Reduction Games, Provability, and Compactness

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Abstract

Hirschfeldt and Jockusch (2016) introduced a two-player game in which winning strategies for one or the other player precisely correspond to implications and non-implications between Π^1_2 principles over ω-models of RCA_0. They also introduced a version of this game that similarly captures provability over RCA_0. We generalize and extend this game-theoretic framework to other formal systems, and establish a certain compactness result that shows that if an implication $\mathsf{Q} \to \mathsf{P}$ between two principles holds, then there exists a winning strategy that achieves victory in a number of moves bounded by a number independent of the specific run of the game. This compactness result generalizes an old proof-theoretic fact noted by H. Wang (1981), and has applications to the reverse mathematics of combinatorial principles.

We also demonstrate how this framework leads to a new kind of analysis of the logical strength of mathematical problems that refines both that of reverse mathematics and that of computability-theoretic notions such as Weihrauch reducibility, allowing for a kind

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of fine-structural comparison between Π^1_2 principles that has both computability-theoretic and proof-theoretic aspects, and can help us distinguish between these, for example by showing that a certain use of a principle in a proof is "purely proof-theoretic", as opposed to relying on its computability-theoretic strength.

We give examples of this analysis to a number of principles at the level of $\mathsf{B}\Sigma^0_2$, uncovering new differences between their logical strengths.

1 Introduction

Reverse mathematics gives us a way to compare the relative strength of theorems by establishing implications and nonimplications over a weak subsystem of second-order arithmetic, typically RCA_0, which corresponds roughly to computable mathematics. (We will assume some familiarity with reverse mathematics and computability theory. Standard resources in these areas include [33] and [34], respectively.) In many cases, nonimplications over RCA_0 are proved using ω-models, i.e., models of RCA_0 with standard first-order part. We say that P is *ω-reducible* to Q, and write $\mathsf{P} \leqslant_\omega \mathsf{Q}$, if every ω-model of $\mathsf{RCA}_0 + \mathsf{Q}$ is a model of P.

Implication over RCA_0 and ω-reducibility are not fine enough for some purposes, so other notions of computability-theoretic reduction between theorems have been extensively studied. These are particularly well-adapted to the following class of theorems, which includes a large proportion of those that have been studied in reverse mathematics: A *Π^1_2-problem* is a sentence $\forall X\,[\Theta(X) \to \exists Y\,\Psi(X,Y)]$ of second-order arithmetic such that Θ and Ψ are arithmetic. The term "problem" reflects a computability-theoretic view that sees such a sentence as a process of finding a suitable Y given X. In line with this view, we say that an *instance* of this problem is an $X \subseteq \omega$ such that $\Theta(X)$ holds, and a *solution* to this problem is a $Y \subseteq \omega$ such that $\Psi(X,Y)$ holds.

For example, the following versions of Ramsey's Theorem are Π^1_2-problems that have been extensively studied in reverse mathematics and computability theory, and will be useful sources of examples for us as well. (We often state Π^1_2-problems in ways that make mention of objects other than natural numbers and sets of natural numbers. We assume these are coded in an appropriate way. For combinatorial objects like the ones below, these codings

are straightforward and do not affect the analysis of these problems.)

Definition 1.1. For a set X, let $[X]^n$ be the collection of n-element subsets of X. A *k-coloring* of $[X]^n$ is a map $c : [X]^n \to k$. A coloring of $[X]^2$ is *stable* if $\lim_{y \in X} c(x, y)$ exists for all $x \in X$. A set $H \subseteq X$ is *homogeneous* for $c : [X]^n \to k$ if there is an i such that $c(s) = i$ for all $s \in [H]^n$. A set $L \subseteq X$ is *limit-homogeneous* for $c : [X]^2 \to k$ if there is an i such that $\lim_{y \in L} c(x, y) = i$ for all $x \in L$.

1. RT^n_k: Every k-coloring of $[\mathbb{N}]^n$ has an infinite homogeneous set.
2. $\mathsf{RT}^n_{<\infty}$: $\forall k\, \mathsf{RT}^n_k$.
3. RT: $\forall n\, \forall k\, \mathsf{RT}^n_k$.
4. SRT^2_k: Every stable k-coloring of $[\mathbb{N}]^2$ has an infinite homogeneous set.
5. D^2_k: Every stable k-coloring of $[\mathbb{N}]^2$ has an infinite limit-homogeneous set.

It is well-known that RT^n_k and $\mathsf{RT}^n_{<\infty}$ are equivalent to ACA_0 for each $n \geqslant 3$ (we always assume $k \geqslant 2$), while RT^1_k is provable in RCA_0. (We will discuss $\mathsf{RT}^1_{<\infty}$ and RT below. For more on the computability theory and reverse mathematics of these principles, see [18].) The question of whether SRT^2_2 implies RT^2_2 motivated a great deal of research since being raised by Cholak, Jockusch, and Slaman [10]. Chong, Slaman, and Yang [12] showed that $\mathsf{RCA}_0 \nvdash \mathsf{SRT}^2_2 \to \mathsf{RT}^2_2$, with a proof that made essential use of non-ω-models. Recently, Monin and Patey [26] have finally shown that $\mathsf{RT}^2_2 \not\leqslant_\omega \mathsf{SRT}^2_2$. The relationship between SRT^2_2 and D^2_2 is also interesting, and will be discussed in Section 6.

Hirschfeldt and Jockusch [19] gave characterizations of both $\mathsf{P} \leqslant_\omega \mathsf{Q}$ and $\mathsf{RCA}_0 \vdash \mathsf{Q} \to \mathsf{P}$ for Π^1_2-problems P and Q in terms of winning strategies in certain games. In this paper, we study further aspects of the latter characterization and generalizations of it, in particular establishing a compactness theorem that shows that certain winning strategies can always be chosen to win in a number of moves bounded by a number independent of the instance of P being considered. As explained below, this theorem can be seen as a generalization of a metatheorem about ACA_0. This metatheorem has been used, for instance, to translate computability-theoretic results of Jockusch [23] into a proof that $\mathsf{ACA}_0 \nvdash \mathsf{RT}$.

The difference between the two game-theoretic characterizations in [19] is that for ω-reducibility, the games are played over the standard natural numbers, while for provability over RCA_0 they are played over possibly nonstandard models of Σ^0_1-PA (the first-order part of RCA_0). We hope to show in this paper that there is a rich theory to be obtained by generalizing computability-theoretic reductions between Π^1_2-problems to models of subsystems of second-order arithmetic with possibly nonstandard first-order parts, and to begin its systematic development. In particular, this theory allows us to conduct a fine-structural comparison between such problems that has both computability-theoretic and proof-theoretic aspects, and can help us distinguish between these, for example by showing that a certain use of a principle in a proof is "purely proof-theoretic", as opposed to relying on its computability-theoretic strength.

Computable reducibility and Weihrauch reducibility are two of the most widely-studied notions of computability-theoretic reducibility between Π^1_2-problems. The latter (in a more general form) has a long history, particularly in computable analysis (see e.g. [7]), while the former was introduced by Dzhafarov [14].

Definition 1.2. Let P and Q be Π^1_2-problems.

We say that P is *computably reducible* to Q, and write $\mathsf{P} \leqslant_{\mathrm{c}} \mathsf{Q}$, if for every instance X of P, there is an X-computable instance $\widehat{X}$ of Q such that, for every solution $\widehat{Y}$ to $\widehat{X}$, there is an $X \oplus \widehat{Y}$-computable solution to X.

We say that P is *Weihrauch reducible* to Q, and write $\mathsf{P} \leqslant_{\mathrm{W}} \mathsf{Q}$, if there are Turing functionals Φ and Ψ such that, for every instance X of P, the set $\widehat{X} = \Phi^X$ is an instance of Q, and for every solution $\widehat{Y}$ to $\widehat{X}$, the set $Y = \Psi^{X \oplus \widehat{Y}}$ is a solution to X.

These two reducibilities allow us to use only a single instance of Q in solving an instance of P. To generalize these notions to allow multiple instances of Q to be used, Hirschfeldt and Jockusch [19] defined the following game.

Definition 1.3. Let P and Q be Π^1_2-problems. The *reduction game* $G(\mathsf{Q} \to \mathsf{P})$ is a two-player game played according to the following rules.

(1) If at any point a player cannot make a move, the opponent wins.

(2) If one of the players wins, the game ends.

(3) On the first move, Player 1 plays an instance X_0 of P. Then Player 2 either plays an X_0-computable solution to X_0 and wins, or plays an X_0-computable instance Y_1 of Q.

(4) For $n > 1$, on the nth move, Player 1 plays a solution X_{n-1} to the instance Y_{n-1} of Q. Then Player 2 either plays an $(X_0 \oplus \cdots \oplus X_{n-1})$-computable solution to X_0 and wins, or plays an $(X_0 \oplus \cdots \oplus X_{n-1})$-computable instance Y_n of Q.

(5) If the game never ends then Player 1 wins.

A winning strategy for Player 2 in this game is a form of generalized computable reduction. Hirschfeldt and Jockusch [19] showed that if $\mathsf{P} \leqslant_\omega \mathsf{Q}$ then Player 2 has a winning strategy for $G(\mathsf{Q} \to \mathsf{P})$, while otherwise Player 1 has a winning strategy for $G(\mathsf{Q} \to \mathsf{P})$, so generalized computable reducibility is actually the same as ω-reducibility. They then defined an analogous notion of generalized Weihrauch reducibility, where $\mathsf{P} \leqslant_{\mathrm{gW}} \mathsf{Q}$ if Player 2 has a uniformly computable winning strategy for $G(\mathsf{Q} \to \mathsf{P})$. (See [19] for the details of this definition.) Neumann and Pauly [28] gave an equivalent definition in terms of an operator $^\diamond$ on the Weihrauch degrees. (See also [38] for some more recent discussion of, and results about, the $^\diamond$ operator.)

We can generalize the notions of instance and solution of a Π^1_2-problem $\mathsf{P} \equiv \forall X\,[\Theta(X) \to \exists Y\,\Psi(X,Y)]$ to possibly nonstandard structures in the language of first-order arithmetic in a natural way. We denote the languages of first- and second-order arithmetic by L_1 and L_2, respectively. Let M be an L_1-structure. We denote the domain of M by $|M|$. For $S \subseteq |M|$, we denote the L_2-structure with first-order part M and second-order part S by (M,S). For an L_1-structure M, an *M-instance of* P is an $X \subseteq |M|$ such that $(M,\{X\}) \vDash \Theta(X)$, and a *solution* to this instance is a $Y \subseteq |M|$ such that $(M,\{X,Y\}) \vDash \Psi(X,Y)$.

Hirschfeldt and Jockusch [19, Section 4.5] noted that reduction games can be extended to possibly nonstandard countable models of Σ^0_1-PA (i.e., first-order parts of models of RCA_0), with Δ^0_1-definability playing the role of computability as follows. For $X_0,\ldots,X_n \subseteq |M|$, we denote by $M[X_0,\ldots,X_n]$ the L_2-structure with first-order part M and second-order part consisting of all $X \subseteq |M|$ that are Δ^0_1-definable over $|M| \cup \{X_0,\ldots,X_n\}$, which means that there are Σ^0_1 formulas $\varphi_0(x)$ and $\varphi_1(x)$ with parameters from $|M| \cup \{X_0,\ldots,X_n\}$ such that $(M,\{X_0,\ldots,X_n\}) \vDash \forall x\,(\varphi_0(x) \leftrightarrow \neg\varphi_1(x))$ and $X = \{n \in |\mathcal{M}| : (M,\{X_0,\ldots,X_n\}) \vDash \varphi_0(n)\}$.

Definition 1.4. Let P and Q be Π^1_2-problems. The RCA_0-*reduction game* $G^{\mathsf{RCA}_0}(\mathsf{Q} \to \mathsf{P})$ is a two-player game played according to the following rules.

(1) If at any point a player cannot make a move, the opponent wins.

(2) If one of the players wins, the game ends.

(3) On the first move, Player 1 plays a countable L_1-structure M and an M-instance X_0 of P such that $M[X_0] \vDash \mathsf{RCA}_0$. Then Player 2 either plays a solution to X_0 in $M[X_0]$ and wins, or plays an M-instance Y_1 of Q in $M[X_0]$.

(4) For $n > 1$, on the nth move, Player 1 plays a solution X_{n-1} to the instance Y_{n-1} of Q such that $M[X_0, \ldots, X_{n-1}] \vDash \mathsf{RCA}_0$. Then Player 2 either plays a solution to X_0 in $M[X_0, \ldots, X_{n-1}]$ and wins, or plays an M-instance Y_n of Q in $M[X_0, \ldots, X_{n-1}]$.

(5) If the game never ends then Player 1 wins.

This definition allows us to capture provability over RCA_0 in terms of winning strategies.

Proposition 1.5 (Hirschfeldt and Jockusch [19])**.** *Let* P *and* Q *be* Π^1_2*-problems. If* $\mathsf{RCA}_0 \vdash \mathsf{Q} \to \mathsf{P}$ *then Player 2 has a winning strategy for* $G^{\mathsf{RCA}_0}(\mathsf{Q} \to \mathsf{P})$*. Otherwise, Player 1 has a winning strategy for* $G^{\mathsf{RCA}_0}(\mathsf{Q} \to \mathsf{P})$*.*

The proof of this proposition is essentially the same as that of the analogous result for games over the standard natural numbers and ω-reducibility in [19, Proposition 4.2]. We will prove a stronger version in Proposition 2.4.

However, it might be that the above definition is not quite the best one. In Section 2, we will discuss a modified game. We will define it for arbitrary subsystems of second-order arithmetic, but in the case of RCA_0, the modified game $\widehat{G}^{\mathsf{RCA}_0}(\mathsf{Q} \to \mathsf{P})$ is defined as above, except that on its first move, Player 1 must play not only a countable L_1-structure M, but a model $\mathcal{M}$ of RCA_0 with countable first-order part (but possibly uncountable second-order part); and from then on, its moves $X_0, X_1, \ldots$ must all come from $\mathcal{M}$. This game makes intuitive sense in that if Player 1 is trying to claim that $\mathsf{RCA}_0 \nvdash \mathsf{Q} \to \mathsf{P}$, then it should be prepared to propose a model of RCA_0 within which to witness this fact. This idea was not noticed in [19] because in the ω-model case after which the original RCA_0-reduction game was modeled,

there is really no issue, since Player 1 always automatically plays within a particular model of RCA_0, namely $(\omega, \mathcal{P}(\omega))$, where $\mathcal{P}(\omega)$ is the full power set of ω. (For a nonstandard model M, of course, the full power set will include a cut, so we cannot add it to M to obtain a model of RCA_0.)

As we will see in Section 2, Proposition 1.5 still holds for this modified game, indeed with the same proof. But we will also be able to prove a stronger version that shows that a certain kind of compactness theorem holds in this case: As shown in [19], for a game $G(\mathsf{Q} \to \mathsf{P})$ over the standard natural numbers, it is possible that Player 2 has a winning strategy but there is no n such that Player 2 has a winning strategy that is guaranteed to win in at most n many moves. As we will show in Section 3, for our modified games over possibly nonstandard models, this will no longer be the case, which makes sense given that these games capture notions of provability, and a proof of $\mathsf{Q} \to \mathsf{P}$ is a finite object.

Theorem 1.6. *Let* P *and* Q *be* Π^1_2*-problems. If* $\mathsf{RCA}_0 \vdash \mathsf{Q} \to \mathsf{P}$ *then there is an* n *such that Player 2 has a winning strategy for* $\widehat{G}^{\mathsf{RCA}_0}(\mathsf{Q} \to \mathsf{P})$ *that ensures victory in at most* n *many moves. Otherwise, Player 1 has a winning strategy for* $\widehat{G}^{\mathsf{RCA}_0}(\mathsf{Q} \to \mathsf{P})$.

We do not know whether the first part of this result holds for the game $G^{\mathsf{RCA}_0}(\mathsf{Q} \to \mathsf{P})$ as well.

Theorem 1.6, whose proof will in fact use the compactness theorem for first-order logic, can be seen as a generalization of the following fact, which appears in Wang [37], where it is said that it is "almost certainly a known theorem in proof theory." For a model-theoretic proof using compactness due to Jockusch, see [18, Section 6.3].

Theorem 1.7 (see Wang [37])**.** *Let* $\mathsf{P} \equiv \forall X\, [\Theta(X) \to \exists Y\, \Psi(X, Y)]$ *be a* Π^1_2*-problem. If* P *is provable in* ACA_0*, then there is an* $n \in \omega$ *such that* ACA_0 *proves* $\forall X\, [\Theta(X) \to \exists Y \in \Sigma^{0,X}_n\, \Psi(X, Y)]$.

As mentioned above, this theorem implies for instance that $\mathsf{ACA}_0 \nvdash \mathsf{RT}$, because Jockusch [23] showed that for each $n \geqslant 2$, there is an instance of RT^n_2 (and hence of RT) with no Σ^0_n solutions. On the other hand, Jockusch also showed that every instance of RT^n_k has a Π^0_n solution, which implies that every ω-model of ACA_0 is a model of RT.

Notice that if we take Q to be the statement that for each X, the Turing jump X' exists, then the provability of P in ACA_0 is equivalent to the provability of $\mathsf{Q} \to \mathsf{P}$ in RCA_0. As part of the proof of Theorem 1.6 in Section 3,

we will prove a theorem that is a direct generalization of Theorem 1.7. Montalbán and Shore [27] also generalized this theorem, in a different way that is particularly suited to problems where each instance has a unique solution, and is indeed equivalent to ours in that case, but is not strong enough for our purposes.

As an example of the application of Theorem 1.6, we will obtain a simple proof that RT^2_2 does not imply $\mathsf{RT}^2_{<\infty}$, even over RCA_0 together with all Π^1_1 formulas true over the natural numbers.

Let Γ be a class of formulas. Recall that $\mathsf{I}\Gamma$ is the axiom scheme stating that induction holds for formulas in Γ. Recall also that the Γ-*bounding axiom scheme* $\mathsf{B}\Gamma$ consists of all formulas of the form

$$\forall n \, [\forall i < n \, \exists k \, \varphi(i,k) \; \rightarrow \; \exists b \, \forall i < n \, \exists k \leqslant b \, \varphi(i,k)]$$

for each formula φ in Γ such that b is not free in φ. Note that φ is allowed to have parameters. The system $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$, which is strictly intermediate between RCA_0 and $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$, has been particularly prominent in reverse mathematics. (In most cases, it is actually $\mathsf{B}\Pi^0_1$ that is used, but $\mathsf{B}\Pi^0_1$ and $\mathsf{B}\Sigma^0_2$ are easily seen to be equivalent over RCA_0.) For example, Hirst [21] showed that $\mathsf{RT}^1_{<\infty}$ is equivalent to $\mathsf{B}\Sigma^0_2$ over RCA_0.

In Section 4, we will consider computable winning strategies and the notion of generalized Weihrauch reducibility over possibly nonstandard models. There is an intriguing connection here with analogs of RCA_0 for intuitionistic logic, first noted in work of Kuyper [25]. We will comment on this connection briefly in that section, but leave further work in this direction to a follow-up paper. In Section 5 we will consider single-instance reducibilities such as computable and Weihrauch reducibility in this context. Our results throughout will apply not only to RCA_0 but also to other systems at the level of computable mathematics, including extensions of RCA_0 by first-order principles, such as $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_n$ or $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_n$, and also restrictions such as RCA^*_0, which roughly speaking is RCA_0 with Σ^0_1-induction replaced by Σ^0_0-induction.

In Sections 6 and 7, we will undertake a case study in the analysis of mathematical principles under Weihrauch and generalized Weihrauch reducibility over possibly nonstandard models, by considering several principles that are equivalent over RCA_0 to Σ^0_2-bounding. We will see how this framework allows us to uncover some hitherto hidden differences between quite similar principles.

2 Reduction games and provability

In this section, we generalize Definition 1.4 from RCA_0 to other axiom systems Γ, modify it as described above, and prove a more general version of Proposition 1.5. Of course, we cannot in general require Player 1's moves to result in models of Γ, since it might be the case that no structure of the form $M[X_0, \ldots, X_{n-1}]$ is a model of Γ. However, we can require that Player 1 never make it impossible for the model built by its moves to be extendable to a model of Γ. Say that an L_2-structure $\mathcal{M}$ is *consistent with* Γ if it is contained in a model $\mathcal{N}$ of Γ with the same first-order part. (Note that if $\mathcal{M}$ is countable, then we can require $\mathcal{N}$ to be countable as well without changing the notion.)

The systems Γ for which we will prove that winning strategies for the following game correspond to provability over Γ will actually have the property that every structure consistent with Γ is in fact a model of Γ. The reason we give the definition in the more general setting is that, when analyzing the provability of $\mathsf{Q} \to \mathsf{P}$ in Γ, we will also want to consider games over $\Gamma + \mathsf{Q}$. We will see that doing so makes no difference in the case of general winning strategies, but does in the case of computable winning strategies.

Definition 2.1. Let Γ be a set of L_2-formulas and let P and Q be Π^1_2-problems. The Γ-*reduction game* $G^\Gamma(\mathsf{Q} \to \mathsf{P})$ is a two-player game played according to the following rules.

(1) If at any point a player cannot make a move, the opponent wins.

(2) If one of the players wins, the game ends.

(3) On the first move, Player 1 plays a countable L_1-structure M and an M-instance X_0 of P such that $M[X_0]$ is consistent with Γ. Then Player 2 either plays a solution to X_0 in $M[X_0]$ and wins, or plays an M-instance Y_1 of Q in $M[X_0]$.

(4) For $n > 1$, on the nth move, Player 1 plays a solution X_{n-1} to the instance Y_{n-1} of Q such that $M[X_0, \ldots, X_{n-1}]$ is consistent with Γ. Then Player 2 either plays a solution to X_0 in $M[X_0, \ldots, X_{n-1}]$ and wins, or plays an M-instance Y_n of Q in $M[X_0, \ldots, X_{n-1}]$.

(5) If the game never ends then Player 1 wins.

We modify this game as follows.

Definition 2.2. Let Γ be a set of L_2-formulas consistent with Δ^0_1-comprehension, and let P and Q be Π^1_2-problems. The *modified* Γ*-reduction game* $\widehat{G}^\Gamma(\mathsf{Q} \to \mathsf{P})$ is a two-player game played according to the following rules.

(1) If at any point a player cannot make a move, the opponent wins.

(2) If one of the players wins, the game ends.

(3) On the first move, Player 1 plays a model (M, S) of Γ such that M is countable and S is closed under Δ^0_1-comprehension, and an M-instance X_0 of P in S. Then Player 2 either plays a solution to X_0 in $M[X_0]$ and wins, or plays an M-instance Y_1 of Q in $M[X_0]$.

(4) For $n > 1$, on the nth move, Player 1 plays a solution X_{n-1} to the instance Y_{n-1} of Q in S. Then Player 2 either plays a solution to X_0 in $M[X_0, \ldots, X_{n-1}]$ and wins, or plays an M-instance Y_n of Q in $M[X_0, \ldots, X_{n-1}]$.

(5) If the game never ends then Player 1 wins.

If Γ is consistent with Δ^0_1-comprehension and $\Gamma \nvdash \mathsf{Q} \to \mathsf{P}$, then Player 1 has winning strategies in both of these games (as we will see in the second part of the proof of Proposition 2.4 below), but we cannot hope in general that the same is the case for Player 2 if $\Gamma \vdash \mathsf{Q} \to \mathsf{P}$, because of that player's restriction to playing computably. However, if Γ is sufficiently well-behaved, then this is no longer an obstacle, and we can obtain a generalization of Proposition 1.5 with essentially the same proof. The key property here is that all axioms of Γ other than Δ^0_1-comprehension be Π^1_1. Of course, this property holds of RCA_0, as well as commonly-studied first-order extensions such as $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_n$ and $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_n$, and restrictions such as RCA^*_0.

In the proof, we will actually use the following properties, but it is not difficult to show that they are equivalent to saying that Γ is a consistent set of L_2-formulas consisting of Δ^0_1-comprehension together with a set of Π^1_1 formulas.

1. Γ is a consistent set of L_2-formulas that includes all instances of Δ^0_1-comprehension.

2. If an L_2-structure is closed under Δ^0_1-definability and is consistent with Γ, then it is a model of Γ.

3. For every countable L_1-structure M and $X_0, X_1, \ldots \subseteq |M|$, if each $M[X_0, \ldots, X_n]$ is a model of Γ, then so is their union $M[X_0, X_1, \ldots]$.

The following simple but important result follows from these properties.

Lemma 2.3. *Let Γ be a consistent extension of Δ^0_1-comprehension by Π^1_1 formulas. Let P and Q be Π^1_2-problems. Let M be an L_1-structure and $X_0, \ldots, X_n \subseteq |M|$ be sets set such that $M[X_0, \ldots, X_n]$ is consistent with Γ. If $\Gamma \vdash \mathsf{Q} \to \mathsf{P}$, then either every instance of P in $M[X_0, \ldots, X_n]$ has a solution in $M[X_0, \ldots, X_n]$, or else Q has an instance in $M[X_0, \ldots, X_n]$.*

Proof. Fix M and $X_0, \ldots, X_n$. Since the L_2-structure $M[X_0, \ldots, X_n]$ is closed under Δ^0_1-comprehension it is in fact a model of Γ, as noted above. If Q has no instance in $M[X_0, \ldots, X_n]$, then $M[X_0, \ldots, X_n]$ trivially satisfies Q. Hence, by assumption, $M[X_0, \ldots, X_n]$ also satisfies P. So every instance of P in $M[X_0, \ldots, X_n]$ has a solution in $M[X_0, \ldots, X_n]$. □

Later on, when we prove a generalization of Theorem 1.6, we will also need to assume that Γ is strong enough to prove the existence of a universal Σ^0_1 formula, but of course that holds of all systems we normally study in reverse mathematics.

We should also expect Γ and $\Gamma + \mathsf{Q}$ to behave similarly here, since there is no difference between saying that $\Gamma \vdash \mathsf{Q} \to \mathsf{P}$ and saying that $\Gamma + \mathsf{Q} \vdash \mathsf{Q} \to \mathsf{P}$. This fact will be of interest below when we consider computable winning strategies.

Proposition 1.5 can be generalized as follows. Notice that of the four games $G^\Gamma(\mathsf{Q} \to \mathsf{P})$, $G^{\Gamma+\mathsf{Q}}(\mathsf{Q} \to \mathsf{P})$, $\widehat{G}^\Gamma(\mathsf{Q} \to \mathsf{P})$, and $\widehat{G}^{\Gamma+\mathsf{Q}}(\mathsf{Q} \to \mathsf{P})$, the first is the hardest one for Player 2 to win, while the last is the hardest one for Player 1 to win.

Proposition 2.4. *Let Γ be a consistent extension of Δ^0_1-comprehension by Π^1_1 formulas. Let P and Q be Π^1_2-problems. If $\Gamma \vdash \mathsf{Q} \to \mathsf{P}$ then Player 2 has a winning strategy for $G^\Gamma(\mathsf{Q} \to \mathsf{P})$ (and hence for each of the three other games above). Otherwise, Player 1 has a winning strategy for $\widehat{G}^{\Gamma+\mathsf{Q}}(\mathsf{Q} \to \mathsf{P})$ (and hence for each of the three other games above).*

Proof. If $\Gamma \vdash \mathsf{Q} \to \mathsf{P}$ then Player 2 can play according to the following strategy. Let M be the L_1-structure played by Player 1 on its first move. At the nth move, if Player 2 has a legal winning move, Player 2 makes that move. Otherwise, it lets $Y_{n,0}, Y_{n,1}, \ldots$ be all M-instances of Q in $M[X_0, \ldots, X_{n-1}]$,

where $X_0, \ldots, X_{n-1}$ are Player 1's first n moves. For the least pair $\langle m, i\rangle$ with $m \leqslant n$ for which Player 2 has not yet acted, it then acts by playing $Y_{m,i}$ (to which Player 1 must reply with a solution to $Y_{m,i}$). Note that Player 2 always has some legal move, by Lemma 2.3. Suppose Player 2 never has a winning move, and Player 1 never fails to have a legal move. By our assumptions on Γ, each $M[X_0, \ldots, X_{n-1}]$ is a model of Γ, and hence so is their union $M[X_0, X_1, \ldots]$. But Player 2's strategy ensures that this structure is also a model of Q, so it must also be a model of P, and hence must contain a solution to X_0. This solution is in $M[X_0, \ldots, X_{n-1}]$ for some n, which gives Player 2 a winning nth move.

If $\Gamma \nvdash \mathsf{Q} \to \mathsf{P}$ then let (M, S) be a model of $\Gamma + \mathsf{Q} + \neg\mathsf{P}$ and let X_0 be an M-instance of P in S with no solution in S. Since (M, S) is a model of Γ, it is closed under Δ^0_1-definability, so as long as Player 1's moves stay inside S, so must Player 2's moves. Furthermore, the fact that (M, S) is a model of Q implies that, as long as Player 2's moves stay inside S, Player 1 will always be able to reply with moves that stay inside S. So Player 1 can simply begin by playing (M, S) and X_0, and then keep playing elements of S, which ensures that the game never ends (unless Player 2 cannot make its first move, in which case it loses immediately). □

Remark 2.5. We can extend the above framework beyond extensions of Δ^0_1-comprehension by Π^1_1 formulas. Let us consider ACA_0, for instance. If we redefine $M[X_0, \ldots, X_{n-1}]$ by replacing Δ^0_1-definability by arithmetic definability, then use this new definition in the definitions of the Γ-reduction game and the modified Γ-reduction game, then Proposition 2.4 carries through essentially unchanged.

There is nothing particularly special about this $\Gamma = \mathsf{ACA}_0$ case. All we need is the existence of a smallest model $M[X_0, \ldots, X_{n-1}]$ of Γ with first-order part M containing $X_0, \ldots, X_{n-1} \subseteq |M|$ (if there is any such model at all), and the requirement that then $\bigcup_n M[X_0, \ldots, X_{n-1}]$ is also a model of Γ (which will happen if Γ is Π^1_2-axiomatizable). For systems Γ that do not have such minimal models, such as WKL_0, we can still extend these ideas by redefining our games in a way that does not affect our results when applied to systems that do have minimal models. For example, $\widehat{G}^\Gamma(\mathsf{Q} \to \mathsf{P})$ can now be played according to the following rules.

(1) If at any point a player cannot make a move, the opponent wins.

(2) If one of the players wins, the game ends.

(3) On the first move, Player 1 plays a model (M, S) of Γ with M countable, an M-instance X_0 of P in S, and a submodel (M, S_0) of (M, S) containing X_0. Then Player 2 either plays a solution to X_0 in (M, S_0) and wins, or plays an M-instance Y_1 of Q in (M, S_0).

(4) For $n > 1$, on the nth move, Player 1 plays a solution X_{n-1} to the instance Y_{n-1} of Q in S and a submodel (M, S_{n-1}) of (M, S) containing X_{n-1}. Then Player 2 either plays a solution to X_0 in (M, S_{n-1}) and wins, or plays an M-instance Y_n of Q in (M, S_{n-1}).

(5) If the game never ends then Player 1 wins.

Theorem 3.1 below remains true for ACA_0, for instance, since in Theorem 3.4 we can replace the eth Turing functional by the eth arithmetical functional. It is not clear how generally Theorem 3.1 holds for other systems, but we will not pursue this further generalization of our framework here.

3 Reduction games and compactness

As mentioned in the introduction, we can improve on Proposition 2.4 by showing that a certain kind of compactness theorem holds, with the very mild extra assumption that Γ proves the existence of a universal Σ^0_1 formula, i.e., that there is a Σ^0_1 formula $\theta(e, n, X)$ such that for every Σ^0_1 formula $\varphi(e, n, X)$, we have $\Gamma \vdash \forall e\, \exists i\, \forall n\, \forall X\, (\theta(i, n, X) \leftrightarrow \varphi(e, n, X))$. In this case, we assume we have fixed such a θ and a bijective pairing function $\langle \cdot, \cdot \rangle$, and write $Y = \Phi^X_e$ to mean that for $e = \langle i, j \rangle$, we have $\forall n\, [\theta(i, n, X) \leftrightarrow \neg\theta(j, n, X)]$ and $\forall n\, [n \in Y \leftrightarrow \theta(i, n, X)]$.

The following result, which we will prove in this section, has Theorem 1.6 as a special case.

Theorem 3.1. *Let Γ be a consistent extension of Δ^0_1-comprehension by Π^1_1 formulas that proves the existence of a universal Σ^0_1 formula. Let P and Q be Π^1_2-problems. If $\Gamma \vdash \mathsf{Q} \to \mathsf{P}$ then there is an n such that Player 2 has a winning strategy for $\widehat{G}^{\Gamma}(\mathsf{Q} \to \mathsf{P})$ (and hence for $\widehat{G}^{\Gamma+\mathsf{Q}}(\mathsf{Q} \to \mathsf{P})$) that ensures victory in at most n many moves. Otherwise, Player 1 has a winning strategy for $\widehat{G}^{\Gamma+\mathsf{Q}}(\mathsf{Q} \to \mathsf{P})$ (and hence for $\widehat{G}^{\Gamma}(\mathsf{Q} \to \mathsf{P})$).*

Notice that if the formulas added to Δ^0_1-comprehension to obtain Γ are true over the standard natural numbers, then a winning strategy for Player

2 for $\widehat{G}^{\Gamma}(\mathsf{Q} \to \mathsf{P})$ that ensures victory in at most n many moves also yields a winning strategy for Player 2 for $G(\mathsf{Q} \to \mathsf{P})$ that ensures victory in at most n many moves, since a run of the latter game is a special case of a run of the former game in which Player 1 begins by playing the model $(\omega, \mathcal{P}(\omega))$. Thus it is not a coincidence that all the examples we have of situations in which $G(\mathsf{Q} \to \mathsf{P})$ can be won by Player 2, but not in a number of moves bounded ahead of time, are ones in which $\mathsf{P} \leqslant_\omega \mathsf{Q}$ but $\mathsf{RCA}_0 \nvdash \mathsf{Q} \to \mathsf{P}$. In fact, the following stronger fact holds, where, as defined in [19], $\mathsf{P} \leqslant^n_\omega \mathsf{Q}$ means that Player 2 has a winning strategy for $G(\mathsf{Q} \to \mathsf{P})$ that ensures victory in at most $n+1$ many moves.

Corollary 3.2. *Let Γ consist of RCA_0 together with all Π^1_1 formulas true over the natural numbers. If $\mathsf{P} \not\leqslant^n_\omega \mathsf{Q}$ for all n, then $\Gamma \nvdash \mathsf{Q} \to \mathsf{P}$.*

Notice that the Γ in this corollary includes full arithmetical induction. An interesting example of the application of this corollary is to take Q to be RT^2_2 and P to be $\mathsf{RT}^2_{<\infty}$. Cholak, Jockusch, and Slaman [10] showed that $\mathsf{RCA}_0 \nvdash \mathsf{RT}^2_k \to \mathsf{RT}^2_{<\infty}$ for all k, but the proof relies on a difference between the first-order parts of these two principles, and hence does not work if we add arithmetical induction to RCA_0. (Note that, with full induction, $\mathsf{RT}^2_{<\infty}$ does in fact follow from RT^2_2.) Patey [29] showed that $\mathsf{RT}^2_{<\infty} \not\leqslant^n_\omega \mathsf{RT}^2_k$ for all n and k, so we have the following.

Corollary 3.3. *Let Γ consist of RCA_0 together with all Π^1_1 formulas true over the natural numbers. Then $\Gamma \nvdash \mathsf{RT}^2_k \to \mathsf{RT}^2_{<\infty}$ for all k.*

We learned from Yokoyama [personal communication] that he and Slaman have recently noticed that this corollary can also be obtained by a more direct model-theoretic argument, still using Patey's result.

The proof of Theorem 3.1 will use the following result, which is of independent interest as a generalization of Theorem 1.7.

Theorem 3.4. *Let Γ be a consistent extension of Δ^0_1-comprehension by Π^1_1 formulas that proves the existence of a universal Σ^0_1 formula. Let P and Q be Π^1_2-problems. For $n \in \omega$, let $\Theta_n(e_0, \dots, e_n, X_0, \dots, X_n, Y_0, \dots, Y_n)$ be a*

formula asserting that

$$\begin{gathered}\textit{if } X_0 \textit{ is a } \mathsf{P}\textit{-instance then } (Y_0 = \Phi_{e_0}^{X_0} \wedge (\textit{either } Y_0 \textit{ is a solution to } X_0 \textit{ or} \\ (Y_0 \textit{ is a } \mathsf{Q}\textit{-instance and if } X_1 \textit{ is a solution to } Y_0 \textit{ then } (Y_1 = \Phi_{e_1}^{X_0 \oplus X_1} \wedge \\ (\textit{either } Y_1 \textit{ is a solution to } X_0 \textit{ or} \\ (Y_1 \textit{ is a } \mathsf{Q}\textit{-instance and if } X_2 \textit{ is a solution to } Y_1 \textit{ then } (Y_2 = \Phi_{e_2}^{X_0 \oplus X_1 \oplus X_2} \wedge \\ (\textit{either } Y_2 \textit{ is a solution to } X_0 \textit{ or } \ldots \\ \vdots \\ \ldots (Y_n = \Phi_{e_n}^{X_0 \oplus \cdots \oplus X_n} \wedge Y_n \textit{ is a solution to } X_0)) \cdots),\end{gathered}$$

and let Δ_n *be*

$$\forall X_0\, \exists e_0, Y_0\, \forall X_1\, \exists e_1, Y_1 \cdots \forall X_n\, \exists e_n, Y_n\, \Theta_n(e_0, \ldots, e_n, X_0, \ldots, X_n, Y_0, \ldots, Y_n).$$

If $\Gamma \vdash \mathsf{Q} \to \mathsf{P}$*, then there exists an* $n \in \omega$ *such that* $\Gamma \vdash \Delta_n$*.*

Proof. Suppose that $\Gamma \vdash \mathsf{Q} \to \mathsf{P}$ but $\Gamma \vdash \neg\Delta_n$ for all n. Extend L_2 to include a function symbol f from first-order objects to second-order objects. Call this new language L_2'. Let $\langle \cdot, \ldots, \cdot \rangle$ be a fixed numbering scheme for finite tuples of numbers.

For each n, there is a model $\mathcal{M} = (M, S)$ of $\Gamma + \neg\Delta_n$. We can turn $\mathcal{M}$ into an L_2'-structure by defining the interpretation $f^{\mathcal{M}}$ by recursion as follows.

There is an $X_0 \in S$ such that

$$\begin{aligned}\mathcal{M} \vDash \forall e_0, Y_0\, \exists X_1\, \forall e_1, Y_1 \cdots \exists X_n\, \forall e_n, Y_n\, \neg\Theta_n(e_0, \ldots, e_n, \\ X_0, \ldots, X_n, Y_0, \ldots, Y_n).\end{aligned}$$

Let $f^{\mathcal{M}}(\langle\rangle) = X_0$.

Assume we have defined $f^{\mathcal{M}}(\langle e_0, \ldots, e_{j-1}\rangle)$, where $j < n$, and have also defined $Y_{\langle e_0\rangle}, Y_{\langle e_0, e_1\rangle}, \ldots, Y_{\langle e_0, \ldots, e_{j-1}\rangle} \in S$ so that

$$\begin{aligned}\mathcal{M} \vDash \forall e_j, Y_j\, \exists X_{j+1}\, \forall e_{j+1}, Y_{j+1} \exists X_{j+2} \cdots \exists X_n\, \forall e_n, Y_n\, \neg\Theta_n(e_0, \ldots, e_n, \\ f^{\mathcal{M}}(\langle\rangle), f^{\mathcal{M}}(\langle e_0\rangle), \ldots, f^{\mathcal{M}}(\langle e_0, \ldots, e_{j-1}\rangle), X_{j+1} \ldots, X_n, \\ Y_{\langle e_0\rangle}, Y_{\langle e_0, e_1\rangle}, \ldots, Y_{\langle e_0, \ldots, e_{j-1}\rangle}, Y_j, \ldots, Y_n).\end{aligned}$$

Given $e_j \in M$, let $Y_{\langle e_0,\ldots,e_j\rangle} = \Phi_{e_j}^{(f^{\mathcal{M}}(\langle\rangle)\oplus f^{\mathcal{M}}(\langle e_0\rangle)\oplus\cdots\oplus f^{\mathcal{M}}(\langle e_0,\ldots,e_{j-1}\rangle))} \in S$. Then there is an $X_{j+1} \in S$ such that

$$\mathcal{M} \vDash \forall e_{j+1}, Y_{j+1}\, \exists X_{j+2}\, \forall e_{j+2}, Y_{j+2} \exists X_{j+3} \cdots \exists X_n\, \forall e_n, Y_n\, \neg\Theta_n(e_0,\ldots,e_n, f^{\mathcal{M}}(\langle\rangle), f^{\mathcal{M}}(\langle e_0\rangle),\ldots, f^{\mathcal{M}}(\langle e_0,\ldots,e_{j-1}\rangle), X_{j+1}\ldots, X_n, Y_{\langle e_0\rangle}, Y_{\langle e_0,e_1\rangle},\ldots, Y_{\langle e_0,\ldots,e_j\rangle}, Y_{j+1},\ldots,Y_n).$$

Let $f^{\mathcal{M}}(\langle e_0,\ldots,e_j\rangle) = X_{j+1}$.

Having defined $f^{\mathcal{M}}$ on all $\langle e_0,\ldots,e_i\rangle$ for $i \leqslant n$, let $f^{\mathcal{M}}(x) = \emptyset$ for all other $x \in M$.

Let

$$\Psi_k \equiv \forall e_0, Y_0 \cdots \forall e_k, Y_k\, \neg\Theta_k(e_0,\ldots,e_k, f(\langle\rangle), f(\langle e_0\rangle),\ldots, f(\langle e_0,\ldots,e_k\rangle), Y_0,\ldots,Y_k).$$

Then $(\mathcal{M}; f^{\mathcal{M}}) \vDash \Psi_n$ by the definition of $f^{\mathcal{M}}$. It is easy to see that this fact implies that $(\mathcal{M}; f^{\mathcal{M}}) \vDash \Psi_k$ for all $k \leqslant n$.

Thus every set $\Gamma \cup \{\Psi_0,\ldots,\Psi_n\}$ is satisfiable, and hence so is the union $\Gamma \cup \{\Psi_0, \Psi_1, \ldots\}$. Let $\mathcal{N} = (N, T)$ be a model of this set. Now we have a winning strategy for Player 1 for $G^{\Gamma}(\mathsf{Q} \to \mathsf{P})$: Player 1 begins by playing N and $f^{\mathcal{N}}(\langle\rangle)$, and if $e_0,\ldots,e_{n-1}$ are indices for Player 2's first n moves, then Player 1 plays $f^{\mathcal{N}}(\langle e_0,\ldots,e_{n-1}\rangle)$ on its next move. By the definition of Ψ_n, Player 2 can never play a solution to $f^{\mathcal{N}}(\langle\rangle)$.

But by Proposition 2.4 and our assumption that $\Gamma \vdash \mathsf{Q} \to \mathsf{P}$, Player 2 must have a winning strategy for $G^{\Gamma}(\mathsf{Q} \to \mathsf{P})$, so we have a contradiction. □

Proof of Theorem 3.1. We use the notation of Theorem 3.4. By Proposition 2.4, it is enough to show that if $\Gamma \vdash \mathsf{Q} \to \mathsf{P}$ then there is an n such that Player 2 has a winning strategy for $\widehat{G}^{\Gamma}(\mathsf{Q} \to \mathsf{P})$ that ensures victory in at most n many moves. So suppose that $\Gamma \vdash \mathsf{Q} \to \mathsf{P}$. Let n be as in Theorem 3.4.

Player 2 can play as follows. Let $\mathcal{M} = (M, S)$ be the model of Γ played by Player 1 on its first move. Since $\mathcal{M}$ is a model of Γ, it is also a model of Δ_n. Let X_0 be Player 1's first move. Since X_0 is in S, there are $e_0 \in M$ and $Y_0 \in S$ such that $\mathcal{M}$ satisfies

$$\forall X_1\, \exists e_1, Y_1\, \forall X_2\, \exists e_2, Y_2 \cdots \forall X_n\, \exists e_n, Y_n\, \Theta_n(e_0,\ldots,e_n, X_0,\ldots,X_n, Y_0,\ldots,Y_n).$$

Now Player 2 plays Y_0. Let X_1 be Player 1's next move. Then there are $e_1 \in M$ and $Y_1 \in S$ such that $\mathcal{M}$ satisfies

$$\forall X_2\, \exists e_2, Y_2\, \forall X_3\, \exists e_3, Y_3 \cdots \forall X_n\, \exists e_n, Y_n\, \Theta_n(e_0, \ldots, e_n, X_0, \ldots, X_n, Y_0, \ldots, Y_n).$$

Now Player 2 plays Y_1.

Continuing in this way, by the definition of Δ_n, some Y_i with $i \leqslant n$ must be a solution to X_0, and thus this strategy ensures victory by Player 2 in at most $n+1$ many moves. □

We do not know whether Theorem 3.1 holds for $G^\Gamma(\mathsf{Q} \to \mathsf{P})$ in general, but normally, if $\Gamma \vdash \mathsf{Q} \to \mathsf{P}$ then the proof allows us to obtain a winning strategy for Player 2 in $\widehat{G}^\Gamma(\mathsf{Q} \to \mathsf{P})$ (and even in $G^\Gamma(\mathsf{Q} \to \mathsf{P})$) that is relatively easy to describe. (The special case of computable winning strategies will be discussed in Section 4.) In such cases, we can show that there is an n such that this particular winning strategy allows Player 2 to win in at most n many moves, not just in $\widehat{G}^\Gamma(\mathsf{Q} \to \mathsf{P})$ but in fact in $G^\Gamma(\mathsf{Q} \to \mathsf{P})$. Here we are thinking of strategies that are first-order definable, but we need to take into account the possibility that there might not be a unique choice of move at a given point (keeping in mind that the idea of choosing the least among the indices of equally good moves is not always available when working over nonstandard models).

Definition 3.5. Let Γ be a consistent set of L_2-formulas and let $\Lambda(X, n, e)$ be an arithmetic formula. Say that Player 2 plays a run of $G^\Gamma(\mathsf{Q} \to \mathsf{P})$ or $\widehat{G}^\Gamma(\mathsf{Q} \to \mathsf{P})$ *according to* Λ if given Player 1's first n moves, M (or (M, S)) and $X_0, \ldots, X_{n-1} \subseteq M$, Player 2 plays $\Phi_e^{X_0 \oplus \cdots \oplus X_{n-1}}$ for some $e \in M$ such that $M[X_0, \ldots, X_{n-1}] \vDash \Lambda(X_0 \oplus \cdots \oplus X_{n-1}, n-1, e)$.

Theorem 3.6. *Let Γ be a consistent extension of Δ^0_1-comprehension that proves the existence of a universal Σ^0_1 formula. Let P and Q be Π^1_2-problems and Λ be an arithmetic formula such that Player 2 wins any run of $\widehat{G}^\Gamma(\mathsf{Q} \to \mathsf{P})$ that it plays according to Λ. Then there is an n such that Player 2 wins any run of $G^\Gamma(\mathsf{Q} \to \mathsf{P})$ that it plays according to Λ in at most n many moves.*

Proof. Let Θ_n be as in Theorem 3.4. Let Ξ_n be a formula asserting that, for all $i \leqslant n$, if X_0 is a P-instance and no Y_j with $j < i$ is a solution to X_0, then $\Lambda(X_0 \oplus \cdots \oplus X_i, i, e_i)$. Let $\widehat{\Theta}_n$ be $\Xi_n \to \Theta_n$, and let Ω_n be

$$\forall X_0\, \forall e_0\, \exists Y_0\, \forall X_1\, \forall e_1\, \exists Y_1 \cdots \forall X_n\, \forall e_n\, \exists Y_n \\ \widehat{\Theta}_n(e_0, \ldots, e_n, X_0, \ldots, X_n, Y_0, \ldots, Y_n).$$

Suppose there is a run of $G^\Gamma(\mathsf{Q} \to \mathsf{P})$ such that Player 2 plays according to Λ but does not win within n moves. Let M and $X_0, \ldots, X_{n-1}$ be Player 1's first n moves in that run. Then $M[X_0, \ldots, X_{n-1}]$ can be extended to a model (M, S) of Γ, and in that model, Ω_{n-1} does not hold. Thus, to establish the theorem, it is enough to show that $\Gamma \vdash \Omega_n$ for some n.

Assume for a contradiction that $\Gamma \nvdash \Omega_n$ for all n. Expand L_2 by adding first-order constant symbols $c_0, c_1, \ldots$ and second-order constant symbols $C_0, C_1, \ldots$. Then a compactness argument as in the proof of Theorem 3.4 shows that there is a model $\mathcal{M}$ of Γ and interpretations $c_0^\mathcal{M}, c_1^\mathcal{M}, \ldots$ and $C_0^\mathcal{M}, C_1^\mathcal{M}, \ldots$ such that each $\Phi_{c_n^\mathcal{M}}^{C_0^\mathcal{M} \oplus \cdots \oplus C_n^\mathcal{M}}$ is total in $\mathcal{M}$, and $\mathcal{M}$ together with these interpretations satisfies

$$\neg\widehat{\Theta}_n(c_0, \ldots, c_n, C_0, \ldots, C_n, \Phi_{c_0}^{C_0}, \ldots, \Phi_{c_n}^{C_0 \oplus \ldots \oplus C_n})$$

for all n. But then there is a run of $\widehat{G}^\Gamma(\mathsf{Q} \to \mathsf{P})$ in which Player 2 plays according to Λ but does not win, namely the one in which Player 1 begins by playing $\mathcal{M}$, then at each move plays $C_n^\mathcal{M}$, and Player 2 responds with $\Phi_{c_n^\mathcal{M}}^{C_0^\mathcal{M} \oplus \cdots \oplus C_n^\mathcal{M}}$, which contradicts our hypothesis. □

For Γ is as in Theorem 3.1, write $\Gamma \vdash^n \mathsf{Q} \to \mathsf{P}$ to mean that Player 2 has a winning strategy for $\widehat{G}^\Gamma(\mathsf{Q} \to \mathsf{P})$ that ensures victory in at most $n+1$ many moves. Then the first part of the theorem can be restated as $\Gamma \vdash \mathsf{Q} \to \mathsf{P} \;\Rightarrow\; \exists n\, [\Gamma \vdash^n \mathsf{Q} \to \mathsf{P}]$. The idea behind this notation is that we can see the least n such that $\Gamma \vdash^n \mathsf{Q} \to \mathsf{P}$ as a measure of the number of applications of Q needed to prove P over Γ. The $n = 0$ case is equivalent to $\Gamma \vdash \mathsf{P}$. We will discuss the $n = 1$ case in Section 5, but make the following remark for now.

Remark 3.7. Recall that $\mathsf{P} \leqslant_\omega^n \mathsf{Q}$ means that Player 2 has a winning strategy for $G(\mathsf{Q} \to \mathsf{P})$ that ensures victory in at most $n+1$ many moves. Hirschfeldt and Jockusch [19] stated that $\mathsf{P} \leqslant_\omega^1 \mathsf{Q}$ is equivalent to $\mathsf{P} \leqslant_\mathrm{c} \mathsf{Q}$, but that is not quite correct, because if P is computably true (i.e., if $\mathsf{P} \leqslant_\omega^0 \mathsf{Q}$) but has an instance that does not compute any instance of Q, then $\mathsf{P} \leqslant_\omega^1 \mathsf{Q}$ but $\mathsf{P} \not\leqslant_\mathrm{c} \mathsf{Q}$. (The same point was made in the context of Weihrauch reducibility by Brattka, Gherardi, and Pauly [7, Section 3].) As this fairly uninteresting case is the only in which the two notions differ, however, we can generally ignore the distinction. We mention it, and make the following remarks, only because an analogous situation will be relevant below.

We can define $\mathsf{P} \leqslant_\omega^{=n} \mathsf{Q}$ to mean that Player 2 has a winning strategy for $G(\mathsf{Q} \to \mathsf{P})$ that ensures victory in exactly $n+1$ many moves. Then $\mathsf{P} \leqslant_{\mathrm{c}} \mathsf{Q}$ is equivalent to $\mathsf{P} \leqslant_\omega^{=1} \mathsf{Q}$. This definition is not otherwise very useful, though, because if Player 2 can win $G(\mathsf{Q} \to \mathsf{P})$ in $m \geqslant 2$ many moves, then it can also win that game in k many moves for any $k > m$, simply by repeating its first move until it is ready to win, except in the case in which Player 2's first move is an instance of Q with no solution (and in this context we are generally not interested in problems that are false over ω as statements of second-order arithmetic).

Note also that $\mathsf{P} \leqslant_\omega^n \mathsf{Q}$ is not quite equivalent to $\exists m \leqslant n\, [\mathsf{P} \leqslant_\omega^{=m} \mathsf{Q}]$, again because of 1-move runs. For example, let P be the Π^1_2-problem whose instances are $\emptyset$ and $\emptyset'$, with unique solutions $\emptyset$ and $\emptyset''$, respectively; and let Q be the Π^1_2-problem whose only instance is $\emptyset'$, with unique solution $\emptyset''$. If Player 1 begins by playing $\emptyset'$, then Player 2 cannot win immediately, but can play $\emptyset'$, to which Player 1 must reply with $\emptyset''$, at which point Player 2 wins by playing $\emptyset''$. So in this case, Player 2 wins in 2 moves. However, if Player 1 plays $\emptyset$, then Player 2 has only one legal move, namely the winning move $\emptyset$. Thus $\mathsf{P} \leqslant_\omega^1 \mathsf{Q}$, but the first case shows that $\mathsf{P} \not\leqslant_\omega^{=0} \mathsf{Q}$, while the second case shows that $\mathsf{P} \not\leqslant_\omega^{=1} \mathsf{Q}$.

Similar considerations hold for the notion of $\mathsf{P} \leqslant_{\mathrm{gW}}^n \mathsf{Q}$ introduced in [19], and for $\Gamma \vdash^n \mathsf{Q} \to \mathsf{P}$. One way around these issues is to replace Q with the problem $\widehat{\mathsf{Q}}$ where an instance is either $\{0\} \cup \{n+1 : n \in X\}$ for an instance X of Q, with a solution to this instance being any solution to X; or $\emptyset$, with the only solution being $\emptyset$ (although if we allow problems Q that have instances with no solutions, we might still have $\mathsf{P} \leqslant_{\mathrm{gW}}^n \widehat{\mathsf{Q}}$ but not have $\exists m \leqslant n\, [\mathsf{P} \leqslant_{\mathrm{gW}}^{=m} \widehat{\mathsf{Q}}]$, because a computable winning strategy might not be able to tell when it is about to play an instance of Q with no solution, and thus instantly win).

The definition of $\Gamma \vdash^n \mathsf{Q} \to \mathsf{P}$ was made in [19] (for $\Gamma = \mathsf{RCA}_0$), but with $G^\Gamma(\mathsf{Q} \to \mathsf{P})$ in place of $\widehat{G}^\Gamma(\mathsf{Q} \to \mathsf{P})$. We have chosen our definition in light of Theorem 3.1, but at least in natural cases, there should be no difference, as shown by the following fact.

Proposition 3.8. *Let Γ be a consistent extension of Δ^0_1-comprehension that proves the existence of a universal Σ^0_1 formula. Let P and Q be Π^1_2-problems and Λ be an arithmetic formula such that Player 2 wins any run of $\widehat{G}^\Gamma(\mathsf{Q} \to \mathsf{P})$ that it plays according to Λ in at most n many moves. Then Player 2*

wins any run of $G^\Gamma(\mathsf{Q} \to \mathsf{P})$ *that it plays according to* Λ *in at most* n *many moves.*

Proof. In the notation of the proof of Theorem 3.6, it is easy to see that $\Gamma \vdash \Omega_{n-1}$, and hence Player 2 has a winning strategy for $G^\Gamma(\mathsf{Q} \to \mathsf{P})$ that ensures victory in at most n many moves as in that proof. □

Remark 3.9. Hirst and Mummert [22] discussed a different potential form of instance-counting, based on a notion of proving a Π^1_2 principle P with one typical use of another Π^1_2 principle Q in a system Γ. While the definition of that notion in their paper is not quite correct [Hirst and Mummert, personal communication], its main significance is that it allowed them to conclude that, in cases of interest, Γ then proves that for every instance X of P, there is an instance Y of Q such that if Y has a solution then so does X. While their paper is mostly concerned with intuitionistic logic, they also gave examples showing that this notion does not seem useful in the context of classical logic. In particular they showed how RT^2_4 can be obtained with one typical use of RT^2_2 over RCA_0, contrary both to our intuition and to the fact that $\mathsf{RCA}_0 \nvdash^1 \mathsf{RT}^2_2 \to \mathsf{RT}^2_4$, which follows from Patey's result [30] that $\mathsf{RT}^2_4 \not\leqslant_\mathrm{c} \mathsf{RT}^2_2$. In fact, as conjectured by J. Miller [Hirst and Mummert, personal communication], this phenomenon is not a particularity of this and other examples mentioned in [22], but is in fact completely general. Indeed, in classical logic, if $\Gamma \vdash \mathsf{Q} \to \mathsf{P}$ then we can always argue in Γ as follows: Let X be an instance of P. Then there are i and Y such that either $i = 0$ and Y is a solution to X, or $i = 1$ and Y is an instance of Q with no solution. If $i = 1$ then we get a contradiction from one use of Q, so $i = 0$ and hence Y is a solution to X.

Perhaps more satisfying than the above argument is the following one, which is directly in the style of the one given in [22] for RT^2_2 and RT^2_4. Let Γ be as in Theorem 3.1, and let P and Q be Π^1_2-problems such that $\Gamma \vdash \mathsf{Q} \to \mathsf{P}$. Let Θ_n and Δ_n be as in Theorem 3.4. By that theorem, there is an n such that $\Gamma \vdash \Delta_n$. The following proof can be carried out in Γ.

Let X_0 be an instance of P. For each $k = 0, \dots, n$ in turn, proceed as follows. Given $X_0, \dots, X_k$, $e_0, \dots, e_{k-1}$, and $Y_0, \dots, Y_{k-1}$, let e_k and Y_k be such that

$$\forall X_{k+1}\, \exists e_{k+1}, Y_{k+1} \cdots \forall X_n\, \exists e_n, Y_n\, \Theta_n(e_0, \dots, e_n, X_0, \dots, X_n, Y_0, \dots, Y_n).$$

If Y_k is a solution to X_0 then let $Y = Y_k$ and let $i = 0$. Otherwise, Y_k is an

instance of Q. Either that instance has a solution or not. If it does not then let $Y = Y_k$ and let $i = 1$. If it does, then let X_{k+1} be such a solution.

By the definition of Θ_n, we must eventually define Y, i, and j. If $i = 1$ then Y is an instance of Q with no solution. But with one application of Q, we can obtain a solution to Y, so we must have $i = 0$, and hence Y is a solution to X_0.

4 Computable winning strategies

We now turn to the notion of generalized Weihrauch reducibility for games over possibly nonstandard models. Let Γ be a set of L_2-formulas consistent with Δ^0_1-comprehensiaon that proves the existence of a universal Σ^0_1 formula. Let P and Q be Π^1_2-problems. A *computable strategy* for Player 2 in $G^\Gamma(\mathsf{Q} \to \mathsf{P})$ or $\widehat{G}^\Gamma(\mathsf{Q} \to \mathsf{P})$ consists of Player 2 playing according to the formula $e = \Phi_k(n-1)$ (in the sense of Definition 3.5) for some $k \in \omega$.

Remark 4.1. To be precise, in the above definition we also need to have a mechanism to distinguish computably when Player 2 has played a winning move. Formally, we can simply slightly alter our games so that a move by Player 2 is either $\{n+1 : n \in Y\}$ where Y is a Q-instance or $\{0\} \cup \{n+1 : n \in Y\}$ where Y is a solution to Player 1's first move X_0.

Combining Theorem 3.6 and Proposition 3.8 gives us the following.

Proposition 4.2. *Let Γ be a consistent extension of Δ^0_1-comprehension that proves the existence of a universal Σ^0_1 formula, and let P and Q be Π^1_2-problems. Then the following are equivalent.*

(1) *Player 2 has a computable winning strategy for $G^\Gamma(\mathsf{Q} \to \mathsf{P})$.*

(2) *Player 2 has a computable winning strategy for $\widehat{G}^\Gamma(\mathsf{Q} \to \mathsf{P})$.*

(3) *There is an $n \in \omega$ such that Player 2 has a computable strategy for $G^\Gamma(\mathsf{Q} \to \mathsf{P})$ that ensures victory in at most n many moves.*

(4) *There is an $n \in \omega$ such that Player 2 has a computable strategy for $\widehat{G}^\Gamma(\mathsf{Q} \to \mathsf{P})$ that ensures victory in at most n many moves.*

Furthermore, n witnesses (3) *iff it witnesses* (4).

If the conditions in this proposition hold, then we say that P is *generalized Weihrauch reducible over* Γ to Q, and write $\mathsf{P} \leqslant_{\mathrm{gW}}^{\Gamma} \mathsf{Q}$. We can of course define an instance-counting version of this notion, writing $\mathsf{P} \leqslant_{\mathrm{gW}}^{\Gamma, n} \mathsf{Q}$ if $n+1$ witnesses that item (3) above holds.

As an example of the application of Proposition 4.2, we can obtain an analog of Corollary 3.3, using the fact that Hirschfeldt and Jockusch [19, Theorem 4.21] showed that $\mathsf{RT}^1_{<\infty} \not\leqslant_{\mathrm{gW}}^{n} \mathsf{RT}^1_k$ for all n, while Patey [29, Theorem 6.0.1] showed that the same holds for higher exponents. (Notice that Corollary 3.3 itself works only for exponent 2, since $\mathsf{RT}^1_{<\infty}$ is provable in $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$, while RT^n_k for $k > 1$ and $\mathsf{RT}^n_{<\infty}$ are both equivalent to ACA_0 over RCA_0 for $n > 2$, as shown by Simpson [32] using work of Jockusch [23].)

Corollary 4.3. *Let Γ consist of RCA_0 together with all Π^1_1 formulas true over the natural numbers. Then $\mathsf{RT}^n_{<\infty} \not\leqslant_{\mathrm{gW}}^{\Gamma} \mathsf{RT}^n_k$ for all n and k.*

Kuyper [25] studied a notion closely related to this kind of instance-counting (though he considered only the case where Γ is RCA_0). We give a slightly different definition that is easily seen to be equivalent to his.

Definition 4.4. Let P and Q be Π^1_2-problems. Say that P *Weihrauch-reduces to the composition of n many copies of Q via $e_0, \dots, e_n$* if for every $X_0, \dots, X_n$,

$$\begin{aligned}
&\text{if } X_0 \text{ is a } \mathsf{P}\text{-instance then}\\
&\qquad \Phi_{e_0}^{X_0} \text{ is a } \mathsf{Q}\text{-instance and if } X_1 \text{ is a solution to } \Phi_{e_0}^{X_0} \text{ then}\\
&\qquad\quad \Phi_{e_1}^{X_0 \oplus X_1} \text{ is a } \mathsf{Q}\text{-instance and if } X_2 \text{ is a solution to } \Phi_{e_1}^{X_0 \oplus X_1} \text{ then}\\
&\qquad\qquad \vdots\\
&\qquad\quad \Phi_{e_{n-1}}^{X_0 \oplus \cdots \oplus X_{n-1}} \text{ is a } \mathsf{Q}\text{-instance and if } X_n \text{ is a solution to } \Phi_{e_{n-1}}^{X_0 \oplus \cdots \oplus X_{n-1}} \text{ then}\\
&\qquad\qquad \Phi_{e_n}^{X_0 \oplus \cdots \oplus X_n} \text{ is a solution to } X_0.
\end{aligned}$$

(Note that in the $n = 0$ case, this statement becomes

$$\text{if } X_0 \text{ is a } \mathsf{P}\text{-instance then } \Phi_{e_0}^{X_0} \text{ is a solution to } X_0.)$$

Kuyper considered the situation where there are $n \in \omega$ and $e_0, \dots, e_n \in \omega$ such that RCA_0 proves that P Weihrauch-reduces to the composition of n many copies of Q via $e_0, \dots, e_n$. For a fixed n, it is not difficult to see that this condition is equivalent to saying that Player 2 has a a computable winning strategy for $G^{\Gamma}(\mathsf{Q} \to \mathsf{P})$ that ensures victory in exactly $n+1$ many

moves, unless it wins earlier by playing an instance of Q with no solution. One might think that this is the same as saying that there is an n such that $\mathsf{P} \leqslant_{\mathrm{gW}}^{\mathsf{RCA}_0,\, n} \mathsf{Q}$, and hence by Proposition 4.2 to $\mathsf{P} \leqslant_{\mathrm{gW}}^{\mathsf{RCA}_0} \mathsf{Q}$, but Remark 3.7 applies here as well. The example given there shows that it is possible to have $\mathsf{P} \leqslant_{\mathrm{gW}}^{\mathsf{RCA}_0,\, 1} \mathsf{Q}$ but not have Kuyper's condition hold. However, Kuyper's condition is equivalent to $\mathsf{P} \leqslant_{\mathrm{gW}}^{\mathsf{RCA}_0} \widehat{\mathsf{Q}}$ for the modified problem $\widehat{\mathsf{Q}}$ defined in that remark, so we we will express it in this form.

Kuyper [25] claimed that his condition is equivalent to a form of intuitionistically provable implication. Uftring [35, 36] found a counterexample that shows that Kuyper's argument is flawed. Kuyper (see [35, 36]) proposed fixing his proof by replacing the condition $\mathsf{P} \leqslant_{\mathrm{gW}}^{\mathsf{RCA}_0} \widehat{\mathsf{Q}}$ with $\mathsf{P} \leqslant_{\mathrm{gW}}^{\mathsf{RCA}_0 + \mathsf{Q}} \widehat{\mathsf{Q}}$. Uftring's example shows that it is possible for Player 2 to have a computable winning strategy for $G^{\mathsf{RCA}_0+\mathsf{Q}}(\mathsf{Q} \to \mathsf{P})$ but not for $G^{\mathsf{RCA}_0}(\mathsf{Q} \to \mathsf{P})$, in contrast with the case for general winning strategies in Proposition 2.4, so we present a version of it now. We will give another example with the same properties in Section 6.

Example 4.5 (Uftring [35, 36])**.** The proof of Gödel's Incompleteness Theorem shows that there is a primitive recursive predicate G such that $G(n)$ holds for all $n \in \omega$ but RCA_0 cannot prove $\forall x\, G(x)$. For $X \neq \emptyset$, write μX for the least element of X. Let

$$\mathsf{P} \equiv \forall X\, \exists Y\, \forall x\, G(x)$$

and

$$\mathsf{Q} \equiv \forall X\, [X \neq \emptyset \to \exists Y\, G(\mu X)].$$

In $G^{\mathsf{RCA}_0+\mathsf{Q}}(\mathsf{Q} \to \mathsf{P})$, Player 1's first move M and X_0 must be such that $M[X_0]$ is consistent with Q, so $M[X_0] \vDash \forall x\, G(x)$, and hence Player 2 can play, say, $\emptyset$ on its first move and win. In $G^{\mathsf{RCA}_0}(\mathsf{Q} \to \mathsf{P})$, however, Player 1 can play an $M \vDash \neg\forall x\, G(x)$, together with, say, $X_0 = \emptyset$. Then this instance of P has no solution, so the only way for Player 2 to win is eventually to play an M-instance of Q with no solution, that is, an X such that $M \vDash \neg G(\mu X)$.

For any model M of Σ^0_1-PA, we can consider a run in which Player 1 plays M and then keeps playing $\emptyset$ until Player 2 either declares victory or wins by playing an M-instance of Q with no solution. (Notice that we can computably determine if the latter case holds, since the condition $G(\mu X)$ is computable.) If Player 2 has a computable winning strategy for $G^{\mathsf{RCA}_0}(\mathsf{Q} \to \mathsf{P})$, then there is a computable procedure that, over any model M of Σ^0_1-PA, simulates the

above run, making Player 2's moves according to this procedure, outputting 0 if Player 2 declares victory, and outputting μX if Player 2 plays the M-instance X of Q with no solution. The output of this procedure is 0 iff $M \vDash \forall x\, G(x)$. Since this procedure works for any model M of Σ^0_1-PA, we have an existential first-order sentence that is provably equivalent to $\forall x\, G(x)$ over RCA_0, which is a contradiction, because any existential first-order sentence true in the standard natural numbers is provable in RCA_0.

For some Π^1_2-problems Q, on the other hand, there is no difference between $G^{\mathsf{RCA}_0+\mathsf{Q}}(\mathsf{Q} \to \mathsf{P})$ and $G^{\mathsf{RCA}_0}(\mathsf{Q} \to \mathsf{P})$ because every countable model of RCA_0 can be extended to a countable model of $\mathsf{RCA}_0 + \mathsf{Q}$ with the same first-order part, and hence the notion of consistency used in Definition 2.1 is the same for RCA_0 and $\mathsf{RCA}_0 + \mathsf{Q}$. (Showing that this is the case for a given Q is typically done to show that Q is Π^1_1-conservative over RCA_0.) Examples include WKL, as shown by Harrington (see [33, Theorem IX.2.1]), COH, as shown by Cholak, Jockusch, and Slaman [10], and AMT, as shown by Hirschfeldt, Shore, and Slaman [20].

As highlighted by the work of Kuyper and Uftring, the connections with intuitionistic provability are rather subtle, and we believe that generalized Weihrauch reducibility over possibly nonstandard models can be useful in clarifying them. However, as the methods and issues are rather different from the ones in this paper, we leave this work to a future one.

5 Single-instance reductions

As noted in Remark 3.7, $\mathsf{P} \leqslant_{\mathrm{c}} \mathsf{Q}$ iff Player 2 has a strategy for $G(\mathsf{Q} \to \mathsf{P})$ that ensures victory in exactly two moves. Similarly, $\mathsf{P} \leqslant_{\mathrm{W}} \mathsf{Q}$ iff Player 2 has a computable strategy for $G(\mathsf{Q} \to \mathsf{P})$ that ensures victory in exactly two moves. We can define the analogous notions for games over possibly nonstandard models. Let us explicitly define these analogs for computable and Weihrauch reducibilities, and then look at several examples involving them. Although we will not work with them in this paper, we also define the analogs of several related notions of computability-theoretic reduction between Π^1_2-problems.

Definition 5.1. Let Γ be a set of L_2-formulas consistent with Δ^0_1-comprehension that proves the existence of a universal Σ^0_1 formula and let P and Q be Π^1_2-problems.

1. We say that P is *computably reducible over* Γ to Q, and write $\mathsf{P} \leqslant_{\mathrm{c}}^{\Gamma} \mathsf{Q}$, if for every model (M, S) of Γ with M countable and S closed under Δ^0_1-comprehension, and every M-instance X of P in S, there is an M-instance $\widehat{X}$ of Q in $M[X]$ such that for every solution $\widehat{Y}$ to $\widehat{X}$ in S, there is a solution to X in $M[X, \widehat{Y}]$.

2. We say that P is *Weihrauch reducible over* Γ to Q, and write $\mathsf{P} \leqslant_{\mathrm{W}}^{\Gamma} \mathsf{Q}$, if there are $e, i \in \omega$ such that for every model (M, S) of Γ with M countable and S closed under Δ^0_1-comprehension, and every M-instance X of P in S, the set $\widehat{X} = \Phi_e^X$ is an M-instance of Q, and for every solution $\widehat{Y}$ to $\widehat{X}$ in S, the set $\Phi_i^{X \oplus \widehat{Y}}$ is a solution to X.

3. We say that P is *strongly computably reducible over* Γ to Q, and write $\mathsf{P} \leqslant_{\mathrm{sc}}^{\Gamma} \mathsf{Q}$, if for every model (M, S) of Γ with M countable and S closed under Δ^0_1-comprehension, and every M-instance X of P in S, there is an M-instance $\widehat{X}$ of Q in $M[X]$ such that for every solution $\widehat{Y}$ to $\widehat{X}$ in S, there is a solution to X in $M[\widehat{Y}]$.

4. We say that P is *strongly Weihrauch reducible over* Γ to Q, and write $\mathsf{P} \leqslant_{\mathrm{sW}}^{\Gamma} \mathsf{Q}$, if there are $e, i \in \omega$ such that for every model (M, S) of Γ with M countable and S closed under Δ^0_1-comprehension, and every M-instance X of P in S, the set $\widehat{X} = \Phi_e^X$ is an M-instance of Q, and for every solution $\widehat{Y}$ to $\widehat{X}$ in S, the set $\Phi_i^{\widehat{Y}}$ is a solution to X.

5. We say that P is *omnisciently computably reducible over* Γ to Q, and write $\mathsf{P} \leqslant_{\mathrm{oc}}^{\Gamma} \mathsf{Q}$, if for every model (M, S) of Γ with M countable and S closed under Δ^0_1-comprehension, and every M-instance X of P in S, there is an M-instance $\widehat{X}$ of Q in S such that for every solution $\widehat{Y}$ to $\widehat{X}$ in S, there is a solution to X in $M[X, \widehat{Y}]$.

6. We say that P is *omnisciently Weihrauch reducible over* Γ to Q, and write $\mathsf{P} \leqslant_{\mathrm{oW}}^{\Gamma} \mathsf{Q}$, if there is an $i \in \omega$ such that for every model (M, S) of Γ with M countable and S closed under Δ^0_1-comprehension, and every M-instance X of P in S, there is an M-instance $\widehat{X}$ of Q in S such that for every solution $\widehat{Y}$ to $\widehat{X}$ in S, the set $\Phi_i^{X \oplus \widehat{Y}}$ is a solution to X.

7. We say that P is *strongly omnisciently computably reducible over* Γ to Q, and write $\mathsf{P} \leqslant_{\mathrm{soc}}^{\Gamma} \mathsf{Q}$, if for every model (M, S) of Γ with M countable and S closed under Δ^0_1-comprehension, and every M-instance X of P

in S, there is an M-instance $\widehat{X}$ of Q in S such that for every solution $\widehat{Y}$ to $\widehat{X}$ in S, there is a solution to X in $M[\widehat{Y}]$.

8. We say that P is *strongly omnisciently Weihrauch reducible over* Γ to Q, and write $\mathsf{P} \leqslant^{\Gamma}_{\mathrm{soW}} \mathsf{Q}$, if there is an $i \in \omega$ such that for every model (M, S) of Γ with M countable and S closed under Δ^0_1-comprehension, and every M-instance X of P in S, there is an M-instance $\widehat{X}$ of Q in S such that for every solution $\widehat{Y}$ to $\widehat{X}$ in S, the set $\Phi_i^{\widehat{Y}}$ is a solution to X.

Remark 5.2. In light of comments made above, it might be more natural to consider versions corresponding to games in which Player 2 can always win in one or two moves, rather than exactly two moves (even if in natural cases, there will be no difference). Rather than introduce more terminology and notation, however, that can be done simply by replacing Q with the problem $\widehat{\mathsf{Q}}$ from Remark 3.7 in the above definitions.

The study of Weihrauch reducibility in this extended setting seems particularly promising, given the extensive theory that has been developed for Weihrauch reducibility over the standard natural numbers. In particular, there are several operators on the Weihrauch degrees whose analogs in this setting should be of interest. One example is the finite parallelization: For a problem P, the problem P^* is the one whose instances consist of finitely many instances $X_0, \dots, X_k$ of P, with a solution consisting of one solution to each X_i. Clearly, $\mathsf{P}^* \leqslant_{\mathrm{gW}} \mathsf{P}$ for any Π^1_2-problem P, but this fact does not hold in our setting, because given an instance $X_0, \dots, X_k$ of P^*, the obvious reduction strategy for Player 2 takes $k+1$ many moves, and k might be nonstandard. The following example will be relevant in the next section.

Example 5.3. Pauly, Fouché, and Davie [31] defined Bound as follows: An instance is an enumeration of a bounded set F, and a solution is a bound on the elements of F. An instance of Bound^* is then a simultaneous enumeration of a finite family $F_0, \dots, F_k$ of bounded sets, and a solution to this instance consists of a bound for each F_i, or, equivalently, a bound b on $\bigcup_{i \leqslant k} F_k$. (This is basically the principle FUF studied by Frittaion and Marcone [16].) It is easy to see that Bound and Bound^* are Weihrauch-equivalent, but that is no longer the case for Weihrauch-equivalence (or even provable equivalence) over RCA_0, since as statements in second-order arithmetic, Bound is trivially true, while Bound^* is a way to state $\mathsf{B}\Pi^0_1$, and hence is equivalent to $\mathsf{B}\Sigma^0_2$ over RCA_0,

as we further discuss in the following section. Thus $\mathsf{RCA}_0 \nvdash \mathsf{Bound} \to \mathsf{Bound}^*$, and hence $\mathsf{Bound}^* \not\leqslant_{\mathrm{gW}}^{\mathsf{RCA}_0} \mathsf{Bound}$.

It is not clear what the correct generalization of the $^\diamond$ operator of Neumann and Pauly [28] to this setting is. However, one would expect that it would still have the property that P^* is reducible to $\mathsf{P}^\diamond$, and hence, by the above example, that it would no longer be equivalent to gW-reducibility.

On the other hand, it is clear that, as for standard Weihrauch reducibility, if $\mathsf{P} \leqslant_{\mathrm{W}}^{\mathsf{RCA}_0} \mathsf{Q}$ then $\mathsf{P}^* \leqslant_{\mathrm{W}}^{\mathsf{RCA}_0} \mathsf{Q}^*$. It is also not difficult to see that, more generally, if $\mathsf{P} \leqslant_{\mathrm{gW}}^{\mathsf{RCA}_0, n} \mathsf{Q}$ then $\mathsf{P}^* \leqslant_{\mathrm{gW}}^{\mathsf{RCA}_0, n} \mathsf{Q}^*$. Thus, by Proposition 4.2, if $\mathsf{P} \leqslant_{\mathrm{gW}}^{\mathsf{RCA}_0} \mathsf{Q}$ then $\mathsf{P}^* \leqslant_{\mathrm{gW}}^{\mathsf{RCA}_0} \mathsf{Q}^*$. (The same holds for other appropriate systems in place of RCA_0, of course.)

An important point here is that while the principles we consider in reverse mathematics are typically true—in the sense that they hold in $(\omega, \mathcal{P}(\omega))$, or equivalently for Π^1_2-problems, that every instance (over the standard natural numbers) has at least one solution—many of them have nontrivial first-order parts. For example, if $\mathsf{B}\Sigma^0_2$ fails in M, then M cannot be the first-order part of a model of $\mathsf{RCA}_0 + \mathsf{RT}^1_{<\infty}$ (or of $\mathsf{RCA}_0 + \mathsf{RT}^n_k$ for any $n, k \geqslant 2$). Furthermore, for any such M there is an instance of $\mathsf{RT}^1_{<\infty}$ (i.e., a $k \in |M|$ together with a function $c : |M| \to \{j \in |M| : j <^M k\}$) with no solutions. The same is true of Bound^*, to give another example.

We want to use notions such as Weihrauch reducibility over RCA_0 and other systems to study these kinds of principles (as we will do in the next two sections), so it is important that our definitions above do not assume that every instance of a problem has a solution. This fact is particularly worth noting for Weihrauch reducibility, because we usually think of (classical) Weihrauch reducibility between Π^1_2-problems as a special case of the general notion from computable analysis, which is defined using partial multifunctions between represented spaces. (See for instance Brattka, Gherardi, and Pauly [7] or Brattka and Pauly [8].) This point is a bit subtle, and was missed, e.g., in the paper Dorais, Dzhafarov, Hirst, Mileti, and Shafer [13], where a proof is given in Corollaries A.3 and A.4 establishing a correspondence between Π^1_2 principles on the one hand and certain classes of partial multifunctions on the other. Indeed, the proof there works only if the Π^1_2 principles in question are assumed to be true, which is not explicitly mentioned.

There is more than one way to formalize the notion of a partial multifunction between spaces X and Y. One is to say that it is simply a relation

$R \subseteq X \times Y$. Then the domain of the multifunction is $\{x \in X : \exists y\,(x, y) \in R\}$. Another is to say that it is a (possibly partial) function from X to the power set of Y. In this case, the domain of the multifunction can include elements that are mapped to no values at all. The first formalization is the one normally used in the definition of Weihrauch reducibility in computable analysis, which is convenient in particular because of the need to use choice functions in working with represented spaces. And indeed, a true Π^1_2-problem P corresponds to the partial multifunction $F : \subseteq 2^\omega \rightrightarrows 2^\omega$ in this sense whose domain is the set of instances of P, and which maps any such instance X to the solutions to X.

This correspondence breaks down for a Π^1_2-problem that has instances with no solutions, however, unless we move to the second formalization of the notion of multifunction, or allow a multifunction to consist of a relation $R \subseteq X \times Y$ together with a set D such that $\{x \in X : \exists y\,(x, y) \in R\} \subseteq D \subseteq X$, where D represents the domain of the function. This distinction operates even at the level of the Weihrauch degrees (equivalence classes under Weihrauch reducibility), because a problem in which some instance has no solutions can never be Weihrauch reducible to one in which every instance has a solution, and if P has a computable instance with no solutions, then every problem is Weihrauch reducible to P. As discussed in [7], and in more detail in [8], this top degree is usually added to the lattice of Weihrauch degrees as a formal object.

The distinction between the two approaches is also relevant to the notion of *extended Weihrauch reducibility* investigated by Bauer [2] (see also [3]), following work by Bauer and Yoshimura [4, 5]. The focus in that work is on comparing universally quantified statements in the setting of constructive mathematics, using a notion called *instance reducibility*, which can also be understood as an extension of the Weihrauch degrees that in particular allows for "questions that do not have an answer" but that are still "valid" for the purposes of considering whether or not they are reducible to other questions (Bauer [1]).

6 Limit-homogeneous sets

In this section and the next, we give some examples of comparisons of Π^1_2-problems using W- and gW-reducibility over possibly nonstandard models, focusing on versions of $\mathsf{B}\Sigma^0_2$. A natural way to think of $\mathsf{B}\Pi^0_1$ as a Π^1_2-problem

is to identify a Π^0_1 formula $\varphi(i,k)$ with a simultaneous enumeration of the sets $\{m : \forall k < m\, \neg\varphi(i,k)\}$ for $i < n$. Then a b as in the definition of $\mathsf{B}\Pi^0_1$ is the same as a common bound for these sets. Thus we arrive at Bound^*, as defined in Example 5.3.

Recall also the Π^1_2-problems SRT^2_2 and D^2_2 from Definition 1.1. Clearly, SRT^2_2 implies D^2_2. Cholak, Jockusch, and Slaman [10] claimed that the converse implication also holds over RCA_0, but their proof actually required $\mathsf{B}\Sigma^0_2$. Chong, Lempp, and Yang [11] closed this gap by showing that D^2_2 implies $\mathsf{B}\Sigma^0_2$ over RCA_0.

The argument in [10] also shows that $\mathsf{SRT}^2_2 \leqslant_{\mathrm{c}} \mathsf{D}^2_2$. Dzhafarov [15] and Brattka and Rakotoniaina [9] showed that $\mathsf{SRT}^2_2 \not\leqslant_{\mathrm{W}} \mathsf{D}^2_2$. Hirschfeldt and Jockusch [19] noted that $\mathsf{SRT}^2_2 \leqslant^2_{\mathrm{gW}} \mathsf{D}^2_2$, however. To consider this reduction in more detail, we define the following Π^1_2-problem.

Definition 6.1. LH: If $c : [\mathbb{N}]^2 \to 2$ is such that $\lim_y c(x,y) = 1$ for all x, then c has an infinite homogeneous set.

This problem is a convenient way to state the principle that for every 2-coloring of pairs, every infinite limit-homogeneous set has an infinite homogeneous subset.

From the reverse-mathematical perspective, LH is equivalent to $\mathsf{B}\Sigma^0_2$.

Proposition 6.2. $\mathsf{RCA}_0 \vdash \mathsf{LH} \leftrightarrow \mathsf{B}\Sigma^0_2$.

Proof. First, assume $\mathsf{B}\Sigma^0_2$. Fix an instance c of LH. Let S be the set of all tuples $(x_0, \dots, x_{n-1}, y)$ such that $x_0 < \cdots < x_{n-1} < y$ and $c(x_m, y) = 1$ for all $m < n$. We claim that for all $x_0 < \cdots < x_{n-1}$, there is a y such that $(x_0, \dots, x_{n-1}, y) \in S$. For each $m < n$ there is a $b_m > x_{n-1}$ such that $c(x_m, y) = 1$ for all $y > b_m$. By $\mathsf{B}\Sigma^0_2$ (or really $\mathsf{B}\Pi^0_1$), there is a $b > x_{n-1}$ such that $c(x_m, y) = 1$ for all $m < n$ and $y > b$. Then $(x_0, \dots, x_{n-1}, b+1) \in S$, which proves our claim. Now we can define a homogeneous set H for c by primitive recursion: Let $h_0 = 0$, let h_{n+1} be the least y such that $(h_0, \dots, h_n, y) \in S$, and let $H = \{h_0, h_1, \dots\}$.

Now assume LH. We prove $\mathsf{RT}^1_{<\infty}$. Assume for a contradiction that $d : \mathbb{N} \to k$ has no infinite homogeneous set. Then for each $i < k$ there is a b such that $d(x) \neq i$ for all $x > b$. Define $c : [\mathbb{N}]^2 \to 2$ by letting $c(x,y) = 0$ if $d(x) = d(y)$ and letting $c(x,y) = 1$ otherwise. Then $\lim_y c(x,y) = 1$ for all x, so by LH, c has an infinite homogeneous set H. Let $x_0 < \cdots < x_k \in H$. Then for all $m < n \leqslant k$, we have that $c(x_m, x_n) = 1$ and hence $d(x_m) \neq d(x_n)$. But then $\{d(x_0), \dots, d(x_k)\}$ has cardinality $k+1$, which is impossible. $\square$

However, the first part of the above proof shows that LH is computability-theoretically trivial, and indeed uniformly computably true, so that $\mathsf{LH} \leqslant^0_{\mathrm{gW}} \mathsf{P}$ for any P, or equivalently $\mathsf{LH} \leqslant_{\mathrm{W}} \mathsf{1}$, where $\mathsf{1}$ is the identity problem for which an instance is any X and the only solution to this instance is X itself. We can obtain SRT^2_2 from D^2_2 as follows: Given a stable coloring $c : \mathbb{N} \to 2$, use D^2_2 to obtain a limit-homogeneous set L. Now an application of RT^1_2 (which is Weihrauch-reducible to D^2_2) yields an i such that $\lim_{y \in L} c(x, y) = i$ for all $x \in L$. We can think of c restricted to L as a coloring of $\mathbb{N}$ by identifying the nth element of L with n. If $i = 0$, we can also replace c by the coloring whose value at (x, y) is $1 - c(x, y)$. We can then apply LH to obtain an infinite homogeneous set for c. Since LH is Weihrauch-trivial, this procedure shows that $\mathsf{SRT}^2_2 \leqslant^2_{\mathrm{gW}} \mathsf{D}^2_2$. (Since the use of RT^1_2 is computably trivial, it also shows that $\mathsf{SRT}^2_2 \leqslant_{\mathrm{c}} \mathsf{D}^2_2$, as mentioned above.)

Over nonstandard models, however, things are different. In the presence of $\mathsf{B}\Sigma^0_2$, the first part of the proof of Proposition 6.2 shows that LH is still Weihrauch-trivial, i.e., $\mathsf{LH} \leqslant^{\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2}_{\mathrm{W}} \mathsf{1}$, and hence $\mathsf{SRT}^2_2 \leqslant^{\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2,\, 2}_{\mathrm{gW}} \mathsf{D}^2_2$. Of course, if P does not imply $\mathsf{B}\Sigma^0_2$ over RCA_0, then we cannot have $\mathsf{LH} \leqslant^{\mathsf{RCA}_0}_{\mathrm{gW}} \mathsf{P}$. But what if we take P to be some form of $\mathsf{B}\Sigma^0_2$? A natural choice is Bound^*, as it is essentially the form of $\mathsf{B}\Pi^0_1$ used in the first part of the proof of Proposition 6.2.

We will show that $\mathsf{LH} \not\leqslant^{\mathsf{RCA}_0}_{\mathrm{gW}} \mathsf{Bound}^*$, but we can actually obtain a stronger result by considering the contrapositive form of $\mathsf{B}\Pi^0_1$: Given a simultaneous enumeration of sets $F_0, \ldots, F_{n-1}$ with no common bound, there is an $i < n$ such that F_i is infinite. Given such an enumeration, we can define an n-coloring c of $\mathbb{N}$ as follows: for each m, wait until a number greater than m is enumerated into some F_i, then give m the color i. From an infinite homogeneous set for c, we can obtain an $i < n$ such that F_i is infinite. Conversely, given an n-coloring c of $\mathbb{N}$, the sets $F_i = \{m : c(m) = i\}$ for $i < n$ have no common bound, and from an $i < n$ such that F_i is infinite, we can obtain an infinite homogeneous set for c. Both of these processes can be carried out over RCA_0, so up to Weihrauch equivalence over RCA_0, the contrapositive form of $\mathsf{B}\Pi^0_1$ is $\mathsf{RT}^1_{<\infty}$, in the form in which it is usually stated as a Π^1_2-problem, in which an instance consists of a k-coloring of $\mathbb{N}$ together with the number k.

Remark 6.3. The above argument (which we heard from Pauly [personal communication]) also gives a simple proof of Hirst's result from [21] (see also [18, Theorem 6.81]) that $\mathsf{B}\Sigma^0_2$ and $\mathsf{RT}^1_{<\infty}$ are equivalent over RCA_0.

There is a stronger form of $\mathsf{RT}^1_{<\infty}$, which we will call $\mathsf{stRT}^1_{<\infty}$, in which the number of colors is not part of the instance. That is, an instance consists of a function $\mathbb{N} \to \mathbb{N}$ with bounded range (and a solution is still an infinite homogeneous set). As shown by Brattka and Rakotoniaina [9], and also noted by Hirschfeldt and Jockusch [19], $\mathsf{RT}^1_{<\infty} <_{\mathrm{W}} \mathsf{stRT}^1_{<\infty}$. In this section, we show that $\mathsf{LH} \not\leqslant^{\mathsf{RCA}_0}_{\mathrm{gW}} \mathsf{stRT}^1_{<\infty}$. We will show in Proposition 7.6 that $\mathsf{Bound}^* \leqslant^{\mathsf{RCA}_0}_{\mathrm{gW}} \mathsf{stRT}^1_{<\infty}$, so this result implies that $\mathsf{LH} \not\leqslant^{\mathsf{RCA}_0}_{\mathrm{gW}} \mathsf{Bound}^*$, but we also give a direct proof of the latter fact, which uses the same technique but is simpler.

Both proofs will use the following notion of forcing.

Definition 6.4. Let N be an L_1-structure. We define a notion of forcing P_N as follows. (If N is the standard natural numbers then we denote this notion by P_ω.) Write $[m]^2$ for the set of $(x, y) \in [|N|]^2$ such that $x, y <^N m$. A condition is an N-finite function of the form $p : [m]^2 \to 2$ for some $m \in |N|$. Say that a condition q extends such a p if q extends p as a function and $q(i, j) = 1$ for all $i <^N m$ and $j \geqslant^N m$ on which it is defined. Define the notion of $c : [|N|]^2 \to 2$ extending p in the same way. (Notice that if for every $m \in |N|$ there is a condition $p : [m]^2 \to 2$ such that c extends p, then c is an N-instance of LH.)

We will also use the following fact. (A 1-*elementary extension* of a structure N is an extension of N that satisfies exactly the same existential sentences with parameters from N.)

Lemma 6.5. *There is a* 1*-elementary extension M of the standard natural numbers such that for the collection S of all subsets of $|M|$ that are Δ^0_1-definable over M,*

1. *(M, S) is a model of RCA_0 and*
2. *for any condition p for the notion of forcing P_M, there is an M-instance of LH in S that extends p (in the sense of Definition 6.4) and has no solution in S.*

Proof. Let N be any nonstandard elementary extension of the standard natural numbers, and let $a \in N$ be a nonstandard element. Then in particular $N \vDash \mathsf{I}\Sigma^0_2$, and so

$$M = \{x \in N : x \text{ is } \Sigma^0_2\text{-definable in } (N, a)\}$$

is a model of $\mathsf{I}\Sigma^0_1 + \neg\mathsf{B}\Sigma^0_2$ which is a 1-elementary (in fact, 2-elementary) substructure of N. (See Hájek and Pudlak [17, Theorem IV.1.33] or Kossak [24, p. 223].) Thus, M is a 1-elementary extension of the standard model, and for S as in the statement, (M, S) is a model of RCA_0. Since $\mathsf{B}\Sigma^0_2$ fails in M, it follows by Proposition 6.2 that LH fails in (M, S). Fix an instance $c : [M]^2 \to 2$ of LH in S with no solution in S. Then given a condition $p : [m]^2 \to 2$ for P_M, we can define $d : [M]^2 \to 2$ by

$$d(x,y) = \begin{cases} p(x,y) & \text{if } x, y < m, \\ 1 & \text{if } x < m \text{ and } y \geqslant m, \\ c(x,y) & \text{otherwise.} \end{cases}$$

Clearly, d is in S and is an instance of LH that extends p. But if H is any solution to d then $\{x \in H : x \geqslant m\}$ is a solution to c, so d cannot have any solution in S. □

Proposition 6.6. $\mathsf{LH} \not\leqslant^{\mathsf{RCA}_0}_{\mathrm{gW}} \mathsf{Bound}^*$.

Proof. Assume for a contradiction that $\mathsf{LH} \leqslant^{\mathsf{RCA}_0}_{\mathrm{gW}} \mathsf{Bound}^*$. By Proposition 4.2, there is an $n \in \omega$ such that Player 2 has a computable strategy for $\widehat{G}^{\mathsf{RCA}_0}(\mathsf{Bound}^* \to \mathsf{LH})$ that ensures victory in at most n many moves. Fix such a strategy.

For a condition $p : [j]^2 \to 2$ for the notion of forcing P_ω, we can consider what happens when our fixed strategy for Player 2 is applied to a run in which Player 1 plays $(\omega, \mathcal{P}(\omega))$ and p as a partial first move. Unless the strategy declares victory on its first move, it must play part of an instance of Bound^*, which is just a simultaneous enumeration of a finite family of sets. We may assume by the usual convention on uses that no number greater than j is enumerated. Let b^p_0 be the least bound on the set of all numbers enumerated in this way. Now, if Player 1 plays b^p_0, then unless our strategy declares victory on its second move, it again must play part of an instance of Bound^*, yielding an analogous bound b^p_1. Continuing in this way, we obtain numbers $b^p_0, b^p_1, \ldots, b^p_k$ for some $k < n$. Let $b^p_i = 0$ for $k < i < n$.

For $i < n$ and $m \in \omega$, let $D_{i,m}$ be the set of conditions p such that $b^p_i \geqslant m$. If some $D_{0,m}$ is not dense then let m_0 be the least such m. In this case, there is a condition $p_0 \in D_{0,m_0-1}$ with no extension in D_{0,m_0}. Notice that $b^q_0 = m_0 - 1$ for all extensions q of p_0. Now, if some $D_{1,m}$ is not dense below p_0 then let m_1 be the least such m. In this case, there is an extension of

p_0 in D_{1,m_1-1} with no extension in D_{1,m_1}. Proceeding in this way, we obtain a condition p such that either m_i is defined for every $i < n$, or there is a $k < n$ such that m_i is defined for all $i < k$ and every $D_{k,m}$ is dense below p. In either case, $b_i^q = m_i - 1$ for all extensions q of p and all i such that m_i is defined.

We claim that the latter case cannot hold. Suppose otherwise. Let c be an instance of LH that extends p and meets every $D_{k,m}$ (i.e., every $D_{k,m}$ contains a q such that c extends q). Then Player 1 can play $(\omega, \mathcal{P}(\omega))$ and c on its first move, and if Player 2 follows our fixed strategy, then the moves $m_0, m_1, \ldots, m_{k-1}$ will be legal for Player 1 (as otherwise some finite portion of c is a condition q extending p with $b_i^q > m_i$ for some $i < k$). But then Player 2's $(k+1)$st move is not an instance of Bound^*.

Thus each m_i for $i < n$ is defined, and we have the following for our fixed condition p:

$$\forall q \, \forall i < n \, [\text{if } q \text{ extends } p \text{ then } b_i^q = m_i - 1]. \tag{6.1}$$

Now let M and S be as in Lemma 6.5. Then p is also a condition for P_M, so there is an M-instance d of LH in S that extends p and has no solution in S. But it is easy to check that (6.1) is a Π^0_1 statement, so since M is a 1-elementary extension of the standard natural numbers, it also holds over M. So Player 1 can play (M, S) and d on its first move, and if Player 2 follows our fixed strategy, then the moves $m_0, m_1, \ldots, m_{n-1}$ will be legal for Player 1 (as otherwise some finite portion of d is a condition q extending p with $b_i^q >^M m_i$ for some $i < n$). But then Player 2 has not won the game by the nth move (since the only way for Player 2 to win this run of the game is to play an M-instance of Bound^* with no solution), contrary to assumption. $\square$

Thus LH and Bound^* constitute a natural example of the phenomenon witnessed by Uftring's Example 4.5.

We can also interpret the fact that $\mathsf{LH} \leqslant_{\mathrm{W}}^{\mathsf{RCA}_0 + \mathsf{Bound}^*} \mathsf{1}$ but $\mathsf{LH} \not\leqslant_{\mathrm{gW}}^{\mathsf{RCA}_0} \mathsf{Bound}^*$ as saying that the use of Bound^* in the first part of the proof of Proposition 6.2 is "purely proof-theoretic". It neither requires a further "computability-theoretic application" of Bound^* nor can be replaced by one or more such applications (in the uniform setting). Uncovering this kind of information seems to be a promising aspect of this approach to calibrating the logical strength of Π^1_2-problems.

Proposition 6.6 does not show that $\mathsf{SRT}^2_2 \not\leqslant_{\mathrm{gW}}^{\mathsf{RCA}_0} \mathsf{D}^2_2$, but it suggests that this might well be the case, which would provide an even more natural version of Example 4.5, and show that the proof of SRT^2_2 from D^2_2 necessarily makes

both computability-theoretic and further proof-theoretic use of D^2_2. Indeed, it even seems possible that $\mathsf{LH} \not\leqslant^{\mathsf{RCA}_0}_{\mathrm{gW}} \mathsf{D}^2_2$.

Question 6.7. Is $\mathsf{SRT}^2_2 \leqslant^{\mathsf{RCA}_0}_{\mathrm{gW}} \mathsf{D}^2_2$? Is $\mathsf{LH} \leqslant^{\mathsf{RCA}_0}_{\mathrm{gW}} \mathsf{D}^2_2$?

We now strengthen Proposition 6.6 as described above.

Proposition 6.8. $\mathsf{LH} \not\leqslant^{\mathsf{RCA}_0}_{\mathrm{gW}} \mathsf{stRT}^1_{<\infty}$.

Proof. Assume for a contradiction that $\mathsf{LH} \leqslant^{\mathsf{RCA}_0}_{\mathrm{gW}} \mathsf{stRT}^1_{<\infty}$. By Proposition 4.2, there is an $n \in \omega$ such that Player 2 has a computable strategy for $\widehat{G}^{\mathsf{RCA}_0}(\mathsf{stRT}^1_{<\infty} \to \mathsf{LH})$ ensuring victory in at most n many moves. There is then also a strategy that ensures victory in exactly n many moves, since Player 2 can extend the length of any game by playing computable (Δ^0_1-definable) instances of $\mathsf{stRT}^1_{<\infty}$ on all its moves from some point on. Fix such a strategy, and for notational convenience, assume $n > 1$.

We begin as in the previous proof by considering games over the standard natural numbers. Note that if Player 2 plays according to its strategy and does not declare victory on some move, then it has to play an instance of $\mathsf{stRT}^1_{<\infty}$ only provided all of Player 1's moves so far have been legal. However, since every set can be viewed as a coloring $\omega \to \omega$ (not necessarily with bounded range), we can always assume that Player 2 plays such a coloring. This coloring may be partial, however, in which case by usual use conventions we can assume it is defined on a finite initial segment of ω.

Fix a condition p for the notion of forcing P_ω. For each $\alpha \in \omega^{\leqslant n-2}$, we define a coloring f^p_α of a finite initial segment of ω. Having done so, we let $H^p_{\alpha^\frown v}$ for each $v \in \omega$ be the set of all x such that $f^p_\alpha(x) = v$. We start with α equal to λ, the empty string. As in the proof of Proposition 6.6, suppose Player 1 plays $(\omega, \mathcal{P}(\omega))$ and p as a partial first move. Since $n > 1$, the strategy for Player 2 makes it play a coloring of a finite initial segment of ω as its partial first move. Let f^p_λ be this coloring. Now, suppose f^p_α has been defined for some α with $|\alpha| < n - 2$, and fix $v \in \omega$. Suppose Player 1 plays $(\omega, \mathcal{P}(\omega))$ and p as a partial first move, and for $0 < k \leqslant |\alpha| + 1$, plays $H^p_{(\alpha^\frown v)\restriction k}$ as a partial $(k+1)$st move, with Player 2 playing according to its fixed strategy. Since $n > |\alpha| + 2$, the strategy for Player 2 makes it again play a coloring of an initial segment as its partial $(|\alpha|+2)$nd move. Let $f^p_{\alpha^\frown v}$ be this coloring.

We now define a finitely branching subtree T of $\omega^{\leqslant n-2}$, and for each $\alpha \in T$, a condition p_α, such that the following properties hold:

1. For all $\alpha, \beta \in T$, if β length-lexicographically precedes α then p_α extends p_β.

2. For every $\alpha^\frown v \in T$ and for every $m \in \omega$, the set of conditions p with $f^p_\alpha(x) = v$ for some $x \geqslant m$ is dense below p_α.

3. For every $\alpha \in T$ and every v such that $\alpha^\frown v \in \omega^{n-1} \setminus T$, if $f^p_\alpha(x) = v$ for some condition p extending p_α and some x, then $x \in \operatorname{dom} f^{p_\alpha}_\alpha$.

We put strings α into T and define p_α simultaneously. Initially, put $\lambda \in T$ and let p_λ be the empty condition. Notice that properties 1–3 hold vacuously at this point.

Next, assume we are at a point in the definition of T at which properties 1–3 hold, and consider the length-lexicographically least $\alpha \in T$ with $|\alpha| < n-2$ such that we have not yet put $\alpha^\frown v$ into T for any v. Let $\beta \in T$ be length-lexicographically largest such that p_β has been defined. Let $c : \omega \to \omega$ extend p and be sufficiently generic for the forcing notion P_ω. If Player 1 plays $(\omega, \mathcal{P}(\omega))$ and c on its first move, then the strategy for Player 2 makes it play an instance f_0 of $\mathsf{stRT}^1_{<\infty}$ in response. By property 2 and the genericity of c, the set of x such that $f_0(x) = \alpha(0)$ is infinite, so $H_0 = \{x : f_0(x) = \alpha(0)\}$ is a legal second move for Player 1. Then the strategy for Player 2 makes it play another instance f_1 of $\mathsf{stRT}^1_{<\infty}$ on its second move, and the set $H_1 = \{x : f_1(x) = \alpha(1)\}$ will be infinite and hence a legal third move for Player 1. Since $n > |\alpha| + 2$, if we continue in this way we analogously define $f_0, \ldots, f_{|\alpha|}$ and $H_0, \ldots, H_{|\alpha|-1}$, with f_k played by Player 2 on its $(k+1)$st move for all $k \leqslant |\alpha|$, and H_k played by Player 1 on its $(k+2)$nd move for all $k < |\alpha|$. Since the strategy for Player 2 is computable and hence continuous, it is easy to see by induction that if q is any condition extended by c then $f^q_{\alpha\restriction k}$ is an initial segment of f_k, and $H^p_{\alpha\restriction(k+1)}$ is an initial segment of H_k. Now, as $f_{|\alpha|}$ is an instance of $\mathsf{stRT}^1_{<\infty}$, there must be a condition q_0 extended by c and a $b \in \omega$ such that $f^r_\alpha(x) < b$ for all x and all r extending q_0.

We now decide for which $v < b$ to add $\alpha^\frown v$ to T and define $p_{\alpha^\frown v}$. Fix v, and suppose we have already decided this for all $w < v$. For notational convenience, assume we have also defined an auxiliary condition q_v extending q_0. If there is a condition r extending q_v such that for every $m \in \omega$, every extension of r has a further extension s such that $f^s_\alpha(x) = v$ for some $x \geqslant m$, then let $\alpha^\frown v \in T$ and let $p_{\alpha^\frown v} = q_{v+1} = r$. Otherwise, there is an extension r of q_v such that for every extension s of r, if $f^s_\alpha(x) = v$ for some x then x is in the domain of f^r_α, and we let $q_{v+1} = r$ and let $\alpha^\frown v \notin T$. It is readily

seen that this process adds $\alpha^\frown v$ to T for at least one v, and for only finitely many v, and that properties 1, 2, and 3 are preserved.

Let $p^* = p_\beta$ for the length-lexicographically largest $\beta \in T$. Let M be as given by Lemma 6.5, and let S be the set of subsets of $|M|$ that are Δ^0_1-definable over M. Every condition for P_ω is also a condition for P_M. So let c be an instance of LH in S that extends p^* and has no solution in S. For every node $\alpha \in T$, let G_α be the following run of a game. Player 1 plays (M, S) and c as its first move, and Player 2 plays according to its strategy. On its $(k+1)$st move for $0 < k \leqslant |\alpha|$, Player 1 always plays the set of all $x \in M$ that are colored $\alpha(k-1)$ by the coloring played by Player 2 on its previous move (assuming it played a total coloring and not just a partial one). We claim that there is an $\alpha \in T$ of length $n-2$ such that Player 1's moves in G_α are all legal. We argue by induction (along the standard number $n-1$) that for each $k < n-1$ there is such an $\alpha \in T$ of length k. Suppose that for some $\alpha \in T$ of length $k-1$, Player 1's moves in G_α are all legal. Then on its $(|\alpha|+1)$st move in G_α, Player 2 plays an instance f of $\mathsf{stRT}^1_{<\infty}$. Now, property 3 in the definition of T is a Π^0_1 statement of arithmetic, so since M is a 1-elementary extension of ω, it must also hold in M. Thus, all the $v \in M$ such that $f^{-1}(v)$ is unbounded in M must be among those for which $\alpha^\frown v \in T$. Since there are only standardly many such v, there must be at least one for which $f^{-1}(v)$ really is unbounded in M, so Player 1's moves in $G_{\alpha^\frown v}$ will all be legal. This establishes the claim. To complete the proof, fix such an α of length $n-2$. All sets played by Player 1 are clearly in S, so when Player 2 declares victory on its nth (i.e., $(|\alpha|+2)$nd) move in G_α it must play a solution to c in S. But there is no such solution by hypothesis, which is a contradiction. □

7 Versions of $\mathbf{\Pi^0_1}$-bounding

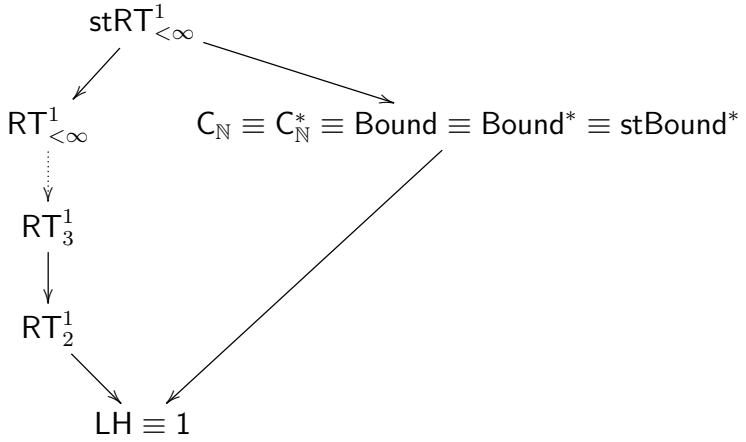
In this section we fill out the picture of implications between versions of $\mathsf{B}\Pi^0_1$ and related principles.

As with $\mathsf{RT}^1_{<\infty}$, we can define a strong form $\mathsf{stBound}^*$ of Bound^* by having the number of sets not be part of the instance. A convenient way to express this problem is to say that an instance is an enumeration of a subset X of $\mathbb{N} \times \mathbb{N}$ such that $\{n : \exists k\,(n,k) \in X\}$ is bounded, and for each n, so is the set $\{k : (n,k) \in X\}$; and a solution is a bound on $\{k : \exists n\,(n,k) \in X\}$. It is easy to see that $\mathsf{stBound}^* \equiv_{\mathrm{W}} \mathsf{Bound}$, but we will see that this equivalence

no longer holds in our setting.

Another problem worth mentioning in this connection is $\mathsf{C}_{\mathbb{N}}$, for which an instance is an enumeration of the complement of a nonempty set X, and a solution is an element of X. The finite parallelization $\mathsf{C}_{\mathbb{N}}^*$ is yet another equivalent of $\mathsf{B}\Sigma_2^0$: In one direction, we can enumerate the sets $\{m : \forall k < m\, \neg\varphi(i,k)\}$ for a given Π_1^0 formula $\varphi(i,k)$, and from a tuple containing an element of the complement of each of these sets, obtain a common bound on the sets. In the other direction, given simultaneous enumerations of the complements of the nonempty sets $F_0, \ldots, F_n$, by $\mathsf{B}\Pi_1^0$, there is a b such that each F_i has an element less than b. Now bounded Π_1^0-comprehension, which holds in RCA_0, gives us the set of all tuples $(a_0, \ldots, a_j)$ with $j \leqslant n$ and $a_i \in F_i$ for all $i \leqslant j$, and set induction shows that there must be such a tuple with $j = n$.

It is easy to see that $\mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}^*$, and Pauly, Fouché, and Davie [31] showed that $\mathsf{Bound} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$, using the Weihrauch equivalence between $\mathsf{C}_{\mathbb{N}}$ and its restriction $\mathsf{UC}_{\mathbb{N}}$ to enumerations of complements of singleton sets, which was proved by Brattka, de Brecht, and Pauly [6]. Brattka and Rakotoniaina [9] showed that $\mathsf{C}_{\mathbb{N}} \mid_{\mathrm{W}} \mathsf{RT}^1_{<\infty}$ and $\mathsf{C}_{\mathbb{N}} <_{\mathrm{W}} \mathsf{stRT}^1_{<\infty}$. Indeed, it is even the case that $\mathsf{RT}^1_2 \not\leqslant_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$; we will prove a stronger version of this fact below. It is also worth noting that $\mathsf{RT}^1_2 <_{\mathrm{W}} \mathsf{RT}^1_3 <_{\mathrm{W}} \cdots$, as shown by Brattka and Rakotoniaina [9] and Hirschfeldt and Jockusch [19]. Thus we have the following picture for Weihrauch reducibility:



(7.1)

Hirschfeldt and Jockusch [19, Proposition 4.7] showed that $\mathsf{RT}^1_{<\infty} \leqslant_{\mathrm{gW}}$

RT^1_2, but their proof in fact shows that $\mathsf{stRT}^1_{<\infty} \leqslant_{\mathrm{gW}} \mathsf{RT}^1_2$. On the other hand, we have the following.

Proposition 7.1. $\mathsf{RT}^1_2 \not\leqslant_{\mathrm{gW}} \mathsf{C}_{\mathbb{N}}$.

Proof. Suppose that $\mathsf{RT}^1_2 \leqslant_{\mathrm{gW}} \mathsf{C}_{\mathbb{N}}$ via a computable strategy P for Player 2. As Player 1, we can begin to build a coloring c by coloring numbers in order, initially giving each number the color 0, and simulate the action of P. We can assume that, even when provided with inputs that do not correspond to a run of $G(\mathsf{C}_{\mathbb{N}} \to \mathsf{RT}^1_2)$, if P does not declare victory at a given move, then it outputs an enumeration of the complement of some set, though in that case the set might be empty.

Let A_i be the set whose complement is being enumerated by P as its $(i+1)$st move (if P has not declared victory at or before that move). We guess at each stage that the least number k_i currently in A_i is a solution to the corresponding instance of $\mathsf{C}_{\mathbb{N}}$ and play that as our $(i+2)$nd move in the simulation. If we ever find that k_i is not in A_i, we restart the simulation (but do not change c on the numbers at which we have already defined it). For the least such i, say that i causes the simulation to restart. If the current simulation is not restarted, then eventually P must declare victory at some move, and declare some number m to be in the set it outputs at that move. We then start to give our numbers the color $1 - c(m)$. If we were to do this forever, then m could not be part of a solution to c, so our current simulation cannot be a true run of the game, and hence eventually some i must cause it to restart.

Thus the simulation is restarted infinitely often. There are now two cases.

If there is a least i that causes the simulation to restart infinitely often, then, by induction, $k_0, \ldots, k_{i-1}$ have final values, and if we play c on our first move, and then play these values in turn, we produce a run of our game in which P's $(i+1)$st move is an enumeration of $\mathbb{N}$, and hence is not an instance of $\mathsf{C}_{\mathbb{N}}$, which is a contradiction.

Otherwise, again by induction, all k_i's have final values, and if we play c on our first move, and then play these values in turn, we produce a run of our game in which P never declares victory, which is again a contradiction. □

So for gW-reducibility, we have the following simpler picture:

$$\begin{array}{c} \mathsf{stRT}^1_{<\infty} \equiv \mathsf{RT}^1_{<\infty} \equiv \mathsf{RT}^1_2 \\ \downarrow \\ \mathsf{C}_{\mathbb{N}} \equiv \mathsf{C}^*_{\mathbb{N}} \equiv \mathsf{Bound} \equiv \mathsf{Bound}^* \equiv \mathsf{stBound}^* \\ \downarrow \\ \mathsf{LH} \equiv 1 \end{array} \tag{7.2}$$

It is easy to check that all the Weihrauch reductions in Diagram (7.1) still work over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$, so that diagram also reflects the relationships between these principles with respect to $\leqslant_{\mathrm{W}}^{\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2}$ (or $\leqslant_{\mathrm{W}}^{\Gamma}$ for any extension Γ of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ by formulas true over the natural numbers). Diagram (7.2), however, does change if we work over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. We still have the equivalence between RT^1_j and RT^1_k for $j, k \geqslant 2$ (which holds even over RCA_0, with the usual proof), but Corollary 4.3 shows that $\mathsf{RT}^1_{<\infty} \not\leqslant_{\mathrm{gW}}^{\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2} \mathsf{RT}^1_k$ for all n and k. Similarly, we have the following.

Proposition 7.2. $\mathsf{C}_{\mathbb{N}} \not\leqslant^n_{\mathrm{gW}} \mathsf{RT}^1_{<\infty}$ *for all* n*, so if we let* Γ *consist of* RCA_0 *together with all* Π^1_1 *formulas true over the natural numbers then* $\mathsf{C}_{\mathbb{N}} \not\leqslant_{\mathrm{gW}}^{\Gamma} \mathsf{RT}^1_{<\infty}$.

Proof. Suppose that $\mathsf{C}_{\mathbb{N}} \leqslant^n_{\mathrm{gW}} \mathsf{RT}^1_{<\infty}$ via a computable strategy P for Player 2. We can assume that, even when provided with inputs that do not correspond to a run of $G(\mathsf{RT}^1_{<\infty} \to \mathsf{C}_{\mathbb{N}})$, if P does not declare victory at a given move, then its output at that move, if nonempty, is a number k together with a possibly partial $c : \mathbb{N} \to k$.

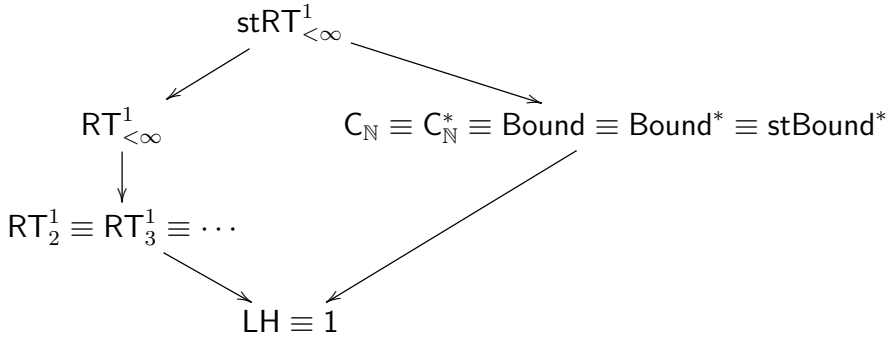
For a possibly partial $c : \mathbb{N} \to k$, let $H_c = \{c^{-1}(0), \ldots, c^{-1}(k-1)\}$. Note that if c is total then at least one element of H_c is a solution to c as an instance of $\mathsf{RT}^1_{<\infty}$. We can start building an instance E of $\mathsf{C}_{\mathbb{N}}$ by initially not enumerating any numbers, and running simulations of possible runs of $G(\mathsf{RT}^1_{<\infty} \to \mathsf{C}_{\mathbb{N}})$ beginning with E, where each time P plays some c, we play a simulation for each possible move for Player 1 in H_c. (Notice that c might not actually be an instance of $\mathsf{RT}^1_{<\infty}$ because this simulation might not correspond to an actual run of the game, but H_c is still finite. This is the reason we could not work with $\mathsf{stRT}^1_{<\infty}$ here, because in that case P would be able to play functions with unbounded range during simulations that do not correspond to actual runs.)

Whenever in any of these simulations P declares victory at or before the $(n+1)$st move with a purported solution m, we enumerate m into E.

Since each H_c is finite, and we consider only finitely many c's during this construction, we enumerate only finitely many numbers into E, and this strategy ensures that there is a run of $G(\mathsf{RT}^1_{<\infty} \to \mathsf{C}_{\mathbb{N}})$ beginning with E in which either P does not declare victory by its $(n+1)$st move, or it does so with a purported solution m that is enumerated into E, and hence is not in fact a solution to E. In either case we have a contradiction.

The second part of the proposition now follows from Proposition 4.2. □

Thus we have the following picture for gW-reducibility over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ (or over any extension of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ by Π^1_1 formulas true over the natural numbers):



(7.3)

When working over RCA_0, things change even further. We do still have $\mathsf{Bound} \equiv^{\mathsf{RCA}_0}_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$, $\mathsf{Bound}^* \equiv^{\mathsf{RCA}_0}_{\mathrm{W}} \mathsf{C}^*_{\mathbb{N}}$, and $\mathsf{C}_{\mathbb{N}} \leqslant^{\mathsf{RCA}_0}_{\mathrm{W}} \mathsf{stRT}^1_{<\infty}$, with essentially the same proofs. The only parts that require a bit of care are $\mathsf{C}_{\mathbb{N}} \leqslant^{\mathsf{RCA}_0}_{\mathrm{W}} \mathsf{Bound}$ and $\mathsf{C}^*_{\mathbb{N}} \leqslant^{\mathsf{RCA}_0}_{\mathrm{W}} \mathsf{Bound}^*$. We prove the latter, as the former is similar but simpler. We argue in RCA_0. Given an enumeration of the complements of nonempty sets $A_0, \ldots, A_n$, constituting an instance of $\mathsf{C}^*_{\mathbb{N}}$, we define enumerations of sets $F_0, \ldots, F_n$ by putting s into F_i whenever the least element m^i_s of A_i at stage s of the enumeration of its complement leaves A_i at that stage. If F_i were unbounded, then so would be the set of numbers m^i_s, since the map taking F_i to this set is injective and computable. But then A_i would be empty. So each F_i is bounded, and hence our enumeration of $F_0, \ldots, F_n$ is an instance of Bound^*. If s is a solution to this instance then for each $i \leqslant n$, the least element of A_i at stage s must be in A_i, so from s we obtain a solution to our instance of $\mathsf{C}^*_{\mathbb{N}}$.

However, every instance of $\mathsf{C}_{\mathbb{N}}$ and Bound in every model of RCA_0 has a solution, while this is not the case for $\mathsf{C}^*_{\mathbb{N}}$ and Bound^*, which are equivalent to $\mathsf{B}\Sigma^0_2$ over RCA_0 as statements of second-order arithmetic. So $\mathsf{C}_{\mathbb{N}}$ is strictly below $\mathsf{C}^*_{\mathbb{N}}$ under both $\leqslant_{\mathrm{W}}^{\mathsf{RCA}_0}$ and $\leqslant_{\mathrm{gW}}^{\mathsf{RCA}_0}$, and similarly for Bound and Bound^*.

We also no longer have a Weihrauch-reduction of $\mathsf{stBound}^*$ to Bound^*, but do have one in two steps, because an instance of Bound^* (or even Bound) can be used to determine the number of sets being enumerated in an instance of $\mathsf{stBound}^*$, allowing us to solve that instance with a second application of Bound^*.

Proposition 7.3. $\mathsf{stBound}^* \leqslant_{\mathrm{gW}}^{\mathsf{RCA}_0,2} \mathsf{Bound}^*$ *but* $\mathsf{stBound}^* \not\leqslant_{\mathrm{W}}^{\mathsf{RCA}_0} \mathsf{Bound}^*$.

Proof. Given an instance X of $\mathsf{stBound}^*$, we can first build an instance of Bound by enumerating n whenever X enumerates (n, k) for some k. Given a solution b to this instance, we can build an instance of Bound^* consisting of enumerations of sets $F_0, \ldots, F_{b-1}$ by enumerating k into F_n whenever X enumerates (n, k). A solution to this instance is also a solution to X.

For the second part, suppose that $\mathsf{stBound}^* \leqslant_{\mathrm{W}}^{\mathsf{RCA}_0} \mathsf{Bound}^*$ via Φ_e and Φ_i. An enumeration E of $\emptyset$ is an instance of $\mathsf{stBound}^*$, so Φ_e^E must be an instance of Bound^*. This instance has a fixed number of sets k, which must be the same standard natural number no matter what model of RCA_0 we are working in, because the convergent computation over the standard natural numbers still exists in any such model. Now let (M, S) be a model of RCA_0 that contains an M-instance D of $\mathsf{stBound}^*$ with no solution. We can delay D to define a new M-instance $\widehat{D}$ of $\mathsf{stBound}^*$ that enumerates the same set as D but agrees with E up to the use of the part of the computation of Φ_e^E that fixes the number of sets at k. Then $\widehat{D}$ has no solution, but $\Phi_e^{\widehat{D}}$ is an instance of Bound^* with a standard number of sets, and hence must have a solution b. But then Φ_i should be able to compute a solution to $\widehat{D}$ from $\widehat{D}$ and b, which is a contradiction. □

We can make the first part of this proposition a bit more precise by using the compositional product from the theory of Weihrauch reducibility: $\mathsf{stBound}^* \leqslant_{\mathrm{W}}^{\mathsf{RCA}_0} \mathsf{Bound}^* \star \mathsf{Bound}$.

The second part of the proposition easily generalizes to establish the following useful principle (which we state for RCA_0 but of course applies to other systems as well).

Proposition 7.4. *Let* P *and* Q *be* Π^1_2*-problems such that*

1. P *has an* ω*-instance* X *such that for any finite initial segment* σ *of* X*, there is a model* (M, S) *of* RCA_0 *and an* M*-instance* Y *of* P *in* S *that extends* σ *and has no solution in* S*; and*

2. *every instance* X *of* Q *includes a parameter* $k_X \in \mathbb{N}$ *such that for every model* (M, S) *of* RCA_0 *and every* M*-instance* X *of* Q *in* S*, if* k_X *is a standard natural number, then* X *has a solution in* S*.*

Then $\mathsf{P} \not\leqslant_{\mathrm{W}}^{\mathsf{RCA}_0} \mathsf{Q}$*.*

As an example of the application of this principle, we have the following.

Corollary 7.5. $\mathsf{LH} \not\leqslant_{\mathrm{W}}^{\mathsf{RCA}_0} \mathsf{RT}^1_{<\infty}$.

We also have the following other example of a W-reducibility that becomes a gW-reducibility in two steps when generalized to models of RCA_0.

Proposition 7.6. $\mathsf{stBound}^* \leqslant_{\mathrm{gW}}^{\mathsf{RCA}_0,2} \mathsf{stRT}^1_{<\infty}$ *but* $\mathsf{Bound}^* \not\leqslant_{\mathrm{W}}^{\mathsf{RCA}_0} \mathsf{stRT}^1_{<\infty}$.

Proof. For the first part, we argue in RCA_0. Given an instance X of $\mathsf{stBound}^*$, let $E_{i,n}$ be the set of k such that (i, k) has been enumerated into X by stage n, and let i_n be the least i that maximizes $\max E_{i,n}$ (which exists because the function taking i to $\max E_{i,n}$ is computable). We first produce an instance of $\mathsf{stRT}^1_{<\infty}$ by giving n the color i_n. Given a solution H to this instance, let i be the color of the elements of H. Now apply Bound (which is W-reducible over RCA_0 to $\mathsf{stRT}^1_{<\infty}$) to obtain a bound b on $\{k : (i, k) \in X\}$. This bound must be a solution to X, because if $(j, k) \in X$ for some j and $k > b$, then once (j, k) is enumerated into X at some stage m, we cannot have $i_n = i$ for $n \geqslant m$.

Now suppose that $\mathsf{Bound}^* \leqslant_{\mathrm{W}}^{\mathsf{RCA}_0} \mathsf{stRT}^1_{<\infty}$ via Φ_e and Φ_i. We work over a model M of Σ^0_1-PA that satisfies Σ^0_2-bounding but not Σ^0_3-bounding. Then there is a Δ^0_2 M-instance $c : |M| \to k$ of $\mathsf{RT}^1_{<\infty}$ with no solution. Say that sets $F_0, \ldots, F_{k-1}$ are acceptable if $c(n) = i$ for every $i < k$ and $n \in F_i$. Notice that in this case, each F_i is bounded, so an enumeration of an acceptable family of sets is an M-instance of Bound^*.

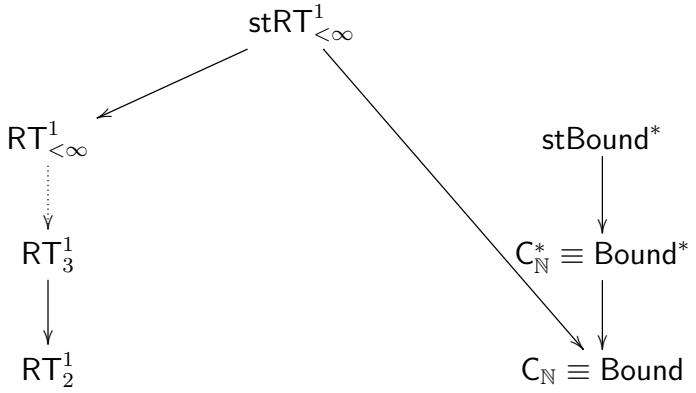
Thinking of M-finite enumerations of acceptable families as a notion of forcing, suppose that for each $j \in M$, the set of such enumerations E for which some element greater than j is in the range of Φ_e^E is dense. Then we can computably build an enumeration D of an acceptable family such that Φ_e^D has unbounded range, and is thus not an instance of $\mathsf{stRT}^1_{<\infty}$. As this

situation cannot happen, there must be a $j \in M$ and an M-finite enumeration E of an acceptable family such that for every enumeration D of an acceptable family extending E, the range of Φ_e^D is bounded by j.

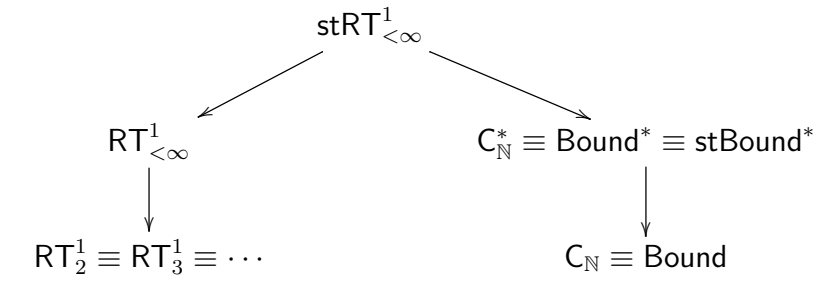
Now we start building such a D by monitoring $\Phi_i^{D \oplus H_p}$ for each $H_p = \{n : \Phi_e^D(n) = p\}$ with $p <^M j$. Whenever we see $\Phi_i^{D \oplus H_p}$ return a number m_p, we enumerate $m_p + 1$ into $F_{c(m_p+1)}$, where $F_0, \ldots, F_{k-1}$ is the family that D is enumerating. The set of $p <^m j$ such that m_p is ever defined is a bounded Σ^0_1 set, and the map taking each p in this set to m_p is computable, so the set of m_p's is M-finite. But then the restriction of c to this set is also M-finite, because the fact that M satisfies Σ^0_2-bounding implies that the intersection of a Δ^0_2 set with an M-finite set is M-finite. So D is an M-finite extension of the M-finite enumeration E, and hence is itself M-finite, and thus Φ_e^D is a computable instance of $\mathsf{stRT}^1_{<\infty}$, and hence must have a solution. But then some H_p with $p <^M j$ must be such a solution, and hence $\Phi_i^{D \oplus H_p}$ must be a solution to D. But we ensured that this is not the case, so we have a contradiction. □

The first part of this proof shows more precisely that $\mathsf{stBound}^* \leqslant_{\mathrm{W}}^{\mathsf{RCA}_0} \mathsf{Bound} \star \mathsf{stRT}^1_{<\infty}$ and that $\mathsf{Bound}^* \leqslant_{\mathrm{W}}^{\mathsf{RCA}_0} \mathsf{Bound} \star \mathsf{RT}^1_{<\infty}$.

Combining the results above with Proposition 6.8 gives us the following pictures of the $\leqslant_{\mathrm{W}}^{\mathsf{RCA}_0}$ and $\leqslant_{\mathrm{gW}}^{\mathsf{RCA}_0}$ cases, respectively.



(7.4)



LH (7.5)

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