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ABSTRACT

In this paper, we provide an analytical study of the transmission eigenvalue problem with two conductivity parameters. We will assume that the underlying physical model is given by the scattering of a plane wave for an isotropic scatterer. In previous studies, this eigenvalue problem was analyzed with one conductive boundary parameter whereas we will consider the case of two parameters. We prove the existence and discreteness of the transmission eigenvalues as well as study the dependence on the physical parameters. We are able to prove monotonicity of the first transmission eigenvalue with respect to the parameters and consider the limiting procedure as the second boundary parameter vanishes. Lastly, we provide extensive numerical experiments to validate the theoretical work.

1. Introduction

In this paper, we study the transmission eigenvalue problem for an acoustic isotropic scatterer with two conductive boundary conditions. Transmission eigenvalues have been a very active field of investigation in the area of inverse scattering. This is due to the fact that these eigenvalues can be recovered from the far-field data, see for e.g. [1, 2], as well as can be used to determine defects in a material [3–7]. In general, one can prove that the transmission eigenvalues depend monotonically on the physical parameters, which implies that they can be used as a target signature for non-destructive testing. Non-destructive testing arises in many applications such as engineering and medical imaging, i.e. one wishes to recover information about the interior structure given exterior measurements. Therefore, by having information or knowledge of the transmission eigenvalues, one can retrieve information about the material properties of the scattering object. Another reason one studies these eigenvalue problems, is their non-linear and non-self-adjoint nature. This makes them mathematically challenging to study. We refer to [8] for a survey on the study of transmission eigenvalue problems.

Deriving accurate numerical algorithms to compute the transmission eigenvalues is an active field of study, see for e.g. [9–16]. As mentioned, here we consider the scalar transmission eigenvalue problem with a two parameter conductive boundary condition denoted λ and η. This problem was first introduced in [17]. The eigenvalue problem with one conductive boundary condition has been studied in [7, 18–20] for the case of acoustic scattering where as in [21, 22] for electromagnetic scatterers. Due to the presence of the second parameter in the conductive boundary condition, the analysis used

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in the aforementioned manuscripts will not work for the problem at hand. Therefore, we will need to use different analytical tools to study our transmission eigenvalue problem.

The rest of the paper is organized as follows. We will derive the transmission eigenvalue problem under consideration from the direct scattering problem in Section 2. Next, in Section 3, we prove that the transmission eigenvalues form a discrete set in the complex plane as well as provide an example via separation of variables to prove that this is a non-self-adjoint eigenvalue problem. Then in Section 4, we prove the existence of infinitely many real transmission eigenvalues as well as study the dependance on the material parameters. Furthermore, in Section 5, we consider the limiting process as $\lambda \to 1$ where we are able to prove that the transmission eigenpairs converge to the eigenpairs for one conductive boundary parameter i.e. with $\lambda = 1$. Numerical examples, using the separation of variables are given in Section 6 to validate the analysis presented in the earlier sections. Further, numerical results are given using boundary integral equations.

2. Formulation of the problem

We now state the transmission eigenvalue problem under consideration by connecting it to the direct scattering problem. To this end, we will formulate the direct scattering problem associated with the transmission eigenvalues in $\mathbb{R}^d$ where $d = 2$ or $d = 3$. Let $D \subset \mathbb{R}^d$ be a simply connected open set with C^2 boundary ∂D where ν denotes the unit outward normal vector. We then assume that the refractive index $n \in L^\infty(D)$ satisfies

$$0 < n_{\min} \leq n(x) \leq n_{\max} < \infty \quad \text{for a.e. } x \in D.$$

We are particularly interested in the case where there are two (conductivity) boundary parameters λ and η as in [17]. These parameters occur e.g. when the scattered medium is enclosed by a thin layer with high conductivity [23]. Therefore, we assume $\eta \in L^\infty(\partial D)$ such that

$$\eta_{\min} \leq \eta(x) \leq \eta_{\max} \quad \text{for a.e. } x \in \partial D$$

and fixed constant $\lambda \neq 1$. The fact that the boundary parameters are real-valued implies that the material covering the boundary is non-absorbing.

We let $u = u^s + u^i$ denote the total field and u^s is the scattered field created by the incident plane wave $u^i := e^{ikx \cdot \hat{y}}$ with wave number $k > 0$ and $\hat{y}$ the incident direction. The direct scattering problem for an isotropic homogeneous scatterer with a two parameter conductive boundary condition can be formulated as: find $u^s \in H^1_{loc}(\mathbb{R}^d)$ satisfying

$$\Delta u^s + k^2 n(x) u^s = k^2 \left(1 - n(x)\right) u^i \quad \text{in } \mathbb{R}^d \backslash \partial D \tag{1}$$

$$u^s_- - u^s_+ = 0 \quad \text{and} \quad \lambda \partial_\nu \left(u^s_- + u^i\right) = \eta(x) \left(u^s_+ + u^i\right) + \partial_\nu \left(u^s_+ + u^i\right) \quad \text{on } \partial D \tag{2}$$

where $\partial_\nu \phi := \nu \cdot \nabla \phi$ for any ϕ. Here $-$ and $+$ corresponds to taking the trace from the interior or exterior of D, respectively (see Figure 1). To close the system, we impose the Sommerfeld radiation condition on the scattered field u^s

$$\partial_r u^s - iku^s = \mathcal{O}\left(\frac{1}{r^{(d+1)/2}}\right) \quad \text{as } r \to \infty$$

which holds uniformly with respect to the angular variable $\hat{x} = x/r$ where $r = |x|$. Here, $|\cdot|$ denotes the Euclidean norm for a vector in $\mathbb{R}^d$.
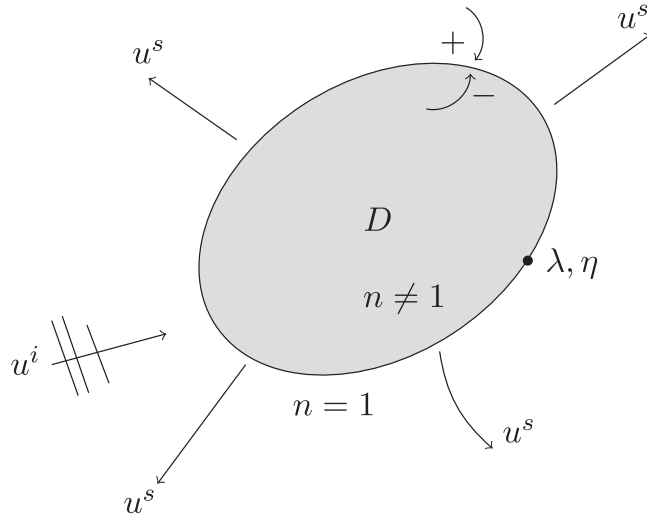
Figure 1. Illustration of the direct scattering problem in $\mathbb{R}^2$.

It has be shown that (1)–(2) is well-posed in [17]. Therefore, we have that the scattered field u^s has the asymptotic behavior (see for e.g. [24, 25])

$$u^s(x, \hat{y}) = \gamma \frac{e^{ik|x|}}{|x|^{(d-1)/2}} \left\{ u^\infty(\hat{x}, \hat{y}) + \mathcal{O}\left(\frac{1}{|x|}\right) \right\} \quad \text{as } |x| \longrightarrow \infty$$

and where the constant γ is given by

$$\gamma = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \quad \text{in } \mathbb{R}^2 \quad \text{and} \quad \gamma = \frac{1}{4\pi} \quad \text{in } \mathbb{R}^3.$$

Here $u^\infty(\hat{x}, \hat{y})$ denotes the far-field pattern depending on the incident direction $\hat{y}$ and the observation direction $\hat{x}$. The far-field pattern for all incident directions defines the far-field operator $F : L^2(\mathbb{S}^{d-1}) \longrightarrow L^2(\mathbb{S}^{d-1})$ given by

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^{d-1}} u^\infty(\hat{x}, \hat{y})g(\hat{y}) \, ds(\hat{y}) \quad \text{for } g \in L^2(\mathbb{S}^{d-1}).$$

Here, $\mathbb{S}^{d-1}$ denotes the unit disk/sphere in $\mathbb{R}^d$. It is also well-known (see [17]) that F is injective with a dense range if and only if there does not exist a nontrivial solution $(w, v) \in H^1(D) \times H^1(D)$ solving:

$$\Delta w + k^2 n(x)w = 0 \quad \text{and} \quad \Delta v + k^2 v = 0 \quad \text{in } D \tag{3}$$

$$w = v \quad \text{and} \quad \lambda \partial_\nu w = \partial_\nu v + \eta(x)v \quad \text{on } \partial D \tag{4}$$

where v takes the form of a Herglotz function

$$v_g(x) := \int_{\mathbb{S}^{d-1}} e^{ikx \cdot \hat{y}} g(\hat{y}) \, ds(\hat{y}), \quad g \in L^2(\mathbb{S}^{d-1}).$$

Now, the values $k \in \mathbb{C}$ for which (3)–(4) has non-trivial solutions are called *transmission eigenvalues*. Due to the fact that, the Herglotz functions are dense in the set of solutions to Helmholtz equation we will consider the transmission eigenvalue problem for any eigenfunction $v \in H^1(D)$. Thus, the goal of this paper is to study this eigenvalue problem as well as possible applications to the inverse spectral problem. We first show that if a set of eigenvalues exists, then this will be a discrete set.

3. Discreteness of eigenvalues

In this section, we study the discreteness of the transmission eigenvalues. In general, sampling methods such as the factorization method [17, 26] do not provide valid reconstructions of D if the wave number k is a transmission eigenvalue. Here, we will assume that the conductivity parameters satisfy either: $\lambda \in (1, \infty)$ and $\eta_{\max} < 0$ or $\lambda \in (0, 1)$ and $\eta_{\min} > 0$. Note, that due to the presence of the parameter $\lambda \neq 1$ in (3)–(4), the discreteness for this problem must be handled differently from the case when $\lambda = 1$ which was proven in [18]. Here, we will use a different variational formulation to study (3)–(4). To this end, we formulate the transmission eigenvalue problem as the problem for the difference $u := w - v \in H_0^1(D)$ and $v \in H^1(D)$. By subtracting the equations and boundary conditions for v and w, we have that the boundary value problem for v and u is given by

$$\lambda(\Delta u + k^2 nu) = (1 - \lambda)\Delta v + k^2(1 - \lambda n)v \quad \text{and} \quad \Delta v + k^2 v = 0 \quad \text{in } D \tag{5}$$

$$\lambda \partial_\nu u = (1 - \lambda)\partial_\nu v + \eta v \quad \text{on } \partial D. \tag{6}$$

Now, in order to analyze (5)–(6), we will employ a variational technique. To do so, we use Green's First Theorem to obtain that

$$\lambda \int_D \nabla u \cdot \nabla \overline{\phi} - k^2 nu\overline{\phi} \, dx = \int_D (1 - \lambda)\nabla v \cdot \nabla \overline{\phi} - k^2(1 - \lambda n)v\overline{\phi} \, dx + \int_{\partial D} \eta v \overline{\phi} \, ds \tag{7}$$

for all $\phi \in H^1(D)$. In addition, we also need to enforce that v is a solution to the Helmholtz equation in D. Therefore, by again appealing to Green's First Theorem, we can have that

$$\int_D \nabla v \cdot \nabla \overline{\psi} \, dx = \int_D k^2 v\overline{\psi} \, dx \quad \text{for all } \psi \in H_0^1(D). \tag{8}$$

We now define the following sesquilinear forms $b(\cdot, \cdot) : H^1(D) \times H_0^1(D) \longrightarrow \mathbb{C}$

$$b(v, \psi) = \int_D \nabla v \cdot \nabla \overline{\psi} \, dx$$

and $a(\cdot, \cdot) : H^1(D) \times H^1(D) \longrightarrow \mathbb{C}$

$$a(v, \phi) = -\frac{1}{\lambda} \int_D (1 - \lambda)\nabla v \cdot \nabla \bar{\phi} \, dx - \frac{1}{\lambda} \int_{\partial D} \eta v \bar{\phi} \, ds.$$

It is clear that by appealing to the Cauchy–Schwarz inequality and the Trace Theorem that both $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are bounded. Defining these sesquilinear forms helps us to write (5)–(6) as a linear eigenvalue problem for $\{(u, v), k\} \in H_0^1(D) \times H^1(D) \times \mathbb{C} \setminus \{0\}$ via the system

$$a(v, \phi) + \overline{b(\phi, u)} = \int_D k^2 nu\overline{\phi} \, dx - \frac{1}{\lambda} \int_D k^2(1 - \lambda n)v\overline{\phi} \, dx \tag{9}$$

$$b(v, \psi) = \int_D k^2 v\overline{\psi} \, dx \quad \text{for all } (\psi, \phi) \in H_0^1(D) \times H^1(D). \tag{10}$$

In the analysis of the equivalent eigenvalue problem (9)–(10), we will consider the corresponding source problem. Therefore, we will make the substitution $k^2 v = g$ and $k^2 u = f$ to define the saddle point problem corresponding to (9)–(10) as

$$a(v, \phi) + \overline{b(\phi, u)} = \left(f, n\phi\right)_{L^2(D)} + \frac{1}{\lambda} \left(g, (\lambda n - 1)\phi\right)_{L^2(D)} \tag{11}$$

$$b(v, \psi) = (g, \psi)_{L^2(D)}. \tag{12}$$

It is clear that there exists constants $C_j > 0$ for $j = 1, 2$ such that

$$\left| (f, n\phi)_{L^2(D)} + \frac{1}{\lambda} \left(g, (\lambda n - 1)\phi \right)_{L^2(D)} \right| \leq C_1 \left\{ \|f\|_{L^2(D)} + \|g\|_{L^2(D)} \right\} \|\phi\|_{H^1(D)}$$

and

$$\left| (g, \psi)_{L^2(D)} \right| \leq C_2 \|g\|_{L^2(D)} \|\psi\|_{H^1(D)}$$

for all $f \in H_0^1(D)$ and $g \in H^1(D)$ since we have assumed that $n \in L^\infty(D)$.

Now, we consider the source problem stated above as: given $(f, g) \in H_0^1(D) \times H^1(D)$ find $(u, v) \in H_0^1(D) \times H^1(D)$ solving (11)–(12). Notice, that in order to prove well-posedness it is sufficient to prove that the sesquilinear form $a(\cdot, \cdot)$ is coercive on $H^1(D)$ and that $b(\cdot, \cdot)$ has the inf–sup condition. Recall, that the inf–sup condition is defined as (see for e.g. [27])

$$\inf_{\psi \in H_0^1(D)} \sup_{v \in H^1(D)} \frac{b(v, \psi)}{\|\psi\|_{H^1(D)} \|v\|_{H^1(D)}} \geq \alpha$$

for some constant $\alpha > 0$. In the following result, we prove that the sesquilinear forms defined above satisfy the aforementioned properties.

Theorem 3.1: *Assume that either $\lambda \in (1, \infty)$ and $\eta_{\max} < 0$ or $\lambda \in (0, 1)$ and $\eta_{\min} > 0$. Then we have that $a(\cdot, \cdot)$ is coercive on $H^1(D)$. Moreover, we have that $b(\cdot, \cdot)$ satisfies the inf–sup condition.*

Proof: We first show that $a(\cdot, \cdot)$ is coercive and we choose to present the case where we assume that $\lambda \in (1, \infty)$ and $\eta_{\max} < 0$. From this, we can now estimate

$$\lambda a(v, v) = -\int_D (1 - \lambda)|\nabla v|^2 \, dx - \int_{\partial D} \eta |v|^2 \, ds$$

$$\geq (\lambda - 1) \int_D |\nabla v|^2 \, dx - \eta_{\max} \int_{\partial D} |v|^2 \, ds$$

$$\geq \min \left\{ (\lambda - 1), |\eta_{\max}| \right\} \left(\int_D |\nabla v|^2 \, dx + \int_{\partial D} |v|^2 \, ds \right).$$

Now, we can use the fact that

$$\| \cdot \|_{H^1(D)}^2 \quad \text{is equivalent to} \quad \int_D |\nabla \cdot|^2 \, dx + \int_{\partial D} |\cdot|^2 \, ds,$$

(see for e.g. [28] Chapter 8) to obtain the estimate

$$|a(v, v)| \geq C\|v\|_{H^1(D)}^2 \quad \text{for some } C > 0.$$

This proves the coercivity for the case when $\lambda \in (1, \infty)$ and $\eta_{\max} < 0$. The case when $\lambda \in (0, 1)$ and $\eta_{\min} > 0$ can be handled in a similar manner.

In order to show that the sesquilinear form $b(\cdot, \cdot)$ satisfies the inf–sup condition, we will use an equivalent definition. Recall, that the inf–sup condition is equivalent to showing that for any $\psi \in$

$H_0^1(D)$ there exists $v_\psi \in H^1(D)$ such that

$$b(v_\psi, \psi) \geq \beta \|\psi\|_{H^1(D)}^2$$

where $\|v_\psi\|_{H^1(D)} \leq C\|\psi\|_{H^1(D)}$ for some constant $\beta > 0$ that is independent of ψ. To this end, we define $v_\psi \in H^1(D)$ to be the solution of the variational problem

$$\int_D \nabla v_\psi \cdot \nabla \overline{\phi} \, dx + \int_{\partial D} v_\psi \overline{\phi} \, ds = \int_D \nabla \psi \cdot \nabla \overline{\phi} \, dx \tag{13}$$

for all $\phi \in H^1(D)$. By appealing to the norm equivalence stated above and the Lax–Milgram Lemma, we have that the mapping $\psi \longmapsto v_\psi$ solving (13) is a well-defined bounded linear operator from $H_0^1(D)$ to $H^1(D)$. Therefore, we have that letting $\phi = \psi$ in (13) gives

$$b(v_\psi, \psi) = \int_D \nabla v_\psi \cdot \nabla \overline{\psi} \, dx = \int_D |\nabla \psi|^2 \, dx \geq \beta \|\psi\|_{H^1(D)}^2$$

by the Poincaré inequality. Note, that we have used the fact that ψ has zero trace on the boundary ∂D. Thus, we have that $b(\cdot, \cdot)$ satisfies the inf–sup condition. ∎

From Theorem 3.1 and the analysis in [27] we have that (11)–(12) is well-posed. Therefore, we can define the bounded linear operator

$$T : H_0^1(D) \times H^1(D) \longrightarrow H_0^1(D) \times H^1(D) \quad \text{such that } T(f, g) = (u, v).$$

By the well-posedness and the estimates on the $L^2(D)$ integrals on the right-hand side of (11)–(12), we have that for some $C > 0$

$$\|T(f, g)\|_{H^1(D) \times H^1(D)} = \|(u, v)\|_{H^1(D) \times H^1(D)} \leq C \left\{ \|f\|_{L^2(D)} + \|g\|_{L^2(D)} \right\}.$$

Now, we have the necessary requirements to prove that the solution operator T is compact using the Rellich–Kondrachov Embedding Theorem.

Theorem 3.2: *Assume that either $\lambda \in (1, \infty)$ and $\eta_{\max} < 0$ or $\lambda \in (0, 1)$ and $\eta_{\min} > 0$. Then the solution operator $T : H_0^1(D) \times H^1(D) \longrightarrow H_0^1(D) \times H^1(D)$ corresponding to (11)–(12) is compact.*

Proof: To prove the claim, we show that for any sequence (f_j, g_j) weakly converging to zero in $H_0^1(D) \times H^1(D)$, then the image $T(f_j, g_j)$ has a subsequence that converges strongly to zero in $H_0^1(D) \times H^1(D)$. Notice, that there exists a subsequence (still denoted with j) that satisfies

$$\|f_j\|_{L^2(D)} + \|g_j\|_{L^2(D)} \to 0 \quad \text{as } j \to \infty$$

by the compact embedding of $H^1(D)$ in $L^2(D)$ see [29]. From this, we have that

$$\|T(f_j, g_j)\|_{H^1(D) \times H^1(D)} \leq C \left\{ \|f_j\|_{L^2(D)} + \|g_j\|_{L^2(D)} \right\} \to 0 \quad \text{as } j \to \infty$$

which proves the claim. ∎

Now, simple calculations show that the relationship between the eigenvalues of T and the transmission eigenvalues k is that $1/k^2 \in \sigma(T)$, where $\sigma(T)$ is the spectrum of the operator T. Therefore, we have related the transmission eigenvalues to the eigenvalues of a compact operator. We can use the compactness of T to prove the following result for the set of transmission eigenvalues independent of the sign of the contrast $n-1$.
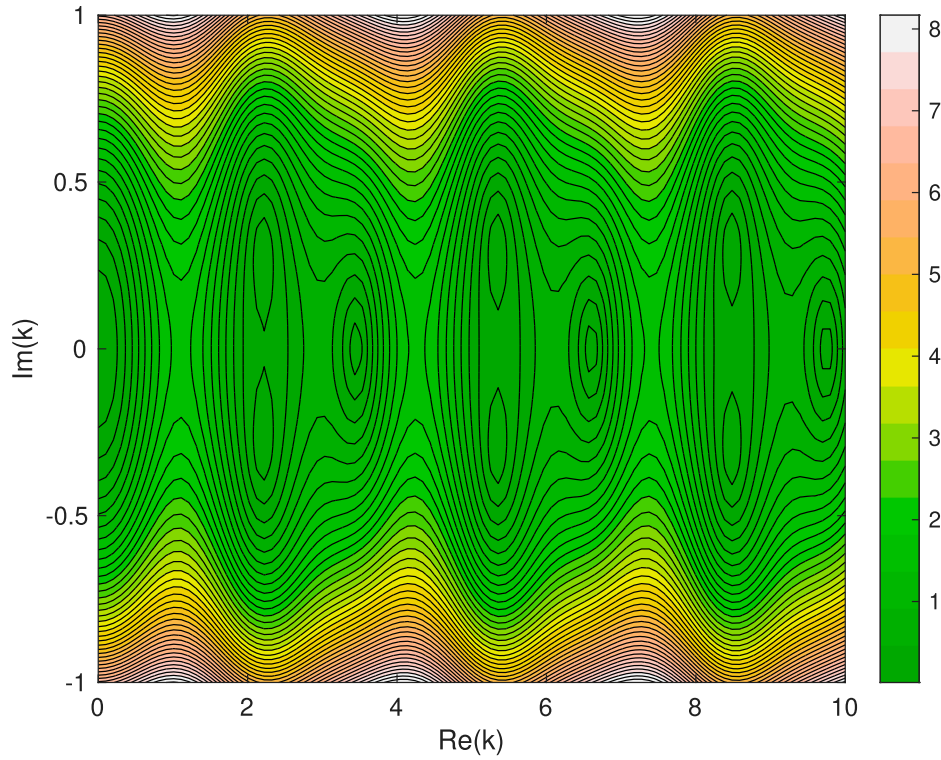
Figure 2. Contour plot of $|d_0(k)|$ on the set $[0, 10] \times [-1, 1]i$ in the complex plane where the parameters are $\lambda = 2$, $n = 4$ and $\eta = -\frac{1}{100}$.

Theorem 3.3: *Assume that either $\lambda \in (1, \infty)$ and $\eta_{\max} < 0$ or $\lambda \in (0, 1)$ and $\eta_{\min} > 0$. Then the set of transmission eigenvalues is discrete with no finite accumulation point.*

Proof: This is a consequence of the fact that k is a transmission eigenvalue implies that $1/k^2 \in \sigma(T)$. Then we exploit that the set $\sigma(T)$ is a discrete set with zero its only possible accumulation point. ∎

An important question is whether or not the operator T is self-adjoint. If so, we would have existence of real transmission eigenvalues by appealing to the Hilbert–Schmidt Theorem. In a similar way with other transmission eigenvalue problems, we have that the operator T is not self-adjoint even when the material parameters are real-valued. To see this fact, we can consider the transmission eigenvalue problem for the unit disk in $\mathbb{R}^2$ with constant coefficients λ, η and n.

Example 3.1: Using separation of variables, we have that k is a transmission eigenvalue provided that $d_m(k) = 0$ for any $m \in \mathbb{Z}$ where

$$d_m(k) := \det \begin{pmatrix} J_m(k\sqrt{n}) & -J_m(k) \\ \lambda J'_m(k\sqrt{n})k\sqrt{n} & -\left(kJ'_m(k) + \eta J_m(k) \right) \end{pmatrix}$$

and $J_m(t)$ are the Bessel functions of the first kind of order m (see Section 6 for details). Therefore, we can plot $|d_0(k)|$ for complex-valued k and determine if there are any complex roots. This is done in Figure 2 using $\lambda = 2$, $n = 4$, and $\eta = -\frac{1}{100}$. We see complex roots at the values $k = 2.2032 \pm 0.2905i$ as well as other points in the set $[0, 10] \times [-1, 1]i$.

More precisely, we obtain 10 interior transmission eigenvalues within the given set for $m = 0$ with MATLAB. They are given to high accuracy as 0.053410, $2.203160 \pm 0.290468i$, 3.456704, $5.338551 \pm 0.305549i$, 6.606526, $8.477827 \pm 0.309699i$, and 9.750981.

From this, we see that there are multiple complex transmission eigenvalues k for this set of parameters. As a result, for this simple example, T has complex eigenvalues since $1/k^2 \in \sigma(T)$ and cannot be self-adjoint. Therefore, we can not rely on standard theory to prove the existence of the transmission eigenvalues. The existence is proven in the next section where we use a similar analysis as in [30]. These techniques are usually used for anisotropic materials. This analysis is utilized due to the fact that the techniques in [18] fail to give a variational formulation for the eigenfunction $u = w - v$ exclusively.

4. Existence of transmission eigenvalues

In this section, we show the existence of the transmission eigenvalues with conductive boundary parameters following a similar analysis as [30]. In our analysis, we will furthermore assume that $\lambda \in (1, \infty)$, $\eta_{\max} < 0$, and $\lambda n_{\max} - 1 < 0$, or $\lambda \in (0, 1)$, $\eta_{\min} > 0$, and $\lambda n_{\min} - 1 > 0$. The goal now is to show the existence of real transmission eigenvalues. To this end, we work with the formulated problem (5)–(6) and the variational formulation (7)

$$\lambda \int_D \nabla u \cdot \nabla \overline{\phi} - k^2 n u \overline{\phi} \, dx = \int_D (1-\lambda) \nabla v \cdot \nabla \overline{\phi} - k^2 (1-\lambda n) v \overline{\phi} \, dx + \int_{\partial D} \eta v \overline{\phi} \, ds$$

for all $\phi \in H^1(D)$. Following the analysis in [30], we consider (5)–(6) as a Robin boundary value problem for $v \in H^1(D)$. This means that for a given $u \in H_0^1(D)$ we need to show that there exists a $v \in H^1(D)$ satisfying (7). We now define the bounded sesquilinear form and the bounded conjugate linear functional from the variational formulation as

$$A(v, \phi) = \int_{\partial D} \eta v \overline{\phi} \, ds + \int_D (1-\lambda) \nabla v \cdot \nabla \overline{\phi} - k^2 (1-\lambda n) v \overline{\phi} \, dx$$

and

$$\ell(\phi) = \lambda \int_D \nabla u \cdot \nabla \overline{\phi} - k^2 n u \overline{\phi} \, dx.$$

Applying the Lax–Milgram Lemma to $A(v, \phi) = \ell(\phi)$ gives us that (5)–(6) is well posed, i.e. there exists a unique solution $v \in H^1(D)$ satisfying (5)–(6) for any given $u \in H_0^1(D)$. Notice, that the coercivity result for $A(v, \phi)$ is proven in a similar manner as the coercivity result for $a(\cdot, \cdot)$ in Section 3. This says that the mapping we have $u \longmapsto v_u$ from $H_0^1(D)$ to $H^1(D)$ is a bounded linear operator. Because the transmission eigenfunction v solves the Helmholtz equation in D, we make sure that v_u is also a solution of the Helmholtz equation in the variational sense. To this end, we use the Riesz Representation Theorem to define $\mathbb{L}_k u$ by

$$(\mathbb{L}_k u, \psi)_{H^1(D)} = \int_D \nabla v_u \cdot \nabla \overline{\psi} - k^2 v_u \overline{\psi} \, dx \quad \forall \psi \in H_0^1(D). \tag{14}$$

Notice, that $\mathbb{L}_k u = 0$ if and only if v_u solves the Helmholtz equation.

We will analyze the null-space of the operator $\mathbb{L}_k : H_0^1(D) \longrightarrow H_0^1(D)$ and connect this to the set of transmission eigenfunctions. To this end, we show that $\mathbb{L}_k$ having a non-trival null-space for a given value of k is equivalent to the transmission eigenvalue problem (3)–(4).

Theorem 4.1: *Assume that either $\lambda \in (1, \infty)$, $\eta_{\max} < 0$, and $\lambda n_{\max} - 1 < 0$ or $\lambda \in (0, 1)$, $\eta_{\min} > 0$, and $\lambda n_{\min} - 1 > 0$. If $(v, w) \in H^1(D) \times H^1(D)$ are non-trivial solutions of (3)–(4), then the non-trivial $u = w - v \in H_0^1(D)$ satisfies that $\mathbb{L}_k u = 0$. Conversely, if for a given value of k we have that $\mathbb{L}_k u = 0$ for a non-trivial $u \in H_0^1(D)$, then $v_u \in H^1(D)$ solving (5)–(6) and $w = u + v_u$ are non-trivial solutions of (3)–(4).*

Proof: The first part of the theorem is given by our construction. Conversely, we assume $\mathbb{L}_k u = 0$ for a given value of k provided that $u \neq 0$ and we let $v = v_u \in H^1(D)$ be the unique solution to (5)–(6), then define $w = u + v \in H^1(D)$. From Equation (5) along with the fact that $\mathbb{L}_k u = 0$ gives that

$$\Delta v + k^2 v = 0 \quad \text{and} \quad \Delta w + k^2 n w = 0 \quad \text{in } D.$$

Similarly, from the boundary condition (6) given by $\lambda \partial_\nu u = (1 - \lambda)\partial_\nu v + \eta v$ on ∂D and using the identity $w = u + v$ we can easily obtain that

$$\lambda \partial_\nu w = \partial_\nu v + \eta v \quad \text{on } \partial D.$$

This proves the claim since $u \in H_0^1(D)$. ∎

We have shown that there exist transmission eigenvalues if and only if the null-space of $\mathbb{L}_k$ is non-trivial. Therefore, we turn our attention to studying this operator. Now, we are going to highlight some properties of the operator $\mathbb{L}_k$ that will help us establish when $\mathbb{L}_k$ has a trivial null-space. From here on, we denote $v_u := v$.

Theorem 4.2: *Assume that either $\lambda \in (1, \infty)$, $\eta_{\max} < 0$, and $\lambda n_{\max} - 1 < 0$ or $\lambda \in (0, 1)$, $\eta_{\min} > 0$, and $\lambda n_{\min} - 1 > 0$. Then, we have the following:*

(1) *the operator $\mathbb{L}_k : H_0^1(D) \longrightarrow H_0^1(D)$ is self-adjoint,*
(2) *the operator $-\mathbb{L}_0$ or $\mathbb{L}_0$ is coercive when $\lambda \in (1, \infty)$ or $\lambda \in (0, 1)$, respectively.*
(3) *and the operator $\mathbb{L}_k - \mathbb{L}_0$ is compact.*

Proof: (1) Now we show that the operator $\mathbb{L}_k$ is self-adjoint. To this end, it is enough to show that the quantity

$$(\mathbb{L}_k u, u)_{H^1(D)} = \int_D \nabla v \cdot \nabla \overline{u} - k^2 v \overline{u} \, dx$$

is real-valued for all u (see for e.g. [31]). Recall, the variational formulation given by (7)

$$\lambda \int_D \nabla u \cdot \nabla \overline{\phi} - k^2 n u \overline{\phi} \, dx = \int_{\partial D} \eta v \overline{\phi} \, ds + \int_D (1 - \lambda)\nabla v \cdot \nabla \overline{\phi} - k^2 (1 - \lambda n) v \overline{\phi} \, dx$$

for any $\phi \in H^1(D)$. Letting $\phi = u$ in (7) implies that

$$\lambda \int_D |\nabla u|^2 - k^2 n |u|^2 \, dx = \int_D (1 - \lambda)\nabla v \cdot \nabla \overline{u} - k^2 (1 - \lambda n) v \overline{u} \, dx. \tag{15}$$

In a similar manner, letting $\phi = v$ in the variational formulation (7), we obtain

$$\lambda \int_D \nabla u \cdot \nabla \overline{v} - k^2 n u \overline{v} \, dx = \int_{\partial D} \eta |v|^2 \, ds + \int_D (1 - \lambda)|\nabla v|^2 - k^2 (1 - \lambda n)|v|^2 \, dx. \tag{16}$$

By the definition of $\mathbb{L}_k$, we have that

$$(\mathbb{L}_k u, u)_{H^1(D)} = \int_D \nabla v \cdot \nabla \overline{u} - k^2 v \overline{u} \, dx$$

$$= \int_D (1 - \lambda)\nabla v \cdot \nabla \overline{u} - k^2 (1 - \lambda n) v \overline{u} \, dx + \lambda \int_D \nabla v \cdot \nabla \overline{u} - k^2 n v \overline{u} \, dx.$$

Using (15) and (16) above, we obtain that

$$(\mathbb{L}_k u, u)_{H^1(D)} = \lambda \int_D |\nabla u|^2 - k^2 n |u|^2 \, dx + \int_{\partial D} \eta |v|^2 \, ds$$

$$+ \int_D (1 - \lambda)|\nabla v|^2 - k^2(1 - \lambda n)|v|^2 \, dx.$$

Thus, all the integrals on the right-hand side are evaluated to be real numbers and that gives us that $\mathbb{L}_k$ is self-adjoint.

(2) Now, we show that $\pm \mathbb{L}_0$ is coercive and we first analyze $-\mathbb{L}_0$. We assume that $\lambda \in (1, \infty)$ and that $\eta_{\max} < 0$. Letting $w = v + u$ in the definition of $\mathbb{L}_k$ gives

$$(\mathbb{L}_k u, u)_{H^1(D)} = \int_D \nabla w \cdot \nabla \overline{u} - k^2 w \overline{u} \, dx - \int_D |\nabla u|^2 - k^2 |u|^2 \, dx.$$

From the variational formulation (7) with $\phi = w$, we have the following equality

$$\int_D \nabla w \cdot \nabla \overline{u} - k^2 w \overline{u} \, dx = \int_D (1 - \lambda)|\nabla w|^2 - k^2(1 - \lambda n)|w|^2 \, dx + \int_{\partial D} \eta |w|^2 \, ds. \tag{17}$$

Now, using (17), we get

$$(\mathbb{L}_k u, u)_{H^1(D)} = \int_D (1 - \lambda)|\nabla w|^2 - k^2(1 - \lambda n)|w|^2 \, dx$$

$$+ \int_{\partial D} \eta |w|^2 \, ds - \int_D |\nabla u|^2 - k^2 |u|^2 \, dx. \tag{18}$$

Therefore, letting $k = 0$, we obtain

$$- (\mathbb{L}_0 u, u)_{H^1(D)} = \int_D (\lambda - 1)|\nabla w|^2 \, dx - \int_{\partial D} \eta |w|^2 \, ds + \int_D |\nabla u|^2 \, dx. \tag{19}$$

By appealing to the assumptions $\lambda \in (1, \infty)$ and $\eta_{\max} < 0$, we see that

$$\int_D (\lambda - 1)|\nabla w|^2 \, dx \geq 0 \quad \text{and} \quad - \int_{\partial D} \eta |w|^2 \, ds \geq 0.$$

From this, we can estimate

$$-(\mathbb{L}_0 u, u)_{H^1(D)} = \int_D (\lambda - 1)|\nabla w|^2 \, dx - \int_{\partial D} \eta |w|^2 ds + \int_D |\nabla u|^2 \, dx$$

$$\geq \int_D |\nabla u|^2 \, dx = \|\nabla u\|^2_{L^2(D)}$$

proving the coercivity of the $-\mathbb{L}_k$ operator in $H^1_0(D)$.

Next, assume that $\lambda \in (0, 1)$ and $\eta_{\min} > 0$ and for this case, we consider the operator $\mathbb{L}_0$. From the definition of $\mathbb{L}_k$, we have that

$$(\mathbb{L}_k u, u)_{H^1(D)} = \int_D \nabla v \cdot \nabla \overline{u} - k^2 v \overline{u} \, dx.$$

Letting $k = 0$ in the variational formulation (7) with $\phi = u$ gives

$$\lambda \int_D |\nabla u|^2 \, dx = \int_D (1 - \lambda)\nabla v \cdot \nabla \overline{u} \, dx. \tag{20}$$

In a similar way, using that $k = 0$ in the variational formulation (7) with $\phi = v$ gives us

$$\lambda \int_D \nabla u \cdot \nabla \overline{v} \, dx = \int_D (1 - \lambda)|\nabla v|^2 \, dx + \int_{\partial D} \eta |v|^2 \, ds. \tag{21}$$

Now, consider $\mathbb{L}_0$ and using (20) and (21) provide independently, and so we get

$$(\mathbb{L}_0 u, u)_{H^1(D)} = \int_D \nabla v \cdot \nabla \overline{u} \, dx$$

$$= \lambda \int_D |\nabla u|^2 \mathrm{d}x + \int_{\partial D} \eta |v|^2 \mathrm{d}s + \int_D (1-\lambda)|\nabla v|^2 \, \mathrm{d}x$$

$$\geq \lambda \int_D |\nabla u|^2 \mathrm{d}x + \int_{\partial D} \eta_{\min} |v|^2 \mathrm{d}s + \int_D (1-\lambda)|\nabla v|^2 \, \mathrm{d}x$$

$$\geq \lambda \|\nabla u\|_{L^2(D)}^2$$

where we have used the assumptions of $\lambda \in (0,1)$ and $\eta_{\min} > 0$. Proving the coercivity in this case.

(3) Now, we turn our attention to proving the compactness of $\mathbb{L}_k - \mathbb{L}_0$. To do so, we assume that we have a weakly convergent sequence $u^j \rightharpoonup 0$ in $H_0^1(D)$. By the well-posedness, there exists $v_k^j \rightharpoonup 0$ and $v_0^j \rightharpoonup 0$ in $H^1(D)$, where these correspond to the solutions of our variational formulation (7). The definition of $\mathbb{L}_k$ gives us that we can define $(\mathbb{L}_k - \mathbb{L}_0)u^j$ in terms of v_k^j and v_0^j. Using the variational formulation (7), we have that

$$\int_{\partial D} \eta v_k^j \overline{\phi} \, \mathrm{d}s + \int_D (1-\lambda)\nabla v_k^j \cdot \nabla \overline{\phi} - k^2(1-\lambda n)v_k^j \overline{\phi} \, \mathrm{d}x = \lambda \int_D \nabla u^j \cdot \nabla \overline{\phi} - k^2 n u^j \overline{\phi} \, \mathrm{d}x$$

and

$$\int_{\partial D} \eta v_0^j \overline{\phi} \, \mathrm{d}s + \int_D (1-\lambda)\nabla v_0^j \cdot \nabla \overline{\phi} \, \mathrm{d}x = \lambda \int_D \nabla u^j \cdot \nabla \overline{\phi} \, \mathrm{d}x$$

for all $\phi \in H^1(D)$. Subtracting both equations gives us that

$$\int_{\partial D} \eta(v_k^j - v_0^j)\overline{\phi} \, \mathrm{d}s + \int_D (1-\lambda)\nabla(v_k^j - v_0^j) \cdot \nabla \overline{\phi} \, \mathrm{d}x = \int_D k^2(1-\lambda n)v_k^j \overline{\phi} - \lambda n k^2 u^j \overline{\phi} \, \mathrm{d}x.$$

We now let $\phi = v_k^j - v_0^j$ and we have the following

$$\int_{\partial D} \eta |v_k^j - v_0^j|^2 \, \mathrm{d}s + \int_D (1-\lambda)|\nabla(v_k^j - v_0^j)|^2 \, \mathrm{d}x = \int_D k^2(1-\lambda n)v_k^j \overline{(v_k^j - v_0^j)} - \lambda n k^2 u^j \overline{(v_k^j - v_0^j)} \, \mathrm{d}x.$$

Notice, that on the left-hand side, we use the fact that

$$\| \cdot \|_{H^1(D)}^2 \quad \text{is equivalent to} \quad \int_D |\nabla \cdot |^2 \, \mathrm{d}x + \int_{\partial D} | \cdot |^2 \, \mathrm{d}s.$$

By the compact embedding of $H^1(D)$ into $L^2(D)$, we have that v_k^j and u^j converge strongly to zero in the $L^2(D)$–norm. Thus, we have that the right-hand side behaves as

$$\|v_k^j - v_0^j\|_{H^1(D)} \leq C \left(\|v_k^j\|_{L^2(D)} + \|u^j\|_{L^2(D)} \right) \longrightarrow 0$$

as $j \to \infty$. Notice that the $C > 0$ above is independent of the parameter j's but does depend on the material parameters. Note that we have used the assumptions on λ and η. Now, observe the following

$$\left((\mathbb{L}_k - \mathbb{L}_0)u^j, \psi \right)_{H^1(D)} = \int_D \nabla v_k^j \cdot \nabla \overline{\psi} - k^2 v_k^j \overline{\psi} \, \mathrm{d}x - \int_D \nabla v_0^j \cdot \nabla \overline{\psi} \, \mathrm{d}x$$

$$= \int_D \nabla(v_k^j - v_0^j) \cdot \nabla \overline{\psi} - k^2 v_k^j \overline{\psi} \, \mathrm{d}x$$

and using the Cauchy–Schwartz inequality, we have

$$\|(\mathbb{L}_k - \mathbb{L}_0)u^j\|_{H^1(D)} \leq C \left(\|v_k^j\|_{L^2(D)} + \|v_k^j - v_0^j\|_{H^1(D)} \right) \longrightarrow 0.$$

Therefore, we have shown that $(\mathbb{L}_k - \mathbb{L}_0)u^j$ tends to zero as j tends to infinity, proving the claim. ∎

We have shown three important properties that will help us establish when our operator $\mathbb{L}_k$ has a trivial null-space. In addition, we want to make the observation that $\mathbb{L}_k$ depends continuously on k by a similar argument as in Theorem 4.2. We continue by showing that the operator $\pm\mathbb{L}_k$ is positive for a range of values which will give a lower bound on the transmission eigenvalues.

Theorem 4.3: *Let $\mu_1(D)$ be the first Dirichlet eigenvalue of $-\Delta$ and let k^2 be a real transmission eigenvalue. Then, we have the following:*

(1) *If $\lambda \in (1, \infty)$, $\eta_{\max} < 0$, and $\lambda n_{\max} - 1 < 0$, then $-\mathbb{L}_k$ is a positive operator for $k^2 < \mu_1(D)$.*
(2) *If $\lambda \in (0, 1)$, $\eta_{\min} > 0$, and $\lambda n_{\min} - 1 > 0$, then $\mathbb{L}_k$ is a positive operator for $k^2 < \frac{\mu_1(D)}{n_{\max}}$.*

Proof: (1) We first assume that $\lambda \in (1, \infty)$, $\eta_{\max} < 0$, and $\lambda n_{\max} - 1 < 0$. Using the definition of $\mathbb{L}_k$ and $w = v + u$, we have that

$$
\begin{aligned}
-(\mathbb{L}_k u, u)_{H^1(D)} &= -\int_D (1 - \lambda)|\nabla w|^2 - k^2(1 - \lambda n)|w^2|\, dx - \int_{\partial D} \eta |w^2|\, ds \\
&\quad + \int_D |\nabla u|^2 - k^2|u|^2\, dx \\
&\geq \int_D (\lambda - 1)|\nabla w|^2 - k^2(\lambda n_{\max} - 1)|w^2|\, dx - \eta_{\max}\int_{\partial D} |w^2|\, ds \\
&\quad + \int_D |\nabla u|^2 - k^2|u|^2\, dx \\
&\geq \int_D |\nabla u|^2 - k^2|u|^2\, dx.
\end{aligned}
$$

Observe, that $u \in H_0^1(D)$ implies that we have the estimate

$$
\|u\|_{L^2(D)}^2 \leq \frac{1}{\mu_1(D)}\|\nabla u\|_{L^2(D)}^2 \quad \text{(i.e. Poincaré inequality)}
$$

where $\mu_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$. This gives that

$$
-(\mathbb{L}_k u, u)_{H^1(D)} \geq \left(1 - \frac{k^2}{\mu_1(D)}\right)\|\nabla u\|_{L^2(D)}^2.
$$

Now if $(1 - \frac{k^2}{\mu_1(D)}) > 0$, we have that $-(\mathbb{L}_k u, u)_{H^1(D)} > 0$ for all $u \neq 0$ which gives us that all real transmission eigenvalues must satisfy that $k^2 \geq \mu_1(D)$.

(2) On the other hand, assume that $\lambda \in (0, 1)$, $\eta_{\min} > 0$, and $\lambda n_{\min} - 1 > 0$. Using our variational formulation (7) and let $\phi = u$ to obtain

$$
\begin{aligned}
(\mathbb{L}_k u, u)_{H^1(D)} &= \lambda\int_D |\nabla u|^2 - k^2 n|u|^2\, dx + \int_D (1 - \lambda)|\nabla v|^2 - k^2(1 - \lambda n)|v|^2\, dx \\
&\quad + \int_{\partial D} \eta |v|^2\, ds \\
&\geq \lambda\int_D |\nabla u|^2 - k^2 n_{\max}|u|^2\, dx + \int_D (1 - \lambda)|\nabla v|^2 + k^2(\lambda n_{\min} - 1)|v|^2\, dx \\
&\quad + \int_{\partial D} \eta_{\min}|v|^2\, ds \\
&\geq \lambda\int_D |\nabla u|^2 - k^2 n_{\max}|u|^2\, dx. \quad\quad\quad\quad\quad\quad\quad\quad\quad (22)
\end{aligned}
$$

By again, appealing to the Poincaré inequality, we have that

$$(\mathbb{L}_k u, u)_{H^1(D)} \geq \lambda \left(1 - n_{\max} \frac{k^2}{\mu_1(D)}\right) \|\nabla u\|_{L^2(D)}^2.$$

Now if $(1 - n_{\max} \frac{k^2}{\mu_1(D)}) > 0$, we conclude that $(\mathbb{L}_k u, u)_{H^1(D)} > 0$ for all $u \neq 0$ which implies that all real transmission eigenvalues must satisfy that $k^2 \geq \frac{\mu_1(D)}{n_{\max}}$. ∎

Theorem 4.3 shows that the operator $\pm\mathbb{L}_k$ is positive for a range of k values. Next, we show one last result to help us establish when the null-space of $\mathbb{L}_k$ is non-trivial. The property that we want to show is that the operator $\pm\mathbb{L}_k$ is non-positive for some k on a subset of $H_0^1(D)$.

Theorem 4.4: *There exists $\tau > 0$ such that $-\mathbb{L}_\tau$, or $\mathbb{L}_\tau$ for $\lambda \in (1, \infty)$, $\eta_{\max} < 0$, and $\lambda n_{\max} - 1 < 0$, or $\lambda \in (0, 1)$, $\eta_{\min} > 0$, and $\lambda n_{\min} - 1 > 0$, respectively, is non-positive on N–dimensional subspaces of $H_0^1(D)$ for any $N \in \mathbb{N}$.*

Proof: We begin with the case when $\lambda \in (1, \infty)$, $\eta_{\max} < 0$, and $\lambda n_{\max} - 1 < 0$. We consider the ball B_ϵ of radius $\epsilon > 0$ such that $B_\epsilon \subset D$. Using separation of variables one can see that there exist transmission eigenvalues for the system (See Section 6)

$$\Delta w_1 + \tau^2 n_{\max} w_1 = 0 \quad \text{and} \quad \Delta v_1 + \tau^2 v_1 = 0 \quad \text{in } B_\epsilon$$

$$w_1 = v_1 \quad \text{and} \quad \lambda \partial_\nu w_1 = \partial_\nu v_1 \quad \text{on } \partial B_\epsilon.$$

Letting u_1 be the difference of the eigenfunctions with corresponding eigenvalue τ gives us the following using (22)

$$\int_{B_\epsilon} |\nabla u_1|^2 - \tau^2 |u_1|^2 \, dx + \int_{B_\epsilon} (\lambda - 1)|\nabla w_1|^2 - \tau^2(\lambda n_{\max} - 1)|w_1|^2 \, dx = 0.$$

Therefore, since $u_1 \in H_0^1(B_\epsilon)$ we can take the extension by zero of u_1 to the whole domain be denoted by $u_2 \in H_0^1(D)$. Now, since $\lambda \in (1, \infty)$, $\eta_{\max} < 0$, and $\lambda n_{\max} - 1 < 0$, we can construct the non-trivial $v_2 \in H^1(D)$ that solves the variational formulation (7) with coefficients λ, n, and η in the domain D and we also let $w_2 = v_2 + u_2$. Using the relationship between v_2 and u_2 and $w_2 = v_2 + u_2$ just as in the proof of Theorem 4.2 we have that

$$\int_D (\lambda - 1)\nabla w_2 \cdot \nabla \overline{\phi} - \tau^2(\lambda n - 1)w_2 \overline{\phi} \, dx - \int_{\partial D} \eta w_2 \overline{\phi} \, ds = -\int_D \nabla u_2 \cdot \nabla \overline{\phi} - \tau^2 u_2 \overline{\phi} \, dx$$

$$= -\int_{B_\epsilon} \nabla u_1 \cdot \nabla \overline{\phi} - \tau^2 u_1 \overline{\phi} \, dx$$

$$= \int_{B_\epsilon} (\lambda - 1)\nabla w_1 \cdot \nabla \overline{\phi}$$

$$\quad - \tau^2(\lambda n_{\max} - 1)w_1 \overline{\phi} \, dx. \quad (23)$$

Letting $\phi = w_2$ in (23) and using the Cauchy–Schwartz inequality because we have an inner product on the right-hand side over the space $H^1(B_\epsilon)$ gives us

$$\int_D (\lambda - 1)|\nabla w_2|^2 - \tau^2(\lambda n - 1)|w_2|^2 \, dx - \int_{\partial D} \eta |w_2|^2 \, ds$$

$$= \int_{B_\epsilon} (\lambda - 1)\nabla w_1 \cdot \nabla \overline{w_2} - \tau^2(\lambda n_{\max} - 1)w_1 \overline{w_2} \, dx$$

$$\leq \left[\int_{B_\epsilon} (\lambda - 1)|\nabla w_1|^2 - \tau^2(\lambda n_{\max} - 1)|w_1|^2 \, dx \right]^{\frac{1}{2}}$$

$$\times \left[\int_{B_\epsilon} (\lambda - 1)|\nabla w_2|^2 - \tau^2(\lambda n_{\max} - 1)|w_2|^2 \, dx \right]^{\frac{1}{2}}.$$

As a consequence of the above inequality, we have that

$$\int_D (\lambda - 1)|\nabla w_2|^2 - \tau^2(\lambda n - 1)|w_2|^2 \, dx - \int_{\partial D} \eta |w_2|^2 \, ds$$

$$\leq \int_{B_\epsilon} (\lambda - 1)|\nabla w_1|^2 - \tau^2(\lambda n_{\max} - 1)|w_1|^2 \, dx.$$

Now, we use the definition of $-\mathbb{L}_\tau$ in (22) with the functions u_2 and w_2 to conclude that

$$- (\mathbb{L}_\tau u_2, u_2)_{H^1(D)} = - \int_D \nabla v_2 \cdot \nabla \overline{u_2} - \tau^2 v_2 \overline{u_2} \, dx$$

$$= \int_D |\nabla u_2|^2 - \tau^2 |u_2|^2 \, dx$$

$$+ \int_D (\lambda - 1)|\nabla w_2|^2 - \tau^2(\lambda n - 1)|w_2|^2 \, dx - \int_{\partial D} \eta |w_2|^2 \, ds$$

by the calculations in Theorem 4.3. Next, using the above inequality, we obtain

$$(\mathbb{L}_\tau u_2, u_2)_{H^1(D)} = \int_{B_\epsilon} |\nabla u_1|^2 - \tau^2 |u_1|^2 \, dx$$

$$+ \int_D (\lambda - 1)|\nabla w_2|^2 - \tau^2(\lambda n - 1)|w_2|^2 \, dx - \int_{\partial D} \eta |w_2|^2 \, ds$$

$$\leq \int_{B_\epsilon} |\nabla u_1|^2 - \tau^2 |u_1|^2 \, dx + \int_{B_\epsilon} (\lambda - 1)|\nabla w_1|^2 - \tau^2(\lambda n_{\max} - 1)|w_1|^2 \, dx$$

$$= 0.$$

Thus, the operator is non-positive on this one dimensional subspace.

We now argue that, for some $\tau > 0$, we can construct an N–dimensional subspace of $H_0^1(D)$ where the operator $-\mathbb{L}_\tau$ is non-positive for any $N \in \mathbb{N}$. To this end, let N be fixed and define $B_j = \{B(x_j, \epsilon) : x_j \in D, \epsilon > 0\} \subset D$ for $j = 1, \ldots, N$ where we assume $B_j \cap B_i = \emptyset$ for all $i \neq j$. We make the assumption that $\lambda \in (1, \infty)$, $\eta_{\max} < 0$, and $\lambda n_{\max} - 1 < 0$ and denoting τ as the smallest transmission eigenvalue for

$$\Delta w_j + \tau^2 n_{\max} w_j = 0 \quad \text{and} \quad \Delta v_j + \tau^2 v_j = 0 \quad \text{in } B_j$$

$$w_j = v_j \quad \text{and} \quad \lambda \partial_\nu w_j = \partial_\nu v_j \quad \text{on } \partial B_j.$$

From this, we let $u_j \in H_0^1(D)$ be the difference of the eigenfunctions w_j and v_j extended to D by zero. Therefore, we have that for $j = 1, \ldots, N$ the supports of u_j and u_i are disjoint, i.e. u_j and u_i are orthogonal to each other for $j \neq i$. Thus, the span$\{u_1, u_2, \ldots, u_N\}$ is a N–dimensional subspace of $H_0^1(D)$. Now, because the support of the basis functions are disjoint and using the same arguments as above, we can show that $-\mathbb{L}_\tau$ is non-positive for any u in the N–dimensional subspace of $H_0^1(D)$ spanned by the u_j's. This proves that claim since N is arbitrary. The same result can be proven for $\mathbb{L}_k$ exactly in a similar way for the case when $\lambda \in (0, 1)$, $\eta_{\min} > 0$, and $\lambda n_{\min} - 1 > 0$. ■

We have shown five important properties that will compile to imply the existence of transmission eigenvalues. This requires appealing to the following theorem first introduced in [30] to study anisotropic transmission eigenvalue problems.

Theorem 4.5: *Assume that we have* $\mathbb{L}_k : H_0^1(D) \longrightarrow H_0^1(D)$ *that satisfies*

(1) $\mathbb{L}_k$ *is self-adjoint and it depends on* $k > 0$ *continuously*
(2) $\pm \mathbb{L}_0$ *is coercive*
(3) $\mathbb{L}_k - \mathbb{L}_0$ *is compact*
(4) *There exists* $\alpha > 0$ *such that* $\mathbb{L}_\alpha$ *is a positive operator*
(5) *There exists* $\beta > 0$ *such that* $\mathbb{L}_\beta$ *is non-positive on an* m *dimensional subspace*

Then there exists m *values* $k_j \in (\alpha, \beta)$ *such that* $\mathbb{L}_{k_j}$ *has a non-trivial subspace.*

Proof: The proof of this result can be found in [30] Theorem 2.6. ∎

By the above result as well as the analysis presented in this section we have the main result of the paper. This gives that there exists infinitely many transmission eigenvalues.

Theorem 4.6: *Assume either* $\lambda \in (1, \infty)$, $\eta_{\max} < 0$, *and* $\lambda n_{\max} - 1 < 0$, *or* $\lambda \in (0, 1)$, $\eta_{\min} > 0$ *and* $\lambda n_{\min} - 1 > 0$ *respectively, then there exists infinitely many real transmission eigenvalues* $k > 0$.

Proof: The proof follows directly by applying Theorem 4.5 where we have proven that our operator satisfies the assumptions in the previous results. ∎

We have shown the existence of real transmission eigenvalues and we now wish to study how they depend on the parameters λ, n, and η. We will show monotonicity results for the first transmission eigenvalue with respect to the parameters n and η. We have two different results with respect to n and η. The first result shows that the first eigenvalue is an increasing function when $\lambda \in (1, \infty)$, $\eta_{\max} < 0$, and $\lambda n_{\max} - 1 < 0$. Then we show that the first eigenvalue is a decreasing function when $\lambda \in (0, 1)$, $\eta_{\min} > 0$, and $\lambda n_{\min} - 1 > 0$.

Theorem 4.7: *Assume that the parameters satisfy* $\lambda \in (1, \infty)$, $\eta_{\max} < 0$, *and* $\lambda n_{\max} - 1 < 0$. *Therefore, we have that:*

(1) *If* $n_1 \leq n_2$ *such that* $\lambda n_j - 1 < 0$, *then* $k_1(n_1) \leq k_1(n_2)$.
(2) *If* $\eta_1 \leq \eta_2$ *such that* $\eta_j < 0$, *then* $k_1(\eta_1) \leq k_1(\eta_2)$.

Here k_1 *corresponds to the first transmission eigenvalue.*

Proof: Here, we will prove part (1) for the theorem and part (2) can be handled in a similar manner. To this end, notice that if $n_1 \leq n_2$, then we have $(1 - \lambda n_2) \leq (1 - \lambda n_1)$. Assume that $\lambda \in (1, \infty)$, $\eta_{\max} < 0$, and $\lambda n_2 - 1 < 0$, and that v_2 and w_2 are the transmission eigenfunctions corresponding to the transmission eigenvalue $k_2 = k_1(n_2, \lambda, \eta)$. Therefore, from (22), we obtain that

$$\int_D |\nabla u_2|^2 - k_2^2 |u_2|^2 \, \mathrm{d}x + \int_D (\lambda - 1)|\nabla w_2|^2 + k_2^2(1 - \lambda n_2)|w_2|^2 \, \mathrm{d}x - \int_{\partial D} \eta |w_2|^2 \, \mathrm{d}s = 0$$

where $u_2 = w_2 - v_2 \in H_0^1(D)$.

Now, we have the existence of $v \in H^1(D)$ that solves the variational problem (7) with $u = u_2$, $n = n_1$, and $k = k_2$. Then, we can define $w = v + u_2$. By rearranging the variational form in (7) and using the definition $w = v + u_2$ we have that

$$\int_D (1 - \lambda)\nabla w \cdot \nabla \overline{\phi} - k_2^2(1 - \lambda n_1)w\overline{\phi} \, dx + \int_{\partial D} \eta w\overline{\phi} \, ds$$

$$= \int_D \nabla u_2 \cdot \nabla \overline{\phi} - k_2^2 u_2 \overline{\phi} \, dx$$

$$= \int_D (1 - \lambda)\nabla w_2 \cdot \nabla \overline{\phi} - k_2^2(1 - \lambda n_2)w_2\overline{\phi} \, dx + \int_{\partial D} \eta w_2 \overline{\phi} \, ds. \tag{24}$$

Letting $\phi = w$ in (24) and using the Cauchy–Schwartz inequality as in the proof of Theorem 4.4, we have that

$$\int_D (\lambda - 1)|\nabla w|^2 - k_2^2(\lambda n_1 - 1)|w|^2 \, dx - \int_{\partial D} \eta |w|^2 \, ds$$

$$\leq \int_D (\lambda - 1)|\nabla w_2|^2 - k_2^2(\lambda n_2 - 1)|w_2|^2 \, dx - \int_{\partial D} \eta |w_2|^2 \, ds.$$

We denote the operator $-\mathbb{L}_\tau$ as the operator with $n = n_1$. By appealing to the calculations in Theorem 4.3 and the above inequality, we have that

$$-(\mathbb{L}_{k_2} u_2, u_2)_{H^1(D)} = -\int_D \nabla v \cdot \nabla \overline{u_2} - k_2^2 v\overline{u_2} \, dx$$

$$= \int_D |\nabla u_2|^2 - k_2^2|u_2|^2 \, dx + \int_D (\lambda - 1)|\nabla w|^2 - k_2^2(\lambda n_1 - 1)|w|^2 \, dx$$

$$- \int_{\partial D} \eta |w|^2 \, ds$$

$$\leq \int_D |\nabla u_2|^2 - k_2^2|u_2|^2 \, dx + \int_D (\lambda - 1)|\nabla w_2|^2 - k_2^2(\lambda n_2 - 1)|w_2|^2 \, dx$$

$$- \int_{\partial D} \eta |w_2|^2 \, ds$$

$$= 0.$$

Since $-\mathbb{L}_{k_2}$ is non-positive on the subspace spanned by u_2 we can conclude that there is an eigenvalue corresponding to n_1 in $(0, k_1(n_2)]$. Therefore, the first transmission eigenvalue $k_1(n_1)$ must satisfy that $k_1(n_1) \in (0, k_1(n_2)]$, proving the claim. ∎

Next, we have a similar monotonicity result with respect to the assumptions on the coefficients that $\lambda \in (0, 1)$, $\eta_{\min} > 0$, and $\lambda n_{\min} - 1 > 0$. Since the proof is similar to what is presented in Theorem 4.7 we omit the proof to avoid repetition.

Theorem 4.8: *Assume that the parameters satisfy $\lambda \in (0, 1)$, $\eta_{\min} > 0$, and $\lambda n_{\min} - 1 > 0$. Therefore, we have that:*

(1) *If $n_1 \leq n_2$ such that $\lambda n_j - 1 > 0$, then $k_1(n_2) \leq k_1(n_1)$,*
(2) *If $\eta_1 \leq \eta_2$ such that $\eta_j > 0$, then $k_1(\eta_2) \leq k_1(\eta_1)$.*

Here k_1 corresponds to the first transmission eigenvalue.

From Theorems 4.7 and 4.8 we can see that the first transmission eigenvalue depends monotonically on some of the material parameters n and η. Notice that we are unable to prove a similar monotonicity result with respect to λ due to showing up in the variational definition of $\mathbb{L}_k$ in different terms with different signs. We will present some numerics for the monotonicity with respect to λ in Section 6.

5. Convergence as the conductivity λ goes to 1

In this section, we study the convergence of the transmission eigenvalues in the sense of whether or not we have that $k(\lambda) \longrightarrow k(1)$ as $\lambda \longrightarrow 1$ where $k(1)$ is the transmission eigenvalue corresponding to $\lambda = 1$. Throughout this section, we will assume that the transmission eigenvalues $k(\lambda) = k_\lambda \in \mathbb{R}_+$ form a bounded set as $\lambda \to 1$. From this, we have that the set will have a limit point as λ tends to one. For the eigenfunctions v_λ and w_λ, we may assume that they are normalized in $H^1(D)$ such that

$$\|v_\lambda\|_{H^1(D)}^2 + \|w_\lambda\|_{H^1(D)}^2 = 1$$

for any $\lambda \in (0,1) \cup (1,\infty)$. As a result, we have that $(k_\lambda, v_\lambda, w_\lambda) \in \mathbb{R}_+ \times H^1(D) \times H^1(D)$ are bounded, so there exists $(\kappa, \hat{v}, \hat{w}) \in \mathbb{R}_+ \times H^1(D) \times H^1(D)$ such that

$$k_\lambda \longrightarrow \kappa$$

as well as

$$w_\lambda \rightharpoonup \hat{w} \quad \text{and} \quad v_\lambda \rightharpoonup \hat{v} \quad \text{in } H^1(D) \quad \text{as } \lambda \longrightarrow 1.$$

Now, our task is to show that the limits $\hat{w}$ and $\hat{v}$ satisfy the transmission eigenvalue problem when we let $\lambda = 1$ with eigenvalue κ. To this end, we begin by showing that the difference of the eigenfunctions $u_\lambda = w_\lambda - v_\lambda$ is bounded with respect to λ in the $H^2(D)$–norm. To this end, by (3) we have that

$$\Delta u_\lambda + k_\lambda^2 n u_\lambda = -k_\lambda^2 (n-1) v_\lambda \quad \text{in } D.$$

Notice, the fact that $u_\lambda \in H^2(D) \cap H_0^1(D)$ is given by appealing to standard elliptic regularity results. Observe that $\|\Delta \cdot\|_{L^2(D)}$ is equivalent to $\|\cdot\|_{H^2(D)}$ in $H^2(D) \cap H_0^1(D)$ (see for e.g. [28]). Therefore, we can bound the $H^2(D)$–norm of u_λ using the above equation such that

$$\|u_\lambda\|_{H^2(D)} \leq C\|\Delta u_\lambda\|_{L^2(D)} \leq C\left\{\|u_\lambda\|_{L^2(D)} + \|v_\lambda\|_{L^2(D)}\right\}.$$

Notice, we have used the fact that $n \in L^\infty(D)$ and that k_λ is bounded with respect to λ. This implies that, u_λ is bounded in $H^2(D) \cap H_0^1(D)$ i.e.

$$u_\lambda \rightharpoonup \hat{u} = \hat{w} - \hat{v} \quad \text{in } H^2(D) \cap H_0^1(D) \quad \text{as } \lambda \to 1.$$

We want to determine which boundary value problem the functions $\hat{u}$ and $\hat{v}$ satisfy. To this end, we take $\phi \in H^1(D)$ and integrate over the region D to obtain

$$\int_D (\Delta u_\lambda + k_\lambda^2 n u_\lambda)\overline{\phi}\,\mathrm{d}x = -k_\lambda^2 \int_D (n-1)v_\lambda\overline{\phi}\,\mathrm{d}x.$$

Notice, that since $k_\lambda^2 \to \kappa^2$ as well as $v_\lambda \to \hat{v}$ in $L^2(D)$ and $u_\lambda \rightharpoonup \hat{u}$ in $H^2(D) \cap H_0^1(D)$ as $\lambda \to 1$ we have that

$$\int_D \overline{\phi}\left[\Delta\hat{u} + \kappa^2 n\hat{u} + \kappa^2(n-1)\hat{v}\right]\mathrm{d}x = 0 \quad \text{for all } \phi \in H^1(D).$$

This implies that

$$\Delta\hat{u} + \kappa^2 n\hat{u} = -\kappa^2(n-1)\hat{v} \quad \text{in } D.$$

Using a similar argument, we have that

$$\Delta \hat{v} + \kappa^2 \hat{v} = 0 \quad \text{in } D.$$

Notice, that $u_\lambda|_{\partial D} = 0$ and by the Trace Theorem, we have that

$$\partial_\nu u_\lambda|_{\partial D} \in H^{1/2}(\partial D), \quad v_\lambda|_{\partial D} \in H^{1/2}(\partial D), \quad \text{and} \quad \partial_\nu v_\lambda|_{\partial D} \in H^{-1/2}(\partial D)$$

are bounded. This implies that the above boundary values weakly converge to the corresponding boundary values for the weak limits. Now, multiplying by $\phi \in H^{1/2}(\partial D)$ and integrating over ∂D in Equation (6) we have that

$$\int_{\partial D} \phi \left[\lambda \partial_\nu u_\lambda - \eta v_\lambda \right] \, ds = (1 - \lambda) \int_{\partial D} \phi \partial_\nu v_\lambda \, ds.$$

We can then estimate

$$\left| \int_{\partial D} \phi \left[\lambda \partial_\nu u_\lambda - \eta v_\lambda \right] \, ds \right| \leq |1 - \lambda| \int_{\partial D} |\phi \partial_\nu v_\lambda| \, ds$$

$$\leq |1 - \lambda| \|\partial_\nu v_\lambda\|_{H^{-1/2}(\partial D)} \|\phi\|_{H^{1/2}(\partial D)}$$

$$\leq C|1 - \lambda| \left\{ \|v_\lambda\|_{H^1(D)} + \|\Delta v_\lambda\|_{L^2(D)} \right\} \|\phi\|_{H^{1/2}(\partial D)}.$$

Notice, that the quantity

$$\|v_\lambda\|_{H^1(D)} + \|\Delta v_\lambda\|_{L^2(D)}$$

is bounded due to the normalization and the fact that v_λ satisfies the Helmholtz equation in D. As we let $\lambda \to 1$, we have that

$$\int_{\partial D} \phi \left[\partial_\nu \hat{u} - \eta \hat{v} \right] \, ds = 0 \quad \text{for all } \phi \in H^{1/2}(\partial D).$$

We can conclude that

$$\partial_\nu \hat{u} = \eta \hat{v} \quad \text{on } \partial D.$$

Which gives the boundary value problem for the limits.

Next, we show that as $\lambda \longrightarrow 1$ we have that $u_\lambda \longrightarrow \hat{u}$ in $H^2(D) \cap H_0^1(D)$. From the above analysis, we have obtained that

$$\Delta u_\lambda + k_\lambda^2 n u_\lambda = -k_\lambda^2 (n-1) v_\lambda \quad \text{and} \quad \Delta v_\lambda + k_\lambda^2 v_\lambda = 0 \quad \text{in } D \tag{25}$$

$$\lambda \partial_\nu u_\lambda = (1 - \lambda) \partial_\nu v_\lambda + \eta v_\lambda \quad \text{on } \partial D \tag{26}$$

as well as

$$\Delta \hat{u} + \kappa^2 n \hat{u} = -\kappa^2 (n-1) \hat{v} \quad \text{and} \quad \Delta \hat{v} + \kappa^2 \hat{v} = 0 \quad \text{in } D \tag{27}$$

$$\partial_\nu \hat{u} = \eta \hat{v} \quad \text{on } \partial D. \tag{28}$$

Notice, that (27)–(28) is the transmission eigenvalue problem for $\lambda = 1$ as studied in [18]. This analysis implies that provided that the weak limits are non-trivial as $\lambda \to 1$ we have that k_λ converges to the transmission eigenvalue for $\lambda = 1$. In order to prove that the weak limits $\hat{u}$ and $\hat{v}$ are non-trivial we need the following results.

Theorem 5.1: *Assume that the coefficients satisfy the assumptions of Theorem 4.6 and $k_\lambda \in \mathbb{R}_+$ forms a bounded set as $\lambda \longrightarrow 1$. Then $u_\lambda \longrightarrow \hat{u}$ in $H^2(D) \cap H_0^1(D)$ as $\lambda \longrightarrow 1$.*

Proof: We subtract (25) from (27) to get the following

$$\Delta(u_\lambda - \hat{u}) = -k_\lambda^2 n(u_\lambda - \hat{u}) + n\hat{u}(k_\lambda^2 - \kappa^2) + (1-n)\left(k_\lambda^2(v_\lambda - \hat{v}) + \hat{v}(k_\lambda^2 - \kappa^2)\right).$$

Recall, that u_λ and $\hat{u} \in H^2(D) \cap H_0^1(D)$. Therefore, by taking $L^2(D)$ norm on both sides we obtain the estimate

$$\|\Delta(u_\lambda - \hat{u})\|_{L^2(D)} \le C\left\{\|u_\lambda - \hat{u}\|_{L^2(D)} + |k_\lambda^2 - \kappa^2| + \|v_\lambda - \hat{v}\|_{L^2(D)}\right\}.$$

Where we have used the triangle inequality and that n and k_λ^2 are both bounded with respect to λ. Again, using the fact that $\|\Delta \cdot \|_{L^2(D)}$ is equivalent to $\| \cdot \|_{H^2(D)}$ in $H^2(D) \cap H_0^1(D)$ gives us

$$\|u_\lambda - \hat{u}\|_{H^2(D)} \le C\left\{\|u_\lambda - \hat{u}\|_{L^2(D)} + |k_\lambda^2 - \kappa^2| + \|v_\lambda - \hat{v}\|_{L^2(D)}\right\}.$$

The above inequality implies that $u_\lambda \longrightarrow \hat{u}$ in $H^2(D) \cap H_0^1(D)$ as $\lambda \longrightarrow 1$ by the compact embedding of $H^1(D)$ into $L^2(D)$. ∎

We will now use the above convergence result to prove that $\hat{u} \in H^2(D) \cap H_0^1(D)$ is non-trivial under some further assumptions.

Theorem 5.2: *Assume that the coefficients satisfy the assumptions of Theorem 4.6 as well as $n - 1 \ne 0$ a.e. in D and $\partial_\nu v_\lambda$ is bounded in $L^2(\partial D)$. Then $\hat{u}$ is non-trivial.*

Proof: For contradiction, assume $\hat{u} = 0$. Now, recall that we have

$$\Delta u_\lambda + k_\lambda^2 n u_\lambda = -k_\lambda^2(n-1)v_\lambda$$

and by the convergence as $\lambda \to 1$ we have that

$$0 = -\kappa^2(n-1)\hat{v} \quad \text{in } D.$$

Now, as we have that k_λ^2 is bounded below as a consequence of Theorem 4.3 and $n - 1 \ne 0$, this implies that $\hat{v} = 0$. Thus, we have that $v_\lambda \rightharpoonup 0$ in $H^1(D)$ and by compact embedding $v_\lambda \to 0$ in $L^2(D)$. We now show that ∇v_λ strongly converges to the zero vector. Recall, that the function $v_\lambda \in H^1(D)$ satisfies Helmholtz equation, i.e. $\Delta v_\lambda + k_\lambda^2 v_\lambda = 0$ in D. Using Green's First Theorem gives

$$\int_{\partial D} \overline{\phi} \partial_\nu v_\lambda \, ds = \int_D \overline{\phi} \Delta v_\lambda + \nabla v_\lambda \cdot \nabla \overline{\phi} \, dx \quad \text{for } \phi \in H^1(D).$$

Letting $\phi = v_\lambda$ in the above equality gives that

$$\int_{\partial D} \overline{v_\lambda} \partial_\nu v_\lambda \, ds = \int_D \overline{v_\lambda} \Delta v_\lambda + |\nabla v_\lambda|^2 \, dx = -\int_D k_\lambda^2 |v_\lambda|^2 \, dx + \int_D |\nabla v_\lambda|^2 \, dx.$$

Observe that

$$\int_D |\nabla v_\lambda|^2 \, dx = \int_D k_\lambda^2 |v_\lambda|^2 \, dx + \int_{\partial D} \overline{v_\lambda} \partial_\nu v_\lambda \, ds.$$

Using the Cauchy–Schwarz inequality, we get that

$$\|\nabla v_\lambda\|_{L^2(D)}^2 \le \|\partial_\nu v_\lambda\|_{L^2(\partial D)} \|v_\lambda\|_{L^2(\partial D)} + k_\lambda^2 \|v_\lambda\|_{L^2(D)}^2$$

which implies that

$$\|\nabla v_\lambda\|_{L^2(D)}^2 \le C\left\{\|v_\lambda\|_{L^2(\partial D)} + \|v_\lambda\|_{L^2(D)}^2\right\}$$

since we have assumed that $\|\partial_\nu v_\lambda\|_{L^2(\partial D)}$ and k_λ are bounded. By the compact embedding of $H^{1/2}(\partial D)$ into $L^2(\partial D)$ we have that

$$v_\lambda \rightharpoonup 0 \text{ in } H^{1/2}(\partial D) \quad \text{implies } v_\lambda \to 0 \quad \text{in } L^2(\partial D).$$

Using the fact that $v_\lambda \to 0$ in $L^2(D)$ we can conclude that $v_\lambda \to 0$ in $H^1(D)$ by the above inequality. Therefore, we have that both u_λ and v_λ converge to zero in $H^1(D)$. Now, because we have that $u_\lambda = w_\lambda - v_\lambda$ we obtain that w_λ converges to zero in $H^1(D)$. This contradicts the normalization

$$\|v_\lambda\|^2_{H^1(D)} + \|w_\lambda\|^2_{H^1(D)} = 1$$

proving the claim. ∎

Now, putting everything together, we are able to state the main result of this section. Here, we have that as $\lambda \to 1$ the transmission eigenvalues will have a limit that corresponds to the standard transmission eigenvalue problem when $\lambda = 1$ under some assumptions.

Theorem 5.3: *Assume that the coefficients satisfy the assumptions of Theorem 4.6 as well as $n - 1 \neq 0$ a.e. in D and $\partial_\nu v_\lambda$ is bounded in $L^2(\partial D)$. Then, we have that $k_\lambda \to k(1)$ as $\lambda \to 1$ where $k(1)$ is a transmission eigenvalue corresponding to $\lambda = 1$.*

Proof: The proof is a simple consequence of the analysis presented in this section. ∎

We note that since k_λ and $k(1)$ are chosen arbitrarily, the above result holds for all transmission eigenvalues, without assuming their exact position in the real spectrum. This means that for the ordered subsequence of real eigenvalues, we have $k_{\lambda,j} \to k_j(1)$ for all $j = 1, 2, \ldots$, where $k_{\lambda,1}$ is the first, $k_{\lambda,2}$ the second etc.

We have shown the monotonicity with respect to n and η where as now we have an understanding of the limiting process as $\lambda \to 1$. In the case of inverse problems, it is very useful to understand how the eigenvalues of a differential operator depend on the coefficients. From an application perspective, this implies that the transmission eigenvalues can be used as a target signature to determine information about the scatterer since the eigenvalues can be recovered from the scattering data.

6. Numerical validation

In this section, we provide some numerical examples that validate the theoretical results from the previous sections. First, we will give some numerical examples of the convergence $k(\lambda) \longrightarrow k(1)$ as $\lambda \to 1$ in Theorem 5.3 for the unit ball with constant coefficients. Here we will consider the convergence and estimate the rate of convergence for the case when $\lambda \in (0, 1)$ and $\lambda \in (1, \infty)$. Then, we will provide some examples for the monotonicity of the eigenvalues with respect to the parameters n and η given in Theorems 4.7 and 4.8. Lastly, we will also report the transmission eigenvalues for other shapes using boundary integral equations.

6.1. Validation on the unit disk for the convergence of λ

Here, we consider the convergence of the k_λ as $\lambda \to 1^\pm$. For this we will assume that $D = B(0, 1) \subset \mathbb{R}^2$ (i.e. the unit disk centered at the origin) and that coefficients n, η, and λ are all constants. Under these assumptions, we recall that the transmission eigenvalue problem is given by

$$\Delta w + k^2 n w = 0 \quad \text{and} \quad \Delta v + k^2 v = 0 \quad \text{in } B(0, 1) \tag{29}$$

$$w = v \quad \text{and} \quad \lambda \partial_r w = \partial_r v + \eta v \quad \text{on } \partial B(0, 1). \tag{30}$$
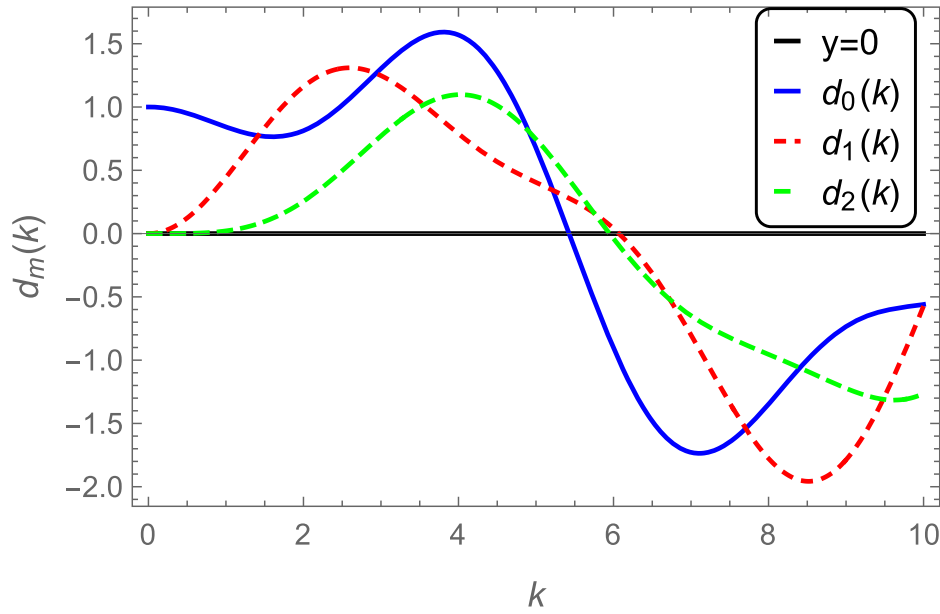
Figure 3. The plots of the determinant function $d_m(k)$ for $m = 0, 1, 2$. Here the parameters are given by $n = 1/6$, $\lambda = 5$, and $\eta = -1$.

Motivated by the separation of variables, we try to find eigenfunctions of the form

$$w(r, \theta) = w_m(r)e^{im\theta} \quad \text{and} \quad v(r, \theta) = v_m(r)e^{im\theta}$$

where $m \in \mathbb{Z}$. From this, we obtain that $w_m(r) = \alpha_m J_m(k\sqrt{n}r)$ and $v_m(r) = \beta_m J_m(kr)$ where both α_m and β_m are constants. Therefore, applying the boundary conditions at $r = 1$ gives that the transmission eigenvalues are given by the roots of $d_m(k)$, defined by

$$d_m(k) := \det \begin{pmatrix} J_m(k\sqrt{n}) & -J_m(k) \\ \lambda J_m'(k\sqrt{n})k\sqrt{n} & -\left(kJ_m'(k) + \eta J_m(k)\right) \end{pmatrix}. \tag{31}$$

Here we let $J_m(t)$ denote the Bessel functions of the first kind of order m (Figure 3).

Letting k_λ be the root(s) of $d_m(k)$, we can see that the eigenfunctions are given by

$$w_\lambda(r, \theta) = J_m(k_\lambda)J_m(k_\lambda\sqrt{n}r)e^{im\theta} \quad \text{and} \quad v_\lambda(r, \theta) = J_m(k_\lambda\sqrt{n})J_m(k_\lambda r)e^{im\theta}.$$

One can easily check that such forms satisfy the boundary conditions and also that if k_λ forms a bounded set then

$$\|\partial_r v_\lambda(1, \theta)\|_{L^2(0, 2\pi)} \quad \text{is bounded with respect to } \lambda.$$

We note that the position of each eigenvalue on the spectrum, is not directly associated with the order m of the determinant $d_m(k)$, of which is a root. This means for e.g. that the lowest eigenvalue k_1 can be the first root of $d_1(k)$ (or of other order) and not $d_0(k)$. As a result, in the examples following, we calculate the roots and sort them in ascending order. We let $k_j(\lambda)$ denote the jth transmission eigenvalue for boundary parameter λ.

Now, we wish to provide some numerical validation of Theorem 5.3. First, we give some examples when we let λ approach 1 from below then we check the case when λ approach 1 from above. The examples are given by considering the first three transmission eigenvalues, as roots of $d_m(k)$, for $m = 0, 1, 2, \ldots$.Therefore, we have that the limiting value as λ tends to 1 of the transmission

Table 1. Convergence of the transmission eigenvalues as $\lambda \longrightarrow 1^-$ for $n = 4$ and $\eta = 1$.

λ	$k_1(\lambda)$	EOC	$k_2(\lambda)$	EOC	$k_3(\lambda)$	EOC
$1 - 1/2$	3.0394	N/A	3.0561	N/A	3.2494	N/A
$1 - 1/4$	2.8388	2.0346	3.1970	1.3241	3.2942	1.8057
$1 - 1/8$	2.7990	1.3774	3.2509	1.2313	3.3048	1.2831
$1 - 1/16$	2.7853	1.1590	3.2723	1.1092	3.3088	1.1223
$1 - 1/32$	2.7794	1.0770	3.2819	1.0516	3.3106	1.0476
$1 - 1/64$	2.7767	1.0415	3.2864	1.0212	3.3114	1.0177
$1 - 1/128$	2.7754	1.0283	3.2886	1.0066	3.3118	0.9823
$1 - 1/256$	2.7747	1.0465	3.2897	0.9934	3.3120	0.9652
$1 - 1/512$	2.7744	1.0728	3.2902	0.9740	3.3121	0.9329
$1 - 1/1024$	2.7742	1.1575	3.2905	0.9494	3.3121	0.8745

Note: Here, the limiting values are $k_1(1) = 2.7741$, $k_2(1) = 3.2908$, and $k_3(1) = 3.3122$.

Table 2. Convergence of the transmission eigenvalues as $\lambda \longrightarrow 1^+$ for $n = 1/3$ and $\eta = -1$.

λ	$k_1(\lambda)$	EOC	$k_2(\lambda)$	EOC	$k_3(\lambda)$	EOC
$1 + 1/2$	7.1094	N/A	7.4849	N/A	7.6108	N/A
$1 + 1/4$	7.0395	1.2433	7.2250	1.1455	7.7774	0.9655
$1 + 1/8$	7.0121	1.1084	7.1108	1.0984	7.8660	1.0189
$1 + 1/16$	6.9998	1.0527	7.0590	1.0513	7.9097	1.0168
$1 + 1/32$	6.9940	1.0268	7.0344	1.0265	7.9311	1.0085
$1 + /64$	6.9912	1.0181	7.0224	1.0165	7.9417	1.0027
$1 + 1/128$	6.9898	1.0157	7.0165	1.0124	7.9470	0.9973
$1 + 1/256$	6.9891	1.0106	7.0136	1.0125	7.9496	0.9892
$1 + 1/512$	6.9887	1.0431	7.0121	1.0304	7.9509	0.9732
$1 + 1/1024$	6.9886	1.1375	7.0114	1.0521	7.9516	0.9582

Note: Here the limiting values are $k_1(1) = 6.9883$, $k_2(1) = 7.0107$ and $k_3(1) = 7.9523$.

eigenvalues, are the corresponding roots for $\lambda = 1$. When $\lambda = 1$, $\eta = 1$ and $n = 4$ we have that $k_1(1) = 2.7741$, $k_2(1) = 3.2908$, and $k_3(1) = 3.3122$ are the first three limiting transmission eigenvalues, coming from $d_1(k)$, $d_0(k)$, and $d_2(k)$ respectively. From this, we show that numerically $k_j(\lambda) \longrightarrow k_j(1)$ as $\lambda \longrightarrow 1^-$ for $j = 1, 2, 3$ and the results are presented in Table 1. We also check the estimated order of convergence (EOC) which is given by

$$\text{EOC} = \log(\epsilon_{\lambda_p}/\epsilon_{\lambda_{p+1}})/\log(2) \quad \text{where } \epsilon_{\lambda_p} = |k_j(\lambda_p) - k_j(1)| \quad \text{for } j = 1, 2, 3$$

$$\text{and} \quad \lambda_p = 1 - \frac{1}{2^p} \quad \text{for } p = 1, 2, 3 \ldots$$

where our calculations suggest first-order convergence as $\lambda \longrightarrow 1^-$. Also, notice that in Table 1, the eigenvalues seem to be monotone with respect to λ. We see that $k_1(\lambda)$ is descending $k_2(\lambda)$ and $k_3(\lambda)$ are ascending with respect to λ.

We now give a numerical example of the convergences when $\lambda \in (1, \infty)$. It is important to remember that for this case, we have that $\eta_{\max} < 0$ and $\lambda n_{\max} - 1 < 0$ as $\lambda \longrightarrow 1^+$. We again compute the EOC with

$$\lambda_p = 1 + \frac{1}{2^p} \quad \text{for } p = 1, 2, 3 \ldots$$

to establish the convergence rate. For Table 2, we choose $n = 1/3$ and $\eta = -1$ following the assumptions on the coefficients given in Theorem 5.3. Again, we compute the lowest three roots of $d_m(k)$ for λ_p. We have that the limiting transmission eigenvalues for $n = 1/3$ and $\eta = -1$ are given by $k_1(1) = 6.9883$, $k_2(1) = 7.0107$, and $k_3(1) = 7.9523$, being the first roots of $d_0(k)$, $d_2(k)$, and $d_1(k)$ respectively.

Table 3. Monotonicity with respect to n where $\lambda = 2$ and $\eta = -3$ for the unit disk.

n	1/6	1/5	1/4	1/3
$k_1(n)$	4.8387	4.9935	5.6504	6.5592
$k_2(n)$	4.8893	5.6474	6.0112	7.3299

Note: Here, k_j are the first two transmission eigenvalues.

Table 4. Monotonicity with respect to n where $\lambda = 1/2$ and $\eta = 1$ for the unit disk.

n	3	4	5	6	7
$k_1(n)$	3.9850	3.0394	2.3699	2.0651	1.6559
$k_2(n)$	4.2464	3.0561	2.5280	2.0706	1.8761

Note: Here, k_j are the first two transmission eigenvalues.

Table 5. Monotonicity with respect to η where $\lambda = 5$ and $n = 1/6$ for the unit disk.

η	-4	-3	-2	-1	$-1/2$
$k_1(\eta)$	4.7141	5.0753	5.4263	5.4283	5.4293
$k_2(\eta)$	5.4220	5.4242	5.7292	5.9486	6.0176

Note: Here, k_j are the first two transmission eigenvalues.

We again notice that, in Table 2, the eigenvalues seem to be monotone with respect to λ. We see that $k_1(\lambda)$ and $k_2(\lambda)$ are increasing where as $k_3(\lambda)$ is decreasing with respect to λ. Although we only showed that there is convergence, we have these numerical examples that seem to suggest monotonicity of the transmission eigenvalues with respect to the parameter λ. Here, we conjecture the monotonicity but due to the variational form studied in the previous section, we are unable to obtain this result theoretically.

6.2. Monotonicity of η and n on the unit disk

Here, we will provide some numerics for the monotonicity with respect to η and n given in Theorems 4.7 and 4.8. Just as in the previous section, we will assume that D is the unit disk with constant coefficients. Therefore, we can again use the fact that k is a transmission eigenvalue provided that it is a root for $d_m(k)$ given by (31).

We first consider the monotonicity with respect to the parameter n. To this end, recall that $\lambda \in (1, \infty), \eta_{\max} < 0$, and $\lambda n_{\max} - 1 < 0$. Therefore, we fix $\lambda = 2$ and $\eta = -3$ and report the transmission eigenvalues $k_j(n)$ for $j = 1, 2$ corresponding to the lowest two roots of $d_m(k)$, in Table 3.

In a similar fashion, we now provide numerical examples for the case when the parameters $\lambda \in (0, 1), \eta_{\min} > 0$, and $\lambda n_{\min} - 1 > 0$ corresponding to Theorem 4.8. Therefore, we again report the first two roots of the functions $d_m(k)$. In Table 4, we fix $\lambda = 1/2$ and $\eta = 1$ for $k_j(n)$ for $j = 1, 2$.

Next, we turn our attention to the monotonicity with respect to η. We first consider the case where we have $\lambda \in (1, \infty), \eta_{\max} < 0$, and $\lambda n_{\max} - 1 < 0$. Recall, that from Theorem 4.7 we expect that the transmission eigenvalues to be increasing with respect to η. In Table 5, we fix $\lambda = 5$ and $n = 1/6$ to compute $k_j(\eta)$ for $j = 1, 2$ and we can see the monotonicity from the reported values.

Now, we focus on case corresponding to Theorem 4.8 where the transmission eigenvalues are decreasing with respect to the parameter η. Therefore, we need the assumptions $\lambda \in (0, 1), \eta_{\min} > 0$, and $\lambda n_{\min} - 1 > 0$ for the result to hold. In Table 6, we fix $\lambda = 1/2$ and $n = 3$ for $k_j(n)$, respectively, for $j = 1, 2$.

Table 6. Monotonicity with respect to η where $\lambda = 1/2$ and $n = 3$ for the unit disk.

η	1	2	3	4	5
$k_1(\eta)$	3.9850	3.6700	3.5212	2.6262	1.6354
$k_2(\eta)$	4.2464	4.0269	3.5409	3.1242	1.9005

Note: Here, k_j are the first two transmission eigenvalues.

6.3. Numerics via boundary integral equations

The derivation of the boundary integral equation to solve the problem follows along the same lines as in [7, Section 3] where one uses a single-layer ansatz for the functions w and v with unknown densities φ and ψ (refer also to [32] for the original idea). Precisely, we use

$$w(x) = \mathrm{SL}_{k\sqrt{n}}\varphi(x) \quad \text{and} \quad v(x) = \mathrm{SL}_k\psi(x), \quad x \in D,$$

where we define the single-layer by

$$\mathrm{SL}_k\phi(x) = \int_{\partial D} \Phi_k(x,y)\phi(y)\,\mathrm{d}s(y), \quad x \in D$$

where

$$\Phi_k(x,y) = \frac{\mathrm{i}}{4}H_0^{(1)}\left(k|x-y|\right), \quad \text{when } x \neq y$$

is the fundamental solution of the Helmholtz equation in two dimensions. Here we let $H_0^{(1)}$ denote the zeroth order first kind Hankel function. On the boundary we have

$$w(x) = \mathrm{S}_{k\sqrt{n}}\varphi(x) \quad \text{and} \quad v(x) = \mathrm{S}_k\psi(x),$$

where the boundary operator S_k is given by

$$\mathrm{S}_k\phi(x) = \int_{\partial D} \Phi_k(x,y)\phi(y)\,\mathrm{d}s(y), \quad x \in \partial D.$$

Likewise, we obtain

$$\partial_\nu w(x) = \left(\frac{1}{2}\mathrm{I} + \mathrm{K}_{k\sqrt{n}}^\top\right)\varphi(x) \quad \text{and} \quad \partial_\nu v(x) = \left(\frac{1}{2}\mathrm{I} + \mathrm{K}_k^\top\right)\psi(x),$$

where

$$\mathrm{K}_k^\top\phi(x) = \int_{\partial D} \partial_{\nu(x)}\Phi_k(x,y)\phi(y)\,\mathrm{d}s(y), \quad x \in \partial D$$

and I denotes the identity. Using the given boundary conditions and assuming that k and $k\sqrt{n}$ are not eigenvalues of $-\Delta$ in D yields

$$\left[\lambda\left(\frac{1}{2}\mathrm{I} + \mathrm{K}_{k\sqrt{n}}^\top\right)\mathrm{S}_{k\sqrt{n}}^{-1} - \left(\frac{1}{2}\mathrm{I} + \mathrm{K}_k^\top\right)\mathrm{S}_k^{-1} - \eta\mathrm{I}\right]w = 0,$$

which is a non-linear eigenvalue problem of the form

$$M(k; n, \eta, \lambda)w = 0. \tag{32}$$

Here, the parameters n, η, and λ are given. Note that we focus on the transpose of this equation since the boundary integral operator

$$K_k\phi(x) = \int_{\partial D} \partial_{\nu(y)}\Phi_k(x,y)\phi(y)\,ds(y), \quad x \in \partial D$$

can be numerically approximated avoiding the singularity (see [33, Section 4.3] for details and the discretization of the boundary integral operators). Then, the non-linear eigenvalue problem is solved with the Beyn's algorithm (see [34] for a detailed description). This algorithm converts a large-scale non-linear eigenvalue problem to a linear eigenvalue problem of smaller size by appealing to complex analysis, i.e. contour integrals in the complex plane. The contour we will choose, will be the disk in the complex plane centered at $\mu \in \mathbb{C}$ for a fixed radius R. From this, Beyn's algorithm will compute the transmission eigenvalues that lie in the interior of the chosen contour.

First, we show that we are able to reproduce the values given in Example 3.1 on page 14 for the unit disk using the material parameters $\lambda = 2$, $n = 4$, $\eta = -1/100$ with the boundary element collocation method. We use 120 collocation nodes (40 pieces) within our algorithm for the discretization of the boundary. For the Beyn algorithm we take the parameters $tol = 10^{-4}$, $\ell = 20$, and $N = 24$ discretization points for the two contour integrals where the contour is a circle with center μ and radius $R = 1/2$. Next, we pick $\mu = 3.5$ and obtain the interior transmission eigenvalue $3.4567 - 0.0000i$ which agrees with the value reported in Example 3.1 to four digits accuracy. This eigenvalue has multiplicity one (it corresponds to $m = 0$). Using $\mu = 2.2$ yields the interior transmission eigenvalue $2.1516 - 0.0000i$ with multiplicity two which is in agreement with the value 2.151602 obtained from the determinant for $m = 4$. Again, we observe that all reported digits are correct. The accuracy does not depend on the multiplicity of the eigenvalue. Finally, we test our boundary element collocation method for a complex-valued interior transmission eigenvalue. Using $\mu = 2.2 + 0.6i$ yields the simple eigenvalue $2.2032 + 0.2905i$ (rounded) which is in agreement to five digits with the value reported in Example 3.1 using the determinant with $m = 0$. In sum, this shows that we are able to compute both real and complex-valued interior transmission eigenvalues to high accuracy. It gives us the flexibility to now compute them also for other scatterers as well.

For an ellipse with semi-axis $a = 1$ and $b = 1.2$ (refer to Figure 4) i.e.

$$\partial D = (\cos(t), 1.2\sin(t)) \quad \text{for } t \in [0, 2\pi)$$

using $\mu = 1/2$ as well as $\mu = 3/2$ and the same material parameters as before, we obtain the first nine real-valued interior transmission eigenvalues

$$0.0420 \quad 0.6036 \quad 0.7165 \quad 1.0830 \quad 1.1136 \quad 1.5244 \quad 1.5311 \quad 1.9494 \quad 1.9507,$$

where we skipped reporting the imaginary eigenvalues. In comparison, the first nine real-valued interior transmission eigenvalues for the unit disk are

$$0.0534 \quad 0.7208 \quad 0.7208 \quad 1.2131 \quad 1.2131 \quad 1.6864 \quad 1.6864 \quad 2.1516 \quad 2.1516.$$

Next, we compute the interior transmission eigenvalues for the kite-shaped domain (refer to Figure 4) using the same parameters as before. Its boundary is given parametrically by

$$\partial D = (0.75\cos(t) + 0.3\cos(2t), \sin(t)) \quad \text{for } t \in [0, 2\pi)$$

(refer to [32]). We use $\mu = 1/2$, $\mu = 3/2$ as well as $\mu = 5/2$ to obtain the first nine real-valued interior transmission eigenvalues

$$0.0523 \quad 0.6868 \quad 0.8514 \quad 1.3452 \quad 1.4398 \quad 1.6348 \quad 2.0181 \quad 2.1439 \quad 2.3494.$$

Now, we consider the ellipse with semi-axis $a = 1$ and $b = 1.2$ and use the material parameters $\eta = 1$ and $n = 4$ and vary λ such that it approaches one from below. We will validate again numerically
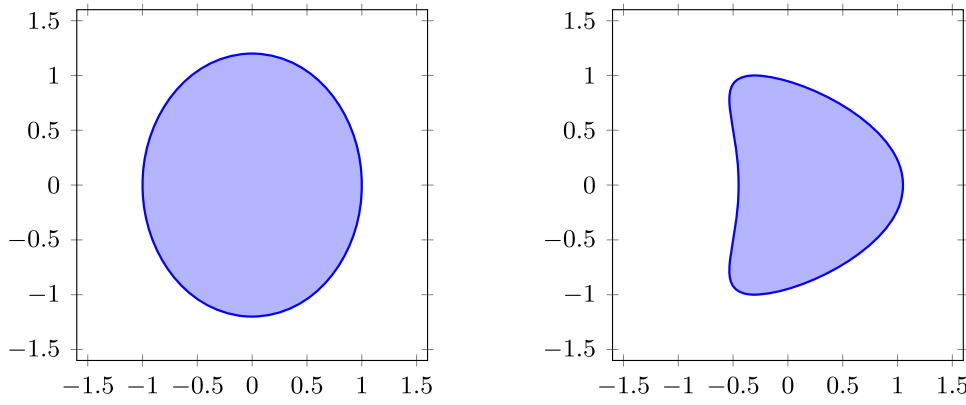
Figure 4. Graphical representation for the elliptical and kite-shaped domains considered in this section.

Table 7. Convergence of the transmission eigenvalues for the ellipse, as $\lambda \longrightarrow 1^-$ for $n = 4$ and $\eta = 1$.

λ	$k_1(\lambda)$	EOC	$k_2(\lambda)$	EOC	$k_3(\lambda)$	EOC
$1 - 1/2$	2.5043	N/A	2.7413	N/A	2.8777	N/A
$1 - 1/4$	2.4701	0.9689	2.7077	0.9693	2.8535	1.0165
$1 - 1/8$	2.4523	0.9867	2.6903	0.9864	2.8416	1.0089
$1 - 1/16$	2.4434	0.9940	2.6815	0.9937	2.8356	1.0026
$1 - 1/32$	2.4388	0.9972	2.6770	0.9970	2.8327	1.0078
$1 - 1/64$	2.4366	0.9987	2.6748	0.9984	2.8312	0.9847
$1 - 1/128$	2.4354	0.9993	2.6737	0.9988	2.8305	1.0107
$1 - 1/256$	2.4349	0.9998	2.6731	0.9994	2.8301	1.0436
$1 - 1/512$	2.4346	1.0021	2.6729	0.9997	2.8299	0.8813
$1 - 1/1024$	2.4344	1.0015	2.6727	0.9968	2.8298	1.0722

Note: Here the limiting values are $k_1(1) = 2.4343$, $k_2(1) = 2.6726$ and $k_3(1) = 2.8300$.

Theorem 5.3 as it was done for the unit disk in Table 1. The results are reported in Table 7. Note that the first three real-valued interior transmission eigenvalues for $\lambda = 1$ are $k_1(1) = 2.4343$, $k_2(1) = 2.6726$, and $k_3(1) = 2.8300$ which we obtained using $\mu = 5/2$ with 240 collocation nodes. As we can see, we obtain the linear convergence for $\lambda \longrightarrow 1^-$ for the given ellipse as expected. Interestingly, we also obtain linear convergence for $\lambda \longrightarrow 1^+$ for $\eta = 1$ and $n = 4$ although theoretically not justified. Refer to Table 8. Again, we also observe a monotonicity of the interior transmission eigenvalues with respect to λ although we have not shown this fact from the theoretical point of view.

Finally, we show some monotonicity results for the kite-shaped domain. We first fix $\lambda = 2$ as well as $\eta = -1$ and vary the index of refraction n. Using 120 collocation nodes within the boundary element collocation method and the same parameters as before for the Beyn method with $\mu = 5$, $\mu = 6$, $\mu = 7$ as well as $\mu = 8$ and $\mu = 8.5$ yields the first three real-valued interior transmission eigenvalues reported in Table 9.

As we can see, the first real-valued interior transmission eigenvalue is monotone with respect to the parameter n as stated in Theorem 4.7 item 1. Interestingly, the same seems to be true for the second and third real-valued interior transmission eigenvalue. In Table 10, we show the monotonicity behavior for fixed material parameter $\lambda = 2$ and $n = 1/6$ and varying η using $\lambda = 4.5$, $\lambda = 5.5$ as well as $\lambda = 6.5$.

We observe the expected monotonicity behavior for the first real-valued interior transmission eigenvalue with respect to the parameter η as stated in Theorem 4.7 item 2. Strikingly, the other interior transmission eigenvalues also show a monotonicity behavior.

Table 8. Convergence of the transmission eigenvalues for the ellipse, as $\lambda \longrightarrow 1^+$ for $n = 4$ and $\eta = 1$.

λ	$k_1(\lambda)$	EOC	$k_2(\lambda)$	EOC	$k_3(\lambda)$	EOC
$1 + 1/2$	2.3601	N/A	2.5995	N/A	2.7844	N/A
$1 + 1/4$	2.3974	1.0101	2.6364	1.0138	2.8067	0.9758
$1 + 1/8$	2.4160	1.0080	2.6546	1.0093	2.8181	0.9880
$1 + 1/16$	2.4252	1.0047	2.6636	1.0052	2.8239	0.9980
$1 + 1/32$	2.4297	1.0025	2.6681	1.0028	2.8268	0.9932
$1 + 1/64$	2.4320	1.0012	2.6703	1.0014	2.8283	0.9966
$1 + 1/128$	2.4331	1.0006	2.6715	1.0009	2.8290	1.0056
$1 + 1/256$	2.4337	1.0002	2.6720	1.0011	2.8294	0.9796
$1 + 1/512$	2.4340	0.9989	2.6723	1.0004	2.8296	0.9780
$1 + 1/1024$	2.4341	0.9982	2.6724	1.0094	2.8297	1.0376

Note: Here the limiting values are $k_1(1) = 2.4343$, $k_2(1) = 2.6726$, and $k_3(1) = 2.8300$.

Table 9. Values of $k_j(n)$'s when n varies using $\lambda = 2$ and $\eta = -1$.

n	1/7	1/6	1/5	1/4	1/3
$k_1(n)$	5.6837	6.0582	6.5231	7.0820	8.1993
$k_2(n)$	6.0870	6.2456	6.5370	7.1497	8.2397
$k_3(n)$	6.6334	6.8110	7.1306	7.7996	8.9628

Note: The first real-valued transmission eigenvalue increases monotonically with respect to the parameter n as stated in Theorem 4.7 item 1 for the kite-shaped domain.

Table 10. Values of $k_j(\eta)$'s when η varies using $\lambda = 2$ and $n = 1/6$.

η	-5	-4	-3	-2	-1
$k_1(\eta)$	4.5272	5.4110	5.7363	5.9202	6.0582
$k_2(\eta)$	5.3892	5.5606	5.7689	6.0044	6.2456
$k_3(\eta)$	5.9899	6.1585	6.3488	6.5702	6.8110

Note: The first real-valued transmission eigenvalue increases monotonically with respect to the parameter η as stated in Theorem 4.7 item 2 for the kite-shaped domain.

Table 11. Values of $k_j(n)$'s when n varies using $\lambda = 1/2$ and $\eta = 1$.

n	3	4	5	6	7
$k_1(n)$	4.6102	3.4720	2.8104	2.4169	2.0606
$k_2(n)$	4.6988	3.4863	2.8713	2.4513	2.2158
$k_3(n)$	5.1191	3.8013	3.0731	2.6823	2.4215

Note: The first real-valued transmission eigenvalue decreases monotonically with respect to the parameter n as stated in Theorem 4.8 item 1 for the kite-shaped domain.

Next, we show numerical results to validate Theorem 4.8. First, we pick the material parameter $\lambda = 1/2$ and $\eta = 1$ and vary n. We use $\mu = 5$, $\mu = 3.5$, and $\mu = 3$ as well as $\mu = 2$ to obtain the results reported in Table 11.

As we can see, we numerically obtain the decreasing behavior for the first real-valued interior transmission eigenvalue as stated in Theorem 4.8 item 1. Interestingly, we also observe a monotonic behavior for the next two interior transmission eigenvalues as well. Now, we show numerical results for the material parameters $\lambda = 1/2$ and $n = 3$ for varying η. Using $\mu = 5$, $\mu = 4.5$, and $\mu = 4$ yields the results that are reported in Table 12.

Again, we observe the proposed monotone decreasing behavior as stated in Theorem 4.8 item 2 for the first real-valued interior transmission eigenvalue for the kite-shaped domain. Strikingly, the same seems to be true for the second and third eigenvalue as well.

Table 12. Values of $k_j(\eta)$'s when η varies using $\lambda = 1/2$ and $n = 3$.

η	1/2	1	2	3	4
$k_1(\eta)$	4.7339	4.6102	4.3089	4.0502	3.8981
$k_2(\eta)$	4.7572	4.6988	4.5914	4.3550	4.0804
$k_3(\eta)$	5.1747	5.1191	4.9526	4.6436	4.4735

Note: The first real-valued transmission eigenvalue decreases monotonically with respect to the parameter η as stated in Theorem 4.8 item 2 for the kite-shaped domain.

7. Summary and outlook

A transmission eigenvalue problem with two conductivity parameters is considered. Existence as well as discreteness of corresponding real-valued interior transmission eigenvalues is proven. Further, it is shown that the first real-valued interior transmission eigenvalue is monotone with respect to the two parameters η and n under certain conditions. Additionally, the linear convergence for λ against one is shown theoretically. Next, the theory is validated by extensive numerical results for a unit disk using Bessel functions. Further, numerical results are presented for more general scatterers using boundary integral equations and its discretization via boundary element collocation method. Interestingly, we can show numerically monotonicity results for cases that are not covered yet by the theory. The existence of complex-valued interior transmission eigenvalues is still open, but it can be shown numerically that they do exist. A worthwhile future project is to study the case when λ is variable.

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