} + \sup_{0\le s\le t}\left|\psi(s)\right|\right). \tag{5.8}$$

Proof. See Appendix C.1.1. □

5.1.2. Application to MFG system

Here and in what follows we will let n be a positive integer such that P is $n+2$ times differentiable; by Assumption 1.1 it is possible to take $n = 2$. Then we deduce that H is $n+1$ times differentiable. A corollary of the results in Section 5.1.1 is the following:

Proposition 5.3. *Let (u,m) be the solution to the mean field games system on a finite or infinite time horizon T, i.e. either of System (4.8) or (1.6). Suppose*

$$r > \max\left\{(2\bar{Q}M_n)^2, 1\right\}\ln(2M_n), \tag{5.9}$$

where $\bar{Q}$ is defined in Equation (4.19) and M_n is the constant from Theorem 5.2. Then for any n such that H is $n+1$ times differentiable, we have

$$\sup_{t\ge 0}\left\|\frac{\partial H}{\partial a}\left(\epsilon, Q^*(t), \frac{\partial u}{\partial x}(\cdot,t)\right)\right\|_n \le D_n(r), \tag{5.10}$$

$$\sup_{t\ge 0}\left\|\frac{\partial H}{\partial Q}\left(\epsilon, Q^*(t), \frac{\partial u}{\partial x}(\cdot,t)\right)\right\|_n \le \epsilon\tilde{D}_n(r), \tag{5.11}$$

where $D(r), \tilde{D}_n(r) \ge 1$ are constants that decrease as r increases.

Proof. See Appendix C.1.2. □

Remark 5.4 *(Constants for $n = 0$).* It is worth noting that in the case $n = 0$, the constants used in this Section are already known. In particular, $M_0 = 1$, $D_0(r) = \bar{Q}$ (see Equation (4.7)), and $\tilde{D}_0(r) = P(0)$ (see Corollary 4.3).

5.2. Assumptions on the data

We will study (5.1) on a time horizon T which could be finite or infinite. When $T < \infty$ we take a final condition $w(x,T) = 0$ and assume that $\epsilon = \epsilon(t)$ satisfies Assumption 4.7. We will denote $\epsilon(0) = \epsilon$. If $T = \infty$ then ϵ is assumed to be constant, and we assume that

$$\lim_{t\to\infty} e^{-\frac{r}{2}t}\left\|w(\cdot,t)\right\|_n = \lim_{t\to\infty} e^{-\frac{r}{2}t}\left\|\frac{\partial w}{\partial x}(\cdot,t)\right\|_n = 0, \quad t \mapsto e^{-\frac{r}{2}t}\left\|\mu(t)\right\|_{-n} \quad \text{is bounded.} \tag{5.12}$$

In addition, we will state many of the following results in terms of an arbitrary positive integer n, which satisfies the restriction that P is $n+2$ times differentiable and therefore H is $n+1$ times differentiable. Assumption 4.18 and Equation (5.9) will be in force throughout this section. Hence Proposition C.1 and its corollaries (5.10) and (5.11) apply.

We now state assumptions on the coefficients $V_1,\dots,V_5$, which are abstracted from the particular cases (5.3) and (5.2).

Assumption 5.5.

(1) $\left\|V_1(\cdot,t)\right\|_n \le D_n(r)$ for all t, where $D_n(r)$ is the same as in Equation (5.10), and we assume without loss of generality that $D_n(r)\ge 1$;
(2) $\left\|V_2(\cdot,t)\right\|_n \le \epsilon\tilde{D}_n(r)$ for all t, where $\tilde{D}_n(r)$ is the same as in Equation (5.11);
(3) $\left\|V_3(\cdot,t)\right\|_n \le D_n(r)$ for all t;
(4) $C_H^{-1} \le V_4(x,t) \le C_H$ for all (x,t);
(5) $V_5(x,t) \in \left[\epsilon\frac{1-\bar{\rho}}{2-\bar{\rho}}, \epsilon\right]$ for all (x,t), and thus $\|V_5\|_0 \le \epsilon \max\left\{\left|\frac{\bar{\rho}-1}{\bar{\rho}-2}\right|, 1\right\} =: \bar{P}\epsilon$.

Lemma 5.6. *Let $V_1,\dots,V_5$ be given using formula (5.3) or (5.2). Then Assumption 5.5 holds.*

Proof. This follows from Corollaries 4.2, 4.3, and 4.6; Equations (5.10) and (5.11); and the a priori estimates from Theorem 4.16. □

Notation: If $g = g(y,t)$ is a function depending on t and other variables y and ρ is a real number, we will denote by g_ρ the function

$$g_\rho(y,t) = e^{-\rho t} g(y,t).$$

The *energy* with parameter ρ is denoted

$$E_\rho(t) = \int_{\mathcal{D}} \left|\frac{\partial w_\rho}{\partial x}(\cdot,t)\right|^2 \mathrm{d}m(t) = \int_{\mathcal{D}} e^{-2\rho t}\left|\frac{\partial w}{\partial x}(\cdot,t)\right|^2 \mathrm{d}m(t). \tag{5.13}$$

This quantity will appear often in our estimates, and we will prove a priori bounds on $\int_0^T E_\rho(t)\,\mathrm{d}t$ in Section 5.4.

5.3. Estimates in X_n and X_n^*

We will denote by X_n^* the dual of the space X_n, and by $\|\cdot\|_{-n}$ the dual norm

$$\left\|\mu\right\|_{-n} = \sup_{\|\phi\|_n\le 1} \langle \phi,\mu\rangle.$$

Note that $\left\|\mu\right\|_{-0} = \left\|\mu\right\|_{TV}$ by the Riesz representation theorem:

$$\|\mu\|_{-0} = \sup_{\|\phi\|_0 \le 1} \int_{\mathcal{D}} \phi(x)\, \mathrm{d}\mu(x) = \|\mu\|_{TV}.$$

In this subsection we provide a priori estimates on $\mu(t)$ in X_n^*, where (w, μ) is a solution of the linearized system. First, we introduce a technical lemma, somewhat reminiscent of Grönwall's inequality. Cf. [21, Lemma 2.1].

Lemma 5.7. *Let $A, B, \delta > 0$ be given constants. Suppose $f, g : [0, \infty) \to [0, \infty)$ are functions that satisfy*

$$f(t_1) \le A f(t_0) + \int_{t_0}^{t_1} (t_1 - s)^{-1/2} \big(B f(s) + g(s)\big)\, \mathrm{d}s \quad \forall 0 \le t_0 \le t_1 \le t_0 + \delta \tag{5.14}$$

Then for any $\lambda > \frac{1}{\delta} \ln(A)$, we have

$$\left(1 - \frac{2\delta^{1/2} B}{1 - Ae^{-\lambda\delta}}\right) \int_0^T e^{-\lambda t} f(t)\, \mathrm{d}t \le \frac{A}{\lambda - \delta^{-1}\ln(A)} f(0) + \frac{2\delta^{1/2}}{1 - Ae^{-\lambda\delta}} \int_0^T e^{-\lambda t} g(t)\, \mathrm{d}t. \tag{5.15}$$

Proof. See Appendix C. □

Lemma 5.8. *Let (w, μ) be a solution of (5.1). Fix $\rho \ge \kappa(r)$, where*

$$\kappa(r) := 32 \left(1 + c(\bar{\rho}, \epsilon) \bar{P} \epsilon\right)^2 D_n(r)^2 M_n^2 \ln(8M_n^2). \tag{5.16}$$

Then we have

$$\int_0^T \|\mu_\rho(t)\|_{-n}^2\, \mathrm{d}t \le \|\mu_0\|_{-n}^2 + \int_0^T \left(\|V_4\|_0^2 E_\rho(s) + \|\nu_\rho(s)\|_{-n}^2\right) \mathrm{d}s. \tag{5.17}$$

Proof. **Step 1:** Fix $t_1 > t_0 \ge 0$ and let $\phi_1 \in X_n$. Define ϕ to be the solution of the Dirichlet problem

$$-\frac{\partial \phi}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 \phi}{\partial x^2}, \quad \phi(0, t) = 0, \quad \phi(x, T) = \phi_1(x).$$

By the reflection principle, a formula for ϕ is

$$\phi(x, t) = \int_0^\infty S(x - y, t_1 - t) \phi_1(y)\, \mathrm{d}y.$$

By applying Theorem 5.2, we get

$$\left\|\phi(\cdot,t)\right\|_n \le M_n \left\|\phi_1\right\|_n \quad \forall t \in [0,t_1]. \tag{5.18}$$

Moreover, by the same argument as in 5.1, we get

$$\left\|\frac{\partial \phi}{\partial x}(\cdot,t)\right\|_n \le M_n \left\|\phi_1\right\|_n (t_1-t)^{-1/2}. \tag{5.19}$$

Now use ϕ as a test function in (5.1)(ii) to get

$$\begin{aligned}\left\langle \phi_1, \mu(t_1)\right\rangle &= \left\langle \phi(t_0), \mu(t_0)\right\rangle + \int_{t_0}^{t_1} \left\langle \frac{\partial \phi}{\partial x}(\cdot,t) V_3(\cdot,t), \mu(t)\right\rangle \mathrm{d}t \\ &\quad - \int_{t_0}^{t_1} \left\langle \frac{\partial \phi}{\partial x}(\cdot,t), \left(V_4(\cdot,t)\frac{\partial w}{\partial x}(\cdot,t) + V_5(\cdot,t)\mathcal{Q}(t)\right) m(t) + \nu(t)\right\rangle \mathrm{d}t.\end{aligned}$$

Applying (5.18) and (5.19) as well as the Cauchy-Schwartz inequality, recalling that $\left\|m(t)\right\|_{TV} \le 1$, we get

$$\begin{aligned}\left|\left\langle \phi_1, \mu(t_1)\right\rangle\right| &\le M_n \left\|\phi_1\right\|_n \left\|\mu(t_0)\right\|_{-n} + M_n \left\|\phi_1\right\|_n \int_{t_0}^{t_1} (t_1-t)^{-1/2} \left\|V_3(\cdot,t)\right\|_n \left\|\mu(t)\right\|_{-n} \mathrm{d}t \\ &\quad + M_n \left\|\phi_1\right\|_n \int_{t_0}^{t_1} (t_1-t)^{-1/2} \left(\|V_4\|_0 E_0(t)^{1/2} + \|V_5\|_0 \left|\mathcal{Q}(t)\right| + \left\|\nu(t)\right\|_{-n}\right) \mathrm{d}t. \end{aligned} \tag{5.20}$$

Step 2: Next, we need to estimate $\mathcal{Q}(t)$ using (5.1)(iii). We get

$$\left|\mathcal{Q}(t)\right| \le c(\bar{\rho},\epsilon)\left(\left\|\nu(t)\right\|_{-n} + \left\|V_3(\cdot,t)\right\|_n \left\|\mu(t)\right\|_{-n} + \|V_4\|_0 E_0(t)^{1/2}\right). \tag{5.21}$$

Plugging (5.21) into (5.20) and using Assumption 5.5, we deduce

$$\begin{aligned}\left|\left\langle \phi_1, \mu(t_1)\right\rangle\right| &\le M_n \left\|\phi_1\right\|_n \left\|\mu(t_0)\right\|_{-n} \\ &\quad + \left(1 + c(\bar{\rho},\epsilon)\bar{P}\epsilon\right) D_n(r) M_n \left\|\phi_1\right\|_n \int_{t_0}^{t_1} (t_1-t)^{-1/2} \left\|\mu(t)\right\|_{-n} \mathrm{d}t \\ &\quad + \left(1 + c(\bar{\rho},\epsilon)\bar{P}\epsilon\right) M_n \left\|\phi_1\right\|_n \int_{t_0}^{t_1} (t_1-t)^{-1/2} \left(\|V_4\|_0 E_0(t)^{1/2} + \left\|\nu(t)\right\|_{-n}\right) \mathrm{d}t.\end{aligned}$$

Taking the supremum over all $\phi_1 \in X_n$, we get

$$\left\|\mu(t_1)\right\|_{-n} \le M_n \left\|\mu(t_0)\right\|_{-n} + \left(1 + c(\bar\rho,\epsilon)\bar P\epsilon\right) D_n(r) M_n \int_{t_0}^{t_1} (t_1 - t)^{-1/2} \left\|\mu(t)\right\|_{-n} \mathrm{d}t$$

$$+ \left(1 + c(\bar\rho,\epsilon)\bar P\epsilon\right) M_n \int_{t_0}^{t_1} (t_1 - t)^{-1/2} \left(\|V_4\|_0 E_0(t)^{1/2} + \left\|\nu(t)\right\|_{-n}\right) \mathrm{d}t, \quad \forall 0 \le t_0 < t_1. \quad (5.22)$$

Step 3: Square both sides of (5.22) and use Cauchy-Schwartz to get

$$\left\|\mu(t_1)\right\|_{-n}^2 \le 4M_n^2 \left\|\mu(t_0)\right\|_{-n}^2 + \tilde B (t_1 - t_0)^{1/2} \int_{t_0}^{t_1} (t_1 - t)^{-1/2} \left\|\mu(t)\right\|_{-n}^2 \mathrm{d}t$$

$$+ \tilde B (t_1 - t_0)^{1/2} \int_{t_0}^{t_1} (t_1 - t)^{-1/2} \left(\|V_4\|_0^2 E_0(t) + \left\|\nu(t)\right\|_{-n}^2\right) \mathrm{d}t, \quad \forall 0 \le t_0 < t_1$$

where $\tilde B := 8\left(1 + c(\bar\rho,\epsilon)\bar P\epsilon\right)^2 D_n(r)^2 M_n^2$. Now we will apply Lemma 5.7 with

$$A = 4M_n^2,\ B = \tilde B \delta^{1/2},\ \delta = (8\tilde B)^{-1},\ f(t) = \left\|\mu(t)\right\|_n^2,$$
$$g(t) = B\left(\|V_4\|_0^2 E_0(t) + \left\|\nu(t)\right\|_{-n}^2\right), \text{ and } \lambda = 2\rho.$$

Comparing the definition in Equation (5.16), we see that

$$\lambda \ge 2\kappa(r) = \delta^{-1}\ln(2A) > \delta^{-1}\ln(A) \quad \Rightarrow \quad 1 - Ae^{-\lambda\delta} \le \frac{1}{2}.$$

We also have $2\delta^{1/2}B = 2\delta\tilde B \le 1/4$, and thus (5.15) implies

$$\frac{1}{2}\int_0^T e^{-\lambda t} f(t)\,\mathrm{d}t \le \frac{A}{\delta^{-1}\ln(2)} f(0) + 4\delta^{1/2}\int_0^T e^{-\lambda t} g(t)\,\mathrm{d}t.$$

By comparing the constants defined above, we deduce

$$\int_0^T e^{-\lambda t} f(t)\,\mathrm{d}t \le f(0) + B^{-1}\int_0^T e^{-\lambda t} g(t)\,\mathrm{d}t,$$

which implies (5.17), as desired. □

Corollary 5.9. *Let (w,μ) be a solution of (5.1), and suppose $\rho \ge \kappa(r)$ with $\kappa(r)$ defined in (5.16). Then*

$$\left(\int_0^T \left|\mathcal{Q}_\rho(t)\right|^2 \mathrm{d}t\right)^{1/2} \leq \hat{D}_n(r)\left(\|\mu_0\|_{-n} + \left(\int_0^T \|\nu_\rho(t)\|_{-n}^2 \mathrm{d}t\right)^{1/2} + \left(\int_0^T E_\rho(t)\,\mathrm{d}t\right)^{1/2}\right)$$

where $\hat{D}_n(r) = c(\bar{\rho}, \epsilon)\left(1 + D_n(r)\right)$.

Proof. Multiply (5.21) by $e^{-\rho t}$, take the $L^2(0,T)$ norm and then apply Lemma 5.8. □

Lemma 5.10. *Let* (w, μ) *be a solution of* (5.1) *with time horizon* T. *There exists a constant* $\kappa_1(r)$, *which depends only on* n, σ, *and* r *and is decreasing with respect to* r, *such that if*

$$\rho \leq r - \kappa_1(r) \tag{5.23}$$

and if

$$\left\|\frac{\partial w_\rho}{\partial x}(\cdot, t)\right\|_n \to 0 \quad as \quad t \to T, \tag{5.24}$$

then the following a estimate holds:

$$\int_0^T \left\|\frac{\partial w_\rho}{\partial x}(\cdot, t)\right\|_n^2 \mathrm{d}t \leq \hat{D}_n(r)^2 \|\mu_0\|_{-n}^2 + \hat{D}_n(r)^2 \int_0^T \left(E_\rho(t) + \|\nu_\rho(t)\|_{-n}^2 + \|f_\rho(\cdot, t)\|_n^2\right) \mathrm{d}t, \tag{5.25}$$

where $\hat{D}_n(r) = c(\bar{\rho}, \epsilon)\left(1 + D_n(r)\right)$.

Proof. Step 1: Fix some $T' < T$, where $T \in (0, \infty]$ is the time horizon. For any function $g = g(y,t)$ depending on t and possibly other variables, let $\tilde{g}(y,t) = g(y, T' - t)$. By reversing time in Equation (5.1)(i), we see that $\tilde{w}_r$ satisfies

$$\frac{\partial \tilde{w}_r}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 \tilde{w}_r}{\partial x^2} + \tilde{V}_1 \frac{\partial \tilde{w}_r}{\partial x} + \tilde{V}_2 \widetilde{\mathcal{Q}}_r(t) - \tilde{f}_r.$$

Since $\tilde{w}_r(0,t) = 0$, we have

$$\tilde{w}_r(x,t) = \int_0^\infty G_1(x, y, t - t_0)\tilde{w}_r(y, t_0)\,\mathrm{d}y$$

$$+ \int_{t_0}^t \int_0^\infty G_1(x, y, t - s)\left(\tilde{V}_1(y,s)\frac{\partial \tilde{w}_r}{\partial y}(y,s) + \tilde{V}_2(y,s)\widetilde{\mathcal{Q}}_r(s) - \tilde{f}_r(y,s)\right)\mathrm{d}y\,\mathrm{d}s \quad \forall t \geq t_0 \geq 0$$

where we define

$$G_{(-1)^n}(x, y, t) = (-1)^n S(x - y, t) - S(x + y, t).$$

Using an argument similar to the proof of Theorem 5.2, we deduce

$$\left\|\frac{\partial \tilde{w}_r}{\partial x}(\cdot,t)\right\|_n \le A_n \left\|\frac{\partial \tilde{w}_r}{\partial x}(\cdot,t_0)\right\|_n + B_{n,\sigma}\left(D_n(r)+\epsilon\tilde{D}_n(r)+1\right)\int_{t_0}^{t}(t-s)^{-1/2}\left(\left\|\frac{\partial \tilde{w}_r}{\partial x}(\cdot,s)\right\|_n + \left|\widetilde{\mathcal{Q}}_r(s)\right| + \left\|\tilde{f}_r(\cdot,s)\right\|_n\right)\mathrm{d}s, \tag{5.26}$$

where A_n depends only on the constants $m_1,\dots,m_n$, $B_{n,\sigma}$ depends only on the constants $m_{1,\sigma},\dots,m_{n,1}$, and $\tilde{D}_n(r)$ is the constant from (5.11).

Step 2: Square both sides of (5.26) to get

$$\left\|\frac{\partial \tilde{w}_r}{\partial x}(\cdot,t)\right\|_n^2 \le \tilde{A}_n \left\|\frac{\partial \tilde{w}_r}{\partial x}(\cdot,t_0)\right\|_n^2 + \tilde{B}_n(t-t_0)^{1/2}\int_{t_0}^{t}(t-s)^{-1/2}\left(\left\|\frac{\partial \tilde{w}_r}{\partial x}(\cdot,s)\right\|_n^2 + \left|\widetilde{\mathcal{Q}}_r(s)\right|^2 + \left\|\tilde{f}_r(\cdot,s)\right\|_n^2\right)\mathrm{d}s,$$

where

$$\tilde{A}_n := 4A_n^2, \quad \tilde{B}_n := 8B_{n,\sigma}^2\left(D_n(r)+\epsilon\tilde{D}_n(r)+1\right)^2.$$

We will apply Lemma 5.7 with

$$\delta = (8\tilde{B}_n)^{-1},\ A=\tilde{A}_n,\ \ B=\tilde{B}_n\delta^{1/2},\ \ g(t)=B\left(\left|\widetilde{\mathcal{Q}}_r(s)\right|^2 + \left\|\tilde{f}_r(\cdot,s)\right\|_n^2\right).$$

We deduce that for every $\lambda \ge \delta^{-1}\ln(2\tilde{A}_n)$,

$$\int_0^{T'} e^{-\lambda t}\left\|\frac{\partial \tilde{w}_r}{\partial x}(\cdot,t)\right\|_n^2 \mathrm{d}t \le \frac{\tilde{A}_n}{4\tilde{B}_n\ln(2)}\left\|\frac{\partial \tilde{w}_r}{\partial x}(\cdot,0)\right\|_n^2 + \int_0^{T'} e^{-\lambda t}\left(\left|\widetilde{\mathcal{Q}}_r(t)\right|^2 + \left\|\tilde{f}_r(\cdot,t)\right\|_n^2\right)\mathrm{d}t.$$

Define

$$\kappa_1(r) := 4\tilde{B}_n\ln(2\tilde{A}_n) = 32B_n^2\left(D_n(r)+\epsilon\tilde{D}_n(r)+1\right)^2\ln(2\tilde{A}_n),$$

which satisfies the hypotheses given in the statement of the lemma. Then set $\rho = r-\frac{\lambda}{2}$; we have define $\kappa_1(r)$ so that $\rho \le r-\kappa_1(r)$ is equivalent to $\lambda \ge \delta^{-1}\ln(2\tilde{A}_n)$. Now make the substitution $t \mapsto T'-t$, then let $T' \to T$ and use (5.24) to get

$$\int_0^T \left\| \frac{\partial w_\rho}{\partial x}(\cdot,t) \right\|_n^2 \mathrm{d}t \le \int_0^T \left(\left| \mathcal{Q}_\rho(t) \right|^2 + \| f_\rho(\cdot,t) \|_n^2 \right) \mathrm{d}t.$$

Finally, we use Corollary 5.9 to get (5.25). □

We can also estimate $\|\mu_\rho(t)\|_{-n}$ pointwise, provided we are willing to include some dependence on $\left\| \frac{\partial w_\rho}{\partial x} \right\|_0$, which will be estimated below.

Lemma 5.11. *Let (w, μ) be a solution of (5.1). Suppose*

$$\rho \ge 36 \left(1 + c(\bar{\rho}, \epsilon)\right)^2 D_n(r)^2 M_n^2 =: \kappa_0(r). \tag{5.27}$$

Then

$$\sup_{0 \le t \le T} \|\mu_\rho(t)\|_{-n} \le 2M_n \|\mu_0\|_{-n} + \sup_{0 \le \tau \le T} \|\nu_\rho(\tau)\|_{-n} + C_n \left\| \frac{\partial w_\rho}{\partial x} \right\|_0^{1/2} \left(\int_0^T E_\rho(s)\,\mathrm{d}s \right)^{1/4}, \tag{5.28}$$

where

$$C_n = 4\left(1 + c(\bar{\rho}, \epsilon)\right)^{1/2} M_n^{1/2}.$$

Proof. Take (5.22) with $t_0 = 0, t_1 = t$, multiply by $e^{-\rho t}$ to get

$$\begin{aligned} \|\mu_\rho(t)\|_{-n} \le M_n \|\mu_0\|_{-n} &+ \left(1 + c(\bar{\rho}, \epsilon)\right) D_n(r) M_n \int_0^t e^{-\rho(t-s)} (t-s)^{-1/2} \|\mu_\rho(s)\|_{-n}\,\mathrm{d}s \\ &+ \left(1 + c(\bar{\rho}, \epsilon)\right) M_n \int_0^t e^{-\rho(t-s)} (t-s)^{-1/2} \left(E_\rho(s)^{1/2} + \|\nu_\rho(s)\|_{-n} \right) \mathrm{d}s. \end{aligned} \tag{5.29}$$

We first use Hölder's inequality to estimate

$$\begin{aligned} \int_0^t e^{-\rho(t-s)} (t-s)^{-1/2} E_\rho(s)^{1/2}\,\mathrm{d}s &\le \left\| \frac{\partial w_\rho}{\partial x} \right\|_0^{1/2} \int_0^t e^{-\rho(t-s)} (t-s)^{-1/2} E_\rho(s)^{1/4}\,\mathrm{d}s \\ &\le \left\| \frac{\partial w_\rho}{\partial x} \right\|_0^{1/2} \left(\int_0^t e^{-\frac{4}{3}\rho(t-s)} (t-s)^{-2/3}\,\mathrm{d}s \right)^{3/4} \left(\int_0^t E_\rho(s)\,\mathrm{d}s \right)^{1/4}. \end{aligned} \tag{5.30}$$

Using the substitution $s \mapsto t - \frac{s}{\rho}$, we find

$$\begin{aligned}\int_0^t e^{-\frac{4}{3}\rho(t-s)}(t-s)^{-2/3}\,\mathrm{d}s &= \rho^{-1/3}\int_0^t e^{-\frac{4}{3}s}s^{-2/3}\,\mathrm{d}s \\ &\le \rho^{-1/3}\left(\int_0^1 s^{-2/3}\,\mathrm{d}s + \int_1^\infty e^{-\frac{4}{3}s}\,\mathrm{d}s\right) \le 4\rho^{-1/3}\end{aligned} \tag{5.31}$$

and also

$$\int_0^t e^{-\rho(t-s)}(t-s)^{-1/2}\,\mathrm{d}s \le 3\rho^{-1/2}. \tag{5.32}$$

Using (5.30), (5.31), and (5.32) in (5.29), we get

$$\begin{aligned}\left\|\mu_\rho(t)\right\|_{-n} \le M_n\left\|\mu_0\right\|_{-n} &+ 3\rho^{-1/2}\left(1+c(\bar{\rho},\epsilon)\right)D_n(r)M_n \sup_{0\le\tau\le T}\left\|\mu_\rho(\tau)\right\|_{-n} \\ &+ 3\rho^{-1/2}\left(1+c(\bar{\rho},\epsilon)\right)M_n \sup_{0\le\tau\le T}\left\|\nu_\rho(\tau)\right\|_{-n} \\ &+ 4\rho^{-1/4}\left(1+c(\bar{\rho},\epsilon)\right)M_n\left\|\frac{\partial w_\rho}{\partial x}\right\|_0^{1/2}\left(\int_0^T E_\rho(s)\,\mathrm{d}s\right)^{1/4}.\end{aligned} \tag{5.33}$$

By the assumption (5.27), (5.33) simplifies to

$$\begin{aligned}\left\|\mu_\rho(t)\right\|_{-n} \le M_n\left\|\mu_0\right\|_{-n} &+ \frac{1}{2}\sup_{0\le\tau\le T}\left\|\mu_\rho(\tau)\right\|_{-n} \\ &+ \frac{1}{2}\sup_{0\le\tau\le T}\left\|\nu_\rho(\tau)\right\|_{-n} + 2\left(1+c(\bar{\rho},\epsilon)\right)^{1/2}M_n^{1/2}\left\|\frac{\partial w_\rho}{\partial x}\right\|_0^{1/2}\left(\int_0^T E_\rho(s)\,\mathrm{d}s\right)^{1/4}.\end{aligned}$$

Take the supremum and rearrange to deduce (5.28). □

From now on we make the following assumption:

Assumption 5.12. We assume $r \ge 2\max\left\{\kappa(r), \kappa_1(r), \kappa_0(r)\right\}$ with $\kappa(r)$ defined in (5.16), $\kappa_1(r)$ defined in (5.23), and $\kappa_0(r)$ defined in (5.27).

Importantly, Assumption 5.12 can always be obtained by choosing r large enough, because $\kappa(r)$, $\kappa_1(r)$, and $\kappa_0(r)$ are all decreasing functions of r.

Remark 5.13. When $n = 0$, Remark 5.4 shows us that $\kappa(r)$, $\kappa_1(r)$, and $\kappa_0(r)$ no longer depend on r. In fact, they have the following formulas, more or less explicit:

$$\kappa(r) := 32\left(1 + c(\bar{\rho}, \epsilon)\bar{P}\epsilon\right)^2 \bar{Q}^2 \ln(8),$$

$$\kappa_1(r) := 32B_0^2\left(\bar{Q} + \epsilon P(0) + 1\right)^2 \ln(2\tilde{A}_0),$$

$$\kappa_0(r) := 36\left(1 + c(\bar{\rho}, \epsilon)\right)^2 \bar{Q}^2.$$

Only the constant B_0 and $\tilde{A}_0$ from the proof of Lemma 5.10 are left undefined, but upon inspection of the proof we can see that $\tilde{A}_0$ and B_0 are constants no greater than, say, 10. Therefore (4.22) is surely an overestimate.

Corollary 5.14 *(Summary of this subsection). Let (w, μ) be a solution of (5.1). Under Assumption 5.12, we have the following a priori estimates:*

$$\int_0^T \left\|\mu_{r/2}(t)\right\|_{-n}^2 \mathrm{d}t \le \left\|\mu_0\right\|_{-n}^2 + \int_0^T \left(\|V_4\|_0^2 E_{r/2}(t) + \left\|\nu_{r/2}(t)\right\|_{-n}^2\right) \mathrm{d}t,$$

$$\left(\int_0^T \left|\mathcal{Q}_{r/2}(t)\right|^2 \mathrm{d}t\right)^{1/2} \le \hat{D}_n(r)\left(\left\|\mu_0\right\|_{-n} + \left(\int_0^T \left\|\nu_{r/2}(t)\right\|_{-n}^2 \mathrm{d}t\right)^{1/2} + \left(\int_0^T E_{r/2}(t)\,\mathrm{d}t\right)^{1/2}\right),$$

$$\int_0^T \left\|\frac{\partial w_{r/2}}{\partial x}(\cdot, t)\right\|_n^2 \mathrm{d}t \le \hat{D}_n(r)^2\left\|\mu_0\right\|_{-n}^2 + \hat{D}_n(r)^2\int_0^T \left(E_{r/2}(t) + \left\|\nu_{r/2}(t)\right\|_{-n}^2 + \left\|f_{r/2}(\cdot, t)\right\|_n^2\right)\mathrm{d}t,$$

$$\sup_{0\le t\le T}\left\|\mu_{r/2}(t)\right\|_{-n} \le 2M_n\left\|\mu_0\right\|_{-n} + \sup_{0\le \tau\le T}\left\|\nu_{r/2}(\tau)\right\|_{-n} + C_n\left\|\frac{\partial w_{r/2}}{\partial x}\right\|_0^{1/2}\left(\int_0^T E_{r/2}(t)\,\mathrm{d}t\right)^{1/4},$$

where $\hat{D}_n(r) = c(\bar{\rho}, \epsilon)\left(1 + D_n(r)\right)$, $C_n = 4\left(1 + c(\bar{\rho}, \epsilon)\right)^{1/2} M_n^{1/2}$, and $D_n(r)$ is the constant appearing in Equation (5.10).

Proof. It suffices to observe that the hypotheses of Lemmas 5.8, 5.10, and 5.11 are all satisfied with $\rho = r/2$. □

5.4. *Energy estimates*

In some mean field games, known as "potential mean field games," the Nash equilibrium can be computed by minimizing a certain energy functional [30,3,5,6]. Because of a formal resemblance, we keep the name "energy estimates" for the estimates derived in this subsection. We divide our results into two lemmas. The first deals with differences of solutions to System (1.6), in which case we assume (5.2) with $f = \nu = 0$, and the second deals with the case (5.3), with no restriction on f, ν. Although it is tempting to view the former as a special case of the latter, there are technical points in the proof in which it is not convenient to do so, and thus the proofs are treated separately. Nevertheless, their basic outline is similar: differentiate the duality pairing $\langle w, \mu \rangle$ with respect to time and use the PDE system to write an identity, then use the assumption on the uniform convexity of H to derive an estimate of the integral $\int_0^T E_{r/2}(t)\,\mathrm{d}t$. (Recall that $E_{r/2}$ is defined by (5.13).)

Lemma 5.15 *(Energy estimates, differences). Let (u, m, Q^*) and $(\hat{u}, \hat{m}, \hat{Q}^*)$ be solutions to System (1.6) with initial conditions m_0 and $\hat{m}_0$, respectively.*

(1) *Assume that ϵ satisfies the smallness condition*

$$4C_H \hat{D}_n(r) \Big(C_H \big(P(0) + 1 \big) + \bar{P} \Big) \epsilon \leq 1, \tag{5.34}$$

where, as in (4.3), $\bar{P} = \max\left\{ \frac{\bar{\rho}-1}{\bar{\rho}-2}, 1 \right\}$. Then

$$\int_0^T \int_{\mathcal{D}} e^{-rt} \left| \frac{\partial u}{\partial x} - \frac{\partial \hat{u}}{\partial x} \right|^2 \big(m(\mathrm{d}x, t) + \hat{m}(\mathrm{d}x, t) \big)\,\mathrm{d}t$$
$$\leq \left\| \hat{m}_0 - m_0 \right\|_{-n}^2 + 2C_H \left\| \hat{u}(\cdot, 0) - u(\cdot, 0) \right\|_n \left\| \hat{m}_0 - m_0 \right\|_{-n}, \tag{5.35}$$

(2) *Assume instead that the demand schedule is linear, i.e. $P(q) = 1 - q$, and that $\epsilon < 2$. Then we have*

$$\int_0^T \int_{\mathcal{D}} e^{-rt} \left| \frac{\partial u}{\partial x} - \frac{\partial \hat{u}}{\partial x} \right|^2 \big(m(\mathrm{d}x, t) + \hat{m}(\mathrm{d}x, t) \big)\,\mathrm{d}t \leq 8 \left\| \hat{u}(\cdot, 0) - u(\cdot, 0) \right\|_n \left\| \hat{m}_0 - m_0 \right\|_{-n}. \tag{5.36}$$

Proof. Step 1: *For a small parameter ϵ.* In this first step, we make no further assumptions on the demand schedule P but instead assume condition (5.34) holds. Multiply $(1.6)_i$(ii) by $u - \hat{u}$ and integrate by parts, then subtract. (See [30, Theorem 2.4].) After rearranging we get

$$\left[\int_{\mathcal{D}} e^{-rt} (u(x,t) - \hat{u}(x,t))(m - \hat{m})(\mathrm{d}x, t) \right]_0^T$$
$$= \int_0^T \int_{\mathcal{D}} e^{-rt} \left(H\left(\epsilon, \hat{Q}^*(t), \frac{\partial \hat{u}}{\partial x}\right) - H\left(\epsilon, Q^*(t), \frac{\partial u}{\partial x}\right) - \frac{\partial H}{\partial a}\left(\epsilon, Q^*(t), \frac{\partial u}{\partial x}\right)\left(\frac{\partial \hat{u}}{\partial x} - \frac{\partial u}{\partial x}\right) \right) m(\mathrm{d}x, t)\,\mathrm{d}t$$
$$+ \int_0^T \int_{\mathcal{D}} e^{-rt} \left(H\left(\epsilon, Q^*(t), \frac{\partial u}{\partial x}\right) - H\left(\epsilon, \hat{Q}^*(t), \frac{\partial \hat{u}}{\partial x}\right) - \frac{\partial H}{\partial a}\left(\epsilon, \hat{Q}^*(t), \frac{\partial \hat{u}}{\partial x}\right)\left(\frac{\partial u}{\partial x} - \frac{\partial \hat{u}}{\partial x}\right) \right) \hat{m}(\mathrm{d}x, t)\,\mathrm{d}t.$$

By Equation (4.2), we deduce

$$\frac{1}{C_H}\int_0^T\int_{\mathcal{D}} e^{-rt}\left|\frac{\partial u}{\partial x}-\frac{\partial \hat u}{\partial x}\right|^2\big(m(\mathrm{d}x,t)+\hat m(\mathrm{d}x,t)\big)\,\mathrm{d}t$$
$$\le \int_0^T\int_{\mathcal{D}} e^{-rt}\left(H\left(\epsilon, Q^*(t),\frac{\partial \hat u}{\partial x}\right)-H\left(\epsilon,\hat Q^*(t),\frac{\partial \hat u}{\partial x}\right)\right)m(\mathrm{d}x,t)\,\mathrm{d}t$$
$$+\int_0^T\int_{\mathcal{D}} e^{-rt}\left(H\left(\epsilon, \hat Q^*(t),\frac{\partial u}{\partial x}\right)-H\left(\epsilon,Q^*(t),\frac{\partial u}{\partial x}\right)\right)\hat m(\mathrm{d}x,t)\,\mathrm{d}t$$
$$+\left[\int_{\mathcal{D}} e^{-rt}(u(x,t)-\hat u(x,t))(m-\hat m)(\mathrm{d}x,t)\right]_0^T. \tag{5.37}$$

Since $u,\hat u$ are bounded and $\int_{\mathcal{D}} m_i(\mathrm{d}x,T)\le 1$ for all T, it follows that

$$\lim_{T\to\infty}\int_{\mathcal{D}} e^{-rT}(u(T,x)-\hat u(T,x))(m-\hat m)(\mathrm{d}x,T)=0.$$

We can rewrite the remaining terms on the right-hand side using the fundamental theorem of calculus. Thus (5.37) becomes, after letting $T\to\infty$,

$$\frac{1}{C_H}\int_0^\infty\int_{\mathcal{D}} e^{-rt}\left|\frac{\partial u}{\partial x}-\frac{\partial \hat u}{\partial x}\right|^2\big(m(\mathrm{d}x,t)+\hat m(\mathrm{d}x,t)\big)\,\mathrm{d}t\le I_0+I_1+I_2, \tag{5.38}$$

where $I_0 := \left|\int_{\mathcal{D}}(u(0,x)-\hat u(0,x))(m-\hat m)(\mathrm{d}x,0)\right|$,

$$I_1 := \int_0^1\int_0^\infty\int_{\mathcal{D}} e^{-rt}\frac{\partial H}{\partial Q}\left(\epsilon, Q_s^*(t),\frac{\partial \hat u}{\partial x}\right)\left(Q^*(t)-\hat Q^*(t)\right)(m-\hat m)(\mathrm{d}x,t)\,\mathrm{d}t\,\mathrm{d}s,\quad \text{and}$$

$$I_2 := \int_0^1\int_0^1\int_0^\infty\int_{\mathcal{D}} e^{-rt}\frac{\partial^2 H}{\partial Q\partial a}\left(\epsilon, Q_s^*(t),\frac{\partial u_{\tilde s}}{\partial x}\right)\left(\frac{\partial \hat u}{\partial x}-\frac{\partial u}{\partial x}\right)\left(Q^*(t)-\hat Q^*(t)\right)m(\mathrm{d}x,t)\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}\tilde s,$$

where $Q_s^*(t) := sQ^*(t)+(1-s)\hat Q^*(t),\ u_s := s\hat u+(1-s)u.$

By using Corollary 4.3 and (4.18), we can estimate

$$\left|\frac{\partial H}{\partial Q}\left(\epsilon, Q_s^*(t),\frac{\partial \hat u}{\partial x}\right)\right|\le\big(P(0)+1\big)\epsilon,$$

$$\left|\frac{\partial^2 H}{\partial Q \partial a}\left(\epsilon, Q_s^*(t), \frac{\partial u_{\tilde{s}}}{\partial x}\right)\left(\frac{\partial \hat{u}}{\partial x} - \frac{\partial u}{\partial x}\right)\right| \leq \bar{P}\epsilon, \quad \forall s, \tilde{s} \in [0,1],$$

where $\bar{P} := \max\left\{\left|\frac{\bar{\rho}-1}{\bar{\rho}-2}\right|, 1\right\}$ is defined in Corollary 4.3. Thus

$$\begin{aligned} |I_1| &\leq \big(P(0)+1\big)\epsilon \int_0^\infty e^{-rt}\left|Q^*(t) - \hat{Q}^*(t)\right| \left\|m(t) - \hat{m}(t)\right\|_{-n} \mathrm{d}t, \\ |I_2| &\leq \bar{P}\epsilon \int_0^\infty \int_{\mathcal{D}} e^{-rt}\left|\frac{\partial \hat{u}}{\partial x} - \frac{\partial u}{\partial x}\right|\left|Q^*(t) - \hat{Q}^*(t)\right| m(\mathrm{d}x,t)\,\mathrm{d}t. \end{aligned} \tag{5.39}$$

Recalling the definitions $w = \hat{u} - u$, $\mu = \hat{m} - m$, and $\mathcal{Q} = \hat{Q} - Q$, using the Cauchy-Schwartz inequality and the fact that m is a sub-probability measure, we deduce the following from (5.39):

$$\begin{aligned} |I_1| &\leq \big(P(0)+1\big)\epsilon \left(\int_0^\infty \left|\mathcal{Q}_{r/2}(t)\right|^2 \mathrm{d}t\right)^{1/2} \left(\int_0^\infty \left\|\mu_{r/2}(t)\right\|_{-n}^2 \mathrm{d}t\right)^{1/2}, \\ |I_2| &\leq \bar{P}\epsilon \left(\int_0^\infty \left|\mathcal{Q}_{r/2}(t)\right|^2 \mathrm{d}t\right)^{1/2} \left(\int_0^\infty E_{r/2}(t)\,\mathrm{d}t\right)^{1/2}. \end{aligned} \tag{5.40}$$

We now apply Corollary 5.14 and Assumption 5.5; here we can assume $\nu = 0$ and $f = 0$. Thus (5.40) implies

$$\begin{aligned} |I_1| &\leq 2\hat{D}_n(r) C_H \left(P(0)+1\right)\epsilon \left(\|\mu_0\|_{-n}^2 + \int_0^\infty E_{r/2}(t)\,\mathrm{d}t\right), \\ |I_2| &\leq 2\hat{D}_n(r)\bar{P}\epsilon \left(\|\mu_0\|_{-n}^2 + \int_0^\infty E_{r/2}(t)\,\mathrm{d}t\right). \end{aligned} \tag{5.41}$$

Plugging (5.41) into (5.38), we deduce

$$\int_0^T \int_{\mathcal{D}} e^{-rt}\left|\frac{\partial u}{\partial x} - \frac{\partial \hat{u}}{\partial x}\right|^2 \big(m(\mathrm{d}x,t) + \hat{m}(\mathrm{d}x,t)\big)\,\mathrm{d}t \leq C_H \hat{C}\epsilon \left(\|\mu_0\|_{-n}^2 + \int_0^\infty E_{r/2}(t)\,\mathrm{d}t\right) + C_H I_0, \tag{5.42}$$

where $\hat{C} = 2\hat{D}_n(r)\Big(C_H\left(P(0)+1\right) + \bar{P}\Big)$. Equation (5.34) can be written

$$2C_H\hat{C}\epsilon \leq 1.$$

Since the left-hand side of (5.42) dominates $\int_0^\infty E_{r/2}(t)\,\mathrm{d}t$, we use (5.34) and rearrange to deduce (5.35).

Step 2: *For a linear demand schedule.* Now we consider the case where $P(q) = 1 - q$ and $\epsilon < 2$. In this case the same series of computations (cf. the proof of Theorem 4.21, see also Equation (5.51) below) now leads to

$$\frac{1}{4}\int_0^{T'}\int_{\mathcal{D}}\left(\frac{\partial w_{r/2}}{\partial x}+\epsilon\mathcal{Q}_{r/2}\right)^2 \mathrm{d}(\hat{m}+m)(t)\,\mathrm{d}t+\epsilon\int_0^{T'}\mathcal{Q}_{r/2}(t)^2\,\mathrm{d}t$$
$$\leq e^{-rT'}\left\|w(\cdot,T')\right\|_0\left\|\mu(T')\right\|_{-0}+\left\|w(\cdot,0)\right\|_n\left\|\mu_0\right\|_{-n}. \quad (5.43)$$

Let $T' \to T$, rearrange the square term in (5.43) and perform standard estimates to deduce

$$\int_0^T\int_{\mathcal{D}}\left(\frac{\partial w_{r/2}}{\partial x}\right)^2 \mathrm{d}(\hat{m}+m)(t)\,\mathrm{d}t \leq 8\left\|w(\cdot,0)\right\|_n\left\|\mu_0\right\|_{-n},$$

which is the same as (5.36). □

Lemma 5.16 *(Energy estimates, all other cases). Let (w,μ) be a solution of the system (5.1), and assume that $V_1,\dots,V_5, f, \nu$ satisfy (5.3).*

(1) *Assume that ϵ is sufficiently small, namely*

$$\epsilon 4\hat{D}_n(r)^2\left(\tilde{D}_n(r)+\bar{P}\right)C_H^2 \leq (4C_H)^{-1}. \quad (5.44)$$

Then

$$\int_0^T E_{r/2}(t)\,\mathrm{d}t \leq 4C_H\left\|w(\cdot,0)\right\|_n\left\|\mu_0\right\|_{-n}+4\left\|\mu_0\right\|_{-n}^2+\hat{C}\int_0^T\left(\left\|f_{r/2}(\cdot,t)\right\|_n^2+\left\|\nu_{r/2}(t)\right\|_{-n}^2\right)\mathrm{d}t, \quad (5.45)$$

where $\hat{C} = 4C_H^2\left(\hat{D}_n(r)^2+C_H^2\right)+1$.

(2) *Assume instead that the demand schedule P is linear, i.e. $P(q) = 1 - q$, and that $\epsilon < 2$. Then*

$$\int_0^T E_{r/2}(s)\,\mathrm{d}s \leq 16\left\|w(\cdot,0)\right\|_n\left\|\mu_0\right\|_{-n}+\left\|\mu_0\right\|_{-n}^2+\hat{C}\int_0^T\left(\left\|\nu_{r/2}(t)\right\|_{-n}^2+\left\|f_{r/2}(\cdot,t)\right\|_n^2\right)\mathrm{d}t, \quad (5.46)$$

where $\hat{C} = \left(32\max\left\{\hat{D}_n(r),C_H\right\}^2+17\right)$.

Proof. Note that the case when (w,μ) is a difference of two solutions to System (1.6), so that (5.2) holds with $f = \nu = 0$, is already proved in Section 4.7.

Step 1: *For a small parameter* ϵ. In this first step, we make no further assumptions on the demand schedule P but instead assume condition (5.44) holds. Note that when

Differentiate $e^{-rt}\int_{\mathcal{D}} w\mu$ with respect to t and integrate by parts to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathcal{D}} w_{r/2}\mu_{r/2}\right) = \int_{\mathcal{D}} f_{r/2}\mu_{r/2} - \mathcal{Q}_{r/2}(t)\int_{\mathcal{D}} V_2\mu_{r/2} + \int_{\mathcal{D}} V_4\left|\frac{\partial w_{r/2}}{\partial x}\right|^2 m + \mathcal{Q}_{r/2}(t)\int_{\mathcal{D}} V_5\frac{\partial w_{r/2}}{\partial x}m + \int_{\mathcal{D}}\frac{\partial w_{r/2}}{\partial x}\nu_{r/2}. \tag{5.47}$$

Let $T' \in (0,T)$ and integrate (5.47) from 0 to T'. Recalling that $V_4 \geq C_H^{-1}$ from Assumption 5.5, we get

$$C_H^{-1}\int_0^{T'} E_{r/2}(t)\,\mathrm{d}t \leq \left\langle w_{r/2}(\cdot,t),\mu_{r/2}(t)\right\rangle\Big|_0^{T'} + \int_0^{T'}\left\|f_{r/2}(\cdot,t)\right\|_n\left\|\mu_{r/2}(t)\right\|_{-n}\mathrm{d}t$$
$$+\int_0^{T'}\left|\mathcal{Q}_{r/2}(t)\right|\left(\|V_2\|_n\left\|\mu_{r/2}(t)\right\|_{-n} + \|V_5\|_0\,E_{r/2}(t)^{1/2}\right)\mathrm{d}t + \int_0^{T'}\left\|\frac{\partial w_{r/2}}{\partial x}(\cdot,t)\right\|_n\left\|\nu_{r/2}(t)\right\|_{-n}\mathrm{d}t.$$

Then let $T' \to T$ and recall that by assumption (5.12), $\lim_{t\to T}\left\langle w_{r/2}(\cdot,t),\mu_{r/2}(t)\right\rangle = 0$. Thus,

$$C_H^{-1}\int_0^{T} E_{r/2}(t)\,\mathrm{d}t \leq \left\|w(\cdot,0)\right\|_n\left\|\mu_0\right\|_{-n} + \int_0^{T}\left\|f_{r/2}(\cdot,t)\right\|_n\left\|\mu_{r/2}(t)\right\|_{-n}\mathrm{d}t$$
$$+\int_0^{T}\left|\mathcal{Q}_{r/2}(t)\right|\left(\|V_2\|_n\left\|\mu_{r/2}(t)\right\|_{-n} + \|V_5\|_0\,E_{r/2}(t)^{1/2}\right)\mathrm{d}t + \int_0^{T}\left\|\frac{\partial w_{r/2}}{\partial x}(\cdot,t)\right\|_n\left\|\nu_{r/2}(t)\right\|_{-n}\mathrm{d}t. \tag{5.48}$$

Now using Corollary 5.14, recalling $\|V_4\|_0 \leq C_H$ (Assumption 5.5), we derive

$$\int_0^T\left(\left\|f_{r/2}(\cdot,t)\right\|_n\left\|\mu_{r/2}(t)\right\|_{-n} + \left\|\frac{\partial w_{r/2}}{\partial x}(\cdot,t)\right\|_n\left\|\nu_{r/2}(t)\right\|_{-n}\right)\mathrm{d}t$$
$$\leq (2C_H)^{-1}\int_0^T E_{r/2}(t)\,\mathrm{d}t + (2C_H)^{-1}\left\|\mu_0\right\|_{-n}^2 + C_1\int_0^T\left(\left\|f_{r/2}(\cdot,t)\right\|_n^2 + \left\|\nu_{r/2}(t)\right\|_{-n}^2\right)\mathrm{d}t,$$

where $C_1 := C_H\left(\hat{D}_n(r)^2 + C_H^2\right)$. Thus (5.48) yields

$$(2C_H)^{-1}\int_0^T E_{r/2}(t)\,\mathrm{d}t \le \|w(\cdot,0)\|_n\|\mu_0\|_{-n} + C_1\int_0^T\left(\|f_{r/2}(\cdot,t)\|_n^2 + \|\nu_{r/2}(t)\|_{-n}^2\right)\mathrm{d}t$$

$$+ (2C_H)^{-1}\|\mu_0\|_{-n}^2 + \int_0^T\left|\mathcal{Q}_{r/2}(t)\right|\left(\|V_2\|_n\|\mu_{r/2}(t)\|_{-n} + \|V_5\|_0 E_{r/2}(t)^{1/2}\right)\mathrm{d}t. \quad (5.49)$$

Also, again using Corollary 5.14 and also Assumption 5.5, we get

$$\int_0^T\left|\mathcal{Q}_{r/2}(t)\right|\left(\|V_2\|_n\|\mu_{r/2}(t)\|_{-n} + \|V_5\|_0 E_{r/2}(t)^{1/2}\right)\mathrm{d}t$$

$$\le \epsilon 4\hat{D}_n(r)^2\left(\tilde{D}_n(r) + \bar{P}\right)\left(\|\mu_0\|_{-n}^2 + \int_0^T\|\nu_{r/2}(t)\|_{-n}^2\,\mathrm{d}t + C_H^2\int_0^T E_{r/2}(t)\,\mathrm{d}t\right)$$

where $\bar{P} = \max\left\{\left|\frac{\bar{\rho}-1}{2-\bar{\rho}}\right|, 1\right\}$. Then by (5.44), Equation (5.49) yields

$$(4C_H)^{-1}\int_0^T E_{r/2}(t)\,\mathrm{d}t \le \|w(\cdot,0)\|_n\|\mu_0\|_{-n} + C_1\int_0^T\left(\|f_{r/2}(\cdot,t)\|_n^2 + \|\nu_{r/2}(t)\|_{-n}^2\right)\mathrm{d}t$$

$$+ C_H^{-1}\|\mu_0\|_{-n}^2 + (4C_H)^{-1}\int_0^T\|\nu_{r/2}(t)\|_{-n}^2\,\mathrm{d}t. \quad (5.50)$$

We rearrange (5.50) to conclude with (5.45).

Step 2: *For a linear demand schedule.* Now we consider the case where $P(q) = 1 - q$ and $\epsilon < 2$, so that the system has the form (4.21). After doing integration by parts and canceling like terms, we get

$$\left(2 + \epsilon(t)\int_{\mathcal{D}}\mathrm{d}m(t)\right)^{-1}\int_{\mathcal{D}}\left|\frac{\partial w}{\partial x}\right|^2\mathrm{d}m(t) + 2\left(2 + \epsilon(t)\int_{\mathcal{D}}\mathrm{d}m(t)\right)^{-1}\left(\int_{\mathcal{D}} q^*(\cdot,t)\,\mathrm{d}\mu(t)\right)^2$$

$$= e^{rt}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-rt}\int_{\mathcal{D}} w(\cdot,t)\,\mathrm{d}\mu(t)\right) + 2\left(2 + \epsilon(t)\int_{\mathcal{D}}\mathrm{d}m(t)\right)^{-1}\int_{\mathcal{D}}\mathrm{d}\nu(t)\int_{\mathcal{D}} q^*(\cdot,t)\,\mathrm{d}\mu(t)$$

$$+ \left(2 + \epsilon(t)\int_{\mathcal{D}}\mathrm{d}m(t)\right)^{-1}\int_{\mathcal{D}}\mathrm{d}\nu(t)\int_{\mathcal{D}}\frac{\partial w}{\partial x}\mathrm{d}m(t) - \int_{\mathcal{D}}\frac{\partial w}{\partial x}\mathrm{d}\nu(t) + \int_{\mathcal{D}} f(\cdot,t)\,\mathrm{d}\mu(t), \quad (5.51)$$

from which we deduce

$$\frac{1}{2}\left(2+\epsilon(t)\int_{\mathcal{D}} \mathrm{d}m(t)\right)^{-1}\int_{\mathcal{D}}\left|\frac{\partial w}{\partial x}\right|^2 \mathrm{d}m(t)+\left(2+\epsilon(t)\int_{\mathcal{D}} \mathrm{d}m(t)\right)^{-1}\left(\int_{\mathcal{D}} q^*(\cdot,t)\,\mathrm{d}\mu(t)\right)^2$$
$$\leq e^{rt}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-rt}\int_{\mathcal{D}} w(\cdot,t)\,\mathrm{d}\mu(t)\right)+\frac{\epsilon^2+2\epsilon}{2}\left(2+\epsilon(t)\int_{\mathcal{D}} \mathrm{d}m(t)\right)^{-1}\left(\int_{\mathcal{D}} \mathrm{d}\nu(t)\right)^2$$
$$-\int_{\mathcal{D}}\frac{\partial w}{\partial x}\,\mathrm{d}\nu(t)+\int_{\mathcal{D}} f(\cdot,t)\,\mathrm{d}\mu(t). \tag{5.52}$$

Multiply (5.52) by e^{-rt}, integrate from 0 to T' and let $T'\to T$ to get

$$\int_0^T E_{r/2}(s)\,\mathrm{d}s \leq 2(2+\epsilon)\|w(\cdot,0)\|_n\|\mu_0\|_{-n}+\frac{\epsilon(2+\epsilon)^2}{2}\int_0^T \|\nu_{r/2}(s)\|_{-n}^2\,\mathrm{d}s$$
$$+2(2+\epsilon)\int_0^T\left(\left\|\frac{\partial w_{r/2}}{\partial x}(\cdot,s)\right\|_n\|\nu_{r/2}(s)\|_{-n}+\|f_{r/2}(\cdot,s)\|_n\|\mu_{r/2}(s)\|_{-n}\right)\mathrm{d}s. \tag{5.53}$$

$$4(2+\epsilon)\int_0^T\left(\left\|\frac{\partial w_{r/2}}{\partial x}(\cdot,t)\right\|_n\|\nu_{r/2}(t)\|_{-n}+\|f_{r/2}(\cdot,t)\|_n\|\mu_{r/2}(t)\|_{-n}\right)\mathrm{d}t$$
$$\leq \|\mu_0\|_{-n}^2+\int_0^T E_{r/2}(t)\,\mathrm{d}t$$
$$+\left(8\max\left\{\hat{D}_n(r),C_H\right\}^2(2+\epsilon)^2+1\right)\int_0^T\left(\|\nu_{r/2}(t)\|_{-n}^2+\|f_{r/2}(\cdot,t)\|_n^2\right)\mathrm{d}t$$

Using Corollary 5.14 and rearranging (5.53), we deduce

$$\int_0^T E_{r/2}(s)\,\mathrm{d}s \leq 4(2+\epsilon)\|w(\cdot,0)\|_n\|\mu_0\|_{-n}+\|\mu_0\|_{-n}^2+\epsilon(2+\epsilon)^2\int_0^T \|\nu_{r/2}(s)\|_{-n}^2\,\mathrm{d}s$$
$$+\left(8\max\left\{\hat{D}_n(r),C_H\right\}^2(2+\epsilon)^2+1\right)\int_0^T\left(\|\nu_{r/2}(t)\|_{-n}^2+\|f_{r/2}(\cdot,t)\|_n^2\right)\mathrm{d}t,$$

which can be rewritten as (5.46), using $\epsilon<2$. □

We now introduce the following condition on ϵ:

Assumption 5.17. We assume either that

$$\epsilon \max\left\{16\hat{D}_n(r)^2\left(\tilde{D}_n(r)+\bar{P}\right)C_H^3, 4C_H\hat{D}_n(r)\left(C_H\left(P(0)+1\right)+\bar{P}\right)\right\}\le 1, \tag{5.54}$$

where $\bar{P}=\max\left\{\frac{\bar{\rho}-1}{\bar{\rho}-2},1\right\}$, or else $P(q)=1-q$ and $\epsilon<2$.

Corollary 5.18. *Let (w,μ) be a solution of (5.1), where either (5.3) or (5.2) holds. Define*

$$J_n(\rho):=\|w(\cdot,0)\|_n\|\mu_0\|_{-n}+\|\mu_0\|_{-n}^2+\int_0^T\left(\|f_\rho(\cdot,s)\|_n^2+\|\nu_\rho(s)\|_{-n}^2\right)\mathrm{d}s, \tag{5.55}$$

$$K_n(\rho):=\|\mu_0\|_{-n}+\sup_{0\le\tau\le T}\|\nu_\rho(\tau)\|_{-n}+\left\|\frac{\partial w_\rho}{\partial x}\right\|_0^{1/2}J_n(\rho)^{1/4}. \tag{5.56}$$

Let Assumptions 5.12 and 5.17 hold. Then there exists a constant C, depending on the data but not on T, such that the following three estimates hold:

$$\int_0^T E_{r/2}(s)\,\mathrm{d}s\le CJ_n(r/2), \tag{5.57}$$

$$\int_0^T\|\mu_{r/2}(t)\|_{-n}^2\,\mathrm{d}t\le CJ_n(r/2), \tag{5.58}$$

$$\int_0^T\left\|\frac{\partial w_{r/2}}{\partial x}(\cdot,t)\right\|_n^2\mathrm{d}t\le CJ_n(r/2), \tag{5.59}$$

$$\int_0^T\left|\mathcal{Q}_{r/2}(t)\right|^2\mathrm{d}t\le CJ_n(r/2), \tag{5.60}$$

$$\sup_{t\in[0,T]}\|\mu_n(t)\|_{-n}\le CK_n(r/2). \tag{5.61}$$

Proof. By Assumption 5.12, taking (5.12) into account, we can apply Lemmas 5.8 and 5.10 with $\rho=r/2$. Apply Lemmas 5.15 and 5.16, we deduce (5.57). Then Equations (5.58), (5.59), (5.60), and (5.61) follow from applying Lemma 5.8, Lemma 5.10, Corollary 5.9, and Lemma 5.11, respectively, using Equation (5.57). □

5.5. Hölder estimates

Recall that $Y_{1+\alpha} := \mathcal{C}^{1+\alpha}_{\diamond}(\mathcal{D})$ is the space of all $\phi \in \mathcal{C}^{1+\alpha}(\overline{\mathcal{D}})$ with the compatibility condition $\phi(0) = 0$. Set $\psi(x) = 1 - e^{-x}$. For $n \geq 2$ we will define $Y_{n+\alpha}$ to be the space of all $\phi \in \mathcal{C}^{1+\alpha}_{\diamond}(\mathcal{D})$ such that $\psi^{j-1}\phi \in \mathcal{C}^{j+\alpha}_{\diamond}(\mathcal{D})$ for $j = 2, \ldots, n$, with norm given by

$$\|\phi\|_{Y_{n+\alpha}} = \sum_{j=1}^{n} \left\|\psi^{j-1}\phi\right\|_{\mathcal{C}^{j+\alpha}}.$$

This defines a Banach space. The following two lemmas provide estimates on solutions to parabolic equations in the spaces $Y_{n+\alpha}$ for $n = 1, 2, 3$.

Lemma 5.19. *Let u be a the solution of*

$$\frac{\partial u}{\partial t} + \lambda u - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + V(x,t)\frac{\partial u}{\partial x} = F, \quad u(0,t) = 0, \quad u(x,0) = u_0(x) \tag{5.62}$$

where λ is any positive constant, F is a bounded continuous function, and $u_0 \in \mathcal{C}^{1+\alpha}_{\diamond}(\mathcal{D})$ (i.e. $u_0 \in \mathcal{C}^{1+\alpha}(\overline{\mathcal{D}})$ with $u_0(0) = 0$). Then

$$\|u\|_{\mathcal{C}^{\alpha,\alpha/2}\left(\overline{\mathcal{D}}\times[0,T]\right)} + \left\|\frac{\partial u}{\partial x}\right\|_{\mathcal{C}^{\alpha,\alpha/2}\left(\overline{\mathcal{D}}\times[0,T]\right)} \leq C\left(\|V\|_0, \alpha, \lambda\right)\left(\|F\|_0 + \|u_0\|_{\mathcal{C}^{1+\alpha}}\right),$$

where $C\left(\|V\|_0, \alpha, \lambda\right)$ is independent of T.

Proof. See [21, Lemma 2.7]. □

Lemma 5.20. *Let u be a solution of (5.62), in which $F, \frac{\partial F}{\partial x}, V, \frac{\partial V}{\partial x} \in \mathcal{C}^{\alpha,\alpha/2}(\overline{\mathcal{D}} \times [0,T])$. Assume also that $u_0 \in Y_{n+\alpha}$ for $n = 2$ or $n = 3$; that is, assume $\psi^{j-1}u_0^{(j)} \in \mathcal{C}^{1+\alpha}_{\diamond}(\mathcal{D})$ for $j = 1, \ldots, n$. Then*

$$\begin{aligned}&\|\psi u\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathcal{D}}\times[0,T])}\\ &\quad\leq C\left(\|V\|_{\mathcal{C}^{\alpha,\alpha/2}}, \lambda, \sigma, \alpha\right)\left(\|\psi u_0\|_{\mathcal{C}^{2+\alpha}} + \|u_0\|_{\mathcal{C}^{1+\alpha}} + \|\psi F\|_{\mathcal{C}^{\alpha,\alpha/2}} + \|F\|_0\right), \quad \textit{and}\end{aligned} \tag{5.63}$$

$$\begin{aligned}\left\|\psi^2\frac{\partial u}{\partial x}\right\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathcal{D}}\times[0,T])} &\leq C\left(\|V\|_{\mathcal{C}^{\alpha,\alpha/2}}, \left\|\frac{\partial V}{\partial x}\right\|_{\mathcal{C}^{\alpha,\alpha/2}}, \lambda, \sigma, \alpha\right)\\ &\times\left(\left\|\psi^2 u_0'\right\|_{\mathcal{C}^{2+\alpha}} + \|\psi u_0\|_{\mathcal{C}^{2+\alpha}} + \left\|\psi^2\frac{\partial F}{\partial x}\right\|_{\mathcal{C}^{\alpha,\alpha/2}} + \|\psi F\|_{\mathcal{C}^{\alpha,\alpha/2}} + \|F\|_0\right).\end{aligned} \tag{5.64}$$

Proof. Multiply (5.62) by ψ to see that $v(x,t) = \psi(x)u(x,t)$ is the solution to

$$\frac{\partial v}{\partial t} + \lambda v - \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2} + V\frac{\partial v}{\partial x} = \psi F - \sigma^2\psi'\frac{\partial u}{\partial x} - \frac{\sigma^2}{2}\psi'' u + V\psi' u,$$
$$v(0,t) = 0, \quad v(x,0) = \psi(x)u_0(x).$$

Note that the compatibility conditions of order 0 and 1 are satisfied. Indeed, the condition of order 0 is trivial: $\psi(0)u_0(0) = 0$. The condition of order 1 is

$$\lambda\psi(0)u_0(0) - \frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\psi u_0)(0) + V(0,0)\frac{\mathrm{d}}{\mathrm{d}x}(\psi u_0)(0)$$
$$= \psi(0)F(0,0) - \sigma^2\psi'(0)u_0'(0) - \frac{\sigma^2}{2}\psi''(0)u_0(0) + V(0,0)\psi'(0)u_0(0),$$

which can be verified by expanding the derivatives and using the fact that $\psi(0) = 0$. Now observe that

$$\left\|\sigma^2\psi'\frac{\partial u}{\partial x} - \frac{\sigma^2}{2}\psi'' u + V\psi' u\right\|_{\mathcal{C}^{\alpha,\alpha/2}} \leq C\left(\|V\|_{\mathcal{C}^{\alpha,\alpha/2}} + 1\right)\left(\|u\|_{\mathcal{C}^{\alpha,\alpha/2}} + \left\|\frac{\partial u}{\partial x}\right\|_{\mathcal{C}^{\alpha,\alpha/2}}\right)$$
$$\leq C\left(\|V\|_{\mathcal{C}^{\alpha,\alpha/2}} + 1\right)\left(\|F\|_0 + \|u_0\|_{\mathcal{C}^{1+\alpha}}\right),$$

where C depends on $\|V\|_0$, α, and λ as in Lemma 5.19. Here we have used the fact that $\left\|\psi^{(n)}\right\|_0 = 1$ for all n. From Lemma 4.12 we have

$$\|v\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathcal{D}}\times[0,T])}$$
$$\leq C\left(\|V\|_{\mathcal{C}^{\alpha,\alpha/2}}, \lambda, \sigma, \alpha\right)\left(\left\|\psi u_0\right\|_{\mathcal{C}^{2+\alpha}} + \|u_0\|_{\mathcal{C}^{1+\alpha}} + \left\|\psi F\right\|_{\mathcal{C}^{\alpha,\alpha/2}} + \|F\|_0 + \left\|\frac{\partial v}{\partial x}\right\|_{\mathcal{C}^{\alpha,\alpha/2}}\right),$$

and Equation (5.63) follows from interpolation.

To derive Equation (5.64), take the derivative with respect to x of (5.62) and multiply by ψ^2. Rearrange to see that $w(x,t) = \psi(x)^2\frac{\partial u}{\partial x}(x,t)$ is the (weak) solution to

$$\frac{\partial w}{\partial t} + \lambda w - \frac{\sigma^2}{2}\frac{\partial^2 w}{\partial x^2} = \psi^2\frac{\partial F}{\partial x} - \left(\psi^2\frac{\partial V}{\partial x} + \sigma^2(\psi')^2 + \sigma^2\psi\psi''\right)\frac{\partial u}{\partial x}$$
$$- \left(\psi^2 V + 2\sigma^2\psi\psi'\right)\frac{\partial^2 u}{\partial x^2},$$
$$w(0,t) = 0, \quad w(x,0) = \psi(x)^2 u_0'(x).$$

Notice that, thanks to the fact that $\psi(0) = 0$, the compatibility conditions of order 0 and 1 are satisfied, by the same reasoning as above. We also have, using Lemma 5.19,

$$\begin{aligned}
&\left\|\psi^2 \frac{\partial F}{\partial x} - \left(\psi^2 \frac{\partial V}{\partial x} + \sigma^2(\psi')^2 + \sigma^2\psi\psi''\right)\frac{\partial u}{\partial x} - \left(\psi^2 V + 2\sigma^2\psi\psi'\right)\frac{\partial^2 u}{\partial x^2}\right\|_{\mathcal{C}^{\alpha,\alpha/2}} \\
&\le \left\|\psi^2 \frac{\partial F}{\partial x}\right\|_{\mathcal{C}^{\alpha,\alpha/2}} + C\left\|\frac{\partial u}{\partial x}\right\|_{\mathcal{C}^{\alpha,\alpha/2}} + C\left\|\psi \frac{\partial^2 u}{\partial x^2}\right\|_{\mathcal{C}^{\alpha,\alpha/2}} \\
&\le \left\|\psi^2 \frac{\partial F}{\partial x}\right\|_{\mathcal{C}^{\alpha,\alpha/2}} + C\left\|\frac{\partial u}{\partial x}\right\|_{\mathcal{C}^{\alpha,\alpha/2}} + C\|u\|_{\mathcal{C}^{\alpha,\alpha/2}} + C\left\|\psi u\right\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}} \\
&\le \left\|\psi^2 \frac{\partial F}{\partial x}\right\|_{\mathcal{C}^{\alpha,\alpha/2}} + C\|F\|_0 + \left\|\psi u\right\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}},
\end{aligned}$$

where C depends on $\|V\|_{\mathcal{C}^{\alpha,\alpha/2}}$ and $\left\|\frac{\partial V}{\partial x}\right\|_{\mathcal{C}^{\alpha,\alpha/2}}$. By Lemma 4.12 and Equation (5.63), we deduce (5.64). □

Lemmas 5.19 and 5.20 have the following consequence in the case $F = 0$:

Corollary 5.21. *Let u be the solution of (5.62) where λ is any positive constant and where $F = 0$. Then*

$$\sup_{t\ge 0}\left\|u(\cdot,t)\right\|_{Y_{n+\alpha}} + \sup_{t_1\neq t_2}\frac{\left\|u(\cdot,t_1) - u(\cdot,t_2)\right\|_{Y_{n+\alpha}}}{|t_1-t_2|^{\alpha/2}} \le C\|u_0\|_{Y_{n+\alpha}},$$

where C depends on α, λ, σ, and on either $\|V\|_0$ (if $n = 1$), $\|V\|_{\mathcal{C}^{\alpha,\alpha/2}}$ (if $n = 2$), or $\|V\|_{\mathcal{C}^{\alpha,\alpha/2}} + \left\|\frac{\partial V}{\partial x}\right\|_{\mathcal{C}^{\alpha,\alpha/2}}$ (if $n = 3$).

Next we wish to establish estimates on the Fokker-Planck equation in the spaces $Y^*_{n+\alpha}$, denoting the dual of $Y_{n+\alpha}$, with regularity in time as well. Note that $\|\cdot\|_n \le \|\cdot\|_{Y_{n+\alpha}}$ and thus $\|\cdot\|_{Y^*_{n+\alpha}} \le \|\cdot\|_{-n}$.

Lemma 5.22. *Let (w,μ) be a solution of (5.1). Suppose Assumption 5.12 holds. Then*

$$\left\|\mu_{r/2}\right\|_{\mathcal{C}^{\alpha/2}([0,T];Y^*_{n+\alpha})} \le C(\alpha,r,\sigma)J_n(r/2)^{1/2}, \quad n = 1,2, \tag{5.65}$$

where J_n is defined in (5.55).

Proof. Step 1: Let $\lambda > 0$ be such that $\lambda < r/2$. Fix $t_1 > 0$, let $\phi_{t_1} \in Y_{n+\alpha}$ with $\left\|\phi_{t_1}\right\|_{Y_{n+\alpha}} \le 1$, and for any $\lambda > 0$ let $\phi^{(\lambda)}$ denote the solution of

$$-\frac{\partial\phi}{\partial t} + \lambda\phi - \frac{\sigma^2}{2}\frac{\partial^2\phi}{\partial x^2} - V_3(x,t)\frac{\partial\phi}{\partial x} = 0, \quad \phi(0,t) = 0, \quad \phi(x,t_1) = \phi_{t_1}(x).$$

Note that we have the relation

$$\phi^{(\lambda_1+\lambda_2)}(x,t) = e^{\lambda_2(t-t_1)}\phi^{(\lambda_1)}(x,t).$$

Now $\|q^*\|_{\mathcal{C}^{\alpha,\alpha/2}}$ and $\left\|\frac{\partial q^*}{\partial x}\right\|_{\mathcal{C}^{\alpha,\alpha/2}}$ can be estimated using the norm $\|u\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}}$, which in turn is estimated by the a priori estimates in Theorem 4.16. By Corollary 5.21 we therefore have

$$\sup_{t\in[0,t_1]}\left\|\phi^{(\lambda)}(t)\right\|_{Y_{n+\alpha}}+\sup_{t_0\in[0,t_1)}\frac{\left\|\phi^{(\lambda)}(t_1)-\phi^{(\lambda)}(t_0)\right\|_{Y_{n+\alpha}}}{(t_1-t_0)^{\alpha/2}}\le C(\alpha,\lambda,\sigma). \tag{5.66}$$

For any $t_0\in[0,t_1]$ we have, using integration by parts,

$$\begin{aligned}\int_{\mathcal{D}}\phi_{t_1}^{(r/2)}(x)\mu_{r/2}(x,t_1)\,\mathrm{d}x&=\int_{\mathcal{D}}\phi^{(r/2)}(x,t_0)\mu_{r/2}(x,t_0)\,\mathrm{d}x\\&-\int_{t_0}^{t_1}\int_{\mathcal{D}}\frac{\partial\phi^{(r/2)}}{\partial x}\left(\left(V_4(x,t)\frac{\partial w_{r/2}}{\partial x}(x,t)+V_5(x,t)\mathcal{Q}_{r/2}(t)\right)m+\nu_{r/2}\right)\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{5.67}$$

Applying (5.66) and Corollary 5.18, using the identity $\phi^{(r/2)}=e^{(\lambda-r/2)(t_1-t)}\phi^{(\lambda)}$, we have

$$\begin{aligned}&\left|\int_{t_0}^{t_1}\int_{\mathcal{D}}\frac{\partial\phi^{(r/2)}}{\partial x}\left(\left(V_4(x,t)\frac{\partial w_{r/2}}{\partial x}(x,t)+V_5(x,t)\mathcal{Q}_{r/2}(t)\right)m+\nu_{r/2}\right)\mathrm{d}x\,\mathrm{d}t\right|\\&\le C(\alpha,\lambda,\sigma)\int_{t_0}^{t_1}e^{(\lambda-r/2)(t_1-t)}\left(E_{r/2}(t)^{1/2}+\left\|\mu_{r/2}(t)\right\|_{-n}+\left\|\nu_{r/2}(t)\right\|_{-n}\right)\mathrm{d}t\\&\le C(\alpha,\lambda,\sigma)\left(\int_0^{t_1-t_0}e^{-2(r/2-\lambda)t}\,\mathrm{d}t\right)^{1/2}\left(\int_{t_0}^{t_1}\left(E_{r/2}(t)+\left\|\mu_{r/2}(t)\right\|_{-n}^2+\left\|\nu_{r/2}(t)\right\|_{-n}^2\right)\mathrm{d}t\right)^{1/2}\\&\le C(\alpha,\lambda,\sigma)\min\left\{(r/2-\lambda)^{-1/2},(t_1-t_0)^{1/2}\right\}J_n(r/2)^{1/2}.\end{aligned} \tag{5.68}$$

Using (5.68) in (5.67) with $t_0=0$ and taking the supremum over all ϕ_{t_1} we deduce the bound

$$\left\|\mu_{r/2}(t_1)\right\|_{Y^*_{n+\alpha}}\le C(\alpha,\lambda,\sigma)(r/2-\lambda)^{-1/2}\left(\left\|\mu_0\right\|_{Y^*_{n+\alpha}}+J_n(r/2)^{1/2}\right)\quad\forall t_1\ge 0. \tag{5.69}$$

On the other hand, subtracting $\int\phi_{t_1}\mu_{r/2}(t_1)$ from both sides of (5.67), we have

$$\begin{aligned}\int_{\mathcal{D}}\phi_{t_1}(x)\left(\mu_{r/2}(x,t_1)-\mu_{r/2}(x,t_0)\right)\mathrm{d}x&=\int_{\mathcal{D}}\left(\phi^{(r/2)}(x,t_0)-\phi^{(r/2)}(x,t_1)\right)\mu_{r/2}(x,t_0)\,\mathrm{d}x\\&-\int_{t_0}^{t_1}\int_{\mathcal{D}}\frac{\partial\phi^{(r/2)}}{\partial x}\left(\left(V_4(x,t)\frac{\partial w}{\partial x}(x,t)+V_5(x,t)\mathcal{Q}(t)\right)m+\nu\right)\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{5.70}$$

Combining (5.66), (5.68), and (5.69) to estimate the right-hand side of (5.70), we deduce that

$$\left\|\mu_{r/2}(t_1)-\mu_{r/2}(t_0)\right\|_{Y^*_{n+\alpha}} \le C(\alpha,\lambda,\sigma)\left(\left\|\mu_0\right\|_{Y^*_{n+\alpha}}+J_n(r/2)^{1/2}\right)(t_1-t_0)^{\alpha/2}. \tag{5.71}$$

It suffices to take $\lambda = r/4$. Then recalling that $\left\|\mu_0\right\|_{Y^*_{n+\alpha}} \le \left\|\mu_0\right\|_{-n} \le J_n(r/2)^{1/2}$, we see that (5.69) and (5.71) imply (5.65). □

5.6. Hölder regularity of the mass function

Let (w,μ) solve (5.1). Our goal is to prove the Hölder regularity of the following functional:

$$\eta_\rho(t) = e^{-\rho t}\left\langle 1, \mu_\rho(t)\right\rangle = e^{-\rho t}\int_{\mathcal{D}} \mu(x,t)\,\mathrm{d}x.$$

This will allow us to estimate $\mathcal{Q}_\rho$ in a Hölder space.

We introduce the space $\mathscr{M}_\alpha^{-n}$, in analogy to the space $\mathscr{M}_\alpha$ defined in Section 2. For any $\mu \in X_n^*$ define the *mass function*

$$\eta^h[\mu](t) := \left\langle \mu, 1\right\rangle - \int_{\mathcal{D}} \left\langle \left(S(x-\cdot,t)-S(x+\cdot,t)\right), \mu\right\rangle \mathrm{d}x, \tag{5.72}$$

cf. (3.8). By Proposition 5.1, we deduce that

$$\left\| \int_{\mathcal{D}} \left(S(x-\cdot,t)-S(x+\cdot,t)\right)\mathrm{d}x \right\|_n \le 2M_n\|1\|_n = 2M_n,$$

and thus we can write (5.72) as

$$\eta^h[\mu](t) := \left\langle \mu, 1\right\rangle - \left\langle \int_{\mathcal{D}} \left(S(x-\cdot,t)-S(x+\cdot,t)\right)\mathrm{d}x, \mu \right\rangle,$$

from which we also deduce

$$\left|\eta^h[\mu](t)\right| \le C_n\left\|\mu\right\|_{-n} \quad \forall t \ge 0.$$

Now we define $\mathscr{M}_\alpha^{-n}$ to be the set of all $\mu \in X_n^*$ such that $\eta^h[\mu]$ is α-Hölder continuous. It is a Banach space endowed with the norm

$$\left\|\mu\right\|_{\mathscr{M}_\alpha^{-n}} = \left\|\mu\right\|_{-n} + \left\|\eta^h[\mu]\right\|_{C^\alpha\left([0,\infty)\right)}.$$

Lemma 5.23. *Let (w,μ) solve (5.1), and suppose Assumption 5.12 holds. Assume that $\alpha \leq 2/5$. There exists a constant C depending only on the data but not on T such that*

$$\left\|\eta_{r/2}\right\|_{\mathcal{C}^{\alpha/2}([0,T])} \leq C\left(\left\|\mu_0\right\|_{\mathscr{M}^{-n}_{\alpha/2}} + \tilde{K}_n(r/2)\right), \tag{5.73}$$

where

$$\tilde{K}_n(r/2) = K_n(r/2) + \left\|\frac{\partial w_{r/2}}{\partial x}\right\|_0^{2/3} J_n(r/2)^{1/6}, \tag{5.74}$$

and where $J_n(r/2)$ and $K_n(r/2)$ are defined in (5.55) and (5.56), respectively.

Proof. Observe that $\|1\|_n = 1$ for all n and $\left\|\xi\right\|_n \leq 2$ for $n = 0, 1, 2$, so we have the bounds

$$\left|\eta_{r/2}(t)\right| \leq \left\|\mu_{r/2}(t)\right\|_{-n}, \quad \left|\zeta_{r/2}(t)\right| \leq \left\|\mu_{r/2}(t)\right\|_{-n}.$$

It remains to prove estimates on the Hölder seminorms.

Step 1: By Duhamel's Principle, we can write

$$\mu(x,t) = I_1(x,t) + I_2(x,t) + I_3(x,t) + I_4(x,t)$$

where

$$I_1(x,t) = -\int_0^t \int_0^\infty \big(S(x-y,t-s) - S(x+y,t-s)\big)(V_3\mu)_y(y,s)\,\mathrm{d}y\,\mathrm{d}s,$$

$$I_2(x,t) = -\int_0^t \int_0^\infty \big(S(x-y,t-s) - S(x+y,t-s)\big)(bm)_y(y,s)\,\mathrm{d}y\,\mathrm{d}s,$$

$$I_3(x,t) = -\int_0^t \int_0^\infty \big(S(x-y,t-s) - S(x+y,t-s)\big)\nu_y(y,s)\,\mathrm{d}y\,\mathrm{d}s,$$

$$I_4(x,t) = \int_0^\infty \big(S(x-y,t) - S(x+y,t)\big)\mu_0(y)\,\mathrm{d}y,$$

$$b(x,t) = V_4(x,t)\frac{\partial w}{\partial x}(x,t) + V_5(x,t)\mathcal{Q}(t).$$

Using integration by parts, we deduce

$$\eta_{r/2}(t) = \sum_{j=1}^4 \eta^j(t)$$

where

$$\eta^1(t) = 2\int_0^t \int_0^\infty e^{-r/2(t-s)} S(x,t-s) V_3(x,s)\mu_{r/2}(x,s)\,\mathrm{d}x\,\mathrm{d}s,$$

$$\eta^2(t) = 2\int_0^t \int_0^\infty e^{-r/2(t-s)} S(x,t-s) b_{r/2}(x,s) m(x,s)\,\mathrm{d}x\,\mathrm{d}s,$$

$$\eta^3(t) = 2\int_0^t \int_0^\infty S(x,t-s) e^{-r/2s} \nu_{r/2}(x,s)\,\mathrm{d}x\,\mathrm{d}s,$$

$$\eta^4(t) = \frac{2}{\sqrt{\pi}} e^{-r/2t} \int_0^\infty \int_{(2\sigma^2 t)^{1/2}x}^\infty e^{-x^2}\mu_0(y)\,\mathrm{d}y\,\mathrm{d}x$$

where we follow the usual convention defining $b_{r/2}(x,t) = e^{-r/2t}b(x,t)$. We use much the same arguments as in Lemma 3.7 to establish Hölder estimates.

Step 2: For the first term, we write

$$\eta^1(t_1) - \eta^1(t_0) = 2\int_{t_0}^{t_1} \int_0^\infty e^{-r/2(t-s)} S(x,t_1-s) V_3(x,s)\mu_{r/2}(x,s)\,\mathrm{d}x\,\mathrm{d}s$$
$$+ 2\int_0^{t_0}\int_0^\infty\int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-r/2(t-s)} S(x,t-s)\right] V_3(x,s)\mu_{r/2}(x,s)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s.$$

Use Corollary 5.18 and Assumption 5.5 to get

$$\left|\eta^1(t_1) - \eta^1(t_0)\right| \le C(n,r)K_n(r/2)\int_{t_0}^{t_1} e^{-r/2(t-s)}\left\|S(\cdot,t_1-s)\right\|_n \mathrm{d}s$$
$$+ C(n,r)K_n(r/2)\int_0^{t_0}\int_{t_0}^{t_1}\left\|\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-r/2(t-s)} S(x,t-s)\right]\right\|_n \mathrm{d}t\,\mathrm{d}s.$$

Use Lemma 3.6 to get

$$\int_{t_0}^{t_1} e^{-r/2(t-s)}\left\|S(\cdot,t_1-s)\right\|_n \mathrm{d}s \le C(n)\int_{t_0}^{t_1}(t_1-s)^{-1/2}\,\mathrm{d}s = C(n)(t_1-t_0)^{1/2}.$$

On the other hand, from Lemma 3.6 we also have

$$\sup_{x,t}\left|x^n t^{3/2}\frac{\partial^{n+2}S}{\partial x^{n+2}}(x,t)\right|<\infty \quad \Rightarrow \left\|\frac{\partial^2 S}{\partial x^2}(x,t)\right\|_n \le C(n)t^{-3/2}$$

for any n. We use this to deduce

$$\int_0^{t_0}\int_{t_0}^{t_1}\left\|\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-r/2(t-s)}S(x,t-s)\right]\right\|_n \mathrm{d}t\,\mathrm{d}s$$

$$= \int_0^{t_0}\int_{t_0}^{t_1} e^{-r/2(t-s)}\left\|\frac{\sigma^2}{2}\frac{\partial^2 S}{\partial x^2}(x,t-s) - r/2S(x,t-s)\right\|_n \mathrm{d}t\,\mathrm{d}s$$

$$\le C(n,\sigma)\int_0^{t_0}\int_{t_0}^{t_1}(t-s)^{-3/2}\,\mathrm{d}t\,\mathrm{d}s + C(n)r/2\int_0^{t_0}\int_{t_0}^{t_1}(t-s)^{-1/2}\,\mathrm{d}t\,\mathrm{d}s \le C(n,\sigma,r)(t_1-t_0)^{1/2} \tag{5.75}$$

so long as $t_1 - t_0 \le 1$. These estimates combine to give

$$\left|\eta^1(t_1)-\eta^1(t_0)\right| \le C(r,\sigma,n)K_n(r/2)(t_1-t_0)^{1/2} \quad \forall 0\le t_0\le t_1\le t_0+1. \tag{5.76}$$

By the very same argument, we also have

$$\left|\eta^3(t_1)-\eta^3(t_0)\right| \le C(r,\sigma,n)K_n(r/2)(t_1-t_0)^{1/2} \quad \forall 0\le t_0\le t_1\le t_0+1. \tag{5.77}$$

Step 3: Next we write

$$\eta^2(t_1)-\eta^2(t_0) = -2\int_{t_0}^{t_1}\int_0^{\infty} e^{-r/2(t-s)}S(x,t_1-s)b_{r/2}(x,s)m(x,s)\,\mathrm{d}x\,\mathrm{d}s$$

$$-2\int_0^{t_0}\int_0^{\infty}\int_{t_0}^{t_1}\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-r/2(t-s)}S(x,t-s)\right]b_{r/2}(x,s)m(x,s)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s.$$

Recall that $b = V_4\frac{\partial w}{\partial x} + V_5\mathcal{Q}$, and recall also the formula (5.1)(iii) for $\mathcal{Q}$. Applying Lemma 5.11, we have

$$\int_0^{\infty}\left|b_{r/2}(x,s)\right|m(x,s)\,\mathrm{d}x \le C(n)K_n(r/2) + C\int_0^{\infty}\left|\frac{\partial w_{r/2}}{\partial x}(x,s)\right|m(x,s)\,\mathrm{d}x$$

$$\le C(n)K_n(r/2) + C\left\|\frac{\partial w_{r/2}}{\partial x}\right\|_0^{2/3} E_{r/2}(s)^{1/6}.$$

Using the same reasoning as in the previous step, we deduce

$$\left|\eta^2(t_1)-\eta^2(t_0)\right| \le C(r,\sigma,n)K_n(r/2)(t_1-t_0)^{1/2}+C(n)\left\|\frac{\partial w_{r/2}}{\partial x}\right\|_0^{2/3}\int_{t_0}^{t_1}(t-s)^{-1/2}E_{r/2}(s)^{1/6}\,\mathrm{d}s$$
$$+C(n,\sigma,r)\left\|\frac{\partial w_{r/2}}{\partial x}\right\|_0^{2/3}\int_0^{t_0}\int_{t_0}^{t_1}\left((t-s)^{-3/2}+(t-s)^{-1/2}\right)E_{r/2}(s)^{1/6}\,\mathrm{d}s,$$

for $0\le t_0\le t_1\le t_0+1$. (Cf. Equation (5.75).) By Hölder's inequality, we compute

$$\int_{t_0}^{t_1}(t-s)^{-1/2}E_{r/2}(s)^{1/6}\,\mathrm{d}s \le C(t_1-t_0)^{1/3}\left(\int_{t_0}^{t_1}E_{r/2}(s)\,\mathrm{d}s\right)^{1/6},$$
$$\int_0^{t_0}\int_{t_0}^{t_1}\left((t-s)^{-3/2}+(t-s)^{-1/2}\right)E_{r/2}(s)^{1/6}\,\mathrm{d}s \le C(t_1-t_0)^{1/5}\left(\int_{t_0}^{t_1}E_{r/2}(s)\,\mathrm{d}s\right)^{1/6}.$$

Combining this with Corollary 5.18, we have

$$\left|\eta^2(t_1)-\eta^2(t_0)\right| \le C(r,\sigma,n)\tilde{K}_n(r/2)(t_1-t_0)^{1/5}, \tag{5.78}$$

where $\tilde{K}_n(r/2)$ is defined in (5.74).

Step 4: For the last term $\eta^4(t)$, we use the definition of $\mathscr{M}_\alpha^{-n}$ and the mass function (5.72) to see that

$$\eta^4(t)=e^{-r/2t}\eta^h[\mu_0](t), \tag{5.79}$$

and so, because $t\mapsto e^{-r/2t}$ is globally Lipschitz with constant $r/2$ on the interval $[0,\infty)$, we deduce

$$\left\|\eta^4\right\|_{\mathcal{C}^{\alpha/2}([0,T])} \le \max\{1,r/2\}\left\|\mu_0\right\|_{\mathscr{M}_{\alpha/2}^{-n}}.$$

Putting together (5.76), (5.78), (5.77), and (5.79), we deduce (5.73). □

Corollary 5.24. *Let (w,μ) be a solution of (5.1) and suppose Assumption 5.12 holds. Assume $\alpha\le 2/5$. Then there exists a constant C, depending only on the data but not on T, such that, for $n=1,2$,*

$$\left\|\mathcal{Q}_{r/2}\right\|_{\mathcal{C}^{\alpha/2}([0,T])} \le C\left(\tilde{J}_n(r/2)+\left\|\frac{\partial w_{r/2}}{\partial x}\right\|_{\mathcal{C}^{\alpha,\alpha/2}}\right), \tag{5.80}$$

where

$$\tilde{J}_n(r/2):=\left\|\langle \nu_{r/2},1\rangle\right\|_{\mathcal{C}^{\alpha/2}([0,T])}+\left\|\mu_0\right\|_{\mathscr{M}_{\alpha/2}^{-n}}+J_n(r/2)^{1/2}+\tilde{K}_n(r/2), \tag{5.81}$$

and J_n and $\tilde{K}_n$ are defined in (5.55) and (5.74), respectively.

Proof. Multiplying Equation (5.1)(iii) by $e^{-rt/2}$, we have

$$\mathcal{Q}_{r/2}(t) = g_1(t)\left(g_2(t) + g_3(t) + g_4(t)\right)$$

where

$$\begin{aligned} g_1(t) &:= \left(1 + \frac{\epsilon}{2}\int_{\mathcal{D}} \mathrm{d}m(t)\right)^{-1}, \\ g_2(t) &:= -\int_{\mathcal{D}} \mathrm{d}\nu_{r/2}(t), \\ g_3(t) &:= \int_{\mathcal{D}} q^*(\cdot,t)\,\mathrm{d}\mu_{r/2}(t), \\ g_4(t) &:= -\frac{1}{2}\int_{\mathcal{D}} \frac{\partial w_{r/2}}{\partial x}(\cdot,t)\,\mathrm{d}m(t). \end{aligned}$$

Using the fact that $m(t)$ is a positive measure-valued process together with the Hölder regularity deduced from Lemma 3.7, we have

$$\|g_1\|_{\mathcal{C}^{\alpha/2}([0,T])} \le C. \tag{5.82}$$

On the other hand,

$$\|g_2\|_{\mathcal{C}^{\alpha/2}([0,T])} = \left\|\langle \nu_{r/2}, 1\rangle\right\|_{\mathcal{C}^{\alpha/2}([0,T])}, \tag{5.83}$$

which is taken as given. Next, we analyze g_3. Set

$$\tilde{q}(x,t) = q^*(x,t) - q^*(0,t),$$

so that

$$g_3(t) = \int_{\mathcal{D}} \tilde{q}(\cdot,t)\,\mathrm{d}\mu_{r/2}(t) + q^*(0,t)\eta_{r/2}(t) =: g_{3,1}(t) + g_{3,2}(t). \tag{5.84}$$

Observe that, since $\tilde{q}(0,t) = 0$ by construction, we have $\tilde{q} \in \mathcal{C}^{\alpha/2}([0,T]; Y_{n+\alpha})$, where by computing the derivatives of q^* we deduce

$$\|\tilde{q}\|_{\mathcal{C}^{\alpha/2}([0,T];Y_{n+\alpha})} \le \|u\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}} + \left\|\psi\,\frac{\partial u}{\partial x}\right\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}} \le C, \quad n = 1, 2.$$

Therefore, using Lemma 5.22, we get

$$\left\| g_{3,1} \right\|_{\mathcal{C}^{\alpha/2}([0,T])} \leq C \left\| \mu_{r/2} \right\|_{\mathcal{C}^{\alpha/2}([0,T];Y^*_{n+\alpha})} \leq C J_n(r/2), \quad n = 1, 2. \tag{5.85}$$

On the other hand,

$$\left\| q^*(0,\cdot) \right\|_{\mathcal{C}^{\alpha/2}([0,T])} \leq C$$

by the Hölder regularity of $\frac{\partial u}{\partial x}$. By Lemma 5.23, we deduce

$$\left\| g_{3,2} \right\|_{\mathcal{C}^{\alpha/2}([0,T])} \leq C \left(\left\| \mu_0 \right\|_{\mathscr{M}^{-n}_{\alpha/2}} + \tilde{K}_n(r/2) \right). \tag{5.86}$$

Finally, we analyze g_4 in a similar way. Write

$$g_4(t) = -\frac{1}{2} \int_{\mathcal{D}} \left(\frac{\partial w_{r/2}}{\partial x}(\cdot,t) - \frac{\partial w_{r/2}}{\partial x}(0,t) \right) \mathrm{d}m(t) - \frac{1}{2} \frac{\partial w_{r/2}}{\partial x}(0,t) \int_{\mathcal{D}} \mathrm{d}m(t).$$

Using Lemmas 3.1 and 3.7 applies to the solution m of System (4.21), we deduce

$$\begin{aligned} \left\| g_4 \right\|_{\mathcal{C}^{\alpha/2}([0,T])} &\leq \left\| \frac{\partial w_{r/2}}{\partial x} \right\|_{\mathcal{C}^{\alpha,\alpha/2}} \left(\| m \|_{\mathcal{C}^{\alpha/2}([0,T];\mathcal{C}^{\alpha}_{\diamond}(\mathcal{D})^*)} + \left\| \int_{\mathcal{D}} \mathrm{d}m(\cdot) \right\|_{\mathcal{C}^{\alpha/2}([0,T])} \right) \\ &\leq C \left\| \frac{\partial w_{r/2}}{\partial x} \right\|_{\mathcal{C}^{\alpha,\alpha/2}}. \end{aligned} \tag{5.87}$$

Combining (5.82), (5.83), (5.84), (5.85), (5.86), and (5.87), we obtain (5.80). □

5.7. Full regularity of w

Multiply Equation (5.1)(i) by $e^{-\rho t}$ to see that w_ρ satisfies

$$\frac{\partial w_\rho}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 w_\rho}{\partial x^2} + V_1(x,t) \frac{\partial w_\rho}{\partial x} + V_2(x,t) \mathcal{Q}_\rho(t) - (r - \rho) w_\rho = f_\rho. \tag{5.88}$$

In this section we will derive an estimate on $w_{r/2}$ in classical Hölder spaces. In particular, let us define $Z_\alpha(T)$ to be the set of all $w \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathcal{D}} \times [0,T])$ such that $\psi \dfrac{\partial w}{\partial x} \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathcal{D}} \times [0,T])$ as well. (As usual, when $T = \infty$ we replace $[0,T]$ with $\big[0,\infty)$.) It is a Banach space with norm

$$\| w \|_{Z_\alpha} = \| w \|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathcal{D}} \times [0,T])} + \left\| \psi \frac{\partial w}{\partial x} \right\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathcal{D}} \times [0,T])}.$$

Theorem 5.25. *Let (w,μ) be a solution of (5.1), with $V_1,\ldots,V_5$ satisfying either (5.3) or (5.2). Then there is a constant $C(r,\sigma,\alpha)$, not depending on T, such that*

$$\|w\|_{Z_\alpha} \le C(r,\sigma,\alpha)\left(\|\mu_0\|_{\mathscr{M}^{-2}_{\alpha/2}} + N(f) + N(\nu)\right), \tag{5.89}$$

where

$$N(f) := \|f_{r/2}\|_{\mathcal{C}^{\alpha,\alpha/2}} + \left\|\psi \frac{\partial f_{r/2}}{\partial x}\right\|_{\mathcal{C}^{\alpha,\alpha/2}} + \left(\int_0^T \|f_{r/2}(\cdot,s)\|_2^2 \,\mathrm{d}s\right)^{1/2} \tag{5.90}$$

and

$$N^*(\nu) := \left\|\langle \nu_{r/2}, 1\rangle\right\|_{\mathcal{C}^{\alpha/2}([0,T])} + \left(\int_0^T \|\nu_{r/2}(s)\|_{-2}^2 \,\mathrm{d}s\right)^{1/2} + \sup_{0\le\tau\le T} \|\nu_{r/2}(\tau)\|_{-2}. \tag{5.91}$$

Proof. Step 1: We will first apply the maximum principle to find a bound on $w_{r/2}$. Let

$$\tilde{w}(x,t) = w_r(x,t) - \int_t^T \left(\|V_2\|_0 \left|\mathcal{Q}_r(s)\right| + \|f_r(\cdot,s)\|_0\right) \mathrm{d}s$$

and differentiate to see that

$$-\frac{\partial \tilde{w}}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 \tilde{w}}{\partial x^2} + V_1(x,t)\frac{\partial \tilde{w}}{\partial x} \le 0.$$

By the maximum principle, using the fact that $\tilde{w}(0,t) \le 0$ for all t and $\tilde{w}(x,T) = 0$ for all x, we have

$$\tilde{w}(x,t) \le 0 \quad \Rightarrow \quad w_r(x,t) \le \int_t^T \left(\|V_2\|_0 \left|\mathcal{Q}_r(s)\right| + \|f_r(\cdot,s)\|_0\right) \mathrm{d}s.$$

Multiply by $e^{rt/2}$ and use the Cauchy-Schwartz inequality to get

$$\begin{aligned} w_{r/2}(x,t) &\le \int_t^T e^{\frac{r}{2}(t-s)} \left(\|V_2\|_0 \left|\mathcal{Q}_{r/2}(s)\right| + \|f_{r/2}(\cdot,s)\|_0\right) \mathrm{d}s \\ &\le r^{-1/2}\left(\int_t^T \left(\|V_2\|_0 \left|\mathcal{Q}_{r/2}(s)\right| + \|f_{r/2}(\cdot,s)\|_0\right)^2 \mathrm{d}s\right)^{1/2} \\ &\le C(r)\left(\int_0^T \left|\mathcal{Q}_{r/2}(s)\right|^2 \mathrm{d}s\right)^{1/2} + C(r)\left(\int_0^T \|f_{r/2}(\cdot,s)\|_0^2 \,\mathrm{d}s\right)^{1/2}. \end{aligned}$$

Applying Corollary 5.18 5.9, we see that

$$w_{r/2}(x,t) \le C J_2(r)^{1/2}.$$

By the same argument applied that $-w$, we deduce

$$\left\| w_{r/2} \right\|_0 \le C J_2(r)^{1/2}. \tag{5.92}$$

Step 2: If we apply Lemma 5.20 to (5.88) with $\rho = r/2$, we obtain an estimate

$$\left\| w_{r/2} \right\|_{Z_\alpha} \le C\left(\|V_1\|_{\mathcal{C}^{\alpha,\alpha/2}}, \left\| \frac{\partial V_1}{\partial x} \right\|_{\mathcal{C}^{\alpha,\alpha/2}}, r, \sigma, \alpha \right) \times \left(\left\| f_{r/2} - V_2 \mathcal{Q}_{r/2} \right\|_{\mathcal{C}^{\alpha,\alpha/2}} + \left\| \psi \left(\frac{\partial f_{r/2}}{\partial x} - \frac{\partial V_2}{\partial x} \mathcal{Q}_{r/2} \right) \right\|_{\mathcal{C}^{\alpha,\alpha/2}} \right). \tag{5.93}$$

The Hölder norms of V_1, V_2, $\frac{\partial V_1}{\partial x}$ and $\frac{\partial V_2}{\partial x}$ are already estimated by the estimates (4.18) from Theorem 4.16. Moreover, $f_{r/2}$ is given. Using Equation (5.80) from Corollary 5.24 in (5.93), we obtain

$$\left\| w_{r/2} \right\|_{Z_\alpha} \le C(r,\sigma,\alpha) \left(\left\| f_{r/2} \right\|_{\mathcal{C}^{\alpha,\alpha/2}} + \left\| \psi \frac{\partial f_{r/2}}{\partial x} \right\|_{\mathcal{C}^{\alpha,\alpha/2}} + \tilde{J}_2(r/2) + \left\| \frac{\partial w_{r/2}}{\partial x} \right\|_{\mathcal{C}^{\alpha,\alpha/2}} \right). \tag{5.94}$$

By using the interpolation inequality

$$\left\| \frac{\partial w_{r/2}}{\partial x} \right\|_{\mathcal{C}^{\alpha,\alpha/2}} \le \varepsilon \left\| w_{r/2} \right\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathcal{D}}\times[0,T])} + C(\varepsilon) \left\| w_{r/2} \right\|_0$$

and applying (5.92), using the fact that $J_2(r/2)^{1/2} \le \tilde{J}_2(r/2)$, estimate (5.94) yields

$$\left\| w_{r/2} \right\|_{Z_\alpha} \le C(r,\sigma,\alpha) \left(\left\| f_{r/2} \right\|_{\mathcal{C}^{\alpha,\alpha/2}} + \left\| \psi \frac{\partial f_{r/2}}{\partial x} \right\|_{\mathcal{C}^{\alpha,\alpha/2}} + \tilde{J}_2(r/2) \right). \tag{5.95}$$

We now return to the definition of $\tilde{J}_2$, Equation (5.81), which can be written

$$\tilde{J}_2(r/2) := \|\mu_0\|_{\mathscr{M}^{-2}_{\alpha/2}} + \left\| \langle \nu_{r/2}, 1 \rangle \right\|_{\mathcal{C}^{\alpha/2}([0,T])} + \sup_{0\le\tau\le T} \left\| \nu_{r/2}(\tau) \right\|_{-2} + J_2(r/2)^{1/2} + \left\| \frac{\partial w_{r/2}}{\partial x} \right\|_0^{1/2} J_2(r/2)^{1/4} + \left\| \frac{\partial w_{r/2}}{\partial x} \right\|_0^{2/3} J_2(r/2)^{1/6}.$$

Now since $\left\| \frac{\partial w_{r/2}}{\partial x} \right\|_0$ is dominated by $\left\| w_{r/2} \right\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathcal{D}}\times[0,T])}$, we apply Young's inequality to (5.81) to get

$$\tilde{J}_2(r/2) \le \|\mu_0\|_{\mathscr{M}^{-2}_{\alpha/2}} + \|\langle \nu, 1\rangle\|_{\mathcal{C}^{\alpha/2}([0,T])} + \sup_{0\le\tau\le T} \|\nu_\rho(\tau)\|_{-2}$$
$$+ C(\varepsilon) J_2(r/2)^{1/2} + \varepsilon \|w_{r/2}\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathcal{D}}\times[0,T])}.$$

Applying (5.81) to (5.95), we derive, using the definition of J_2 in (5.55),

$$\|w_{r/2}\|_{Z_\alpha} \le C(r,\sigma,\alpha)\left(\|\mu_0\|_{\mathscr{M}^{-2}_{\alpha/2}} + \|\mu_0\|_{-2} + \|w(\cdot,0)\|_2^{1/2}\|\mu_0\|_{-2}^{1/2} + N^*(\nu) + N(f)\right), \tag{5.96}$$

where $N(f)$ and $N^*(\nu)$ are defined in (5.90) and (5.91), respectively. Using the fact that $\|w(\cdot,0)\|_2$ is dominated by $\|w\|_{Z_\alpha}$, we apply Young's inequality to (5.96) and rearrange to deduce (5.89). □

5.8. An existence theorem for the linearized system

Before formulating the main result of this section, let us collect assumptions on r and ϵ so that all of the a priori estimates of this section hold. We will formulate two alternatives, one for a linear demand schedule, and one for a more general case where ϵ must be small.

Assumption 5.26 *(r big, ϵ small).* Let r^* be a number large enough to satisfy Assumption 5.12, Equation (5.9) for $n=2$, and

$$2\sqrt{\frac{2}{\sigma^2 r}}H(0,0,0) < P(0) \quad \forall r \ge r^*.$$

Let $\epsilon^* > 0$ be small enough to satisfy (5.54) and

$$M = 2\sqrt{\frac{2}{\sigma^2 r^*}}H(0,0,0) < P\left(\epsilon^*\bar{Q}\right).$$

We assume that $r \ge r^*$ and $0 < \epsilon \le \epsilon^*$.

We remark that Assumption 5.26 implies Assumption 4.18; see Remark 4.19.

An alternative assumption is as follows.

Assumption 5.27 *(r big, P linear).* We assume that $P(q) = 1 - q$ and that $0 < \epsilon < 2$. Let r^* be a number large enough to satisfy Assumption 5.12, Equation (5.9) for $n = 2$, and

$$2\sqrt{\frac{2}{\sigma^2 r}}H(0,0,0) < 1 - \frac{\epsilon}{2} \quad \forall r \ge r^*.$$

We assume that $r \ge r^*$.

Theorem 5.28. *Let $T > 0$ be a fixed time horizon. Suppose that Assumption 5.26 or 5.27 holds. Then System (5.1) has a unique solution (w,μ) satisfying $w(x,T) = 0$ with regularity*

- $w_{r/2} \in Z_\alpha(T)$,
- $\mu_{r/2} \in \mathcal{C}^{\alpha/2}\left([0,T]; Y^*_{n+\alpha}\right) \cap L^\infty\left((0,T); X^*_2\right) =: \tilde{Z}_\alpha(T)$.

There exists a constant $C(r,\sigma,\alpha)$, not depending on T, such that

$$\left\|w_{r/2}\right\|_{Z_\alpha} + \left(\int_0^T \left\|\frac{\partial w_{r/2}}{\partial x}(\cdot,t)\right\|_2^2 \mathrm{d}t\right)^{1/2} + \left\|\mu_{r/2}\right\|_{\mathcal{C}^{\alpha/2}\left([0,T];Y^*_{2+\alpha}\right)} + N^*(\mu) \leq C(r,\sigma,\alpha)\left(\left\|\mu_0\right\|_{\mathscr{M}^{-2}_{\alpha/2}} + N(f) + N^*(\nu)\right), \quad (5.97)$$

where $N(f)$ and $N^*(\nu)$ are defined in (5.90) and (5.91), respectively.

Proof. First we assume the data are smooth. Then existence of solutions follows from the Leray-Schauder fixed point theorem, along the same lines as in the proof of Theorem 4.15. The a priori estimates (5.97) follow from Lemmas 5.25 and 5.22 (Equations (5.89) and (5.65)). A similar argument is also found in [4, Lemma 3.3.1]. To see that the solution is unique, note that the system is linear, so the a priori bounds also imply uniqueness. □

Theorem 5.29. *Suppose that Assumption 5.26 or 5.27 holds. Then System (5.1) has a unique solution (w,μ) satisfying*

- $w_{r/2} \in Z_\alpha(\infty)$,
- $\mu_{r/2} \in \mathcal{C}^{\alpha/2}\left([0,\infty); Y^*_{2+\alpha}\right) \cap L^\infty\left((0,\infty); X^*_2\right) =: \tilde{Z}_\alpha(\infty)$,
- $\lim_{t\to\infty}\left\|w_{r/2}(\cdot,t)\right\|_2 = \lim_{t\to\infty}\left\|\frac{\partial w_{r/2}}{\partial x}(\cdot,t)\right\|_2 = 0.$

The estimate (5.97) holds with $T=\infty$.

Proof. For each $T>0$, let (w^T,μ^T) be the solution to the finite time horizon problem on $[0,T]$ given by Theorem 5.28. We extend (w^T,μ^T) in time such that $w^T(x,t)=0$ for all $t \geq T$ and such that the a priori estimate (5.97) implies that $(w^T_{r/2},\mu^T_{r/2})$ is bounded in $(Z_\alpha(\infty),\tilde{Z}_\alpha(\infty))$. Then by standard compactness arguments there exists a subsequence $T_n \to \infty$ such that $(w^{T_n}_{r/2},\mu^{T_n}_{r/2})$ converges to some $(w_{r/2},\mu_{r/2})$ in $Z_\beta(\infty) \times \mathcal{C}^{\beta/2}\left([0,T]; Y^*_{2+\alpha}\right)$ for $\beta<\alpha$. Moreover, (w,μ) satisfies (5.97) with $T=\infty$. Passing to the limit in the system satisfied by (w^{T_n},μ^{T_n}), we deduce that (w,μ) satisfies System (5.1). It follows that (w,μ) is a solution. To see that $\lim_{t\to\infty}\left\|w_{r/2}(\cdot,t)\right\|_2 = \lim_{t\to\infty}\left\|\frac{\partial w_{r/2}}{\partial x}(\cdot,t)\right\|_2 = 0$, we observe that since $w^{T_n}_{r/2} \to w_{r/2}$ in $Z_\beta(\infty)$, it follows that $w^{T_n}_{r/2} \to w_{r/2}$ and $\frac{\partial w^{T_n}_{r/2}}{\partial x} \to \frac{\partial w_{r/2}}{\partial x}$ in $\mathcal{C}\left([0,\infty); X_2\right)$. Then the fact that $w^{T_n}(x,t)=0$ for all $t \geq T_n$ implies the desired limit. Finally, the a priori bounds together with the linearity of the system imply uniqueness. □

6. The solution to the master equation

For each m_0, define $U(m_0, x) = u(x, 0)$ where (u, m) is the solution of (1.6) given initial condition m_0. We refer to $U(m_0, x)$ as the *master field*. We will prove that it satisfies the master equation (1.3). All the hypotheses of Theorem 4.16 plus Assumption 5.26 or 5.27 are in force.

6.1. Continuity and differentiability of the master field

In this subsection we show that $U(m_0, x)$ is Lipschitz continuous and differentiable with respect to the measure variable m_0. To do this, we appeal to the estimates found in Section 5.

Theorem 6.1. *There exists a constant C such that*

$$\left\| U(\tilde{m}_0, \cdot) - U(m_0, \cdot) \right\|_{Y_{n+1+\alpha}} \leq C \left\| \tilde{m}_0 - m_0 \right\|_{\mathscr{M}^{-2}_{\alpha/2}} \quad \forall \tilde{m}_0, m_0 \in \mathscr{M}^{-2}_{\alpha/2}.$$

Proof. We may assume that $\tilde{m}_0, m_0 \in \mathscr{M}_{\alpha/2}$; then by density of this set in $\mathscr{M}^{-2}_{\alpha/2}$, we deduce the result. We have $U(m_0, x) = u(x, 0)$ and $U(\tilde{m}_0, x) = \tilde{u}(x, 0)$, where (u, m) is the solution of (4.21) given initial condition m_0 and $(\tilde{u}, \tilde{m})$ is the solution of (1.6) given initial condition $\tilde{m}_0$. Let $(w, \mu) = (\tilde{u} - u, \tilde{m} - m)$. Then (w, μ) solves the linearized system (5.1) with $f = 0$, $\nu = 0$, and $V_1, \ldots, V_5$ defined in (5.2). Observe that

$$\left\| U(\tilde{m}_0, \cdot) - U(m_0, \cdot) \right\|_{Y_{3+\alpha}} = \left\| w(\cdot, 0) \right\|_{Y_{3+\alpha}} \leq \|w\|_{Z_\alpha}.$$

We conclude by appealing to Theorem 5.25. □

Before proving that U is differentiable with respect to m, we provide a candidate for the derivative in the following lemma.

Lemma 6.2. *Let $f = 0$ and $\nu = 0$. There exists a map $K(m_0, x, y)$ such that K is thrice differentiable with respect to x and twice differentiable with respect to y, such that*

$$\left\| \frac{\partial^\ell K}{\partial y^\ell}(m_0, \cdot, y) \right\|_{Y_{3+\alpha}} \leq C \max\left\{ |y|^{-\alpha-\ell}, 1 \right\}, \tag{6.1}$$

and such that if (w, μ) is the solution of System (5.1), then

$$w(x, 0) = \left\langle K(m_0, x, \cdot), \mu_0 \right\rangle. \tag{6.2}$$

Moreover, K and its derivatives in (x, y) are continuous with respect to the topology on $\mathcal{M}_+(\mathcal{D}) \times \mathcal{D} \times \mathcal{D}$.

Proof. The proof is very similar to that of [4, Corollary 3.4.2]: for $\ell = 0, 1, 2$ and $y > 0$ let the pair $(w^{(\ell)}(\cdot, \cdot, y), \mu^{(\ell)}(\cdot, \cdot, y))$ be the solution of (5.1) with $f = 0$, $\nu = 0$, $V_1, \ldots, V_5$ given by (5.3) and initial condition $\mu_0 = D^\ell \delta_y$, where δ_y is the Dirac delta mass concentrated at y and $D^\ell \delta_y$ is its ℓth derivative in the sense of distributions. Then set $K(m_0, x, y) = w^{(0)}(x, 0, y)$.

Notice that by the density of empirical measures, (6.2) follows for any solution (w, μ) of System (5.1). Moreover, one can check by induction that

$$\frac{\partial^\ell K}{\partial y^\ell}(m_0, x, y) = (-1)^\ell w^{(\ell)}(x, 0, y).$$

To prove (6.1), we use the estimates (5.97) from Theorem 5.29, which imply in particular that

$$\left\| w^{(\ell)}(\cdot, 0, y) \right\|_{Y_{3+\alpha}} \leq C \left\| D^\ell \delta_y \right\|_{\mathscr{M}^{-2}_{\alpha/2}}.$$

It remains only to estimate $D^\ell \delta_y$ in $\mathscr{M}^{-2}_{\alpha/2}$. First, we see that

$$\left\langle \phi, D^\ell \delta_y \right\rangle = \frac{\mathrm{d}^\ell \phi}{\mathrm{d} y^\ell}(y) \leq \|\phi\|_2 \max\left\{ |y|^{-\ell}, 1 \right\} \quad \forall \phi \in X_2 \quad \Rightarrow \quad \left\| D^\ell \delta_y \right\|_{-2} \leq \max\left\{ |y|^{-\ell}, 1 \right\}.$$

Next, we plug $\mu = D^\ell \delta_y$ into (5.72) to get

$$\eta^h[D^\ell \delta_y](t) = \int_0^\infty \left((-1)^{\ell+1} \frac{\partial^\ell S}{\partial x^\ell}(x - y, t) + \frac{\partial^\ell S}{\partial x^\ell}(x + y, t) \right) \mathrm{d}x = -2 \frac{\partial^{\ell-1} S}{\partial y^{\ell-1}}(y, t), \tag{6.3}$$

where we define

$$\frac{\partial^{-1} S}{\partial y^{-1}}(y, t) = \int_0^y S(x, t)\, \mathrm{d}x.$$

Taking the derivative with respect to t in (6.3), we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \eta^h[D^\ell \delta_y](t) = -\sigma^2 \frac{\partial^{\ell+1} S}{\partial y^{\ell+1}}(y, t).$$

Let $p > 1$. Applying Lemma 3.6 we estimate

$$\int_0^\infty \left| \frac{\mathrm{d}}{\mathrm{d}t} \eta^h[D^\ell \delta_y](t) \right|^p \mathrm{d}t \leq C(\sigma) \int_0^\infty t^{-p(\ell+2)/2} P_{\ell+1}\left(\frac{|y|}{\sqrt{t}} \right)^p \exp\left\{ -\frac{p y^2}{2\sigma^2 t} \right\} \mathrm{d}t.$$

Here $P_{\ell+1}$ is a polynomial of degree $\ell + 1$. Using the substitution $s = \frac{py^2}{t}$ we obtain

$$\left(\int_0^\infty \left| \frac{\mathrm{d}}{\mathrm{d}t} \eta^h[D^\ell \delta_y](t) \right|^p \mathrm{d}t \right)^{1/p}$$

$$\leq C(\sigma,p) y^{\frac{2}{p}-(\ell+2)} \left(\int_0^\infty s^{\frac{p(\ell+2)}{2}-2} P_{\ell+1}(s^{1/2})^p \exp\left\{-\frac{s}{2\sigma^2}\right\} \mathrm{d}s \right)^{1/p},$$

where the integral on the right-hand side converges; hence

$$\left(\int_0^\infty \left| \frac{\mathrm{d}}{\mathrm{d}t} \eta^h[D^\ell \delta_y](t) \right|^p \mathrm{d}t \right)^{1/p} \leq C(\sigma,p) y^{-\frac{2}{p'}-\ell}, \quad p' := p/(p-1).$$

By Hölder's inequality,

$$\left| \eta^h[D^\ell \delta_y](t_1) - \eta^h[D^\ell \delta_y](t_2) \right| \leq C(\sigma,p) y^{-\frac{2}{p'}-\ell} |t_1-t_2|^{1/p'}$$

Cf. the proof of Lemma 3.4. Choosing $p = \frac{2}{2-\alpha}$, we now deduce

$$\left\| \eta^h[D^\ell \delta_y] \right\|_{\mathcal{C}^{\alpha/2}\left([0,\infty)\right)} \leq C(\sigma,\alpha) y^{-\alpha-\ell}.$$

Therefore,

$$\left\| D^\ell \delta_y \right\|_{\mathscr{M}^{-2}_{\alpha/2}} \leq C(\sigma,\alpha) \max\left\{ |y|^{-\alpha-\ell}, 1 \right\},$$

from which we deduce (6.1). The remaining details are the same as in [4, Corollary 3.4.2]. □

Lemma 6.3. *Let (u,m) and $(\hat{u},\hat{m})$ be the solutions of (1.6) with initial conditions m_0 and $\hat{m}_0$, respectively. Let (w,μ) be the solution of (5.1) with initial condition $\hat{m}_0 - m_0$. Then*

$$\left\| \hat{u}(\cdot,0) - u(\cdot,0) - w(\cdot,0) \right\|_{Y_{3+\alpha}} \leq C \left\| \hat{m}_0 - m_0 \right\|^2_{\mathscr{M}^{-2}_{\alpha/2}}. \tag{6.4}$$

As a corollary, $U(m_0,x)$ is differentiable with respect to m_0 with

$$\frac{\delta U}{\delta m}(m_0,x,y) = K(m_0,x,y), \tag{6.5}$$

where K is defined in Lemma 6.2. Moreover, (6.4) reads

$$\left\| U(\hat{m}_0,\cdot) - U(m_0,\cdot) - \int_{\mathcal{D}} \frac{\delta U}{\delta m}(m_0,x,y)\, \mathrm{d}(\hat{m}_0 - m_0)(y) \right\|_{Y_{3+\alpha}} \leq C \left\| \hat{m}_0 - m_0 \right\|^2_{\mathscr{M}^{-2}_{\alpha/2}}. \tag{6.6}$$

Proof. Let f and ν be defined by (5.4). We follow the same steps as in [4, Chapter 3] to find an estimate

$$N(f)+N^*(\nu)\le C\left\|\hat{m}_0-m_0\right\|^2_{\mathscr{M}^{-2}_{\alpha/2}}.$$

By Theorem 5.29, this proves (6.4). Combined with Lemma 6.2, we deduce (6.5) and (6.6). □

6.2. The master field satisfies the master equation

In this subsection we prove Theorem 1.5.

Theorem 6.4. *Suppose that Assumption 5.26 or 5.27 holds. For all $m_0\in\mathcal{M}^{2+\alpha}$, $x\in\mathcal{D}$, System (1.3)-(1.4) is satisfied. Moreover, U is the unique continuously differentiable function satisfying*

$$\left\|\frac{\partial^\ell}{\partial y^\ell}\frac{\delta U}{\delta m}(m_0,\cdot,y)\right\|_{Y_{3+\alpha}}\le C\max\left\{|y|^{-\alpha-\ell},1\right\}, \tag{6.7}$$

such that System (1.3)-(1.4) holds for all $m_0\in\mathcal{M}^{2+\alpha}$, $x\in\mathcal{D}$.

Proof. Let (u,m) be the solution to the mean field game system with initial condition $m_0\in\mathcal{M}^{2+\alpha}$. Set $m_s:=sm(t)+(1-s)m_0$ for $0\le s\le 1$. Then for any $t>0$ we have

$$\begin{aligned}u(x,t)-u(x,0)=U(m(t),x)-U(m_0,x)&=\int_0^1\int_{\mathcal{D}}\frac{\delta U}{\delta m}(m_s,x,y)\,\mathrm{d}\big(m(t)-m_0\big)(y)\,\mathrm{d}s\\&=\int_0^1\int_0^t\int_{\mathcal{D}}\Bigg(\frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}\frac{\delta U}{\delta m}(m_s,x,y)\\&\quad+\frac{\partial H}{\partial a}\left(\epsilon,Q^*(\tau),\frac{\partial u}{\partial x}(y,\tau)\right)\frac{\partial}{\partial y}\frac{\delta U}{\delta m}(m_s,x,y)\Bigg)\,\mathrm{d}m(\tau)(y)\,\mathrm{d}\tau\,\mathrm{d}s,\end{aligned} \tag{6.8}$$

using the Fokker-Planck equation satisfied by m. To see that the last integral converges, first note that (6.7) holds by Lemmas 6.2 and 6.3. Then we note that by the assumption $m_0\in\mathcal{M}^{2+\alpha}$ together with Lemma 3.8,

$$\int_{\mathcal{D}}\left(1+x^{-(2+\alpha)}\right)m(\mathrm{d}x,t)\le Ce^{Ct}\int_{\mathcal{D}}\left(1+x^{-(2+\alpha)}\right)m_0(\mathrm{d}x).$$

Combining this with (6.7), we deduce that (6.8) holds. Now divide by t and let $t\to 0$ to get

$$\begin{aligned}&\frac{\partial u}{\partial t}(x,0)\\&=\int_{\mathcal{D}}\left(\frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}\frac{\delta U}{\delta m}(m_0,x,y)+\frac{\partial H}{\partial a}\left(\epsilon,Q^*(0),\frac{\partial u}{\partial x}(y,0)\right)\frac{\partial}{\partial y}\frac{\delta U}{\delta m}(m_0,x,y)\right)\mathrm{d}m_0(y).\end{aligned}$$

By substituting for $\frac{\partial u}{\partial t}(x,0)$ using Equation (1.6)(i), we get

$$-\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(x,0) - H\left(\epsilon, Q^*(0), \frac{\partial u}{\partial x}(x,0)\right) + ru(x,0)$$
$$= \int_{\mathcal{D}} \left(\frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}\frac{\delta U}{\delta m}(m_0,x,y) + \frac{\partial H}{\partial a}\left(\epsilon, Q^*(0), \frac{\partial u}{\partial x}(y,0)\right)\frac{\partial}{\partial y}\frac{\delta U}{\delta m}(m_0,x,y)\right) \mathrm{d}m_0(y),$$

which becomes Equation (1.3) after defining $Q^* = Q^*(0)$. Equation (1.4) follows from (1.6)(iii).

To see that U is unique, we follow the same argument as in [4]. By using the Leray-Schauder fixed point theorem and the estimates we have established, it is straightforward to show the existence of a solution to the Fokker-Planck equation

$$\frac{\partial m}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 m}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial a}\left(\epsilon, Q^*(m(t)), \frac{\partial U}{\partial x}(m(t),x)\right)\right) = 0,$$

where $Q^*(m)$ is defined using (1.4). Set $u(x,t) = U(m(t),x)$. Using condition (6.7) together with Lemma 3.8, as above, we can differentiate u with respect to time. Then using the fact that (1.3) holds, we deduce that (u,m) is the solution of (1.6), which is unique. It follows that $U(m,x)$ is uniquely determined. □

Data availability

No data was used for the research described in the article.

Appendix A. Proofs of results from Section 3

Proof of Lemma 3.1. *Uniqueness:* Let us start by observing that uniqueness of weak solutions holds. This follows from a proof by duality, cf. [23, Proposition B.1] and [34, Corollary 3.5], which also provide the basic estimate (3.3).

Existence: We thus turn our attention to existence and estimates. By linearity we can assume that $m_0(\mathcal{D}) = 1$, i.e. m_0 is a probability measure, without loss of generality.

Assume for now that b is infinitely smooth and bounded, and that $m_0 \in \mathcal{P}_1(\mathcal{D})$ is in fact a smooth density such that $m_0 \in \mathcal{C}_c^\infty(\overline{\mathcal{D}})$. Then classical theory [29, Theorems IV.5.2, IV.9.1] implies that (3.1) has a smooth solution m whose derivatives are also in L^p for arbitrarily large p. We have the following probabilistic interpretation: for any continuous function on $\overline{\mathcal{D}}$ satisfying

$$\left|\phi(x)\right| \le C\left(1 + |x|\right),$$

we have

$$\int_0^\infty \phi(x) m(x,t)\,\mathrm{d}x = \mathbb{E}\left[\phi(X_t)\mathbb{1}_{t<\tau}\right] \tag{A.1}$$

where X_t is the diffusion process given by

$$\mathrm{d}X_t = -b(X_t,t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \;\; X_0 \sim m_0, \tag{A.2}$$

W_t is a standard Brownian motion with respect to a filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}_t)$, and

$$\tau := \min\left\{\inf\{t \geq 0 : X_t \leq 0\}, T\right\}.$$

In particular the complementary mass function $\bar{\eta}(t)$ can be written

$$\bar{\eta}(t) = \mathbb{P}(t < \tau).$$

The continuity of this function follows from probabilistic arguments, which can be found in [25] and [23].

It remains to establish (3.4). Fix $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Pick any $\phi : \overline{\mathcal{D}} \to \mathbb{R}$ that is α-Hölder continuous (or Lipschitz, in the case $\alpha = 1$) such that $\phi(0) = 0$ and $[\phi]_{\mathcal{C}^\alpha} \leq 1$. Let X_t be a solution to (A.2). Then by (A.1) we have

$$\begin{aligned}
\left|\int_0^\infty \phi(x)\left(m(x,t_1) - m(x,t_2)\right) \mathrm{d}x\right| &= \left|\mathbb{E}\left[\phi(X_{t_1})\mathbb{1}_{t_1<\tau} - \phi(X_{t_2})\mathbb{1}_{t_2<\tau}\right]\right| \\
&\leq \mathbb{E}\left[\left|X_{t_1}\right|^\alpha \mathbb{1}_{t_1<\tau\leq t_2} + \left|X_{t_1} - X_{t_2}\right|^\alpha \mathbb{1}_{t_2<\tau}\right] \\
&= \mathbb{E}\left[\left|-\int_{t_1}^{\tau} b(X_t,t)\,\mathrm{d}t + \sigma(W_\tau - W_{t_1})\right|^\alpha \mathbb{1}_{t_1<\tau\leq t_2}\right] \\
&\quad + \mathbb{E}\left[\left|-\int_{t_1}^{t_2} b(X_t,t)\,\mathrm{d}t + \sigma(W_{t_2} - W_{t_1})\right|^\alpha \mathbb{1}_{t_2<\tau}\right] \\
&\leq \mathbb{E}\left[\|b\|_\infty^\alpha |\tau - t_1|^\alpha \mathbb{1}_{t_1<\tau\leq t_2}\right] + \sigma^\alpha \mathbb{E}\left[\left|W_\tau - W_{t_1}\right|^\alpha \mathbb{1}_{t_1<\tau\leq t_2}\right] \\
&\quad + \|b\|_\infty^\alpha |t_2 - t_1|^\alpha + \sigma^\alpha \mathbb{E}\left[\left|W_{t_2} - W_{t_1}\right|^\alpha\right] \\
&\leq 2\|b\|_\infty^\alpha |t_2 - t_1|^\alpha + 2\sigma^\alpha |t_2 - t_1|^{\alpha/2}.
\end{aligned}$$

Taking $t_1 = t$ and $t_2 = 0$, we get

$$\begin{aligned}
\left|\int_0^\infty \phi(x)m(x,t)\,\mathrm{d}x\right| &\leq \left|\int_0^\infty \phi(x)m_0(x)\,\mathrm{d}x\right| + 2\|b\|_\infty^\alpha t^\alpha + 2\sigma^\alpha t^{\alpha/2} \\
&\leq \int_0^\infty x^\alpha m_0(x)\,\mathrm{d}x + 2\|b\|_\infty^\alpha t^\alpha + 2\sigma^\alpha t^{\alpha/2}.
\end{aligned}$$

Finally, to get existence for general data, let b_n be a sequence of smooth functions converging uniformly to b and let $m_{0,n}$ be a sequence of measures with smooth densities converging to m_0 in $\mathcal{M}_{1,+}$. Letting m_n be the solution corresponding to $b_n, m_{0,n}$, we have that m_n is uniformly Hölder continuous in the $\mathrm{Lip}_\diamond(\mathcal{D})^*$ metric, hence by Arzelá-Ascoli we have a subsequence converging to m in $\mathcal{C}^0([0,T];\mathcal{M}_{1,+})$. We deduce that m is a weak solution, i.e. it satisfies (3.2). □

Proof of Lemma 3.5. For each $n \in \mathbb{N}$ define

$$\phi_n(x) = \begin{cases} n^\alpha x & \text{if } 0 < x \le n^{-1}, \\ x^{-\alpha} & \text{if } x > n^{-1}. \end{cases} \tag{A.3}$$

Set $\Phi_n^{(0)}(x) = \phi_n(x)$, and inductively define

$$\Phi_n^{(j)}(x) = \int_0^x \Phi_n^{(j-1)}(t)\,\mathrm{d}t, \quad j = 1, 2, 3, \ldots.$$

By induction we have that

$$\left|\Phi_n^{(j)}(x)\right| \le C(j,\alpha) x^{j-\alpha} \quad \forall x > 0. \tag{A.4}$$

Since ϕ_n is a bounded, continuous function, we have

$$\begin{aligned} \int_{\mathcal{D}} \phi_n(x) m(\mathrm{d}x, t) &= \int_{\mathcal{D}} \phi_n(x) \int_{\mathcal{D}} \big(S(x-y,t) - S(x+y,t)\big)\, m_0(\mathrm{d}y)\,\mathrm{d}x \\ &= \int_{\mathcal{D}} \int_{\mathcal{D}} \phi_n(x) \big(S(x-y,t) - S(x+y,t)\big)\,\mathrm{d}x\; m_0(\mathrm{d}y) \end{aligned} \tag{A.5}$$

using Fubini's Theorem. Our goal now is to prove that

$$\int_{\mathcal{D}} \phi_n(x) \big(S(x-y,t) - S(x+y,t)\big)\,\mathrm{d}x \le C(\alpha) y^{-\alpha} \quad \forall y > 0. \tag{A.6}$$

By plugging (A.6) into (A.5) and then applying the Monotone Convergence Theorem, (3.10) follows.

To prove (A.6), start by noting

$$\begin{aligned} \int_{\mathcal{D}} \phi_n(x) \big(S(x-y,t) - S(x+y,t)\big)\,\mathrm{d}x &\le \int_0^\infty \phi_n(x) S(x-y,t)\,\mathrm{d}x \\ &\le \int_0^{y/2} \phi_n(x) S(x-y,t)\,\mathrm{d}x \end{aligned} \tag{A.7}$$

$$+ (y/2)^{-\alpha} \int_{y/2}^{\infty} \phi_n(x) S(x-y,t)\,\mathrm{d}x$$
$$\leq \int_0^{y/2} \phi_n(x) S(x-y,t)\,\mathrm{d}x + 2^\alpha y^{-\alpha},$$

using the fact that $S(\cdot,t)$ is a probability density. Now for any $j > \alpha - 1$, integrate by parts j times to get

$$\int_0^{y/2} \phi_n(x) S(x-y,t)\,\mathrm{d}x$$
$$= \sum_{i=0}^{j-1} (-1)^i \Phi_n^{(i+1)}(y/2) \frac{\partial^i S}{\partial x^i}(-y/2,t) + (-1)^j \int_0^{y/2} \Phi_n^{(j)}(x) \frac{\partial^j S}{\partial x^j}(x-y,t)\,\mathrm{d}x. \quad \text{(A.8)}$$

Applying (A.4) and Lemma 3.6 to Equation (A.8), we obtain

$$\begin{aligned} \int_0^{y/2} \phi_n(x) S(x-y,t)\,\mathrm{d}x &\leq C(j,\alpha) \left(\sum_{i=0}^{j-1} y^{i+1-\alpha} y^{-(i+1)} + \int_0^{y/2} x^{j-\alpha} \left| x-y \right|^{-(j+1)} \mathrm{d}x \right) \\ &\leq C(j,\alpha) y^{-\alpha}. \end{aligned} \quad \text{(A.9)}$$

Take $j = \lfloor \alpha \rfloor$ and combine (A.9) with (A.7) to obtain (A.6), which completes the proof. □

Proof of 3.7. First, note that $\left|\eta(t)\right| \leq \|m_0\|_{TV} \leq \|m_0\|_{\mathscr{M}_\alpha}$ for all $t \geq 0$, using Lemma 3.1. Thus, it suffices to prove estimates of the Hölder constant for η. We will assume the data are sufficiently regular so that the solution is smooth. The claim then follows from a density argument.

We have, by Duhamel's principle,

$$m(x,t) = \int_0^\infty \big(S(x-y,t) - S(x+y,t)\big) m_0(y)\,\mathrm{d}y$$
$$+ \int_0^t \int_0^\infty \big(S(x-y,t-s) - S(x+y,t-s)\big) (bm)_y(y,s)\,\mathrm{d}y\,\mathrm{d}s,$$

which becomes

$$m(x,t) = \int_0^\infty \big(S(x-y,t) - S(x+y,t)\big) m_0(y)\,\mathrm{d}y$$

$$+\int_0^t\int_0^\infty\left(\frac{\partial S}{\partial x}(x-y,t-s)+\frac{\partial S}{\partial x}(x+y,t-s)\right)(bm)(y,s)\,\mathrm{d}y\,\mathrm{d}s, \quad (A.10)$$

using integration by parts. Integrating in x and using Fubini's Theorem, we get

$$\eta(t)=\eta^h[m_0](t)+\eta_2(t),$$

where $\eta^h[m_0](t)$ is defined in (3.8) and

$$\eta_2(t)=\int_0^t\int_0^\infty\int_0^\infty\left(\frac{\partial S}{\partial x}(x-y,t-s)+\frac{\partial S}{\partial x}(x+y,t-s)\right)(bm)(y,s)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$=-2\int_0^t\int_0^\infty S(y,t-s)(bm)\big(y,s\big)\,\mathrm{d}y\,\mathrm{d}s.$$

By definition of the norm in $\mathscr{M}_\alpha$,

$$\left\|\eta^h[m_0]\right\|_{\mathcal{C}^\alpha([0,T])}\le\|m_0\|_{\mathscr{M}_\alpha}. \quad (A.11)$$

It remains to derive Hölder estimates for η_2. Let $t_2>t_1\ge 0$. Then $\eta_2(t_2)-\eta_2(t_1)=-2\,(I_1+I_2)$ where

$$I_1=\int_{t_1}^{t_2}\int_0^\infty S(y,t_2-s)b(y,s)m(\mathrm{d}y,s)\,\mathrm{d}s,$$

$$I_2=\int_0^{t_1}\int_0^\infty\big(S(y,t_2-s)-S(y,t_1-s)\big)\,b(y,s)m(\mathrm{d}y,s)\,\mathrm{d}s.$$

In the first place, we have

$$|I_1|\le(2\sigma^2\pi)^{-1/2}\|b\|_\infty\int_{t_1}^{t_2}(t_2-s)^{-1/2}\,\mathrm{d}s=2(2\sigma^2\pi)^{-1/2}\|b\|_\infty\,(t_2-t_1)^{1/2}.$$

In the second place, we write

$$I_2=\int_0^{t_1}\int_0^\infty\int_{t_1}^{t_2}\frac{\partial S}{\partial t}(y,\tau-s)\,\mathrm{d}\tau\;b(y,s)m(\mathrm{d}y,s)\,\mathrm{d}s.$$

Since $\dfrac{\partial S}{\partial t}=\dfrac{\sigma^2}{2}\dfrac{\partial^2 S}{\partial x^2}$, Lemma 3.6 implies

$$|I_2| \le C(\sigma)\|b\|_\infty \int_0^{t_1}\int_{t_1}^{t_2} (\tau - s)^{-3/2}\, \mathrm{d}\tau\, \mathrm{d}s. \tag{A.12}$$

By Fubini's Theorem,

$$\int_0^{t_1}\int_{t_1}^{t_2} (\tau - s)^{-3/2}\, \mathrm{d}\tau\, \mathrm{d}s = 2\int_{t_1}^{t_2} \left((\tau - t_1)^{-1/2} - \tau^{-1/2}\right) \mathrm{d}\tau \le 4(t_2 - t_1)^{1/2}. \tag{A.13}$$

Combining (A.12) and (A.13), we get

$$\left|\eta_2(t_1) - \eta_2(t_2)\right| \le C(\sigma)\|b\|_\infty |t_1 - t_2|^{1/2}. \tag{A.14}$$

Equation (3.12) follows from combining (A.11) and (A.14). □

Proof of Lemma 3.8. We start from Equation (A.10) and multiply by $\phi_n(x)$, which is defined in (A.3). Then integrate and use Lemma 3.5 to get

$$\begin{aligned}\int_{\mathcal{D}} \phi_n(x) m(\mathrm{d}x, t) \le{}& C(\alpha) \int_{\mathcal{D}} |x|^{-\alpha}\, m_0(\mathrm{d}x) \\ &+ \|b\|_\infty \int_0^t \int_{\mathcal{D}} \left| \int_0^\infty \phi_n(x) \left(\frac{\partial S}{\partial x}(x - y, t - s) + \frac{\partial S}{\partial x}(x + y, t - s)\right) \mathrm{d}x \right| m(\mathrm{d}y, s)\, \mathrm{d}s. \end{aligned} \tag{A.15}$$

Let $j = \lfloor \alpha \rfloor$. Integrating by parts j times as in the proof of Lemma 3.5, we get

$$\begin{aligned}\int_0^{y/2} \phi_n(x) \frac{\partial S}{\partial x}(x \pm y, t - s)\, \mathrm{d}x ={}& \sum_{i=1}^{j} (-1)^{i-1} \Phi_n^{(i)}(y/2) \frac{\partial^i S}{\partial x^i}(y/2 \pm y, t - s) \\ &+ (-1)^j \int_0^{y/2} \Phi_n^{(j)}(x) \frac{\partial^{j+1} S}{\partial x^{j+1}}(x \pm y, t - s)\, \mathrm{d}x. \end{aligned}$$

Using Lemma 3.6 and Equation (A.4), we deduce

$$\begin{aligned}\left| \int_0^{y/2} \phi_n(x) \frac{\partial S}{\partial x}(x \pm y, t - s)\, \mathrm{d}x \right| \le{}& \sum_{i=1}^{j} C(i, \alpha) \left|y\right|^{i-\alpha} \left|y\right|^{-i} \sigma^{-1}(t - s)^{-1/2} \\ &+ C(j, \alpha) \int_0^{y/2} |x|^{j-\alpha} \left|x \pm y\right|^{-j} \sigma^{-1}(t - s)^{-1/2}\, \mathrm{d}x. \end{aligned} \tag{A.16}$$

For $0 \le x \le y/2$ we have $\left|x \pm y\right| \ge y/2$, and thus (A.16) yields

$$\left| \int_0^{y/2} \phi_n(x) \frac{\partial S}{\partial x}(x \pm y, t-s)\,\mathrm{d}x \right| \le C(j,\alpha)\sigma^{-1}\left|y\right|^{-\alpha}(t-s)^{-1/2}. \tag{A.17}$$

On the other hand, using Lemma 3.6 it follows that $\int_0^\infty t^{1/2}\left|\frac{\partial S}{\partial x}(x,t)\right|\mathrm{d}x \le C$ for all t, and thus

$$\left| \int_{y/2}^{\infty} \phi_n(x) \frac{\partial S}{\partial x}(x \pm y, t-s)\,\mathrm{d}x \right| \le C\left|y\right|^{-\alpha}(t-s)^{-1/2}. \tag{A.18}$$

Combining (A.17) and (A.18) into (A.15), then letting $n \to \infty$, we derive

$$\int_{\mathcal{D}} |x|^{-\alpha} m(\mathrm{d}x,t) \le C(\alpha)\int_{\mathcal{D}} |x|^{-\alpha} m_0(\mathrm{d}x) + C(\alpha,\sigma)\|b\|_\infty \int_0^t \int_{\mathcal{D}} \left|y\right|^{-\alpha} m(\mathrm{d}y,s)(t-s)^{-1/2}\,\mathrm{d}s. \tag{A.19}$$

For $\lambda > 0$ let

$$f_\lambda(t) = e^{-\lambda t}\int_{\mathcal{D}} |x|^{-\alpha} m(\mathrm{d}x,t).$$

Multiply (A.19) by $e^{-\lambda t}$ to derive

$$\begin{aligned} f_\lambda(t) &\le C(\alpha) f_\lambda(0) + C(\alpha,\sigma)\|b\|_\infty \int_0^t e^{-\lambda(t-s)}(t-s)^{-1/2} f_\lambda(s)\,\mathrm{d}s \\ &\le C(\alpha) f_\lambda(0) + C(\alpha,\sigma)\|b\|_\infty \lambda^{-1/2} \sup_{\tau\ge 0} f_\lambda(\tau) \end{aligned} \tag{A.20}$$

where by a change of variables we have computed

$$\int_0^t e^{-\lambda(t-s)}(t-s)^{-1/2}\,\mathrm{d}s = \lambda^{-1/2}\int_0^{\lambda t} e^{-s}s^{-1/2}\,\mathrm{d}s \le \lambda^{-1/2}\left(\int_0^1 s^{-1/2}\,\mathrm{d}s + \int_1^\infty e^{-s}\,\mathrm{d}s\right) \le 3\lambda^{-1/2}.$$

(As usual, the value of $C(\alpha,\sigma)$ might have changed from line to line.) Let $\lambda = \left(2C(\alpha,\sigma)\|b\|_\infty\right)^2$. Take the supremum in (A.20) to deduce

$$\sup_{t\ge 0} f_\lambda(t) \le C(\alpha) f_\lambda(0) + \frac{1}{2}\sup_{t\ge 0} f_\lambda(t) \quad \Rightarrow \sup_{t\ge 0} f_\lambda(t) \le 2C(\alpha) f_\lambda(0). \tag{A.21}$$

Equation (A.21) implies (3.13), as desired. □

Appendix B. Proofs of results from Section 4

B.1. Proofs of results from Section 4.1

We will actually show that all of the results of this section hold on a larger domain. Set $p_\infty := \lim_{q\to\infty} P(q)$. Note that $p_\infty < 0$ because there exists a finite saturation point (Assumption 1.1). Recall that the profit function π is defined as

$$\pi(\epsilon, q, Q, a) = \begin{cases} q\left(P(\epsilon Q + q) - a\right) & \text{if } q > 0, \\ 0 & \text{if } q = 0. \end{cases}$$

In the following the domain of π is defined to be $\left[0,\infty\right)^3 \times (p_\infty, \infty)$. Thus the domain of $H(\epsilon, Q, a) := \sup_{q\geq 0} \pi(\epsilon, q, Q, a)$ is $\left[0,\infty\right)^2 \times (p_\infty, \infty)$. All the statements about the regularity of H hold on this larger domain. This remark will be useful in Lemma B.1 below.

Proof of Lemma 4.1. We first compute

$$\frac{\partial \pi}{\partial q}(\epsilon, q, Q, a) := qP'(\epsilon Q + q) + P(\epsilon Q + q) - a$$

and

$$\frac{\partial^2 \pi}{\partial q^2}(\epsilon, q, Q, a) = qP''(\epsilon Q + q) + 2P'(\epsilon Q + q) = -\left(q\frac{\rho(\epsilon Q + q)}{\epsilon Q + q} - 2\right)P'(\epsilon Q + q).$$

By Assumption 1.2 we deduce

$$\frac{\partial^2 \pi}{\partial q^2}(\epsilon, q, Q, a) \leq -\left(\bar{\rho} - 2\right)P'(\epsilon Q + q) < 0,$$

i.e. π is strictly concave with respect to q. On the other hand, since $P' \leq 0$ we also have

$$\limsup_{q\to\infty} \frac{\partial \pi}{\partial q}(\epsilon, q, Q, a) \leq \lim_{q\to\infty} P(\epsilon Q + q) - a = p_\infty - a < 0.$$

Thus if $\frac{\partial \pi}{\partial q}(\epsilon, 0, Q, a) = P(\epsilon Q) - a > 0$ there must exist a unique $q^* > 0$ such that $\frac{\partial \pi}{\partial q}(\epsilon, q^*, Q, a) = 0$, and hence q^* maximizes $\pi(\epsilon, \cdot, Q, a)$. We also compute

$$\begin{aligned} \frac{\partial^2 \pi}{\partial Q \partial q}(\epsilon, q, Q, a) &= \epsilon q P''(\epsilon Q + q) + \epsilon P'(\epsilon Q + q) \\ &= -\epsilon\left(q\frac{\rho(\epsilon Q + q)}{\epsilon Q + q} - 1\right)P'(\epsilon Q + q), \\ \frac{\partial^2 \pi}{\partial a \partial q}(\epsilon, q, Q, a) &= -1, \end{aligned}$$

$$\frac{\partial^2 \pi}{\partial \epsilon \partial q}(\epsilon, q, Q, a) = QqP''(\epsilon Q + q) + QP'(\epsilon Q + q)$$
$$= -Q\left(q\frac{\rho(\epsilon Q + q)}{\epsilon Q + q} - 1\right)P'(\epsilon Q + q)$$

By the implicit function theorem, we deduce that q^* is differentiable function of (ϵ, Q, a) in the region where $P(\epsilon Q) > a$, with

$$\begin{aligned}
\frac{\partial q^*}{\partial Q} &= -\frac{\frac{\partial^2 \pi}{\partial Q \partial q}(\epsilon, q^*, Q, a)}{\frac{\partial^2 \pi}{\partial q^2}(\epsilon, q^*, Q, a)} = -\epsilon\left(1 - \left(2 - \frac{q^*\rho(\epsilon Q + q^*)}{\epsilon Q + q^*}\right)^{-1}\right), \\
\frac{\partial q^*}{\partial a} &= \frac{1}{\frac{\partial^2 \pi}{\partial q^2}(\epsilon, q^*, Q, a)} < 0, \\
\frac{\partial q^*}{\partial \epsilon} &= -\frac{\frac{\partial^2 \pi}{\partial \epsilon \partial q}(\epsilon, q^*, Q, a)}{\frac{\partial^2 \pi}{\partial q^2}(\epsilon, q^*, Q, a)} = -Q\left(1 - \left(2 - \frac{q^*\rho(\epsilon Q + q^*)}{\epsilon Q + q^*}\right)^{-1}\right)
\end{aligned} \tag{B.1}$$

Note that (4.1) follows from (B.1).

In this region we also compute

$$\frac{\partial H}{\partial \epsilon} = \frac{\partial q^*}{\partial \epsilon}\left(P(\epsilon Q + q^*) - a\right) + q^* P'(\epsilon Q + q^*)\left(Q + \frac{\partial q^*}{\partial \epsilon}\right) = Qq^* P'(\epsilon Q + q^*), \tag{B.2}$$

$$\frac{\partial H}{\partial Q} = \frac{\partial q^*}{\partial Q}\left(P(\epsilon Q + q^*) - a\right) + q^* P'(\epsilon Q + q^*)\left(\epsilon + \frac{\partial q^*}{\partial Q}\right) = \epsilon q^* P'(\epsilon Q + q^*), \tag{B.3}$$

and

$$\frac{\partial H}{\partial a} = \frac{\partial q^*}{\partial a}\left(P(\epsilon Q + q^*) - a\right) + q^*\left(P'(\epsilon Q + q^*)\frac{\partial q^*}{\partial a} - 1\right) = -q^*. \tag{B.4}$$

On the other hand, if $P(\epsilon Q) \le a$ it follows that the unique maximizer is $q^* = 0$. Because P is continuous and monotone decreasing, the interior of this region is the set where $P(\epsilon Q) < a$, while its boundary is where $P(\epsilon Q) = a$. It remains to show that as (ϵ, Q, a) approaches this boundary set, the derivative of q^* remains bounded. By (B.1) it is enough to show that $\frac{\partial^2 \pi}{\partial q^2}(\epsilon, q^*(Q, a), Q, a)$ remains bounded away from zero. For this we observe that as (ϵ, Q, a) approaches the set where $P(\epsilon Q) = a$, $q^*(\epsilon, Q, a) \to 0$ and thus $\frac{\partial^2 \pi}{\partial q^2}(\epsilon, q^*, Q, a) \to 2P'(\epsilon Q)$, which is bounded away from zero for bounded values of Q. □

Proof of Corollary 4.2. For $(\epsilon, Q, a) \in [0, \bar{\epsilon}] \times [0, \bar{Q}] \times [0, \bar{a}]$ we have that $a \le \bar{a} < P(\bar{\epsilon}\bar{Q}) \le P(\epsilon Q)$, since P is decreasing. By differentiating (B.2), (B.3), and (B.4) in the proof of Lemma 4.1, and using (B.1), we see that H is n times continuously differentiable in this region. These derivatives are Lipschitz on this domain because $P^{(n)}$ is locally Lipschitz by Assumption 1.1. In particular,

$$\frac{\partial^2 H}{\partial a^2}(\epsilon, Q, a) = -\frac{\partial q^*}{\partial a}(\epsilon, Q, a) > 0.$$

The claim follows from compactness of the region. □

Proof of Corollary 4.3. From Equation (B.3) and the first-order condition for optimality, using the fact that $P' < 0$, we have

$$\left|\frac{\partial H}{\partial Q}\right| = -\epsilon q^* P'(\epsilon Q + q^*) = \epsilon(P(\epsilon Q + q^*) - a),$$

from which the first estimate in (4.3) follows. The second estimate follows from (4.1) and (B.3). □

Proof of Lemma 4.4. Let $f(Q) = Q - \int_{\mathcal{D}} q^*(\epsilon, Q, \phi(x))\,\mathrm{d}m(x)$. We claim that $f(Q^*) = 0$ for a unique $Q^* \geq 0$. Note that $f(0) \leq 0$ because $q^* \geq 0$. By Lemma 4.1 and Assumption 1.2 we have

$$f'(Q) = 1 - \int_{\mathcal{D}} \frac{\partial q^*}{\partial Q}(\epsilon, Q, \phi(x))\,\mathrm{d}m(x) \geq 1 - \epsilon\frac{\bar{\rho} - 1}{2 - \bar{\rho}}\int_{\mathcal{D}} \mathrm{d}m(x) \geq \frac{2 + \epsilon - (1 + \epsilon)\bar{\rho}}{2 - \bar{\rho}} > 0$$

if $\bar{\rho} \geq 1$; otherwise we get simply $f'(Q) \geq 1$. The claim follows, and we deduce (4.4). To derive estimate (4.5), we use the lower bound on f' to deduce

$$Q^* \leq c(\bar{\rho}, \epsilon)\big(f(Q^*) - f(0)\big) = c(\bar{\rho}, \epsilon)\int_{\mathcal{D}} q^*(\epsilon, 0, \phi(x))\,\mathrm{d}m(x). \tag{B.5}$$

Now because $\pi(\epsilon, q, 0, a) = \pi(0, q, 0, a)$ for all ϵ, a, it follows that $q^*(\epsilon, 0, \phi(x)) = q^*(0, 0, \phi(x))$. Then, since q^* is decreasing in the last variable and $\int_{\mathcal{D}} \mathrm{d}m(x) \leq 1$, we use (B.5) to deduce (4.5).

We now prove (4.6). Without loss of generality we will assume $Q_1^* \geq Q_2^*$. First, observe that

$$Q_1^* - Q_2^* \leq c(\bar{\rho}, \epsilon)\left(\int_{\mathcal{D}} q^*(\epsilon_1, Q_2^*, \phi_1(x))\,\mathrm{d}m_1(x) - \int_{\mathcal{D}} q^*(\epsilon_2, Q_2^*, \phi_2(x))\,\mathrm{d}m_2(x)\right). \tag{B.6}$$

To see this, note that (4.1) implies

$$\begin{aligned} Q_1^* - Q_2^* &= \int_{\mathcal{D}} q^*(\epsilon_1, Q_1^*, \phi_1(x))\,\mathrm{d}m_1(x) - \int_{\mathcal{D}} q^*(\epsilon_2, Q_2^*, \phi_2(x))\,\mathrm{d}m_2(x) \\ &\leq -\epsilon_1\frac{1 - \bar{\rho}}{2 - \bar{\rho}}(Q_1^* - Q_2^*)\int_{\mathcal{D}} \mathrm{d}m_1(x) + \int_{\mathcal{D}} q^*(\epsilon_1, Q_2^*, \phi_1(x))\,\mathrm{d}m_1(x) - \int_{\mathcal{D}} q^*(\epsilon_2, Q_2^*, \phi_2(x))\,\mathrm{d}m_2(x). \end{aligned}$$

Then one obtains (B.6) by rearranging and using the fact that $\int_{\mathcal{D}} \mathrm{d}m_1(x) \le 1$ and $c(\bar\rho, \epsilon)$ is increasing in ϵ. Next, appealing to (4.5) and the fact that q^* is locally Lipschitz, recalling once more that $\int_{\mathcal{D}} \mathrm{d}m_1(x) \le 1$, (B.6) becomes

$$\begin{aligned} Q_1^* - Q_2^* &\le C\left(|\epsilon_1 - \epsilon_2| + \int_{\mathcal{D}} \left|\phi_1(x) - \phi_2(x)\right| \mathrm{d}m_1(x)\right) \\ &\quad + C\int_{\mathcal{D}} q^*(\epsilon_2, Q_2^*, \phi_2(x))\, \mathrm{d}(m_1 - m_2)(x) \\ &\le C\left(|\epsilon_1 - \epsilon_2| + \|\phi_1 - \phi_2\|_\infty\right) + C\sup_x \left|\frac{\mathrm{d}q^*}{\mathrm{d}a}(\epsilon_2, Q_2^*, \phi_2(x))\frac{\mathrm{d}\phi_2}{\mathrm{d}x}(x)\right| \mathbf{d}_1(m_1, m_2) \\ &\quad + q^*(\epsilon_2, Q_2^*, \phi_2(0))\int_{\mathcal{D}} \mathrm{d}(m_1 - m_2)(x), \end{aligned}$$

which implies (4.6). □

Proof of Corollary 4.6. We use Lemma 4.1 to get

$$q^*(\epsilon, Q^*, \phi(x)) \le q^*(\epsilon, 0, 0) + \epsilon\frac{\bar\rho - 1}{2 - \bar\rho} Q^*. \tag{B.7}$$

We recall that $q^*(\epsilon, 0, 0) = q^*(0, 0, 0)$. To derive (4.7), it suffices to plug (4.5) into (B.7) and use the definition of $c(\bar\rho, \epsilon)$. □

B.2. Proofs of results from Section 4.3

Proof of Lemma 4.10. First let $v = e^{-rt}u$. Then v satisfies

$$\frac{\partial v}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2} + e^{-rt} H\left(\epsilon(t), Q^*(t), e^{rt}\frac{\partial v}{\partial x}\right) = 0,\ x \in \mathcal{D},\ t > 0.$$

Using the fact that $H \ge 0$ and $v(0, t) = 0$, the maximum principle (see [36, Proposition 2.1]) implies

$$\min_{x\in\mathcal{D}, 0\le t\le T} v(x, t) = \min_{x\in\mathcal{D}} v(x, T) = e^{-rT}\min_{x\in\mathcal{D}} u(x, T) = 0 \quad \Rightarrow v \ge 0 \Rightarrow u \ge 0. \tag{B.8}$$

It also follows that $u(0, t) = \min u$ and so $u_x(0, t) \ge 0$.

We now use the fact that H is decreasing in all variables to deduce

$$0 \le H\left(\epsilon(t), Q^*(t), e^{rt}\frac{\partial v}{\partial x}\right) \le H(0, 0, 0)$$

and thus

$$-\frac{\partial v}{\partial t}-\frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2}\leq e^{-rt}H(0,0,0).$$

Set $\tilde{v}(x,t)=v(x,t)-\int_t^T e^{-rs}H(0,0,0)\,\mathrm{d}s=v(x,t)+\frac{1}{r}H(0,0,0)\left(e^{-rT}-e^{-rt}\right)$. It follows that

$$-\frac{\partial \tilde{v}}{\partial t}-\frac{\sigma^2}{2}\frac{\partial^2 \tilde{v}}{\partial x^2}\leq 0$$

and thus

$$\max_{x\in\mathcal{D},0\leq t\leq T}\tilde{v}(x,t)=\max_{t=T\text{ or }x=0}\tilde{v}(x,t)\leq c_1e^{-rT}$$

since $\tilde{v}(x,T)=e^{-rT}u_T(x)\leq c_1e^{-rT}$ and $\tilde{v}(0,t)=\frac{1}{r}H(0,0,0)\left(e^{-rT}-e^{-rt}\right)\leq 0$. Together with (B.8) we deduce that

$$0\leq v(x,t)\leq\left(\frac{1}{r}H(0,0,0)+c_1\right)e^{-rt}\ \Rightarrow\ 0\leq u(x,t)\leq\frac{1}{r}H(0,0,0)+c_1. \tag{B.9}$$

To get an estimate on u_x, we now use a Bernstein type argument, cf. [29, Section VI.3]. Notice that

$$-\frac{\partial u}{\partial t}-\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}\leq H(0,0,0).$$

Set $\tilde{u}(x,t)=u(x,t)+M_\lambda e^{-\lambda x}$, where $M_\lambda>0$ and $\lambda>0$ are defined below in (B.12) and (B.14). The constants M_λ and λ have to be chosen so that, for all $t\leq T$ and all $x\in[0,\ell]$ for $\ell>0$ to be specified later, we have

$$\begin{aligned}H(0,0,0)-\frac{\sigma^2}{2}\lambda^2M_\lambda e^{-\lambda x}&\leq 0,\\ c_3&\leq M_\lambda\lambda e^{-\lambda x},\\ \frac{1}{r}H(0,0,0)+c_1+M_\lambda e^{-\lambda x}&\leq M_\lambda.\end{aligned}\tag{B.10}$$

Then one can check that

$$-\frac{\partial \tilde{u}}{\partial t}-\frac{\sigma^2}{2}\frac{\partial^2 \tilde{u}}{\partial x^2}\leq 0\text{ in }(0,\ell)\times(0,T),$$

$\tilde{u}(x,T)\leq M_\lambda$ for all $x\in[0,\ell]$ (using the fact that $\max u_T'\leq c_3$), $\tilde{u}(\ell,t)\leq M_\lambda$ (using (B.9)), and $\tilde{u}(0,t)=M_\lambda$. By the maximum principle, it follows that $\tilde{u}(x,t)\leq M_\lambda$ for all $x\in[0,\ell]$, $t\in[0,T]$. This means $\tilde{u}(0,t)=\max_{0\leq x\leq\ell}\tilde{u}(x,t)$, which implies $\tilde{u}_x(0,t)\leq 0$ and thus $u_x(0,t)\leq$

$M_\lambda \lambda$. Finally, we can take the derivative of Equation (4.9) to see that the maximum principle applies to u_x, and thus

$$\max_{x \in \mathcal{D}, 0 \le t \le T} u_x(x,t) = \max_{t=T \text{ or } x=0} u_x(x,t) \le \max\{M_\lambda \lambda, c_3\}. \tag{B.11}$$

To satisfy (B.10), we choose

$$M_\lambda = \max\left\{\frac{2}{\sigma^2\lambda^2} H(0,0,0) e^{\lambda\ell}, \frac{c_3}{\lambda} e^{\lambda\ell}, \frac{1}{1-e^{-\lambda\ell}}\left(\frac{1}{r} H(0,0,0) + c_1\right)\right\}. \tag{B.12}$$

If we set $J = \max\left\{\frac{2}{\sigma^2\lambda^2} H(0,0,0), \frac{c_3}{\lambda}\right\}$, then (B.12) becomes

$$M_\lambda = \max\left\{J e^{\lambda\ell}, \frac{1}{1-e^{-\lambda\ell}}\left(\frac{1}{r} H(0,0,0) + c_1\right)\right\}. \tag{B.13}$$

To minimize the value of $M_\lambda \lambda$, we first choose the constant ℓ so as to minimize the maximum appearing in (B.13); it suffices to choose it so that the two maximands are equal, because the first is increasing in ℓ while the second is decreasing. This is achieved by setting

$$\ell = \frac{1}{\lambda} \ln\left(1 + \frac{1}{rJ} H(0,0,0) + \frac{c_1}{J}\right) \Rightarrow M_\lambda = J e^{\lambda\ell} = J + \frac{1}{r} H(0,0,0) + c_1.$$

We therefore have

$$M_\lambda \lambda = \max\left\{\frac{2}{\sigma^2\lambda} H(0,0,0), c_3\right\} + \frac{\lambda}{r} H(0,0,0) + \lambda c_1.$$

The minimum possible value of the right-hand side is attained by setting

$$\lambda = \min\left\{\frac{\sqrt{2rH(0,0,0)}}{\sigma\sqrt{H(0,0,0) + rc_1}}, \frac{2}{\sigma^2 c_3} H(0,0,0)\right\}, \tag{B.14}$$

and its minimum value is given by $M_\lambda \lambda = M$ where M is defined in (4.11).

Put together (B.9), and (B.11) to get (4.10). □

B.3. Proof of result from Section 4.4

Proof of Lemma 4.11. Estimate (4.12) follows from Lemmas 4.4, 4.1, 4.10, and 3.1. Note that a direct application of Lemma 4.4 would put the constant $c(\bar{\rho}, \epsilon(t))$ in place of $c(\bar{\rho}, \epsilon(0))$; however, $c(\bar{\rho}, \epsilon)$ defined in (4.5) is an increasing function of ϵ, and since Assumption 4.7 implies $\epsilon(t) \le \epsilon(0)$, we have replaced $c(\bar{\rho}, \epsilon(t))$ with $c(\bar{\rho}, \epsilon(0))$ to get an upper bound that is uniform in time.

Now we turn to estimate (4.13). By Lemma 4.4, there exists a constant $C = C(\epsilon(0), \bar{\rho}, M)$ such that

$$\left|Q^*(t_1) - Q^*(t_2)\right| \le C\left(\left|\epsilon(t_1) - \epsilon(t_2)\right| + \left\|\frac{\partial u}{\partial x}(\cdot, t_1) - \frac{\partial u}{\partial x}(\cdot, t_2)\right\|_\infty\right) + C\left(\left\|\frac{\partial^2 u}{\partial x^2}\right\|_\infty \mathbf{d}_1(m(t_1), m(t_2)) + \left|\int_{\mathcal{D}} \mathrm{d}(m(t_1) - m(t_2))(x)\right|\right)$$

Now suppose $m_0 \in \mathscr{M}_\alpha$. Appealing to Lemma 3.1 and also Assumption 4.7, we have

$$\left|Q^*(t_1) - Q^*(t_2)\right| \le C\left(|t_1 - t_2| + \left\|\frac{\partial u}{\partial x}\right\|_{\mathcal{C}^{\alpha,\alpha/2}} |t_1 - t_2|^{\alpha/2}\right) + C\left(\left\|\frac{\partial^2 u}{\partial x^2}\right\|_\infty \left(\|b\|_\infty + \sigma\right)|t_1 - t_2|^{1/2} + C(\sigma)\left(\|m_0\|_{\mathscr{M}_{\alpha/2}} + \|b\|_\infty\right)|t_1 - t_2|^{\alpha/2}\right)$$

for any $|t_1 - t_2| \le 1$. Here $b = \frac{\partial H}{\partial a}\left(\epsilon, Q^*, \frac{\partial u}{\partial x}\right)$. By Lemmas 4.1 and 4.10 together with (4.12), we deduce there exists $C = C(\bar{\rho}, \epsilon(0), M)$ such that $\|b\|_\infty \le C$. We deduce that there exists $C = C(\bar{\rho}, \epsilon(0), M, \sigma, \|m_0\|_{\mathscr{M}_{\alpha/2}})$ such that

$$\left|Q^*(t_1) - Q^*(t_2)\right| \le C\left(\left\|\frac{\partial u}{\partial x}\right\|_{\mathcal{C}^{\alpha,\alpha/2}} + \left\|\frac{\partial^2 u}{\partial x^2}\right\|_\infty + 1\right)|t_1 - t_2|^{\alpha/2}, \quad |t_1 - t_2| \le 1,$$

and since Q^* is bounded according to (4.12), Equation (4.13) follows. □

B.4. *Proofs of results from Section 4.5*

Proof of Lemma 4.12. We begin by taking $u_0 = 0$. First we let $g(x,t) = e^{rt} f(x,t)$ and consider

$$\frac{\partial v}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2} = g, \ \forall x \in \mathcal{D}, t > 0; \ v(0,t) = 0 \ \forall t > 0; \ \ v(x,0) = 0 \ \forall x \in \mathcal{D}. \tag{B.15}$$

By [29, Theorem IV.6.1], (B.15) is uniquely solvable in $\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathcal{D}} \times [0,T])$ for arbitrary $T > 0$. Also, by the maximum principle, we have

$$\left|v(x,t)\right| \le \frac{1}{r} e^{rt} \|f\|_{\mathcal{C}^{0,0}} \ \forall x \in \overline{\mathcal{D}}, t \in \left[0, \infty\right).$$

To see this, first let $\tilde{v}(x,t) = v(x,t) - \frac{e^{rt}-1}{r}\|f\|_{\mathcal{C}^{0,0}}$ and observe that

$$\frac{\partial \tilde{v}}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 \tilde{v}}{\partial x^2} \le 0, \ \ \tilde{v}(0,t) \le 0, \ \ \tilde{v}(x,0) \le 0.$$

By the maximum principle, $\tilde{v} \le 0$, which implies $v(x,t) \le \frac{1}{r} e^{rt} \|f\|_{\mathcal{C}^{0,0}}$. The opposite inequality is similarly proved.

Now we let $u(x,t) = e^{-rt} v(x,t)$. Then u satisfies (4.14) and

$$\|u\|_{\mathcal{C}^{0,0}(\overline{\mathcal{D}}\times[0,\infty))} \le \frac{1}{r} \|f\|_{\mathcal{C}^{0,0}}. \tag{B.16}$$

Moreover, appealing again to [29, Theorem IV.6.1], we have an estimate

$$\left[\frac{\partial^2 u}{\partial x^2}\right]_{\alpha,\alpha/2} + \left[\frac{\partial u}{\partial t}\right]_{\alpha,\alpha/2} \le C(\sigma)\left([f]_{\alpha,\alpha/2} + r\,[u]_{\alpha,\alpha/2}\right), \tag{B.17}$$

where $C(\sigma)$ does not depend on T. By interpolation, see [29, Lemma II.3.2], we can find a constant $C(\alpha)$ such that for arbitrary $\delta > 0$ we have

$$[u]_{\alpha,\alpha/2} \le C(\alpha)\left(\delta^{-\alpha}\|u\|_{\mathcal{C}^{0,0}(\overline{\mathcal{D}})} + \delta^2\left(\left[\frac{\partial^2 u}{\partial x^2}\right]_{\alpha,\alpha/2} + \left[\frac{\partial u}{\partial t}\right]_{\alpha,\alpha/2}\right)\right). \tag{B.18}$$

Combining (B.16), (B.17), and (B.18) with δ a sufficiently small multiple of $r^{-1/2}$, we deduce that (4.16) holds for $u_0 = 0$.

Now suppose $f = 0$ and let $u_0 \in \mathcal{C}^{2+\alpha}(\overline{\mathcal{D}})$ be given. Then appealing to [29, Theorem IV.5.1], (4.14) is uniquely solvable, and moreover by the potential estimates from [29, Section IV.2] we have

$$\left[\frac{\partial^2 u}{\partial x^2}\right]_{\alpha,\alpha/2} + \left[\frac{\partial u}{\partial t}\right]_{\alpha,\alpha/2} \le C(\sigma)\left(\|u_0\|_{\mathcal{C}^{2+\alpha}(\overline{\mathcal{D}})} + r\,[u]_{\alpha,\alpha/2}\right),$$

where again $C(\sigma)$ does not depend on time. Using the maximum principle, we get $\|u\|_0 \le \|u_0\|_0$. Arguing as before, we deduce (4.16) for $f = 0$.

The general case now follows from linearity.

B.5. Proofs of results from Section 4.6

Proof of Lemma 4.13. Let $f = f(x,t) = H\left(\epsilon(t), Q^*(t), \frac{\partial u}{\partial x}(x,t)\right)$. From Lemma 4.12 we have

$$\|u\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathcal{D}}\times[0,T])} \le C(\sigma,\alpha)\left([f]_{\alpha,\alpha/2} + r^{\frac{\alpha}{2}}\|f\|_0 + C_\alpha + r^{1+\frac{\alpha}{2}} c_1\right). \tag{B.19}$$

We now estimate f in $\mathcal{C}^{\alpha,\alpha/2}$. First, because H is decreasing in all variables, we (again) deduce

$$0 \le f(x,t) \le H(0,0,0).$$

Because H is locally Lipschitz by Lemma 4.1, and because ϵ, Q^*, and $\frac{\partial u}{\partial x}$ are bounded with estimates given in Assumption 4.7, Lemma 4.10 and Lemma 4.11, we have a constant $C = C(\bar{\rho}, \epsilon(0), \sigma, M, \alpha)$ such that

$$\| f \|_{\mathcal{C}^{\alpha,\alpha/2}} \le C \left(1 + \| Q^* \|_{\mathcal{C}^{\alpha/2}} + \left\| \frac{\partial u}{\partial x} \right\|_{\mathcal{C}^{\alpha,\alpha/2}} \right),$$

where $\|\epsilon\|_{\mathcal{C}^{\alpha/2}}$ is also estimated using Assumption 4.7. Using Lemma 4.11 and interpolation on Hölder spaces, we see that for an arbitrary $\delta > 0$, there exists $C_\delta = C(\delta, \bar{\rho}, \epsilon(0), \sigma, M, \|m_0\|_{\mathscr{M}_{\alpha/2}}, \alpha)$ such that

$$\| f \|_{\mathcal{C}^{\alpha,\alpha/2}} \le \delta \|u\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}} + C_\delta.$$

Taking $\delta > 0$ small enough, (B.19) becomes

$$\|u\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathcal{D}}\times[0,T])} \le C(\sigma, \alpha) \left(C_\delta + r^{\frac{\alpha}{2}} H(0,0,0) + C_\alpha + r^{1+\frac{\alpha}{2}} c_1 \right),$$

which proves (4.17). □

Before getting to the proof of Theorem 4.15, we establish the following lemma:

Lemma B.1. *Let $Q^* \in \mathcal{C}^{\alpha/2}\left([0,T]; [0,\infty)\right)$ be given, and let Assumptions 4.7 and 4.8 hold. Then there exists a unique solution u to the PDE*

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + H\left(\epsilon(t), Q^*(t), \frac{\partial u}{\partial x}\right) - ru = 0, \quad u(0,t) = 0, \quad u(x,T) = u_T(x). \tag{B.20}$$

This solution satisfies $\frac{\partial u}{\partial x} \ge 0$ and the a priori estimate

$$\|u\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}} \le C\left(\sigma, r, \alpha, H(0,0,0)\right) \left(1 + \| Q^* \|_{\mathcal{C}^{\alpha/2}} + \|u_T\|_{\mathcal{C}^{2+\alpha}}\right). \tag{B.21}$$

Proof. As above we set $p_\infty := \lim_{q\to\infty} P(q) < 0$. Fix a $\mathcal{C}^\infty$ function $\psi : \mathbb{R} \to (p_\infty, \infty)$ such that $p(a) = a$ for all $a \ge 0$. Define $X = \mathcal{C}^{2,1}\left(\overline{\mathcal{D}} \times [0,T]\right)$. Let $v \in X$ and $\lambda \in [0,1]$. By Lemma 4.12, Assumption 4.8, and the local Lipschitz property of H, we get a unique solution u to the equation

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + \lambda H\left(\epsilon(t), Q^*(t), \psi\left(\frac{\partial v}{\partial x}\right)\right) - ru = 0, \quad u(0,t) = 0, \quad u(x,T) = \lambda u_T(x), \tag{B.22}$$

and u satisfies

$$\|u\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}} \le C\left(\| Q^* \|_\infty, \left\| \frac{\partial v}{\partial x} \right\|_\infty\right) \left(1 + \| Q^* \|_{\mathcal{C}^{\alpha/2}} + \left\| \frac{\partial v}{\partial x} \right\|_{\mathcal{C}^{\alpha,\alpha/2}}\right).$$

This defines a map $\mathcal{T} : X \times [0,1] \to X$. We claim that $\mathcal{T}$ is continuous and compact. Suppose $\{v_n, \lambda_n\}$ is a bounded sequence in $X \times [0,1]$ and let $u_n = \mathcal{T}(v_n, \lambda_n)$. Then $\{u_n\}$ is bounded in $\mathcal{C}^{2+\alpha,1+\alpha/2}$, which is compactly embedded in X, so it has a subsequence that converges to some u in X. To conclude that $\mathcal{T}$ is both continuous and compact, it is enough to show that whenever $(v_n, \lambda_n) \to (v, \lambda)$ in $X \times [0,1]$, then $u = \mathcal{T}(v, \lambda)$. But this can be deduced from plugging v_n, λ_n into (B.22) in place of v, λ, then passing to the limit using the local Lipschitz property of H.

Notice that $\mathcal{T}(v, 0) = 0$. To apply the Leray-Schauder fixed point theorem, it remains to find an a priori bound on solutions to the fixed point equation $u = \mathcal{T}(u, \lambda)$. Note that for any such fixed point, $w = \frac{\partial u}{\partial x}$ satisfies, in a weak sense,

$$\frac{\partial w}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 w}{\partial x^2} + \lambda \frac{\partial H}{\partial a}\left(\epsilon(t), Q^*(t), \psi\left(\frac{\partial u}{\partial x}\right)\right)\psi'\left(\frac{\partial u}{\partial x}\right)\frac{\partial w}{\partial x} - rw = 0, \quad w(x,T) = \lambda u_T'(x).$$

Since $u_T' \geq 0$, by the maximum principle we deduce $\frac{\partial u}{\partial x} = w \geq 0$. It follows that u satisfies

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + \lambda H\left(\epsilon(t), Q^*(t), \frac{\partial u}{\partial x}\right) - ru = 0, \quad u(0,t) = 0, \quad u(x,T) = \lambda u_T(x),$$

Lemma 4.10 establishes an a priori bound on u; combined with Lemma 4.12 and using interpolation, we deduce that (B.21) holds for any u satisfying $u = \mathcal{T}(u, \lambda)$. By the Leray-Schauder fixed point theorem [16, Theorem 11.6], there exists $u \in X$ such that $u = \mathcal{T}(u, 1)$, which means u is a solution to (B.20). Uniqueness follows from the maximum principle by standard arguments. □

Proof of Theorem 4.15. Set X to be the set of all $(v, Q) \in \mathcal{C}^{2,1}\left(\overline{\mathcal{D}} \times [0,T]\right) \times \mathcal{C}^0([0,T])$ such that $\frac{\partial v}{\partial x} \geq 0$ and $Q \geq 0$, and define $\mathcal{T} : X \times [0,1] \to X$ as follows. Let $(v, Q) \in X, \lambda \in [0,1]$. From Lemma 4.1 we know that the function $\frac{\partial H}{\partial a}\left(\epsilon(t), Q(t), \frac{\partial v}{\partial x}\right)$ is bounded and continuous with

$$\left\|\frac{\partial H}{\partial a}\left(\epsilon(t), Q(t), \frac{\partial v}{\partial x}\right)\right\|_\infty \leq C\left(\epsilon(0), \|Q\|_\infty, \left\|\frac{\partial v}{\partial x}\right\|_\infty\right)$$

By Lemma 3.1, there exists a unique solution m satisfying

$$\frac{\partial m}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 m}{\partial x^2} + \frac{\partial}{\partial x}\left(\lambda \frac{\partial H}{\partial a}\left(\epsilon(t), Q(t), \frac{\partial v}{\partial x}\right) m\right) = 0, \; m|_{x=0} = 0, \; m|_{t=0} = m_0, \tag{B.23}$$

and moreover we have Hölder estimates (3.4) and (3.12). Now by Lemma 4.4 we can define $Q^*(t)$ by

$$Q^*(t) = -\int_0^\infty \lambda \frac{\partial H}{\partial a}\left(\epsilon(t), Q^*(t), \frac{\partial v}{\partial x}\right) \mathrm{d}m(t), \tag{B.24}$$

and combining (4.5), (4.6), (3.4) and (3.12), we have

$$\|Q^*\|_{\mathcal{C}^{\alpha/2}([0,T])} \leq C\left(\epsilon(0), \|Q\|_\infty, \left\|\frac{\partial v}{\partial x}\right\|_\infty, \|m_0\|_{\mathscr{M}_{\alpha/2}}, \sigma, \alpha\right)\left(\|v\|_{\mathcal{C}^{2,1}} + 1\right). \tag{B.25}$$

Setting $f(x,t) = \lambda H\left(\epsilon(t), Q^*(t), \frac{\partial v}{\partial x}\right)$, we have, as in the proof of Lemma 4.13,

$$\|f\|_{\mathcal{C}^{\alpha,\alpha/2}} \leq C\left(\|Q^*\|_\infty, \left\|\frac{\partial v}{\partial x}\right\|_\infty\right)\left(1 + \|Q^*\|_{\mathcal{C}^{\alpha/2}} + \left\|\frac{\partial v}{\partial x}\right\|_{\mathcal{C}^{\alpha,\alpha/2}}\right).$$

Thus, by Lemma B.1 there exists a unique solution u of

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + \lambda H\left(\epsilon(t), Q^*(t), \frac{\partial u}{\partial x}\right) - ru = 0, \; u|_{x=0} = 0, \; u|_{t=T} = \lambda u_T \tag{B.26}$$

satisfying (B.21), which in this case can be written

$$\|u\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}} \leq C\left(\epsilon(0), \|Q\|_\infty, \|v\|_{\mathcal{C}^{2,1}}, \|m_0\|_{\mathscr{M}_{\alpha/2}}, \sigma, \alpha\right). \tag{B.27}$$

Then we set $\mathcal{T}(v, Q, \lambda) = (u, Q^*) \in X$. We need to show that $\mathcal{T}$ is continuous and compact. Suppose $\{(v_n, Q_n, \lambda_n)\}$ is a sequence in $X \times [0,1]$, and let $(u_n, Q_n^*) = \mathcal{T}(v_n, Q_n, \lambda_n)$. Note that by (B.25) and (B.27), (u_n, Q_n^*) must have a subsequence converging to $(u, Q^*) \in X$, because $\mathcal{C}^{2+\alpha,1+\alpha/2} \times \mathcal{C}^{\alpha/2}$ is compactly embedded in $\mathcal{C}^{2,1} \times \mathcal{C}^0$. We now show that if $(v_n, Q_n, \lambda_n) \to (v, Q, \lambda)$, then $(u, Q^*) = \mathcal{T}(v, Q, \lambda)$. First let m_n be the solution of (B.23) corresponding to (v_n, Q_n, λ_n). By Lemma 3.1 we have that m_n is uniformly Hölder in the $\mathbf{d}_1$ metric, hence by passing to a subsequence it converges to some m in $\mathcal{C}\left([0,T]; \mathcal{M}_{1,+}\right)$. Since $\frac{\partial H}{\partial a}$ is locally Lipschitz, we have that $\lambda_n \frac{\partial H}{\partial a}\left(\epsilon(t), Q_n(t), \frac{\partial v_n}{\partial x}\right) \to \lambda \frac{\partial H}{\partial a}\left(\epsilon(t), Q(t), \frac{\partial v}{\partial x}\right)$ uniformly. Combining these facts we deduce that m is really the solution to (B.23) and Q^* is the solution of (B.24). Finally, we deduce that u is really the solution to (B.26) by taking the corresponding equation for u_n and passing to the limit. We have thus proved that $\mathcal{T}$ is continuous and compact.

It remains to show there exists a constant C such that whenever $\mathcal{T}(u, Q^*, \lambda) = (u, Q^*)$, then

$$\left\|(u, Q^*)\right\|_X \leq C.$$

But this is a consequence of Lemmas 4.11 and 4.13, since λH and λu_T satisfy all the same estimates as H and u_T. Now we can apply the Leray-Schauder fixed point theorem, which says that there exists (u, Q^*) such that $\mathcal{T}(u, Q^*, 1) = (u, Q^*)$. Letting m now be defined by solving (4.8)(ii), we deduce that (u, m) solves the system (4.8). The regularity of this solution follows by once more appealing to Lemmas 3.1 and 4.13. □

Appendix C. Proof of the integral estimate used in Section 5

The following proof is more or less the same as that of [21, Lemma 2.1]. We include it for completeness.

Proof of Lemma 5.7. Set $h(t) = Bf(t) + g(t)$, so that (5.14) reads simply

$$f(t_1) \le Af(t_0) + \int_{t_0}^{t_1} (t_1 - s)^{-1/2} h(s)\,\mathrm{d}s \quad \forall 0 \le t_0 \le t_1 \le t_0 + \delta \tag{C.1}$$

For arbitrary $t > 0$ let $n = \left\lfloor \frac{t}{\delta} \right\rfloor$. Use (C.1) $n + 1$ times to get

$$f(t) \le A^{n+1} f(0) + \sum_{j=0}^{n} A^j \int_{(t-(j+1)\delta)_+}^{t-j\delta} (t - j\delta - s)^{-1/2} h(s)\,\mathrm{d}s, \tag{C.2}$$

where $s_+ := \max\{s, 0\}$. Note that

$$t - (j+1)\delta < s \le t - j\delta \Rightarrow j = \left\lfloor \frac{t-s}{\delta} \right\rfloor$$

So we define $\phi(s) = \left(s - \left\lfloor \frac{s}{\delta} \right\rfloor \delta\right)^{-1/2}$. Then (C.2) implies

$$f(t) \le A^{\frac{t}{\delta}+1} f(0) + \sum_{j=0}^{n} \int_{(t-(j+1)\delta)_+}^{t-j\delta} A^{\frac{t-s}{\delta}} \phi(t-s) h(s)\,\mathrm{d}s = A^{\frac{t}{\delta}+1} f(0) + \int_0^t A^{\frac{t-s}{\delta}} \phi(t-s) h(s)\,\mathrm{d}s. \tag{C.3}$$

Let $\lambda > \frac{1}{\delta}\ln(A)$ and set $\kappa = \lambda - \frac{1}{\delta}\ln(A) > 0$. Multiply (C.3) by $e^{-\lambda t}$, then integrate from 0 to T to get

$$\begin{aligned} \int_0^T e^{-\lambda t} f(t)\,\mathrm{d}t &\le \frac{A}{\kappa} f(0) + \int_0^T \int_0^t e^{-\kappa(t-s)} \phi(t-s) e^{-\lambda s} h(s)\,\mathrm{d}s\,\mathrm{d}t \\ &= \frac{A}{\kappa} f(0) + \int_0^T \int_0^{T-s} e^{-\kappa t} \phi(t) e^{-\lambda s} h(s)\,\mathrm{d}t\,\mathrm{d}s. \end{aligned} \tag{C.4}$$

We now observe that

$$\begin{aligned}\int_0^\infty e^{-\kappa t}\phi(t)\,\mathrm{d}t &= \sum_{n=0}^\infty \int_{n\delta}^{(n+1)\delta} e^{-\kappa t}(t-n\delta)^{-1/2}\,\mathrm{d}t \\ &= \sum_{n=0}^\infty e^{-n\kappa\delta}\int_0^\delta e^{-\kappa t}t^{-1/2}\,\mathrm{d}t \\ &\le \frac{1}{1-e^{-\kappa\delta}}\int_0^\delta t^{-1/2}\,\mathrm{d}t \\ &= \frac{2\delta^{1/2}}{1-e^{-\kappa\delta}} = \frac{2\delta^{1/2}}{1-Ae^{-\lambda\delta}}\end{aligned} \tag{C.5}$$

Applying (C.5) to (C.4), we get

$$\int_0^T e^{-\lambda t}f(t)\,\mathrm{d}t \le \frac{A}{\kappa}f(0) + \frac{2\delta^{1/2}}{1-Ae^{-\lambda\delta}}\int_0^T e^{-\lambda s}\big(Bf(s)+g(s)\big)\,\mathrm{d}s,$$

which implies (5.15). □

C.1. Proofs of results from Section 5.1

C.1.1. Proofs of results from Section 5.1.1

Proof of Proposition 5.1. Step 1: For a fixed $x>0$ set $z=d(x)/2$. We have chosen z so that $x-y\ge z$ for all $y\in[0,z]$. Integrate by parts n times to get

$$\frac{\partial^n u}{\partial x^n}(x,t) = \int_0^z \frac{\partial^n S}{\partial x^n}(x-y,t)u_0(y)\,\mathrm{d}y + \sum_{j=1}^n \frac{\partial^{n-j}S}{\partial x^{n-j}}(x-z,t)u_0^{(j-1)}(z) + \int_z^\infty S(x-y,t)u_0^{(n)}(y)\,\mathrm{d}y.$$

Now multiply by z^n:

$$z^n\frac{\partial^n u}{\partial x^n}(x,t) = \frac{1}{z}\int_0^z z^{n+1}\frac{\partial^n S}{\partial x^n}(x-y,t)u_0(y)\,\mathrm{d}y + \sum_{j=0}^{n-1} z^{n-j}\frac{\partial^{n-j-1}S}{\partial x^{n-j-1}}(x-z,t)z^j u_0^{(j)}(z) + \int_z^\infty S(x-y,t)z^n u_0^{(n)}(y)\,\mathrm{d}y.$$

By Corollary 3.6 and the fact that $S(x-\cdot,t)$ is a density, we get

$$\left|z^n \frac{\partial^n u}{\partial x^n}(x,t)\right| \le m_n \|u_0\|_0 + \sum_{j=0}^{n-1} m_{n-j-1} \left\|d^j u_0^{(j)}\right\|_0 + \left\|d^n u_0^{(n)}\right\|_0.$$

Taking the supremum over all x, we get

$$\left\|d^n \frac{\partial^n u}{\partial x^n}(\cdot,t)\right\|_0 \le 2^n \left(m_n \|u_0\|_0 + \sum_{j=0}^{n-1} m_{n-j-1} \left\|d^j u_0^{(j)}\right\|_0 + \left\|d^n u_0^{(n)}\right\|_0\right). \tag{C.6}$$

Step 2: We proceed similarly to estimate v, but first we define

$$F(y,s) := \int_0^y f(\xi,s)\,\mathrm{d}\xi.$$

By integration by parts we have

$$v(x,t) = \int_0^t \int_0^\infty \frac{\partial S}{\partial x}(x-y,t-s) F(y,s)\,\mathrm{d}y\,\mathrm{d}s.$$

Calculating as before, we get

$$\begin{aligned} z^n \frac{\partial^n v}{\partial x^n}(x,t) &= \int_0^t \frac{1}{z} \int_0^z z^{n+1} \frac{\partial^{n+1} S}{\partial x^{n+1}}(x-y,t-s) F(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &+ \int_0^t \sum_{j=0}^{n-1} z^{n-j} \frac{\partial^{n-j} S}{\partial x^{n-j}}(x-z,t-s) z^j \frac{\partial^j F}{\partial x^j}(z,s)\,\mathrm{d}s + \int_0^t \int_z^\infty \frac{\partial S}{\partial x}(x-y,t) z^n \frac{\partial^n F}{\partial x^n}(y,s)\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{C.7}$$

Now applying Corollary 3.6 in (C.7), we get

$$\begin{aligned} \left|z^n \frac{\partial^n v}{\partial x^n}(x,t)\right| \le \left(m_{n+1,\sigma} + m_{n,1}\right) \int_0^t (t-s)^{-1/2} \sup_{0\le y\le 1} \left|F(y,s)\right| \mathrm{d}s \\ + \sum_{j=1}^n \int_0^t m_{n-j,\sigma} (t-s)^{-1/2} \left\|d^j \frac{\partial^{j-1} f}{\partial x^{j-1}}(\cdot,s)\right\|_0. \end{aligned}$$

Thus,

$$\left\| d^n \frac{\partial^n v}{\partial x^n}(\cdot,t)\right\|_0 \le 2^n \left(m_{n+1,\sigma}+m_{n,1}\right)\int_0^t (t-s)^{-1/2} \sup_{0\le y\le 1}\left|\int_0^y f(\xi,s)\,\mathrm{d}\xi\right| \mathrm{d}s$$

$$+ 2^n \sum_{j=1}^{n}\int_0^t m_{n-j,\sigma}(t-s)^{-1/2}\left\| d^j \frac{\partial^{j-1} f}{\partial x^{j-1}}(\cdot,s)\right\|_0 . \quad \text{(C.8)}$$

Step 3: Finally,

$$d(x)^n \frac{\partial^n w}{\partial x^n}(x,t) = -2\int_0^t d(x)^n \frac{\partial^{n+1} S}{\partial x^{n+1}}(x,t-s)\psi(s)\,\mathrm{d}s.$$

By induction we can establish a formula

$$\frac{\partial^{n+1} S}{\partial x^{n+1}}(x,t) = S(x,t)\sum_{j=0}^{\left\lfloor \frac{n+1}{2}\right\rfloor} (\sigma^2 t)^{j-n-1} c_{n+1,j}\, x^{n+1-2j},$$

where $c_{n,j}$ are coefficients defined recursively with respect to n. Multiply by x^n to get, using (3.5),

$$x^n \frac{\partial^{n+1} S}{\partial x^{n+1}}(x,t) = (2\pi)^{-1/2}(\sigma^2 t)^{-3/2} x \exp\left\{-\frac{|x|^2}{2\sigma^2 t}\right\}\sum_{j=0}^{\left\lfloor \frac{n+1}{2}\right\rfloor} c_{n+1,j}\left(\frac{x^2}{\sigma^2 t}\right)^{n-j}$$

and thus

$$\int_0^\infty \left| x^n \frac{\partial^{n+1} S}{\partial x^{n+1}}(x,t)\right| \mathrm{d}t \le \sum_{j=0}^{\left\lfloor \frac{n+1}{2}\right\rfloor} \left|c_{n+1,j}\right| \int_0^\infty (2\pi)^{-1/2}(\sigma^2 t)^{-3/2} x \left(\frac{x^2}{\sigma^2 t}\right)^{n-j} \exp\left\{-\frac{|x|^2}{2\sigma^2 t}\right\} \mathrm{d}t.$$

Use the substitution $t = \frac{x^2}{\sigma^2 s}$ to get

$$\int_0^\infty (2\pi)^{-1/2}(\sigma^2 t)^{-3/2} x \left(\frac{x^2}{\sigma^2 t}\right)^{n-j} \exp\left\{-\frac{|x|^2}{2\sigma^2 t}\right\} \mathrm{d}t$$

$$= \sigma^{-2}\int_0^\infty (2\pi)^{-1/2} s^{n-j-1/2} \exp\left\{-\frac{s}{2}\right\} \mathrm{d}s < \infty.$$

We deduce that for some constant ι_n, not depending on x,

$$2\int_0^\infty \left| x^n \frac{\partial^{n+1} S}{\partial x^{n+1}}(x,t) \right| \mathrm{d}t \le \iota_n,$$

and thus

$$\left\| d^n \frac{\partial^n w}{\partial x^n}(\cdot,t) \right\|_0 \le \iota_n \sup_{0\le s\le t} \left|\psi(s)\right|. \tag{C.9}$$

The estimates (C.6), (C.8), and (C.9) result in (5.6). □

Proof of Theorem 5.2. Define

$$u_1(x,t) = \int_0^\infty S(x-y,t)u_0(y)\,\mathrm{d}y,$$

$$u_2(x,t) = \int_0^t \int_0^\infty S(x-y,t-s)f(y,s)\,\mathrm{d}y\,\mathrm{d}s,$$

$$u_3(x,t) = -2\int_0^t \frac{\partial S}{\partial x}(x,t-s)\big(\psi(s) - u_1(0,s) - u_2(0,s)\big)\,\mathrm{d}s.$$

Then by classical arguments (cf. [29, Section IV.1]) we see that $u = u_1 + u_2 + u_3$ is a solution to (5.7). By the maximum principle, this solution is unique.

By Proposition 5.1, we have

$$\begin{aligned} \left\|u_1(\cdot,t)\right\|_n &\le M_n \|u_0\|_n, \\ \left\|u_2(\cdot,t)\right\|_n &\le M_n \int_0^t (t-s)^{-1/2} \|f(\cdot,s)\|^*_{n-1,1}\,\mathrm{d}s, \\ \left\|w(\cdot,t)\right\|_n &\le M_n \sup_{0\le s\le t}\left|\psi(s) - u_1(0,s) - u_2(0,s)\right|. \end{aligned} \tag{C.10}$$

It also follows from Proposition 5.1 that

$$\begin{aligned} &\sup_{0\le s\le t}\left|u_1(0,s)\right| \le M_n\|u_0\|_n, \\ &\sup_{0\le s\le t}\left|u_2(0,s)\right| \le \sup_{0\le s\le t} M_n \int_0^s (s-s')^{-1/2}\|f(\cdot,s')\|^*_{n-1,1}\,\mathrm{d}s' = 2M_n t^{1/2} \sup_{0\le s\le t}\|f(\cdot,s)\|^*_{n-1,1}. \end{aligned} \tag{C.11}$$

Combining (C.10) and (C.11), then modifying the constant M_n, we deduce (5.8). □

C.1.2. Proofs of results from Section 5.1.2

Let (u, m) be the solution to the finite or infinite time-horizon problem, i.e. to System (4.8) or (1.6). For a finite time-horizon we assume $u(x, T) = u_T(x)$ satisfies Assumption 4.8. In addition, we will impose that $\|u_T\|_{\mathcal{C}^n} \le \tilde{C}_n$ for each $n = 1, 2, \dots$. (For $n = 1, 2$, this is not a new assumption. For larger n, it is always possible to impose this restriction at the same time as Assumption 4.8.) We again take Assumption 4.7, and we denote $\epsilon = \epsilon(0)$.

If H is $n + 1$ times differentiable, then, under Assumption 4.18, by Corollary 4.2 we have

$$C_\ell := \max_{0 \le \tilde{\epsilon} \le \epsilon, 0 \le Q \le \bar{Q}, 0 \le a \le M} \left| \frac{\partial^{\ell+1} H}{\partial a^{\ell+1}} (\tilde{\epsilon}, Q, a) \right| < \infty \quad \forall \ell \le n, \tag{C.12}$$

where $\bar{Q}$ is given by (4.19), M is given in Lemma 4.10, and c_2 is the constant from Assumption 4.8 and can be made arbitrarily small. In particular, by Corollary 4.6, we have that C_0 can be made arbitrarily close to $\bar{Q}$. By the a priori bounds proved in Section 4 (see Theorem 4.16), we have the following point-wise bound:

$$\left| \frac{\partial^{\ell+1} H}{\partial a^{\ell+1}} \left(\epsilon, Q^*(t), \frac{\partial u}{\partial x} \right) \right| \le C_\ell.$$

Proposition C.1. *Let (u, m) be the solution to the mean field games system on a finite or infinite time horizon T, i.e. either of System (4.8) or (1.6). Suppose (5.9) holds. Then for any n such that H is $n + 1$ times differentiable, we have*

$$\sup_{t \in [0,T]} \left\| \frac{\partial u}{\partial x} (\cdot, t) \right\|_n \le B_n(r) \tag{C.13}$$

where $B_n(r)$ is a decreasing function of r that depends on the constants C_ℓ for $\ell = 0, 1, \dots, n$.

Proof. Assume first that (u, m) solves the finite horizon problem. We proceed by induction. In the first step we prove the base case $n = 1$, and in the second step we prove the inductive step. In the final step we extend the result to the infinite-horizon case. Note that, by taking c_2 small enough in (C.12), the condition (5.9) implies

$$r > \max\left\{ (2C_0 M_n)^2, 1 \right\} \ln(2M_n).$$

Step 1: Define

$$w(x, t) = e^{rt} \frac{\partial u}{\partial x} (x, T - t), \qquad f(x, t) = e^{rt} \frac{\partial}{\partial x} \left(H \left(\epsilon(T - t), Q^*(T - t), \frac{\partial u}{\partial x} (x, T - t) \right) \right).$$

Then w satisfies

$$\frac{\partial w}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial x^2} + f(x, t).$$

We first calculate

$$\begin{aligned}
&\left|\int_0^x f(y,t)\,\mathrm{d}y\right| \\
&= e^{rt}\left|H\left(\epsilon(T-t), Q^*(T-t), \frac{\partial u}{\partial x}(x,T-t)\right) - H\left(\epsilon(T-t), Q^*(T-t), \frac{\partial u}{\partial x}(0,T-t)\right)\right| \\
&\le 2e^{rt}H(0,0,0),
\end{aligned} \tag{C.14}$$

using the fact that H is decreasing in all its variables (Lemma 4.1). Next, since

$$f(x,t) = \frac{\partial H}{\partial a}\left(\epsilon(T-t), Q^*(T-t), \frac{\partial u}{\partial x}(x,T-t)\right)\frac{\partial w}{\partial x}(x,t),$$

we have

$$\left|d(x)f(x,t)\right| \le C_0\left\|w(\cdot,t)\right\|_1. \tag{C.15}$$

By (C.14) and (C.15), we deduce

$$\left\|f(\cdot,t)\right\|_{0,1}^* \le C_0\left\|w(\cdot,t)\right\|_1 + 2e^{rt}H(0,0,0).$$

We also know that $\left|w(0,t)\right| \le Me^{rt}$. Now we apply Theorem 5.2 to get

$$\left\|w(\cdot,t)\right\|_1 \le M_1\left(\left\|w(\cdot,t_0)\right\|_1 + (t-t_0)^{1/2}C_0\sup_{t_0\le s\le t}\left\|w(\cdot,s)\right\|_1 + A_1e^{rt}\right)$$

for all $0 \le t_0 \le t \le t_0+1$, where

$$A_1 := 2H(0,0,0) + M,$$

which can be made arbitrarily close to $2H(0,0,0)+M$. Set $\delta = \min\left\{(2C_0M_1)^{-2}, 1\right\}$. Then for any $0 \le t_0 \le t \le t_0+\delta$, we deduce

$$\sup_{t_0\le s\le t}\left\|w(\cdot,s)\right\|_1 \le 2M_1\left(\left\|w(\cdot,t_0)\right\|_1 + A_1e^{rt}\right). \tag{C.16}$$

By using (C.16) repeatedly, we deduce

$$\left\|w(\cdot,t)\right\|_1 \le (2M_1)^{\left\lfloor \frac{t}{\delta}\right\rfloor+1}\left\|w(\cdot,0)\right\|_1 + \sum_{j=0}^{\left\lfloor \frac{t}{\delta}\right\rfloor}(2M_1)^{j+1}A_1e^{r(t-j\delta)}$$

$$= (2M_1)^{\left\lfloor \frac{t}{\delta}\right\rfloor+1}\left\|u_T'\right\|_1 + 2M_1A_1e^{rt}\frac{1-(2M_1e^{-r\delta})^{\left\lfloor \frac{t}{\delta}\right\rfloor+1}}{1-2M_1e^{-r\delta}}.$$

We use the assumption

$$r > \frac{\ln(2M_1)}{\delta} = \max\left\{(2C_0M_1)^2, 1\right\}\ln(2M_1)$$

and divide by e^{rt} to deduce

$$\left\|\frac{\partial u}{\partial x}(\cdot, T-t)\right\|_1 \le 2M_1\|u_T\|_1 + \frac{2M_1A_1}{1-2M_1e^{-r(2C_0M_1)^{-2}}},$$

and since $\left\|u_T'\right\|_1 \le \|u_T\|_{C^2} \le \tilde{C}_2$ we deduce

$$\sup_{t\in[0,T]}\left\|\frac{\partial u}{\partial x}(\cdot,t)\right\|_1 \le 2M_1\tilde{C}_2 + \frac{2M_1A_1}{1-2M_1e^{-r(2C_0M_1)^{-2}}} =: B_1(r),$$

which is the base case.

Step 2: Suppose for now that (C.13) holds for $n-1$; we will prove it holds for n. By using the chain and product rules, we have

$$\frac{\partial^{m-1} f}{\partial x^{m-1}}(x,t) = e^{rt}\frac{\partial^m}{\partial x^m}\left(H\left(\epsilon(T-t), Q^*(T-t), \frac{\partial u}{\partial x}(x,T-t)\right)\right)$$

$$= e^{rt}\sum_{\ell=0}^{m-1}\sum_{1\le k_\ell<k_{\ell-1}<\cdots<k_1<k_0=m}\prod_{j=0}^{\ell-1}\binom{k_j-1}{k_{j+1}}\frac{\partial^{k_j-k_{j+1}+1}u}{\partial x^{k_j-k_{j+1}+1}}(x,T-t)\frac{\partial^{k_\ell+1}u}{\partial x^{k_\ell+1}}(x,T-t)$$

$$\times\frac{\partial^{\ell+1}H}{\partial a^{\ell+1}}\left(\epsilon(T-t), Q^*(T-t), \frac{\partial u}{\partial x}(x,T-t)\right) \quad \forall m=1,\dots,n,$$

where we interpret an empty product as equal to 1. Then using Equation (C.12) we have

$$\left|d^m(x)\frac{\partial^{m-1} f}{\partial x^{m-1}}(x,t)\right| \le$$

$$e^{rt}\sum_{\ell=0}^{m-1}\sum_{1\le k_\ell<\cdots<k_0=m}C_\ell\prod_{j=0}^{\ell-1}\binom{k_j-1}{k_{j+1}}\left|d(x)^{k_j-k_{j+1}}\frac{\partial^{k_j-k_{j+1}+1}u}{\partial x^{k_j-k_{j+1}+1}}(x,T-t)d(x)^{k_\ell}\frac{\partial^{k_\ell+1}u}{\partial x^{k_\ell+1}}(x,T-t)\right|$$

$$\le C_0\left\|w(\cdot,t)\right\|_m + e^{rt}\sum_{\ell=1}^{m-1}\sum_{1\le k_\ell<\cdots<k_0=m}C_\ell\prod_{j=0}^{\ell-1}\binom{k_j-1}{k_{j+1}}\left\|\frac{\partial u}{\partial x}(\cdot,T-t)\right\|_{k_j-k_{j+1}}\left\|\frac{\partial u}{\partial x}(\cdot,T-t)\right\|_{k_\ell}.$$

We deduce that there exists some constant $A_n(r)$, depending only on C_ℓ and $B_\ell(r)$ for $\ell \le n-1$ as well as the constant appearing in estimate (C.14), such that

$$\left\| f(\cdot,t) \right\|^*_{n-1,1} \le C_0 \left\| w(\cdot,t) \right\|_n + (A_n(r) - M) e^{rt}.$$

Since $B_\ell(r)$ is decreasing with respect to r for $\ell \le n-1$, the same holds for $A_n(r)$. We apply Theorem 5.2 again to get

$$\left\| w(\cdot,t) \right\|_2 \le M_n \left(\left\| w(\cdot,t_0) \right\|_n + (t-t_0)^{1/2} C_0 \sup_{t_0 \le s \le t} \left\| w(\cdot,s) \right\|_n + A_n(r) e^{rt} \right)$$

for all $0 \le t_0 \le t \le t_0 + 1$. We will now use the assumption (5.9), and the exactly same argument as before yields

$$\sup_{t \in [0,T]} \left\| \frac{\partial u}{\partial x}(\cdot,t) \right\|_n \le 2M_n \tilde{C}_{n+1} + \frac{2M_n A_n(r)}{1 - 2M_n e^{-r(2C_0 M_n)^{-2}}} =: B_n(r).$$

Since $A_n(r)$ is decreasing with respect to r, so is $B_n(r)$.

For the infinite horizon case, if (u^T, m^T) denotes the solution to the finite time-horizon problem, then its limit as $T \to \infty$ is the solution (u,m) to System (1.6). We deduce that (u,m) satisfies (C.13), with $[0,T]$ replaced by $\left[0,\infty\right)$. □

As a corollary, we derive (5.10) and (5.11). To prove (5.10), observe that

$$\begin{aligned}
&\frac{\partial^n}{\partial x^n}\left(\frac{\partial H}{\partial a}\left(\epsilon, Q^*(t), \frac{\partial u}{\partial x}(x,t)\right)\right) \\
&\qquad = \sum_{\ell=0}^{n-1} \sum_{1 \le k_\ell < \cdots < k_1 < k_0 = n} \prod_{j=0}^{\ell-1} \binom{k_j - 1}{k_{j+1}} \frac{\partial^{k_j - k_{j+1}+1} u}{\partial x^{k_j - k_{j+1}+1}}(x,t) \frac{\partial^{k_\ell+1} u}{\partial x^{k_\ell+1}}(x,t) \\
&\qquad\qquad \times \frac{\partial^{\ell+2} H}{\partial a^{\ell+2}}\left(\epsilon, Q^*(t), \frac{\partial u}{\partial x}(x,t)\right),
\end{aligned}$$

so that

$$\begin{aligned}
&\left| d(x)^n \frac{\partial^n}{\partial x^n}\left(\frac{\partial H}{\partial a}\left(\epsilon, Q^*(t), \frac{\partial u}{\partial x}(x,t)\right)\right)\right| \\
&\le \sum_{\ell=0}^{n-1} \sum_{1 \le k_\ell < \cdots < k_1 < k_0 = n} C_{\ell+1} \prod_{j=0}^{\ell-1} \binom{k_j - 1}{k_{j+1}} \left| d(x)^{k_j - k_{j+1}} \frac{\partial^{k_j - k_{j+1}+1} u}{\partial x^{k_j - k_{j+1}+1}}(x,t) d(x)^{k_\ell} \frac{\partial^{k_\ell+1} u}{\partial x^{k_\ell+1}}(x,t) \right| \\
&\le \sum_{\ell=0}^{n-1} \sum_{1 \le k_\ell < \cdots < k_1 < k_0 = n} C_{\ell+1} \prod_{j=0}^{\ell-1} \binom{k_j - 1}{k_{j+1}} \left\| \frac{\partial u}{\partial x}(\cdot,t) \right\|_{k_j - k_{j+1}} \left\| \frac{\partial u}{\partial x}(\cdot,t) \right\|_{k_\ell}.
\end{aligned}$$

Thus (5.10) follows from (C.13). The proof of (5.11) is similar: use the formulas (B.3) and (B.4), taking successive derivatives and applying Equation (5.10).

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