DOI 10.1007/s11141-022-10171-6
Radiophysics and Quantum Electronics, Vol. 64, No. 10, March, 2022
(Russian Original Vol. 64, No. 10, October, 2021)

CHAOS AND GEOMETRICAL OPTICS

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Chaotic evolution of dynamical systems is caused by the divergence of nearby orbits, i. e., by the intrinsic instability of the dynamics. The best way to see how the divergence of orbits may occur is to consider the orbits as the rays of light, i. e., within the framework of the geometrical optics. We discuss the basic mechanisms of chaos and demonstrate how the discovery of these mechanisms allowed one to enrich the geometrical optics by some new fundamental ideas and notions.

1. INTRODUCTION

A fundamental principle of classical mechanics states that the orbits of conservative systems follow geodesics on a manifold of constant energy [1]. Hadamard [2] was the first to observe the instability of the dynamics if a manifold of constant energy has a negative curvature. Exponential instability (i. e., the chaotic behavior) and ergodicity of geodesic flows on surfaces of constant negative curvature were proved by Hedlund and Hopf [3–5]. It was known even before that geodesic flows on surfaces with positive curvature demonstrate a regular, and sometimes even integrable (e. g., on ellipsoids) dynamics.

These results formed a very natural intuition in physics and mathematical communities that the positive curvature always generates stable dynamics, while the negative curvature is the only cause of instability, when the neughboring orbits tend to diverge in the phase space of a conservative mechanical system. Moreover, bearing in mind this very natural and seemingly so "obvious" ideology, Hopf [6] even claimed that if the positive curvature is confined on a "small" piece of a manifold, then a geodesic flow still will be unstable and ergodic (i. e., chaotic in modern terminology) because the orbits will spend much more time on the negative curvature part than on the part with positive curvature.

Following these ideas, N. S. Krylov made a major breakthrough in the fundamentals of statistical mechanics by establishing that the dynamics of a gas of hard spheres is also exponentially unstable, as the dynamics of geodesic flows on surfaces of negative curvature [7]. His idea was that the dispersing boundary of spheres plays a role analogous to the negative curvature for geodesic flows forcing the initially close orbits to diverge fast in the phase space of a system. This idea was essentially generalized and put on a firm mathematical ground by Ya. G. Sinai [8, 9].

In [8], a class of the so-called Sinai billiards was introduced. A billiard is a dynamical system generated by the uniform motion of a point particle within some domain Q (called a billiard table), which undergoes elastic reflections (i. e., the angle of reflection equals the angle of incidence) off the boundary ∂Q. Analogously, one can describe the dynamics of billiards as the evolution of the rays of light in a billiard table Q with the boundary ∂Q formed by mirrors. In the Sinai billiards, the boundary is smooth and dispersing, i. e., ∂Q is convex inward the billiard table Q. In a more general case, where the boundary is not smooth, the corresponding billiards are called dispersing. The mechanism which generates the chaotic dynamics

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Georgia Institute of Technology, Atlanta, USA. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Radiofizika, Vol. 64, No. 10, pp. 769–776, October 2021. Russian DOI: 10.52452/00213462_2021_64_10_769
Original article submitted July 8, 2021; accepted October 28, 2021.

of geodesic flows on the surface with negative curvature and in dispersing billiards is naturally called the mechanism of dispersing.

In [8], Sinai has laid a foundation of the mathematical theory of billiards, which is now at the center of the modern theory of dynamical systems. Moreover, soon after the publication of work [8], the theory of billiards took a leading role in the theory of dynamical systems, which was played before by geodesic flows. This happened because of the discovery of a new fundamental mechanism of chaos [10–12], which was called the mechanism of defocusing by late B. V. Chirikov.

To a great surprise to both physicist and mathematician communities, it was proved [10, 11] that a focusing boundary (i. e., a boundary that is convex outward a billiard table) can generate chaotic dynamics. Even having rigorous mathematical proof, the physicist community could not believe that such a phenomenon, which contradicts a very clear, natural, and seemingly unbeatable intuition, may exist. Only after numerous computer experiments, physicists agreed that mathematicians can still discover new fundamental physical phenomena.

A natural definition of a dispersing mirror is that any parallel beam of rays after reflection off this mirror becomes divergent. On the contrary, a parallel beam of rays becomes convergent (focuses) after reflection off a focusing mirror. A mirror is called neutral if it has a zero curvature, i. e., any parallel beam of rays remains parallel after reflection off such a mirror.

The essence of the mechanism of defocusing is that after the focusing (convergence) a beam of rays may defocus and become divergent (dispersing). If after the defocusing this beam travels for a longer time than the time of covergence before the defocusing, then, as a result, divergence beats convergence and there occurs a local instability of the dynamics, which generates chaos. A large class of chaotic billiards with a focusing boundary was built in [10–12].

The mechanism of defocusing is more general than the mechanism of dispersing. Indeed, dispersing demands that the orbits always diverge, while defocusing allows orbits also to converge, and divergence must beat convergence just on average (in time). There is no surprise that after the discovery of the mechanism of defocusing, examples of chaotic geodesic flows on surfaces with pieces of positive curvature were built [13–15]. All these papers followed the construction of chaotic billiards with a focusing boundary. So, billiards got into the forefront of the theory of dynamical systems. Indeed, Hopf's idea [6] is absolutely nonconstructive. In fact, till now there are no examples of chaotic geodesic flows on surfaces/manifolds with pieces of positive curvature, which demonstrate that this idea may work.

Chaotic focusing billiards introduced in [10–12] used only focusing components with a constant curvature (arcs of circles). Therefore, a natural question was what types of focusing mirrors (arcs) could be pieces of a boundary for chaotic billiards. First, in [16] it was demonstrated that small perturbations of circular arcs would work. Then, in [17, 18], two dual classes of focusing mirrors were studied and proved to work.

Finally, in [19], a class of absolutely focusing mirrors was introduced (actually, this name was coined later [20]). The further studies have shown that the property to be absolutely focusing is a sufficient [21–23] and necessary [24] condition for a focusing arc to be part of a chaotic billiard table.

Absolute focusing is a new notion in the geometrical optics, which has been inspired by the studies of the dynamics of billiards. Observe that a narrow parallel beam of rays cannot have two consecutive reflections off a dispersing (or planar) mirror, while it may have any number of consecutive reflections off a focusing mirror.

A mirror is focusing if it focuses any planar beam of rays falling on this mirror after the first (just one!) reflection. The focusing mirror is absolutely focusing if any parallel beam of rays leaves this mirror as a focusing (convergent) beam after the last reflection in any series of consecutive reflections off this mirror [19, 20]. There are of course focusing but not absolutely focusing mirrors [20, 23]. Independently, in [23], a more restrictive class of focusing mirrors was formally introduced. However, it was proved [20] that this class coincides with the class of absolutely focusing mirrors. In fact, this formally more restrictive but actually equivalent property was used in [21] in the studies of ergodic two-dimensional billiards.

In the next section, we present the properties and examples of the absolutely focusing mirrors. The last section deals with high-dimensional billiards (in dimensions greater than two), where astigmatism, i. e., another optical phenomenon, plays a leading role.

2. ABSOLUTELY FOCUSING MIRRORS

We start with a formal definition of billiards and explain that the dynamics of these systems can completely be described by two classical formulas of the geometrical optics in dimension two, and by an additional classical formula in higher dimensions.

A billiard table Q is a bounded domain in a d-dimensional Euclidean space with the boundary ∂Q consisting of a finite number of smooth manifolds (curves if $d = 2$). The minimum required smoothness is C^2, but one needs C^3 if statistical (ergodic) properties are of interest. A ray (or a point particle) moves by inertia within a billiard table Q and is reflected off the boundary according to the law of elastic reflections, i. e., the angle of incidence equals the angle of reflection. Therefore, billiards are Hamiltonian systems, where the potential is equal to zero within Q and to infinity on the boundary. It is also clear that the billiard orbits are broken lines, i. e., consist of a finite (if the orbit is periodic) or infinite number of straight segments.

The best and clearest characterization of the billiard dynamics is in terms of wave fronts orthogonal to narrow beams of rays. These local beams are characterized by a curvature of the front on the "central" ray in a local (i. e., narrow) beam of rays.

The dynamics of billiards consists in propagation with a constant speed (which can be put equal to unity) and reflections off the boundary ∂Q. In the process of propagation within the billiard table, the curvature of a narrow beam of rays changes according to the following classical formula of the geometrical optics:

$$\kappa_t^{-1} = \kappa_0^{-1} + t, \tag{1}$$

where κ_0 is the initial curvature of the front and κ_t is its curvature at the time t.

It immediately follows from Eq. (1) that a divergent (dispersing) front corresponding to $\kappa_0 > 0$ always remains dispersing. However, a focusing (convergent) front corresponding to $\kappa_0 < 0$ becomes divergent at the time instant equal to $|\kappa_0^{-1}|$. Therefore, a natural basic idea to construct chaotic billiards with focusing components is to put them sufficiently far apart from the other boundary components in order to give beams of rays an ample time to defocus and diverge.

Reflections of wave fronts at the boundary are described by another classical formula of the geometrical optics, called the mirror formula,

$$\kappa_+ = \kappa_- + \frac{2k}{\cos\phi}, \tag{2}$$

where κ_- and κ_+ are the curvatures of the front of a beam of rays right before and after reflection off the boundary, respectively, ϕ is the angle of reflection, and k is the curvature of the boundary at the point of reflection. It is clear that the dispersing fronts in dispersing billiards (where the curvature of the boundary is strictly positive) forever remain dispersing, which leads to the chaotic behavior by the mechanism of dispersing. However, if at least one boundary component of a billiard table is focusing, then the situation becomes much more subtle and, in fact, richer.

Indeed, it is commonly known that billiards in circles are integrable. Indeed, this fact follows from tangency of the orbits in such billiards to one and the same circle, which is concentric to the boundary.

The first example of a chaotic focusing billiard [10, 11] was obtained from the circle by cutting out its piece by a chord, which is shorter than the diameter (Fig. 1*b*). In the integrable circle billiard, the convergence and divergence of rays are in balance (it is integrability). The billiard in Fig. 1*b* allows one to make an average path between two reflections off the focusing part of the boundary longer. As a result, we get a chaotic billiard where the time of divergence of the orbits (rays) is on average longer than the time of their convergence.

It was already mentioned that all focusing components (mirrors) at the boundary of chaotic billiards should be absolutely focusing, i. e., focus all parallel beams of rays falling on these components after the

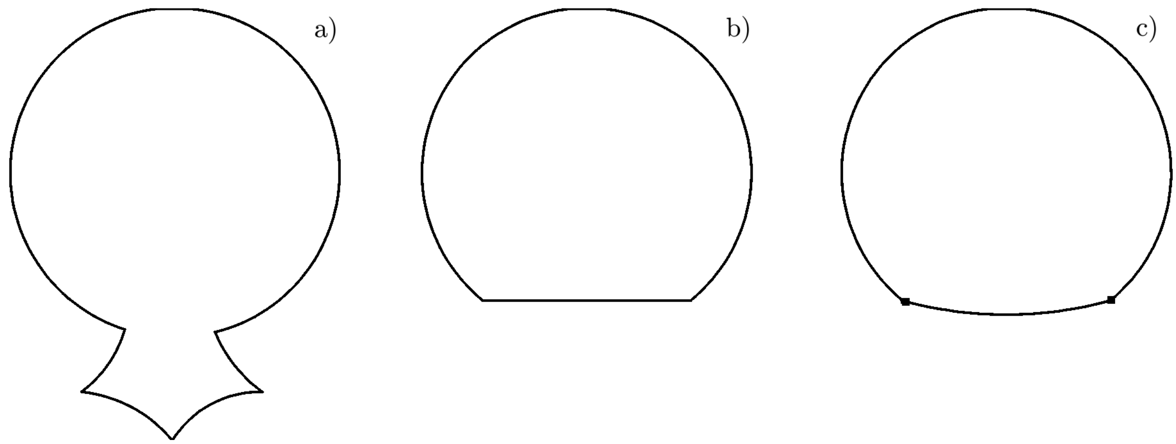

Fig. 1. "Perturbed" dispersing billiard (*a*), the first chaotic focusing billiard (*b*), and the skewed lemon (*c*), which is a perturbation of the billiard on panel (*b*).

last reflection off this mirror. All arcs of circles are absolutely focusing [11]. Also, any focusing mirror could be cut into absolutely focusing mirrors, i. e., any sufficiently short focusing mirror is also absolutely focusing [23]. To get a non-absolutely focusing mirror, we should move away from the circles. For instance, a semi-ellipse cut over its minor axis remains absolutely focusing if its eccentricity is less than $\sqrt{2}$, but it becomes non-absolutely focusing when the eccentricity exceeds $\sqrt{2}$ [20, 23].

By far, the so-called stadium billiard, which is the most famous among chaotic focusing billiards, can immediately be obtained from that in Fig. 1*b*. To do this, we just use a famous geometric-optical method of constructing images. According to it, if a ray hits a planar (zero curvature) mirror, then, instead of reflecting the ray from such a planar mirror, one can reflect the entire region (resonator or the billiard table) with respect to such a planar mirror and continue the ray as a straight (oriented) line. After such a reflection, we get, from the billiard table in Fig. 1*b*, a new billiard table which has the shape of a figure eight. We now take two common tangents to the two halves of eight. Then we will get a billiard table with two semicircles connected by two segments tangent to them, i. e., a "stadium." It is worthwhile to mention that this billiard is a singular ("degenerate") example because there exists a continuous family of orbits of period two (bouncing ball orbits, as physicists call them). Actually, the singularity of this billiard makes the study more complicated than, e. g., the analysis of squash billiards where the arcs of two circles with different radii are connected by two tangent segments.

The discovery of the mechanism of defocusing came out of a standard idea to perturb a dispersing billiard by adding small focusing components (Fig. 1*a*). It turned out though [11] that the dispersing part can be completely cut out, but the billiard in Fig. 1*a* remains to be chaotic.

After understanding the mechanism of defocusing, a natural idea (algorithm) how to construct chaotic billiards with focusing components immediately comes. Indeed, we just make all the focusing components absolutely focusing and put them sufficiently far apart. This idea was realized in, e. g., [21, 22]. For a long time, it was considered the only way of how focusing billiards could become chaotic.

However, a new surprise has recently come, which shows that the mechanism of defocusing is much more general and ubiquitous than we thought before. It was proved in [25] that the billiard in Fig. 1*c* is chaotic. Observe that the boundary of billiards of this class consists of two arcs of two circles which are not far from each other. Instead, they are very close. Indeed, each of these two circles contains the entire billiard table.

This billiard is called a skewed lemon (or a squeezed lemon). It can naturally be considered as a "small" perturbation of the first example of a chaotic focusing billiard in Fig. 1*b*. Indeed, this billiard is chaotic if the curvature of a larger circle is sufficiently small [25]. Therefore, the defocusing mechanism works in a much more general setting than the standard way of putting focusing mirrors sufficiently far apart. Clearly, we still do not understand this mechanism of chaos well enough.

3. HIGH-DIMENSIONAL BILLIARDS AND ASTIGMATISM

A fundamental paper [8] was dealing with two-dimensional Sinai billiards. Higher-dimensional Sinai billiards were studied in another Sinai's paper [9]. By combining Eqs. (1) and (2), one gets a continued fraction which describes the curvature of a wave front after n reflections off the boundary [8]. This continued fraction has $2n$ elements defined by the lengths of free paths between reflections off the boundary and by the curvatures and incidence angles at the points of reflection [8]. For billiard tables with dimension $d \geq 3$, the orbits are described by operator-valued continued fractions [9], which are analogous in structure to continued fractions in the case of dimension two, but the curvature is substituted by the curvature operator (also called the second fundamental form) of the boundary at the point of reflections. In the case of Sinai and dispersing billiards, the curvature operators are strictly positive and the analysis of the corresponding continued fractions requires just technical efforts.

However, if the boundary of a billiard table has at least one focusing component (mirror), then the well-known optical phenomenon of astigmatism becomes to be a major obstacle to chaos (i. e., to divergence of rays in the process of dynamics).

To explain why and how it happens, let us consider reflection of a narrow beam of rays off a spherical mirror. Take at first the plain P which contains (and therefore uniquely defines) the center of the corresponding (to the mirror) sphere and the velocity vector of the central ray in the beam under consideration. It is easy to see that during the entire series of consecutive reflections off the mirror, the curvature of the section of the beam front by the plain P will be described by Eqs. (1) and (2). Indeed, the velocity vector will belong to the two-dimensional plane P during the entire series of consecutive reflections off the mirror. Therefore, it is the case of dimension 2.

Consider now the plane P' which also contains the velocity vector but is orthogonal to the plane P. The evolution of the curvature of the section of the front by the orthogonal plane P' is described by another classical formula of the geometrical optics, which is called Coddington's formula.

Coddington's formula has the following form:

$$\kappa'_+ = \kappa'_- + 2k \cos\phi, \tag{3}$$

where the notations are the same as in the mirror formula (2), but κ' corresponds to the curvature of the section of the front by the plane P'.

At first sight, there is a very little difference between the mirror formula (2) and Coddington's formula (3). Indeed, the only difference is in the last term, where $\cos\phi$ is in the denominator of Eq. (2) and in the numerator of Eq. (3). But in fact the mirror formula (2) shows that the curvature of the front at the moment of reflection undergoes a jump (by at least $2k$), while in the orthogonal plane P', the change in the curvature depends on the incidence angle ϕ. In particular, this change can be arbitrarily small if the incidence angle of the beam is almost tangent to the mirror at the point of reflection.

This phenomenon of the different strength of focusing under different angles of incidence is called the astigmatism. Because of the astigmatism, there were claims that the mechanism of defocusing works (and exists) only in dimension two.

These claims turned out to be false, and the examples of high-dimensional (in fact, arbitrarily dimensional) chaotic billiards with focusing components were proved to exist [26–28]. However, one has to pay for astigmatism. Recall that in two-dimensional chaotic billiards, a focusing component of the boundary can be arbitrarily close to the entire circle (see, e. g., Fig. 1*b*). In a dimension greater than two, focusing mirrors (focusing components of the boundary of a billiard table) cannot exceed a relatively small piece of a sphere. Consider, e. g., a spherical cap, i. e., a section of a sphere by some plane, and take the smallest of the two pieces. Consider now a cone with the vertex at the center of the sphere, which consists of segments connecting the center of the sphere to the boundary of the spherical cap. It could be derived from Coddington's formula that in order to ensure that a billiard is chaotic, the angle of this cone cannot exceed $\pi/2$. So, due to astigmatism, spherical caps in chaotic high-dimensional billiards should be relatively small.

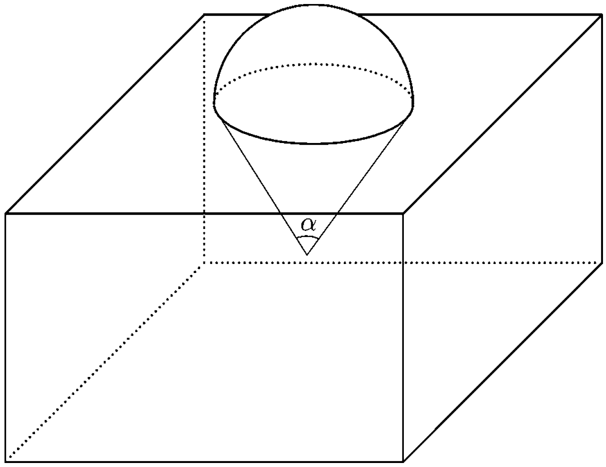

Fig. 2. A nowhere dispersing three-dimensional chaotic billiard.

The simplest example of a nowhere dispersing three-dimensional chaotic billiard is shown in Fig. 2. Here, a proper (not too large) spherical cap is put on the top of a sufficiently large cube [26]. This type of billiards is proved to be chaotic in any finite dimension [27].

The notion (and meaning) of absolutely focusing mirrors does not depend on dimension. However, so far only relatively small spherical caps have been shown to be absolutely focusing mirrors. Finding other examples of high-dimensional absolutely focusing mirrors could be of interest for the geometrical optics and certainly for various applications.

Another interesting problem, first of all for the chaos theory, is to find a high-dimensional analog of skewed lemons (see Fig. 1*c*).

4. CONCLUDING REMARKS

Discovery of the mechanism of defocusing and the introduction of the notion of absolutely focusing mirrors allowed one to extend the chaos theory to a much larger class of dynamical systems. Quite a few optical devices have been built in many physical laboratories mainly to analyze the phenomenon of quantum chaos.

However, many interesting problems in the billiard dynamics and its natural counterpart dealing with the dynamics of rays in resonators, the illumination of domains (billiard tables), etc., still remain open. The situation is completely understood only in dimension two. In higher dimensions, there are relatively few results on the dynamics of billiards with focusing components.

First of all, the only example of absolutely focusing mirrors is provided by sufficiently small spherical caps [26, 27]. Therefore, a natural question is whether or not other absolutely focusing mirrors exist in dimensions greater than two.

Moreover, all known examples of nowhere dispersing chaotic billiards with dimensions greater than two also have flat (zero-curvature) components of the boundary. Is it possible to build a chaotic billiard with all focusing components of the boundary of the corresponding billiard table? In dimension two, the answer is yes, but it is not known in higher dimensions. A related, but much more difficult question is whether or not there exist chaotic billiards with convex billiard tables in dimensions greater than two.

Answering these questions will essentially advance our understanding of possible types of the dynamics of rays in resonators.

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