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Stochastic Galerkin Methods for Time-Dependent Radiative Transfer Equations with Uncertain Coefficients

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Abstract

The generalized polynomial chaos (gPC) method is one of the most popular method for uncertainty quantification. Being essentially a spectral approach, the gPC method exhibits the spectral convergence rate which heavily depends on the regularity of the solution in the random space. Many regularity studies have been made for stochastic elliptic and parabolic equations while regularities studies of stochastic hyperbolic equations has long been infeasible due to its intrinsic difficulties. In this paper, we investigate the impact of uncertainty on the time-dependent radiative transfer equation (RTE) with nonhomogeneous boundary conditions, which sits somewhere between hyperbolic and parabolic equations. We theoretically prove the a-priori bound of the solution, its continuity with respect to the scattering coefficient, and its regularity in the random space. These studies can serve as a building block in understanding the influence of uncertainties in the passage from hyperbolic to parabolic equations. Moreover, we vigorously justify the validity of the gPC expansion ansatz based on the regularity study. Then the stochastic Galerkin method of the gPC approach is employed to discretize the random variable. We further conduct a delicate analysis to show the exponential decay rate of the gPC coefficients and establish the error estimates of the stochastic Galerkin approximation for both one-dimensional and multi-dimensional random space cases. Numerical tests are presented to verify our analytical results.

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1 Introduction

The radiative transfer equation (RTE) [6, 7] is a fundamental model for light propagation. It is a model equation for a class of kinetic equations, whose solutions are probability distribution functions of particles in the phase space. RTE, like other kinetic equations, describes the dynamics of photons in a given optical environment. The equation has wide applications in many areas such as astrophysics, inertial confinement fusion, optical molecular imaging, shielding, atmospheric science, and remote sensing, etc..

In practical applications, the RTE almost always contains uncertainties. It is very common that the optical environment contains uncertain parameters, rendering a parameterized equation that is otherwise deterministic. For each fixed configuration of the parameters, solving the deterministic RTE is a classical topic. At the forefront of stochastic computation, most numerical techniques focus on capturing the stochastic behavior of the solution when parameters vary. Among many choices of numerical methods, the polynomial chaos (PC) methods have received intensive attention. The term "polynomial chaos" was coined by Wiener in [36], where the decomposition of Gaussian random processes was studied with Hermite polynomials serving as an orthogonal basis in the random space. Inspired by Wiener's work, the original PC method using Hermite polynomials was developed by Ghanem and his collaborators for many engineering problems where uncertainties are mostly modeled by Gaussian stochastic processes. We refer to [17] for an overview of the method. Later, Xiu and Karniadakis proposed the generalized polynomial chaos (gPC) method in [39], where different types of orthogonal polynomials are chosen as basis functions according to the features of the random inputs, extending Gaussian process to a more general setting.

Being essentially a spectral approach, the gPC method exhibits the spectral convergence with the specific rate depending on the regularity of the solution in the random space. As a result, justifying the regularity of the solution lies at the core of numerical validation. Many regularity studies have been made for stochastic elliptic and parabolic equations; see [2, 3, 9, 10, 40]. These studies explore the solutions' properties in the random space, and based on these properties, new numerical methods were developed to further incorporate the solutions' structure [1, 8, 12, 20, 21, 32–34]. However, these studies have mostly been confined to elliptic/parabolic type equations since this regularity argument for stochastic hyperbolic equations usually breaks down [4, 12]: The solutions develop nonsmooth structures rendering the failure of the spectral accuracy.

Interestingly, kinetic equations sit somewhere between hyperbolic and parabolic systems. The equations describe dynamics of particles, and thus naturally contain terms that characterize transport phenomena, resembling most hyperbolic systems. However, particles sometimes interact, either with each other or with the environment, bringing in a dissipative feature that eventually sends the system to equilibrium. In this sense, it is closer to a parabolic system. The feature the system shows largely depends on the regime the equation gets presented on. Typically one adjusts the scaling between temporal and spatial variables, and this scaling is presented by the Knudsen number. In the large time regime where temporal scale dominates and the Knudsen number is small, the system dissipates fast into an equilibrium, demonstrating a parabolic feature. This interesting phenomenon attracted growing interest in kinetic equations in recent years, especially when the system contains uncertainty. These studies serve as a building block in understanding the influence of uncertainties in the passage from hyperbolic to parabolic equations. Uniform (in the Knudsen number) regularity in the random space and the uniform spectral convergence of the stochastic Galerkin method were proved for the transport equation in [23] and for the semiconductor Boltzmann equation in [24]. The authors in [30] carried out uniform regularity analysis based on hypocoercivity [13] for stochastic linear kinetic equations in the random space where the regularity result is independent of the form of the collision operator, the probability distribution of the random variables, or the regime in which the system is in, and can be applied to a wide range of linear kinetic equations and to different regimes including kinetic, diffusive, and high field. It was followed by [16] where uniform error estimates of the bi-fidelity method was conducted for linear transport equations. We refer readers to the asymptotic-preserving stochastic Galerkin methods for kinetic equations in [22, 25, 26, 28, 42], etc.. We also refer readers to the monograph [27] for more details regarding uncertainty quantification for hyperbolic and kinetic equations.

All of these results focused on *preserving* the regularity in time based on the assumption that the solution has a gPC expansion ansatz in the random space. However, why the solution can be represented by random variables and be formed as a well-defined function is left unjustified. The study in this paper can be regarded as a complement to these previous results. It provides a recipe, using the time-dependent RTE as an example, to justify the gPC expansion ansatz. By establishing the a-prior estimate of the RTE solution, we prove the regularity of the solution in the random space for the RTE with nonhomogeneous boundary conditions. Based on these studies, we rigorously show that the solution can be represented as a power series with nontrivial radius of convergence and thus is well-defined in the random space. Therefore, the stochastic Galerkin method can be adopted to approximate the random variables. Moreover, a delicate analysis is conducted to prove the exponential decay rate of the gPC coefficients, and the error estimates are established for the stochastic Galerkin approximation. It is worth mentioning that, for the time-dependent RTE with homogeneous boundary conditions, the regularity of the solution in the random space was proved in the [23]. It is also a special case of linear kinetic equations with random inputs studied in [30] where uniform regularity was proved based on the hypocoercivity depending on four assumptions regarding microscopic coercivity, macroscopic coercivity, orthogonality, and boundedness of auxiliary operator. Although almost all the kinetic equations satisfy these four assumptions as checked in [13], it can be verified that, for the RTE with nonhomogeneous boundary conditions, the transport operator is not skew symmetric and thus breaks the assumption of macroscopic coercivity that the uniform regularity analysis relies on. We also refer readers to the regularity and convergence study for the stationary RTE investigated in [41].

The rest of paper is organized as follows: In Sect. 2, we present the equation and some theoretical aspects, including the a-prior bound in Sect. 2.2, continuity with respect to the scattering coefficient in Sect. 2.3, and the regularity in the random space in Sect. 2.4. Section 3 is devoted to the study of the stochastic Galerkin approximation with the method presented in Sect. 3.1, the decay rate analysis of the gPC coefficients carried out in Sect. 3.2, and the error estimates for the stochastic Galerkin approximation established in Sect. 3.3. Numerical tests are provided in Sect. 4 and the paper is concluded with summaries and remarks in Sect. 5.

2 Theoretical Aspects of the RTE Solution

In this section, we introduce the equation and investigate theoretical aspects of the RTE solution including the a-priori bound, its continuity with respect to the scattering coefficient, and its regularity in the random space.

The time-dependent RTE is given by

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \mathcal{L}_\sigma f, & (x, v) \in \mathcal{R} \times \mathbb{S}, \\ f(t, x, v) = \psi(t, x, v), & (x, v) \in \Gamma_-, \\ f(0, x, v) = f_0(x, v), & (x, v) \in \mathcal{R} \times \mathbb{S}, \end{cases} \tag{2.1}$$

where $f(t, x, v)$ is the probability distribution function of particles at position $x \in \mathcal{R}$ with velocity $v \in \mathbb{S}$ at time $t > 0$. Here $\mathcal{R}$ denotes some bounded Lipschitz domain, and $\mathbb{S}$ is the unit sphere ($\int_{\mathbb{S}} \mathrm{d}v = 1$). The boundary $\partial\mathcal{R} \times \mathbb{S}$ can be split into two parts

$$\Gamma_\pm = \{(x, v) \in \partial\mathcal{R} \times \mathbb{S} : \pm v \cdot n(x) > 0\}, \tag{2.2}$$

where Γ_+ and Γ_- are the so-called outflow and inflow boundary, respectively, and $n(x)$ denotes the outward unit normal vector at $x \in \partial\mathcal{R}$. The collision operator $\mathcal{L}_\sigma$, defined by

$$\mathcal{L}_\sigma f = \int_{\mathbb{S}} \sigma(x, v, v') f(t, x, v') \mathrm{d}v' - \sigma_a f(t, x, v)\,, \tag{2.3}$$

describes the interaction of particles with the media (the mutual interactions between particles are ignored). Here $\sigma_a > 0$ is the total cross-section of the medium, and $\sigma > 0$ is the scattering kernel. $\sigma_s = \int \sigma(x, v, v')\mathrm{d}v'$ is the scattering coefficient.

Without loss of generality, we focus on the critical case with $\sigma_a = \sigma_s$. The scattering coefficient σ_s is further assumed to be independent of the velocity v and involves uncertainties. Thus, (2.3) can be rewritten as

$$\mathcal{L}_\sigma f = \sigma_s(x, \omega) \left(\int_{\mathbb{S}} f \mathrm{d}v' - f \right) \triangleq \sigma_s \mathcal{L} f, \tag{2.4}$$

where $\omega \in \Omega$ denotes the random variables, and

$$\mathcal{L} f = \int_{\mathbb{S}} f \mathrm{d}v - f. \tag{2.5}$$

2.1 Preliminaries

In this section, we unify and introduce notations. To study the model problem (2.1) with (2.4), we first represent the random variable $\omega \in \Omega$ with more concrete parameters. By applying the well-known Karhunen-Loève (KL) expansion [31], the coefficient σ_s can be rewritten as

$$\sigma_s(x, \omega) = \bar{\sigma}_s(x) + \sum_{j=1}^{d} y_j(\omega) \phi_j(x), \tag{2.6}$$

where $\{y_j\}_{j=1,2,\ldots,d}$ are mutually uncorrelated random variables, and $\{\phi_j\}_{j=1,2,\ldots,d}$ are orthogonal functions in $L^2(\mathcal{R})$. Here we assume that $\|y_j\|_{L^\infty(\Omega)} = \sup_{\omega \in \Omega} |y_j(\omega)| = 1,\ j = 1, \ldots, d$, up to a renormalization of the function ϕ_j. It is worth mentioning that the standard KL expansion has an eigenvalue term as the coefficient of $y_j \phi_j$, which is also absorbed into ϕ_j. The eigenvalue and the associated orthogonal eigenfunctions can be derived from the eigenvalue problem of the KL expansion. More details can be found in [31]. Here d is the number of random variables, which could be infinity but is truncated here with a finite approximation under a preset error tolerance.

Now the solution is viewed as a function $f(t, x, v, y)$, where $y = (y_1, \ldots, y_d)$ are the random variables. We unify the notations and rewrite the RTE with uncertainty as

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \sigma_s \mathcal{L} f, \quad f = f(t, x, v, y), \quad (t, x, v, y) \in (0, T) \times \mathcal{R} \times \mathbb{S} \times U, \\ f(t, x, v) = \psi(t, x, v), \quad (t, x, v) \in (0, T) \times \Gamma_-, \\ f(0, x, v) = f_0(x, v), \quad (x, v) \in \mathcal{R} \times \mathbb{S} \end{cases} \tag{2.7}$$

with $U = [-1, 1]^d$, σ_s given in (2.6), and $\mathcal{L}$ defined in (2.5). It describes the evolution of the probability distribution function of particles f at time t on phase space (x, v) and subject to a set of random variables $\{y_j\}$. Without abuse of notations, here and after, we adopt simpler notations such as f or $f(v)$ or $f(y)$, σ_s, etc.. It is worth emphasizing that f is viewed as $f(t, x, v, y)$, σ_s as $\sigma_s(x, y)$, ψ as the deterministic boundary $\psi(t, x, v)$, and f_0 as the initial condition $f_0(x, v)$.

Let $\rho(y)$ be the probability density function of y. We define the norm in the Hilbert space $L^2(U \times \mathcal{R} \times \mathbb{S}; \rho(y)\mathrm{d}y\mathrm{d}x\mathrm{d}v)$ as

$$\|f\|^2_{L^2(U\times\mathcal{R}\times\mathbb{S};\ \rho(y)\mathrm{d}y\mathrm{d}x\mathrm{d}v)} = \int_U \|f\|_2^2 \, \rho(y)\mathrm{d}y, \tag{2.8}$$

where $\|f\|$ is the standard L^2 norm of f in the phase space $(x, v) \in \mathcal{R} \times \mathbb{S}$ given by

$$\|f\|_2^2 = \int_{\mathcal{R}\times\mathbb{S}} f^2 \mathrm{d}x\mathrm{d}v, \quad \|f\|^2_{2,\Gamma_-} = \int_{\Gamma_-} |v \cdot n(x)| \, f^2 \mathrm{d}x\mathrm{d}v. \tag{2.9}$$

We further define the dual norm of the operator $\mathcal{L}$ in (2.5) by

$$\|\mathcal{L}\|_2 = \sup_{\|f\|_2=1} \sqrt{\int_{\mathbb{S}} (\mathcal{L}f)^2 \, \mathrm{d}v}.$$

The bound of the dual norm is presented in the following lemma, which plays an important role for the theoretical study of the solution. The proof of this lemma can be found in [41] and thus is omitted here.

Lemma 2.1 *The dual norm of $\mathcal{L}$ is bounded by 1, meaning $\|\mathcal{L}\|_2 \leq 1$.*

For all the lemmas and theorems presented in this paper, we assume that $\sigma_s \geq 0$, $\sigma_s \in L^\infty(U \times \mathcal{R})$, and $f_0 \in L^2(\mathcal{R} \times \mathbb{S})$. We also assume that there exists a positive constant C_ψ independent of t such that

$$\|\psi\|_{2,\Gamma_-} \leq C_\psi < \infty, \tag{2.10}$$

which can be achieved by assuming for simplicity that ψ is independent of time t.

2.2 The A-priori Estimate of the RTE Solution

The existence and uniqueness for this initial boundary value problem were first proved for the L^1 case in [5] under the assumption that the scattering coefficient is bounded away from the absorption coefficient. Under the assumption that $\sigma_s \geq 0$ and $\sigma_s \in L^\infty(\mathcal{R})$, the existence and uniqueness (in the weak sense) for this initial boundary value problem with absorbing (homogeneous) boundary conditions ($\psi = 0$) were proved for general L^p case with semi-group argument in [11] where a remark on nonhomogeneous boundary conditions was also given by transforming the nonhomogeneous problem to its homogeneous counterpart with the help of the lifting of ψ. More details can be found in the monograph [11] (Chapter XXI, Section 2). We also refer to [15] for the study on a class of Galerkin schemes for the time-dependent RTE with homogeneous boundary conditions and to [14] for the a-priori estimates in the L^p sense for the stationary RTE.

In the following theorem, we provide the a-priori estimates for the RTE solution with source terms and nonhomogeneous boundary conditions. For simplicity, we omit the argument y in this subsection.

Theorem 2.1 *Suppose $S \in L^2(\mathcal{R} \times \mathbb{S})$, the following initial boundary value problem*

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \sigma_s(x)\mathcal{L}f + S, \quad (x,v) \in \mathcal{R} \times \mathbb{S}, \\ f(t,x,v) = \psi(t,x,v), \quad (x,v) \in \times\Gamma_-, \\ f(0,x,v) = f_0(x,v), \quad (x,v) \in \mathcal{R} \times \mathbb{S} \end{cases} \tag{2.11}$$

has a unique solution which satisfies

$$\|f\|_2^2 \le e^{Ct}\left(\|f_0\|_2^2 + \int_0^t e^{-C\tau}\left(\|S(\tau)\|_2^2 + \|\psi(\tau)\|_{2,\Gamma_-}^2\right)\mathrm{d}\tau\right),$$

where $C = 2\|\sigma_s\|_\infty + 1$.

Proof Multiplying both sides of the equation by f and integrating over $\mathcal{R} \times \mathbb{S}$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{R}\times\mathbb{S}} f^2 \mathrm{d}x\mathrm{d}v + \frac{1}{2}\int_{\partial\mathcal{R}\times\mathbb{S}} f^2 v\cdot n \mathrm{d}x\mathrm{d}v = \int_{\mathcal{R}\times\mathbb{S}} \sigma_s(x)\,(\mathcal{L}f)\, f \mathrm{d}x\mathrm{d}v + \int_{\mathcal{R}\times\mathbb{S}} Sf\mathrm{d}x\mathrm{d}v, \tag{2.12}$$

where the second term on the left hand side is obtained by

$$\int_{\mathcal{R}\times\mathbb{S}} f\,(v\cdot\nabla_x f)\,\mathrm{d}x\mathrm{d}v = \frac{1}{2}\int_{\mathcal{R}\times\mathbb{S}} v\cdot\nabla_x f^2 \mathrm{d}x\mathrm{d}v = \frac{1}{2}\int_{\partial\mathcal{R}\times\mathbb{S}} f^2 v\cdot n\mathrm{d}x\mathrm{d}v.$$

It follows from Lemma 2.1 and Young's inequality that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{R}\times\mathbb{S}} f^2\mathrm{d}x\mathrm{d}v + \tfrac{1}{2}\textstyle\int_{\Gamma_+} f^2 v\cdot n\mathrm{d}x\mathrm{d}v &\le \|\sigma_s\|_\infty\|f\|_2^2 + \tfrac{1}{2}\left(\|f\|_2^2 + \|S\|_2^2\right) \\ &\quad -\tfrac{1}{2}\textstyle\int_{\Gamma_-} f^2 v\cdot n\mathrm{d}x\mathrm{d}v, \end{aligned} \tag{2.13}$$

where the boundary term $\partial\mathcal{R} \times \mathbb{S}$ is split into two parts $\Gamma_\pm$. According to the definition of $\Gamma_\pm$ in (2.2), it is easy to verify that

$$\int_{\Gamma_+} f^2 v\cdot n\mathrm{d}x\mathrm{d}v \ge 0, \qquad -\frac{1}{2}\int_{\Gamma_-} f^2 v\cdot n\mathrm{d}x\mathrm{d}v = \|\psi\|_{2,\Gamma_-}^2,$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_2^2 \le C\|f\|_2^2 + \|S\|_2^2 + \|\psi\|_{2,\Gamma_-}^2, \tag{2.14}$$

with $C = 2\|\sigma_s\|_\infty + 1$. The proof is completed by directly applying the Gronwall's inequality. □

Corollary 2.1 *The solution to the RTE (2.7) satisfies*

$$\|f\|_2^2 \le e^{Ct}\left(\|f_0\|_2^2 + C_\psi^2\right), \tag{2.15}$$

where $C = 2\|\sigma_s\|_\infty + 1$, and C_ψ is defined in (2.10).

Proof For (2.7) with null source, we apply Theorem 2.1 with $S = 0$ and obtain

$$\|f\|_2^2 \le e^{Ct}\left(\|f_0\|_2^2 + \int_0^t e^{-C\tau}\left(\|\psi(\tau)\|_{2,\Gamma_-}^2\right)\mathrm{d}\tau\right) \le e^{Ct}\left(\|f_0\|_2^2 + C_\psi^2\right) \tag{2.16}$$

based on the assumption on ψ and the fact that $\int_0^t e^{-C\tau}\mathrm{d}\tau \le \frac{1}{C}(1 - e^{-Ct}) \le \frac{1}{C} \le 1$. □

2.3 Continuity with Respect to σ_s

In this subsection, we show that the solution f is continuous with respect to the scattering coefficient σ_s. Namely, for small changes in σ_s, we will show that the solution f responds with a small change as well.

Theorem 2.2 (continuity) *The solution f to the RTE (2.7) is continuous in the L^2 sense with respect to the scattering coefficient σ_s. That is, if f and $\tilde{f}$ satisfy the following equations*

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \sigma_s \mathcal{L} f, \\ f|_{\Gamma_-} = \psi, \\ f|_{t=0} = f_0, \end{cases} \qquad \begin{cases} \partial_t \tilde{f} + v \cdot \nabla_x \tilde{f} = \tilde{\sigma}_s \mathcal{L} \tilde{f}, \\ \tilde{f}|_{\Gamma_-} = \psi, \\ \tilde{f}|_{t=0} = f_0, \end{cases}$$

respectively, then the error $E = f - \tilde{f}$ satisfies

$$\|E\|_2^2 \le e^{Ct} t \|\sigma_s - \tilde{\sigma}_s\|_\infty^2 C_1,$$

where $C = 2\|\sigma_s\|_\infty + 1$ and $C_1 = \|f_0\|_2^2 + C_\psi^2$.

Proof Clearly, the error function E satisfies the following equation

$$\begin{cases} \partial_t E + v \cdot \nabla_x E = \sigma_s \mathcal{L} E + (\sigma_s - \tilde{\sigma}_s)\mathcal{L}\tilde{f}, \\ E|_{\Gamma_-} = 0, \\ E|_{t=0} = 0. \end{cases}$$

By Theorem 2.1 and Lemma 2.1, we have

$$\|E\|_2^2 \le e^{Ct}\int_0^t e^{-C\tau}\|(\sigma_s - \tilde{\sigma}_s)\mathcal{L}\tilde{f}\|_2^2 \mathrm{d}\tau \le e^{Ct}\|\sigma_s - \tilde{\sigma}_s\|_\infty^2 \int_0^t e^{-C\tau}\|\tilde{f}(\tau,\cdot,\cdot)\|_2^2\mathrm{d}\tau,$$

which, together with Corollary 2.1, yields

$$\|E\|_2^2 \le e^{Ct} t \|\sigma_s - \tilde{\sigma}_s\|_\infty^2 (\|f_0\|_2^2 + C_\psi^2).$$

Thus, the proof is complete. □

2.4 Differentiability and A-priori Estimates for Derivatives

We further study the dependence of f on $y = (y_1, \dots, y_d)$, which is the representation of the perturbation in σ_s, by taking the derivatives of (2.7) with respect to y.

We show in the following theorem the existence and L^2 bound of the first order derivative.

Theorem 2.3 (differentiability) *The solution f to the RTE (2.7) is differentiable with respect to y, and $\frac{\partial f}{\partial y_m}$ satisfies the following equation*

$$\begin{cases} \partial_t \left(\dfrac{\partial f}{\partial y_m} \right) + v \cdot \nabla_x \dfrac{\partial f}{\partial y_m} = \sigma_s \mathcal{L} \dfrac{\partial f}{\partial y_m} + \phi_m \mathcal{L} f, \\ \dfrac{\partial f}{\partial y_m} |_{\Gamma_-} = 0, \\ \dfrac{\partial f}{\partial y_m} |_{t=0} = 0. \end{cases} \tag{2.17}$$

Moreover,

$$\| \partial f / \partial y_m \|_2^2 \le e^{Ct} t \| \phi_m \|_\infty^2 C_1, \tag{2.18}$$

where $C = 2\|\sigma_s\|_\infty + 1$ and $C_1 = \| f_0 \|_2^2 + C_\psi^2$.

Proof Let f and f_h be the solutions to (2.7) with $\sigma_s(x) = \bar{\sigma}_s + \sum_{k \ge 1} y_k \phi_k(x)$ and $\tilde{\sigma}_s(x) = \bar{\sigma}_s + \sum_{k \ge 1} y_k \phi_k(x) + h\phi_m(x)$, respectively. Then, $f_h - f$ satisfies

$$\begin{cases} \partial_t (f_h - f) + v \cdot \nabla_x (f_h - f) = \sigma_s \mathcal{L} (f_h - f) + (\tilde{\sigma}_s - \sigma_s) \mathcal{L} f_h, \\ (f_h - f) |_{\Gamma_-} = 0, \\ (f_h - f) |_{t=0} = 0. \end{cases}$$

Denote $w = \frac{1}{h} (f_h - f)$, we have

$$\begin{cases} \partial_t w + v \cdot \nabla_x w = \sigma_s \mathcal{L} w + \phi_m(x) \mathcal{L} f_h, \\ w|_{\Gamma_-} = 0, \\ w|_{t=0} = 0. \end{cases}$$

For every $y \in [-1, 1]$, $\|w\|_2$ is bounded according to Theorem 2.1, which implies that

$$\| f_h - f \|_2 \to 0, \quad \text{as } h \to 0, \text{ a.s.} .$$

Therefore, by Lemma 2.1, $\mathcal{L} f_h \to \mathcal{L} f$, as $h \to 0$, and we conclude that the solution is differentiable with respect to y by letting $\frac{\partial f}{\partial y_m} = \lim_{h \to 0} w = \lim_{h \to 0} \frac{1}{h} (f - f_h)$.

Moreover, it follows from Theorem 2.1 and Lemma 2.1 that

$$\begin{aligned} \left\| \frac{\partial f}{\partial y_m} \right\|_2^2 &\le e^{Ct} \int_0^t e^{-C\tau} \| \phi_m(x) \mathcal{L} f \|_2^2 \mathrm{d}\tau \\ &\le e^{Ct} \| \phi_m \|_\infty^2 \int_0^t e^{-C\tau} \| f \|_2^2 \mathrm{d}\tau \le e^{Ct} t \| \phi_m \|_\infty^2 (\| f_0 \|_2^2 + C_\psi^2). \end{aligned}$$

□

We now generalize the results in Theorem 2.3 to higher order derivatives with respect to $y = (y_1, \ldots, y_d)$. Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a multi-index with the length and factorial defined as

$$|\alpha| = \sum_{j=1}^d \alpha_j, \quad \alpha! = \prod_{j=1}^d \alpha_j!,$$

respectively. We further define $b = (b_1, \ldots, b_d)$ with $b_j > 0$ to the power of α as

$$b^\alpha = \prod_{1 \le j \le d} b_j^{\alpha_j}.$$

Theorem 2.4 *The high order derivatives of the solution f to the RTE (2.7) are governed by the following equation*

$$\begin{cases} \partial_t\left(\partial_y^\alpha f\right)+v\cdot\nabla_x\partial_y^\alpha f=\sigma_s\mathcal{L}\left(\partial_y^\alpha f\right)+\sum\limits_{j,\alpha_j\neq 0}\alpha_j\phi_j\mathcal{L}\left(\partial_y^{\alpha-e_j}f\right), \\ \partial_y^\alpha f|_{\Gamma_-}=0, \\ \partial_y^\alpha f|_{t=0}=0, \end{cases} \tag{2.19}$$

and satisfy

$$\left\|\partial_y^\alpha f\right\|_2^2\leq e^{Ct}d^{|\alpha|-1}\alpha!b^{2\alpha}t^{|\alpha|}C_1, \tag{2.20}$$

where $e_j\in\mathbb{N}^d$ is the multi-index whose j-th entry is 1 and all other entries are zero. Here $C=2\|\sigma_s\|_\infty+1$, $C_1=\|f_0\|_2^2+C_\psi^2$, and $b=(b_1,\ldots,b_d)$ is a d-dimensional vector with $b_j=\|\phi_j\|_\infty$.

Proof We will prove (2.19) and (2.20) by induction on $|\alpha|$. Clearly (2.19) and (2.20) hold for $|\alpha|=1$ by Theorem 2.3. For $|\alpha|>1$ with the m-th entry $\alpha_m\neq 0$, let $\tilde\alpha=\alpha-e_m$. Then $\tilde\alpha_m=\alpha_m-1$, $\tilde\alpha_j=\alpha_j$ for $j\neq m$, and $|\tilde\alpha|=|\alpha|-1$. By the induction hypothesis, we have

$$\partial_t\left(\partial_y^{\tilde\alpha} f\right)+v\cdot\nabla_x\partial_y^{\tilde\alpha} f=\sigma_s\mathcal{L}\left(\partial_y^{\tilde\alpha} f\right)+\sum_{j,\tilde\alpha_j\neq 0}\tilde\alpha_j\phi_j\mathcal{L}\left(\partial_y^{\tilde\alpha-e_j}f\right), \tag{2.21}$$

and

$$\left\|\partial_y^{\tilde\alpha} f\right\|_2^2\leq e^{Ct}d^{|\tilde\alpha|-1}\tilde\alpha!b^{2\tilde\alpha}t^{|\tilde\alpha|}C_1, \tag{2.22}$$

where $C=2\|\sigma_s\|_\infty+1$, $C_1=\|f_0\|_2^2+C_\psi^2$, and $b=(b_1,\ldots,b_d)$ is a d-dimensional vector with $b_j=\|\phi_j\|_\infty$.

Taking derivative of (2.21) with respect to y_m, we obtain

$$\begin{aligned}
\partial_t\left(\partial_y^\alpha f\right)+v\cdot\nabla_x\partial_y^\alpha f&=\sigma_s\mathcal{L}\left(\partial_y^\alpha f\right)+\partial_{y_m}\sigma_s\mathcal{L}\left(\partial_y^{\tilde\alpha} f\right)+\partial_{y_m}\left(\sum_{j,\tilde\alpha_j\neq 0}\tilde\alpha_j\phi_j\mathcal{L}\left(\partial_y^{\tilde\alpha-e_j}f\right)\right)\\
&=\sigma_s\mathcal{L}\left(\partial_y^\alpha f\right)+\phi_m\mathcal{L}\left(\partial_y^{\alpha-e_m}f\right)+(\alpha_m-1)\phi_m\mathcal{L}\left(\partial_y^{\alpha-e_m}f\right)\\
&\quad+\sum_{j\neq m,\alpha_j\neq 0}\alpha_j\phi_j\mathcal{L}\left(\partial_y^{\alpha-e_j}f\right)\\
&=\sigma_s\mathcal{L}\left(\partial_y^\alpha f\right)+\sum_{j,\alpha_j\neq 0}\alpha_j\phi_j\mathcal{L}\left(\partial_y^{\alpha-e_j}f\right),
\end{aligned}$$

which is exactly (2.19). It follows from Theorem 2.1 and Lemma 2.1 that

$$\begin{aligned}
\|\partial_y^\alpha f\|_2^2&\leq e^{Ct}\int_0^t e^{-C\tau}\left\|\sum_{j,\alpha_j\neq 0}\alpha_j\phi_j\mathcal{L}\left(\partial_y^{\alpha-e_j}f\right)\right\|_2^2\mathrm{d}\tau\\
&\leq e^{Ct}\int_0^t e^{-C\tau}\left(\sum_{j,\alpha_j\neq 0}\left\|\alpha_j\phi_j\mathcal{L}\left(\partial_y^{\alpha-e_j}f\right)\right\|_2\right)^2\mathrm{d}\tau\\
&\leq e^{Ct}\int_0^t e^{-C\tau}\left(\sum_{j,\alpha_j\neq 0}\alpha_j\|\phi_j\|_\infty\|\partial_y^{\alpha-e_j}f\|_2\right)^2\mathrm{d}\tau,
\end{aligned} \tag{2.23}$$

which, together with the hypothesis induction (2.22), yields

$$\begin{aligned}
\|\partial_y^\alpha f\|_2^2 &\le e^{Ct}\int_0^t e^{-C\tau}\left(\sum_{j,\alpha_j\neq 0} e^{\frac{C\tau}{2}} d^{\frac{|\alpha|-2}{2}}\alpha_j b_j\sqrt{(\alpha-e_j)!}b^{\alpha-e_j}\tau^{\frac{|\alpha|-1}{2}}\sqrt{C_1}\right)^2 \mathrm{d}\tau\\
&= e^{Ct}d^{|\alpha|-2}\left(\sum_{j,\alpha_j\neq 0}\sqrt{\alpha_j}\right)^2 \alpha! b^{2\alpha}C_1\int_0^t \tau^{|\alpha|-1}\mathrm{d}\tau\\
&= e^{Ct}d^{|\alpha|-2}\frac{\left(\sum_{j,\alpha_j\neq 0}\sqrt{\alpha_j}\right)^2}{|\alpha|}\alpha! b^{2\alpha}t^{|\alpha|}C_1\\
&\le e^{Ct}d^{|\alpha|-1}\alpha! b^{2\alpha}t^{|\alpha|}C_1.
\end{aligned}$$

We complete the proof. □

It is worth mentioning that the existence of the first derivative shown in proof of Theorem 2.3 is the limit of $(f - f_h)/h$. The procedure can be generalized to the existence of the high order derivatives. Likewise, Eq. 2.17 can be obtained by directly taking derivatives with respect to y_m in (2.7).

It is tempting to use these regularity results to expand f pointwisely as a power series

$$f(t,x,v,y) = \sum_{k=0}^{\infty}\sum_{|\alpha|=k}\left.\frac{\partial_y^\alpha f(t,x,v,y)}{\alpha!}\right|_{y=y_0}(y-y_0)^\alpha, \tag{2.24}$$

and to obtain an upper bound of f as

$$\|f\|_2(y) \le \sum_{k=0}^{\infty}\sum_{|\alpha|=k}\frac{|y-y_0|^\alpha}{\alpha!}\|\partial_y^\alpha f\|_2(y_0)$$

by directly taking L^2-norm in (x, v). However, the expression (2.24) is only formal and requires the assumption that y belongs to the convergence radius of y_0 for fixed (t, x, v). Although Theorem 2.4 provides the bound for its convergence radius and ensures the validity of the power series for the $L^2(\mathcal{R}\times\mathbb{S})$ norm of f, i.e., $\|f\|_2$, a more rigorous analysis writes out the expansion of $F(y) = \|f\|_2^2$ directly as

$$F(y) = \sum_{k=0}^{\infty}\sum_{|\alpha|=k}\frac{\partial_y^\alpha F(y_0)}{\alpha!}(y-y_0)^\alpha$$

and requires an estimate of $\partial_y^\alpha F(y_0)$. In fact, for multi-indices $\alpha = (\alpha_1,\ldots,\alpha_d)$ and $\kappa = (\kappa_1,\ldots,\kappa_d)$, and fixed $y_0\in U = [-1,1]^d$, we have

$$\begin{aligned}
\left|\partial_y^\alpha F(y_0)\right| = \left|\partial_y^\alpha \|f\|_2^2(y_0)\right| &\le \int_{\mathcal{R}\times\mathbb{S}}\left|\partial_y^\alpha f^2(t,x,v,y_0)\right|\mathrm{d}x\mathrm{d}v\\
&= \int_{\mathcal{R}\times\mathbb{S}}\left|\sum_{\kappa\le\alpha}\binom{\alpha}{\kappa}\left(\partial_y^\kappa f(t,x,v,y_0)\right)\left(\partial_y^{\alpha-\kappa} f(t,x,v,y_0)\right)\right|\mathrm{d}x\mathrm{d}v\\
&\le \sum_{\kappa\le\alpha}\binom{\alpha}{\kappa}\left\|\partial_y^\kappa f\right\|_2(y_0)\left\|\partial_y^{\alpha-\kappa}\right\|_2(y_0),
\end{aligned}$$

where the partial order $\alpha \leq \kappa$ is defined as $\alpha_i \leq \kappa_i$, for $i = 1, \ldots, d$, and the binomial coefficient is defined as

$$\binom{\alpha}{\kappa} = \binom{\alpha_1}{\kappa_1}\binom{\alpha_2}{\kappa_2}\cdots\binom{\alpha_d}{\kappa_d} = \frac{\alpha!}{\kappa!\,(\alpha-\kappa)!}.$$

Thus, it follows from Theorem 2.4 directly to get the bound for $\partial_y^\alpha F(y_0)$ and to further claim that F is a well-defined function that can be represented by a power series. Instead of engaging in details of this issue, we focus our effort and move on to analyze the stochastic Galerkin method (the spectral approach).

3 Stochastic Galerkin Approximation

In this section, we apply the stochastic Galerkin method to the RTE based on based on the vadility of justification discussed in previous section. We first briefly review the stochastic Galerkin method in Sect. 3.1. This amounts to rewriting the solution in the random space using the expansion of orthogonal polynomials. The coefficients for the expansion enjoy certain decay, and such decay analysis is presented in Sect. 3.2, in which we further justify the convergence of the expansion series. We finally establish the error estimate of the stochastic Galerkin approximation in Sect. 3.3.

3.1 The Stochastic Galerkin Method

The stochastic Galerkin method is essentially the spectral method in the random space. This is to write the solution in the random space as an expansion of orthogonal polynomials of random variables.

The gPC basis functions, denoted as $p_n(y)$, are orthogonal polynomials with the probability density function $\rho(y)$ serving as the weight function, i.e.,

$$\langle p_m, p_n \rangle_y \equiv \int_U p_m(y) p_n(y) \rho(y) \mathrm{d}y = \delta_{mn}, \quad \forall m, n, \tag{3.1}$$

where δ_{mn} is the Kronecker delta function. Here the polynomials are normalized and single-indexed even for multivariate functions. These polynomial basis p_m depends on the distribution ρ. The commonly used ones are Legendre polynomials associated with uniform distributions, and Hermite polynomials associated with Gaussian distributions [38, 39]. To simplify the presentation, we set $U = [-1, 1]^d$ and ρ as the uniform distribution on U. As such the tensor product of one-dimensional Legendre polynomials in each direction forms the gPC basis in the d-dimensional random space U.

As a start, we form the ansatz to the solution $f(t, x, v, y)$ to (2.7) as

$$f(t, x, v, y) = \sum_{n=0}^{\infty} \hat{f}_n(t, x, v) p_n(y), \tag{3.2}$$

where the coefficient $\hat{f}_n$ is defined as

$$\hat{f}_n(t, x, v) = \langle f(t, x, v, y), p_n(y) \rangle_y. \tag{3.3}$$

The N-th degree gPC projection of f is defined by

$$\mathcal{P}_N f = \sum_{n=0}^{N} \hat{f}_n p_n. \tag{3.4}$$

It is worth emphasizing that that the validity of the ansatz (3.2) is roughly justified in the end of Sect. 2.4 and will be delicately justified in Lemma 3.1. Based on these converging seriers, it is straightforward to verify that the coefficients satisfy the following infinite system

$$\partial_t \hat{f}_n + v \cdot \nabla_x \hat{f}_n = \sum_{m=0}^{\infty} A_{n,m} \mathcal{L} \hat{f}_n, \quad n = 0, 1, 2, \ldots, \tag{3.5}$$

where

$$A_{n,m} = \langle \sigma_s(y) p_m(y), p_n(y) \rangle_y . \tag{3.6}$$

The stochastic Galerkin methods truncates the expansion into a finite dimensional space, and use this finite gPC expansion g_N given by

$$g_N(t, x, v, y) = \sum_{n=0}^{N} \hat{g}_n p_n(y) . \tag{3.7}$$

as a numerical approximation, such that the residue of (2.7) is orthogonal to the subspace spanned by the first $N + 1$ gPC basis polynomials. That is,

$$\partial_t \hat{g}_n + v \cdot \nabla_x \hat{g}_n = \sum_{m=0}^{N} A_{n,m} \mathcal{L} \hat{g}_n, \quad n = 0, 1, 2, \ldots, N, \tag{3.8}$$

with $A_{n,m}$ defined in (3.6).

3.2 Decay Rate Analysis of the gPC Coefficients

The solution f to the RTE has been shown to be in H^k for all k in Sect. 2.4. The coefficients $\hat{f}_n$ for $f \in H^k$ under the spectral expansion are supposed to decay exponentially by the classical theory of spectral methods, when all derivatives of f are bounded by a constant independent of the order of the derivatives. However, it can be observed from Theorem 2.4 that the bounds of the derivatives of the RTE solution grow in a factorial fashion as the order increases. Therefore, in this section, we establish a more delicate analysis for the decay rate of the gPC coefficients, which can be used as a preparation for error estimates of the stochastic Galerkin approximation in Sect. 3.3.

This section is organized as follows: We first prove that f can be represented as a power series for all $y \in (-1, 1)$ in Lemma 3.1. Then with the help of the integral formula given in Lemma 3.2, and an integral estimate given in Lemma 3.3, we obtain the decay rate of the gPC coefficients $\hat{f}_n$ in Theorem 3.1. Finally we conclude this section with the extension to multi-dimensional random space.

As did in [3] for the stochastic elliptic equation and in [41] for stationary radiative transfer equation with random coefficients, we first show the procedure for one KL mode with σ_s written as

$$\sigma_s = \bar{\sigma_s} + y\phi, \tag{3.9}$$

where $y \in [-1, 1]$ and ϕ is the corresponding KL mode.

Lemma 3.1 *For $y_0 \in (-1, 1)$, the solution f to (2.7) can be represented as a power series in the form of $f = \sum_{j=0}^{\infty} \tilde{f}_j (y - y_0)^j$ with $\tilde{f}_j (j \geq 1)$ and $\tilde{f}_0$ satisfying*

$$\begin{cases} \partial_t \tilde{f}_j + v \cdot \nabla_x \tilde{f}_j = \sigma_0 \mathcal{L} \tilde{f}_j + \phi \mathcal{L} \tilde{f}_{j-1} \\ \tilde{f}_j \Big|_{\Gamma_-} = 0, \\ \tilde{f}_j \Big|_{t=0} = 0, \end{cases} \quad \text{and} \quad \begin{cases} \partial_t f_0 + v \cdot \nabla_x \tilde{f}_0 = \sigma_0 \mathcal{L} \tilde{f}_0 \\ \tilde{f}_0 \Big|_{\Gamma_-} = \psi, \\ \tilde{f}_0 \Big|_{t=0} = f_0, \end{cases} \tag{3.10}$$

respectively. Here $\sigma_0 = \sigma_s (y_0)$.

Proof To begin with, we show the ansatz $f = \sum_{j=0}^{\infty} \tilde{f}_j (y - y_0)^j$ with $\tilde{f}_j$ satisfying (3.10) is a solution to (2.7). It follows from (3.10) that

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f &= \partial_t \left(\sum_{j=0}^{\infty} \tilde{f}_j (y - y_0)^j \right) + v \cdot \nabla_x \left(\sum_{j=0}^{\infty} \tilde{f}_j (y - y_0)^j \right) \\ &= \partial_t f_0 + v \cdot \nabla_x f_0 + \sum_{j=1}^{\infty} (y - y_0)^j \left(\partial_t \tilde{f}_j + v \cdot \nabla_x \tilde{f}_j \right) \\ &= \partial_t f_0 + v \cdot \nabla_x f_0 + \sum_{j=1}^{\infty} (y - y_0)^j \left(\sigma_0 \mathcal{L} \tilde{f}_j + \phi \mathcal{L} \tilde{f}_{j-1} \right) \\ &= \sigma_0 \mathcal{L} \tilde{f}_0 + \sum_{j=1}^{\infty} (y - y_0)^j \sigma_0 \mathcal{L} \tilde{f}_j + \sum_{j=0}^{\infty} (y - y_0)^{j+1} \phi \mathcal{L} \tilde{f}_j \\ &= (\sigma_0 + (y - y_0)\phi) \, \mathcal{L} \left(\sum_{j=0}^{\infty} \tilde{f}_j (y - y_0)^j \right) \\ &= \sigma_s \mathcal{L} f, \end{aligned}$$

where the last equality is due to the fact $\sigma_0 + (y - y_0)\phi = \bar{\sigma}_s + y_0\phi + (y - y_0)\phi = \sigma_s$ by (3.9).

We further show that the ansatz $f = \sum_{j=0}^{\infty} \tilde{f}_j (y - y_0)^j$ is valid by checking its L^2 bound. By Lemma 2.1, Theorem 2.1, and Corollary 2.1, we have

$$\begin{aligned} \|\tilde{f}_j\|_2^2 &\leq e^{C_0 t} \int_0^t e^{-C_0 t_j} \left\| \phi \mathcal{L} \tilde{f}_{j-1} \right\|_2^2 \mathrm{d}t_j \leq e^{C_0 t} \|\phi\|_\infty^2 \int_0^t e^{-C_0 t_j} \left\| \tilde{f}_{j-1} \right\|_2^2 \mathrm{d}t_j \\ &\leq e^{C_0 t} \|\phi\|_\infty^{2j} \int_0^t \int_0^{t_j} \cdots \int_0^{t_2} e^{-C_0 t_1} \|\tilde{f}_0(t_1)\|_2^2 \mathrm{d}t_1 \mathrm{d}t_2 \cdots \mathrm{d}t_j \\ &\leq e^{C_0 t} \|\phi\|_\infty^{2j} \frac{t^j}{j!} \left(\|f_0\|_2^2 + C_\psi^2 \right) \end{aligned}$$

with $C_0 = 2\|\sigma_0\|_\infty + 1$. Next we show

$$\|f\|_2 \leq \sum_{j=0}^{\infty} \|\tilde{f}_j\|_2 (y - y_0)^j \leq e^{C_0 t/2} \sqrt{\|f_0\|_2^2 + C_\psi^2} \sum_{j=0}^{\infty} \|\phi\|_\infty^j \frac{t^{j/2}}{\sqrt{j!}} (y - y_0)^j < \infty$$

by applying d'Alembert's ratio test, where one compute the limit

$$\lim_{j \to \infty} \frac{t^{(j+1)/2} \sqrt{j!} \|\phi\|_\infty (y - y_0)}{\sqrt{(j+1)!} t^{j/2}} = \lim_{j \to \infty} \sqrt{\frac{t}{j+1}} \|\phi\|_\infty (y - y_0) = 0 < 1.$$

We conclude that the solution f can be represented as a power series in $y - y_0$ for all $y \in (-1, 1)$. □

Remark 3.1 One special property of the converging series for the stationary RTE studied in [41] is that the series converges only in a small ball around y_0. One then needs to patch up all balls to cover the entire U to show the convergence of the series over the whole random space. This difficulty is not encountered for the time-dependent RTE discussed in this paper, since it follows from the proof of Lemma 3.1 that the power series converges uniformly for all $y \in (-1, 1)$.

Remark 3.2 In Sect. 2, the bound of the solution or the derivatives depends on the constant C which are related with $\|\sigma_s\|_\infty$. The discussion is conducted point-wise for random variables and $\|\sigma_s\|_\infty$ is taken as $\max_{x\in\mathcal{R}} \sigma_s(x)$ as the standard L^∞ norm in the space $\mathcal{R}$ for the deterministic case, while for the stochastic case, one needs to take $\|\sigma_s\|_\infty = \sup_{y_0\in(-1,1)} \max_{x\in\mathcal{R}} \sigma_s(x, y_0)$ with y_0 defined in Lemma 3.1.

Lemma 3.2 *There exists a positive constant β such that*

$$\|\hat{f}_n\|_2 \le e^{Ct/2} \frac{\sqrt{2n+1}\sqrt{C_1}}{2^n} \int_{-1}^{1} \left(\frac{1-y^2}{1+y+\beta} \right)^n \mathrm{d}y, \tag{3.11}$$

where $C = 2\|\sigma_s\|_\infty + 1$ and $C_1 = \|f_0\|_2^2 + C_\psi^2$.

Proof With our assumption that y is uniformly distributed on $[-1, 1]$, the probability density function $\rho(y) = \frac{1}{2}$, and the gPC basis function are the Legendre polynomials which can be written in the form of

$$p_n(y) = \frac{\sqrt{2n+1}}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}y^n} (1-y^2)^n, \quad n = 0, 1, \dots. \tag{3.12}$$

Then the gPC coefficients $\hat{f}_n$ defined in (3.3) can be written as

$$\hat{f}_n = \frac{1}{2} \int_{-1}^{1} f p_n \mathrm{d}y = \frac{(-1)^n}{n!} \frac{\sqrt{2n+1}}{2^{n+1}} \int_{-1}^{1} \frac{\mathrm{d}^n}{\mathrm{d}y^n} f(y) \left(1-y^2\right)^n \mathrm{d}y \tag{3.13}$$

after integrating by parts n times.

It follows from Lemma 3.1 that f can be represented as a power series and is point-wise analytic for $y \in (-1, 1)$. Thus for each fixed $y \in (-1, 1)$, f can be analytically extended to the complex domain with the complex variable η. By Cauchy's formula, we have

$$\frac{\mathrm{d}^n}{\mathrm{d}y^n} f(y) = \frac{n!(-1)^n}{2\pi \mathrm{i}} \int_{\gamma_y} \frac{f(\eta)}{(\eta - y)^{n+1}} \mathrm{d}\eta,$$

where i is the imaginary unit satisfying $\mathrm{i}^2 = -1$, and γ_y is a positively oriented closed circumference with the center at the real point $y \in (-1, 1)$ and the radius $R(y) > 0$. Here we set the radius of γ_y by $R(y) = 1 - |y| + \beta > 0$ where β is a positive constant.

By Corollary 2.1, we get

$$\left\| \frac{\mathrm{d}^n}{\mathrm{d}y^n} f \right\|_2 \le \frac{n!}{2\pi} \int_{\gamma_y} \frac{\|f(\eta)\|_2}{|\eta - y|^{n+1}} |\mathrm{d}\eta| \le \frac{n!}{2\pi} \frac{e^{Ct/2}\sqrt{C_1}}{R(y)^{n+1}} 2\pi R(y) = \frac{e^{Ct/2}\sqrt{C_1} n!}{R(y)^n}, \tag{3.14}$$

which, combining with (3.13), yields

$$
\begin{aligned}
\|\hat{f}_n\|_2 &\le \frac{e^{Ct/2}\sqrt{2n+1}\sqrt{C_1}}{2^{n+1}} \int_{-1}^{1} \left(\frac{1-y^2}{1-|y|+\beta}\right)^n \mathrm{d}y \\
&= \frac{e^{Ct/2}\sqrt{2n+1}\sqrt{C_1}}{2^{n}} \int_{-1}^{0} \left(\frac{1-y^2}{1-|y|+\beta}\right)^n \mathrm{d}y \\
&= \frac{e^{Ct/2}\sqrt{2n+1}\sqrt{C_1}}{2^{n}} \int_{-1}^{0} \left(\frac{1-y^2}{1+y+\beta}\right)^n \mathrm{d}y \\
&\le \frac{e^{Ct/2}\sqrt{2n+1}\sqrt{C_1}}{2^{n}} \int_{-1}^{1} \left(\frac{1-y^2}{1+y+\beta}\right)^n \mathrm{d}y.
\end{aligned}
$$

Thus the proof is complete. □

Lemma 3.3 *(integral estimate from [3, 19]) Let* $\xi < -1$, *then*

$$
\int_{-1}^{1} \left(\frac{1-y^2}{y-\xi}\right)^n \mathrm{d}y = (2r)^n 2^{n+1} \frac{n!}{(2n+1)!!} \Phi_{n,0}\left(r^2\right), \tag{3.15}
$$

where $r = \frac{1}{|\xi|+\sqrt{\xi^2-1}}$ *and* $\Phi_{n,0}\left(r^2\right)$ *is the Gauss hypergeometric function. Moreover, we have*

$$
\Phi_{n,0}\left(r^2\right) = \sqrt{1-r^2} + \mathcal{O}\left(1/n^{1/3}\right),
$$

uniformly with respect to $0 < r < 1$.

By Lemma 3.1 and Lemma 3.3, together with the asymptotic equivalence $\frac{2^n n!}{(2n-1)!!} \sim \sqrt{\pi n}$ as $n \to \infty$, we finally establish in the following theorem the estimate for the gPC coefficients $\hat{f}_n$.

Theorem 3.1 *There exists* $\beta > 0$ *such that*

$$
\|\hat{f}_n\|_2 \lesssim e^{Ct/2}\sqrt{\pi C_1}\left(\sqrt{1-r^2} + \mathcal{O}\left(1/n^{1/3}\right)\right) r^n
$$

with $0 < r = \frac{1}{|\xi|+\sqrt{\xi^2-1}} < 1$ *and* $\xi = -1-\beta < -1$. *Here* $C = 2\|\sigma_s\|_\infty + 1$ *and* $C_1 = \|f_0\|_2^2 + C_\psi^2$.

Remark 3.3 As shown in the proof of Lemma 3.2, there exists a positive number r such that the radius $R(y)$ is positive. Accordingly, the decay rate r in Theorem 3.1 satisfies $0 < r < 1$. The specific value of r depends on time t and the diffusive scaling ϵ, as shown in the numerical results in Sect. 4, where the values of r are approximated by the decay rate of the numerical data. The dependence on ϵ is also numerically observed in [28, 41].

Remark 3.4 The bound for the gPC coefficients can also be achieved by applying $\left\| \frac{\mathrm{d}^n}{\mathrm{d}y^n} f \right\|_2$ given in Theorem 2.4. By this approach, we have

$$
\begin{aligned}
\|\hat{f}_n\|_2 &\le \frac{\sqrt{2n+1}}{2^{n+1}n!} \int_{-1}^{1} \left\| \frac{\mathrm{d}^n f}{\mathrm{d}y^n} \right\|_2 (1-y^2)^n \mathrm{d}y \\
&\le \frac{\sqrt{2n+1}}{2^{n+1}\sqrt{n!}} e^{\frac{Ct}{2}} \|\phi\|_\infty^n t^{\frac{n}{2}} \sqrt{C_1} \int_{-1}^{1} (1-y^2)^n \mathrm{d}y \\
&= e^{\frac{Ct}{2}} \sqrt{C_1} (\|\phi\|_\infty \sqrt{t})^n \frac{2^n}{\sqrt{2n+1}} \frac{(n!)^{\frac{3}{2}}}{(2n)!} \\
&\le e^{\frac{Ct}{2}} \sqrt{C_1} (\|\phi\|_\infty \sqrt{t})^n \frac{2^n}{\sqrt{2n+1}} \frac{(\sqrt{2\pi n} n^n e^{-n+\frac{1}{12n}})^{\frac{3}{2}}}{\sqrt{4\pi n}(2n)^{2n} e^{-2n}} \\
&\lesssim e^{\frac{Ct}{2}} \sqrt{C_1} n^{-\frac{1}{4}} \left(\frac{\|\phi\|_\infty \sqrt{et}}{2\sqrt{n}} \right)^n
\end{aligned}
$$

with C and C_1 defined in Theorem 2.4. Here we use the fact that $\int_{-1}^{1}(1-y^2)\mathrm{d}y = \frac{2^{2n+1}(n!)^2}{(2n+1)!}$ in the third equality and the Stirling formula $\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n+\frac{1}{12n+1}} < n! < \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n+\frac{1}{12n}}$ in the forth inequality. As a consequence, we also obtain the decay rate of the gPC coefficients $\hat{f}_n$.

It is worth mentioning that although the decay rate analysis above is carried out for one KL mode, i.e., one-dimensional random variable, it can be extended to multi-dimension as well. We now present the two-dimensional case to show the derivation. Let $y = (y_1, y_2) \in [-1, 1]^2$ denote the two-dimensional random variable and $\hat{f}_{n_1 n_2}$ be the gPC coefficient associated with the basis function $p_{n_1}(y) p_{n_2}(y)$, where $\{p_n\}$ are the one-dimensional Legendre polynomials given in (3.12). Similar as (3.13), $\hat{f}_{n_1 n_2}$ can be rewritten as

$$
\begin{aligned}
\hat{f}_{n_1 n_2} &= \frac{1}{2^2} \int_{-1}^{1} \int_{-1}^{1} f p_{n_1} p_{n_2} \mathrm{d}y_1 \mathrm{d}y_2 \\
&= \frac{(-1)^{n_1}(-1)^{n_2}}{n_1! n_2!} \frac{\sqrt{(2n_1+1)(2n_2+1)}}{2^{n_1+n_2+2}} \\
&\quad \int_{-1}^{1} \int_{-1}^{1} \frac{\partial^{n_1+n_2}}{\partial y_1^{n_1} \partial y_2^{n_2}} f(y_1, y_2) \left(1-y_1^2\right)^{n_1} \left(1-y_2^2\right)^{n_2} \mathrm{d}y_1 \mathrm{d}y_2
\end{aligned}
$$

after integrating by parts n_1 times with respect to y_1 and n_2 times with respect to y_2.

It follows from Lemma 3.1 that $f(y_1, y_2)$ is analytic with respect to one argument while fixing the other, which combining with the classical complex theory yields that f is analytic with respect to $(y_1, y_2) \in [-1, 1]^2$. Thus f can be analytically extended to the complex domain with several complex variables (η_1, η_2). Similar to one-dimensional case, by applying Cauchy's formula, we have

$$
\frac{\partial^{n_1+n_2}}{\partial y_1^{n_1} \partial y_2^{n_2}} f(y_1, y_2) = \frac{n_1! n_2! (-1)^{n_1+n_2}}{(2\pi \mathrm{i})^2} \int_{\gamma_{y_2}} \int_{\gamma_{y_1}} \frac{f(\eta_1, \eta_2)}{(\eta_1 - y_1)^{n_1+1} (\eta_2 - y_2)^{n_2+1}} \mathrm{d}\eta_1 \mathrm{d}\eta_2,
$$

where γ_{y_ℓ} $(\ell = 1, 2)$ are positively oriented closed circumferences with the center at the real point y_ℓ and the radius set as $R(y_\ell) = 1 - |y_\ell| + \beta > 0$. Here β is a positive constant. By

Corollary 2.1, we obtain

$$\left\| \frac{\partial^{n_1+n_2}}{\partial y_1^{n_1} \partial y_2^{n_2}} f \right\|_2 \le \frac{n_1! n_2!}{(2\pi)^2} \int_{\gamma_{y_2}} \int_{\gamma_{y_2}} \frac{\| f(\eta_1, \eta_2) \|_2}{|\eta_1 - y_1|^{n_1+1} \, |\eta_2 - y_2|^{n_2+1}} |\mathrm{d}\eta_1| \, |\mathrm{d}\eta_2|$$
$$\le e^{\frac{Ct}{2}} \sqrt{C_1} \frac{n_1! n_2!}{R(y_1)^{n_1+1} R(y_2)^{n_2+1}},$$

which further yields the estimates for the gPC coefficient as

$$\| \hat{f}_{n_1 n_2} \|_2 \le \frac{\sqrt{(2n_1+1)(2n_2+1)}}{2^{n_1+n_2}} \int_{-1}^{1} \int_{-1}^{1} \left(\frac{1 - y_1^2}{1 + y_1 + \beta} \right)^{n_1} \left(\frac{1 - y_2^2}{1 + y_2 + \beta} \right)^{n_2} \mathrm{d}y_1 \mathrm{d}y_2.$$

The decay rate of $\| \hat{f}_{n_1 n_2} \|_2$ can be obtained by estimating integrals by Lemma 3.3 for each variable.

3.3 Error Estimates of the Stochastic Galerkin Method

In this section, we investigate the error between the exact solution f to the RTE (2.7) and the numerical solution g_N obtained by the stochastic Galerkin method presented in Sect. 3.1. The error can be separated into the following two parts

$$f - g_N = \underbrace{f - \mathcal{P}_N f}_{\text{truncation error}} + \underbrace{\mathcal{P}_N f - g_N}_{\text{scheme error}}$$

with $\mathcal{P}_N f$ defined in (3.4). We will first show the estimate for the truncation error $f - \mathcal{P}_N f$ in Sect. 3.3.1. In Sect. 3.3.2, we study the scheme error $\mathcal{P}_N f - g_N$ and establish the main results about $f - g_N$.

3.3.1 Error Estimate of $f - \mathcal{P}_N f$

In this section, we establish the error estimate of $f - \mathcal{P}_N f$ for one-dimensional and multi-dimensional random variables. We first show in the following lemma the error estimate for the case when the random variable y is one-dimensional.

Lemma 3.4 *Let $U = [-1, 1]$. There exists a positive constant $\beta > 0$ such that*

$$\| f - \mathcal{P}_N f \|_{L^2(U \times \mathcal{R} \times \mathbb{S}; \rho(y) \mathrm{d}y \mathrm{d}x \mathrm{d}v)} \lesssim e^{Ct/2} \sqrt{\pi C_1} \left(\sqrt{1 - r^2} + \mathcal{O}(1/N^{1/3}) \right) \frac{r^{N+1}}{\sqrt{1 - r^2}}, \tag{3.16}$$

where $0 < r = \frac{1}{|\xi| + \sqrt{\xi^2 - 1}} < 1$ with $\xi = -1 - \beta < -1$, $C = 2\|\sigma_s\|_\infty + 1$, and $C_1 = \| f_0 \|_2^2 + C_\psi^2$.

Proof By the definition of the norm in (2.8), Equation (3.2), and the orthonormal property of the gPC basis polynomials, we have

$$\| f - \mathcal{P}_N f \|^2_{L^2(U \times \mathcal{R} \times \mathbb{S}; \rho(y) \mathrm{d}y \mathrm{d}x \mathrm{d}v)} = \int_{-1}^{1} \| f - \mathcal{P}_N f \|_2^2 \, \rho(y) \mathrm{d}y = \sum_{n=N+1}^{\infty} \| \hat{f}_n \|_2^2, \tag{3.17}$$

which, together with Theorem 3.1, yields

$$
\begin{aligned}
\|f-\mathcal{P}_N f\|^2_{L^2(U\times\mathcal{R}\times\mathbb{S};\rho(y)\mathrm{d}y\mathrm{d}x\mathrm{d}v)} &\lesssim e^{Ct}\pi C_1 \sum_{n=N+1}^{\infty}\left(\sqrt{1-r^2}+\mathcal{O}(1/n^{1/3})\right)^2 r^{2n}\\
&\lesssim e^{Ct}\pi C_1 \left(\sqrt{1-r^2}+\mathcal{O}(1/N^{1/3})\right)^2 \sum_{n=N+1}^{\infty} r^{2n}\\
&\lesssim e^{Ct}\pi C_1 \left(\sqrt{1-r^2}+\mathcal{O}(1/N^{1/3})\right)^2 \frac{r^{2N+2}}{1-r^2}.
\end{aligned}
\tag{3.18}
$$

Thus the proof is complete. □

For the case with d-dimensional random variable $y=(y_1,y_2,\ldots,y_d)$, $\mathcal{P}_N f$ is now multi-indexed with $N=(N_1,N_2,\ldots,N_d)$ where N_k is the order of the gPC basis polynomial for y_k. The error estimate for this case in the following theorem can be obtained by applying the analysis in Lemma 3.4 direction by direction.

Theorem 3.2 *Let $U=[-1,1]^d$. There exists positive constants $\beta_1,\ldots,\beta_d>0$ such that*

$$
\|f-\mathcal{P}_N f\|_{L^2(U\times\mathcal{R}\times\mathbb{S};\ \rho(y)\mathrm{d}y\mathrm{d}x\mathrm{d}v)} \lesssim e^{Ct/2}\sqrt{\pi C_1}\sum_{k=1}^{d}\left(\sqrt{1-r_k^2}+\mathcal{O}(1/N_k^{1/3})\right)\frac{r_k^{N_k+1}}{\sqrt{1-r_k^2}},
\tag{3.19}
$$

where $0<r_k=\frac{1}{|\xi_k|+\sqrt{\xi_k^2-1}}<1$ with $\xi_k=-1-\beta_k<-1$ for $k=1,\ldots,d$. Here $C=2\|\sigma_s\|_\infty+1$ and $C_1=\|f_0\|_2^2+C_\psi^2$.

3.3.2 Error Estimate of $f-g_N$: One-dimensional Random Space

In this section, we investigate the error estimates of $\mathcal{P}_N f-g_N$ and establish the main results for the error estimates of $f-g_N$ for one-dimensional random variables.

By the orthonormal property of the gPC basis polynomials, the error estimate for $\mathcal{P}_N f-g_N$ are

$$
\|\mathcal{P}_N f-g_N\|^2_{L^2(U\times\mathcal{R}\times\mathbb{S};\ \rho(y)\mathrm{d}y\mathrm{d}x\mathrm{d}v)}=\sum_{n=0}^{N}\|\hat f_n-\hat g_n\|_2^2.
$$

Let $\hat f=(\hat f_0,\hat f_1,\ldots,\hat f_N,\ldots)^T$ and $\hat g=(\hat g_0,\hat g_1,\ldots,\hat g_N)^T$. Then (3.5) and (3.8) can be rewritten as

$$
\partial_t\hat f+v\cdot\nabla_x\hat f=A\mathcal{L}\hat f, \tag{3.20}
$$

$$
\partial_t\hat g+v\cdot\nabla_x\hat g=A^{11}\mathcal{L}\hat g, \tag{3.21}
$$

where A is an infinite matrix with each entry defined in (3.6), and A^{11} is the top left $(N+1)\times(N+1)$ block of A. Furthermore, we split $\hat f$ into the following two vectors

$$
\hat f^{(1)}=(\hat f_0,\hat f_1,\ldots,\hat f_N)^T,\quad \hat f^{(2)}=(\hat f_{N+1},\hat f_{N+2},\ldots)^T
$$

with which (3.20) can be written as

$$
\partial_t\begin{bmatrix}\hat f^{(1)}\\ \hat f^{(2)}\end{bmatrix}+v\cdot\nabla_x\begin{bmatrix}\hat f^{(1)}\\ \hat f^{(2)}\end{bmatrix}=A\begin{bmatrix}\mathcal{L}\hat f^{(1)}\\ \mathcal{L}\hat f^{(2)}\end{bmatrix}=\begin{bmatrix}A^{11} & A^{12}\\ A^{21} & A^{22}\end{bmatrix}\begin{bmatrix}\mathcal{L}\hat f^{(1)}\\ \mathcal{L}\hat f^{(2)}\end{bmatrix}. \tag{3.22}
$$

Comparing it with (3.21), the error vector $\varepsilon = \hat{f}^{(1)} - \hat{g}$ satisfies the following equation

$$\partial_t \varepsilon + v \cdot \nabla_x \varepsilon = A^{11} \mathcal{L} \varepsilon + A^{12} \mathcal{L} \hat{f}^{(2)}. \tag{3.23}$$

The coupling matrix A^{11} is symmetric positive definite as shown in [18, 37]. It has $N+1$ real eigenvalues $\{\bar{\sigma}_s + \zeta_j\}_{j=0}^{N}$ where $\{\zeta_j\}$ are Legendre-Gauss quadrature points in $[-1, 1]$. We address these properties of A^{11} in the following lemma. The proof is similar to the stationary RTE case discussed in [41] and thus is omitted.

Lemma 3.5 *A^{11} is a symmetric positive definite matrix. Denote $\lambda(A)$ as the collection of the eigenvalues of A^{11}. Then*

$$\|\lambda_k\|_\infty \le C_\sigma, \quad \lambda_k \in \lambda(A)$$

holds with the constant $C_\sigma = \|\bar{\sigma}_s\|_\infty + 1$.

Lemma 3.6 *Let $U = [-1, 1]$. There exists a positive constant $\beta > 0$ such that*

$$\|\mathcal{P}_N f - g_N\|_{L^2(U \times \mathcal{R} \times \mathbb{S};\ \rho(y)\mathrm{d}y\mathrm{d}x\mathrm{d}v)} \lesssim e^{C_{\max} t/2} \|\sigma_s\|_\infty \sqrt{\pi N C_1 t} \left(\sqrt{1-r^2} + \mathcal{O}(1/N^{1/3}) \right) \frac{r^{N+1}}{1-r},$$

where $0 < r = \frac{1}{|\xi| + \sqrt{\xi^2 - 1}} < 1$ with $\xi = -1 - \beta < -1$, $C_1 = \|f_0\|_2^2 + C_\psi^2$, and $C_{\max} = \max(C, 2C_\sigma + 1)$ with C defined in Theorem 3.1 and C_σ defined in Lemma 3.5.

Proof By Lemma 3.5, A^{11} can be diagonalized by a unitary matrix Q, i.e., $A^{11} = Q^{-1} \Lambda Q$ with $\Lambda = \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_N)$ being a diagonal matrix and λ_i, $i = 0, 1, \ldots, N$ being eigenvalues of A^{11}.

Let $\tilde{\varepsilon} = Q\varepsilon$, then (3.23) becomes

$$\partial_t \tilde{\varepsilon} + v \cdot \nabla_x \tilde{\varepsilon} = \Lambda \mathcal{L} \tilde{\varepsilon} + Q A^{12} \mathcal{L} \hat{f}^{(2)}, \tag{3.24}$$

which can be further written component-wisely as

$$\partial_t \tilde{\varepsilon}_m + v \cdot \nabla_x \tilde{\varepsilon}_m = \lambda_m \mathcal{L} \tilde{\varepsilon}_m + s_m, \quad m = 0, 1, \ldots, N, \tag{3.25}$$

with s_m being the m-th entry of $QA^{12}\mathcal{L}\hat{f}^{(2)}$. The boundary and initial conditions for (3.25) are

$$\tilde{\varepsilon}_m|_{\Gamma^-} = 0, \quad \tilde{\varepsilon}_m|_{t=0} = 0, \quad m = 0, 1, \ldots, N, \tag{3.26}$$

since the boundary and initial conditions of $\hat{f}$ and $\hat{g}$ are

$$\hat{f}_0\Big|_{\Gamma^-} = \hat{g}_0\big|_{\Gamma^-} = \psi, \quad \hat{f}_k\Big|_{\Gamma^-} = \hat{g}_k\big|_{\Gamma^-} = 0, \quad k = 1, 2, \ldots, N, \ldots,$$
$$\hat{f}_0\Big|_{t=0} = \hat{g}_0\big|_{t=0} = f_0, \quad \hat{f}_k\Big|_{t=0} = \hat{g}_0\big|_{t=0} = 0, \quad k = 1, 2, \ldots, N, \ldots,$$

based on the given boundary and initial conditions $f|_{\Gamma^-} = \psi(t, x, v)$, $f|_{t=0} = f_0(x, v)$.

It follows from Theorem 2.1 that the solution $\tilde{\varepsilon}_m$ to (3.25) with (3.26) satisfies

$$\|\tilde{\varepsilon}_m(t)\|_2^2 \le e^{C_m t} \int_0^t e^{-C_m \tau} \|s_m(\tau)\|_2^2 \mathrm{d}\tau, \tag{3.27}$$

where $C_m = 2\|\lambda_m\|_\infty + 1$. The estimate on s_m is required in order to get the bound for $\tilde{\varepsilon}_m(t)$. Clearly $\|Q\|_2 = \sqrt{\lambda_{max}(Q^T Q)} = 1$ since Q is a unitary matrix, and each entry of A satisfies

$$\|A_{mn}\|_\infty \le \|\sigma_s\|_\infty \tag{3.28}$$

according to the definition in (3.6). Combining with Lemma 2.1 and Theorem 3.1, we have

$$\begin{aligned}
\|s_m(t)\|_2 &= \left\| \sum_{k=0}^{N} \sum_{n=N+1}^{\infty} Q_{mk} A_{kn} \mathcal{L} \hat{f}_n \right\|_2 \\
&\leq \|\sigma_s\|_\infty \|Q\|_2 \|\mathcal{L}\|_2 \sum_{n=N+1}^{\infty} \|\hat{f}_n\|_2 \\
&\lesssim e^{Ct/2} \|\sigma_s\|_\infty \sqrt{\pi C_1} \sum_{n=N+1}^{\infty} \left(\sqrt{1-r^2} + \mathcal{O}\left(1/n^{1/3}\right) \right) r^n \\
&\lesssim e^{Ct/2} \|\sigma_s\|_\infty \sqrt{\pi C_1} \left(\sqrt{1-r^2} + \mathcal{O}\left(1/N^{1/3}\right) \right) \frac{r^{N+1}}{1-r},
\end{aligned} \tag{3.29}$$

where $0 < r = \frac{1}{|\xi| + \sqrt{\xi^2 - 1}} < 1$ with $\xi = -1 - \beta < -1$, $C = 2\|\sigma_s\|_\infty + 1$, and $C_1 = \|f_0\|_2^2 + C_\psi^2$.

By substituting (3.29) into (3.27), we have

$$\|\tilde{\varepsilon}\|_2^2 = \sum_{m=0}^{N} \|\tilde{\varepsilon}_m(t)\|_2^2 \lesssim \|\sigma_s\|_\infty^2 \pi C_1 \left(\sqrt{1-r^2} + \mathcal{O}(1/N^{1/3}) \right)^2 \frac{r^{2N+2}}{(1-r)^2} \sum_{m=0}^{N} e^{C_m t} \int_0^t e^{(-C_m + C)\tau} \mathrm{d}\tau.$$

Denote $\tilde{C}_\sigma = 2C_\sigma + 1$ with C_σ defined in Lemma 3.5. Clearly $C_m \leq \tilde{C}_\sigma$ for $m = 0, \cdots, N$. It follows from the fact that $\frac{e^{Ct} - e^{C_m t}}{C - C_m}$ is increasing with respect to C_m and $e^x \geq x + 1$ that

$$\sum_{m=0}^{N} e^{C_m t} \int_0^t e^{(-C_m + C)\tau} \mathrm{d}\tau = \sum_{m=0}^{N} \frac{e^{Ct} - e^{C_m t}}{C - C_m} \leq \sum_{m=0}^{N} \frac{e^{Ct} - e^{\tilde{C}_\sigma t}}{C - \tilde{C}_\sigma} \leq N t e^{C_{\max} t},$$

where $C_{\max} = \max(C, \tilde{C}_\sigma)$. Therefore,

$$\|\tilde{\varepsilon}\|_2 \lesssim e^{C_{\max} t/2} \|\sigma_s\|_\infty \sqrt{\pi N C_1 t} \left(\sqrt{1-r^2} + \mathcal{O}(1/N^{1/3}) \right) \frac{r^{N+1}}{1-r},$$

which, together with the orthonormal property of the gPC basis polynomials and the fact that $Q^T Q = I$, yields

$$\|\mathcal{P}_N f - g_N\|_{L^2(U \times \mathcal{R} \times \mathbb{S};\ \rho(y)\mathrm{d}y\mathrm{d}x\mathrm{d}v)} = \|\varepsilon(t)\|_2 = \|Q^T \tilde{\varepsilon}\|_2 = \|\tilde{\varepsilon}\|_2.$$

The proof is complete. □

We end this subsection by presenting the main theorem regarding the error estimate of $f - g_N$ for one-dimensional random variables.

Theorem 3.3 *Let* $U = [-1, 1]$. *There exists a positive constant* $\beta > 0$ *such that*

$$\begin{aligned}
&\|f - g_N\|_{L^2(U \times \mathcal{R} \times \mathbb{S}; \rho(y)\mathrm{d}y\mathrm{d}x\mathrm{d}v)} \\
&\quad \lesssim \sqrt{\pi C_1} \left(\|\sigma_s\|_\infty \sqrt{Nt} e^{C_{\max} t/2} + e^{Ct/2} \right) \left(\sqrt{1-r^2} + \mathcal{O}(1/N^{1/3}) \right) \frac{r^{N+1}}{1-r},
\end{aligned} \tag{3.30}$$

where $0 < r = \frac{1}{|\xi|+\sqrt{\xi^2-1}} < 1$ *with* $\xi = -1 - \beta < -1$, $C = 2\|\sigma_s\|_\infty + 1$, *and* $C_1 = \|f_0\|_2^2 + C_\psi^2$. *Here* $C_{\max}$ *is the same as defined in Lemma 3.6.*

Proof According to the definition of $\|\cdot\|_{L^2(U\times\mathcal{R}\times\mathbb{S};\rho(y)\mathrm{d}y\mathrm{d}x\mathrm{d}v)}$, we have

$$\begin{aligned}\|f - g_N\|^2_{L^2(U\times\mathcal{R}\times\mathbb{S};\rho(y)\mathrm{d}y\mathrm{d}x\mathrm{d}v)} &= \sum_{n=0}^{N} \|\hat{f}_n - \hat{g}_n\|_2^2 + \sum_{n=N+1}^{\infty} \|\hat{f}_n\|_2^2 \\ &= \|\mathcal{P}_N f - g_N\|^2_{L^2(U\times\mathcal{R}\times\mathbb{S};\rho(y)\mathrm{d}y\mathrm{d}x\mathrm{d}v)} \\ &\quad + \|f - \mathcal{P}_N f\|^2_{L^2(U\times\mathcal{R}\times\mathbb{S};\rho(y)\mathrm{d}y\mathrm{d}x\mathrm{d}v)},\end{aligned}$$

which, together with Lemma 3.4 and Lemma 3.6, completes the proof. □

3.3.3 Error Estimate of $f - g_N$: Multi-dimensional Random Space

With the multi-indices $m = (m_1, \ldots, m_d)$, $n = (n_1, \ldots, n_d)$, and $N = (N_1, \ldots, N_d)$, the gPC expansion is written as

$$f(t, x, v, y) = \sum_{n_1,n_2,\ldots,n_d=0}^{\infty} \hat{f}_{n_1,\ldots,n_d}(t, x, v) p_{n_1}(y_1)\cdots p_{n_d}(y_d), \tag{3.31}$$

and the first N-term gPC projection is

$$\mathcal{P}_N f = \sum_{n_1,\ldots,n_d=0}^{N_1\ldots,N_d} \hat{f}_{n_1,\ldots,n_d}(t, x, v) p_{n_1}(y_1)\cdots p_{n_d}(y_d). \tag{3.32}$$

The numerical solution g_N obtained by the stochastic Galerkin method is

$$g_N(t, x, v, y) = \sum_{n_1,\ldots,n_d=0}^{N_1\ldots,N_d} \hat{g}_{n_1,\ldots,n_d}(t, x, v) p_{n_1}(y_1)\cdots p_{n_d}(y_d). \tag{3.33}$$

The gPC coefficients for f and g_N still satisfies Eqs. (3.5) and (3.8), respectively, with each entry of A defined by

$$A_{n,m} = \left\langle \sigma_s(y) p_{m_1}(y_1)\cdots p_{m_d}(y_d), p_{n_1}(y_1)\cdots p_{n_d}(y_d)\right\rangle_y. \tag{3.34}$$

According to the KL expansion of σ_s in (2.6), then

$$A = \bar{\sigma}_s \otimes_{k=1}^{d} \mathbb{I}_k + A_1 \otimes_{k=2}^{d} \mathbb{I}_k + \mathbb{I}_1 \otimes A_2 \otimes_{k=3}^{d} \mathbb{I}_k + \ldots + \otimes_{k=1}^{d-1}\mathbb{I}_k \otimes A_d,$$

where $\otimes$ denotes the Kronecker product, A_k is an infinite matrix with the $m_k n_k$-th entry given by $\langle y_k\phi_k p_{m_k}(y_k), p_{n_k}(y_k)\rangle_{y_k}$, and I_k is the infinity identity matrix, for $k = 1, \ldots, d$. A's top left block is

$$A^{11} = \bar{\sigma}_s \otimes_{k=1}^{d} \mathbb{I}_k^{11} + A_1^{11} \otimes_{k=2}^{d} \mathbb{I}_k^{11} + \mathbb{I}_1^{11} \otimes A_2^{11} \otimes_{k=3}^{d} \mathbb{I}_k^{11} + \ldots + \otimes_{k=1}^{d-1}\mathbb{I}_k^{11} \otimes A_d^{11},$$

where A_k^{11} and $\mathbb{I}_k^{11}$ are the top left $N_k \times N_k$ block in y_k direction of matrices A_k and $\mathbb{I}_k$, respectively. Then the eigenvalues of A^{11} satisfies

$$\lambda_{\max}\left(A^{11}\right) \le \bar{\sigma}_s + \sum_{k=1}^{d} \lambda_{\max}\left(A_k^{11}\right),$$

since the eigenvalues of the Kronecker product matrix are the products of the eigenvalues of the sub-matrices; see [29] for more details. Moreover, $\lambda(A_k^{11})$ is bounded for $k = 1, 2. \dots, d$, as discussed in the one-dimensional case. Thus $\lambda(A^{11})$ is bounded under simple algebraic operations. Namely, there exists a positive constant C_σ that depends only on σ_s such that

$$\|\lambda\|_\infty \le C_\sigma \tag{3.35}$$

holds for all the eigenvalues of A^{11}.

With the bound (3.35), the error estimate of $\mathcal{P}_N f - g_N$ in Lemma 3.6 for one-dimensional random variable can be extended to the multi-dimensional case by replacing $C_m = 2\|\lambda_m\|_\infty + 1$ in (3.27) with $2C_\sigma + 1$. The extension involves much more tedious notations but no technique difficulties. Combining with the error estimate of $f - \mathcal{P}_N f$ for multi-dimensional random variable in Theorem 3.2, we finally present in the following theorem the main result for the error estimate of $f - g_N$ in d-dimensional random space.

Theorem 3.4 *Let $U = [-1, 1]^d$. Denote the d-dimensional random variable as $y = (y_1, y_2, \dots, y_d)$ with $N = (N_1, N_2, \dots, N_d)$, where N_k is the order of the gPC basis polynomial for y_k. There exists a positive constant $\beta > 0$ such that*

$$\begin{aligned}&\|f - g_N\|_{L^2(U\times\mathcal{R}\times\mathbb{S};\rho(y)\mathrm{d}y\mathrm{d}x\mathrm{d}v)}\\ &\quad\lesssim \sqrt{\pi C_1}\sum_{k=1}^{d}\left(\|\sigma_s\|_\infty\sqrt{N_k t}\,e^{C_{\max}t/2} + e^{Ct/2}\right)\left(\sqrt{1-r_k^2} + \mathcal{O}(1/N_k^{1/3})\right)\frac{r_k^{N_k+1}}{1-r_k},\end{aligned}$$

where $0 < r_k = \frac{1}{|\xi_k| + \sqrt{\xi_k^2 - 1}} < 1$ with $\xi_k = -1 - \beta < -1$, $C = 2\|\sigma_s\|_\infty + 1$, $C_1 = \|f_0\|_2^2 + C_\psi^2$, and
$C_{\max} = \max(C, 2C_\sigma + 1)$ with C_σ given in (3.35).

Remark 3.5 The values $\{r_k\}$ here is the multi-dimensional extension of that in Theorem 3.1. Similar to Remark 3.3, here we only show the existence of this decay rate r with $0 < r < 1$. The specific value of r depends on the final time t and the diffusive scaling. It is challenging to spell out the details. We refer interested readers to [2, 3, 40] for the original derivations that uses this argument.

4 Numerical Tests

In this section, we verify our theoretical results with the following time-dependent RTE

$$\epsilon\partial_t f + v\cdot\nabla_x f = \frac{\sigma(x,y)}{\epsilon}\left(\frac{1}{2}\int_{-1}^{1} f(v)\mathrm{d}v - f\right), \quad x\in[0,1], v\in[-1,1], \tag{4.1}$$

with the random coefficient $\sigma(x, y)$. The initial condition is given by

$$f(t=0,x,v,y) = 0, \quad x\in[0,1],\ v\in[-1,1], \tag{4.2}$$

and boundary conditions are

$$f(t, x=0, v, y) = 1, \quad v > 0; \qquad f(t, x=1, v, y) = 0, \quad v < 0.$$

We use the upwind finite difference method to discretize the x variable with N_x uniform points in [0, 1]. v variable is discretized with the Legendre-Gauss nodes on $[-1, 1]$, and 16

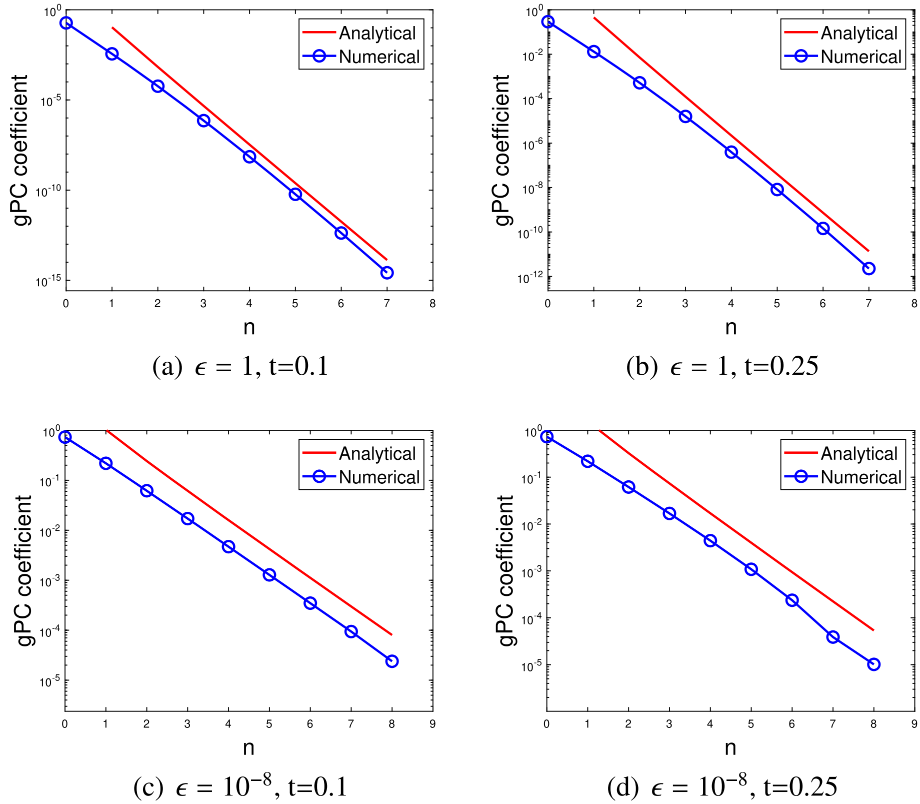

Fig. 1 Decay rate of gPC coefficients for $\sigma(x, y) = 2 + y$ at $t = 0.1$ (left) and $t = 0.25$ (right). Top: $\epsilon = 1$ with $N_x = 80$. Bottom: $\epsilon = 10^{-8}$ with $N_x = 100$

points are used in our numerical tests. We apply the third order strong-stability-preserving Runge–Kutta method [35] for temporal discretization with the time step set as $\Delta t = 0.035\Delta x$.

We first test one-dimensional random space with $\sigma(x, y) = 2 + y$, where $y \in [-1, 1]$ is uniformly distributed. Figure 1 plots the decay rate of the gPC coefficients with different diffusive scaling at $t = 0.1$ and $t = 0.25$. The analytical result is computed by

$$\|\hat{f}_n\|_2 \sim e^{Ct/2}\sqrt{\pi C_1}(\sqrt{1-r^2} + \frac{2}{n^{1/3}})r^n \tag{4.3}$$

by Theorem 3.1 with $C = \max_{\{x,y\}} |\sigma(x, y)| = 7$, and $C_\psi = 1/4$. For $\epsilon = 1$, we choose $r = 0.008$ for $t = 0.1$ and $r = 0.019$ for $t = 0.25$. For $\epsilon = 10^{-8}$, we choose $r = 0.27$ for $t = 0.1$ and $r = 0.24$ for $t = 0.25$. $N_x = 80$ for $\epsilon = 1$, while a finer mesh $N_x = 100$ is taken for $\epsilon = 10^{-8}$ to obtain better resolution. It is worth mentioning that the asymptotic preserving method [28] is adopted for the case with $\epsilon = 10^{-8}$.

We now test the multi-dimensional case where the random field $\sigma(x, y)$ has the following form

$$\sigma(x, y) = 1 + \sigma \sum_{k=1}^{d} \cos(2\pi kx) y_k \tag{4.4}$$

with $\sigma = \frac{1}{d+1}$ and $\{y_k\}$ being a set of mutually independent uniformly distributed random variables in $[-1, 1]$. We set $d = 6$ and plot the decay rate of the gPC coefficients with

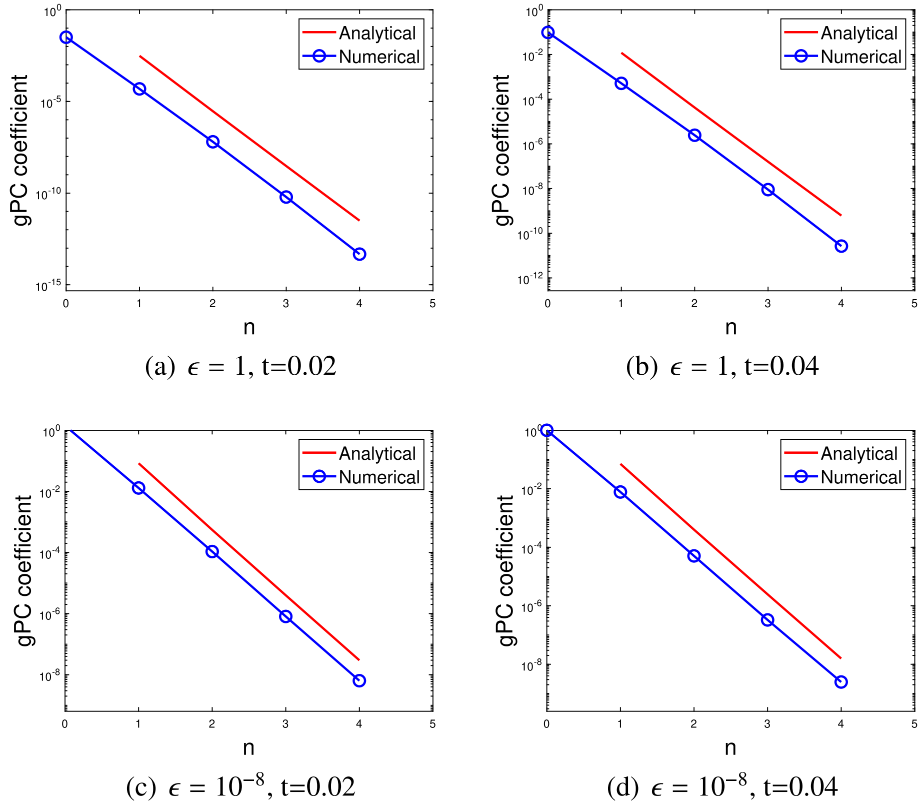

Fig. 2 Decay rate of gPC coefficients for σ_s in (4.4) with $d = 6$ at $t = 0.02$ (left) and $t = 0.04$ (right). Top: $\epsilon = 1$ with $N_x = 40$. Bottom: $\epsilon = 10^{-8}$ with $N_x = 100$

different diffusive scaling at $t = 0.02$ and $t = 0.04$ in Fig. 2. The analytic result is computed by (4.3) with $C = 2(1 + \frac{d}{d+1}) + 1 = \frac{33}{7}$ and $C_\psi = 1/4$. For $\epsilon = 1$, we choose $r = 0.001$ for $t = 0.02$ and $r = 0.004$ for $t = 0.04$. For $\epsilon = 10^{-8}$, we choose $r = 0.008$ for $t = 0.02$ and $r = 0.006$ for $t = 0.04$. Again, we use $N_x = 40$ for $\epsilon = 1$ and a finer mesh $N_x = 100$ for $\epsilon = 10^{-8}$. Again asymptotic preserving technique [28] is applied for the case $\epsilon = 10^{-8}$.

5 Conclusion

In this paper, we investigate the impact of uncertainty for the time-dependent radiative transfer equation with nonhomogeneous boundary condition through the stochastic Galerkin approximation. We theoretically prove the a-priori bound of the solution, its continuity with respect to the coefficient σ_s, as well as the regularity in the random space. Based on the theoretical study of the regularity of the solution, the stochastic Galerkin method of the gPC approach is adopted. Besides the regularity study for the case with nonhomogeneous boundary conditions, our main contribution is that we provide a recipe for a more fundamental question: why is the gPC expansion even a converging series? Furthermore, we prove the exponential decay rate of the gPC coefficients and establish the error estimates of the stochastic Galerkin method. Our analysis shows that the error decay exponentially with the rate r^n. Numerical tests are conducted based on our analytical results.

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Declarations

Conflict of interest Both authors have no relevant financial or non-financial interests to disclose.

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