Topological bounds on hyperkähler manifolds*

Justin Sawon

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Abstract

We conjecture that certain curvature invariants of compact hyperkähler manifolds are positive/negative. We prove the conjecture in complex dimension four, give an "experimental proof" in higher dimensions, and verify it for all known hyperkähler manifolds up to dimension eight. As an application, we show that our conjecture leads to a bound on the second Betti number in all dimensions.

1 Introduction

In [22] Rozansky and Witten introduced new invariants of hyperkähler manifolds. These invariants are defined by taking copies of the curvature tensor and the holomorphic symplectic form of the hyperkähler manifold, and contracting indices to get a form that can be integrated over the manifold, giving an $\mathbb{R}$-valued invariant. The pattern by which one contracts the indices is encoded in a trivalent graph Γ, and the corresponding invariant is written b_Γ.

These Rozansky-Witten invariants were studied by the author in [25]. Let M be a hyperkähler manifold of complex dimension $2n$ (in this article we will always assume our hyperkähler manifolds are compact and irreducible, i.e., simply connected and with $h^{2,0} = 1$; and by dimension we will always mean the complex dimension). Let Θ^n be the trivalent graph consisting of n disjoint copies of the trivalent graph

with two vertices. Together with Hitchin [25, 11], the author proved that

$$b_{\Theta^n}(M) = \frac{n!}{(4\pi^2 n)^n} \frac{\|R\|^{2n}}{(\mathrm{vol}M)^{n-1}},$$

where $\|R\|^2$ is the $\mathcal{L}^2$-norm of the curvature and $\mathrm{vol}M$ is the volume of M. In particular, $b_{\Theta^n}(M)$ is positive. More recent investigations have led to the following conjecture. Let Θ_k denote the trivalent graph with $2k$ vertices consisting of a cycle of k small loops; thus $\Theta_1 = \Theta$ and

$\Theta_2 =$  $\Theta_3 =$  $\Theta_4 =$  $\Theta_5 =$  etc.

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Conjecture 1 *Let M be a hyperkähler manifold of complex dimension $2n$. Then for $2 \leq k \leq n$ the Rozansky-Witten invariant $b_{\Theta^{n-k}\Theta_k}(M)$ is negative if k is even and positive if k is odd.*

Our evidence for this conjecture takes three forms.

1. We prove the conjecture in dimension $2n = 4$, i.e., we prove that $b_{\Theta_2}(M)$ is always negative.

2. We give an "experimental proof" in higher dimensions.

3. We verify the conjecture for all currently known hyperkähler manifolds in dimensions six and eight.

Using results of Salamon [23] and Verbitsky [31], Guan [10] bounded the Betti numbers of four-dimensional hyperkähler manifolds. The Rozansky-Witten invariants can be computed in terms of the Betti numbers and this suffices to show that $b_{\Theta_2}(M) < 0$.

By "experimental proof" we mean the following. We generate a random multi-array of complex numbers with the same symmetries as the curvature tensor of a hyperkähler manifold. We take copies of this multi-array and contract indices according to the trivalent graph Θ_2. Repeating this computation hundreds of times, we always get negative real numbers. This suggests that $b_{\Theta^{n-2}\Theta_2}(M)$ is negative because the integrand of this invariant is negative at all points of M. We perform similar computations for Θ_3, always getting positive real numbers, and therefore expect that the integrand of $b_{\Theta^{n-3}\Theta_3}(M)$ is positive at all points of M.

In dimensions six and eight we have Hilbert schemes of points on K3 surfaces and generalized Kummer varieties; their Rozansky-Witten invariants were calculated by the author [25], and one can readily verify the conjecture is true for these examples. In dimension six there is one additional example due to O'Grady [21], whose Hodge numbers were calculated by Mongardi, Rapagnetta, and Saccà [18]. In six dimensions the Rozansky-Witten invariants can be computed in terms of the Hodge numbers, and this leads to a verification of the conjecture for O'Grady's example too.

As an application, we prove that if $b_{\Theta^{n-2}\Theta_2}(M) < 0$ then the second Betti number b_2 is bounded. For this result we use a recent result of Jiang [13] which gives an upper bound on $b_{\Theta^n}(M)$. The degree two cohomology group $\mathrm{H}^2(M, \mathbb{Z})$ plays a central role in hyperkähler geometry. For example, Theorem 4.3 of Huybrechts [12] states that in each dimension there exist only finitely many deformation types of irreducible hyperkähler manifolds such that the lattice $(\mathrm{H}^2(M, \mathbb{Z}), \tilde{q})$ is isomorphic to a given one, where $\tilde{q}$ denotes a certain positive integral multiple of the Beauville-Bogomolov quadratic form q. Bounding b_2 therefore represents a first step toward proving finiteness of hyperkähler manifolds. Note however that q and $\tilde{q}$ are not unimodular in general, so even for fixed b_2 infinitely many lattices could arise. A different approach to bounding b_2 is to analyze the Looijenga-Lunts-Verbitsky decomposition of the cohomology of the hyperkähler manifold. Under a hypothesis on the irreducible representations that appear in this decomposition, the author [30] and Kim and Laza [15] proved that in six dimensions $b_2 \leq 23$ (which would be a sharp bound) and in eight dimensions $b_2 \leq 24$. Kim and Laza also proved bounds in higher dimensions, again assuming certain restrictions on the Looijenga-Lunts-Verbitsky decomposition.

Finally, we compare our results to recent work of Cao and Jiang [3] and Jiang [13]. They investigated the Todd class of a hyperkähler manifold and showed that it is quasi-effective,

which translates into positivity of certain Rozansky-Witten invariants. Our conjecture appears to be stronger, and it implies their results in some cases.

Soon after the preprint of this paper was posted on the arXiv, Beckmann and Song posted a preprint with similar results [2]. They arrived independently at the conjectural bound $b_{\Theta^{n-2}\Theta_2}(M) < 0$ and explored its consequences, particularly for the Riemann-Roch polynomial of M. In dimension four they also considered orbifold examples, particularly those described by Fu and Menet [6].

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2 Rozansky-Witten invariants

Rozansky-Witten invariants were defined in [22], and an equivalent formulation was given by Kapranov [14] (see also Kontsevich [16]). We will briefly describe the latter, which is the approach most closely followed in Sawon [25] and Hitchin and Sawon [11]. Let M be a compact hyperkähler manifold of complex dimension $2n$. We fix a compatible complex structure and regard M as a Kähler manifold. Using the holomorphic symplectic form σ to identify T and Ω^1, we can regard the Riemann curvature tensor as a section in

$$\Omega^{1,1}(M, \mathrm{End}T) = \Omega^{0,1}(M, \Omega^1 \otimes \Omega^1 \otimes T) \cong \Omega^{0,1}(M, \Omega^1 \otimes \Omega^1 \otimes \Omega^1).$$

The fact that M is hyperkähler implies additional symmetries, so that we get

$$\Phi \in \Omega^{0,1}(M, \mathrm{Sym}^3\Omega^1).$$

Let Γ be an oriented trivalent graph with $2n$ vertices. Taking $2n$ copies of Φ and $3n$ copies of the dual $\tilde{\sigma}$ of the symplectic form, we can contract indices according to Γ to get a section in $(\Omega^{0,1})^{\otimes 2n}(M)$. Anti-symmetrizing gives

$$\Gamma(\Phi) \in \Omega^{0,2n}(M).$$

Example 2 *Denote by Θ and Θ_k, $k \geq 2$, the trivalent graphs*

$\Theta =$  $\Theta_2 =$  $\Theta_3 =$  $\Theta_4 =$  etc.

Then

$$\Theta(\Phi)_{\bar{d}\bar{h}} = \Phi_{abc\bar{d}}\Phi_{efg\bar{h}}\tilde{\sigma}^{ae}\tilde{\sigma}^{bf}\tilde{\sigma}^{cg} \in \Omega^{0,2}(M),$$

and

$$\Theta_2(\Phi)_{\bar{d}\bar{h}\bar{l}\bar{p}} = \Phi_{abc\bar{d}}\Phi_{efg\bar{h}}\Phi_{ijk\bar{l}}\Phi_{mno\bar{p}}\tilde{\sigma}^{ae}\tilde{\sigma}^{bf}\tilde{\sigma}^{im}\tilde{\sigma}^{jn}\tilde{\sigma}^{ck}\tilde{\sigma}^{go} \in \Omega^{0,4}(M).$$

The orientation of Γ is defined in a way that takes care of any ambiguity in how to order the terms and their indices.

Finally, we can multiply by $\sigma^n \in \Omega^{2n,0}(M)$ and integrate the resulting $(2n, 2n)$-form over M.

Definition *The Rozansky-Witten invariant of the hyperkähler manifold M corresponding to the trivalent graph Γ is*

$$b_\Gamma(M) = \frac{1}{(8\pi^2)^n n!}\int_M \Gamma(\Phi)\sigma^n.$$

We now assume that M is irreducible, i.e., simply connected and with $h^{2,0} = 1$. Then $\mathrm{H}^{0,2k}_{\bar\partial}(M)$ is generated by the class of $\bar\sigma^k$ for $k = 1, \ldots, n$. In particular, if Γ is a trivalent graph with $2k$ vertices then

$$[\Gamma(\Phi)] = \beta_\Gamma[\bar\sigma^k] \in \mathrm{H}^{0,2k}_{\bar\partial}(M)$$

for some constant β_Γ. Hitchin and Sawon [11] proved the following result.

Proposition 3

$$\beta_\Theta = \frac{1}{2n}\frac{\|R\|^2}{\mathrm{vol}M}$$

where $\|R\|^2$ is the $\mathcal{L}^2$-norm of the curvature tensor of M and $\mathrm{vol}M$ is its volume. In particular, $\beta_\Theta > 0$.

We can now calculate the Rozansky-Witten invariant associated to Θ^n, the disjoint union of n copies of the graph Θ. We find

$$b_{\Theta^n}(M) = \frac{1}{(8\pi^2)^n n!}\int_M \Theta(\Phi)^n\sigma^n = \frac{1}{(8\pi^2)^n n!}\beta_\Theta^n\int_M \sigma^n\bar\sigma^n = \frac{n!}{(4\pi^2 n)^n}\frac{\|R\|^{2n}}{(\mathrm{vol}M)^{n-1}} > 0.$$

Here we used the fact that

$$\int_M \sigma^n\bar\sigma^n = \frac{1}{\binom{2n}{n}}\int_M(\sigma+\bar\sigma)^{2n} = 2^{2n}(n!)^2\int_M\frac{\omega_J^{2n}}{(2n)!} = 2^{2n}(n!)^2\mathrm{vol}M.$$

In Sawon [24] it was conjectured that this invariant is related to Chern numbers, and this was proved in Hitchin and Sawon [11].

Theorem 4

$$b_{\Theta^n}(M) = 48^n n!\hat{A}^{1/2}[M]$$

where

$$\hat{A}^{1/2} = 1 + \frac{1}{24}c_2 + \frac{1}{5760}(7c_2^2 - 4c_4) + \frac{1}{967680}(31c_2^3 - 44c_2c_4 + 16c_6) + \ldots$$

is the characteristic class given by the square root of the $\hat{A}$-polynomial.

Generalizing $\beta_\Theta > 0$, we make the following conjecture.

Conjecture 5 *For all irreducible hyperkähler manifolds M of complex dimension $2n$ and for $2 \le k \le n$ we have*

$$\beta_{\Theta_k}\begin{cases} < & 0 \quad \textit{if } k \textit{ is even,} \\ > & 0 \quad \textit{if } k \textit{ is odd.}\end{cases}$$

Equivalently

$$b_{\Theta^{n-k}\Theta_k}(M)\begin{cases} < & 0 \quad \textit{if } k \textit{ is even,} \\ > & 0 \quad \textit{if } k \textit{ is odd.}\end{cases}$$

The second statement is equivalent to the first because

$$b_{\Theta^{n-k}\Theta_k}(M) = \frac{1}{(8\pi^2)^n n!}\beta_\Theta^{n-k}\beta_{\Theta_k}\int_M \sigma^n\bar{\sigma}^n = \frac{n!}{(2\pi^2)^n}\beta_\Theta^{n-k}\beta_{\Theta_k}\mathrm{vol}M$$

and $\beta_\Theta > 0$. For $k = 2$ we have the following result, proved on pages 67 to 72 of Sawon [25].

Theorem 6 *The invariant $b_{\Theta^{n-2}\Theta_2}(M)$ is a linear combination of Chern numbers, which can be computed explicitly in any dimension. In dimensions four and six we have*

$$b_{\Theta_2}(M) = -\frac{4}{5}(c_2^2 - 2c_4)[M]$$

and

$$b_{\Theta\Theta_2}(M) = -\frac{8}{35}(11c_2^3 - 26c_2c_4 + 12c_6)[M].$$

Theorem 17 of Sawon [25] also gives the following.

Theorem 7 *For $k > 2$ and M an irreducible hyperkähler manifold of dimension $2n$, the invariant $b_{\Theta^{n-k}\Theta_k}(M)$ is a rational function of Chern numbers.*

Remark 8 *Up to dimension six all Rozansky-Witten invariants are linear combinations of Chern numbers; in particular, this is true for $b_{\Theta_3}(M)$. However, this is no longer true for a general Rozansky-Witten invariant in higher dimensions. For example, Theorem 22 of Sawon [25] (summarized in Section 5 of Sawon [26]) says that in dimension eight $b_{\Theta_2^2}(M)$ is not a linear combination of Chern numbers, and it follows that nor is $b_{\Theta\Theta_3}(M)$.*

We end this section by briefly recalling Rozansky and Witten's original formulation, which will be used in the calculations of Section 4. If we denote by TM the real tangent bundle of a hyperkähler manifold M, then there is a decomposition

$$TM \otimes_{\mathbb{R}} \mathbb{C} = V \otimes S$$

into the tensor product of a (complex) rank $2n$ vector bundle V and a trivializable rank 2 vector bundle S. The Levi-Civita connection on $TM \otimes_{\mathbb{R}} \mathbb{C}$ reduces to an $\mathrm{Sp}(n)$-connection on V, and the holomorphic symplectic form σ and Riemann curvature tensor Φ are equivalent to a symplectic form ϵ on V and a section Ω of $\mathrm{Sym}^4 V$. Rozansky and Witten take copies of ϵ and Ω and contract indices according to the trivalent graph, then integrate to produce $b_\Gamma(M)$, much like before (see equations 3.41 and 3.43 of [22]). For example, for $\Gamma = \Theta$ in dimension $2n = 2$ we would take

$$\Omega_{ABCD}\Omega_{EFGH}\epsilon^{AE}\epsilon^{BF}\epsilon^{CG}.$$

Anti-symmetrizing over the free indices D and H is basically equivalent to contracting with the skew form ϵ^{DH}, and thus

$$b_\Theta(M) = \frac{1}{(2\pi)^2}\int_M \Omega_{ABCD}\Omega_{EFGH}\epsilon^{AE}\epsilon^{BF}\epsilon^{CG}\epsilon^{DH}d\mathrm{vol}.$$

For $\Gamma = \Theta_2$ in dimension $2n = 4$ we would take

$$\Omega_{ABCD}\Omega_{EFGH}\Omega_{IJKL}\Omega_{MNOP}\epsilon^{AE}\epsilon^{BF}\epsilon^{IM}\epsilon^{JN}\epsilon^{CK}\epsilon^{OG}.$$

Anti-symmetrizing over the free indices D, H, L, and P is basically equivalent to contracting with the totally skew form

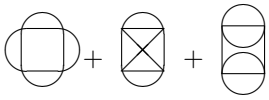
$$\epsilon^{DHLP} := \epsilon^{DH}\epsilon^{LP} + \epsilon^{DL}\epsilon^{PH} + \epsilon^{DP}\epsilon^{HL},$$

and thus

$$b_{\Theta_2}(M) = \frac{1}{(2\pi)^4}\int_M \Omega_{ABCD}\Omega_{EFGH}\Omega_{IJKL}\Omega_{MNOP}\epsilon^{AE}\epsilon^{BF}\epsilon^{IM}\epsilon^{JN}\epsilon^{CK}\epsilon^{OG}\epsilon^{DHLP}d\text{vol}.$$

Remark 9 *It is interesting that the final contractions with ϵ^{DH} and ϵ^{DHLP} mean that we are really using a 4-valent graph to contract indices. For $b_\Theta(M)$ this would be the 4-valent graph*

while for $b_{\Theta_2}(M)$ it would be the sum of the 4-valent graphs

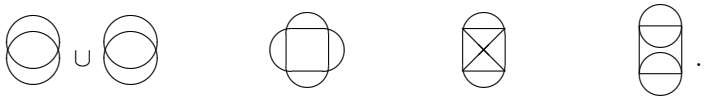


These 4-valent graphs are really encoding the invariant theory of $\mathrm{Sp}(n)$-representations. The bundle V corresponds to the standard representation $\mathbb{C}^{2n}$ of $\mathrm{Sp}(n)$, whereas Ω is a section of $\mathrm{Sym}^4 V$ which corresponds to $\mathrm{Sym}^4\mathbb{C}^{2n}$. If we decompose $\mathrm{Sym}^2(\mathrm{Sym}^4\mathbb{C}^{2n})$ into irreducible $\mathrm{Sp}(n)$-representations, we find the trivial representation appears with multiplicity one. Indeed, we can identify $\mathbb{C}^{2n}$ with its dual using the symplectic form, and then the invariant subspace of $\mathrm{Sym}^2(\mathrm{Sym}^4\mathbb{C}^{2n})$ is

$$\mathrm{Sym}^2(\mathrm{Sym}^4\mathbb{C}^{2n})^{\mathrm{Sp}(n)} \cong \mathrm{Hom}_{\mathrm{Sp}(n)}(\mathrm{Sym}^4\mathbb{C}^{2n}, \mathrm{Sym}^4\mathbb{C}^{2n}),$$

which is one-dimensional by Schur's lemma applied to the irreducible *representation $\mathrm{Sym}^4\mathbb{C}^{2n}$. This means there is a unique quadratic invariant of Ω, which gives $b_\Theta(M)$.*

The decomposition of $\mathrm{Sym}^4(\mathrm{Sym}^4\mathbb{C}^{2n})$ contains four copies of the trivial representation[1]*, and thus there are four quartic invariants. These correspond to the graphs*



The first gives the Rozansky-Witten invariant $b_{\Theta^2}(M)$, and the sum of the other three gives $b_{\Theta_2}(M)$. In fact, each individual graph gives a way of contracting indices and thus produces a curvature integral, but it is not clear what these individual curvature integrals represent. Although Conjecture 5 states that $b_{\Theta_2}(M) < 0$ in dimension four, and $\beta_{\Theta_2} < 0$ in higher dimensions, we do not expect that these individual curvature integrals are all negative. Indeed, computations like those in Section 4 suggest that they are not (more precisely, computations suggest that the three terms in the integrand for β_{Θ_2} are not individually pointwise negative; only their sum is pointwise negative).

There is also a unique cubic invariant of Ω; it corresponds to the graph

[1] We use the form interface to LiE at wwwmathlabo.univ-poitiers.fr/∼maavl/LiE/form.html.

and looks like

$$\Omega_{ABCD}\Omega_{EFGH}\Omega_{IJKL}\epsilon^{AE}\epsilon^{BF}\epsilon^{CI}\epsilon^{DJ}\epsilon^{GK}\epsilon^{HL}.$$

Again, it is not clear what the resulting curvature integral represents. For more relations between graphs and the invariant theory of $\mathrm{Sp}(n)$*, see Garoufalidis and Nakamura [8].*

3 Dimension four

In this section we prove Conjecture 5 in dimension four.

Theorem 10 *Let M be an irreducible hyperkähler manifold of complex dimension four. Then*

$$b_{\Theta_2}(M) < 0.$$

Proof In dimension four we can write the Rozansky-Witten invariants in terms of Chern numbers, which can be written in terms of Betti numbers. Then we can apply Guan's bounds on the Betti numbers [10]. Specifically, from Sawon [25] (or Hitchin and Sawon [11]) we have

$$\begin{aligned} b_{\Theta^2}(M) &= \tfrac{4}{5}(7c_2^2 - 4c_4)[M] \\ b_{\Theta_2}(M) &= -\tfrac{4}{5}(c_2^2 - 2c_4)[M], \end{aligned}$$

and from Section 8 of [28] we have

$$\begin{aligned} c_2^2[M] &= 736 + 4b_2 - b_3 \\ c_4[M] &= 48 + 12b_2 - 3b_3. \end{aligned}$$

Therefore

$$b_{\Theta_2}(M) = -512 + 16b_2 - 4b_3.$$

Guan [10] proved that b_2 and b_3 can take only finitely many values; the maximum value of $b_{\Theta_2}(M)$ occurs when $b_2 = 23$ and $b_3 = 0$, and thus

$$b_{\Theta_2}(M) \leq -144.$$

□

Remark 11 *We also have*

$$b_{\Theta^2}(M) = 3968 - 16b_2 + 4b_3,$$

whose minimum value according to Guan's bounds [10] again occurs when $b_2 = 23$ and $b_3 = 0$, and thus

$$b_{\Theta^2}(M) \geq 3600.$$

These values are realized by the Hilbert scheme $S^{[2]}$, and the bounds are therefore sharp. The maximum value according to Guan's bounds [10] occurs when $b_2 = 3$ and $b_3 = 68$ (not realized by any currently known hyperkähler manifold), and thus

$$b_{\Theta^2}(M) \leq 4192.$$

In terms of $\hat{A}^{1/2}[M] = \frac{1}{48^2 2!} b_{\Theta^2}(M)$ these bounds become

$$\frac{25}{32} \leq \hat{A}^{1/2}[M] \leq \frac{131}{144}.$$

4 "Experimental proof"

The Maple code for our "experimental proof" of Conjecture 5 is in Appendix A, and a dictionary for the variables is in Table 1.

$E[i,j]$	ϵ^{IJ}
$EE[i,j,k,l]$	$\epsilon^{IJKL} := \epsilon^{IJ}\epsilon^{KL} + \epsilon^{IK}\epsilon^{LJ} + \epsilon^{IL}\epsilon^{JK}$
$EEE[i,j,k,l,m,p]$	ϵ^{IJKLMP}
$M[i,j,k,l]$	Ω_{IJKL}
$V[i,j,k,l]$	$\Psi_{IJKL} := \Omega_{ABIJ}\Omega_{CDKL}\epsilon^{AC}\epsilon^{BD}$
Th	$\Theta(\Omega)_{JL}\epsilon^{JL} = \Psi_{IJKL}\epsilon^{IK}\epsilon^{JL} = \Omega_{ABIJ}\Omega_{CDKL}\epsilon^{AC}\epsilon^{BD}\epsilon^{IK}\epsilon^{JL}$
$W[i,j,k,l]$	$\Psi_{AIBJ}\Psi_{CKDL}\epsilon^{AC}\epsilon^{DB}$
$Th2$	$\Theta_2(\Omega)_{IJKL}\epsilon^{IJKL} = \Psi_{AIBJ}\Psi_{CKDL}\epsilon^{AC}\epsilon^{DB}\epsilon^{IJKL}$
$U[i,j,k,l,m,p]$	$\Psi_{AIBJ}\Psi_{CKDL}\Psi_{EMFP}\epsilon^{AD}\epsilon^{CF}\epsilon^{EB}$
$Th3$	$\Theta_3(\Omega)_{IJKLMP}\epsilon^{IJKLMP} = \Psi_{AIBJ}\Psi_{CKDL}\Psi_{EMFP}\epsilon^{AD}\epsilon^{CF}\epsilon^{EB}\epsilon^{IJKLMP}$

Table 1: Dictionary for Maple code

The first step is to specify the dimension (in the given code we select $2n = 6$). In a suitable basis we can assume that ϵ^{IJ} is the standard skew form, so we define $E[i,j]$ accordingly. Similarly, $EE[i,j,k,l]$ and $EEE[i,j,k,l,m,p]$ are defined to be the standard totally antisymmetric forms.

Next we generate a random multi-array of complex numbers to represent the curvature tensor Ω_{IJKL} of a hyperkähler manifold. We first set $R[i,j,k,l]$ to be a totally symmetric form with random complex entries satisfying $-1 \leq \mathcal{R}e \leq 1$ and $-1 \leq \mathcal{I}m \leq 1$. The curvature tensor Ω_{IJKL} must satisfy a reality condition, and thus we define $M[i,j,k,l]$ by adding $R[i,j,k,l]$ to its conjugate under the appropriate reality condition.

Next we take copies of $M[i,j,k,l]$ and $E[i,j]$ and contract indices according to a trivalent graph Γ. There is one intermediate step: we compute $V[i,j,k,l]$, which corresponds to the tensor

$$\Psi_{IJKL} := \Omega_{ABIJ}\Omega_{CDKL}\epsilon^{AC}\epsilon^{BD}$$

given by contracting according to the unitrivalent graph  . This piece appears in all of our graphs Θ, Θ_2, and Θ_3, etc., and by computing $V[i,j,k,l]$ once and reusing it in later computations we can speed up the algorithm considerably.

The code outputs the values Th, $Th2$, and $Th3$. We can iterate this computation hundreds of times and verify that Th and $Th3$ are always positive and $Th2$ is always negative. Of course, we know that Th must be positive because at each point of M

$$\Omega_{ABCD}\Omega_{EFGH}\epsilon^{AE}\epsilon^{BF}\epsilon^{CG}\epsilon^{DH}$$

is the $\mathcal{L}^2$-norm of the curvature tensor, and likewise Th is the $\mathcal{L}^2$-norm of $M[i,j,k,l]$. As observed earlier, this implies that $b_\Theta(M) > 0$ in two dimensions, and $\beta_\Theta > 0$ in arbitrary dimension, because up to a universal positive constant β_Θ is the integral over M of

$$\Omega_{ABCD}\Omega_{EFGH}\epsilon^{AE}\epsilon^{BF}\epsilon^{CG}\epsilon^{DH}.$$

Similarly, the computational verification that $Th2$ is always negative suggests that

$$\Omega_{ABCD}\Omega_{EFGH}\Omega_{IJKL}\Omega_{MNOP}\epsilon^{AE}\epsilon^{BF}\epsilon^{IM}\epsilon^{JN}\epsilon^{CK}\epsilon^{OG}$$

should be a pointwise negative function on M. This would imply that $b_{\Theta_2}(M) < 0$ in four dimensions, and $\beta_{\Theta_2} < 0$ in arbitrary dimension. Finally, the verification that $Th3$ is always positive suggests that $b_{\Theta_3}(M) > 0$ in six dimensions, and $\beta_{\Theta_3} > 0$ in arbitrary dimension.

Remark 12 *The value of this "experimental proof" is that it suggests an approach to a rigorous proof of the conjecture. Namely, it suggests that these particular Rozansky-Witten invariants are positive (respectively, negative) because their integrands are pointwise positive (respectively, negative), and moreover, that this should be the consequence of a purely algebraic computation. If successful, this would mean that we could obtain valuable topological information about hyperkähler manifolds from purely algebraic identities.*

5 Dimensions six and eight

In this section we verify Conjecture 5 for all currently known hyperkähler manifolds in complex dimensions six and eight. In dimension six these are the Hilbert scheme of points on a K3 surface $S^{[3]}$, the generalized Kummer variety $K_3(A)$ (see Beauville [1]), and O'Grady's example OG6 [21]. In dimension eight these are $S^{[4]}$ and $K_4(A)$. The Rozansky-Witten invariants of $S^{[n]}$ and $K_n(A)$ up to dimension eight were calculated in Sawon [25] (see Appendix E.1); the results are reproduced below.

$2n$		$S^{[n]}$	$K_n(A)$	OG6
2	b_Θ	48		
4	b_{Θ^2}	3600	3888	
	b_{Θ_2}	-144	-432	
6	b_{Θ^3}	373248	442368	442368
	$b_{\Theta\Theta_2}$	-13824	-36864	-36864
	b_{Θ_3}	512	2560	3072
8	b_{Θ^4}	49787136	64800000	
	$b_{\Theta^2\Theta_2}$	-1693440	-4320000	
	$b_{\Theta_2^2}$	57600	288000	
	$b_{\Theta\Theta_3}$	56448	240000	
	b_{Θ_4}	-1824	-12000	
	b_Ξ	348	-1500	

The last invariant in dimension eight, b_Ξ, corresponds to the trivalent graph which is the skeleton of a cube,

$\Xi =$  .

We have also included the invariants of OG6 in this table; let us explain how they are calculated. The Hodge numbers were calculated by Mongardi, Rapagnetta, and Saccà [18]; they are

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & 1 & & 6 & & 1 & & & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 1 & & 12 & & 173 & & 12 & & 1 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
1 & & 6 & & 173 & & 1144 & & 173 & & 6 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & & 1 & & 12 & & 173 & & 12 & & 1 & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & & & 1 & & 6 & & 1 & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & & & 1 & & & & & &
\end{array}.$$

From this we can compute the Hirzebruch χ_y-genus

$$\chi_y = \sum_{p,q=0}^{6} (-1)^q h^{p,q} y^p = 4 - 24y + 348y^2 - 1168y^3 + 348y^4 - 24y^5 + 4y^6.$$

By Riemann-Roch, the coefficients $\chi^p = \sum_{q=0}^{6}(-1)^q h^{p,q}$ can be written in terms of Chern numbers, and these relations can be inverted to give

$$\begin{array}{rcrcl}
c_2^3[M] & = & 7272\chi^0 - 184\chi^1 - 8\chi^2 & = & 30720 \\
c_2c_4[M] & = & 1368\chi^0 - 208\chi^1 - 8\chi^2 & = & 7680 \\
c_6[M] & = & 36\chi^0 - 16\chi^1 + 4\chi^2 & = & 1920
\end{array}$$

(see Appendix B of [25]). Finally, in dimension six we can compute all Rozansky-Witten invariants in terms of Chern numbers, and we find

$$\begin{array}{rcrcl}
b_{\Theta^3}(M) & = & \frac{24}{35}(31c_2^3 - 44c_2c_4 + 16c_6)[M] & = & 442368 \\
b_{\Theta\Theta_2}(M) & = & -\frac{8}{35}(11c_2^3 - 26c_2c_4 + 12c_6)[M] & = & -36864 \\
b_{\Theta_3}(M) & = & \frac{8}{35}(c_2^3 - 3c_2c_4 + 3c_6)[M] & = & 3072.
\end{array}$$

Remark 13 *It is perhaps surprising that some of the Rozansky-Witten invariants of $K_3(A)$ and* OG6 *are the same. Here is an explanation for this. In dimension ten, there is a relation between $S^{[5]}$ and O'Grady's example* OG10*: they can be deformed to Lagrangian fibrations that are locally isomorphic away from singular fibres. More precisely, on the complement of the singular fibres one fibration is a torsor over the other (see Sawon [27], or de Cataldo, Rapagnetta, and Saccà [4] where this relation was used to calculate the Hodge numbers of* OG10*). In particular, the two fibrations have the same discriminant locus and their fibres have the same polarization type. By Theorem 2 of Sawon [28], $\sqrt{\hat{A}}[M]$ (equivalently, $b_{\Theta^n}(M)$) can be calculated from the degree of the discriminant locus and the polarization type, and therefore $b_{\Theta^5}(S^{[5]}) = b_{\Theta^5}(\text{OG10})$.*

Wu [32] described a similar relation between $K_3(A)$ and OG6*, i.e., after deforming to Lagrangian fibrations, one is a (compactified) torsor over the other. As the two fibrations will again have the same discriminant locus and polarization type, we can conclude as above that $b_{\Theta^3}(K_3(A)) = b_{\Theta^3}(\text{OG6})$. The relation $b_{\Theta\Theta_2}(M) = 2^{12}3^3 - \frac{1}{3}b_{\Theta^3}(M)$ (proved in the next section) then gives $b_{\Theta\Theta_2}(K_3(A)) = b_{\Theta\Theta_2}(\text{OG6})$ too.*

6 Applications

Our first goal in this section is to prove the following inequality involving Rozansky-Witten invariants and the second Betti number. This generalizes Lemma 3 of Guan [10] to arbitrary dimension; it was known to Guan (see the Remark following Lemma 3 of [10]) but he did not publish an explicit proof (though see Lemma 1 of Kurnosov [17]). The key idea is to substitute a decomposition of the second Chern class $c_2 = \mu q + p^\perp$ into some formulae for Rozansky-Witten invariants appearing in Hitchin and Sawon [11]. This decomposition of c_2 and some of the calculations in the proof of Theorem 14 were also used by Cao and Jiang to prove Lemma 3.3 of [3].

Theorem 14 *Let M be an irreducible hyperkähler manifold of complex dimension $2n$. Then*

$$b_{\Theta^n}(M) \geq -(b_2 + 2n - 2) b_{\Theta^{n-2}\Theta_2}(M).$$

In this section we will denote the Beauville-Bogomolov quadratic form of M by $Q \in \mathrm{Sym}^2\mathrm{H}^2(M,\mathbb{Z})^*$, and denote its dual by $q \in \mathrm{Sym}^2\mathrm{H}^2(M,\mathbb{Q}) \subset \mathrm{H}^4(M,\mathbb{Q})$.

Lemma 15 *Let u be an arbitrary element of $\mathrm{H}^2(M,\mathbb{R})$. Then*

$$(2n-1)(b_{\Theta^n}(M)+2b_{\Theta^{n-2}\Theta_2}(M)) \left(\int_M q u^{2n-2}\right)^2 \geq (2n-3) b_{\Theta^n}(M) \left(\int_M q^2 u^{2n-4}\right) \left(\int_M u^{2n}\right).$$

Proof The second Chern class c_2 of M can be written as $p + p^\perp$, with $p \in \mathrm{Sym}^2\mathrm{H}^2(M,\mathbb{Q})$ and $p^\perp$ in the primitive cohomology $\mathrm{H}^4_{\mathrm{prim}}(M,\mathbb{Q})$. Up to scale, q is the unique element of $\mathrm{Sym}^2\mathrm{H}^2(M,\mathbb{Q})$ that remains of type $(2,2)$ with respect to all complex structures on M. Since c_2 also remains of type $(2,2)$ with respect to all complex structures, its first component p must be a multiple μ of q, i.e.,

$$c_2 = \mu q + p^\perp.$$

The constant μ can be determined by multiplying by $\sigma^{n-1}\bar\sigma^{n-1}$ and integrating

$$\int_M c_2 \sigma^{n-1}\bar\sigma^{n-1} = \mu \int_M q \sigma^{n-1}\bar\sigma^{n-1}.$$

Moreover

$$c_2^2 = \mu^2 q^2 + 2\mu q p^\perp + (p^\perp)^2.$$

Multiplying by $\sigma^{n-2}\bar\sigma^{n-2}$ and integrating yields

$$\begin{aligned} \int_M c_2^2 \sigma^{n-2}\bar\sigma^{n-2} &= \mu^2 \int_M q^2 \sigma^{n-2}\bar\sigma^{n-2} + 0 + \int_M (p^\perp)^2 \sigma^{n-2}\bar\sigma^{n-2} \\ &\geq \mu^2 \int_M q^2 \sigma^{n-2}\bar\sigma^{n-2}. \end{aligned}$$

On the first line we have used the fact that $q\sigma^{n-2}\bar\sigma^{n-2} \in \mathrm{Sym}^{2n-2}\mathrm{H}^2(M,\mathbb{Q})$ and $p^\perp \in \mathrm{H}^4_{\mathrm{prim}}(M,\mathbb{Q})$ are orthogonal, and on the second line we have used the Hodge-Riemann bilinear relations. Substituting the formula for μ into the inequality yields

$$\left(\int_M c_2^2 \sigma^{n-2}\bar\sigma^{n-2}\right)\left(\int_M q\sigma^{n-1}\bar\sigma^{n-1}\right)^2 \geq \left(\int_M c_2 \sigma^{n-1}\bar\sigma^{n-1}\right)^2 \left(\int_M q^2\sigma^{n-2}\bar\sigma^{n-2}\right).$$

The terms involving c_2 can be rewritten as Rozansky-Witten invariants. Specifically, Equation 8 of Hitchin and Sawon [11] states that

$$\beta_\Theta = \frac{16\pi^2 n \int_M c_2 \sigma^{n-1}\bar{\sigma}^{n-1}}{\int_M \sigma^n \bar{\sigma}^n}.$$

Similarly, using Chern-Weil theory (see Section 4 of [11] or Chapter 2 of Sawon [25]) we find that

$$\beta_\Theta^2 + 2\beta_{\Theta_2} = \frac{(16\pi^2)^2 n(n-1) \int_M c_2^2 \sigma^{n-2}\bar{\sigma}^{n-2}}{\int_M \sigma^n \bar{\sigma}^n}.$$

Substituting these into the inequality and simplifying gives

$$n(\beta_\Theta^2 + 2\beta_{\Theta_2}) \left(\int_M q\sigma^{n-1}\bar{\sigma}^{n-1}\right)^2 \geq (n-1)\beta_\Theta^2 \left(\int_M q^2\sigma^{n-2}\bar{\sigma}^{n-2}\right)\left(\int_M \sigma^n\bar{\sigma}^n\right).$$

Multiplying both sides by $\frac{1}{(8\pi^2)^n n!}\beta_\Theta^{n-2}\int \sigma^n\bar{\sigma}^n > 0$ converts the terms involving βs into genuine Rozansky-Witten invariants,

$$n(b_{\Theta^n}(M)+2b_{\Theta^{n-2}\Theta_2}(M)) \left(\int_M q\sigma^{n-1}\bar{\sigma}^{n-1}\right)^2 \geq (n-1)b_{\Theta^n}(M) \left(\int_M q^2\sigma^{n-2}\bar{\sigma}^{n-2}\right)\left(\int_M \sigma^n\bar{\sigma}^n\right).$$

The terms involving σ and $\bar{\sigma}$ can be rewritten so that they only depend on the combination $u := \sigma + \bar{\sigma}$; for example

$$\int_M q\sigma^{n-1}\bar{\sigma}^{n-1} = \frac{\int_M q(\sigma+\bar{\sigma})^{2n-2}}{\binom{2n-2}{n-1}}.$$

After simplifying, this gives

$$(2n-1)(b_{\Theta^n}(M)+2b_{\Theta^{n-2}\Theta_2}(M)) \left(\int_M qu^{2n-2}\right)^2 \geq (2n-3)b_{\Theta^n}(M) \left(\int_M q^2u^{2n-4}\right)\left(\int_M u^{2n}\right).$$

Since the resulting inequality will remain true under a deformation of the complex structure, it must be true for an arbitrary element $u \in \mathrm{H}^2(M,\mathbb{R})$. □

The proof of Theorem 14 now proceeds by rewriting the integrals involving u in terms of the Beauville-Bogomolov form $Q(u)$. As we will see, the second Betti number b_2 naturally appears.

Proof The Fujiki relation [7] says that there is a constant λ depending only on M such that

$$\int_M u^{2n} = \lambda Q(u)^n \tag{1}$$

for all $u \in \mathrm{H}^2(M,\mathbb{C})$. Moreover, because q remains of type $(2,2)$ under deformations of the complex structure, Corollary 23.17 of Huybrechts' notes in [9] implies that there exists a constant λ_q such that

$$\int_M qu^{2n-2} = \lambda_q Q(u)^{n-1} \tag{2}$$

for all $u \in \mathrm{H}^2(M, \mathbb{C})$. Similarly, q^2 remains of type $(4,4)$ under deformations of the complex structure, so there exists a constant λ_{q^2} such that

$$\int_M q^2 u^{2n-4} \quad = \quad \lambda_{q^2} Q(u)^{n-2} \tag{3}$$

for all $u \in \mathrm{H}^2(M, \mathbb{C})$. We need to determine λ_q and λ_{q^2}.

Let $\{e_1, \ldots, e_{b_2}\}$ be an orthonormal basis for $\mathrm{H}^2(M, \mathbb{C})$ with respect to the Beauville-Bogomolov form Q; then

$$q = \sum_{i=1}^{b_2} e_i^2.$$

Substituting $u = e_i + te_j + se_k$, for distinct i, j, and k, into the Fujiki relation gives

$$\int_M (e_i + te_j + se_k)^{2n} = \lambda Q(e_i + te_j + se_k)^n = \lambda(1 + t^2 + s^2)^n.$$

By comparing the constant, t^{2n-2}, t^{2n-4}, and $t^2 s^{2n-4}$ terms we find that

$$\int_M e_i^{2n} = \lambda, \qquad \binom{2n}{2} \int_M e_i^2 e_j^{2n-2} = \lambda \binom{n}{1}, \qquad \binom{2n}{4} \int_M e_i^4 e_j^{2n-4} = \lambda \binom{n}{2},$$

and

$$\binom{2n}{2, 2, 2n-4} \int_M e_i^2 e_j^2 e_k^{2n-4} = \lambda \binom{n}{1, 1, n-2}.$$

To calculate λ_q we substitute $q = \sum e_i^2$ and $u = e_j$ into Equation 2, which gives

$$\begin{aligned}
\lambda_q &= \lambda_q Q(e_j)^{n-1} \\
&= \int_M \left(\sum e_i^2 \right) e_j^{2n-2} \\
&= \int_M e_j^{2n} + \sum_{i \neq j} \int_M e_i^2 e_j^{2n-2} \\
&= \lambda + (b_2 - 1) \frac{\lambda \binom{n}{1}}{\binom{2n}{2}} \\
&= \lambda \left(\frac{b_2 + 2n - 2}{2n - 1} \right).
\end{aligned}$$

To calculate λ_{q^2} we substitute $q^2 = (\sum e_i^2)^2 = \sum e_i^4 + \sum_{i \neq j} e_i^2 e_j^2$ and $u = e_k$ into Equation 3,

which gives

$$\begin{aligned}
\lambda_{q^2} &= \lambda_{q^2} Q(e_k)^{n-2} \\
&= \int_M \left(\sum e_i^4 + \sum_{i \neq j} e_i^2 e_j^2 \right) e_k^{2n-4} \\
&= \int_M e_k^{2n} + \sum_{i \neq k} \int_M e_i^4 e_k^{2n-4} + \sum_{i,j,k \text{ distinct}} \int_M e_i^2 e_j^2 e_k^{2n-4} + \sum_{i \neq k} \int_M e_i^2 e_k^{2n-2} + \sum_{j \neq k} \int_M e_j^2 e_k^{2n-2} \\
&= \lambda + (b_2 - 1) \frac{\lambda \binom{n}{2}}{\binom{2n}{4}} + (b_2 - 1)(b_2 - 2) \frac{\lambda \binom{n}{1,1,n-2}}{\binom{2n}{2,2,2n-4}} + 2(b_2 - 1) \frac{\lambda \binom{n}{1}}{\binom{2n}{2}} \\
&= \lambda \frac{(b_2 + 2n - 2)(b_2 + 2n - 4)}{(2n-1)(2n-3)}.
\end{aligned}$$

In particular, Equations 1, 2, and 3 hold for $u \in \mathrm{H}^2(M, \mathbb{R})$. Substituting them into the inequality of Lemma 15, and using the above calculations of λ_q and λ_{q^2}, we find that the factors of λ and $Q(u)$ on each side cancel. After simplifying we are left with

$$(b_{\Theta^n}(M) + 2b_{\Theta^{n-2}\Theta_2}(M))(b_2 + 2n - 2) \geq b_{\Theta^n}(M)(b_2 + 2n - 4),$$

which can be rearranged to give

$$b_{\Theta^n}(M) \geq -(b_2 + 2n - 2) b_{\Theta^{n-2}\Theta_2}(M).$$

□

Our main conjecture, Conjecture 5, asserts that $b_{\Theta^{n-2}\Theta_2}(M) < 0$, which would give the following.

Corollary 16 *If $b_{\Theta^{n-2}\Theta_2}(M) < 0$ then the second Betti number b_2 is bounded above.*

Proof Because $b_{\Theta^{n-2}\Theta_2}(M)$ is a (rational) linear combination of Chern numbers, if it is negative then it must be $\leq -1/C_n$ where C_n is a positive integer depending only on n. On the other hand, Jiang proved that $\hat{A}^{1/2}[M]$ is strictly less than 1 (Corollary 5.5 of [13]), and this implies that

$$b_{\Theta^n}(M) = 48^n n! \hat{A}^{1/2}[M] < 48^n n!.$$

Theorem 14 then gives

$$48^n n! > b_{\Theta^n}(M) \geq -(b_2 + 2n - 2) b_{\Theta^{n-2}\Theta_2}(M) \geq \frac{b_2 + 2n - 2}{C_n},$$

so b_2 is bounded above by $48^n n! C_n - 2n + 2$. □

Example 17 *In dimension four we already have the sharp bound $b_2 \leq 23$, due to Guan [10]. In dimension six we have*

$$\hat{A}^{1/2}[M] = \frac{1}{967680}(31c_2^3 - 44c_2c_4 + 16c_6)[M],$$

so $\hat{A}^{1/2}[M] < 1$ implies

$$b_{\Theta^3}(M) \leq 48^3 3! \frac{967679}{967680} = \frac{23224296}{35}.$$

In addition

$$b_{\Theta\Theta_2}(M) = -\frac{8}{35}(11c_2^3 - 26c_2c_4 + 12c_6)[M],$$

so $b_{\Theta\Theta_2}(M) < 0$ *would imply* $b_{\Theta\Theta_2}(M) \leq -\frac{8}{35}$. *Therefore*

$$\frac{23224296}{35} \geq b_{\Theta^3}(M) \geq -(b_2+4)b_{\Theta\Theta_2}(M) \geq (b_2+4)\frac{8}{35},$$

which simplifies to give

$$2903033 \geq b_2.$$

In fact, we can improve on this. The first step is the following surprising relation.

Lemma 18 *Let* M *be an irreducible hyperkähler manifold of complex dimension six. Then*

$$b_{\Theta^3}(M) + 3b_{\Theta\Theta_2}(M) = 2^{10}3^4\chi(\mathcal{O}_M) = 2^{12}3^4.$$

Proof Writing the left-hand side as a linear combination of Chern numbers, we find

$$b_{\Theta^3}(M) + 3b_{\Theta\Theta_2}(M) = \frac{48}{35}(10c_2^3 - 9c_2c_4 + 2c_6)[M].$$

By the Riemann-Roch theorem, $\chi(\mathcal{O}_M)$ can also be written as a linear combination of Chern numbers, and a calculation reveals agreement with the above expression up to the constant $2^{10}3^4$. Finally, one notes that $\chi(\mathcal{O}_M) = 4$ for an irreducible hyperkähler manifold of complex dimension six. □

Our Conjecture 5 immediately yields a lower bound for $b_{\Theta^3}(M)$.

Corollary 19 *If* $b_{\Theta\Theta_2}(M) < 0$ *then* $b_{\Theta^3}(M) > 2^{12}3^4$. *Equivalently,*

$$\hat{A}^{1/2}[M] = \frac{1}{48^3 3!}b_{\Theta^3}(M) > \frac{1}{2}.$$

Now we can improve the upper bound on the second Betti number b_2.

Theorem 20 *If* $b_{\Theta\Theta_2}(M) < 0$ *then*

$$b_2 \leq 1451519.$$

Remark 21 *We expect that this bound must be extremely crude. There are three known examples of irreducible hyperkähler manifolds in dimension six, and their second Betti numbers are* 7, 8, *and* 23. *The results of Sawon [30] and Kim and Laza [15] suggest that* 23 *is most likely the largest possible value for* b_2.

Proof Combining Theorem 14, $b_{\Theta\Theta_2}(M) < 0$, and Lemma 18 gives

$$b_2 + 4 \leq \frac{b_{\Theta^3}(M)}{-b_{\Theta\Theta_2}(M)} = \frac{3b_{\Theta^3}(M)}{b_{\Theta^3}(M) - 2^{12}3^4}.$$

The right-hand side is a decreasing function of $b_{\Theta^3}(M)$, so it is maximized when $b_{\Theta^3}(M)$ is smallest. Now $b_{\Theta^3}(M) > 2^{12}3^4$ and

$$b_{\Theta^3}(M) = \frac{24}{35}(31c_2^3 - 44c_2c_4 + 16c_6)[M] \in \frac{24}{35}\mathbb{Z},$$

therefore $b_{\Theta^3}(M)$ is at least $2^{12}3^4 + \frac{24}{35}$, and substituting this value gives

$$b_2 + 4 \leq 1451523.$$

□

We also obtain an upper bound for $b_{\Theta^3}(M)$.

Proposition 22 *If $b_{\Theta\Theta_2}(M) < 0$ then $b_{\Theta^3}(M) \leq 2^{10}3^47$. Equivalently,*

$$\hat{A}^{1/2}[M] = \frac{1}{48^3 3!} b_{\Theta^3}(M) \leq \frac{7}{8}.$$

Remark 23 *This would improve on Jiang's bound $\hat{A}^{1/2}[M] < 1$ (though note that Jiang's bound holds in all dimensions, not just dimension six).*

Proof The second Betti number b_2 must be at least three, and therefore the inequality at the start of the last proof gives

$$7 \leq \frac{3b_{\Theta^3}(M)}{b_{\Theta^3}(M) - 2^{12}3^4}.$$

Rearranging gives $b_{\Theta^3}(M) \leq 2^{10}3^47$. □

Example 24 *For the Hilbert scheme $S^{[3]}$ of three points on a K3 surface we have $b_{\Theta^3}(M) = 373248$ and $b_2 = 23$. This implies that we have equality in*

$$b_2 + 4 \leq \frac{3b_{\Theta^3}(M)}{b_{\Theta^3}(M) - 2^{12}3^4}.$$

It follows that in the decomposition of the second Chern class $c_2 = p + p^\perp$ in the proof of Lemma 15, we must have $p^\perp = 0$. In other words, $c_2 = p$ lies in the part $\mathrm{Sym}^2\mathrm{H}^2(M, \mathbb{Q})$ of $\mathrm{H}^4(M, \mathbb{Q})$ which is generated by $\mathrm{H}^2(M, \mathbb{Q})$.

Example 25 *For the generalized Kummer variety $K_3(A)$ and O'Grady's example* OG6 *we have $b_{\Theta^3}(M) = 442368$ and $b_2 = 7$ and 8, respectively. This implies that we have equality in*

$$b_2 + 4 \leq \frac{3b_{\Theta^3}(M)}{b_{\Theta^3}(M) - 2^{12}3^4}$$

for OG6*, but strict inequality for $K_3(A)$. Thus for* OG6*, $p^\perp = 0$ and c_2 lies in $\mathrm{Sym}^2\mathrm{H}^2(M, \mathbb{Q})$, but for $K_3(A)$ this is no longer true.*

7 Comparisons to other work

Cao and Jiang proposed (Conjecture 3.6 in [3]), and Jiang later proved (Corollary 5.3 in [13]), that Todd classes of hyperkähler manifolds are quasi-effective, in the following sense.

Theorem 26 *Let X be a hyperkähler manifold of complex dimension $2n$ and L a nef and big line bundle on X. Then $\int_M \mathrm{td}_{2n-2i} L^{2i} > 0$ for all $0 \leq i \leq n$.*

This is equivalent to proving $\int_M \mathrm{td}_{2n-2i}\sigma^i\bar{\sigma}^i > 0$, or in other words, proving that certain Rozansky-Witten invariants are positive. For example, consider the quasi-effectivity of td_4, which is equivalent to

$$\int_M \mathrm{td}_4\sigma^{n-2}\bar{\sigma}^{n-2} = \int_M \frac{1}{720}(3c_2^2 - c_4)\sigma^{n-2}\bar{\sigma}^{n-2} > 0.$$

This case was already proved by Cao and Jiang, Theorem 3.2 in [3], by writing

$$\begin{aligned}\mathrm{td}_4 &= (\mathrm{td}^{1/2})_0(\mathrm{td}^{1/2})_4 + (\mathrm{td}^{1/2})_2(\mathrm{td}^{1/2})_2 + (\mathrm{td}^{1/2})_4(\mathrm{td}^{1/2})_0 \\ &= 2(\mathrm{td}^{1/2})_4 + (\mathrm{td}^{1/2})_2^2 \\ &= \frac{1}{2880}(7c_2^2 - 4c_4) + \frac{1}{576}c_2^2,\end{aligned}$$

and then showing that both

$$\int_M (7c_2^2 - 4c_4)\sigma^{n-2}\bar{\sigma}^{n-2} \qquad \text{and} \qquad \int_M c_2^2\sigma^{n-2}\bar{\sigma}^{n-2}$$

are positive.

Now as already observed in the proof of Lemma 15, using Chern-Weil theory (see Section 4 of Hitchin and Sawon [11] or Chapter 2 of Sawon [25]) we can show that

$$\beta_\Theta^2 + 2\beta_{\Theta_2} = \frac{(8\pi^2)^2 n(n-1)\int_M (s_2^2)\sigma^{n-2}\bar{\sigma}^{n-2}}{\int_M \sigma^n\bar{\sigma}^n} = \frac{(8\pi^2)^2 n(n-1)\int_M 4c_2^2\sigma^{n-2}\bar{\sigma}^{n-2}}{\int_M \sigma^n\bar{\sigma}^n}.$$

In the same way we find that

$$\frac{5}{2}\beta_{\Theta_2} = \frac{(8\pi^2)^2 n(n-1)\int_M (-s_4)\sigma^{n-2}\bar{\sigma}^{n-2}}{\int_M \sigma^n\bar{\sigma}^n} = -\frac{(8\pi^2)^2 n(n-1)\int_M (2c_2^2 - 4c_4)\sigma^{n-2}\bar{\sigma}^{n-2}}{\int_M \sigma^n\bar{\sigma}^n}.$$

Here $s_2 = 2!ch_2 = -2c_2$ and $s_4 = 4!ch_4 = 2c_2^2 - 4c_4$. Therefore we observe that the three inequalities

$$\int_M \mathrm{td}_4\sigma^{n-2}\bar{\sigma}^{n-2} > 0, \qquad \int_M (\mathrm{td}^{1/2})_4\sigma^{n-2}\bar{\sigma}^{n-2} > 0, \qquad \text{and} \qquad \beta_{\Theta_2} < 0$$

are equivalent to

$$3\int_M c_2^2\sigma^{n-2}\bar{\sigma}^{n-2} > \int_M c_4\sigma^{n-2}\bar{\sigma}^{n-2}, \qquad \frac{7}{4}\int_M c_2^2\sigma^{n-2}\bar{\sigma}^{n-2} > \int_M c_4\sigma^{n-2}\bar{\sigma}^{n-2},$$

and

$$\frac{1}{2}\int_M c_2^2\sigma^{n-2}\bar{\sigma}^{n-2} > \int_M c_4\sigma^{n-2}\bar{\sigma}^{n-2},$$

respectively. Because $\int_M c_2^2\sigma^{n-2}\bar{\sigma}^{n-2}$ is positive these inequalities are ordered weakest to strongest. In particular, our Conjecture 5 that $\beta_{\Theta_2} < 0$ is strictly stronger than the inequality $\int_M (\mathrm{td}^{1/2})_4\sigma^{n-2}\bar{\sigma}^{n-2} > 0$, which in turn is strictly stronger than Cao and Jiang's result that $\int_M \mathrm{td}_4\sigma^{n-2}\bar{\sigma}^{n-2} > 0$.

For higher degrees $k \geq 3$ we suspect that Jiang's theorem that $\int_M \mathrm{td}_{2k}\sigma^{n-k}\bar{\sigma}^{n-k} > 0$ may also follow from our Conjecture 5 that $(-1)^{k+1}\beta_{\Theta_k} > 0$, though we probably need to

combine it with some inequality analogous to the positivity of $\int_M c_2^2 \sigma^{n-2} \bar{\sigma}^{n-2}$ in the $k = 2$ case.

Finally, let us mention positivity of Chern numbers. The Chern numbers of Hilbert schemes of K3 surfaces can be calculated using the methods of Ellingsrud, Göttsche, and Lehn [5]. Those of generalized Kummer varieties were calculated up to dimension eight and twenty by Sawon [24] and Nieper-Wißkirchen [19], respectively, in both cases using Rozansky-Witten invariants. Those of OG6 were calculated by Mongardi, Rapagnetta, and Saccà [18] (see Section 5). These and other calculations have provoked the following questions (see Appendix B of [19], Conjecture 5.8 of Jiang [13], and Questions 4.7 and 4.8 of Oberdieck, Song, and Voisin [20]).

Question Let M be an irreducible hyperkähler manifold of complex dimension $2n$ and let $n = \sum_{i=1}^{t} k_i$ be a partition. Is it true that $c_{2k_1} \cdots c_{2k_t}[M]$ is positive?

Question Let M be an irreducible hyperkähler manifold of complex dimension $2n$ and let $n = \sum_{i=1}^{t} k_i$ be a partition. Is it true that $(-1)^n ch_{2k_1} \cdots ch_{2k_t}[M]$ is positive?

For example, Proposition 3.7 of [20] gives an affirmative answer to the second question for generalized Kummer varieties in all dimensions.

Now every Chern number $c_{2k_1} \cdots c_{2k_t}[M]$ and $ch_{2k_1} \cdots ch_{2k_t}[M]$ can be written in terms of Rozansky-Witten invariants, but Conjecture 5 only concerns Rozansky-Witten invariants involving the particular trivalent graphs Θ_k. Up to dimension six these are, in fact, all Rozansky-Witten invariants; but in higher dimensions the Chern numbers $c_{2k_1} \cdots c_{2k_t}[M]$ and $ch_{2k_1} \cdots ch_{2k_t}[M]$ *cannot* be written as linear combinations of Rozansky-Witten invariants involving only the graphs Θ_k. Consequently it is unclear how Conjecture 5 is related to the above questions, if at all.

A Maple code

```
> n := 3 :

> E := Array(1..2·n, 1..2·n, datatype = float[8]) :
  for i from 1 to n do
   E[2·i − 1, 2·i] := 1 :
   E[2·i, 2·i − 1] := -1 :
  od:

> EE := Array(1..2·n, 1..2·n, 1..2·n, 1..2·n, datatype = float[8]) :
  for i from 1 to 2·n do
   for j from 1 to 2·n do
    for k from 1 to 2·n do
     for l from 1 to 2·n do
      EE[i, j, k, l] := E[i, j]·E[k, l] + E[i, k]·E[l, j] + E[i, l]·E[j, k] :
     od:
    od:
   od:
  od:

> EEE := Array(1..2·n, 1..2·n, 1..2·n, 1..2·n, 1..2·n, 1..2·n, datatype = float[8]) :
  for i from 1 to 2·n do
   for j from 1 to 2·n do
    for k from 1 to 2·n do
     for l from 1 to 2·n do
      for m from 1 to 2·n do
       for p from 1 to 2·n do
        EEE[i, j, k, l, m, p] := E[i, j]·EE[k, l, m, p] + E[i, k]·EE[l, m, p, j] + E[i, l]·EE[m, p, j, k]
          + E[i, m]·EE[p, j, k, l] + E[i, p]·EE[j, k, l, m] :
       od:
      od:
     od:
    od:
   od:
  od:

> R := Array( 1..2·n, 1..2·n, 1..2·n, 1..2·n, datatype = complex[8]) :
  for i from 1 to 2·n do
   for j from 1 to 2·n do
    for k from 1 to 2·n do
     for l from 1 to 2·n do
      if i > j then R[i, j, k, l] := R[j, i, k, l]
       elif j > k then R[i, j, k, l] := R[i, k, j, l]
       elif k > l then R[i, j, k, l] := R[i, j, l, k]
       else R[i, j, k, l] := 2·rand( )/10^12 − 1.0 + (2·rand( )/10^12 − 1.0)·I
      end if:
     od:
```

```
 od:
od:
od:

> M := Array(1..2·n, 1..2·n, 1..2·n, 1..2·n, datatype = complex[8]):
for i from 1 to 2·n do
 for j from 1 to 2·n do
  for k from 1 to 2·n do
   for l from 1 to 2·n do
    t := 0:
    if i mod 2 = 1 then a := i + 1
     else a := i - 1: t := t + 1
    end if:
    if j mod 2 = 1 then b := j + 1
     else b := j - 1: t := t + 1
    end if:
    if k mod 2 = 1 then c := k + 1
     else c := k - 1: t := t + 1
    end if:
    if l mod 2 = 1 then d := l + 1
     else d := l - 1: t := t + 1
    end if:
    M[i, j, k, l] := R[i, j, k, l] + (-1)^t·conjugate(R[a, b, c, d]):
   od:
  od:
 od:
od:

> V := Array(1..2·n, 1..2·n, 1..2·n, 1..2·n, datatype = complex[8]):
for i from 1 to 2·n do
 for j from 1 to 2·n do
  for k from 1 to 2·n do
   for l from 1 to 2·n do
    V[i, j, k, l] := add(add(add(add(M[a, b, i, j]·M[c, d, k, l]·E[a, c]·E[b, d], a = 1..2·n), b
      = 1..2·n), c = 1..2·n), d = 1..2·n):
   od:
  od:
 od:
od:

> Th := add(add(add(add(V[i, j, k, l]·E[i, k]·E[j, l], i = 1..2·n), j = 1..2·n), k = 1..2·n), l = 1
      ..2·n);

                                Th := 1881.99949351646 + 0. I                    (1)

> W := Array(1..2·n, 1..2·n, 1..2·n, 1..2·n, datatype = complex[8]):
for i from 1 to 2·n do
 for j from 1 to 2·n do
```

```
 for k from 1 to 2·n do
  for l from 1 to 2·n do
   W[i,j,k,l] := add(add(add(add(V[a,i,b,j]·V[c,k,d,l]·E[a,c]·E[d,b], a = 1..2·n), b
     = 1..2·n), c = 1..2·n), d = 1..2·n) :
  od:
 od:
od:
```

```
> Th2 := add(add(add(add(W[i,j,k,l]·EE[i,j,k,l], i = 1..2·n), j = 1..2·n), k = 1..2·n), l = 1
     ..2·n);
```

$$Th2 := -2.49382174560616\ 10^{5} + 1.77635683940025\ 10^{-13}\ I \qquad \textbf{(2)}$$

```
> U := Array(1..2·n, 1..2·n, 1..2·n, 1..2·n, 1..2·n, 1..2·n, datatype = complex[8]) :
for i from 1 to 2·n do
 for j from 1 to 2·n do
  for k from 1 to 2·n do
   for l from 1 to 2·n do
    for m from 1 to 2·n do
     for p from 1 to 2·n do
      U[i,j,k,l,m,p] := add(add(add(add(add(add(V[a,i,b,j]·V[c,k,d,l]·V[e,m,f,p]
      ·E[a,d]·E[c,f]·E[e,b], a = 1..2·n), b = 1..2·n), c = 1..2·n), d = 1..2·n), e = 1..2·n), f
      = 1..2·n) :
     od:
    od:
   od:
  od:
 od:
od:
```

```
> Th3 := add(add(add(add(add(add(U[i,j,k,l,m,p]·EEE[i,j,k,l,m,p], i = 1..2·n), j = 1..2
     ·n), k = 1..2·n), l = 1..2·n), m = 1..2·n), p = 1..2·n);
```

$$Th3 := 9.06073037332295\ 10^{6} - 8.36735125631094\ 10^{-11}\ I \qquad \textbf{(3)}$$

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Department of Mathematics
University of North Carolina
Chapel Hill NC 27599-3250
USA

sawon@email.unc.edu
sawon.web.unc.edu