The Annals of Statistics
2023, Vol. 51, No. 1, 362–390
https://doi.org/10.1214/22-AOS2255

UNIFORM CONSISTENCY IN NONPARAMETRIC MIXTURE MODELS

BY BRYON ARAGAM[1,a] AND RUIYI YANG[2,b]

[1]*Booth School of Business, University of Chicago,* [a]*bryon@chicagobooth.edu*
[2]*Program in Applied and Computational Mathematics, Princeton University,* [b]*ry8311@princeton.edu*

We study uniform consistency in nonparametric mixture models as well as closely related mixture of regression (also known as mixed regression) models, where the regression functions are allowed to be nonparametric and the error distributions are assumed to be convolutions of a Gaussian density. We construct uniformly consistent estimators under general conditions while simultaneously highlighting several pain points in extending existing pointwise consistency results to uniform results. The resulting analysis turns out to be nontrivial, and several novel technical tools are developed along the way. In the case of mixed regression, we prove L^1 convergence of the regression functions while allowing for the component regression functions to intersect arbitrarily often, which presents additional technical challenges. We also consider generalizations to general (i.e., nonconvolutional) nonparametric mixtures.

1. Introduction. Mixture models are a classical approach to modeling heterogeneous populations composed of many subpopulations, and have found a variety of applications in prediction and classification [9, 10, 16, 17], clustering [29, 64] and latent variable models [1, 31, 55, 56]. Mixture models can also be used as a flexible tool for density estimation [33, 34, 58] and arise in the study of empirical Bayes [27, 74] and deconvolution [26, 65, 92]. When covariates are involved, mixtures can be used to model heterogeneous dependencies between an observation Y and some covariate(s) X, in which the conditional distribution $\mathbb{P}[Y \mid X = x]$ arises as a mixture of multiple (noisy) regression curves. Despite their relevance and usefulness in applications, mixture models can be notoriously difficult to analyze: Except in special cases, mixture models are a classical example of a nonidentifiable, irregular statistical model. As one might imagine, this situation is exacerbated for *nonparametric* mixtures, to the extent that even fundamental properties such as identifiability and consistency remain only partially addressed.

For parametric mixture models, many of these issues have been carefully addressed. We now have optimal estimators for Gaussian mixtures [20, 39, 83], a detailed understanding of the EM algorithm for mixtures [5, 8] and efficient algorithms for mixed linear regression models [60, 89]. The situation for nonparametric mixtures, however, is quite different. Here and in the sequel, by a "nonparametric mixture" we mean a finite mixture whose mixture components belong to a nonparametric family of distributions. For both vanilla nonparametric mixtures and mixtures of nonparametric regressions, much less is known despite many decades of work. For example, although there is a substantial body of work focused on core identifiability and estimation problems, *uniform* consistency has been comparatively understudied; see Section 2 for a more detailed review of previous work.

Motivated by this disparity, in this paper we study uniform consistency in nonparametric mixture models and highlight several subtleties that arise when constructing uniformly consistent estimators and that appear to be peculiar to the setting of nonparametric mixtures.

Received August 2021; revised October 2022.
MSC2020 subject classifications. Primary 62G05; secondary 62J02, 62G20.
Key words and phrases. Mixture models, mixed regression, nonparametric estimation, uniform consistency.