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FULLY DISCRETE FINITE ELEMENT METHODS FOR NONLINEAR STOCHASTIC ELASTIC WAVE EQUATIONS WITH MULTIPLICATIVE NOISE

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Dedicated to the memory of Professor Roland Glowinski

Abstract. This paper is concerned with fully discrete finite element methods for approximating variational solutions of nonlinear stochastic elastic wave equations with multiplicative noise. A detailed analysis of the properties of the weak solution is carried out and a fully discrete finite element method is proposed. Strong convergence in the energy norm with rate $\mathscr{O}(k+h^r)$ is proved, where k and h denote respectively the temporal and spatial mesh sizes, and $r(\geq 1)$ is the order of the finite element. Numerical experiments are provided to test the efficiency of proposed numerical methods and to validate the theoretical error estimate results.

Keywords. Stochastic elastic wave equations; multiplicative noise; Itô stochastic integral; finite element method; error estimates; quantities of stochastic interests.

1. INTRODUCTION

This paper is concerned with numerical approximations of the following stochastic elastic wave equations with multiplicative noise of Itô type:

$$\mathbf{u}_{tt} - \operatorname{div}(\sigma(\mathbf{u})) = \mathbf{F}[\mathbf{u}] + \mathbf{G}[\mathbf{u}]\xi \qquad \text{in } \mathscr{D}_T := (0,T) \times \mathscr{D}, \tag{1.1}$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0, \quad \mathbf{u}_t(0,\cdot) = \mathbf{v}_0 \quad \text{in } \mathscr{D}, \tag{1.2}$$

$$\mathbf{u}(t,\cdot) = 0 \qquad \text{on } \partial\mathscr{D}_T := (0,T) \times \partial\mathscr{D}, \tag{1.3}$$

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where $\mathbf{u}_t = \frac{d\mathbf{u}}{dt}$, $\xi = \dot{\mathbf{W}} = \frac{d\mathbf{W}}{dt}$ is the white noise, $\mathscr{D} \subset \mathbb{R}^d$ ($d = 2, 3$) is a bounded domain, $(\mathbf{u}_0, \mathbf{v}_0)$ is an $\mathbf{H}_0^1 \times \mathbf{L}^2$-valued random variable, and

$$\sigma(\mathbf{u}) = (\lambda \operatorname{div} \mathbf{u})\mathbf{I} + \mu \varepsilon(\mathbf{u}), \tag{1.4}$$

$$\varepsilon(\mathbf{u}) = \frac{1}{2}\big(\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathbf{T}}\big), \tag{1.5}$$

$$\mathbf{F}[\mathbf{u}] = \mathbf{F}(\mathbf{u}, \nabla \mathbf{u}), \tag{1.6}$$

$$\mathbf{G}[\mathbf{u}] = \mathbf{G}(\mathbf{u}, \nabla \mathbf{u}). \tag{1.7}$$

Here $\mathbf{I}$ denotes the unit matrix. $\mathbf{F}[\mathbf{u}]$ and $\mathbf{G}[\mathbf{u}]$ are two given nonlinear mappings satisfying some structure conditions. The multiplicative noise $\mathbf{G}[\mathbf{u}]\xi$ has the following three cases:

Case 1. W is a $\mathbb{R}$-valued Wiener process which is defined on the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{0 \le t \le T}, \mathbb{P})$, and $\mathbf{G}[\mathbf{u}]$ is d-dimensional nonlinear mapping;

Case 2. $\mathbf{W}$ is a $\mathbb{R}^d$-valued Wiener process, and $G[\mathbf{u}]$ is a scalar nonlinear mapping;

Case 3. $\mathbf{W}$ is a $\mathbb{R}^l$-valued Wiener process, $\mathbf{G}[\mathbf{u}]$ is a $d \times l$ matrix, then $\mathbf{G}\dot{\mathbf{W}}$ is a d-dimensional multiplicative noise.

For the sake of presentation clarity, we only consider *Case 1*, For the other two cases, it can be shown that the same results still hold.

Wave propagation is a fundamental physical phenomenon, and it arises from various applications in geophysics, engineering, medical science, biology, etc. There is a large amount of literature on numerical methods for deterministic acoustic wave equations, we refer the reader to [1, 3, 4, 5, 7, 8, 16, 17, 21, 22, 29, 32, 33, 35, 38, 40] and the references therein for a detailed account. Moreover, numerical methods for stochastic acoustic wave equations have also been intensively developed in the last few years, see [2, 10, 12, 14, 15, 18, 20, 23, 24, 26, 27, 30, 37]. Similarly, the elastic wave equations are also of great importance and find applications in geoscience for modeling seismic waves and in medical science for tumor detection as well as in materials science for non-destructive testing. Although there is a large literature in numerical methods for deterministic elastic wave equations, see [9, 19, 25, 28, 34, 36, 31] and the references therein, there is barely any work on numerical analysis of stochastic elastic wave equations in the literature, which motivates us to carry out the work of this paper.

The primary goal of this paper is to develop some semi-discrete (in time) scheme and fully discrete finite element methods for the stochastic elastic wave equation with multiplicative noise. The highlight of the paper is the establishment of strong norm convergence and error estimates for both semi-discrete and fully discrete methods. To achieve this goal, we first need to establish some stability and Hölder continuity estimates for the (variational) weak solution of the stochastic wave equations. These results will be crucially used to derive the desired error estimates for the semi-discrete scheme. We next need to establish various (energy) stability estimates for the semi-discrete numerical solution, which are necessary for deriving the desired error estimates for the fully discrete finite element methods.

The rest of the paper is organized as follows. In Section 2, we introduce a variational weak formulation for problem (1.1)–(1.3). The stability and Hölder continuity estimates in the $\mathbf{L}^2$-, $\mathbf{H}^1$-, and $\mathbf{H}^2$-norm are established for the strong solution. In Section 3, we propose a semi-discrete in time numerical scheme for problem(1.1)–(1.3). It is proved that the semi-discrete solution is energy stable. Moreover, we prove the convergence with rates $O(k)$ in the $\mathbf{L}^2$-norm and $O(k^{\frac{1}{2}})$ in the $\mathbf{H}^1$-norm for the displacement approximations. In Section 4, we propose a

fully discrete finite element method to discretize the semi-discrete scheme in space and derive its error estimates, which show that for the linear finite element, the $\mathbf{L}^2$-norm of the error converges with $O(h^2)$ rate and the $\mathbf{H}^1$-norm converges with $O(h)$ rate. In Section 5, we present two two-dimensional numerical experiments to test the efficiency of the proposed numerical methods and to validate the theoretical error estimate results. Finally, we conclude the paper with a short summary given in Section 6.

2. Preliminaries

Standard notations for functions and spaces are adopted in this paper. For example, $\mathbf{L}^p$ denotes $(L^p(\mathscr{D}))^d$ for $1 \leq p \leq \infty$, $(\cdot, \cdot)$ denotes the standard $\mathbf{L}^2(\mathscr{D})$-inner product and $\mathbf{H}^m(\mathscr{D})$ denotes the Sobolev space of order m. Throughout this paper, C will be used to denote a generic positive constant which is independent of the mesh parameters k and h.

2.1. **Assumptions.** The following structural conditions will be imposed on the mappings $\mathbf{F}[\cdot]$ and $\mathbf{G}[\cdot]$:

$$\|\mathbf{F}[0]\|_{\mathbf{L}^2} + \|\mathbf{G}[0]\|_{\mathbf{L}^2} \leq C_A, \tag{2.1}$$

$$\|\nabla_{\mathbf{u}}\mathbf{F}[\cdot]\|_{\mathbf{L}^\infty} + \|\nabla_{\mathbf{u}}\mathbf{G}[\cdot]\|_{\mathbf{L}^\infty} \leq C_A, \tag{2.2}$$

$$\|\mathbf{F}_{\mathbf{u}_i\mathbf{u}_j}[\cdot]\|_{\mathbf{L}^\infty} \leq C_A, \quad 1 \leq i, j \leq d, \tag{2.3}$$

$$\|\mathbf{F}[\mathbf{v}] - \mathbf{F}[\mathbf{w}]\|_{\mathbf{L}^2} \leq C_B\left(\lambda\|\mathrm{div}(\mathbf{v}-\mathbf{w})\|_{\mathbf{L}^2}^2 + \mu\|\varepsilon(\mathbf{v}-\mathbf{w})\|_{\mathbf{L}^2}^2 + \|\mathbf{v}-\mathbf{w}\|_{\mathbf{L}^2}^2\right)^{\frac{1}{2}}, \tag{2.4}$$

$$\|\mathbf{G}[\mathbf{v}] - \mathbf{G}[\mathbf{w}]\|_{\mathbf{L}^2} \leq C_B\left(\lambda\|\mathrm{div}(\mathbf{v}-\mathbf{w})\|_{\mathbf{L}^2}^2 + \mu\|\varepsilon(\mathbf{v}-\mathbf{w})\|_{\mathbf{L}^2}^2 + \|\mathbf{v}-\mathbf{w}\|_{\mathbf{L}^2}^2\right)^{\frac{1}{2}}, \tag{2.5}$$

where $\mathbf{F}_{\mathbf{u}_i\mathbf{u}_j}[\cdot]$ denotes the second derivative of $\mathbf{F}$ with respect to $\mathbf{u}_i, \mathbf{u}_j$, and C_A and C_B are two positive constants.

2.2. **Variational weak formulation and properties of weak solutions.** In this subsection, we first give the definition of variational weak formulation and weak solutions for problem (1.1)–(1.3). We then establish several technical lemmas that will be used in the subsequent sections.

Equations (1.1)–(1.3) can be written as

$$d\mathbf{u} = \mathbf{v}dt, \tag{2.6}$$

$$d\mathbf{v} = (\mathscr{L}\mathbf{u} + \mathbf{F}[\mathbf{u}])dt + \mathbf{G}[\mathbf{u}]dW(t), \quad \mathscr{L}\mathbf{u} := \mathrm{div}\,\sigma(\mathbf{u}), \tag{2.7}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{v}(0) = \mathbf{v}_0, \tag{2.8}$$

$$\mathbf{u}(t, \cdot) = 0. \tag{2.9}$$

Definition 2.1. The weak formulation for problem (2.6)–(2.9) is defined as seeking $(\mathbf{u}, \mathbf{v}) \in \mathbf{L}^2\big(\Omega; \mathbf{C}([0,T]; \mathbf{L}^2) \cap \mathbf{L}^2((0,T), \mathbf{H}_0^1)\big) \times \mathbf{L}^2\big(\Omega; \mathbf{C}([0,T], \mathbf{L}^2)\big)$ such that

$$\big(\mathbf{u}(t), \phi\big) = \int_0^t \big(\mathbf{v}(s), \phi\big)ds + (\mathbf{u}_0, \phi) \qquad\qquad \forall \phi \in \mathbf{L}^2, \tag{2.10}$$

$$\big(\mathbf{v}(t), \psi\big) = -\int_0^t \lambda\big(\mathrm{div}\,(\mathbf{u}(s)), \mathrm{div}\,(\psi)\big)ds - \int_0^t \mu\big(\varepsilon(\mathbf{u}(s)), \varepsilon(\psi)\big)ds \tag{2.11}$$

$$+ \int_0^t \big(\mathbf{F}[\mathbf{u}(s)], \psi\big)ds + \int_0^t \big(\mathbf{G}[\mathbf{u}(s)]dW(s), \psi\big) + (\mathbf{v}_0, \psi) \qquad \forall \psi \in \mathbf{H}_0^1$$

for all $(\phi, \psi) \in \mathbf{L}^2 \times \mathbf{H}_0^1$. Such a pair $(\mathbf{u}, \mathbf{v})$, if it exists, is called a (variational) weak solution to problem (2.6)–(2.9). Moreover, if a weak solution $(\mathbf{u}, \mathbf{v})$ belongs to $\mathbf{L}^2\big(\Omega; \mathbf{C}([0,T]; \mathbf{H}^2 \cap \mathbf{H}_0^1)\big) \times \mathbf{L}^2\big(\Omega; \mathbf{C}([0,T]; \mathbf{H}_0^1)\big)$, then $(\mathbf{u}, \mathbf{v})$ is called a strong solution to problem (2.6)–(2.9).

Remark 2.2. The well-posedness of problem (2.6)–(2.9) can be proved using the same technique (i.e., the Galerkin method) as done in [6] for the acoustic stochastic wave equation with multiplicative noise. The only markable difference is that to verify the coercivity (or ellipticity) in $\mathbf{H}^1(\mathscr{D})$ of the operator $\mathscr{L}$, we need to use the well-known Korn's (second) inequality.

We now state and prove the stability properties of the strong solution $(\mathbf{u}, \mathbf{v})$ of problem (2.6)–(2.9). Those bounds will be used to prove Hölder continuity in time in this section. They are also useful in establishing rates of convergence for the numerical schemes.

Lemma 2.3. *Let* $(\mathbf{u}, \mathbf{v})$ *be a strong solution to equations* (2.6)–(2.9). *Under the assumptions* (2.1)–(2.5), *there hold*

$$\sup_{0 \le t \le T} \mathbb{E}\left[\|\mathbf{v}\|_{\mathbf{L}^2}^2\right] + \sup_{0 \le t \le T} \mathbb{E}\left[\lambda \|\mathrm{div}\,(\mathbf{u})\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{u})\|_{\mathbf{L}^2}^2\right] \le C_{s1}, \tag{2.12}$$

$$\sup_{0 \le t \le T} \mathbb{E}\left[\lambda \|\mathrm{div}\,(\mathbf{v})\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{v})\|_{\mathbf{L}^2}^2\right] + \sup_{0 \le t \le T} \mathbb{E}\left[\|\mathscr{L}\mathbf{u}\|_{\mathbf{L}^2}^2\right] \le C_{s2}, \tag{2.13}$$

$$\sup_{0 \le t \le T} \mathbb{E}\left[\|\partial_{x_j}\mathbf{v}\|_{\mathbf{L}^2}^2\right] + \sup_{0 \le t \le T} \mathbb{E}\left[\lambda \|\mathrm{div}\,(\partial_{x_j}\mathbf{u})\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\partial_{x_j}\mathbf{u})\|_{\mathbf{L}^2}^2\right] \le C_{s3} \tag{2.14}$$

for $1 \le j \le d$, *and*

$$C_{s1} = \left(\mathbb{E}\left[\|\mathbf{v}_0\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[\lambda \|\mathrm{div}\,(\mathbf{u}_0)\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{u}_0)\|_{\mathbf{L}^2}^2\right] + 4C_A^2\right)e^{CC_B^2},$$

$$C_{s2} = \left(\mathbb{E}\left[\lambda \|\mathrm{div}\,(\mathbf{v}_0)\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{v}_0)\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[\|\mathscr{L}\mathbf{u}_0\|_{\mathbf{L}^2}^2\right] + CC_A^2 C_{s1}\right)e^T,$$

$$C_{s3} = \left(\mathbb{E}\left[\|\partial_{x_j}\mathbf{v}_0\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[\lambda \|\mathrm{div}\,(\partial_{x_j}\mathbf{u}_0)\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\partial_{x_j}\mathbf{u}_0)\|_{\mathbf{L}^2}^2\right]\right)e^{CC_A^2}.$$

Proof. Step 1: Applying Itô's formula to the functional $\Phi_1(\mathbf{v}(\cdot)) = \|\mathbf{v}(\cdot)\|_{\mathbf{L}^2}^2$ yields

$$\|\mathbf{v}(t)\|_{\mathbf{L}^2}^2 = \|\mathbf{v}_0\|_{\mathbf{L}^2}^2 + \int_0^t \mathrm{D}\Phi_1\big(\mathbf{v}(s)\big)\big(\mathscr{L}\mathbf{u} + \mathbf{F}[\mathbf{u}]\big)\mathrm{d}s \tag{2.15}$$

$$+ \int_0^t \mathrm{Tr}\Big(\mathrm{D}^2\Phi_1\big(\mathbf{v}(s)\big)\big(\mathbf{G}[\mathbf{u}], \mathbf{G}[\mathbf{u}]\big)\Big)\mathrm{d}s + \int_0^t \mathrm{D}\Phi_1\big(\mathbf{v}(s)\big)\big(\mathbf{G}[\mathbf{u}]\mathrm{d}W(s)\big).$$

The expressions of $\mathrm{D}\Phi_1(\mathbf{v}(\cdot))$ and $\mathrm{D}^2\Phi_1(\mathbf{v}(\cdot))$ are as follows:

$$\mathrm{D}\Phi_1(\mathbf{v})(\mathbf{w}_1) = 2(\mathbf{v}, \mathbf{w}_1) \qquad \forall \mathbf{w}_1 \in C_0^\infty, \tag{2.16}$$

$$\mathrm{D}^2\Phi_1(\mathbf{v})(\mathbf{w}_1, \mathbf{w}_2) = 2(\mathbf{w}_1, \mathbf{w}_2) \qquad \forall \mathbf{w}_1, \mathbf{w}_2 \in C_0^\infty.$$

Substituting the expressions of $\mathrm{D}\Phi_1(\mathbf{v}(\cdot))$ and $\mathrm{D}^2\Phi_1(\mathbf{v}(\cdot))$ into (2.15), we get

$$\|\mathbf{v}(t)\|_{\mathbf{L}^2}^2 + \lambda \|\mathrm{div}\,(\mathbf{u}(t))\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{u}(t))\|_{\mathbf{L}^2}^2 \tag{2.17}$$

$$= \|\mathbf{v}_0\|_{\mathbf{L}^2}^2 + \lambda \|\mathrm{div}\,(\mathbf{u}_0)\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{u}_0)\|_{\mathbf{L}^2}^2$$

$$+ 2\int_0^t \big(\mathbf{F}[\mathbf{u}], \mathbf{v}\big)\mathrm{d}s + \int_0^t \|\mathbf{G}[\mathbf{u}]\|_{\mathbf{L}^2}^2 \,\mathrm{d}s + 2\int_0^t \big(\mathbf{G}[\mathbf{u}], \mathbf{v}\big)\mathrm{d}W(s)$$

$$:= \|\mathbf{v}_0\|_{\mathbf{L}^2}^2 + \lambda \|\mathrm{div}\,(\mathbf{u}_0)\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{u}_0)\|_{\mathbf{L}^2}^2 + I_1 + I_2 + I_3.$$

For the first term I_1, let $\bar{\mathbf{u}} = 0$ in equation (2.4). Using equation (2.1), the Poincaré inequality, and the Korn's inequality, we have

$$2\int_0^t \left(\mathbf{F}[\mathbf{u}], \mathbf{v}\right)\mathrm{d}s \le \int_0^t \|\mathbf{F}[\mathbf{u}]\|_{\mathbf{L}^2}^2 + \|\mathbf{v}\|_{\mathbf{L}^2}^2 \,\mathrm{d}s \tag{2.18}$$

$$\le 2C_B^2 \int_0^t \lambda \|\mathrm{div}\,(\mathbf{u})\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{u})\|_{\mathbf{L}^2}^2 + \|\mathbf{u}\|_{\mathbf{L}^2}^2 \,\mathrm{d}s + 2C_A^2 + \int_0^t \left(\|\mathbf{v}\|_{\mathbf{L}^2}^2\right)\mathrm{d}s$$

$$\le CC_B^2 \int_0^t \lambda \|\mathrm{div}\,(\mathbf{u})\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{u})\|_{\mathbf{L}^2}^2 \,\mathrm{d}s + 2C_A^2 + \int_0^t \|\mathbf{v}\|_{\mathbf{L}^2}^2 \,\mathrm{d}s.$$

Taking the expectation on both sides of (2.18), we obtain

$$2\mathbb{E}\left[\int_0^t \left(\mathbf{F}[\mathbf{u}], \mathbf{v}\right)\mathrm{d}s\right] \le CC_B^2 \mathbb{E}\left[\int_0^t \lambda \|\mathrm{div}\,(\mathbf{u})\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{u})\|_{\mathbf{L}^2}^2 \,\mathrm{d}s\right] \tag{2.19}$$

$$+ 2C_A^2 + \mathbb{E}\left[\int_0^t \|\mathbf{v}\|_{\mathbf{L}^2}^2 \,\mathrm{d}s\right].$$

Similarly, using equations (2.1) and (2.5), the Poincaré inequality, and the Korn's inequality, the expectation of the second term I_2 can be bounded by

$$\mathbb{E}\left[\int_0^t \|\mathbf{G}[\mathbf{u}]\|_{\mathbf{L}^2}^2 \,\mathrm{d}s\right] \le 2C_B^2 \mathbb{E}\left[\int_0^t \lambda \|\mathrm{div}\,(\mathbf{u})\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{u})\|_{\mathbf{L}^2}^2 + \|\mathbf{u}\|_{\mathbf{L}^2}^2 \,\mathrm{d}s\right] + 2C_A^2 \tag{2.20}$$

$$\le CC_B^2 \mathbb{E}\left[\int_0^t \lambda \|\mathrm{div}\,(\mathbf{u})\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{u})\|_{\mathbf{L}^2}^2 \,\mathrm{d}s\right] + 2C_A^2.$$

The third term I_3 is a martingale, and $\mathbb{E}[I_3] = 0$. Taking the expectation on both sides of (2.17) and using the Gronwall's inequality, we get

$$\mathbb{E}\left[\|\mathbf{v}\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[\lambda \|\mathrm{div}\,(\mathbf{u})\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{u})\|_{\mathbf{L}^2}^2\right] \tag{2.21}$$

$$\le \left(\mathbb{E}\left[\|\mathbf{v}_0\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[\lambda \|\mathrm{div}\,(\mathbf{u}_0)\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{u}_0)\|_{\mathbf{L}^2}^2\right] + 4C_A^2\right)e^{CC_B^2}.$$

Step 2: Again, by applying Itô's formula to $\Phi_2\big(\mathbf{u}(\cdot)\big) = \|\mathscr{L}\mathbf{u}(\cdot)\|_{\mathbf{L}^2}^2$, we obtain

$$\|\mathscr{L}\mathbf{u}(t)\|_{\mathbf{L}^2}^2 = \|\mathscr{L}\mathbf{u}_0\|_{\mathbf{L}^2}^2 + 2\int_0^t \left(\mathscr{L}\mathbf{u}(s), \mathscr{L}\mathbf{v}(s)\right)\mathrm{d}s. \tag{2.22}$$

Applying Itô's formula to $\Phi_3\big(\mathbf{v}(\cdot)\big) = \lambda \left\|\mathrm{div}\,\big(\mathbf{v}(\cdot)\big)\right\|_{\mathbf{L}^2}^2 + \mu \left\|\varepsilon\big(\mathbf{v}(\cdot)\big)\right\|_{\mathbf{L}^2}^2$ leads to

$$\lambda \|\mathrm{div}\,(\mathbf{v}(t))\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{v}(t))\|_{\mathbf{L}^2}^2 = \lambda \|\mathrm{div}\,(\mathbf{v}_0)\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{v}_0)\|_{\mathbf{L}^2}^2 \tag{2.23}$$

$$+ 2\int_0^t \left(\lambda \big(\mathrm{div}\,(\mathscr{L}\mathbf{u}), \mathrm{div}\,(\mathbf{v})\big) + \mu \big(\varepsilon(\mathscr{L}\mathbf{u}), \varepsilon(\mathbf{v})\big)\right)\mathrm{d}s$$

$$+ 2\int_0^t \left(\lambda \big(\mathrm{div}\,(\mathbf{F}[\mathbf{u}]), \mathrm{div}\,(\mathbf{v})\big) + \mu \big(\varepsilon(\mathbf{F}[\mathbf{u}]), \varepsilon(\mathbf{v})\big)\right)\mathrm{d}s$$

$$+ \int_0^t \left(\lambda \|\mathrm{div}\,(\mathbf{G}[\mathbf{u}])\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{G}[\mathbf{u}])\|_{\mathbf{L}^2}^2\right)\mathrm{d}s$$

$$+ 2\int_0^t \left(\lambda \big(\mathrm{div}\,(\mathbf{G}[\mathbf{u}]\mathrm{d}W(s)), \mathrm{div}\,(\mathbf{v})\big) + \mu \big(\varepsilon(\mathbf{G}[\mathbf{u}]\mathrm{d}W(s)), \varepsilon(\mathbf{v})\big)\right)\mathrm{d}s.$$

Adding (2.22) and (2.23) gives

$$\|\mathscr{L}\mathbf{u}(t)\|_{\mathbf{L}^2}^2 + \lambda\,\|\mathrm{div}\,(\mathbf{v}(t))\|_{\mathbf{L}^2}^2 + \mu\,\|\varepsilon(\mathbf{v}(t))\|_{\mathbf{L}^2}^2 = \lambda\,\|\mathrm{div}\,(\mathbf{v}_0)\|_{\mathbf{L}^2}^2 + \mu\,\|\varepsilon(\mathbf{v}_0)\|_{\mathbf{L}^2}^2 \qquad (2.24)$$

$$+ \|\mathscr{L}\mathbf{u}_0\|_{\mathbf{L}^2}^2 + 2\int_0^t \Big(\lambda\,\big(\mathrm{div}\,(\mathbf{F}[\mathbf{u}]),\mathrm{div}\,(\mathbf{v})\big) + \mu\,\big(\varepsilon(\mathbf{F}[\mathbf{u}]),\varepsilon(\mathbf{v})\big)\Big)\mathrm{d}s$$

$$+ \int_0^t \big(\lambda\,\|\mathrm{div}\,(\mathbf{G}[\mathbf{u}])\|_{\mathbf{L}^2}^2 + \mu\,\|\varepsilon(\mathbf{G}[\mathbf{u}])\|_{\mathbf{L}^2}^2\big)\mathrm{d}s$$

$$+ 2\int_0^t \Big(\lambda\,\big(\mathrm{div}\,(\mathbf{G}[\mathbf{u}]\mathrm{d}W(s)),\mathrm{div}\,(\mathbf{v})\big) + \mu\,\big(\varepsilon(\mathbf{G}[\mathbf{u}]\mathrm{d}W(s)),\varepsilon(\mathbf{v})\big)\Big)\mathrm{d}s$$

$$:= \lambda\,\|\mathrm{div}\,(\mathbf{v}_0)\|_{\mathbf{L}^2}^2 + \mu\,\|\varepsilon(\mathbf{v}_0)\|_{\mathbf{L}^2}^2 + \|\mathscr{L}\mathbf{u}_0\|_{\mathbf{L}^2}^2 + I_1 + I_2 + I_3.$$

By equation (2.2) and the Korn's inequality, the first term I_1 can be bounded by

$$2\int_0^t \Big(\lambda\,\big(\mathrm{div}\,(\mathbf{F}[\mathbf{u}]),\mathrm{div}\,(\mathbf{v})\big) + \mu\,\big(\varepsilon(\mathbf{F}[\mathbf{u}]),\varepsilon(\mathbf{v})\big)\Big)\mathrm{d}s \qquad (2.25)$$

$$\le \int_0^t \big(\lambda\,\|\mathrm{div}\,(\mathbf{F}[\mathbf{u}])\|_{\mathbf{L}^2}^2 + \mu\,\|\varepsilon(\mathbf{F}[\mathbf{u}])\|_{\mathbf{L}^2}^2\big)\mathrm{d}s + \int_0^t \big(\lambda\,\|\mathrm{div}\,(\mathbf{v})\|_{\mathbf{L}^2}^2 + \mu\,\|\varepsilon(\mathbf{v})\|_{\mathbf{L}^2}^2\big)\mathrm{d}s$$

$$\le CC_A^2\int_0^t \big(\lambda\,\|\mathrm{div}\,(\mathbf{u})\|_{\mathbf{L}^2}^2 + \mu\,\|\varepsilon(\mathbf{u})\|_{\mathbf{L}^2}^2\big)\mathrm{d}s + \int_0^t \big(\lambda\,\|\mathrm{div}\,(\mathbf{v})\|_{\mathbf{L}^2}^2 + \mu\,\|\varepsilon(\mathbf{v})\|_{\mathbf{L}^2}^2\big)\mathrm{d}s.$$

Similarly, by equation (2.2) and the Korn's inequality, the second term I_2 can be bounded by

$$\int_0^t \big(\lambda\,\|\mathrm{div}\,(\mathbf{G}[\mathbf{u}])\|_{\mathbf{L}^2}^2 + \mu\,\|\varepsilon(\mathbf{G}[\mathbf{u}])\|_{\mathbf{L}^2}^2\big)\mathrm{d}s \le CC_A^2\int_0^t \big(\lambda\,\|\mathrm{div}\,(\mathbf{u})\|_{\mathbf{L}^2}^2 + \mu\,\|\varepsilon(\mathbf{u})\|_{\mathbf{L}^2}^2\big)\mathrm{d}s. \quad (2.26)$$

The third term I_3 is a martingale, and $\mathbb{E}[I_3] = 0$. Using equation (2.12), taking the expectation on both sides of (2.24) and using the Gronwall's inequality, we have

$$\mathbb{E}\Big[\|\mathscr{L}\mathbf{u}\|_{\mathbf{L}^2}^2\Big] + \mathbb{E}\Big[\lambda\,\|\mathrm{div}\,(\mathbf{v})\|_{\mathbf{L}^2}^2 + \mu\,\|\varepsilon(\mathbf{v})\|_{\mathbf{L}^2}^2\Big] \qquad (2.27)$$

$$\le \big(\mathbb{E}\Big[\lambda\,\|\mathrm{div}\,(\mathbf{v}_0)\|_{\mathbf{L}^2}^2 + \mu\,\|\varepsilon(\mathbf{v}_0)\|_{\mathbf{L}^2}^2\Big] + \mathbb{E}\Big[\|\mathscr{L}\mathbf{u}_0\|_{\mathbf{L}^2}^2\Big] + CC_A^2 C_{s1}\big)e^T.$$

Step 3: Similar to Step 1, by applying Itô's formula to the functional $\big\|\partial_{x_j}\mathbf{v}(\cdot)\big\|_{\mathbf{L}^2}^2$ for $1 \le j \le d$, we get

$$\big\|\partial_{x_j}\mathbf{v}(t)\big\|_{\mathbf{L}^2}^2 + \lambda\,\big\|\mathrm{div}\,(\partial_{x_j}\mathbf{u}(t))\big\|_{\mathbf{L}^2}^2 + \mu\,\big\|\varepsilon\big(\partial_{x_j}\mathbf{u}(t)\big)\big\|_{\mathbf{L}^2}^2 \qquad (2.28)$$

$$= \big\|\partial_{x_j}\mathbf{v}_0\big\|_{\mathbf{L}^2}^2 + \lambda\,\big\|\mathrm{div}\,(\partial_{x_j}\mathbf{u}_0)\big\|_{\mathbf{L}^2}^2 + \mu\,\big\|\varepsilon(\partial_{x_j}\mathbf{u}_0)\big\|_{\mathbf{L}^2}^2 + \int_0^t \big\|\partial_{x_j}\mathbf{G}[\mathbf{u}]\big\|_{\mathbf{L}^2}^2\,\mathrm{d}s$$

$$+ 2\int_0^t \big(\partial_{x_j}\mathbf{F}[\mathbf{u}],\partial_{x_j}\mathbf{v}\big)\mathrm{d}s + 2\int_0^t \big(\partial_{x_j}\mathbf{G}[\mathbf{u}],\partial_{x_j}\mathbf{v}\big)\mathrm{d}W(s)$$

$$:= \big\|\partial_{x_j}\mathbf{v}_0\big\|_{\mathbf{L}^2}^2 + \lambda\,\big\|\mathrm{div}\,(\partial_{x_j}\mathbf{u}_0)\big\|_{\mathbf{L}^2}^2 + \mu\,\big\|\varepsilon(\partial_{x_j}\mathbf{u}_0)\big\|_{\mathbf{L}^2}^2 + I_1 + I_2 + I_3.$$

Using equation (2.2) and Korn's inequality, we bound I_1 and I_2 by

$$2\mathbb{E}\Big[\int_0^t \big(\partial_{x_j}\mathbf{F}[\mathbf{u}],\partial_{x_j}\mathbf{v}\big)\mathrm{d}s\Big] \le CC_A^2\mathbb{E}\Big[\int_0^t \lambda\big\|\mathrm{div}(\partial_{x_j}\mathbf{u})\big\|_{\mathbf{L}^2}^2 \qquad (2.29)$$

$$+ \mu\big\|\varepsilon(\partial_{x_j}\mathbf{u})\big\|_{\mathbf{L}^2}^2\mathrm{d}s\Big] + \mathbb{E}\Big[\int_0^t \big\|\partial_{x_j}\mathbf{v}\big\|_{\mathbf{L}^2}^2\,\mathrm{d}s\Big],$$

$$\mathbb{E}\Big[\int_0^t \big\|\partial_{x_j}\mathbf{G}[\mathbf{u}]\big\|_{\mathbf{L}^2}^2\,\mathrm{d}s\Big] \le CC_A^2\mathbb{E}\Big[\int_0^t \lambda\big\|\mathrm{div}(\partial_{x_j}\mathbf{u})\big\|_{\mathbf{L}^2}^2 + \mu\big\|\varepsilon(\partial_{x_j}\mathbf{u})\big\|_{\mathbf{L}^2}^2\mathrm{d}s\Big]. \qquad (2.30)$$

The third term I_3 is a martingale, and $\mathbb{E}[I_3] = 0$. Taking the expectation on both sides of (2.17) and using Gronwall's inequality, we get

$$\mathbb{E}\left[\left\|\partial_{x_j}\mathbf{v}\right\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[\lambda \left\|\operatorname{div}\left(\partial_{x_j}\mathbf{u}\right)\right\|_{\mathbf{L}^2}^2 + \mu \left\|\varepsilon(\partial_{x_j}\mathbf{u})\right\|_{\mathbf{L}^2}^2\right] \tag{2.31}$$
$$\leq \left(\mathbb{E}\left[\left\|\partial_{x_j}\mathbf{v}_0\right\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[\lambda \left\|\operatorname{div}\left(\partial_{x_j}\mathbf{u}_0\right)\right\|_{\mathbf{L}^2}^2 + \mu \left\|\varepsilon(\partial_{x_j}\mathbf{u}_0)\right\|_{\mathbf{L}^2}^2\right]\right) e^{CC_A^2}.$$

The proof of Lemma 2.3 is complete. □

The following lemma is also needed to establish another stability property of the strong solution $(\mathbf{u}, \mathbf{v})$, which will be given in Lemma 2.5.

Lemma 2.4. *Let* $(\mathbf{u}, \mathbf{v})$ *be a strong solution to equations* (2.6)–(2.9)*. Under the assumptions* (2.1)–(2.5)*, there hold*

$$\mathbb{E}\left[\|\mathscr{L}\mathbf{F}[\mathbf{u}]\|_{\mathbf{L}^2}^2\right] \leq CC_A^2 C_{s1} + CC_A^2 C_{s3}, \tag{2.32}$$
$$\mathbb{E}\left[\|\mathscr{L}\mathbf{G}[\mathbf{u}]\|_{\mathbf{L}^2}^2\right] \leq CC_A^2 C_{s1} + CC_A^2 C_{s3}. \tag{2.33}$$

Proof. Notice that

$$\mathscr{L}\mathbf{F}[\mathbf{u}] = \operatorname{div}\left((\lambda \operatorname{div}\mathbf{F}[\mathbf{u}])\mathbf{I}\right) + \operatorname{div}\left(\mu\varepsilon(\mathbf{F}[\mathbf{u}])\right) := I_1 + I_2. \tag{2.34}$$

The first term I_1 on the right-hand side of (2.34) can be written as

$$I_1 = \operatorname{div}\left((\lambda(\nabla_{\mathbf{u}}\mathbf{F})^T : \nabla\mathbf{u})\mathbf{I}\right) \tag{2.35}$$
$$= \lambda \nabla\left((\nabla_{\mathbf{u}}\mathbf{F})^T : \nabla\mathbf{u}\right)$$
$$= \lambda \sum_{j=1}^d \left[(\nabla((\nabla_{\mathbf{u}}\mathbf{F})^T)_j)^T(\nabla\mathbf{u})_j + (\nabla(\nabla\mathbf{u})_j)^T((\nabla_{\mathbf{u}}\mathbf{F})^T)_j\right],$$

where $(\nabla_{\mathbf{u}}\mathbf{F})^T : \nabla\mathbf{u}$ denotes the element-wise product of two matrices, and $((\nabla_{\mathbf{u}}\mathbf{F})^T)_j$ and $(\nabla\mathbf{u})_j$ denote the j-th columns of $(\nabla_{\mathbf{u}}\mathbf{F})^T$ and $\nabla\mathbf{u}$, respectively.

By equations (2.2)–(2.3), equations (2.12)–(2.14), and the Korn's inequality, we have

$$\mathbb{E}\left[\|I_1\|_{\mathbf{L}^2}^2\right] \leq CC_A^2 C_{s1} + CC_A^2 C_{s3}. \tag{2.36}$$

Define A and B by their ij-th components as below:

$$A_{ij} = \frac{1}{2}\nabla_{\mathbf{u}}\mathbf{F}^j \cdot \mathbf{u}_{x_i} \qquad 1 \leq i, j \leq d, \tag{2.37}$$

$$B_{ij} = \frac{1}{2}\nabla_{\mathbf{u}}\mathbf{F}^i \cdot \mathbf{u}_{x_j} \qquad 1 \leq i, j \leq d, \tag{2.38}$$

where $\mathbf{F}^i$ and $\mathbf{F}^j$ denote the i-th and j-th components of $\mathbf{F}$, respectively.

Then the second term I_2 on the right-hand side of (2.34) can be written as

$$I_2 = \mu \operatorname{div}(A + B). \tag{2.39}$$

By equations (2.2)–(2.3), equations (2.12)–(2.14), and the Korn's inequality, we have

$$\mathbb{E}\left[\|I_2\|_{\mathbf{L}^2}^2\right] \leq C\mathbb{E}\left[\sum_{i=1}^d \sum_{j=1}^d \sum_{k=1}^d \|(\nabla_{\mathbf{u}}\mathbf{F}^i \cdot \mathbf{u}_{x_j})_{x_k}\|_{\mathbf{L}^2}^2\right] \tag{2.40}$$
$$\leq CC_A^2 C_{s1} + CC_A^2 C_{s3}.$$

Combining equations (2.36) and (2.40), the conclusion (2.32) is proved. The conclusion (2.33) can be similarly proved. □

Lemma 2.5. *Let* $(\mathbf{u}, \mathbf{v})$ *be a strong solution to equations* (2.6)–(2.9). *Under the assumptions* (2.1)–(2.5), *there holds*

$$\sup_{0 \le t \le T} \mathbb{E}\left[\|\mathscr{L}\mathbf{v}\|^2_{\mathbf{L}^2}\right] + \sup_{0 \le t \le T} \mathbb{E}\left[\lambda \|\mathrm{div}\left(\mathscr{L}\mathbf{u}\right)\|^2_{\mathbf{L}^2} + \mu \|\varepsilon(\mathscr{L}\mathbf{u})\|^2_{\mathbf{L}^2}\right] \le C_{s4}, \tag{2.41}$$

where

$$C_{s4} = \left(\mathbb{E}\left[\|\mathscr{L}\mathbf{v}_0\|^2_{\mathbf{L}^2}\right] + \mathbb{E}[\lambda \|\mathrm{div}(\mathscr{L}\mathbf{u}_0)\|^2_{\mathbf{L}^2} + \mu \|\varepsilon(\mathscr{L}\mathbf{u}_0)\|^2_{\mathbf{L}^2}] + CC_A^2(C_{s1} + C_{s3})\right)e^T.$$

Proof. Applying Itô's formula to $\Phi_4\big(\mathbf{u}(\cdot), \mathbf{v}(\cdot)\big) = \lambda \|\mathrm{div}\left(\mathscr{L}\mathbf{u}(\cdot)\right)\|^2_{\mathbf{L}^2} + \mu \|\varepsilon(\mathscr{L}\mathbf{u}(\cdot))\|^2_{\mathbf{L}^2} + \|\mathscr{L}\mathbf{v}(\cdot)\|^2_{\mathbf{L}^2}$, we obtain

$$\lambda \|\mathrm{div}\left(\mathscr{L}\mathbf{u}(t)\right)\|^2_{\mathbf{L}^2} + \mu \|\varepsilon(\mathscr{L}\mathbf{u}(t))\|^2_{\mathbf{L}^2} + \|\mathscr{L}\mathbf{v}(t)\|^2_{\mathbf{L}^2} \tag{2.42}$$

$$= \lambda \|\mathrm{div}\left(\mathscr{L}\mathbf{u}_0\right)\|^2_{\mathbf{L}^2} + \mu \|\varepsilon(\mathscr{L}\mathbf{u}_0)\|^2_{\mathbf{L}^2} + \|\mathscr{L}\mathbf{v}_0\|^2_{\mathbf{L}^2} + 2\int_0^t \left(\mathscr{L}\mathbf{F}[\mathbf{u}], \mathscr{L}\mathbf{v}\right)\mathrm{d}s$$

$$+ \int_0^t \|\mathscr{L}\mathbf{G}[\mathbf{u}]\|^2_{\mathbf{L}^2}\,\mathrm{d}s + 2\int_0^t \left(\mathscr{L}\mathbf{G}[\mathbf{u}]\mathrm{d}W(s), \mathscr{L}\mathbf{v}\right)$$

$$:= \lambda \|\mathrm{div}\left(\mathscr{L}\mathbf{u}_0\right)\|^2_{\mathbf{L}^2} + \mu \|\varepsilon(\mathscr{L}\mathbf{u}_0)\|^2_{\mathbf{L}^2} + \|\mathscr{L}\mathbf{v}_0\|^2_{\mathbf{L}^2} + I_1 + I_2 + I_3.$$

By equation (2.32), the expectation of the first term I_1 can be bounded by

$$\mathbb{E}\left[2\int_0^t \left(\mathscr{L}\mathbf{F}[\mathbf{u}], \mathscr{L}\mathbf{v}\right)\mathrm{d}s\right] \le \int_0^t \mathbb{E}\left[\|\mathscr{L}\mathbf{F}[\mathbf{u}]\|^2_{\mathbf{L}^2}\right]ds + \int_0^t \mathbb{E}\left[\|\mathscr{L}\mathbf{v}\|^2_{\mathbf{L}^2}\right]\mathrm{d}s \tag{2.43}$$

$$\le CC_A^2 C_{s1} + CC_A^2 C_{s3} + \int_0^t \mathbb{E}\left[\|\mathscr{L}\mathbf{v}\|^2_{\mathbf{L}^2}\right]\mathrm{d}s.$$

By equation (2.33), the expectation of the second term I_2 can be bounded by

$$\mathbb{E}\left[\int_0^t \|\mathscr{L}\mathbf{G}[\mathbf{u}]\|^2_{\mathbf{L}^2}\,\mathrm{d}s\right] \le CC_A^2 C_{s1} + CC_A^2 C_{s3}. \tag{2.44}$$

The third term I_3 is a martingale, and $\mathbb{E}[I_3] = 0$. Taking the exception on the both side of (2.42) and using Gronwall's inequality, we have

$$\mathbb{E}\left[\lambda \|\mathrm{div}\left(\mathscr{L}\mathbf{u}\right)\|^2_{\mathbf{L}^2} + \mu \|\varepsilon(\mathscr{L}\mathbf{u})\|^2_{\mathbf{L}^2}\right] + \mathbb{E}\left[\|\mathscr{L}\mathbf{v}\|^2_{\mathbf{L}^2}\right] \tag{2.45}$$

$$\le \left(\mathbb{E}\left[\|\mathscr{L}\mathbf{v}_0\|^2_{\mathbf{L}^2}\right] + \mathbb{E}[\lambda \|\mathrm{div}(\mathscr{L}\mathbf{u}_0)\|^2_{\mathbf{L}^2} + \mu \|\varepsilon(\mathscr{L}\mathbf{u}_0)\|^2_{\mathbf{L}^2}] + CC_A^2(C_{s1} + C_{s3})\right)e^T.$$

The proof of Lemma 2.3 is complete. □

Based on Lemma 2.3, we can establish Hölder continuity in time of the solution $\mathbf{u}$ in $\mathbf{L}^2$-norm, $\mathbf{H}^1$-seminorm, and $\mathbf{H}^2$-seminorm. These results play a key role in the error analysis.

Lemma 2.6. *Let* $(\mathbf{u}, \mathbf{v})$ *be a strong solution to problem* (2.6)–(2.9). *Under the assumptions* (2.1)–(2.5), *for any* $s, t \in [0, T]$, *we have*

$$\mathbb{E}\left[\|\mathbf{u}(t) - \mathbf{u}(s)\|_{\mathbf{L}^2}^{2p}\right] \leq C_{s1}|t-s|^{2p}, \qquad (2.46)$$

$$\mathbb{E}\left[\lambda\left\|\mathrm{div}\left(\mathbf{u}(t) - \mathbf{u}(s)\right)\right\|_{\mathbf{L}^2}^{2p} + \mu\left\|\varepsilon\left(\mathbf{u}(t) - \mathbf{u}(s)\right)\right\|_{\mathbf{L}^2}^{2p}\right] \leq C_{s2}|t-s|^{2p}, \qquad (2.47)$$

$$\mathbb{E}\left[\left\|\mathscr{L}\left(\mathbf{u}(t) - \mathbf{u}(s)\right)\right\|_{\mathbf{L}^2}^{2}\right] \leq C_{s4}|t-s|^{2}. \qquad (2.48)$$

Proof. Note that $\mathbf{u}(t) - \mathbf{u}(s) = \int_s^t \mathbf{v}(\xi)\mathrm{d}\xi$, and then we get

$$\mathbb{E}\left[\|\mathbf{u}(t) - \mathbf{u}(s)\|_{\mathbf{L}^2}^{2p}\right] \leq \mathbb{E}\left[\left\|\left(\int_s^t |\mathbf{v}(\xi)|^2 \mathrm{d}\xi\right)^{\frac{1}{2}}\left(\int_s^t 1\mathrm{d}\xi\right)^{\frac{1}{2}}\right\|_{\mathbf{L}^2}^{2p}\right] \qquad (2.49)$$

$$\leq |t-s|^p\mathbb{E}\left[\left(\int_s^t \|\mathbf{v}(\xi)\|_{\mathbf{L}^2}^2\,\mathrm{d}\xi\right)^p\right]$$

$$\leq \sup_{0 \leq t \leq T}\mathbb{E}\left[\|\mathbf{v}(t)\|_{\mathbf{L}^2}^2\right]|t-s|^{2p}.$$

Hence, (2.46) holds when applying (2.12) in Lemma 2.3. The inequality (2.47) and (2.48) could be derived similarly. $\qquad\square$

The next Lemma establishes some Hölder continuity in time results for the solution $\mathbf{v}$ with respect to $\mathbf{L}^2$-norm, $\mathbf{H}^1$-seminorm and $\mathbf{H}^2$-seminorm.

Lemma 2.7. *Let* $(\mathbf{u}, \mathbf{v})$ *be a strong solution to problem* (2.6)–(2.9). *Under the assumptions* (2.1)–(2.5), *for any* $s, t \in [0, T]$, *we have*

$$\mathbb{E}\left[\|\mathbf{v}(t) - \mathbf{v}(s)\|_{\mathbf{L}^2}^{2}\right] \leq C_{s5}|t-s|, \qquad (2.50)$$

$$\mathbb{E}\left[\lambda\left\|\mathrm{div}\left(\mathbf{v}(t) - \mathbf{v}(s)\right)\right\|_{\mathbf{L}^2}^{2} + \mu\left\|\varepsilon\left(\mathbf{v}(t) - \mathbf{v}(s)\right)\right\|_{\mathbf{L}^2}^{2}\right] \leq C_{s6}|t-s|, \qquad (2.51)$$

$$\mathbb{E}\left[\left\|\mathscr{L}\left(\mathbf{v}(t) - \mathbf{v}(s)\right)\right\|_{\mathbf{L}^2}^{2}\right] \leq C_{s7}|t-s|, \qquad (2.52)$$

where

$$C_{s5} = e^T(CC_B^2C_{s1} + 4C_A^2),$$
$$C_{s6} = e^T(CC_A^2C_{s1} + CC_B^2C_{s1} + 2C_A^2),$$
$$C_{s7} = CC_A^2(C_{s1} + C_{s2} + C_{s4}).$$

Proof. Step 1: Fix $s \geq 0$. Applying Itô's formula to $\Phi_5(\mathbf{u}(\cdot)) := \lambda\|\mathrm{div}\left(\mathbf{u}(\cdot) - \mathbf{u}(s)\right)\|_{\mathbf{L}^2}^2 + \mu\|\varepsilon\left(\mathbf{u}(\cdot) - \mathbf{u}(s)\right)\|_{\mathbf{L}^2}^2$, we have

$$\lambda\left\|\mathrm{div}\left(\mathbf{u}(t) - \mathbf{u}(s)\right)\right\|_{\mathbf{L}^2}^{2} + \mu\left\|\varepsilon\left(\mathbf{u}(t) - \mathbf{u}(s)\right)\right\|_{\mathbf{L}^2}^{2} \qquad (2.53)$$

$$= 2\int_s^t \left(\lambda\left(\mathrm{div}\left(\mathbf{u}(\xi) - \mathbf{u}(s)\right), \mathrm{div}\left(\mathbf{v}(\xi)\right)\right) + \mu\left(\varepsilon(\mathbf{u}(\xi) - \mathbf{u}(s)), \varepsilon(\mathbf{v}(\xi))\right)\right)\mathrm{d}\xi.$$

Applying Itô's formula to $\Phi_6(\mathbf{v}(\cdot)) := \|\mathbf{v}(\cdot) - \mathbf{v}(s)\|_{\mathbf{L}^2}^2$ and using (2.53), we get

$$
\begin{aligned}
\|\mathbf{v}(t) - \mathbf{v}(s)\|_{\mathbf{L}^2}^2 & + \lambda \left\|\mathrm{div}\left(\mathbf{u}(t) - \mathbf{u}(s)\right)\right\|_{\mathbf{L}^2}^2 + \mu \left\|\varepsilon\left(\mathbf{u}(t) - \mathbf{u}(s)\right)\right\|_{\mathbf{L}^2}^2 \\
& = 2 \int_s^t \left(\mathbf{F}[\mathbf{u}(\xi)], \mathbf{v}(\xi) - \mathbf{v}(s)\right) \mathrm{d}\xi + \int_s^t \|\mathbf{G}[\mathbf{u}(\xi)]\|_{\mathbf{L}^2}^2 \, \mathrm{d}\xi \\
& \quad + 2 \int_s^t \left(\mathbf{G}[\mathbf{u}(\xi)], \mathbf{v}(\xi) - \mathbf{v}(s)\right) \mathrm{d}W(\xi) \\
& := I_1 + I_2 + I_3.
\end{aligned}
\tag{2.54}
$$

Similar to the analysis in (2.18)–(2.20), and by equation (2.12), we have

$$
\begin{aligned}
\int_s^t \|\mathbf{F}[\mathbf{u}(\xi)]\|_{\mathbf{L}^2}^2 \, \mathrm{d}\xi & + \int_s^t \|\mathbf{G}[\mathbf{u}(\xi)]\|_{\mathbf{L}^2}^2 \, \mathrm{d}\xi \\
& \leq C C_B^2 \int_s^t \left(\lambda \|\mathrm{div}\,(\mathbf{u})\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{u})\|_{\mathbf{L}^2}^2\right) \mathrm{d}\xi + 4 C_A^2 |t - s| \\
& \leq (C C_B^2 C_{s1} + 4 C_A^2)|t - s|.
\end{aligned}
\tag{2.55}
$$

Then the first term and the second term on the right-hand side of (2.54) can be bounded by

$$
\begin{aligned}
2 \int_s^t \left(\mathbf{F}[\mathbf{u}(\xi)], \mathbf{v}(\xi) - \mathbf{v}(s)\right) \mathrm{d}\xi & + \int_s^t \|\mathbf{G}[\mathbf{u}(\xi)]\|_{\mathbf{L}^2}^2 \, \mathrm{d}\xi \\
& \leq \int_s^t \|\mathbf{v}(\xi) - \mathbf{v}(s)\|_{\mathbf{L}^2}^2 \, \mathrm{d}\xi + \int_s^t \left(\|\mathbf{F}[\mathbf{u}(\xi)]\|_{\mathbf{L}^2}^2 + \|\mathbf{G}[\mathbf{u}(\xi)]\|_{\mathbf{L}^2}^2\right) \mathrm{d}\xi \\
& \leq \int_s^t \|\mathbf{v}(\xi) - \mathbf{v}(s)\|_{\mathbf{L}^2}^2 \, \mathrm{d}\xi + (C C_B^2 C_{s1} + 4 C_A^2)|t - s|.
\end{aligned}
\tag{2.56}
$$

The third term is a martingale and $\mathbb{E}\left[\int_s^t \left(\mathbf{G}[\mathbf{u}(\xi)], \mathbf{v}(\xi) - \mathbf{v}(s)\right)\right] = 0$. Taking the expectation on both sides of (2.54), we get

$$
\begin{aligned}
\mathbb{E}\left[\|\mathbf{v}(t) - \mathbf{v}(s)\|_{\mathbf{L}^2}^2\right] & + \mathbb{E}\left[\lambda \left\|\mathrm{div}\left(\mathbf{u}(t) - \mathbf{u}(s)\right)\right\|_{\mathbf{L}^2}^2 + \mu \left\|\varepsilon\left(\mathbf{u}(t) - \mathbf{u}(s)\right)\right\|_{\mathbf{L}^2}^2\right] \\
& \leq \int_s^t \mathbb{E}\left[\|\mathbf{v}(\xi) - \mathbf{v}(s)\|_{\mathbf{L}^2}^2\right] \mathrm{d}\xi + (C C_B^2 C_{s1} + 4 C_A^2)|t - s|.
\end{aligned}
\tag{2.57}
$$

Using Gronwall's inequality yields

$$
\begin{aligned}
\mathbb{E}\left[\|\mathbf{v}(t) - \mathbf{v}(s)\|_{\mathbf{L}^2}^2\right] & + \mathbb{E}\left[\lambda \left\|\mathrm{div}\left(\mathbf{u}(t) - \mathbf{u}(s)\right)\right\|_{\mathbf{L}^2}^2 + \mu \left\|\varepsilon\left(\mathbf{u}(t) - \mathbf{u}(s)\right)\right\|_{\mathbf{L}^2}^2\right] \\
& \leq e^T (C C_B^2 C_{s1} + 4 C_A^2)|t - s|.
\end{aligned}
\tag{2.58}
$$

Step 2: Fix $s \geq 0$. Applying Itô's formula to $\Phi_7(\mathbf{u}(\cdot)) = \|\mathscr{L}\mathbf{u}(\cdot) - \mathscr{L}\mathbf{u}(s)\|_{\mathbf{L}^2}^2$ and using integration by parts yield

$$
\|\mathscr{L}\mathbf{u}(t) - \mathscr{L}\mathbf{u}(s)\|_{\mathbf{L}^2}^2 = 2 \int_s^t \left(\mathscr{L}\mathbf{u}(\xi) - \mathscr{L}\mathbf{u}(s), \mathscr{L}\mathbf{v}(\xi)\right) \mathrm{d}\xi.
\tag{2.59}
$$

Applying Itô's formula to $\Phi_8(\mathbf{v}(\cdot)) = \lambda \left\| \mathrm{div} \left(\mathbf{v}(\cdot) - \mathbf{v}(s) \right) \right\|_{\mathbf{L}^2}^2 + \mu \left\| \varepsilon \left(\mathbf{v}(\cdot) - \mathbf{v}(s) \right) \right\|_{\mathbf{L}^2}^2$ and using (2.59) yield

$$\left\| \mathscr{L} \mathbf{u}(t) - \mathscr{L} \mathbf{u}(s) \right\|_{\mathbf{L}^2}^2 + \lambda \left\| \mathrm{div} \left(\mathbf{v}(\cdot) - \mathbf{v}(s) \right) \right\|_{\mathbf{L}^2}^2 + \mu \left\| \varepsilon \left(\mathbf{v}(\cdot) - \mathbf{v}(s) \right) \right\|_{\mathbf{L}^2}^2 \tag{2.60}$$

$$\leq 2 \int_s^t \Big(\lambda \left(\mathrm{div} \left(\mathbf{v}(\xi) - \mathbf{v}(s) \right), \mathrm{div} \left(\mathbf{F}[\mathbf{u}(\xi)] \right) \right)$$

$$+ \mu \left(\varepsilon(\mathbf{v}(\xi) - \mathbf{v}(s)), \varepsilon(\mathbf{F}[\mathbf{u}(\xi)]) \right) \Big) \mathrm{d}\xi + \int_s^t \left\| \mathbf{G}[\mathbf{u}(\xi)] \right\|_{\mathbf{L}^2}^2 \mathrm{d}\xi$$

$$+ 2 \int_s^t \lambda \left(\mathrm{div} \left(\mathbf{v}(\xi) - \mathbf{v}(s) \right) \mathrm{d}W(\xi), \mathrm{div} \left(\mathbf{G}[\mathbf{u}(\xi)] \right) \right) \mathrm{d}\xi$$

$$+ 2 \int_s^t \mu \left(\varepsilon \left(\mathbf{v}(\xi) - \mathbf{v}(s) \right) \mathrm{d}W(\xi), \varepsilon(\mathbf{G}[\mathbf{u}(\xi)]) \right) \mathrm{d}\xi$$

$$:= I_1 + I_2 + I_3 + I_4.$$

By equations (2.2) and (2.12), and the Korn's inequality, the first term I_1 can be bounded by

$$2 \int_s^t \Big(\lambda \left(\mathrm{div} \left(\mathbf{v}(\xi) - \mathbf{v}(s) \right), \mathrm{div} \left(\mathbf{F}[\mathbf{u}(\xi)] \right) \right) + \mu \left(\varepsilon(\mathbf{v}(\xi) - \mathbf{v}(s)), \varepsilon(\mathbf{F}[\mathbf{u}(\xi)]) \right) \Big) \mathrm{d}\xi \tag{2.61}$$

$$\leq \int_s^t \lambda \left\| \mathrm{div} \left(\mathbf{F}[\mathbf{u}(\xi)] \right) \right\|_{\mathbf{L}^2}^2 \mathrm{d}\xi + \int_s^t \mu \left\| \varepsilon \left(\mathbf{F}[\mathbf{u}(\xi)] \right) \right\|_{\mathbf{L}^2}^2 \mathrm{d}\xi$$

$$+ \int_s^t \left(\lambda \left\| \mathrm{div} \left(\mathbf{v}(\xi) - \mathbf{v}(s) \right) \right\|_{\mathbf{L}^2}^2 + \mu \left\| \varepsilon \left(\mathbf{v}(\xi) - \mathbf{v}(s) \right) \right\|_{\mathbf{L}^2}^2 \right) \mathrm{d}\xi$$

$$\leq C C_A^2 \int_s^t \left(\lambda \left\| \mathrm{div} \left(\mathbf{u}(\xi) \right) \right\|_{\mathbf{L}^2}^2 + \mu \left\| \varepsilon \left(\mathbf{u}(\xi) \right) \right\|_{\mathbf{L}^2}^2 \right) \mathrm{d}\xi$$

$$+ \int_s^t \lambda \left\| \mathrm{div} \left(\mathbf{v}(\xi) - \mathbf{v}(s) \right) \right\|_{\mathbf{L}^2}^2 \mathrm{d}\xi + \int_s^t \mu \left\| \varepsilon \left(\mathbf{v}(\xi) - \mathbf{v}(s) \right) \right\|_{\mathbf{L}^2}^2 \mathrm{d}\xi$$

$$\leq \int_s^t \left(\lambda \left\| \mathrm{div} \left(\mathbf{v}(\xi) - \mathbf{v}(s) \right) \right\|_{\mathbf{L}^2}^2 + \mu \left\| \varepsilon \left(\mathbf{v}(\xi) - \mathbf{v}(s) \right) \right\|_{\mathbf{L}^2}^2 \right) \mathrm{d}\xi + C C_A^2 C_{s1} |t - s|.$$

Similar to equation (2.20), by the Poincaré inequality, Korn's inequality, and equations (2.1), (2.5) and (2.12), the second term I_2 can be bounded by

$$\int_s^t \left\| \mathbf{G}[\mathbf{u}] \right\|_{\mathbf{L}^2}^2 \mathrm{d}s \leq C C_B^2 \int_s^t \lambda \left\| \mathrm{div} \left(\mathbf{u} \right) \right\|_{\mathbf{L}^2}^2 + \mu \left\| \varepsilon(\mathbf{u}) \right\|_{\mathbf{L}^2}^2 \mathrm{d}s + 2 C_A^2 |t - s| \tag{2.62}$$

$$\leq (C C_B^2 C_{s1} + 2 C_A^2) |t - s|.$$

The last two terms I_3 and I_4 are martingales, so $\mathbb{E}[I_3 + I_4] = 0$. Taking the expectation on both sides of equation (2.60) and using (2.61)-(2.62) give

$$\mathbb{E} \left[\left\| \mathscr{L} \mathbf{u}(t) - \mathscr{L} \mathbf{u}(s) \right\|_{\mathbf{L}^2}^2 \right] + \mathbb{E} \left[\lambda \left\| \mathrm{div} \left(\mathbf{v}(\cdot) - \mathbf{v}(s) \right) \right\|_{\mathbf{L}^2}^2 + \mu \left\| \varepsilon \left(\mathbf{v}(\cdot) - \mathbf{v}(s) \right) \right\|_{\mathbf{L}^2}^2 \right] \tag{2.63}$$

$$\leq \int_s^t \mathbb{E} \left[\lambda \left\| \mathrm{div} \left(\mathbf{v}(\xi) - \mathbf{v}(s) \right) \right\|_{\mathbf{L}^2}^2 + \mu \left\| \varepsilon \left(\mathbf{v}(\xi) - \mathbf{v}(s) \right) \right\|_{\mathbf{L}^2}^2 \right] \mathrm{d}\xi$$

$$+ (C C_A^2 C_{s1} + C C_B^2 C_{s1} + 2 C_A^2) |t - s|.$$

Then by Gronwall's inequality we obtain

$$
\mathbb{E}\left[\lambda \left\|\operatorname{div}\left(\mathbf{v}(t)-\mathbf{v}(s)\right)\right\|_{\mathbf{L}^2}^2 + \mu \left\|\varepsilon\left(\mathbf{v}(t)-\mathbf{v}(s)\right)\right\|_{\mathbf{L}^2}^2\right] \tag{2.64}
$$
$$
\leq e^T (CC_A^2 C_{s1} + CC_B^2 C_{s1} + 2C_A^2)|t-s|.
$$

Step 3: Fix $s \geq 0$. Applying Itô's formula to

$$
\Phi_9\left(\mathbf{u}(\cdot)\right) = \lambda \left\|\operatorname{div}\left(\mathscr{L}\mathbf{u}(\cdot)-\mathscr{L}\mathbf{u}(s)\right)\right\|_{\mathbf{L}^2}^2 + \mu \left\|\varepsilon\left(\mathscr{L}\mathbf{u}(\cdot)-\mathscr{L}\mathbf{u}(s)\right)\right\|_{\mathbf{L}^2}^2,
$$

and then we get

$$
\lambda \left\|\operatorname{div}\left(\mathscr{L}\mathbf{u}(t)-\mathscr{L}\mathbf{u}(s)\right)\right\|_{\mathbf{L}^2}^2 + \mu \left\|\varepsilon\left(\mathscr{L}\mathbf{u}(t)-\mathscr{L}\mathbf{u}(s)\right)\right\|_{\mathbf{L}^2}^2 \tag{2.65}
$$
$$
= 2\int_s^t \Big(\lambda\left(\operatorname{div}\left(\mathscr{L}\mathbf{u}(\xi)-\mathscr{L}\mathbf{u}(s)\right),\operatorname{div}\left(\mathscr{L}\mathbf{v}(\xi)\right)\right)
$$
$$
+ \mu\left(\varepsilon(\mathscr{L}\mathbf{u}(\xi)-\mathscr{L}\mathbf{u}(s)),\varepsilon(\mathscr{L}\mathbf{v}(\xi))\right)\Big)d\xi.
$$

Applying Itô's formula to $\Phi_{10}\left(\mathbf{v}(\cdot)\right) = \|\mathscr{L}\mathbf{v}(\cdot)-\mathscr{L}\mathbf{v}(s)\|_{\mathbf{L}^2}^2$, and using integration by parts and equation (2.65), we have

$$
\|\mathscr{L}\mathbf{v}(t)-\mathscr{L}\mathbf{v}(s)\|_{\mathbf{L}^2}^2 + \lambda \left\|\operatorname{div}\left(\mathscr{L}\mathbf{u}(t)-\mathscr{L}\mathbf{u}(s)\right)\right\|_{\mathbf{L}^2}^2 + \mu \left\|\varepsilon\left(\mathscr{L}\mathbf{u}(t)-\mathscr{L}\mathbf{u}(s)\right)\right\|_{\mathbf{L}^2}^2 \tag{2.66}
$$
$$
= 2\int_s^t \left(\mathscr{L}\mathbf{v}(\xi)-\mathscr{L}\mathbf{v}(s),\mathscr{L}\mathbf{F}[\mathbf{u}(\xi)]\right)d\xi + \int_s^t \|\mathscr{L}\mathbf{G}[\mathbf{u}(\xi)]\|_{\mathbf{L}^2}^2 d\xi
$$
$$
+ 2\int_s^t \left(\mathscr{L}\mathbf{v}(\xi)-\mathscr{L}\mathbf{v}(s),\mathscr{L}\mathbf{G}[\mathbf{u}(\xi)]dW(\xi)\right)
$$
$$
:= I_1 + I_2 + I_3.
$$

Similar to the estimation of equation (2.44). we have

$$
\int_s^t \|\mathscr{L}\mathbf{F}[\mathbf{u}(\xi)]\|_{\mathbf{L}^2}^2 + \|\mathscr{L}\mathbf{G}[\mathbf{u}(\xi)]\|_{\mathbf{L}^2}^2 ds \tag{2.67}
$$
$$
\leq CC_A^2 \int_s^t \lambda \|\operatorname{div}\left(\mathscr{L}\mathbf{u}\right)\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathscr{L}\mathbf{u})\|_{\mathbf{L}^2}^2 + \|\mathscr{L}\mathbf{u}\|_{\mathbf{L}^2}^2 ds
$$
$$
+ CC_A^2 \int_s^t \lambda \|\operatorname{div}\left(\mathbf{u}\right)\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{u})\|_{\mathbf{L}^2}^2 ds
$$
$$
\leq CC_A^2 (C_{s1} + C_{s2} + C_{s4})|t-s|.
$$

By equation (2.67), the first and second terms on the right-hand side of (2.66) could be written as

$$
2\int_s^t \left(\mathscr{L}\mathbf{v}(\xi)-\mathscr{L}\mathbf{v}(s),\mathscr{L}\mathbf{F}[\mathbf{u}(\xi)]\right)d\xi + \int_s^t \|\mathscr{L}\mathbf{G}[\mathbf{u}(\xi)]\|_{\mathbf{L}^2}^2 d\xi \tag{2.68}
$$
$$
\leq \int_s^t \|\mathscr{L}\mathbf{F}[\mathbf{u}(\xi)]\|_{\mathbf{L}^2}^2 + \|\mathscr{L}\mathbf{G}[\mathbf{u}(\xi)]\|_{\mathbf{L}^2}^2 d\xi + \int_s^t \|\mathscr{L}\mathbf{v}(\xi)-\mathscr{L}\mathbf{v}(s)\|_{\mathbf{L}^2}^2 d\xi
$$
$$
\leq CC_A^2 (C_{s1} + C_{s2} + C_{s4})|t-s| + \int_s^t \|\mathscr{L}\mathbf{v}(\xi)-\mathscr{L}\mathbf{v}(s)\|_{\mathbf{L}^2}^2 d\xi.
$$

The third term I_3 is a martingale and $\mathbb{E}[I_3] = 0$. Taking the exceptation on both sides of (2.66), we get

$$\mathbb{E}\left[\|\mathscr{L}\mathbf{v}(t) - \mathscr{L}\mathbf{v}(s)\|_{\mathbf{L}^2}^2\right] \tag{2.69}$$

$$+ \mathbb{E}\left[\lambda\|\mathrm{div}\,(\mathscr{L}\mathbf{u}(t) - \mathscr{L}\mathbf{u}(s))\|_{\mathbf{L}^2}^2 + \mu\|\varepsilon(\mathscr{L}\mathbf{u}(t) - \mathscr{L}\mathbf{u}(s))\|_{\mathbf{L}^2}^2\right]$$

$$\leq CC_A^2(C_{s1} + C_{s2} + C_{s4})|t - s| + \int_s^t \mathbb{E}\left[\|\mathscr{L}\mathbf{v}(\xi) - \mathscr{L}\mathbf{v}(s)\|_{\mathbf{L}^2}^2\right]\mathrm{d}\xi.$$

Then the Gronwall's inequality yields

$$\mathbb{E}\left[\|\mathscr{L}\mathbf{v}(t) - \mathscr{L}\mathbf{v}(s)\|_{\mathbf{L}^2}^2\right] \leq CC_A^2(C_{s1} + C_{s2} + C_{s4})|t - s|. \tag{2.70}$$

The proof is complete. $\qquad\square$

3. SEMI-DISCRETIZATION IN TIME

In this section we propose a time semi-discrete scheme to approximate the nonlinear stochastic elastic wave equations. The goals are to prove some stability results and to establish the error estimates.

3.1. **Time semi-discrete scheme.** Let $0 < k << T$ and $t_n := nk$ for $n = 0, 1, 2, \cdots, N$ be uniform meshes with size k on the interval $[0, T]$.

Scheme 1. *Let $\mathbf{u}^0$ be an $\mathscr{F}_{t_0}$-measurable and $\mathbf{H}_0^1$-valued random variable and $\mathbf{v}^0$ be an $\mathscr{F}_{t_0}$-measurable and $\mathbf{H}_0^1$-valued random variable. For each $n \geq 1$, find $(\mathbf{H}_0^1 \times \mathbf{L}^2)$-valued and $\{\mathscr{F}_{t_{n+1}}\}$-measurable random variables $(\mathbf{u}^{n+1}, \mathbf{v}^{n+1})$ such that $\mathbb{P}$-a.s.*

$$\left(\mathbf{u}^{n+1} - \mathbf{u}^n, \phi\right) = k\left(\mathbf{v}^{n+1}, \phi\right) \qquad \forall \phi \in \mathbf{L}^2, \tag{3.1}$$

$$(\mathbf{v}^{n+1} - \mathbf{v}^n, \psi) = -k\lambda\left(\mathrm{div}\,(\mathbf{u}^{n+\frac{1}{2}}), \mathrm{div}\,(\psi)\right) - k\mu\left(\varepsilon(\mathbf{u}^{n+\frac{1}{2}}), \varepsilon(\psi)\right) \tag{3.2}$$

$$+ \left(\mathbf{G}[\mathbf{u}^n]\Delta W_n, \psi\right) + k\left(\mathbf{F}[\mathbf{u}^n], \psi\right) \qquad \forall \psi \in \mathbf{H}_0^1,$$

where $\Delta W_n := W(t_{n+1}) - W(t_n)$ and $\mathbf{u}^{n+\frac{1}{2}} := \frac{1}{2}\left(\mathbf{u}^{n+1} + \mathbf{u}^n\right)$.

Remark 3.1. (a) At each time step, the above scheme is a nonlinear random PDE system for $(\mathbf{u}^{n+1}, \mathbf{v}^{n+1})$ whose well-posedness can be proved by a standard fixed point argument based on the stability estimates to be given in the next subsection.

(b) Following [16], a possible improvement to Scheme 1 is the following modified scheme: Seeking $(\mathbf{H}_0^1 \times \mathbf{L}^2)$-valued and $\mathscr{F}_{t_{n+1}}$-measurable random variables $(\mathbf{u}^{n+1}, \mathbf{v}^{n+1})$ such that $\mathbb{P}$-a.s.

$$\left(\mathbf{u}^{n+1} - \mathbf{u}^n, \phi\right) = k\left(\mathbf{v}^{n+1}, \phi\right) \qquad \forall \phi \in \mathbf{L}^2, \tag{3.3}$$

$$(\mathbf{v}^{n+1} - \mathbf{v}^n, \psi) = -k\lambda\left(\mathrm{div}\,(\mathbf{u}^{n,\frac{1}{2}}), \mathrm{div}\,(\psi)\right) - k\mu\left(\varepsilon(\mathbf{u}^{n,\frac{1}{2}}), \varepsilon(\psi)\right) \tag{3.4}$$

$$+ \left(\mathbf{G}[\mathbf{u}^n]\Delta W_n, \psi\right) + k\left(\mathbf{F}[\mathbf{u}^n], \psi\right) \qquad \forall \psi \in \mathbf{H}_0^1,$$

where $\mathbf{u}^{n,\frac{1}{2}} := \frac{1}{2}\left(\mathbf{u}^{n+1} + \mathbf{u}^{n-1}\right)$.

However, such an improvement could not be realized unless some more involved higher order treatment of the noise term is adopted as demonstrated in [18] for the corresponding stochastic acoustic wave equations.

3.2. **Stability analysis of the time semi-discrete scheme.** Definie the following energy functional:

$$J^n = J(\mathbf{u}^n, \mathbf{v}^n) := \frac{1}{2} \left(\|\mathbf{v}^n\|_{\mathbf{L}^2}^2 + \lambda \|\mathrm{div}\,(\mathbf{u}^n)\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{u}^n)\|_{\mathbf{L}^2}^2 \right). \tag{3.5}$$

The following lemma gives the estimate for the expectation of the above energy functional.

Lemma 3.2. *Let $I_k := \{t_n\}_{n=0}^N$ be uniform meshes with size k satisfying $0 < k \leq k_0 < T$ and $(\mathbf{u}^0, \mathbf{v}^0) = (\mathbf{u}_0, \mathbf{v}_0) \in (\mathbf{H}^2 \cap \mathbf{H}_0^1) \times \mathbf{H}_0^1$ be given. Then there holds*

$$\max_{1 \leq n \leq N} \mathbb{E}\left[J(\mathbf{u}^n, \mathbf{v}^n) \right] \leq \left(\mathbb{E}\left[J(\mathbf{u}^0, \mathbf{v}^0) \right] + 4C_A^2 \right) e^{CC_B^2}. \tag{3.6}$$

Proof. Fix $\omega \in \Omega$ and choose $\psi = \mathbf{v}^{n+1}$ in (3.2), and then we get

$$\frac{1}{2} \left[\|\mathbf{v}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{v}^n\|_{\mathbf{L}^2}^2 + \|\mathbf{v}^{n+1} - \mathbf{v}^n\|_{\mathbf{L}^2}^2 \right] \tag{3.7}$$
$$= k\left(\mathscr{L}\mathbf{u}^{n+\frac{1}{2}}, \mathbf{v}^{n+1} \right) + \left(\mathbf{G}[\mathbf{u}^n]\Delta W_n, \mathbf{v}^{n+1} \right) + k\left(\mathbf{F}[\mathbf{u}^n], \mathbf{v}^{n+1} \right).$$

Choose $\phi = -\mathscr{L}\mathbf{u}^{n+\frac{1}{2}}$ in (3.1), and then we get

$$(\mathbf{u}^{n+1} - \mathbf{u}^n, -\mathscr{L}\mathbf{u}^{n+\frac{1}{2}}) = -k\left(\mathscr{L}\mathbf{u}^{n+\frac{1}{2}}, \mathbf{v}^{n+1} \right). \tag{3.8}$$

Combining (3.7) and (3.8) and taking the expectation on both sides yield

$$\mathbb{E}\left[(\mathbf{u}^{n+1} - \mathbf{u}^n, -\mathscr{L}\mathbf{u}^{n+\frac{1}{2}}) \right] + \frac{1}{2} \left(\mathbb{E}\left[\|\mathbf{v}^{n+1}\|_{\mathbf{L}^2}^2 \right] - \mathbb{E}\left[\|\mathbf{v}^n\|_{\mathbf{L}^2}^2 \right] \right. \tag{3.9}$$
$$\left. + \mathbb{E}\left[\|\mathbf{v}^{n+1} - \mathbf{v}^n\|_{\mathbf{L}^2}^2 \right] \right)$$
$$= \mathbb{E}\left[\left(\mathbf{G}[\mathbf{u}^n]\Delta W_n, \mathbf{v}^{n+1} \right) \right] + \mathbb{E}\left[k\left(\mathbf{F}[\mathbf{u}^n], \mathbf{v}^{n+1} \right) \right] := I_1 + I_2.$$

For the first term on the right-hand side, note that $\mathbb{E}\left[\left(\mathbf{G}[\mathbf{u}^n]\Delta W_n, \mathbf{v}^n \right) \right] = 0$. Similar to the estimation of (2.20), and by Itô's isometry, we obtain

$$\mathbb{E}\left[\left(\mathbf{G}[\mathbf{u}^n]\Delta W_n, \mathbf{v}^{n+1} \right) \right] = \mathbb{E}\left[\left(\mathbf{G}[\mathbf{u}^n]\Delta W_n, \mathbf{v}^{n+1} - \mathbf{v}^n \right) \right] \tag{3.10}$$
$$\leq \frac{1}{4}\mathbb{E}\left[\|\mathbf{v}^{n+1} - \mathbf{v}^n\|_{\mathbf{L}^2}^2 \right] + k\mathbb{E}\left[\|\mathbf{G}[\mathbf{u}^n]\|_{\mathbf{L}^2}^2 \right]$$
$$\leq \frac{1}{4}\mathbb{E}\left[\|\mathbf{v}^{n+1} - \mathbf{v}^n\|_{\mathbf{L}^2}^2 \right] + CC_B^2 k\mathbb{E}\left[\lambda \|\mathrm{div}\,(\mathbf{u}^n)\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{u}^n)\|_{\mathbf{L}^2}^2 \right] + 2C_A^2 k.$$

Similar to the estimation of (2.18), the second term I_2 can be bounded by

$$k\mathbb{E}\left[\left(\mathbf{F}[\mathbf{u}^n], \mathbf{v}^{n+1} \right) \right] \tag{3.11}$$
$$\leq k\mathbb{E}\left[\|\mathbf{v}^{n+1}\|_{\mathbf{L}^2}^2 \right] + CC_B^2 k\mathbb{E}\left[\lambda \|\mathrm{div}\,(\mathbf{u}^n)\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{u}^n)\|_{\mathbf{L}^2}^2 \right] + 2C_A^2 k.$$

The left-hand side of (3.9) can be written as

$$\frac{1}{2}\mathbb{E}\Big[\big(\|\mathbf{v}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{v}^n\|_{\mathbf{L}^2}^2 + \|\mathbf{v}^{n+1} - \mathbf{v}^n\|_{\mathbf{L}^2}^2\big)\Big] - \mathbb{E}\Big[(\mathbf{u}^{n+1} - \mathbf{u}^n, \mathscr{L}\mathbf{u}^{n+\frac{1}{2}})\Big] \qquad (3.12)$$

$$= \frac{1}{2}\mathbb{E}\Big[\big(\|\mathbf{v}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{v}^n\|_{\mathbf{L}^2}^2 + \|\mathbf{v}^{n+1} - \mathbf{v}^n\|_{\mathbf{L}^2}^2\big)\Big]$$

$$+ \mathbb{E}\Big[\frac{1}{2}\big(\lambda\|\mathrm{div}\,(\mathbf{u}^{n+1})\|_{\mathbf{L}^2}^2 + \mu\|\varepsilon(\mathbf{u}^{n+1})\|_{\mathbf{L}^2}^2\big)\Big]$$

$$- \mathbb{E}\Big[\frac{1}{2}\big(\lambda\|\mathrm{div}\,(\mathbf{u}^n)\|_{\mathbf{L}^2}^2 + \mu\|\varepsilon(\mathbf{u}^n)\|_{\mathbf{L}^2}^2\big)\Big]$$

$$= \mathbb{E}\Big[J(\mathbf{u}^{n+1}, \mathbf{v}^{n+1})\Big] - \mathbb{E}\Big[J(\mathbf{u}^n, \mathbf{v}^n)\Big] + \frac{1}{2}\mathbb{E}\Big[\|\mathbf{v}^{n+1} - \mathbf{v}^n\|_{\mathbf{L}^2}^2\Big].$$

Putting (3.10)-(3.12) together yields

$$\mathbb{E}\big[J(\mathbf{u}^{n+1}, \mathbf{v}^{n+1})\big] + \frac{1}{4}\sum_{l=0}^n \mathbb{E}\Big[\|\mathbf{v}^{l+1} - \mathbf{v}^l\|_{\mathbf{L}^2}^2\Big] \qquad (3.13)$$

$$\leq \mathbb{E}\big[J(\mathbf{u}^0, \mathbf{v}^0)\big] + 4C_A^2 + CC_B^2 k \sum_{l=0}^n \mathbb{E}\big[J(\mathbf{u}^l, \mathbf{v}^l)\big].$$

By the discrete Gronwall's inequality, we obtain

$$\mathbb{E}\big[J(\mathbf{u}^{n+1}, \mathbf{v}^{n+1})\big] \leq \big(\mathbb{E}\big[J(\mathbf{u}^0, \mathbf{v}^0)\big] + 4C_A^2\big)e^{CC_B^2}. \qquad (3.14)$$

The proof is complete. $\qquad\qquad\square$

Since previous stability results are not sufficient to establish the convergence results of fully discrete finite element methods, we will prove some stability results in stronger norms. Consider the following energy functional:

$$\widetilde{J}(\mathbf{u}^n, \mathbf{v}^n) = \frac{1}{2}\big(\|\mathscr{L}\mathbf{u}^n\|_{\mathbf{L}^2}^2 + \lambda\|\mathrm{div}\,(\mathbf{v}^n)\|_{\mathbf{L}^2}^2 + \mu\|\varepsilon(\mathbf{v}^n)\|_{\mathbf{L}^2}^2\big), \qquad (3.15)$$

we will establish the stability for the expectation of the above energy functional.

Lemma 3.3. *Let* $(\mathbf{u}^0, \mathbf{v}^0) = (\mathbf{u}_0, \mathbf{v}_0) \in (\mathbf{H}^2 \cap \mathbf{H}_0^1) \times \mathbf{H}_0^1$ *be given. The solution* $\{(\mathbf{u}^n, \mathbf{v}^n); 1 \leq n \leq N\}$ *of* (3.1)–(3.2) *satisfies*

$$\max_{1 \leq n \leq N} \mathbb{E}\Big[\widetilde{J}(\mathbf{u}^n, \mathbf{v}^n)\Big] \leq \Big(CC_A^2 C_{s1} + \mathbb{E}\Big[\widetilde{J}(\mathbf{u}^0, \mathbf{v}^0)\Big]\Big)e^T. \qquad (3.16)$$

Proof. By choosing $\phi = \mathscr{L}^2\mathbf{u}^{n+\frac{1}{2}}$ in (3.1), we get

$$(\mathscr{L}\mathbf{u}^{n+1} - \mathscr{L}\mathbf{u}^n, \mathscr{L}\mathbf{u}^{n+\frac{1}{2}}) = k(\mathscr{L}\mathbf{v}^{n+1}, \mathscr{L}\mathbf{u}^{n+\frac{1}{2}}). \qquad (3.17)$$

By choosing $\psi = -\mathscr{L}\mathbf{v}^{n+1}$ in (3.2), we get

$$(\mathbf{v}^{n+1} - \mathbf{v}^n, -\mathscr{L}\mathbf{v}^{n+1}) = k(\mathscr{L}\mathbf{u}^{n+\frac{1}{2}}, -\mathscr{L}\mathbf{v}^{n+1}) \qquad (3.18)$$

$$+ \lambda\big(\mathrm{div}\,(\mathbf{G}[\mathbf{u}^n]\Delta W_n), \mathrm{div}\,(\mathbf{v}^{n+1})\big) + \mu\big(\varepsilon(\mathbf{G}[\mathbf{u}^n]\Delta W_n), \varepsilon(\mathbf{v}^{n+1})\big)$$

$$+ k\lambda\big(\mathrm{div}\,(\mathbf{F}[\mathbf{u}^n]), \mathrm{div}\,(\mathbf{v}^{n+1})\big) + \mu\big(\varepsilon(\mathbf{F}[\mathbf{u}^n]), \varepsilon(\mathbf{v}^{n+1})\big).$$

Combining (3.17) and (3.18) yields

$$
\begin{aligned}
\widetilde{J}\big(\mathbf{u}^{n+1},\mathbf{v}^{n+1}\big) - \widetilde{J}\big(\mathbf{u}^n,\mathbf{v}^n\big) & \hspace{4cm} (3.19) \\
+ \frac{1}{2}\Big(\lambda\,\big\|\mathrm{div}\,(\mathbf{v}^{n+1}-\mathbf{v}^n)\big\|_{\mathbf{L}^2}^2 + \mu\,\big\|\varepsilon(\mathbf{v}^{n+1}-\mathbf{v}^n)\big\|_{\mathbf{L}^2}^2\Big) & \\
= \lambda\big(\mathrm{div}\,(\mathbf{G}[\mathbf{u}^n]\Delta W_n),\mathrm{div}\,(\mathbf{v}^{n+1})\big) + \mu\big(\varepsilon(\mathbf{G}[\mathbf{u}^n]\Delta W_n),\varepsilon(\mathbf{v}^{n+1})\big) & \\
+ k\lambda\big(\mathrm{div}\,(\mathbf{F}[\mathbf{u}^n]),\mathrm{div}\,(\mathbf{v}^{n+1})\big) + \mu\big(\varepsilon(\mathbf{F}[\mathbf{u}^n]),\varepsilon(\mathbf{v}^{n+1})\big) & \\
:= I_1 + I_2 + I_3 + I_4. &
\end{aligned}
$$

Note that

$$
\mathbb{E}\big[\lambda\big(\mathrm{div}\,(\mathbf{G}[\mathbf{u}^n]\Delta W_n),\mathrm{div}\,(\mathbf{v}^n)\big) + \mu\big(\varepsilon(\mathbf{G}[\mathbf{u}^n]\Delta W_n),\varepsilon(\mathbf{v}^n)\big)\big] = 0. \qquad (3.20)
$$

Similar to the estimation of equation (2.26), and by Itô's isometry, we have

$$
\begin{aligned}
I_1 + I_2 & \le \frac{k}{2}\Big(\lambda\,\big\|\mathrm{div}\,(\mathbf{G}[\mathbf{u}^n])\big\|_{\mathbf{L}^2}^2 + \mu\,\big\|\varepsilon(\mathbf{G}[\mathbf{u}^n])\big\|_{\mathbf{L}^2}^2\Big) \hspace{2cm} (3.21) \\
& \quad + \frac{1}{2}\Big(\lambda\,\big\|\mathrm{div}\,(\mathbf{v}^{n+1}-\mathbf{v}^n)\big\|_{\mathbf{L}^2}^2 + \mu\,\big\|\varepsilon(\mathbf{v}^{n+1}-\mathbf{v}^n)\big\|_{\mathbf{L}^2}^2\Big) \\
& \le \frac{1}{2}\Big(\lambda\,\big\|\mathrm{div}\,(\mathbf{v}^{n+1}-\mathbf{v}^n)\big\|_{\mathbf{L}^2}^2 + \mu\,\big\|\varepsilon(\mathbf{v}^{n+1}-\mathbf{v}^n)\big\|_{\mathbf{L}^2}^2\Big) + CC_A^2 C_{s1} k.
\end{aligned}
$$

Similar to the estimation of equation (2.25), we have

$$
I_3 + I_4 \le \frac{k}{2}\Big(\lambda\,\big\|\mathrm{div}\,(\mathbf{v}^{n+1})\big\|_{\mathbf{L}^2}^2 + \mu\,\big\|\varepsilon(\mathbf{v}^{n+1})\big\|_{\mathbf{L}^2}^2\Big) + CC_A^2 C_{s1} k. \qquad (3.22)
$$

Taking the expectation and summation over n from 0 to l on the both sides of (3.19), using equations (3.20)–(3.22), and then switching n and l, we have

$$
\mathbb{E}\big[\widetilde{J}(\mathbf{u}^{n+1},\mathbf{v}^{n+1})\big] \le CC_A^2 C_{s1} + \mathbb{E}\big[\widetilde{J}(\mathbf{u}^0,\mathbf{v}^0)\big] + k\sum_{l=0}^{n}\mathbb{E}\big[\widetilde{J}(\mathbf{u}^l,\mathbf{v}^l)\big]. \qquad (3.23)
$$

Then the discrete Gronwall's inequality yields

$$
\mathbb{E}\big[\widetilde{J}(\mathbf{u}^{n+1},\mathbf{v}^{n+1})\big] \le \Big(CC_A^2 C_{s1} + \mathbb{E}\big[\widetilde{J}(\mathbf{u}^0,\mathbf{v}^0)\big]\Big)e^T. \qquad (3.24)
$$

The proof is complete. $\hspace{8cm}\square$

3.3. **Error estimates for the time semi-discrete scheme.** In this section, we derive error estimates in both $\mathbf{H}^1$-norm and $\mathbf{L}^2$-norm for the time semi-discrete scheme.

Theorem 3.4. *Let* $(\mathbf{u},\mathbf{v})$ *be the solution of* (2.10)–(2.11) *and* $\{(\mathbf{u}^n,\mathbf{v}^n); 1 \le n \le N\}$ *be the solution of* (3.1)–(3.2). *Under the assumptions* (2.1)–(2.5), *there holds*

$$
\max_{1\le n\le N}\mathbb{E}\Big[\|\mathbf{u}(t_n)-\mathbf{u}^n\|_{\mathbf{H}^1}^2 + \|\mathbf{v}(t_n)-\mathbf{v}^n\|_{\mathbf{L}^2}^2\Big] \le C_{s6}e^{CC_B^2}k. \qquad (3.25)
$$

Proof. By (2.10)–(2.11), we have

$$\big(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n), \phi\big) = \Big(\int_{t_n}^{t_{n+1}} \mathbf{v}(s)\mathrm{d}s, \phi \Big), \tag{3.26}$$

$$\big(\mathbf{v}(t_{n+1}) - \mathbf{v}(t_n), \psi\big) = \int_{t_n}^{t_{n+1}} \big(\mathscr{L}(\mathbf{u}(s)), \psi\big)\mathrm{d}s + \int_{t_n}^{t_{n+1}} \big(\mathbf{F}[\mathbf{u}(s)], \psi\big)\mathrm{d}s \tag{3.27}$$
$$+ \Big(\int_{t_n}^{t_{n+1}} \mathbf{G}[\mathbf{u}(s)]\mathrm{d}W(s), \psi \Big).$$

By (3.1)–(3.2), we have

$$\big(\mathbf{u}^{n+1} - \mathbf{u}^n, \phi\big) = k\big(\mathbf{v}^{n+1}, \phi\big) \qquad\qquad \forall \phi \in \mathbf{L}^2, \tag{3.28}$$

$$\big(\mathbf{v}^{n+1} - \mathbf{v}^n, \psi\big) = -k\lambda\big(\mathrm{div}\,(\mathbf{u}^{n+\frac{1}{2}}), \mathrm{div}\,(\psi)\big) - k\mu\big(\varepsilon(\mathbf{u}^{n+\frac{1}{2}}), \varepsilon(\psi)\big) \tag{3.29}$$
$$+ \big(\mathbf{G}[\mathbf{u}^n]\Delta_{n+1}W, \psi\big) + k\big(\mathbf{F}[\mathbf{u}^n], \psi\big) \qquad \forall \psi \in \mathbf{H}_0^1.$$

Define notations $\mathbf{e}_{\mathbf{u}}^n$, $\mathbf{e}_{\mathbf{v}}^n$, and $\mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}}$ by

$$\mathbf{e}_{\mathbf{u}}^n := \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}, \quad \mathbf{e}_{\mathbf{v}}^n := \mathbf{v}(t_{n+1}) - \mathbf{v}^{n+1},$$

$$\mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}} := \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2} - \mathbf{u}^{n+\frac{1}{2}} = \frac{\mathbf{e}_{\mathbf{u}}^{n+1} + \mathbf{e}_{\mathbf{u}}^n}{2}.$$

Subtracting (3.28) from (3.26) and (3.29) from (3.27) yield

$$\big(\mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^n, \phi\big) = \Big(\int_{t_n}^{t_{n+1}} \big(\mathbf{v}(s) - \mathbf{v}(t_{n+1})\big)\mathrm{d}s, \phi \Big) + k\big(\mathbf{e}_{\mathbf{v}}^{n+1}, \phi\big), \tag{3.30}$$

$$\big(\mathbf{e}_{\mathbf{v}}^{n+1} - \mathbf{e}_{\mathbf{v}}^n, \psi\big) = \int_{t_n}^{t_{n+1}} \Big(\mathscr{L}\Big(\mathbf{u}(s) - \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}\Big), \psi \Big)\mathrm{d}s \tag{3.31}$$
$$+ k\big(\mathscr{L}\mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}}, \psi\big) + \int_{t_n}^{t_{n+1}} \big(\mathbf{F}[\mathbf{u}(s)] - \mathbf{F}[\mathbf{u}^n], \psi\big)\mathrm{d}s$$
$$+ \Big(\int_{t_n}^{t_{n+1}} \big(\mathbf{G}[\mathbf{u}(s)] - \mathbf{G}[\mathbf{u}^n]\big)\mathrm{d}W(s), \psi \Big).$$

By choosing $\phi = -\mathscr{L}\mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}}$ in (3.30), we have

$$k\big(\mathscr{L}\mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}}, \mathbf{e}_{\mathbf{v}}^{n+1}\big) = \big(\mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^n, \mathscr{L}\mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}}\big) - \Big(\int_{t_n}^{t_{n+1}} \big(\mathbf{v}(s) - \mathbf{v}(t_{n+1})\big)\mathrm{d}s, \mathscr{L}\mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}} \Big). \tag{3.32}$$

By choosing $\psi = \mathbf{e}_{\mathbf{v}}^{n+1}$ in (3.31), we obtain

$$\big(\mathbf{e}_{\mathbf{v}}^{n+1} - \mathbf{e}_{\mathbf{v}}^n, \mathbf{e}_{\mathbf{v}}^{n+1}\big) = \int_{t_n}^{t_{n+1}} \Big(\mathscr{L}\Big(\mathbf{u}(s) - \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}\Big), \mathbf{e}_{\mathbf{v}}^{n+1} \Big)\mathrm{d}s \tag{3.33}$$
$$+ k\big(\mathscr{L}\mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}}, \mathbf{e}_{\mathbf{v}}^{n+1}\big) + \int_{t_n}^{t_{n+1}} \big(\mathbf{F}[\mathbf{u}(s)] - \mathbf{F}[\mathbf{u^n}], \mathbf{e}_{\mathbf{v}}^{n+1}\big)\mathrm{d}s$$
$$+ \Big(\int_{t_n}^{t_{n+1}} \big(\mathbf{G}[\mathbf{u}(s)] - \mathbf{G}[\mathbf{u}^n]\big)\mathrm{d}W(s), \mathbf{e}_{\mathbf{v}}^{n+1} \Big).$$

Combining (3.32)-(3.33) yields

$$
J(\mathbf{e_u}^{n+1}, \mathbf{e_v}^{n+1}) - J(\mathbf{e_u}^n, \mathbf{e_v}^n) + \frac{1}{2} \left\| \mathbf{e_v}^{n+1} - \mathbf{e_v}^n \right\|_{\mathbf{L}^2}^2 \tag{3.34}
$$

$$
= - \int_{t_n}^{t_{n+1}} \left(\mathscr{L} \mathbf{e_u}^{n+\frac{1}{2}}, \mathbf{v}(s) - \mathbf{v}(t_{n+1}) \right) ds
$$

$$
+ \int_{t_n}^{t_{n+1}} \left(\mathscr{L} \left(\mathbf{u}(s) - \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2} \right), \mathbf{e_v}^{n+1} \right) ds
$$

$$
+ k \left(\mathbf{F}[\mathbf{u}(t_n)] - \mathbf{F}[\mathbf{u}^n], \mathbf{e_v}^{n+1} \right) + \int_{t_n}^{t_{n+1}} \left(\mathbf{F}[\mathbf{u}(s)] - \mathbf{F}[\mathbf{u}(t_n)], \mathbf{e_v}^{n+1} \right) ds
$$

$$
+ \left(\int_{t_n}^{t_{n+1}} \left(\mathbf{G}[\mathbf{u}(s)] - \mathbf{G}[\mathbf{u}^n] \right) dW(s), \mathbf{e_v}^{n+1} \right)
$$

$$
:= I_1 + I_2 + I_3 + I_4 + I_5.
$$

By equation (2.51), the expectation of the first term I_1 can be bounded by

$$
\mathbb{E}[I_1] \leq 2 \int_{t_n}^{t_{n+1}} \mathbb{E} \left[\lambda \left\| \mathrm{div} \left(\mathbf{v}(s) - \mathbf{v}(t_{n+1}) \right) \right\|_{\mathbf{L}^2}^2 \right. \tag{3.35}
$$

$$
+ \mu \left(\varepsilon \left(\mathbf{v}(s) - \mathbf{v}(t_{n+1}) \right) \right\|_{\mathbf{L}^2}^2 \right] ds
$$

$$
+ \frac{k}{4} \mathbb{E} \left[\lambda \left\| \mathrm{div} \, (\mathbf{e_u}^{n+1}) \right\|_{\mathbf{L}^2}^2 + \mu \left\| \varepsilon(\mathbf{e_u}^{n+1}) \right\|_{\mathbf{L}^2}^2 \right]
$$

$$
+ \frac{k}{4} \mathbb{E} \left[\lambda \left\| \mathrm{div} \, (\mathbf{e_u}^n) \right\|_{\mathbf{L}^2}^2 + \mu \left\| \varepsilon(\mathbf{e_u}^n) \right\|_{\mathbf{L}^2}^2 \right]
$$

$$
\leq C_{s6} k^2 + \frac{k}{4} \mathbb{E} \left[\lambda \left\| \mathrm{div} \, (\mathbf{e_u}^{n+1}) \right\|_{\mathbf{L}^2}^2 + \mu \left\| \varepsilon(\mathbf{e_u}^{n+1}) \right\|_{\mathbf{L}^2}^2 \right]
$$

$$
+ \frac{k}{4} \mathbb{E} \left[\lambda \left\| \mathrm{div} \, (\mathbf{e_u}^n) \right\|_{\mathbf{L}^2}^2 + \mu \left\| \varepsilon(\mathbf{e_u}^n) \right\|_{\mathbf{L}^2}^2 \right].
$$

By equation (2.48), the expectation of the second term I_2 can be bounded by

$$
\mathbb{E}[I_2] \leq \frac{1}{8} \int_{t_n}^{t_{n+1}} \mathbb{E} \left[\left\| \mathscr{L} \left(\mathbf{u}(s) - \mathbf{u}(t_n) \right) \right\|_{\mathbf{L}^2}^2 \right] ds \tag{3.36}
$$

$$
+ \frac{1}{8} \int_{t_n}^{t_{n+1}} \mathbb{E} \left[\left\| \mathscr{L} \left(\mathbf{u}(s) - \mathbf{u}(t_{n+1}) \right) \right\|_{\mathbf{L}^2}^2 \right] ds + k \mathbb{E} \left[\left\| \mathbf{e_v}^{n+1} \right\|_{\mathbf{L}^2}^2 \right]
$$

$$
\leq \frac{1}{12} C_{s4} k^3 + k \mathbb{E} \left[\left\| \mathbf{e_v}^{n+1} \right\|_{\mathbf{L}^2}^2 \right].
$$

By the Poincaré inequality, the Korn's inequality, and equations (2.46) and (2.47), we have

$$\mathbb{E}[I_3] \leq \frac{k}{4}\mathbb{E}\left[\|\mathbf{F}[\mathbf{u}(t_n)] - \mathbf{F}[\mathbf{u}^n]\|_{\mathbf{L}^2}^2\right] + k\mathbb{E}\left[\|\mathbf{e}_\mathbf{v}^{n+1}\|_{\mathbf{L}^2}^2\right] \tag{3.37}$$

$$\leq \frac{k}{4}C_B^2\mathbb{E}\left[\lambda\|\mathrm{div}\,(\mathbf{e}_\mathbf{u}^n)\|_{\mathbf{L}_2}^2 + \mu\|\varepsilon(\mathbf{e}_\mathbf{u}^n)\|_{\mathbf{L}_2}^2 + \|\mathbf{e}_\mathbf{u}^n\|_{\mathbf{L}_2}^2\right] + k\mathbb{E}\left[\|\mathbf{e}_\mathbf{v}^{n+1}\|_{\mathbf{L}^2}^2\right]$$

$$\leq kCC_B^2\mathbb{E}\left[\lambda\|\mathrm{div}\,(\mathbf{e}_\mathbf{u}^n)\|_{\mathbf{L}_2}^2 + \mu\|\varepsilon(\mathbf{e}_\mathbf{u}^n)\|_{\mathbf{L}_2}^2\right] + k\mathbb{E}\left[\|\mathbf{e}_\mathbf{v}^{n+1}\|_{\mathbf{L}^2}^2\right],$$

$$\mathbb{E}[I_4] \leq \int_{t_n}^{t_{n+1}}\mathbb{E}\left[\|\mathbf{F}[\mathbf{u}(s)] - \mathbf{F}[\mathbf{u}(t_n)]\|_{\mathbf{L}^2}^2\right]\mathrm{d}s + k\mathbb{E}\left[\|\mathbf{e}_\mathbf{v}^{n+1}\|_{\mathbf{L}^2}^2\right] \tag{3.38}$$

$$\leq \frac{1}{4}C_B^2\int_{t_n}^{t_{n+1}}\left(\mathbb{E}\left[\lambda\|\mathrm{div}\,(\mathbf{u}(s)) - \mathrm{div}\,(\mathbf{u}(t_n))\|_{\mathbf{L}^2}^2\right]\right.$$

$$\left.+ \mathbb{E}\left[\mu\|\varepsilon(\mathbf{u}(s)) - \varepsilon(\mathbf{u}(t_n))\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[\|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{L}^2}^2\right]\right)ds + k\mathbb{E}\left[\|\mathbf{e}_\mathbf{v}^{n+1}\|_{\mathbf{L}^2}^2\right]$$

$$\leq CC_B^2k^3(C_{s4} + C_{s2}) + k\mathbb{E}\left[\|\mathbf{e}_\mathbf{v}^{n+1}\|_{\mathbf{L}^2}^2\right].$$

For the term I_5, note that $\mathbb{E}\left[\left(\int_{t_n}^{t_{n+1}}\left(\mathbf{G}[\mathbf{u}(s)] - \mathbf{G}[\mathbf{u}^n]\right)\mathrm{d}W(s), \mathbf{e}_\mathbf{v}^n\right)\right] = 0$. Then by the Poincaré inequality, the Korn's inequality, Itô's isometry, and equations (2.46) and (2.47), we have

$$\mathbb{E}[I_5] = \mathbb{E}\left[\left((\mathbf{G}[\mathbf{u}(t_n)] - \mathbf{G}[\mathbf{u}^n])\Delta W_n, \mathbf{e}_\mathbf{v}^{n+1} - \mathbf{e}_\mathbf{v}^n\right)\right] \tag{3.39}$$

$$+ \mathbb{E}\left[\left(\left(\int_{t_n}^{t_{n+1}}\left(\mathbf{G}[\mathbf{u}(s)] - \mathbf{G}[\mathbf{u}(t_n)]\right)\mathrm{d}W(s), \mathbf{e}_\mathbf{v}^{n+1} - \mathbf{e}_\mathbf{v}^n\right)\right)\right]$$

$$\leq \frac{1}{4}\mathbb{E}\left[\|\mathbf{e}_\mathbf{v}^{n+1} - \mathbf{e}_\mathbf{v}^n\|_{\mathbf{L}^2}^2\right] + 2kC_B^2\mathbb{E}\left[\lambda\|\mathrm{div}\,(\mathbf{e}_\mathbf{u}^n)\|_{\mathbf{L}^2}^2 + \mu\|\varepsilon(\mathbf{e}_\mathbf{u}^n)\|_{\mathbf{L}^2}^2 + \|\mathbf{e}_\mathbf{u}^n\|_{\mathbf{L}^2}^2\right]$$

$$+ 2C_B^2\int_{t_n}^{t_{n+1}}\mathbb{E}\left[\lambda\|\mathrm{div}\,(\mathbf{u}(s) - \mathbf{u}(t_n))\|_{\mathbf{L}^2}^2 + \mu\|\varepsilon(\mathbf{u}(s) - \mathbf{u}(t_n))\|_{\mathbf{L}^2}^2\right]\mathrm{d}s$$

$$+ 2C_B^2\int_{t_n}^{t_{n+1}}\mathbb{E}\left[\|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{L}^2}^2\right]\mathrm{d}s$$

$$\leq \frac{1}{4}\mathbb{E}\left[\|\mathbf{e}_\mathbf{v}^{n+1} - \mathbf{e}_\mathbf{v}^n\|_{\mathbf{L}^2}^2\right] + CC_B^2k\mathbb{E}[J(\mathbf{e}_\mathbf{u}^n, \mathbf{e}_\mathbf{v}^n)] + CC_B^2k^3(C_{s1} + C_{s2}).$$

Combining (3.34)-(3.39) together, we have

$$\mathbb{E}\left[J(\mathbf{e}_\mathbf{u}^{n+1}, \mathbf{e}_\mathbf{v}^{n+1})\right] - \mathbb{E}[J(\mathbf{e}_\mathbf{u}^n, \mathbf{e}_\mathbf{v}^n)] + \frac{1}{4}\mathbb{E}\left[\|\mathbf{e}_\mathbf{v}^{n+1} - \mathbf{e}_\mathbf{v}^n\|_{\mathbf{L}^2}^2\right] \leq CC_B^2k\mathbb{E}[J(\mathbf{e}_\mathbf{u}^n, \mathbf{e}_\mathbf{v}^n)] + C_{s6}k^2. \tag{3.40}$$

Then the discrete Gronwall's inequality yields

$$\mathbb{E}\left[J(\mathbf{e}_\mathbf{u}^{n+1}, \mathbf{e}_\mathbf{v}^{n+1})\right] \leq C_{s6}e^{CC_B^2}k. \tag{3.41}$$

where we use the fact that $\mathbb{E}\left[J(\mathbf{u}^0, \mathbf{v}^0)\right] = 0$. Then the theorem is proved. $\qquad\square$

Theorem 3.4 states that the time semi-discrete convergence order in $\mathbf{H}^1$-norm is $\frac{1}{2}$. The next theorem establishes the time semi-discrete $\mathbf{L}^2$-error of $\mathbf{u}^n$, and it states that the time semi-discrete convergence order in $\mathbf{L}^2$-norm is 1. Its proof is inspired by a similar proof for the stochastic scalar wave equation given in [18].

Theorem 3.5. *Let* $(\mathbf{u}, \mathbf{v})$ *be the solution of* (2.10)–(2.11) *and* $\{\mathbf{u}^n, \mathbf{v}^n\}_n$ *be the solution of* (3.1)–(3.2). *Under the assumptions* (2.1)–(2.5) *and the assumptions that*

$$\|\mathbf{G}[\mathbf{u}(0)]\|_{L^2}^2 \leq Ck \qquad and \qquad \|k\mathbf{v}_0 - (\mathbf{u}^1 - \mathbf{u}^0)\|_{L^2}^2 \leq Ck^4,$$

there holds

$$\max_{1 \leq n \leq N} \mathbb{E}\left[\|\mathbf{u}(t_n) - \mathbf{u}^n\|_{\mathbf{L}^2}^2\right] \leq CC_{s4}k^2. \tag{3.42}$$

Proof. Note that $d_t\mathbf{u}^l = \frac{(\mathbf{u}^l - \mathbf{u}^{l-1})}{k}$. Then by (3.2), we have

$$\left(d_t\mathbf{u}^{l+1} - d_t\mathbf{u}^l, \psi\right) + k\lambda\left(\operatorname{div}\mathbf{u}^{l+\frac{1}{2}}, \operatorname{div}\psi\right) + k\mu\left(\varepsilon(\mathbf{u}^{l+\frac{1}{2}}), \varepsilon(\psi)\right) \tag{3.43}$$
$$= \left(\mathbf{G}[\mathbf{u}^l]\Delta W_l, \psi\right) + k\left(\mathbf{F}[\mathbf{u}^l], \psi\right).$$

Denote $\bar{\mathbf{u}}^{n+\frac{1}{2}} = \sum_{l=1}^n \mathbf{u}^{l+\frac{1}{2}}$. Taking the summation over l from $l = 1$ to $l = n$ and multiplying k on both sides of (3.43) yield

$$\left(\mathbf{u}^{n+1} - \mathbf{u}^n, \psi\right) - k^2\left(\mathscr{L}\bar{\mathbf{u}}^{n+\frac{1}{2}}, \psi\right) - k(d_t\mathbf{u}^1, \psi) \tag{3.44}$$
$$= k\left(\sum_{l=1}^n \mathbf{G}[\mathbf{u}^l]\Delta W_l, \psi\right) + k^2\left(\sum_{l=1}^n \mathbf{F}[\mathbf{u}^l], \psi\right).$$

Besides, using (2.10) and (2.11), we get

$$\left(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n), \psi\right) - \int_{t_n}^{t_{n+1}} \int_0^s \left(\mathscr{L}\mathbf{u}(\xi), \psi\right)\mathrm{d}\xi\,\mathrm{d}s - k\left(\mathbf{v}_0, \psi\right) \tag{3.45}$$
$$= \left(\int_{t_n}^{t_{n+1}} \int_0^s \mathbf{G}[\mathbf{u}(\xi)]\mathrm{d}W(\xi)\mathrm{d}s, \psi\right) + \int_{t_n}^{t_{n+1}} \int_0^s \left(\mathbf{F}[\mathbf{u}(\xi)], \psi\right)\mathrm{d}\xi\,\mathrm{d}s.$$

Subtracting (3.44) from (3.45) leads to

$$\left(\mathbf{e}_\mathbf{u}^{n+1} - \mathbf{e}_\mathbf{u}^n, \psi\right) - k^2\left(\mathscr{L}\bar{\mathbf{e}}_\mathbf{u}^{n+\frac{1}{2}}, \psi\right) \tag{3.46}$$
$$= \left(\int_{t_n}^{t_{n+1}} \int_0^s \mathscr{L}\mathbf{u}(\xi)\mathrm{d}\xi\,\mathrm{d}s - k^2\sum_{l=1}^n \mathscr{L}\left(\frac{\mathbf{u}(t_{l+1}) + \mathbf{u}(t_l)}{2}\right), \psi\right)$$
$$+ \left(\int_{t_n}^{t_{n+1}} \int_0^s \mathbf{G}[\mathbf{u}(\xi)]\mathrm{d}W(\xi)\mathrm{d}s - k\sum_{l=1}^n \mathbf{G}[\mathbf{u}^l]\Delta W_l, \psi\right)$$
$$+ \left(\int_{t_n}^{t_{n+1}} \int_0^s \mathbf{F}[\mathbf{u}(\xi)]\mathrm{d}\xi\,\mathrm{d}s - k^2\sum_{l=1}^n \mathbf{F}[\mathbf{u}^l], \psi\right)$$
$$- \left(k\mathbf{v}_0 - (\mathbf{u}^1 - \mathbf{u}^0), \psi\right).$$

Borrowing an idea of [18], we write $\int_0^s d\xi = \sum\limits_{l=0}^{n-1} \int_{t_l}^{t_{l+1}} d\xi + \int_{t_n}^s d\xi$ and choosing $\psi = \mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}}$ in (3.46) yield

$$
\frac{1}{2} \left(\left\| \mathbf{e}_{\mathbf{u}}^{n+1} \right\|_{\mathbf{L}^2}^2 - \left\| \mathbf{e}_{\mathbf{u}}^{n} \right\|_{\mathbf{L}^2}^2 \right) \tag{3.47}
$$
$$
+ \frac{k^2\lambda}{2} \left(\left\| \operatorname{div}(\bar{\mathbf{e}}_{\mathbf{u}}^{n+\frac{1}{2}}) \right\|_{\mathbf{L}^2}^2 - \left\| \operatorname{div}(\bar{\mathbf{e}}_{\mathbf{u}}^{n-\frac{1}{2}}) \right\|_{\mathbf{L}^2}^2 + \left\| \operatorname{div}(\mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}}) \right\|_{\mathbf{L}^2}^2 \right)
$$
$$
+ \frac{k^2\mu}{2} \left(\left\| \varepsilon(\bar{\mathbf{e}}_{\mathbf{u}}^{n+\frac{1}{2}}) \right\|_{\mathbf{L}^2}^2 - \left\| \varepsilon(\bar{\mathbf{e}}_{\mathbf{u}}^{n-\frac{1}{2}}) \right\|_{\mathbf{L}^2}^2 + \left\| \varepsilon(\mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}}) \right\|_{\mathbf{L}^2}^2 \right)
$$
$$
:= I_1 + I_2 + \cdots + I_{15},
$$

where

$$
I_1 + \cdots + I_4 = \frac{1}{2} \int_{t_n}^{t_{n+1}} \sum_{l=1}^{n} \int_{t_{l-1}}^{t_{l+1}} \left(\mathscr{L}\mathbf{u}(\xi) - \mathscr{L}\left(\frac{\mathbf{u}(t_{l+1}) + \mathbf{u}(t_l)}{2}\right), \mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}} \right) d\xi\, ds \tag{3.48}
$$
$$
+ \frac{1}{2} \int_{t_n}^{t_{n+1}} \int_{t_0}^{t_1} \left(\mathscr{L}\mathbf{u}(\xi), \mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}} \right) d\xi\, ds + \frac{1}{2} \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \left(\mathscr{L}\mathbf{u}(\xi), \mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}} \right) d\xi\, ds
$$
$$
- \frac{1}{2} \int_{t_n}^{t_{n+1}} \int_{s}^{t_{n+1}} \left(\mathscr{L}\mathbf{u}(\xi), \mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}} \right) d\xi\, ds,
$$

$$
I_5 + \cdots + I_9 := \frac{1}{2} \int_{t_n}^{t_{n+1}} \sum_{l=1}^{n} \int_{t_{l-1}}^{t_{l+1}} \left(\mathbf{F}[\mathbf{u}(\xi)] - \mathbf{F}[\mathbf{u}(t_l)], \mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}} \right) d\xi\, ds \tag{3.49}
$$
$$
+ k^2 \left(\sum_{l=1}^{n} \left(\mathbf{F}[\mathbf{u}(t_l)] - \mathbf{F}[\mathbf{u}^l] \right), \mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}} \right) + \frac{1}{2} \int_{t_n}^{t_{n+1}} \int_{t_0}^{t_1} \left(\mathbf{F}[\mathbf{u}(\xi)], \mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}} \right) d\xi\, ds
$$
$$
+ \frac{1}{2} \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \left(\mathbf{F}[\mathbf{u}(\xi)], \mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}} \right) d\xi\, ds - \frac{1}{2} \int_{t_n}^{t_{n+1}} \int_{s}^{t_{n+1}} \left(\mathbf{F}[\mathbf{u}(\xi)], \mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}} \right) d\xi\, ds,
$$

and

$$
I_{10} + \cdots + I_{14} := \left(k \sum_{l=1}^{n} \left(\mathbf{G}[\mathbf{u}(t_l)] - \mathbf{G}[\mathbf{u}^l] \right) \Delta W_l, \mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}} \right) \tag{3.50}
$$
$$
+ \left(\int_{t_n}^{t_{n+1}} \sum_{\ell=1}^{n} \int_{t_\ell}^{t_{\ell+1}} \mathbf{G}[\mathbf{u}(\xi)] - \mathbf{G}[\mathbf{u}(t_l)] dW(\xi) ds, \mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}} \right)
$$
$$
- \left(\int_{t_n}^{t_{n+1}} \int_{s}^{t_{n+1}} \mathbf{G}[\mathbf{u}(\xi)] - \mathbf{G}[\mathbf{u}(t_n)] dW(\xi) ds, \mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}} \right)
$$
$$
- \left(\int_{t_n}^{t_{n+1}} \int_{s}^{t_{n+1}} \mathbf{G}[\mathbf{u}(t_n)] dW(\xi) ds, \mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}} \right) + \left(\int_{t_n}^{t_{n+1}} \int_{t_0}^{t_1} \mathbf{G}[\mathbf{u}(\xi)] dW(\xi) ds, \mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}} \right).
$$

By equation (2.48), the first term I_1 can be bounded by

$$
\mathbb{E}\left[I_1\right] \leq \int_{t_n}^{t_{n+1}} \sum_{l=1}^{n} \int_{t_{l-1}}^{t_{l+1}} \mathbb{E}\left[\left(\mathscr{L}\left(\frac{\mathbf{u}(\xi)-\mathbf{u}(t_{l+1})+\mathbf{u}(\xi)-\mathbf{u}(t_l)}{2}\right), \mathbf{e_u}^{n+\frac{1}{2}}\right)\right] \mathrm{d}\xi\,\mathrm{d}s \tag{3.51}
$$

$$
\leq \int_{t_n}^{t_{n+1}} \sum_{l=1}^{n} \int_{t_{l-1}}^{t_{l+1}} \left(\mathbb{E}\left[\left\|\mathscr{L}\left(\frac{\mathbf{u}(\xi)-\mathbf{u}(t_{l+1})+\mathbf{u}(\xi)-\mathbf{u}(t_l)}{2}\right)\right\|_{\mathbf{L}^2}^2\right]\right)^{\frac{1}{2}}
$$

$$
\left(\mathbb{E}\left[\left\|\mathbf{e_u}^{n+\frac{1}{2}}\right\|_{\mathbf{L}^2}^2\right]\right)^{\frac{1}{2}} \mathrm{d}\xi\,\mathrm{d}s
$$

$$
\leq \int_{t_n}^{t_{n+1}} \sum_{l=1}^{n} \int_{t_{l-1}}^{t_{l+1}} \left(CC_{s4}k^2 \mathbb{E}\left[\left\|\mathbf{e_u}^{n+\frac{1}{2}}\right\|_{\mathbf{L}^2}^2\right]\right)^{\frac{1}{2}} \mathrm{d}\xi\,\mathrm{d}s
$$

$$
\leq k\mathbb{E}\left[\left\|\mathbf{e_u}^{n+1}\right\|_{\mathbf{L}^2}^2\right] + k\mathbb{E}\left[\left\|\mathbf{e_u}^{n}\right\|_{\mathbf{L}^2}^2\right] + CC_{s4}k^3.
$$

By equation (2.48), the second term I_2 can be bounded by

$$
\mathbb{E}\left[I_2\right] \leq \frac{1}{2}\int_{t_n}^{t_{n+1}} \mathbb{E}\left[\int_{t_0}^{t_1}\left(\mathscr{L}\left(\mathbf{u}(\xi)-\mathbf{u}(t_0)\right), \mathbf{e_u}^{n+\frac{1}{2}}\right)\mathrm{d}\xi\right]\mathrm{d}s \tag{3.52}
$$

$$
+ \frac{1}{2}\int_{t_n}^{t_{n+1}} \mathbb{E}\left[\int_{t_0}^{t_1}\left(\mathscr{L}\left(\mathbf{u}(t_0)\right), \mathbf{e_u}^{n+\frac{1}{2}}\right)\mathrm{d}\xi\right]\mathrm{d}s
$$

$$
\leq k\mathbb{E}\left[\left\|\mathbf{e_u}^{n+1}\right\|_{\mathbf{L}^2}^2\right] + k\mathbb{E}\left[\left\|\mathbf{e_u}^{n}\right\|_{\mathbf{L}^2}^2\right] + CC_{s4}k^5 + Ck^3.
$$

By equation (2.48), the summation of the third term and the fourth term can be bounded by

$$
\mathbb{E}\left[I_3+I_4\right] \leq \frac{1}{2}\int_{t_n}^{t_{n+1}} \int_{t_n}^{s}\left(\mathscr{L}\left(\mathbf{u}(\xi)-\mathbf{u}(t_n)\right), \mathbf{e_u}^{n+\frac{1}{2}}\right)\mathrm{d}\xi\,\mathrm{d}s \tag{3.53}
$$

$$
- \frac{1}{2}\int_{t_n}^{t_{n+1}} \int_{s}^{t_{n+1}}\left(\mathscr{L}\left(\mathbf{u}(\xi)-\mathbf{u}(t_n)\right), \mathbf{e_u}^{n+\frac{1}{2}}\right)\mathrm{d}\xi\,\mathrm{d}s
$$

$$
+ \frac{1}{2}\left(\mathscr{L}\left(\mathbf{u}(t_n)\right), \mathbf{e_u}^{n+\frac{1}{2}}\right)\left(\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\mathrm{d}\xi\,\mathrm{d}s - \int_{t_n}^{t_{n+1}}\int_{s}^{t_{n+1}}\mathrm{d}\xi\,\mathrm{d}s\right)
$$

$$
\leq k\mathbb{E}\left[\left\|\mathbf{e_u}^{n+1}\right\|_{\mathbf{L}^2}^2\right] + k\mathbb{E}\left[\left\|\mathbf{e_u}^{n}\right\|_{\mathbf{L}^2}^2\right] + CC_{s4}k^5.
$$

By equations (2.2) and (2.46), we have

$$
\mathbb{E}\left[I_5\right] \leq C\int_{t_n}^{t_{n+1}} \sum_{l=1}^{n} \int_{t_{l-1}}^{t_{l+1}} \mathbb{E}\left[\left\|\mathbf{F}[\mathbf{u}(\xi)]-\mathbf{F}[\mathbf{u}(t_l)]\right\|_{\mathbf{L}^2}^2\right]\mathrm{d}\xi\,\mathrm{d}s \tag{3.54}
$$

$$
+ k\mathbb{E}\left[\left\|\mathbf{e_u}^{n+1}\right\|_{\mathbf{L}^2}^2\right] + k\mathbb{E}\left[\left\|\mathbf{e_u}^{n}\right\|_{\mathbf{L}^2}^2\right]
$$

$$
\leq C\int_{t_n}^{t_{n+1}} \sum_{l=1}^{n} \int_{t_{l-1}}^{t_{l+1}} \left[\left\|\mathbf{u}(\xi)-\mathbf{u}(t_l)\right\|_{\mathbf{L}^2}^2 \mathrm{d}\xi\,\mathrm{d}s\right.
$$

$$
+ k\mathbb{E}\left[\left\|\mathbf{e_u}^{n+1}\right\|_{\mathbf{L}^2}^2\right] + k\mathbb{E}\left[\left\|\mathbf{e_u}^{n}\right\|_{\mathbf{L}^2}^2\right]
$$

$$
\leq CC_{s1}k^3 + k\mathbb{E}\left[\left\|\mathbf{e_u}^{n+1}\right\|_{\mathbf{L}^2}^2\right] + k\mathbb{E}\left[\left\|\mathbf{e_u}^{n}\right\|_{\mathbf{L}^2}^2\right].
$$

By equation (2.2), the sixth term can be bounded by

$$
\begin{aligned}
\mathbb{E}\left[I_6\right] &= k^2 \mathbb{E}\left[\left(\sum_{l=1}^{n}\left(\mathbf{F}[\mathbf{u}(t_l)] - \mathbf{F}[\mathbf{u}^l]\right), \mathbf{e_u}^{n+\frac{1}{2}}\right)\right] \\
&\leq k^2 \sum_{l=1}^{n}\left(\frac{T}{4}\mathbb{E}\left[\left\|\mathbf{F}[\mathbf{u}(t_l)] - \mathbf{F}[\mathbf{u}^l]\right\|_{\mathbf{L}^2}^2\right] + \frac{1}{T}\mathbb{E}\left[\left\|\mathbf{e_u}^{n+\frac{1}{2}}\right\|_{\mathbf{L}^2}^2\right]\right) \\
&\leq Ck^2 \sum_{l=1}^{n}\mathbb{E}\left[\left\|\mathbf{u}(t_l) - \mathbf{u}^l\right\|_{\mathbf{L}^2}^2\right] + k\mathbb{E}\left[\left\|\mathbf{e_u}^{n+\frac{1}{2}}\right\|_{\mathbf{L}^2}^2\right] \\
&\leq Ck^2 \sum_{l=1}^{n}\mathbb{E}\left[\left\|\mathbf{e_u}^l\right\|_{\mathbf{L}^2}^2\right] + k\mathbb{E}\left[\left\|\mathbf{e_u}^{n+1}\right\|_{\mathbf{L}^2}^2\right] + k\mathbb{E}\left[\left\|\mathbf{e_u}^n\right\|_{\mathbf{L}^2}^2\right].
\end{aligned}
\tag{3.55}
$$

By equations (2.2) and (2.46), the seventh term I_7 can be bounded by

$$
\begin{aligned}
\mathbb{E}\left[I_7\right] &\leq \frac{1}{2}\int_{t_n}^{t_{n+1}}\mathbb{E}\left[\int_{t_0}^{t_1}\left(\mathbf{F}[\mathbf{u}(\xi)] - \mathbf{F}[\mathbf{u}(t_0)], \mathbf{e_u}^{n+\frac{1}{2}}\right)\mathrm{d}\xi\right]\mathrm{d}s \\
&\quad + \frac{1}{2}\int_{t_n}^{t_{n+1}}\mathbb{E}\left[\int_{t_0}^{t_1}\left(\mathbf{F}[\mathbf{u}(t_0)], \mathbf{e_u}^{n+\frac{1}{2}}\right)\mathrm{d}\xi\right]\mathrm{d}s \\
&\leq k\mathbb{E}\left[\left\|\mathbf{e_u}^{n+1}\right\|_{\mathbf{L}^2}^2\right] + k\mathbb{E}\left[\left\|\mathbf{e_u}^n\right\|_{\mathbf{L}^2}^2\right] + CC_{s1}k^5 + Ck^3.
\end{aligned}
\tag{3.56}
$$

By equation (2.2), the summation of the third term and the fourth term can be bounded by

$$
\begin{aligned}
\mathbb{E}\left[I_8 + I_9\right] &\leq \frac{1}{2}\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\left(\mathbf{F}[\mathbf{u}(\xi)] - \mathbf{F}[\mathbf{u}(t_n)], \mathbf{e_u}^{n+\frac{1}{2}}\right)\mathrm{d}\xi\,\mathrm{d}s \\
&\quad - \frac{1}{2}\int_{t_n}^{t_{n+1}}\int_{s}^{t_{n+1}}\left(\mathbf{F}[\mathbf{u}(\xi)] - \mathbf{F}[\mathbf{u}(t_n)], \mathbf{e_u}^{n+\frac{1}{2}}\right)\mathrm{d}\xi\,\mathrm{d}s \\
&\quad + \frac{1}{2}\left(\mathbf{F}[\mathbf{u}(t_n)], \mathbf{e_u}^{n+\frac{1}{2}}\right)\left(\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\mathrm{d}\xi\,\mathrm{d}s - \int_{t_n}^{t_{n+1}}\int_{s}^{t_{n+1}}\mathrm{d}\xi\,\mathrm{d}s\right) \\
&\leq k\mathbb{E}\left[\left\|\mathbf{e_u}^{n+1}\right\|_{\mathbf{L}^2}^2\right] + k\mathbb{E}\left[\left\|\mathbf{e_u}^n\right\|_{\mathbf{L}^2}^2\right] + CC_{s4}k^5.
\end{aligned}
\tag{3.57}
$$

By Young's inequality and equation (2.2), the tenth term I_{10} can be bounded by

$$
\begin{aligned}
\mathbb{E}\left[I_{10}\right] &= \mathbb{E}\left[k\left(\sum_{l=1}^{n}\left(\mathbf{G}[\mathbf{u}(t_l)] - \mathbf{G}[\mathbf{u}^l]\right)\Delta W_l, \mathbf{e_u}^{n+\frac{1}{2}}\right)\right] \\
&\leq k\mathbb{E}\left[\left\|\mathbf{e_u}^{n+\frac{1}{2}}\right\|_{\mathbf{L}^2}^2\right] + Ck\mathbb{E}\left[\left\|\sum_{l=1}^{n}\left(\mathbf{G}[\mathbf{u}(t_l)] - \mathbf{G}[\mathbf{u}^l]\right)\Delta W_l\right\|_{\mathbf{L}^2}^2\right] \\
&\leq k\mathbb{E}\left[\left\|\mathbf{e_u}^{n+1}\right\|_{\mathbf{L}^2}^2\right] + k\mathbb{E}\left[\left\|\mathbf{e_u}^n\right\|_{\mathbf{L}^2}^2\right] + Ck^2\sum_{l=1}^{n}\mathbb{E}\left[\left\|\mathbf{e_u}^l\right\|_{\mathbf{L}^2}^2\right].
\end{aligned}
\tag{3.58}
$$

For the eleventh term I_{11}, by Itô's isometry, Young's inequality, and equations (2.2) and (2.46), we get

$$
\begin{aligned}
\mathbb{E}[I_{11}] &\leq C \int_{t_n}^{t_{n+1}} \sum_{l=1}^{n} \int_{t_l}^{t_{l+1}} \mathbb{E}\left[\|\mathbf{G}[\mathbf{u}(\xi)] - \mathbf{G}[\mathbf{u}(t_l)]\|_{\mathbf{L}^2}^2\right] d\xi ds \\
&\quad + k\mathbb{E}\left[\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2\right] + k\mathbb{E}\left[\|\mathbf{e}_{\mathbf{u}}^{n}\|_{\mathbf{L}^2}^2\right] \\
&\leq C \int_{t_n}^{t_{n+1}} \sum_{l=1}^{n} \int_{t_l}^{t_{l+1}} \mathbb{E}\left[\|\mathbf{u}(\xi) - \mathbf{u}(t_l)\|_{\mathbf{L}^2}^2\right] d\xi ds \\
&\quad + k\mathbb{E}\left[\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2\right] + k\mathbb{E}\left[\|\mathbf{e}_{\mathbf{u}}^{n}\|_{\mathbf{L}^2}^2\right] \\
&\leq CC_{s1}k^3 + k\mathbb{E}\left[\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2\right] + k\mathbb{E}\left[\|\mathbf{e}_{\mathbf{u}}^{n}\|_{\mathbf{L}^2}^2\right].
\end{aligned}
\tag{3.59}
$$

By Itô's isometry, Young's inequality, and equations (2.2) and (2.46), the term I_{12} can be bounded by

$$
\begin{aligned}
\mathbb{E}[I_{12}] &\leq C \int_{t_n}^{t_{n+1}} \int_{s}^{t_{n+1}} \mathbb{E}\left[\|\mathbf{G}[\mathbf{u}(\xi)] - \mathbf{G}[\mathbf{u}(t_n)]\|_{\mathbf{L}^2}^2\right] d\xi ds \\
&\quad + k\mathbb{E}\left[\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2\right] + k\mathbb{E}\left[\|\mathbf{e}_{\mathbf{u}}^{n}\|_{\mathbf{L}^2}^2\right] \\
&\leq C \int_{t_n}^{t_{n+1}} \int_{s}^{t_{n+1}} \mathbb{E}\left[\|\mathbf{u}(\xi) - \mathbf{u}(t_n)\|_{\mathbf{L}^2}^2\right] d\xi ds \\
&\quad + k\mathbb{E}\left[\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2\right] + k\mathbb{E}\left[\|\mathbf{e}_{\mathbf{u}}^{n}\|_{\mathbf{L}^2}^2\right] \\
&\leq CC_{s1}k^4 + k\mathbb{E}\left[\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2\right] + k\mathbb{E}\left[\|\mathbf{e}_{\mathbf{u}}^{n}\|_{\mathbf{L}^2}^2\right].
\end{aligned}
\tag{3.60}
$$

By equations (3.30) and (2.50), and Theorem 3.4, the term I_{13} can be bounded by

$$
\begin{aligned}
\mathbb{E}[I_{13}] &= \mathbb{E}\left[\mathbf{G}[\mathbf{u}(t_n)]\left(\int_{t_n}^{t_{n+1}} W(t_{n+1}) - W(s)ds, \mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}}\right)\right] \\
&= \frac{1}{2}\mathbb{E}\left[\left(\mathbf{G}[\mathbf{u}(t_n)]\left(\int_{t_n}^{t_{n+1}} W(t_{n+1}) - W(s)ds, \mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^{n}\right)\right] \\
&= \frac{1}{2}\mathbb{E}\left[\left(\mathbf{G}[\mathbf{u}(t_n)]\int_{t_n}^{t_{n+1}} W(t_{n+1}) - W(s)ds, k\mathbf{e}_{\mathbf{v}}^{n+1}\right)\right] \\
&\quad + \frac{1}{2}\mathbb{E}\left[\left(\mathbf{G}[\mathbf{u}(t_n)]\int_{t_n}^{t_{n+1}} W(t_{n+1}) - W(s)ds, \int_{t_n}^{t_{n+1}} v(s) - v(t_{n+1})ds\right)\right] \\
&\leq Ck^3 + k^2\|\mathbf{e}_{\mathbf{v}}^{n+1}\|_{L^2}^2 + k\int_{t_n}^{t_{n+1}} \|v(s) - v(t_{n+1})\|_{L^2}^2 ds \\
&\leq Ck^3 + C_{s6}e^{CC_B^2}k^3 + C_{s5}k^3.
\end{aligned}
\tag{3.61}
$$

By Itô's isometry, Young's inequality, and equations (2.2) and (2.46), the term I_{14} can be bounded by

$$\mathbb{E}[I_{14}] = \left(\int_{t_n}^{t_{n+1}} \int_{t_0}^{t_1} \mathbf{G}[\mathbf{u}(\xi)] - \mathbf{G}[\mathbf{u}(0)]dW(\xi)ds, \mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}} \right) \tag{3.62}$$

$$+ \left(\int_{t_n}^{t_{n+1}} \int_{t_0}^{t_1} \mathbf{G}[\mathbf{u}(0)]dW(\xi)ds, \mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}} \right)$$

$$\leq CC_{s1}k^4 + k\|\mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}}\|_{L^2}^2 + Ck^2\|\mathbf{G}[\mathbf{u}(0)]\|_{L^2}^2$$

$$\leq k\|\mathbf{e}_{\mathbf{u}}^{n+\frac{1}{2}}\|_{L^2}^2 + Ck^3.$$

Combining (3.47)-(3.62) and taking the summation, we have

$$\mathbb{E}\left[\left\|\mathbf{e}_{\mathbf{u}}^{n+1}\right\|_{\mathbf{L}^2}^2\right] \leq Ck\sum_{l=0}^{n}\mathbb{E}\left[\left\|\mathbf{e}_{\mathbf{u}}^{l+1}\right\|_{\mathbf{L}^2}^2\right] + CC_{s4}k^2 + CC_{s1}k^2 + Ck^2 \tag{3.63}$$

$$+ C_{s6}e^{CC_B^2}k^2 + C_{s5}k^2.$$

Then the discrete Gronwall's inequality yields

$$\mathbb{E}\left[\left\|\mathbf{e}_{\mathbf{u}}^{n+1}\right\|_{\mathbf{L}^2}^2\right] \leq CC_{s4}k^2, \tag{3.64}$$

where we use $\mathbf{e}_{\mathbf{u}}^0 = \mathbf{0}$. $\qquad\qquad\square$

4. FINITE ELEMENT DISCRETIZATION IN SPACE

In this section we discretize the time semi-discrete scheme in space using the finite element methods and give detailed error analysis for the fully discrete scheme.

4.1. Finite element fully discrete scheme.
Let $\mathscr{T}_h$ be a quasi-uniform triangulation of $\mathscr{D}$ with diameter h. We consider the finite element spaces

$$\mathbb{U}_h^{r_1} = \left\{ \mathbf{u}_h \in \mathbf{H}^1 : \mathbf{u}_h|_K \in \mathbf{P}_{r_1}(K) \quad \forall K \in \mathscr{T}_h \right\},$$
$$\mathbb{V}_h^{r_2} = \left\{ \mathbf{v}_h \in \mathbf{H}^1 : \mathbf{v}_h|_K \in \mathbf{P}_{r_2}(K) \quad \forall K \in \mathscr{T}_h \right\},$$

where $\mathbf{P}_r(K)$ donates the space of polynomials with degree not exceeding a given integer r on $K \in \mathscr{T}_h$. Next, we define two types of projection as follows.

Definition 4.1. (1) The $\mathbf{L}^2$-projection $\mathscr{P}_h : \mathbf{L}^2 \to \mathbb{U}_h^{r_1}$ is defined by

$$(\mathscr{P}_h\mathbf{w}, \mathbf{v}_h) = (\mathbf{w}, \mathbf{v}_h),$$

for all $\mathbf{w} \in \mathbf{L}^2$ and $\mathbf{v}_h \in \mathbb{U}_h^{r_1}$.

(2) The $\mathbf{H}^1$-projection $\mathscr{R}_h : \mathbf{H}_0^1 \to \mathbb{V}_h^{r_2}$ is defined by

$$\lambda\left(\mathrm{div}\,(\mathscr{R}_h\mathbf{w}), \mathrm{div}\,(\mathbf{v}_h)\right) + \mu\left(\varepsilon(\mathscr{R}_h\mathbf{w}), \varepsilon(\mathbf{v}_h)\right) = \lambda\left(\mathrm{div}\,(\mathbf{w}), \mathrm{div}\,(\mathbf{v}_h)\right) + \mu\left(\varepsilon(\mathbf{w}), \varepsilon(\mathbf{v}_h)\right),$$

for all $\mathbf{w} \in \mathbf{H}_0^1$ and $\mathbf{v}_h \in \mathbb{V}_h^{r_2}$.

The following error estimate results are well-known [4, 13].

$$\|\mathbf{w} - \mathscr{P}_h\mathbf{w}\|_{\mathbf{L}^2} \leq C_{\mathscr{P}_h} h^{\min(r_1+1,s)} \|\mathbf{w}\|_{\mathbf{H}^s}, \tag{4.1}$$

$$\|\mathbf{w} - \mathscr{R}_h\mathbf{w}\|_{\mathbf{L}^2} \leq C_{\mathscr{R}_h} h^{\min(r_2+1,s)} \|\mathbf{w}\|_{\mathbf{H}^s}, \tag{4.2}$$

$$\left(\lambda \|\mathrm{div}\,(\mathbf{w} - \mathscr{R}_h\mathbf{w})\|_{\mathbf{L}^2}^2 + \mu \|\varepsilon(\mathbf{w} - \mathscr{R}_h\mathbf{w})\|_{\mathbf{L}^2}^2\right)^{\frac{1}{2}} \leq C_{\mathscr{R}_h} h^{\min(r_2,s-1)} \|\mathbf{w}\|_{\mathbf{H}^s}. \tag{4.3}$$

Scheme 2. Let $(\mathbf{u}_h^0, \mathbf{v}_h^0) = (\mathscr{R}_h\mathbf{u}^0, \mathscr{P}_h\mathbf{v}^0)$. Seeking a $\mathbb{U}_h^{r_1} \times \mathbb{V}_h^{r_2}$-valued $\{\mathscr{F}_{t_{n+1}}\}$-adapted solution $(\mathbf{u}_h^{n+1}, \mathbf{v}_h^{n+1})$ such that $\mathbb{P}$-almost surely

$$\left(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n, \phi_h\right) = k\left(\mathbf{v}_h^{n+1}, \phi_h\right), \tag{4.4}$$

$$\left(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n, \psi_h\right) = k\left(\mathscr{L}_h\mathbf{u}_h^{n+\frac{1}{2}}, \psi_h\right) + \left(\mathbf{G}[\mathbf{u}_h^n]\Delta_{n+1}W, \psi_h\right) + k\left(\mathbf{F}[\mathbf{u}_h^n], \psi_h\right), \tag{4.5}$$

for all $(\phi_h, \psi_h) \in \mathbb{U}_h^{r_1} \times \mathbb{V}_h^{r_2}$.

Remark 4.2. At each time step, the above scheme is a nonlinear random algebraic system for $(\mathbf{u}_h^{n+1}, \mathbf{v}_h^{n+1})$ whose well-posedness can be proved by a standard fixed point argument based on the stability estimates of the next lemma.

Since the proof of the next lemma is similar to that of Lemma 3.2, we will omit it to save space.

Lemma 4.3. *Assume* $0 < k << k_0 < T$ *and choose* $(\mathbf{u}_h^0, \mathbf{v}_h^0) = (\mathscr{R}_h\mathbf{u}^0, \mathscr{P}_h\mathbf{v}^0)$. *Then there holds*

$$\max_{1 \leq n \leq N-1} \mathbb{E}\left[J(\mathbf{u}_h^{n+1}, \mathbf{v}_h^{n+1})\right] \leq \left(\mathbb{E}\left[J(\mathbf{u}^0, \mathbf{v}^0)\right] + 4C_A^2\right)e^{CC_B^2}. \tag{4.6}$$

4.2. Error estimates for the finite element fully discrete scheme. The linear finite elements are used in this section, and $r_1 = r_2 = 1$ are chosen in the finite element space. We start with the discretization of the operator $\mathscr{L}$.

Definition 4.4. The discrete operator $\mathscr{L}_h : \mathbb{U}_h^1 \to \mathbb{V}_h^1$ is defined by

$$(-\mathscr{L}_h\mathbf{w}_h, \mathbf{v}_h) = \lambda\left(\mathrm{div}\,(\mathbf{w}_h), \mathrm{div}\,(\mathbf{v}_h)\right) + \mu\left(\varepsilon(\mathbf{w}_h), \varepsilon(\mathbf{v}_h)\right), \tag{4.7}$$

for all $(\mathbf{w}_h, \mathbf{v}_h) \in \mathbb{U}_h^1 \times \mathbb{V}_h^1$.

Define $\mathbf{E}_{\mathbf{u}}^n := \mathbf{u}^n - \mathbf{u}_h^n$ and $\mathbf{E}_{\mathbf{v}}^n := \mathbf{v}^n - \mathbf{v}_h^n$, we now derive the estimates for $\mathbf{E}_{\mathbf{u}}^n$ and $\mathbf{E}_{\mathbf{v}}^n$.

Theorem 4.5. *Suppose* $\{\mathbf{u}^n, \mathbf{v}^n\}_n$ *solves* (3.1)–(3.2) *and* $\{\mathbf{u}_h^n, \mathbf{v}_h^n\}$ *solves* (4.4)–(4.5). $(\mathbf{u}_h^0, \mathbf{v}_h^0) = (\mathscr{R}_h\mathbf{u}^0, \mathscr{P}_h\mathbf{v}^0)$ *and* $(\mathbf{u}^0, \mathbf{v}^0) \in \mathbf{H}^2 \times \mathbf{H}^1$ *are given. Under the assumptions* (2.1)–(2.5), *we have*

$$\max_{1 \leq n \leq N} \mathbb{E}\left[\|\mathbf{u}^n - \mathbf{u}_h^n\|_{\mathbf{H}^1}^2 + \|\mathbf{v}^n - \mathbf{v}_h^n\|_{\mathbf{L}^2}^2\right] \leq C\mathbb{E}\left[\|\mathbf{u}^n\|_{\mathbf{H}^2}^2\right]h^2. \tag{4.8}$$

Proof. Subtracting (4.4) from (3.1) leads to

$$\left(\mathbf{E}_{\mathbf{u}}^{n+1} - \mathbf{E}_{\mathbf{u}}^n, \phi_h\right) = k\left(\mathbf{E}_{\mathbf{v}}^{n+1}, \phi_h\right), \tag{4.9}$$

$$\left(\mathbf{E}_{\mathbf{v}}^{n+1} - \mathbf{E}_{\mathbf{v}}^n, \psi_h\right) = k\left(\mathscr{L}\mathbf{E}_{\mathbf{u}}^{n+\frac{1}{2}}, \psi_h\right) + \left((\mathbf{G}[\mathbf{u}^n] - \mathbf{G}[\mathbf{u}_h^n])\Delta W_n, \psi_h\right) \tag{4.10}$$
$$+ k\left(\mathbf{F}[\mathbf{u}^n] - \mathbf{F}[\mathbf{u}_h^n], \psi_h\right).$$

Choosing $\phi_h = -\mathscr{L}_h \mathscr{R}_h \mathbf{E_u}^{n+\frac{1}{2}}$ in (4.9) yields

$$\left(\mathbf{E_u}^{n+1} - \mathbf{E_u}^n, \mathscr{L}_h \mathscr{R}_h \mathbf{E_u}^{n+\frac{1}{2}}\right) = k\left(\mathbf{E_v}^{n+1}, \mathscr{L}_h \mathscr{R}_h \mathbf{E_u}^{n+\frac{1}{2}}\right). \tag{4.11}$$

Choosing $\psi_h = \mathscr{P}_h \mathbf{E_v}^{n+1}$ in (4.10), we have

$$\left(\mathbf{E_v}^{n+1} - \mathbf{E_v}^n, \mathscr{P}_h \mathbf{E_v}^{n+1}\right) = k\left(\mathscr{L} \mathbf{E_u}^{n+\frac{1}{2}}, \mathscr{P}_h \mathbf{E_v}^{n+1}\right) \tag{4.12}$$
$$+ \left((\mathbf{G}[\mathbf{u}^n] - \mathbf{G}[\mathbf{u}_h^n])\Delta W_n, \mathscr{P}_h \mathbf{E_v}^{n+1}\right) + k\left(\mathbf{F}[\mathbf{u}^n] - \mathbf{F}[\mathbf{u}_h^n], \mathscr{P}_h \mathbf{E_v}^{n+1}\right).$$

Based on the definitions of $\mathscr{P}_h$, $\mathscr{L}$, and $\mathscr{L}_h$, we have

$$\left(\mathscr{L} \mathbf{E_u}^{n+\frac{1}{2}}, \mathscr{P}_h \mathbf{E_v}^{n+1}\right) = \left(\mathscr{L} \mathscr{R}_h \mathbf{E_u}^{n+\frac{1}{2}}, \mathscr{P}_h \mathbf{E_v}^{n+1}\right) \tag{4.13}$$
$$= \left(\mathscr{L}_h \mathscr{R}_h \mathbf{E_u}^{n+\frac{1}{2}}, \mathscr{P}_h \mathbf{E_v}^{n+1}\right)$$
$$= \left(\mathscr{L}_h \mathscr{R}_h \mathbf{E_u}^{n+\frac{1}{2}}, \mathbf{E_v}^{n+1}\right).$$

Besides, the left-hand side of (4.12) can be written as

$$\left(\mathbf{E_v}^{n+1} - \mathbf{E_v}^n, \mathscr{P}_h \mathbf{E_v}^{n+1}\right) \tag{4.14}$$
$$= \left(\mathscr{P}_h \mathbf{E_v}^{n+1} - \mathscr{P}_h \mathbf{E_v}^n, \mathscr{P}_h \mathbf{E_v}^{n+1}\right)$$
$$= \frac{1}{2}\|\mathscr{P}_h \mathbf{E_v}^{n+1}\|_{\mathbf{L}^2}^2 - \frac{1}{2}\|\mathscr{P}_h \mathbf{E_v}^n\|_{\mathbf{L}^2}^2 + \frac{1}{2}\|\mathscr{P}_h \mathbf{E_v}^{n+1} - \mathscr{P}_h \mathbf{E_v}^n\|_{\mathbf{L}^2}^2.$$

Combining (4.11)-(4.14), we have

$$J(\mathscr{R}_h \mathbf{E_u}^{n+1}, \mathscr{P}_h \mathbf{E_v}^{n+1}) - J(\mathscr{R}_h \mathbf{E_u}^n, \mathscr{P}_h \mathbf{E_v}^n) + \frac{1}{2}\mathbb{E}\left[\|\mathscr{P}_h \mathbf{E_v}^{n+1} - \mathscr{P}_h \mathbf{E_v}^n\|_{\mathbf{L}^2}^2\right] \tag{4.15}$$
$$= k\left(\mathbf{F}[\mathbf{u}^n] - \mathbf{F}[\mathbf{u}_h^n], \mathscr{P}_h \mathbf{E_v}^{n+1}\right) + \left((\mathbf{G}[\mathbf{u}^n] - \mathbf{G}[\mathbf{u}_h^n])\Delta W_n, \mathscr{P}_h \mathbf{E_v}^{n+1}\right)$$
$$:= I_1 + I_2.$$

By equation (4.2), the first term I_1 could be bounded by

$$\mathbb{E}\left[k\left(\mathbf{F}[\mathbf{u}^n] - \mathbf{F}[\mathbf{u}_h^n], \mathscr{P}_h \mathbf{E_v}^{n+1}\right)\right] \tag{4.16}$$
$$\leq k\mathbb{E}\left[\|\mathbf{F}[\mathbf{u}^n] - \mathbf{F}[\mathbf{u}_h^n]\|_{\mathbf{L}^2}^2\right] + \frac{k}{4}\mathbb{E}\left[\|\mathscr{P}_h \mathbf{E_v}^{n+1}\|_{\mathbf{L}^2}^2\right]$$
$$\leq kC_B^2\mathbb{E}\left[\lambda\|\text{div}\,(\mathbf{E_u}^n)\|_{\mathbf{L}^2}^2 + \mu\|\varepsilon(\mathbf{E_u}^n)\|_{\mathbf{L}^2}^2 + \|\mathbf{E_u}^n\|_{\mathbf{L}^2}^2\right] + \frac{k}{4}\mathbb{E}\left[\|\mathscr{P}_h \mathbf{E_v}^{n+1}\|_{\mathbf{L}^2}^2\right]$$
$$\leq kC_B^2\mathbb{E}\left[\lambda\|\text{div}\,(\mathscr{R}_h \mathbf{E_u}^n)\|_{\mathbf{L}^2}^2 + \mu\|\varepsilon(\mathscr{R}_h \mathbf{E_u}^n)\|_{\mathbf{L}^2}^2 + \|\mathscr{R}_h \mathbf{E_u}^n\|_{\mathbf{L}^2}^2\right]$$
$$+ kC_B^2\mathbb{E}\left[\lambda\|\text{div}\,(\mathbf{u}^n - \mathscr{R}_h \mathbf{u}^n)\|_{\mathbf{L}^2}^2 + \mu\|\varepsilon(\mathbf{u}^n - \mathscr{R}_h \mathbf{u}^n)\|_{\mathbf{L}^2}^2\right.$$
$$+ \left.\|\mathbf{u}^n - \mathscr{R}_h \mathbf{u}^n\|_{\mathbf{L}^2}^2\right] + \frac{k}{4}\mathbb{E}\left[\|\mathscr{P}_h \mathbf{E_v}^{n+1}\|_{\mathbf{L}^2}^2\right]$$
$$\leq kCC_B^2\mathbb{E}\left[\lambda\|\text{div}\,(\mathscr{R}_h \mathbf{E_u}^n)\|_{\mathbf{L}^2}^2 + \mu\|\varepsilon(\mathscr{R}_h \mathbf{E_u}^n)\|_{\mathbf{L}^2}^2\right]$$
$$+ kC_B^2 C_{\mathscr{R}_h}^2 h^2 \mathbb{E}\left[\|\mathbf{u}^n\|_{\mathbf{H}^2}^2\right] + \frac{k}{4}\mathbb{E}\left[\|\mathscr{P}_h \mathbf{E_v}^{n+1}\|_{\mathbf{L}^2}^2\right].$$

Let $M_t = \left((\mathbf{G}[\mathbf{u}^n] - \mathbf{G}[\mathbf{u}_h^n])\Delta W_n, \mathscr{R}_h \mathbf{E}_{\mathbf{v}}^n\right)$, M_t is a martingale and $\mathbb{E}[M_t] = 0$. Similar to the estimation of (4.16), the second term I_2 goes to

$$\mathbb{E}\left[\left((\mathbf{G}[\mathbf{u}^n] - \mathbf{G}[\mathbf{u}_h^n])\Delta W_n, \mathscr{P}_h \mathbf{E}_{\mathbf{v}}^{n+1}\right)\right] \tag{4.17}$$

$$\leq k\mathbb{E}\left[\|\mathbf{G}[\mathbf{u}^n] - \mathbf{G}[\mathbf{u}_h^n]\|_{\mathbf{L}^2}^2\right] + \frac{1}{4}\mathbb{E}\left[\left\|\mathscr{P}_h \mathbf{E}_{\mathbf{v}}^{n+1} - \mathscr{P}_h \mathbf{E}_{\mathbf{v}}^n\right\|_{\mathbf{L}^2}^2\right]$$

$$\leq kCC_B^2\mathbb{E}\left[\lambda\|\operatorname{div}(\mathscr{R}_h \mathbf{E}_{\mathbf{u}}^n)\|_{\mathbf{L}^2}^2 + \mu\|\varepsilon(\mathscr{R}_h \mathbf{E}_{\mathbf{u}}^n)\|_{\mathbf{L}^2}^2\right]$$

$$+ kC_B^2 C_{\mathscr{P}_h}^2 h^4\mathbb{E}\left[\|\mathbf{u}^n\|_{\mathbf{H}^2}^2\right] + \frac{1}{4}\mathbb{E}\left[\left\|\mathscr{P}_h \mathbf{E}_{\mathbf{v}}^{n+1} - \mathscr{P}_h \mathbf{E}_{\mathbf{v}}^n\right\|_{\mathbf{L}^2}^2\right].$$

Combining (4.15)–(4.17), we have

$$\mathbb{E}\left[J(\mathscr{R}_h \mathbf{E}_{\mathbf{u}}^{n+1}, \mathscr{P}_h \mathbf{E}_{\mathbf{v}}^{n+1})\right] - \mathbb{E}\left[J(\mathscr{R}_h \mathbf{E}_{\mathbf{u}}^n, \mathscr{P}_h \mathbf{E}_{\mathbf{v}}^n)\right] \tag{4.18}$$

$$\leq kCC_B^2\mathbb{E}\left[J(\mathscr{R}_h \mathbf{E}_{\mathbf{u}}^n, \mathscr{P}_h \mathbf{E}_{\mathbf{v}}^n)\right] + kC_B^2 C_{\mathscr{P}_h}^2 h^4\mathbb{E}\left[\|\mathbf{u}^n\|_{\mathbf{H}^2}^2\right].$$

Then the discrete Gronwall's inequality yields

$$\mathbb{E}\left[J(\mathscr{R}_h \mathbf{E}_{\mathbf{u}}^{n+1}, \mathscr{P}_h \mathbf{E}_{\mathbf{v}}^{n+1})\right] \leq C_B^2 C_{\mathscr{P}_h}^2 h^4\mathbb{E}\left[\|\mathbf{u}^n\|_{\mathbf{H}^2}^2\right]e^{CC_B^2}. \tag{4.19}$$

Combining equations (4.19), (4.1), and (4.3), we get the conclusion. $\qquad\square$

Theorem 4.5 states that the linear finite element discrete convergence order in $\mathbf{H}^1$-norm is 1. The next theorem states that the linear finite element discrete convergence order in $\mathbf{L}^2$-norm is 2.

Theorem 4.6. *Let $\{\mathbf{u}^n, \mathbf{v}^n\}_n$ solves (3.1)–(3.2), $(\mathbf{u}_h^0, \mathbf{v}_h^0) = (\mathscr{R}_h \mathbf{u}^0, \mathscr{P}_h \mathbf{v}^0)$ and $(\mathbf{u}^0, \mathbf{v}^0) \in \mathbf{H}^2 \times \mathbf{H}^1$ be given. Assume that $\mathbf{F}[\mathbf{u}]$ and $\mathbf{G}[\mathbf{u}]$ are linear in $\mathbf{u}$ and conditions (2.1)–(2.5) hold, then we have*

$$\max_{1 \leq n \leq N}\mathbb{E}\left[\|\mathbf{u}^n - \mathbf{u}_h^n\|_{\mathbf{L}^2}^2\right] \leq e^{CC_B^2}\max_{1 \leq l \leq n}\mathbb{E}\left[\|\mathbf{u}^n\|_{\mathbf{H}^2}^2\right]h^4. \tag{4.20}$$

Proof. Plugging $\mathbf{v}_h^l = d_t\mathbf{u}_h^l := k^{-1}(\mathbf{u}_h^l - \mathbf{u}_h^{l-1})$ into (4.5), we have

$$\left(d_t\mathbf{u}_h^{l+1} - d_t\mathbf{u}_h^l, \phi_h\right) + k\lambda\left(\operatorname{div}(\mathbf{u}_h^{l+\frac{1}{2}}), \operatorname{div}(\phi_h)\right) + k\mu\left(\varepsilon(\mathbf{u}_h^{l+\frac{1}{2}}), \varepsilon(\phi_h)\right) \tag{4.21}$$

$$= \left(\mathbf{G}[\mathbf{u}_h^l]\Delta W_l, \phi_h\right) + k\left(\mathbf{F}[\mathbf{u}_h^l], \phi_h\right).$$

Setting $\bar{\mathbf{u}}_h^n = \sum_{l=1}^n \mathbf{u}_h^l$ and taking the summation give

$$\left(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n, \phi_h\right) = k^2\left(\mathscr{L}\bar{\mathbf{u}}_h^{n+\frac{1}{2}}, \phi_h\right) + \left(\mathbf{u}_h^1 - \mathbf{u}_h^0, \phi_h\right) \tag{4.22}$$

$$+ k\left(\sum_{l=1}^n \mathbf{G}[\mathbf{u}_h^l]\Delta W_l, \phi_h\right) + k^2\left(\sum_{l=1}^n \mathbf{F}[\mathbf{u}_h^l], \phi_h\right).$$

Subtracting (4.22) from (3.44) leads to

$$\left(\mathbf{E}_{\mathbf{u}}^{n+1} - \mathbf{E}_{\mathbf{u}}^n, \phi_h\right) + k^2\lambda\left(\operatorname{div}(\bar{\mathbf{E}}_{\mathbf{u}}^{n+\frac{1}{2}}), \operatorname{div}(\phi_h)\right) + \mu\left(\varepsilon(\bar{\mathbf{E}}_{\mathbf{u}}^{n+\frac{1}{2}}), \varepsilon(\phi_h)\right) \tag{4.23}$$

$$= k\left(\sum_{l=1}^n (\mathbf{G}[\mathbf{u}^l] - \mathbf{G}[\mathbf{u}_h^l])\Delta W_l, \phi_h\right) + k^2\left(\sum_{l=1}^n (\mathbf{F}[\mathbf{u}^l] - \mathbf{F}[\mathbf{u}_h^l]), \phi_h\right).$$

Choosing $\phi_h = \mathscr{P}_h \mathbf{E_u}^{n+1}$ in (4.23), the first term on the left-hand side of (4.23) can be written as

$$
\left(\mathbf{E_u}^{n+1} - \mathbf{E_u}^n, \mathscr{P}_h \mathbf{E_u}^{n+1} \right) \tag{4.24}
$$
$$
= \left(\mathscr{P}_h \mathbf{E_u}^{n+1} - \mathscr{P}_h \mathbf{E_u}^n, \mathscr{P}_h \mathbf{E_u}^{n+1} \right)
$$
$$
= \frac{1}{2} \| \mathscr{P}_h \mathbf{E_u}^{n+1} \|_{\mathbf{L}^2}^2 - \frac{1}{2} \| \mathscr{P}_h \mathbf{E_u}^n \|_{\mathbf{L}^2}^2 + \frac{1}{2} \| \mathscr{P}_h \mathbf{E_u}^{n+1} - \mathscr{P}_h \mathbf{E_u}^n \|_{\mathbf{L}^2}^2.
$$

Besides, note that

$$
\lambda \left(\mathrm{div}\,(\bar{\mathbf{E}}_{\mathbf{u}}^{n+\frac{1}{2}}), \mathrm{div}\,(\mathscr{P}_h \mathbf{E_u}^{n+1}) \right) + \mu \left(\varepsilon(\bar{\mathbf{E}}_{\mathbf{u}}^{n+\frac{1}{2}}), \varepsilon(\mathscr{P}_h \mathbf{E_u}^{n+1}) \right) \tag{4.25}
$$
$$
= \lambda \left(\mathrm{div}\,(\mathscr{R}_h \bar{\mathbf{E}}_{\mathbf{u}}^{n+\frac{1}{2}}), \mathrm{div}\,(\mathscr{P}_h \mathbf{E_u}^{n+1}) \right) + \mu \left(\varepsilon(\mathscr{R}_h \bar{\mathbf{E}}_{\mathbf{u}}^{n+\frac{1}{2}}), \varepsilon(\mathscr{P}_h \mathbf{E_u}^{n+1}) \right)
$$
$$
= - \left(\mathscr{L}_h \mathscr{R}_h \bar{\mathbf{E}}_{\mathbf{u}}^{n+\frac{1}{2}}, \mathscr{P}_h \mathbf{E_u}^{n+1} \right)
$$
$$
= - \left(\mathscr{L}_h \mathscr{R}_h \bar{\mathbf{E}}_{\mathbf{u}}^{n+\frac{1}{2}}, \mathbf{E_u}^{n+1} \right)
$$
$$
= - \left(\mathscr{L}_h \mathscr{R}_h \bar{\mathbf{E}}_{\mathbf{u}}^{n+\frac{1}{2}}, \mathscr{R}_h (\bar{\mathbf{E}}_{\mathbf{u}}^{n+1} - \bar{\mathbf{E}}_{\mathbf{u}}^n) \right).
$$

Define another energy functional $\mathscr{E}(\mathbf{E_u}^n)$ by

$$
\mathscr{E}(\mathbf{E_u}^n) = \frac{1}{2} \| \mathscr{P}_h \mathbf{E_u}^n \|_{\mathbf{L}^2}^2 + \frac{k^2}{2} \left(\lambda \left\| \mathrm{div}\,(\mathscr{R}_h \bar{\mathbf{E}}_u^n) \right\|_{\mathbf{L}^2}^2 + \mu \left\| \varepsilon(\mathscr{R}_h \bar{\mathbf{E}}_u^n) \right\|_{\mathbf{L}^2}^2 \right). \tag{4.26}
$$

Choosing $\phi_h = \mathscr{P}_h \mathbf{E_u}^{n+1}$ in (4.23) and using (4.24)–(4.25) yield

$$
\mathscr{E}(\mathbf{E_u}^{n+1}) - \mathscr{E}(\mathbf{E_u}^n) + \frac{1}{2} \left\| \mathscr{P}_h \mathbf{E_u}^{n+1} - \mathscr{P}_h \mathbf{E_u}^n \right\|_{\mathbf{L}^2}^2 \tag{4.27}
$$
$$
= k^2 \left(\sum_{l=1}^n (\mathbf{F}[\mathbf{u}^l] - \mathbf{F}[\mathbf{u}_h^l]), \mathscr{P}_h \mathbf{E_u}^{n+1} \right)
$$
$$
+ k \left(\sum_{l=1}^n (\mathbf{G}[\mathbf{u}^l] - \mathbf{G}[\mathbf{u}_h^l]) \Delta W_n, \mathscr{P}_h \mathbf{E_u}^{n+1} \right)
$$
$$
:= I_1 + I_2.
$$

Using the linearity assumption of F and G in u, the I_1 term can be bounded by

$$
\mathbb{E}\,[I_1] = \mathbb{E}\left[k^2 \big(\sum_{l=1}^n (\mathbf{F}[\mathbf{u}^l] - \mathbf{F}[\mathbf{u}_h^l]), \mathscr{P}_h \mathbf{E_u}^{n+1} \big) \right] \tag{4.28}
$$
$$
= \mathbb{E}\left[k^2 \big(\mathbf{F}[\bar{\mathbf{u}}^l] - \mathbf{F}[\bar{\mathbf{u}}_h^l], \mathscr{P}_h \mathbf{E_u}^{n+1} \big) \right]
$$
$$
\leq C C_B^2 k^3 \mathbb{E}\left[\left\| \bar{\mathbf{E}}_u^n \right\|_{\mathbf{L}^2}^2 \right] + k \mathbb{E}\left[\left\| \mathscr{P}_h \mathbf{E_u}^{n+1} \right\|_{\mathbf{L}^2}^2 \right]
$$
$$
\leq C C_B^2 k^3 \mathbb{E}\left[\lambda \left\| \mathrm{div}\,(\mathscr{R}_h \bar{\mathbf{E}}_{\mathbf{u}}^n) \right\|_{\mathbf{L}^2}^2 + \mu \left\| \varepsilon(\mathscr{R}_h \bar{\mathbf{E}}_{\mathbf{u}}^n) \right\|_{\mathbf{L}^2}^2 \right]
$$
$$
+ C C_B^2 k h^4 \max_{1 \leq l \leq n} \mathbb{E}\left[\| \mathbf{u}^n \|_{\mathbf{H}^2}^2 \right] + k \mathbb{E}\left[\left\| \mathscr{P}_h \mathbf{E_u}^{n+1} \right\|_{\mathbf{L}^2}^2 \right].
$$

Let $M_t = k \left(\sum\limits_{l=0}^{n-1} (\mathbf{G}[\mathbf{u}^l] - \mathbf{G}[\mathbf{u}_h^l]) \Delta W_n, \mathscr{P}_h \mathbf{E}_{\mathbf{u}}^n \right)$. Then M_t is a martingale and $\mathbb{E}[M_t] = 0$. Similarly, using the linearity assumption of F and G in u, the I_2 term can be bounded by

$$\mathbb{E}[I_2] = \mathbb{E}\left[k \left(\sum_{l=0}^{n-1} (\mathbf{G}[\mathbf{u}^l] - \mathbf{G}[\mathbf{u}_h^l]) \Delta W_n, \mathscr{P}_h \mathbf{E}_{\mathbf{u}}^{n+1} \right) \right] \tag{4.29}$$

$$= \mathbb{E}\left[k \left((\mathbf{G}[\bar{\mathbf{u}}^l] - \mathbf{G}[\bar{\mathbf{u}}_h^l]) \Delta W_n, \mathscr{P}_h \mathbf{E}_{\mathbf{u}}^{n+1} - \mathscr{P}_h \mathbf{E}_{\mathbf{u}}^n \right) \right]$$

$$\leq CC_B^2 k^3 \mathbb{E}\left[\left\| \bar{\mathbf{E}}_u^n \right\|_{\mathbf{L}^2}^2 \right] + \frac{1}{4} \mathbb{E}\left[\left\| \mathscr{P}_h \mathbf{E}_{\mathbf{u}}^{n+1} - \mathscr{P}_h \mathbf{E}_{\mathbf{u}}^n \right\|_{\mathbf{L}^2}^2 \right]$$

$$\leq CC_B^2 k^3 \mathbb{E}\left[\lambda \left\| \mathrm{div}\,(\mathscr{R}_h \bar{\mathbf{E}}_{\mathbf{u}}^n) \right\|_{\mathbf{L}^2}^2 + \mu \left\| \varepsilon(\mathscr{R}_h \bar{\mathbf{E}}_{\mathbf{u}}^n) \right\|_{\mathbf{L}^2}^2 \right]$$

$$+ CC_B^2 k h^4 \max_{1 \leq l \leq n} \mathbb{E}\left[\|\mathbf{u}^n\|_{\mathbf{H}^2}^2 \right] + \frac{1}{4} \mathbb{E}\left[\left\| \mathscr{P}_h \mathbf{E}_{\mathbf{u}}^{n+1} - \mathscr{P}_h \mathbf{E}_{\mathbf{u}}^n \right\|_{\mathbf{L}^2}^2 \right].$$

Combining (4.27)-(4.29), we obtain

$$\mathbb{E}\left[\mathscr{E}(\mathbf{E}_{\mathbf{u}}^{n+1}) \right] - \mathbb{E}\left[\mathscr{E}(\mathbf{E}_{\mathbf{u}}^n) \right] + \frac{1}{4} \mathbb{E}\left[\left\| \mathscr{P}_h \mathbf{E}_{\mathbf{u}}^{n+1} - \mathscr{P}_h \mathbf{E}_{\mathbf{u}}^n \right\|_{\mathbf{L}^2}^2 \right] \tag{4.30}$$

$$\leq CC_B^2 k \mathbb{E}\left[\mathscr{E}(\mathbf{E}_{\mathbf{u}}^n) \right] + CC_B^2 k h^4 \max_{1 \leq l \leq n} \mathbb{E}\left[\|\mathbf{u}^n\|_{\mathbf{H}^2}^2 \right].$$

Then the discrete Gronwall's inequality yields

$$\mathbb{E}\left[\mathscr{E}(\mathbf{E}_{\mathbf{u}}^{n+1}) \right] \leq e^{CC_B^2} \max_{1 \leq l \leq n} \mathbb{E}\left[\|\mathbf{u}^n\|_{\mathbf{H}^2}^2 \right] h^4. \tag{4.31}$$

The proof is complete. $\square$

5. NUMERICAL EXPERIMENTS

In this section, we present two two-dimensional numerical experiments using the fully discrete Scheme 2 to test and validate our theoretical results. Our computations are done using the software package FEniCS [11].

Test 1. We consider the following stochastic elastic wave equation with linear multiplicative noise:

$$\begin{cases} \mathbf{u}_{tt}(t, \mathbf{x}) - \mathrm{div}\,(\sigma(\mathbf{u}(t, \mathbf{x}))) = \mathbf{F}[\mathbf{u}] + \mathbf{G}[\mathbf{u}] \dfrac{d\mathbf{W}}{dt} & \text{in } \mathscr{D} \times [0, T], \\ \mathbf{u}(t, \mathbf{x}) = \mathbf{0} & \text{on } \partial\mathscr{D} \times [0, T], \\ \mathbf{u}(0, \mathbf{x}) = \mathbf{u_0}, \ \mathbf{u_t}(0, \mathbf{x}) = \mathbf{u_1} & \text{in } [0, T], \end{cases}$$

where $\mathscr{D} = [0,1] \times [0,1]$, $\mathbf{F}[\mathbf{u}] = |\mathbf{u}|^2 \mathbf{u}$, and $\mathbf{G}[\mathbf{u}] = \delta \mathbf{u}$. Notice that $\mathbf{G}$ is linear in $\mathbf{u}$.

The datum functions are chosen as

$$\begin{cases} \mathbf{u_0} = \begin{pmatrix} \sin^2 \pi x_1 \sin 2\pi x_2, \\ \sin 2\pi x_1 \sin^2 \pi x_2 \end{pmatrix}, \\ \mathbf{u_1} = -0.3\mathbf{u_0}. \end{cases}$$

Choose the noise intensity parameter $\delta = 0.1$, final time $T = 0.5$, and the numerical solution with $k_{ref} = 1/500$ as the numerical exact solution for verifying convergence orders in time. Table 1 and Figure 1 list the temporal error and convergence order results of Test 1.
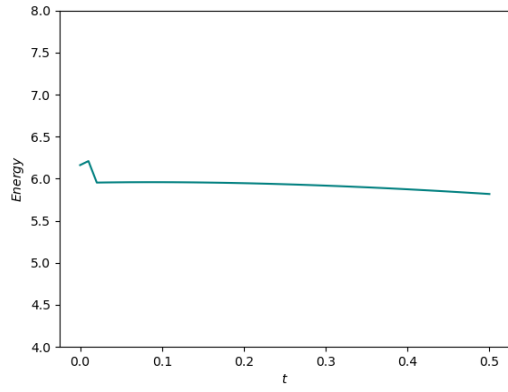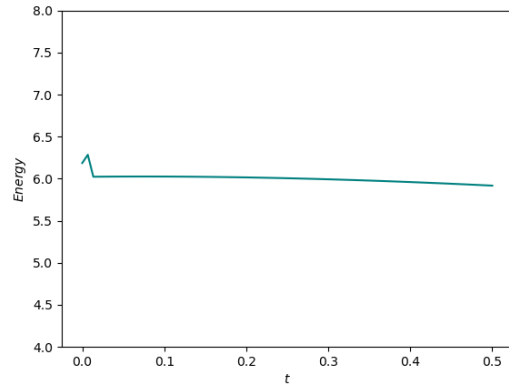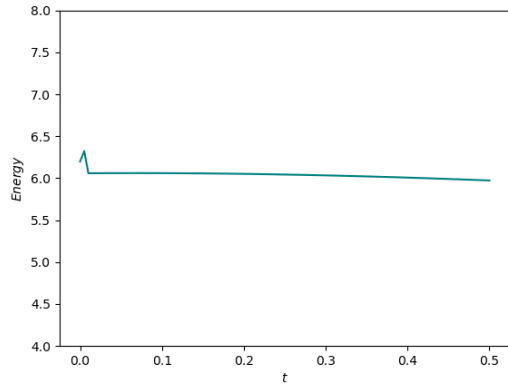
TABLE 1. Temporal errors of Test 1. Uniform meshes with $h = 1/256$ at $T = 0.5$, and 500 samples are selected.

Scheme 2		$\mathbb{E}\left[\left\|\mathbf{u}(T) - \mathbf{u}_h^N\right\|\right]$				$\mathbb{E}\left[\left\|\mathbf{v}(T) - \mathbf{v}_h^N\right\|\right]$	
h	k	L^2 error	order	H^1 error	order	L^2 error	order
1/256	1/50	9.84×10^{-3}	——	8.11×10^{-2}	——	1.69×10^{-2}	——
	1/75	6.88×10^{-3}	0.88	6.80×10^{-2}	0.44	1.46×10^{-2}	0.37
	1/100	5.17×10^{-3}	0.99	5.97×10^{-2}	0.45	1.30×10^{-2}	0.39
	1/150	3.43×10^{-3}	1.01	4.87×10^{-2}	0.50	1.09×10^{-2}	0.44
	1/200	2.51×10^{-3}	1.09	4.16×10^{-2}	0.55	9.57×10^{-3}	0.45
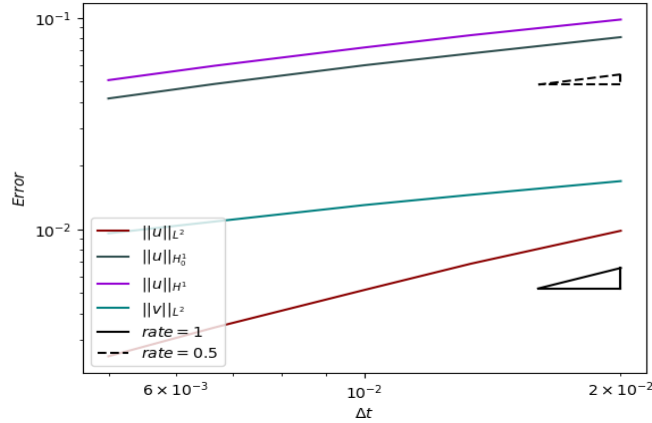
FIGURE 1. Rates of the temporal error convergence of Test 1 in $\mathbf{L}^2$- and $\mathbf{H}^1$-norm. Here $T = 0.5$, $h = 1/256$, and $k \in \{50^{-1}, 75^{-1}, 100^{-1}, 150^{-1}, 200^{-1}\}$.

Now we fix $k_{ref} = 1/500$ and choose $h_{ref} = 1/512$. Table 2 and Figure 2 list the spatial error and convergence order results of Test 1.

TABLE 2. Spatial errors of Test 1. The reference solution is given by $h_{ref} = 1/512$ and 500 samples are selected.

Scheme 2		$\mathbb{E}\left[\left\|\mathbf{u}(T) - \mathbf{u}_h^N\right\|\right]$			
k	h	L^2 error	order	H_0^1 error	order
1/500	1/64	9.38×10^{-5}	——	1.76×10^{-3}	——
	1/128	2.37×10^{-5}	1.98	9.52×10^{-4}	0.89
	1/256	5.71×10^{-6}	2.06	4.44×10^{-4}	1.10
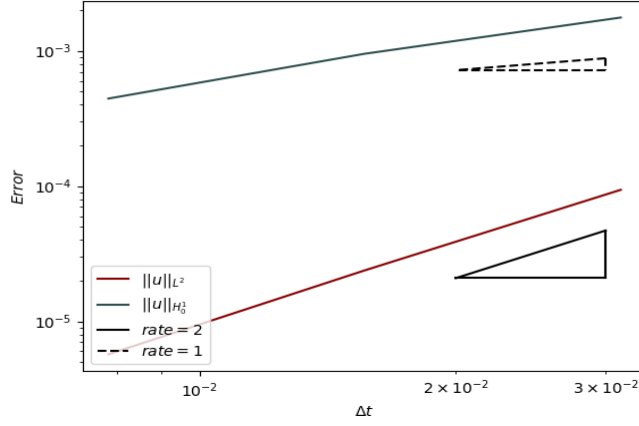
FIGURE 2. The spatial convergence rates of Test 1 in $\mathbf{L}^2$- and $\mathbf{H}^1$-norm. Here $T = 0.5$, $k = 1/500$, and $h \in \{2^{(-6+i)}, i = 0, 1, 2\}$.

Finally, to verify the stability of the proposed numerical methods, we compute the energy norm numerically and draw the graph of the energy norm over time, as shown in Figure 3.

Test 2. We consider the stochastic elastic wave equations with nonlinear multiplicative noise as given below.

$$\begin{cases} \mathbf{u}_{tt}(t,\mathbf{x}) - \operatorname{div}\left(\sigma(\mathbf{u}(\mathbf{t},\mathbf{x}))\right) = \mathbf{F}[\mathbf{u}] + \mathbf{G}[\mathbf{u}]\dfrac{\mathbf{dW}}{\mathbf{dt}} & \text{in } \mathscr{D} \times [0,T], \\ \mathbf{u}(t,\mathbf{x}) = \mathbf{0} & \text{on } \partial\mathscr{D} \times [0,T], \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{u_0},\ \mathbf{u_t}(\mathbf{0},\mathbf{x}) = \mathbf{u_1} & \text{in } [0,T], \end{cases}$$

where $\mathscr{D} = [0,1] \times [0,1]$, $\mathbf{F}[\mathbf{u}] = |\mathbf{u}|^2\mathbf{u}$ and $\mathbf{G}[\mathbf{u}] = \delta(|\mathbf{u}|^2 + 1)\mathbf{u}$. Notice that both $\mathbf{F}$ and $\mathbf{G}$ are nonlinear in $\mathbf{u}$.

The datum functions are chosen as

$$\begin{cases} \mathbf{u_0} = \begin{pmatrix} \sin 3\pi x_1 \sin 2\pi x_2, \\ \sin 2\pi x_1 \sin 3\pi x_2 \end{pmatrix}, \\ \mathbf{u_1} = \mathbf{0}. \end{cases}$$

Choose the noise intensity parameter $\delta = 0.1$, final time $T = 0.5$, and the numerical solution with $k_{ref} = 1/500$ as the numerical exact solution for verifying the time convergence order. Table 3 and Figure 4 list the temporal error and convergence order results of Test 1.
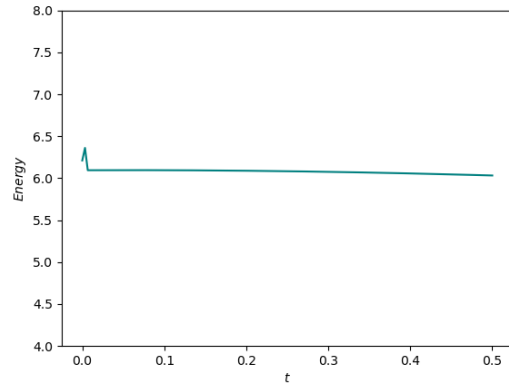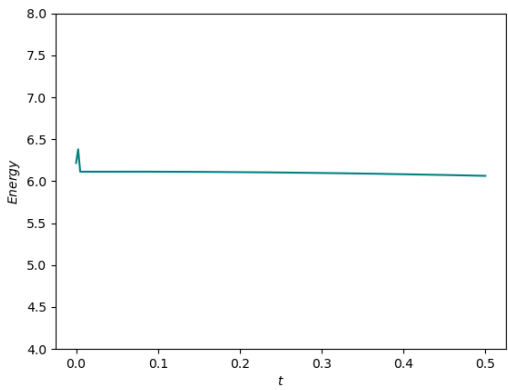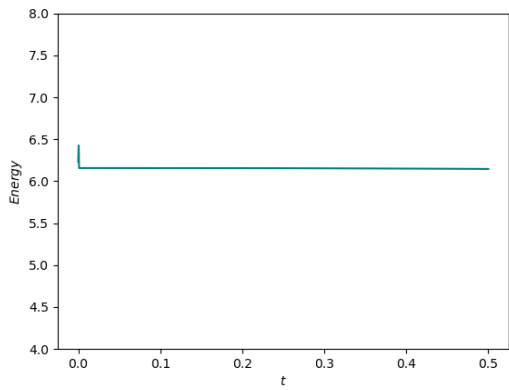
(a) $k = 1/50$

(b) $k = 1/75$

(c) $k = 1/100$

(d) $k = 1/150$

(e) $k = 1/200$

(f) $k = k_{ref}$

FIGURE 3. *Energy norm in Test 1 over time based on different time step sizes.*

TABLE 3. Temporal errors of Test 2. Uniform meshes with $h = 1/256$ at $T = 0.5$, and 500 samples are selected.

Scheme 2		$\mathbb{E}\left[\left\|\mathbf{u}(T) - \mathbf{u}_h^N\right\|\right]$				$\mathbb{E}\left[\left\|\mathbf{v}(T) - \mathbf{v}_h^N\right\|\right]$	
h	k	L^2 error	order	H_0^1 error	order	L^2 error	order
	1/50	1.32×10^{-1}	——	1.65×10^0	——	4.42×10^{-1}	——
	1/75	9.46×10^{-2}	0.82	1.41×10^0	0.39	3.76×10^{-1}	0.40
1/256	1/100	7.27×10^{-2}	0.92	1.24×10^0	0.43	3.32×10^{-1}	0.43
	1/150	4.88×10^{-2}	0.98	1.03×10^0	0.47	2.79×10^{-1}	0.43
	1/200	3.65×10^{-2}	1.01	8.96×10^{-1}	0.50	2.41×10^{-1}	0.51
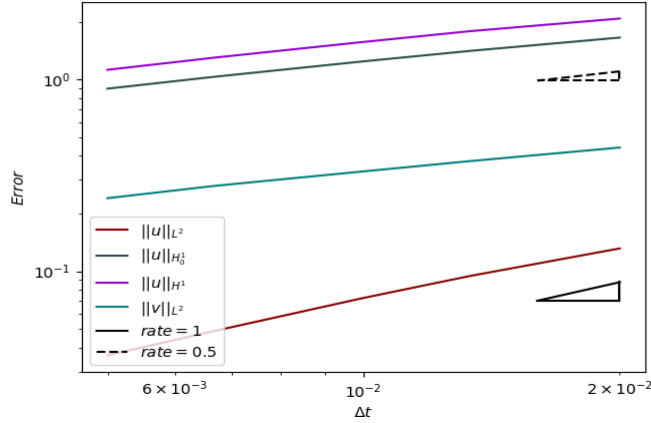
FIGURE 4. Rates of the temporal error convergence of Test 2 in $\mathbf{L}^2$- and $\mathbf{H}^1$-norm. Here $T = 0.5$, $h = 1/256$, and $k \in \{50^{-1}, 75^{-1}, 100^{-1}, 150^{-1}, 200^{-1}\}$.

Now we fix $k_{ref} = 1/500$ and choose $h_{ref} = 1/512$. Table 4 and Figure 5 list the spatial error and convergence order results of Test 2.

TABLE 4. Spatial errors of Test 2. The reference solution is computed by using $h_{ref} = 1/512$ and 500 samples are selected.

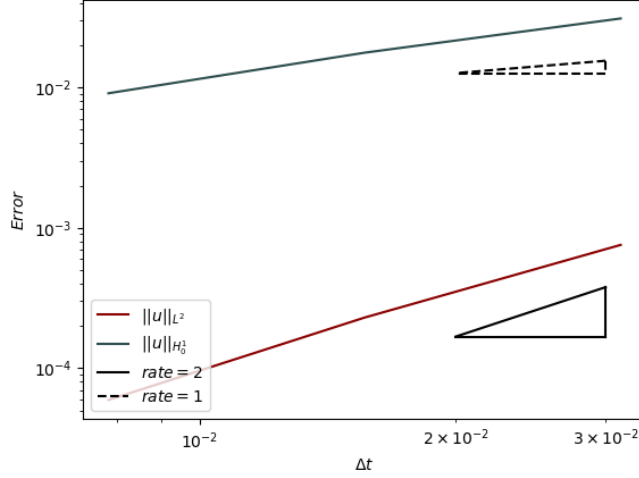| Scheme 2		$\mathbb{E}\left[\left\|\mathbf{u}(T) - \mathbf{u}_h^N\right\|\right]$			
k	h	L^2 error	order	H_0^1 error	order
	1/64	7.56×10^{-4}	——	3.10×10^{-2}	——
1/500	1/128	2.30×10^{-4}	1.72	1.77×10^{-2}	0.81
	1/256	5.94×10^{-5}	1.95	9.10×10^{-3}	0.96

FIGURE 5. Rates of the spatial error convergence of Test 2 in the $\mathbf{L}^2$- and $\mathbf{H}^1$-norm. $T = 0.5$, $k = 1/500$ and $h \in \{2^{(-6+i)}, i = 0, 1, 2\}$.

Finally, to verify the stability of the proposed numerical method, we compute the energy norm numerically and draw the graph of the energy norm over time, as shown in Figure 6.

6. CONCLUSION

In this paper, we proposed a semi-discrete in time scheme and a fully discrete finite element method for nonlinear stochastic elastic wave equations with multiplicative noise. We proved various properties for the strong solution such as stability and Hölder continuity estimates. We also established the stability and error estimates for the semi-discrete numerical solution, which show that the $\mathbf{L}^2$-norm of the temporal error has first order convergence, while the $\mathbf{H}^1$-norm of the error has one half order convergence. Moreover, we proved that, for the linear finite element, the $\mathbf{L}^2$-norm of the spatial error has second order convergence, and the $\mathbf{H}^1$-norm of the error has first order convergence. Two-dimensional numerical experiments were also presented using the proposed numerical methods to validate the theoretical results proved in the paper.

To the best of our knowledge, no numerical analysis result for stochastic elastic wave equations has been reported in the literature. The work of this paper fills the void in this area. Like in the numerical analysis of stochastic parabolic PDEs, a key idea and technique for overcoming the difficulty caused by the noise is to establish and make use of the Hölder continuity in time of the weak solution in various spatial norms. We also note that the results of this paper could be improved in light of the recent work [18]. In particular, it is possible to construct better time-stepping schemes which can achieve the optimal $\frac{3}{2}$ order of convergence for the displacement approximation, which in turn requires higher order approximation for the multiplicative noise term. In addition, the classical Monte Carlo (MC) method is computationally inefficient, more efficient quasi-MC methods could be used to speed up the computation of the expected values. Moreover, other numerical methods such as the stochastic spectral method [39] could be used to process and analyze the noise term. Finally, the analysis techniques of this paper may not be able to handle more general nonlinear functions $\mathbf{F}[\mathbf{u}]$ and $\mathbf{G}[\mathbf{u}]$, new analysis techniques are needed for the job. We plan to continue addressing those issues in a future work.
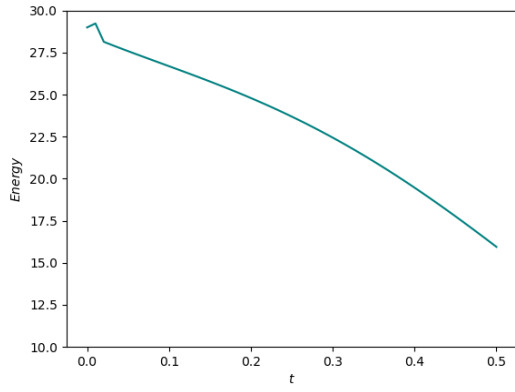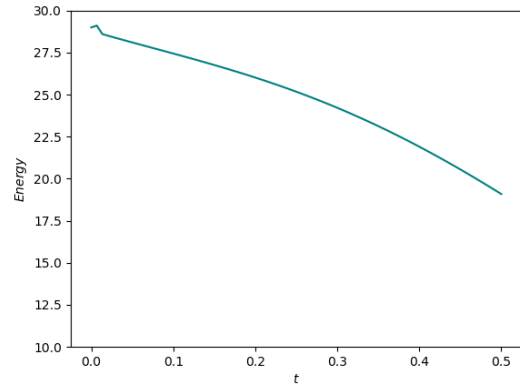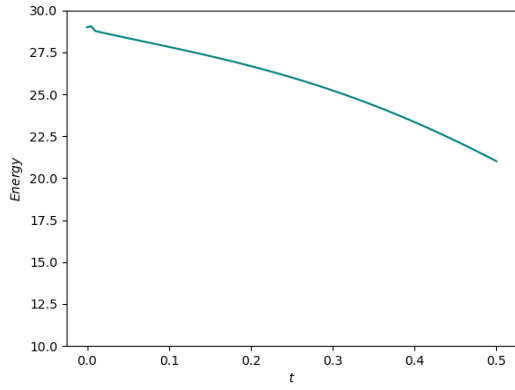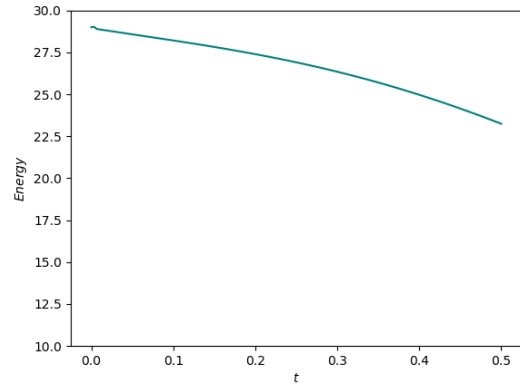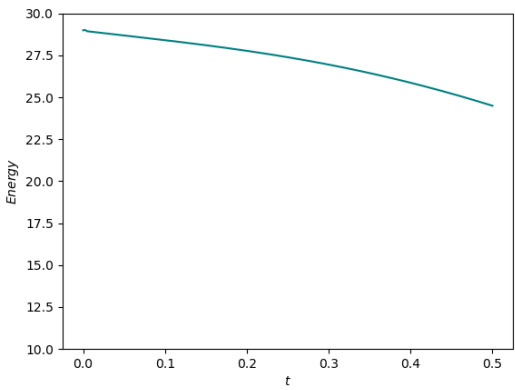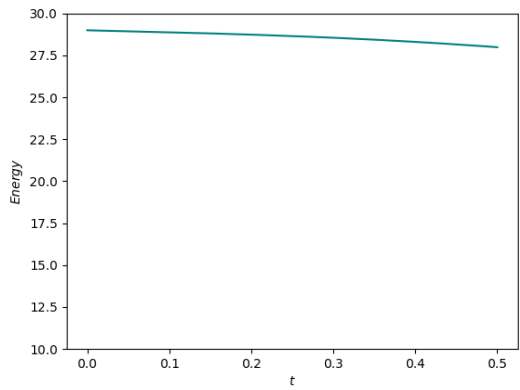
(a) $k = 1/50$

(b) $k = 1/75$

(c) $k = 1/100$

(d) $k = 1/150$

(e) $k = 1/200$

(f) $k = \Delta t_{ref}$

FIGURE 6. *Energy norm of Test 2 over time, using different time step sizes.*

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