SYNCHRONICITY FOR QUANTUM NON-LOCAL GAMES

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Abstract. We introduce concurrent quantum non-local games, quantum output mirror games and concurrent classical-to-quantum non-local games, as quantum versions of synchronous non-local games, and provide tracial characterisations of their perfect strategies belonging to various correlation classes. We define *-algebras and C*-algebras of concurrent classical-to-quantum and concurrent quantum non-local games, and algebraic versions of the orthogonal rank of a graph. We show that quantum homomorphisms of quantum graphs can be viewed as entanglement assisted classical homomorphisms of the graphs, and give descriptions of the perfect quantum commuting and the perfect approximately quantum strategies for the quantum graph homomorphism game. We specialise the latter results to the case where the inputs of the game are based on a classical graph.

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1. Introduction

Over the past decade, the theory of non-local games has undergone a flurry of development and is now a fundamental branch of modern quantum information theory, with deep applications to many areas of mathematics, physics, and computer science, including operator algebras, noncommutative geometry, quantum non-locality, entanglement, and quantum complexity theory. Mathematically, a (two-player) non-local game consists of a tuple $\mathcal{G} = (X, Y, A, B, \lambda)$, where X, Y, A, B are finite sets, and $\lambda : X \times Y \times A \times B \to \{0, 1\}$ is a function. The game is played cooperatively by two spatially separated non-communicating players, Alice and Bob, against a referee. During each round of the game, the referee samples a pair of "questions" $(x, y) \in X \times Y$, and sends question x to Alice, and quesiton y to Bob. Alice is then required to supply an "answer" $a \in A$, and Bob – an answer $b \in B$, to the referee. Alice and Bob win the round of the game if and only if the rule function λ evaluates to 1 on this question-answer combination, that is, if the condition $\lambda(x, y, a, b) = 1$ is satisfied.

The fact that the players Alice and Bob are not allowed to communicate during play makes it difficult to win each round of a non-local game with high probability. On the other hand, it is precisely this nature of non-local games that makes them interesting as both theoretical and practical tools in quantum information. The idea here is that, in certain scenarios, Alice and Bob can utilise the phenomenon of quantum entanglement to help correlate their answers in a much stronger way than what the resources of classical physics allow.

A prototypical example of a non-local game is the graph homomorphism game: Given a pair of finite simple graphs G and H with vertex sets $V(G)$, $V(H)$ and edge sets $E(G)$, $E(H)$, respectively, the (G, H)-homomorphism game is the non-local game $\mathcal{G}$ with $X = Y = V(G)$, $A = B = V(H)$ and $\lambda(x, y, a, b) = 0$ if either (i) $x = y$ and $a = b$ or (ii) $(x, y) \in E(G)$ and $(a, b) \notin E(H)$. Clearly the graph homomorphism game captures, in the operational language of non-local games, the notion of a graph homomorphism $G \to H$: Any winning strategy for this game would serve to convince an observer that there exists such a graph homomorphism $G \to H$.

Graph homomorphism games form an interesting class of non-local games for several reasons. First, they give rise to quantum analogues of graph parameters, including quantum chromatic numbers and quantum independence numbers [18, 24]. These parameters can be genuinely different than the corresponding classical versions, thus providing new manifestations of the fundamental Bell Theorem. Second, they provide some of the simplest examples of pseudo-telepathy games – ones which can be perfectly won only with the help of quantum entanglement as a resource [18, 11, 24]. Third, and perhaps most importantly, graph homomorphism games belong to the particularly important class of synchronous non-local games introduced in [24] (see also [12]). Recall that a non-local game $\mathcal{G} = (X, Y, A, B, \lambda)$ is called sychronous if $X = Y$, $A = B$, and $\lambda(x,x,a,b) = 0$ for all $x \in X$ and $a \neq b \in A$. This means that in order for Alice and Bob to win a round of G, they must "sychronise" their answers whenever they both receive the same question from the referee. This seemingly innocuous constraint on a game G turns out to have very interesting quantum information theoretic and operator algebraic consequences. For example, the problem of finding perfect quantum strategies for a synchronous game G amounts to finding tracial states on a certain game $*$-algebra $\mathcal{A}(G)$ associated to G [12]. The algebras $\mathcal{A}(G)$ play the role of a non-commutative analogue of the algebras of coordinate functions on spaces of perfect deterministic (classical) strategies for G, and are therefore of significant interest from several perspectives in non-commutative geometry, quantum groups [28, 4], and von Neumann algebra theory [13]. It follows from the breakthrough work [13] that there exists a synchronous non-local game G whose game $*$-algebra $\mathcal{A}(G)$ admits a tracial state τ for which the generated von Neumann algebra $M = \pi_\tau(\mathcal{A}(G))''$ fails to embed into an ultraproduct of the hyperfinite II_1-factor – yielding a (n al-beit non-constructive) counter-example to the Connes Embedding Problem in operator algebras and to the equivalent [14] strong Tsirelson Problem in quantum physics.

The purpose of the present paper is to introduce and study generalisations of synchronous non-local games within the framework of quantum non-local games – non-local games where the questions and answers are allowed to be quantum states, or possibly mixtures of classical and quantum states. In this paper, we use the language of quantum no-signalling (QNS) correlations and quantum non-local games recently introduced by two of the present authors [30]. Classically, in the course of a non-local game $G = (X, Y, A, B, \lambda)$, Alice and Bob's behaviour is described by a family $p = (p(a,b|x,y))_{(a,b,x,y)\in A\times B\times X\times Y}$ of conditional probability distributions, which can, in a canonical way, be viewed as a noisy information channel $\mathcal{N} : X \times Y \to A \times B$ with well-defined marginal channels. In the quantum setting, one replaces the classical state spaces X, Y, A, B by their quantum analogues (i.e. the Hilbert spaces $\mathbb{C}^{|X|}$, $\mathbb{C}^{|Y|}$, etc.), and the classical channel $\mathcal{N} : X \times Y \to A \times B$ by a quantum channel $\Gamma : M_X \otimes M_Y \to M_A \otimes M_B$, where, for any finite set Z, we have let $M_Z = \mathcal{B}(\mathbb{C}^{|Z|})$ be the matrix algebra of linear maps on $\mathbb{C}^{|Z|}$. In this framework, the rule function λ can be generalized by replacing it with a zero-preserving, join-preserving map-ping φ from the projection lattice on $\mathcal{P}_{XY}$ in $M_X \otimes M_Y$ to the projection lattice $\mathcal{P}_{AB}$ in $M_A \otimes M_B$. A winning strategy for a quantum non-local game $\varphi : \mathcal{P}_{XY} \to \mathcal{P}_{AB}$ is then given by a QNS correlation Γ satisfying the trace-orthogonality relation

$$\langle \Gamma(P), \varphi(P)^{\perp} \rangle = 0, \quad P \in \mathcal{P}_{XY};$$

the latter condition constrains the supports of the output states of Γ according to the supports of its input states (see Section 3.1 for further motivation and details).

We note that non-local games with quantum inputs and/or outputs have been previously studied in [7] and [27]. The strategies used in the latter papers are the elements from the quantum QNS correlation class. Since our main interest lies in the characterisation of the perfect strategies of a game and their applications, we have adopted the present approach, where we only specify the rules of the non-local game, without fixing a probability distribution on the questions (or a quantum version thereof).

One of our main achievements in the present work is the introduction of quantum analogues of synchronous non-local games (called herein concurrent quantum games), as well as classical input-quantum output versions of the mirror games introduced in [17]. Classically, synchronous games form a special class of mirror games, and both of these classes of games have the remarkable property that "Alice's quantum behaviour completely determine Bob's quantum behaviour" when considering perfect strategies for the games; moreover, such perfect strategies can always be described in terms of correlations coming from tracial states on a particular game algebra. We show that such a paradigm persists in the quantum case by associating *-algebras and C*-algebras to concurrent quantum and to concurrent classical-to-quantum games. Our main results in this direction (cf. Theorem 3.2, Corollary 3.7, Theorem 4.1, Corollary 4.4) provide an operational interpretation of the tracial QNS correlations introduced in [30] in terms of perfect strategies of concurrent and quantum mirror games, and their associated game algebras.

One of our long-term motivations for the present work is to develop tools that may eventually be useful for gaining a better understanding of the work [13], which, as mentioned above, implicitly constructs a game G, whose game algebra is a witness to the failure of the Connes Embedding Problem. At present, the game constructed in [13] is not well understood, and involves very large input/output sets. There is some hope that quantum non-local games may provide additional flexibility in the construction of game algebras with pathological operator algebraic properties. A particularly interesting and tractable source of examples in this more general framework are the quantum graph homomorphism games. Quantum graphs have achieved a lot of attention in recent years, as objects that arise in a variety of areas (e.g. zero-error quantum information theory, quantum error correction, quantum groups, quantum teleportation schemes, and subfactor theory) [3, 4, 21, 29, 33]. In Section 5, we study the quantum graph homomorphism game in detail, extending previous work of the authors [5, 30] in the classical-quantum hybrid setting, and also making connections with the work of Stahlke [29] and the algebraic work of Musto-Reutter-Verdon [21] on quantum graph homomorphisms.

The paper is organised as follows. Section 2 introduces some necessary notation and background that will be used throughout the paper. Section 3 recalls the notions related to QNS correlations and their various subclasses (quantum commuting, approximately quantum, quantum, local), examines in detail the case of classical to quantum non-local games, introducing the aforementioned semi-quantised mirror games and concurrent games, and studies them as operational realisations of tracial QNS correlations. In Section 4, we consider the fully quantum concurrent games, proving tracial characterisations of perfect strategies of these games. Finally, in Section 5, we focus on the quantum graph homomorphism game, and describe connections with the prior work of Stahlke [29] on entanglement assisted quantum graph homomorphisms, as well as with our prior works [5, 30]. We show that the perfect quantum strategies of the quantum graph homomorphism game can, in a rigorous sense, be thought of as entanglement assisted perfect classical strategies for this game, and extract characterisations of the corresponding quantum commuting and approximately quantum strategies in terms of natural inclusion relations relating the two quantum graphs. Our results are further specialised in the case where the inputs are based on a classical graph, leading to separation results on the algebraic and C^*-algebraic versions of the orthogonal rank of a graph (cf. Propositions 5.16 and 5.17).

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Note on related work. After the first draft of this paper was completed, we learnt from Piotr Soltan that characterisations of concurrent correlations from the quantum commuting class, closely related to the ones described in Subsection 4.1, were independently obtained by Bochniak-Kasprzak-Soltan in the recently posted preprint [2]; more specifically, [2, Theorem 6.6] generalises the first statement within Theorem 4.1 in the present paper.

2. Preliminary notions and results

For a finite set X, let M_X be the algebra of all complex matrices indexed by $X \times X$; we identify M_X with the algebra of all linear transformations on the Hilbert space $\mathbb{C}^X := \oplus_{x\in X}\mathbb{C}$. If $T \in M_A$, we write T^t for the transpose matrix, and set $\overline{T} = (T^*)^t$. We let $\mathcal{D}_X$ be the subalgebra of M_X of all diagonal matrices, and $\Delta_X : M_X \to \mathcal{D}_X$ be the conditional expectation. We write $M_{XY} = M_X \otimes M_Y$, $\mathcal{P}_{XY}$ for the projection lattice of M_{XY}, and $\mathcal{P}^d_{XY}$ for the projection lattice of $\mathcal{D}_{XY}$. We let $\xi\xi^*$ be the rank one operator given by $\xi\xi^*(\eta) = \langle \eta, \xi\rangle \xi$, and $(e_x)_{x\in X}$ be the canonical orthonormal basis of $\mathbb{C}^X$. For a Hilbert space H and vectors $\xi, \eta \in H$, we write $\xi \perp \eta$ if $\langle \xi, \eta\rangle = 0$. Let H^d be the dual (Banach) space of H and $d : H \to H^d$ be the map, given by $d(\xi)(\eta) = \langle \eta, \xi\rangle$; we write $\xi^d = d(\xi)$. Note that $(\lambda\xi)^d = \overline{\lambda}\xi^d$, $\lambda \in \mathbb{C}$, and that, if $T \in \mathcal{L}(H)$, then the dual operator $T^d : H^d \to H^d$ satisfies the relation

$$T^{d*} = (T^{*})^{d}, \quad T \in L(H). \tag{1}$$

Let $\omega \in M_X$. Writing f_ω for the functional on M_X given by $f_\omega(\rho) = \operatorname{Tr}(\rho\omega)$, we have that the map $\omega \to f_\omega$ is a complete order isomorphism from M_X onto the dual operator system M_X^d (see e.g. [26, Theorem 6.2]). On the other hand, the map $\omega^d \to \omega^t$ is a *-isomorphism from $L((\mathbb{C}^X)^d)$ onto M_X. The composition of these maps, $\omega^d \to f_{\omega^t}$, is thus a complete order isomorphism from $L\left(\left(\mathbb{C}^X\right)^d\right)$ onto M_X^d. In the sequel, we identify these two spaces; note that, via this identification,

$$\langle \rho, \omega^d \rangle = \langle \rho, \omega^t \rangle = \operatorname{Tr}(\rho\omega), \quad \rho, \omega \in M_X. \tag{2}$$

If $P \in M_X$ is a projection, we write $P^\perp$ for the projection in M_X^d on the annihilator in $\left(\mathbb{C}^X\right)^d$ of the range of P.

Write $\epsilon_{x,x'} = e_x e_{x'}^*$ for the matrix unit in M_X, corresponding to the pair (x, x') of indices. Set

$$J_X = \frac{1}{|X|} \sum_{x,x' \in X} \epsilon_{x,x'} \otimes \epsilon_{x,x'};$$

if $m_X = \frac{1}{\sqrt{|X|}} \sum_{x \in X} e_x \otimes e_x$ is the maximally entangled unit vector in $\mathbb{C}^X \otimes \mathbb{C}^X$, then $J_X = m_X m_X^*$ is its corresponding rank one projection. Set also

$$J_X^{cl} = \sum_{x \in X} \epsilon_{x,x} \otimes \epsilon_{x,x},$$

and note that $\Delta_{XX}(J_X) = \frac{1}{|X|} J_X^{cl}$. Heuristically, J_X^{cl} is the (normalised) part of J_X that can be seen by a classical observer.

Recall [30] that a quantum non-local game is a join-preserving map $\varphi : \mathcal{P}_{XY} \to \mathcal{P}_{AB}$ with $\varphi(0) = 0$, while a classical-to-quantum (cq) non-local game is a join-preserving map $\varphi : \mathcal{P}_{XY}^{cl} \to \mathcal{P}_{AB}$ with $\varphi(0) = 0$. Similarly, a classical non-local game is a join-preserving and zero-preserving map $\varphi : \mathcal{P}_{XY}^{cl} \to \mathcal{P}_{AB}^{cl}$.

Recall also that a non-local game on the quadruple (X, Y, A, B) is a function $\lambda : X \times Y \times A \times B \to \{0, 1\}$. In [30], we associated to such λ the classical non-local game $\varphi_\lambda : \mathcal{P}_{XY}^{cl} \to \mathcal{P}_{AB}^{cl}$ given by

$$\varphi_\lambda\left(\sum_{(x,y) \in \kappa} \epsilon_{x,x} \otimes \epsilon_{y,y}\right) = \bigvee \{\epsilon_{a,a} \otimes \epsilon_{b,b} : \exists\, (x, y) \in \kappa \text{ s.t. } \lambda(x, y, a, b) = 1\},$$

after recalling that projections in $\mathcal{P}_{XY}^{cl}$ correspond to subsets $\kappa \subseteq X \times Y$.

A non-local game (X, Y, A, B, λ) is called

- a mirror game [17] if there exist functions $f : X \to Y$ and $g : Y \to X$ such that for every $x \in X$ (resp. $y \in Y$) the set

$$\{(a, b) \in A \times B : \lambda(x, f(x), a, b) = 1\}$$

(resp.
$$\{(a,b)\in A\times B : \lambda(g(y),y,a,b)=1\})$$
is the graph of a bijection, and

- a synchronous game [24] (see also [12]) if $X=Y$, $A=B$ and
$$a,b\in A,\ a\neq b \implies \lambda(x,x,a,b)=0.$$

Mirror games include the subclass of unique games (that is, games for which the set $\{(a,b)\in A\times B : \lambda(x,y,a,b)=1\}$ is the graph of a bijection for every $(x,y)\in X\times Y$ [31]); in particular, they form a class, strictly larger than that of synchronous games.

Set $B=A$ and recall the standard (linear) identification of matrices in M_A with vectors in $\mathbb{C}^A\otimes\mathbb{C}^B$, which associates to the matrix unit $\epsilon_{a,b}$ the vector $e_a\otimes e_b$ (see e.g. [32, Section 1.1.2]). Write $\tilde{\xi}_T\in\mathbb{C}^A\otimes\mathbb{C}^B$ for the vector corresponding to $T\in M_A$ and set $\xi_T=\frac{\tilde{\xi}_T}{\|\tilde{\xi}_T\|}$; we have that $\xi_{I_A}=m_A$. We note the relations [32, Section 1.1.1]

$$(R\otimes S)\tilde{\xi}_T=\tilde{\xi}_{RTS^t},\quad R,S,T\in M_A. \tag{3}$$

If $\sigma: A\to B$ is a bijection, let $P_\sigma=\sum_{a\in A}\epsilon_{a,a}\otimes\epsilon_{\sigma(a),\sigma(a)}$; clearly, $P_\sigma\in\mathcal{P}_{AB}$.

Remark 2.1. A non-local game λ is

(i) synchronous if and only if $\Gamma(\mathcal{J}^{cl}_X)\subseteq\mathcal{J}^{cl}_A$;

(ii) mirror if and only if there exist functions $f: X\to Y$, $g: Y\to X$ and bijections $\sigma_x,\tau_y: A\to B$, $x\in X$, $y\in Y$, such that

$$\Gamma(\epsilon_{x,x}\otimes\epsilon_{f(x),f(x)})=P_{\sigma_x} \text{ and } \Gamma(\epsilon_{g(y),g(y)}\otimes\epsilon_{y,y})=P_{\tau_y^{-1}},\ x\in X,\ y\in Y.$$

Proof. (i) If λ is synchronous then, clearly,
$$\Gamma(\epsilon_{x,x}\otimes\epsilon_{x,x})\in\mathcal{J}^{cl}_A \text{ for all } x\in X;$$
taking the span over all x, we get $\Gamma(\mathcal{J}^{cl}_X)\subseteq\mathcal{J}^{cl}_A$. Conversely, the condition $\Gamma(\mathcal{J}^{cl}_X)\subseteq\mathcal{J}^{cl}_A$ implies in particular $\Gamma(\epsilon_{x,x}\otimes\epsilon_{x,x})\in\mathcal{J}^{cl}_A$, which is equivalent to $\lambda(x,x,a,b)=0$ whenever $a\neq b$. Claim (ii) is equally straightforward. $\square$

Remark 2.1 motivates the following versions of mirror and synchronous games, where the inputs are still classical, while the outputs are allowed to be quantum. We assume that $|A|=|B|$ but continue to use different symbols to denote the sets A and B for clarity. If $\omega\in M_A$, let $L_\omega: M_{AB}\to M_B$ be the slice map, given by $L_\omega(S\otimes T)=\langle S,\omega\rangle T$ and write $\mathrm{Tr}_A=L_{I_A}$ for the partial trace; the slice map $L^{\omega}: M_{AB}\to M_A$, for $\omega\in M_B$, and the partial trace Tr_B, are defined similarly. Call a rank one projection $P\in M_{AB}$ bijective if

$$e,f\in\mathbb{C}^A,\ e\perp f \implies L_{ee^*}(P)\perp L_{ff^*}(P) \tag{4}$$

(note that the orthogonality is understood in terms of the trace in M_B). Bijective projections can be thought of as quantum versions of bijections; in fact, if $\sigma: A\to B$ is a bijection then $P=P_\sigma$ satisfies (4) when e and f are taken to be elements of the standard basis.

Lemma 2.2. A rank one projection $P \in M_{AB}$ is bijective if and only if $P = \xi_U\xi_U^*$ for some unitary operator $U \in M_A$.

Proof. Let $P = \xi\xi^*$ for some $\xi \in \mathbb{C}^A$, and $U \in M_A$ be an operator with $\xi = \xi_U$. Let $e = \sum_{a\in A} e_a \in \mathbb{C}^A$ and write $\bar{e} = \sum_{a\in A} \bar{e}_a$. Set $r = \|\xi_U\|$. We have

$$\begin{aligned}
\langle L_{ee^*}(\xi\xi^*), e_ae_b^*\rangle &= \langle \xi\xi^*, ee^* \otimes e_ae_b^*\rangle = \mathrm{Tr}\left((\xi\xi^*)(ee^* \otimes e_ae_b^*)^t\right)\\
&= \mathrm{Tr}\left((\xi\xi^*)((ee^*)^t \otimes (e_ae_b^*)^t)\right) = \mathrm{Tr}\left((\xi\xi^*)(\bar{e}\bar{e}^* \otimes e_be_a^*)\right)\\
&= \mathrm{Tr}\left((\xi\xi^*)(\bar{e}\otimes e_b)(\bar{e}\otimes e_a)^*\right) = \langle \bar{e}\otimes e_b, \xi\rangle\overline{\langle \bar{e}\otimes e_a, \xi\rangle}\\
&= r^{-2}\sum_{c\in A}\langle Ue_a, e_c\rangle\overline{\sum_{d\in A}\langle Ue_b, e_d\rangle}\\
&= r^{-2}\langle Ue_a, \bar{e}\rangle\overline{\langle Ue_b, \bar{e}\rangle} = r^{-2}\langle e_a, U^*\bar{e}\rangle\langle U^*\bar{e}, e_b\rangle\\
&= r^{-2}\langle (U^*\bar{e})(U^*\bar{e})^*e_a, e_b\rangle = r^{-2}\overline{\langle (U^*\bar{e})(U^*\bar{e})^*, e_be_a^*\rangle}\\
&= r^{-2}\langle (U^te)(U^te)^*, e_ae_b^*\rangle;
\end{aligned}$$

thus, $L_{ee^*}(\xi\xi^*) = r^{-2}(U^te)(U^te)^*$. It follows that P is bijective if and only if U is a multiple of a unitary operator, that is, if and only if μU is unitary for some $\mu \in \mathbb{C}$. Clearly, $P = \xi_{\mu U}\xi_{\mu U}^*$. ∎

A projection $P \in M_{AB}$ of rank r will be called bijective if there exist partial isometries U_i, $i = 1, \dots, r$, such that $\sum_{i=1}^r U_iU_i^* = \sum_{i=1}^r U_i^*U_i = I$ and $P = \sum_{i=1}^r \xi_{U_i}\xi_{U_i}^*$. Note that, if $\sigma : A \to B$ is a bijection and $P = P_\sigma$, then P is bijective of rank $|A|$ with corresponding partial isometries $\epsilon_{\sigma(a),a}$, $a \in A$.

Definition 2.3. Let $\varphi : \mathcal{P}^{cl}_{XY} \to \mathcal{P}_{AB}$ be a classical-to-quantum non-local game and $\Gamma : \mathcal{P}_{XY} \to \mathcal{P}_{AB}$ be a quantum non-local game.

(i) φ is called a quantum output mirror game if there exists functions $f : X \to Y$, $g : Y \to X$ such that the projections $\varphi(\epsilon_{x,x} \otimes \epsilon_{f(x),f(x)})$ and $\varphi(\epsilon_{g(y),g(y)} \otimes \epsilon_{y,y})$ are bijective, $x \in X$, $y \in Y$;

(ii) φ is called concurrent if $\varphi(J^{cl}_X) = J_A$;

(iii) Γ is called concurrent if $\Gamma(J_X) = J_A$.

In view of Remark 2.1, we consider quantum output mirror games as a quantum version of mirror games, and concurrent games – as quantum versions of synchronous games.

3. Classical-to-quantum games

This section contains characterisations of the prefect strategies of quantum output mirror games and classical-to-quantum concurrent games, and their applications to quantum orthogonal ranks of graphs. We start with recalling the main classes of quantum no-signalling correlations introduced in [30] that will be used subsequently.

3.1. **Quantum no-signalling correlations.** If $\mathcal{A}$ is a C*-algebra, we denote by $\mathcal{A}^{op}$ its opposite C*-algebra. As a set, $\mathcal{A}^{op}$ can be identified with $\mathcal{A}$ and we write $\mathcal{A}^{op} = \{z^{op} : z \in \mathcal{A}\}$; the C*-algebra $\mathcal{A}^{op}$ has the same norm, additive and involutive structure as $\mathcal{A}$, and its multiplication is given by letting $z_1^{op} z_2^{op} = (z_2 z_1)^{op}$, $z_1, z_2 \in \mathcal{A}$.

Let $\mathcal{V}_{X,A}$ be the ternary ring, generated by elements $v_{a,x}$, $x \in X$, $a \in A$, such that the matrix $V = (v_{a,x})_{a\in A, x\in X}$ satisfies the condition of an isometry, that is,

$$\sum_{a\in A} v_{a'',x''} v_{a,x}^* v_{a,x'} = \delta_{x,x'} v_{a'',x''}, \quad x, x', x'' \in X, a'' \in A.$$

Let $\mathcal{C}_{X,A}$ be the unital *-algebra, generated by the set $\{v_{a,x}^* v_{a',x'} : x, x' \in X, a, a' \in A\}$, and set $e_{x,x',a,a'} = v_{a,x}^* v_{a',x'}$ for brevity. Further, let $\mathcal{V}_{X,A}$ be the universal ternary ring of operators (TRO) of the isometry V, and let $\mathcal{C}_{X,A}$ be its right C*-algebra; thus, $\mathcal{C}_{X,A}$ is generated, as a C*-algebra, by $e_{x,x',a,a'}$, $x, x' \in X$, $a, a' \in A$ (see [30]). We write

$$E = (e_{x,x',a,a'})_{x,x',a,a'} \quad \text{and} \quad E^{op} = (e^{op}_{x',x,a',a})_{x,x',a,a'};$$

thus, $E \in M_{XA} \otimes \mathcal{C}_{X,A}$ and $E^{op} \in M_{XA} \otimes \mathcal{C}^{op}_{X,A}$.

A stochastic operator matrix acting on a Hilbert space H is a positive block operator matrix $\tilde{E} = (E_{x,x',a,a'})_{x,x',a,a'} \in M_{XA}(\mathcal{B}(H))$ such that $\mathrm{Tr}_A \tilde{E} = I$. Stochastic operator matrices $\tilde{E}$ acting on H correspond to unital *-representations $\pi : \mathcal{C}_{X,A} \to \mathcal{B}(H)$ by via the assignment $\pi(e_{x,x',a,a'}) = E_{x,x',a,a'}$, $x, x' \in X$, $a, a' \in A$ [30].

Let X, Y, A and B be finite sets. A quantum no-signalling (QNS) correlation [10] is a quantum channel (that is, a completely positive trace preserving map) $\Gamma : M_{XY} \to M_{AB}$ such that

(5) $\quad \mathrm{Tr}_A\, \Gamma(\omega_X \otimes \omega_Y) = 0$ whenever $\omega_X \in M_X$ and $\mathrm{Tr}(\omega_X) = 0$, and

(6) $\quad \mathrm{Tr}_B\, \Gamma(\omega_X \otimes \omega_Y) = 0$ whenever $\omega_Y \in M_Y$ and $\mathrm{Tr}(\omega_Y) = 0$.

A QNS correlation $\Gamma : M_{XY} \to M_{AB}$ is quantum commuting if there exist a Hilbert space H, a unit vector $\xi \in H$ and stochastic operator matrices $\tilde{E} = (E_{x,x',a,a'})_{x,x',a,a'}$ and $\tilde{F} = (F_{y,y',b,b'})_{y,y',b,b'}$ on H such that

$$E_{x,x',a,a'} F_{y,y',b,b'} = F_{y,y',b,b'} E_{x,x',a,a'}$$

for all $x, x' \in X$, $y, y' \in Y$, $a, a' \in A$, $b, b' \in B$, and the Choi matrix of Γ coincides with

$$\left(\langle E_{x,x',a,a'} F_{y,y',b,b'} \xi, \xi \rangle\right)^{y,y',b,b'}_{x,x',a,a'} \in M_{XYAB}(\mathcal{B}(H)). \tag{7}$$

Quantum QNS correlations are defined as in (7), but using tensor products of stochastic operator matrices acting on finite dimensional Hilbert spaces (that is, ones having the form $E_{x,x',a,a'} \otimes F_{y,y',b,b'}$). Approximately quantum QNS correlations are limits of quantum QNS correlations, while local QNS correlations are defined as in (7) by requiring that the entries of $\tilde{E}$ (resp. $\tilde{F}$) pairwise commute.

We write $\mathcal{Q}_{qc}$ (resp. $\mathcal{Q}_{qa}$, $\mathcal{Q}_{q}$, $\mathcal{Q}_{loc}$) for the (convex) set of all quantum commuting (resp. approximately quantum, quantum, local) QNS correlations. It was shown in [30] that $\Gamma \in \mathcal{Q}_{qc}$ precisely when there exists a state $s : \mathcal{C}_{X,A} \otimes_{\max} \mathcal{C}_{Y,B} \to \mathbb{C}$ such that $\Gamma = \Gamma_s$, where Γ_s is given by

$$\Gamma_s(\epsilon_{x,x'} \otimes \epsilon_{y,y'}) = \sum_{a,a' \in A} \sum_{b,b' \in B} s(e_{x,x',a,a'} \otimes e_{y,y',b,b'})\, \epsilon_{a,a'} \otimes \epsilon_{b,b'},$$

where $x, x' \in X$, $y, y' \in Y$. Similarly, $\Gamma \in \mathcal{Q}_{qa}$ precisely when $\Gamma = \Gamma_s$ for some state s of $\mathcal{C}_{X,A} \otimes_{\min} \mathcal{C}_{Y,B}$, and $\Gamma \in \mathcal{Q}_{q}$ (resp. $\Gamma \in \mathcal{Q}_{loc}$) if and only if $\Gamma = \Gamma_s$ for some state s of $\mathcal{C}_{X,A} \otimes_{\min} \mathcal{C}_{Y,B}$ that factors through a finite dimensional (resp. abelian) representation of the latter C*-algebra. We point out that the elements of $\mathcal{Q}_{loc}$ are precisely the quantum channels of the form $\Gamma = \sum_{i=1}^{k} \lambda_i \Phi_i \otimes \Psi_i$ as a convex combination (where $\Phi_i : M_X \to M_A$ and $\Psi_i : M_Y \to M_B$ are quantum channels, $i = 1, \dots, k$).

Let $\mathcal{B}_{X,A}$ (resp. $\mathcal{B}_{X,A}$) be the algebraic (resp. the C*-algebraic) free product $M_A *_1 \cdots *_1 M_A$, and $\mathcal{A}_{X,A}$ (resp. $\mathcal{A}_{X,A}$) be the algebraic (resp. the C*-algebraic) free product $\mathcal{D}_A *_1 \cdots *_1 \mathcal{D}_A$, both having $|X|$ terms and amalgamated over the units. We denote by $e_{x,a,a'}$, $a, a' \in A$, the matrix units of the x-th copy of M_A in $\mathcal{B}_{X,A}$, and by $e_{x,a}$, $a \in A$, the canonical basis of the x-th copy of $\mathcal{D}_A$ in $\mathcal{A}_{X,A}$. Set $E_{cq} = (e_{x,a,a'})_{x,a,a'} \in \mathcal{D}_X \otimes M_A \otimes \mathcal{B}_{X,A}$ and $E_{cq}^{op} = (e^{op}_{x,a,a'})_{x,a,a'} \in \mathcal{D}_X \otimes M_A \otimes \mathcal{B}^{op}_{X,A}$; similarly, let $E_{cl} = (e_{x,a})_{x,a} \in \mathcal{D}_{XA} \otimes \mathcal{A}_{X,A}$ and $E^{op}_{cl} = (e^{op}_{x,a})_{x,a} \in \mathcal{D}_{XA} \otimes \mathcal{A}^{op}_{X,A}$.

A classical-to-quantum no-signalling (CQNS) correlation is a channel $\mathcal{E} : \mathcal{D}_{XY} \to M_{AB}$ such that (5) and (6) hold true for (traceless) elements $\omega_X \in \mathcal{D}_X$ and $\omega_Y \in \mathcal{D}_Y$. A semi-classical stochastic operator matrix acting on a Hilbert space H is a positive block operator matrix $\tilde{E} = (E_{x,a,a'})_{x,a,a'} \in \mathcal{D}_X \otimes M_A(\mathcal{B}(H))$ with $\mathrm{Tr}_A \tilde{E} = I$. A CQNS correlation $\mathcal{E}$ is quantum commuting if its Choi matrix is given as in (7) but employing semi-classical stochastic operator matrices; this is equivalent to the requirement that its canonical extension to a QNS correlation $M_{XY} \to M_{AB}$ is quantum commuting, as well as to the existence of a state s of $\mathcal{B}_{X,A} \otimes_{\max} \mathcal{B}_{X,A}$ such that $\mathcal{E} = \mathcal{E}_s$, where $\mathcal{E}_s$ is the CQNS correlation given by

$$\mathcal{E}_s(\epsilon_{x,x} \otimes \epsilon_{y,y}) = \sum_{a,a' \in A} \sum_{b,b' \in B} s(e_{x,a,a'} \otimes e_{y,b,b'})\, \epsilon_{a,a'} \otimes \epsilon_{b,b'}.$$

Similarly, approximately quantum (resp. quantum, local) CQNS correlations have the form $\mathcal{E}_s$, where s is a state of $\mathcal{B}_{X,A} \otimes_{\min} \mathcal{B}_{X,A}$ (which in addition gives rise to a finite dimensional and abelain GNS representation, respectively). We denote by $\mathcal{CQ}_{qc}$ (resp. $\mathcal{CQ}_{qa}$, $\mathcal{CQ}_{q}$, $\mathcal{CQ}_{loc}$) the (convex) set of all quantum commuting (resp. approximately quantum, quantum, local) QNS correlations.

Let $\varphi : \mathcal{P}_{XY} \to \mathcal{P}_{AB}$ be a quantum non-local game. A QNS correlation $\Gamma : M_{XY} \to M_{AB}$ is called a perfect strategy for φ if

$$\langle \Gamma(P), \varphi(P)^{\perp} \rangle = 0, \quad P \in \mathcal{P}_{XY}.$$

Perfect strategies for classical-to-quantum non-local games are defined analogously [30].

3.2. Quantum output mirror games. We first describe the perfect strategies of quantum output mirror games that lie in the various correlation classes. In the sequel, we fix finite sets X, Y, A and B, and for clarity denote the canonical generators of $\mathcal{B}_{Y,B}$ by $f_{y,b,b'}$, $y \in Y$, $b, b' \in B$. We fix a quantum output mirror game $\varphi : \mathcal{P}^{cl}_{XY} \to \mathcal{P}_{AB}$ and let $f : X \to Y$ and $g : Y \to X$ be as in Definition 2.3. We write

$$\varphi\left(\epsilon_{x,x} \otimes \epsilon_{f(x),f(x)}\right) = \sum_{i=1}^{r(x)} \xi_{U_i^x} \xi_{U_i^x}^*, \quad x \in X,$$

where U_i^x, $i = 1, \dots, r(x)$, $x \in X$, are partial isometries satisfying the relations

$$\sum_{i=1}^{r(x)} (U_i^x)^* U_i^x = \sum_{i=1}^{r(x)} U_i^x (U_i^x)^* = I. \tag{8}$$

Let $D_x := \sum_{i=1}^{r(x)} U_i^x$; the relations (8) imply that D_x is unitary.

Lemma 3.1. Let s be a state of $\mathcal{B}_{X,A} \otimes_{\max} \mathcal{B}_{Y,B}$ such that $\Gamma_s : \mathcal{D}_{XY} \to M_{AB}$ is a perfect quantum commuting CQNS strategy for φ. Let $\pi_1 : \mathcal{B}_{X,A} \to \mathcal{B}(H)$ and $\pi_2 : \mathcal{B}_{Y,B} \to \mathcal{B}(H)$ be *-representations with commuting ranges and $\xi \in H$ be a unit vector such that

$$s(u_1 \otimes u_2) = \langle \pi_1(u_1)\pi_2(u_2)\xi, \xi \rangle, \quad u_1 \in \mathcal{B}_{X,A}, u_2 \in \mathcal{B}_{Y,B},$$

$E_x = (\pi_1(e_{x,a,a'}))_{a,a' \in A}$ and $F_y = (\pi_2(f_{y,b,b'}))_{b,b' \in B}$. Then

$$(U_i^x \otimes I)^* E_x (e_a \otimes \xi) = F^t_{f(x)} (U_i^x \otimes I)^* (e_a \otimes \xi), \quad i = 1, \dots, r(x), \ a \in A.$$

Proof. Set $P_{i,x} = U_i^x (U_i^x)^*$ and $Q_{i,x} = (U_i^x)^* U_i^x$; thus,

$$\sum_{i=1}^{r(x)} P_{i,x} = \sum_{i=1}^{r(x)} Q_{i,x} = I,$$

that is, $(P_{i,x})_{i=1}^{r(x)}$ and $(Q_{i,x})_{i=1}^{r(x)}$ are PVM's (in M_A) for every $x \in X$. We have that, if $\Gamma := \Gamma_s$ then

$$\Gamma\left(\epsilon_{x,x} \otimes \epsilon_{f(x),f(x)}\right) = \left(\sum_{i=1}^{r(x)} \xi_{U_i^x} \xi_{U_i^x}^*\right) \Gamma\left(\epsilon_{x,x} \otimes \epsilon_{f(x),f(x)}\right) \left(\sum_{i=1}^{r(x)} \xi_{U_i^x} \xi_{U_i^x}^*\right). \tag{9}$$

Taking traces in (9), we obtain

$$
\begin{aligned}
1 &= \operatorname{Tr}\left(\epsilon_{x,x}\otimes\epsilon_{f(x),f(x)}\right)\\
&= \sum_{i,j=1}^{r(x)}\sum_{a,b,a',b'} s(e_{x,a,a'}\otimes f_{f(x),b,b'})\operatorname{Tr}\left((\zeta_{U_j^x}\zeta_{U_j^x}^*)(\epsilon_{a,a'}\otimes\epsilon_{b,b'})(\zeta_{U_i^x}\zeta_{U_i^x}^*)\right)\\
&= \sum_{i,j=1}^{r(x)}\sum_{a,b,a',b'} s(e_{x,a,a'}\otimes f_{f(x),b,b'})\langle\zeta_{U_i^x}, e_{a'}\otimes e_{b'}\rangle\langle e_a\otimes e_b, \zeta_{U_j^x}\rangle\langle\zeta_{U_j^x},\zeta_{U_i^x}\rangle\\
&= \sum_{i=1}^{r(x)}\sum_{a,b,a',b'} s(e_{x,a,a'}\otimes f_{f(x),b,b'})\langle\zeta_{U_i^x}, e_{a'}\otimes e_{b'}\rangle\langle e_a\otimes e_b, \zeta_{U_i^x}\rangle,
\end{aligned}
$$

where we have used the fact that $\langle\zeta_{U_j^x},\zeta_{U_i^x}\rangle = 0$ whenever $i \neq j$. Recall that $s(e_{x,a,a'}\otimes e_{y,b,b'}) = \left\langle E_{x,a,a'}F_{y,b,b'}\xi,\xi\right\rangle$ for all $x\in X$, $y\in Y$, $a,a'\in A$, $b,b'\in B$. In the sequel, we denote by $T_{a,b}$ the (a,b)-entry of a (possibly block operator) matrix T. Noting that $\zeta_{U_i^x} = \left(\sum_{a,b}(U_i^x)_{a,b}e_a\otimes e_b\right)/\|U_i^x\|_2$ and $\|U_i^x\|_2^2 = \operatorname{Tr}(U_i^x(U_i^x)^*)$ is the rank $r_i(x)$ of the projection $P_{i,x}$, we obtain

$$
\left\langle\zeta_{U_i^x}, e_{a'}\otimes e_{b'}\right\rangle = \frac{(U_i^x)_{a',b'}}{r_i(x)^{1/2}} \quad\text{and}\quad \left\langle e_a\otimes e_b, \zeta_{U_i^x}\right\rangle = \frac{\overline{(U_i^x)_{a,b}}}{r_i(x)^{1/2}}.
$$

Setting $\tilde{U}_i^x = U_i^x\otimes I$, $i = 1,\dots,r(x)$, $x\in X$, we therefore have

$$
\begin{aligned}
1 &= \sum_{i=1}^{r(x)}\sum_{a,b,a',b'}\left\langle E_{x,a,a'}F_{f(x),b,b'}\xi,\xi\right\rangle\left\langle\zeta_{U_i^x}, e_{a'}\otimes e_{b'}\right\rangle\left\langle e_a\otimes e_b,\zeta_{U_i^x}\right\rangle\\
&= \sum_{i=1}^{r(x)}\frac{1}{r_i(x)}\sum_{a,b,a',b'}\left\langle E_{x,a,a'}(U_i^x)_{a',b'}F_{f(x),b,b'}\overline{(U_i^x)_{a,b}}\,\xi,\xi\right\rangle\\
&= \sum_{i=1}^{r(x)}\frac{1}{r_i(x)}\sum_{a\in A}\left\langle\left(E_x\tilde{U}_i^xF_{f(x)}^t(\tilde{U}_i^x)^*\right)_{a,a}\xi,\xi\right\rangle\\
&= \sum_{i=1}^{r(x)}\frac{1}{r_i(x)}\sum_{a\in A}\left\langle E_x\tilde{U}_i^xF_{f(x)}^t(\tilde{U}_i^x)^*(e_a\otimes\xi), e_a\otimes\xi\right\rangle\\
&= \sum_{i=1}^{r(x)}\frac{1}{r_i(x)}\sum_{a\in A}\left\langle F_{f(x)}^t(\tilde{U}_i^x)^*(e_a\otimes\xi), (\tilde{U}_i^x)^*E_x(e_a\otimes\xi)\right\rangle.
\end{aligned}
$$

By the Cauchy-Schwartz inequality,

$$
\begin{aligned}
(10)\quad 1 &\le \sum_{i=1}^{r(x)} \frac{1}{r_i(x)} \sum_{a\in A} \left\langle F^t_{f(x)}(\tilde U_i^x)^*(e_a\otimes\xi), (\tilde U_i^x)^* E_x(e_a\otimes\xi)\right\rangle\\
&\le \left(\sum_{i=1}^{r(x)}\sum_{a\in A}\frac{1}{r_i(x)}\|F^t_{f(x)}(\tilde U_i^x)^*(e_a\otimes\xi)\|\|(\tilde U_i^x)^*E_x(e_a\otimes\xi)\|\right)^2\\
&\le \left(\sum_{i=1}^{r(x)}\sum_{a\in A}\frac{1}{r_i(x)}\|(\tilde U_i^x)^*E_x(e_a\otimes\xi)\|^2\right)\left(\sum_{i=1}^{r(x)}\sum_{a\in A}\frac{1}{r_i(x)}\|F^t_{f(x)}(\tilde U_i^x)^*(e_a\otimes\xi)\|^2\right)\\
&= \left(\sum_{a\in A}\left\langle\sum_{i=1}^{r(x)}\frac{1}{r_i(x)}E_x^*\tilde U_i^x(\tilde U_i^x)^*E_x(e_a\otimes\xi), e_a\otimes\xi\right\rangle\right)\\
&\quad\times\left(\sum_{a\in A}\left\langle\sum_{i=1}^{r(x)}\frac{1}{r_i(x)}\tilde U_i^x(F^t_{f(x)})^*F^t_{f(x)}(\tilde U_i^x)^*(e_a\otimes\xi), e_a\otimes\xi\right\rangle\right).
\end{aligned}
$$

Since $(P_{i,x})_{i=1}^{r(x)}$ is a PVM, there exist a partition $(S_i)_{i=1}^{r(x)}$ of A with $|S_i| = r_i(x)$ and a unitary V_x in M_A such that $V_x^*P_{i,x}V_x$ coincides with the projection P_{S_i} onto $\mathrm{span}\{e_a : a\in S_i\}$, $i = 1,\dots,r(x)$. Let $\tilde E_x = V_x^*E_xV_x$, and write $\tilde E_x = \sum_{a,b}\epsilon_{a,b}\otimes\tilde E_{x,a,b}$. As V_x is unitary and $E_{x,a,b}E_{x,a',b'} = \delta_{b,a'}E_{x,a,b'}$, we also have $\tilde E_{x,a,b}\tilde E_{x,a',b'} = \delta_{b,a'}\tilde E_{x,a,b'}$. Thus, we have

$$
\begin{aligned}
\tilde E_x^*P_{S_i}\tilde E_x &= \left(\sum_{a,b}\epsilon_{a,b}\otimes\tilde E_{x,a,b}\right)\left(\sum_{c\in S_i}\epsilon_{c,c}\otimes 1\right)\left(\sum_{a',b'}\epsilon_{a',b'}\otimes\tilde E_{x,a',b'}\right)\\
&= r_i(x)\tilde E_x.
\end{aligned}
$$

Let, similarly, $(R_i)_{i=1}^{r(x)}$ be a partition of B with $|R_i| = r_i(x)$ and W_x be a unitary such that $W_x^*Q_{x,i}W_x = P_{R_i}$, $i = 1,\dots,r(x)$. Setting $\tilde F_{f(x)} = W_x^*F_{f(x)}W_x$, we have that $(\tilde F^t_{f(x)})^*P_{R_i}\tilde F^t_{f(x)} = r_i(x)\tilde F^t_{f(x)}$. This implies that the last product in (10) is equal to

$$
\begin{aligned}
&\left(\sum_{a\in A}\langle E_x(e_a\otimes\xi), e_a\otimes\xi\rangle\right)\left(\sum_{a\in A}\langle F^t_{f(x)}(e_a\otimes\xi), e_a\otimes\xi\rangle\right)\\
&= \left\langle\sum_{a\in A}E_{x,a,a}\xi,\xi\right\rangle\left\langle\sum_{a\in A}F_{f(x),a,a}\xi,\xi\right\rangle = 1.
\end{aligned}
$$

Hence we have equalities in all chains of inequalities which implies that there exist scalars λ_x such that

$$F^t_{f(x)}(\tilde{U}_i^x)^*(e_a \otimes \xi) = \lambda_x(\tilde{U}_i^x)^* E_x(e_a \otimes \xi), \quad i = 1, \dots, r(x), \ a \in A.$$

Summing up over i, we obtain that $(D_x \otimes I)F^t_{f(x)}(D_x^* \otimes I)(e_a \otimes \xi) = \lambda_x E_x(e_a \otimes \xi)$ for all $a \in A$. After applying Tr_A, we conclude that $\lambda_x = 1$, which yields the desired result. $\square$

Theorem 3.2. Let $\varphi : \mathcal{P}^{cl}_{XY} \to \mathcal{P}_{AB}$ be a quantum output mirror game and $\Gamma : \mathcal{D}_{XY} \to M_{AB}$ be a perfect quantum commuting CQNS strategy for φ. Then there exists a tracial state $\tau : \mathcal{B}_{X,A} \to \mathbb{C}$ and a *-homomorphism $\pi : \mathcal{B}_{Y,B} \to \mathcal{B}_{X,A}$ such that

$$\Gamma(\epsilon_{x,x} \otimes \epsilon_{y,y}) = \left(\tau(e_{x,a,a'}\pi(f_{y,b',b}))\right)_{a,a',b,b'}, \quad x, y \in X. \tag{11}$$

Proof. We choose $f : X \to Y$ and $g : Y \to X$ as in Definition 2.3, and write

$$\varphi(\epsilon_{x,x} \otimes \epsilon_{f(x),f(x)}) = \sum_{i=1}^{r(x)} \xi_{U_i^x}\xi_{U_i^x}^*, \quad x \in X,$$

for partial isometries U_i^x, $i = 1, \dots, r(x)$, $x \in X$, such that

$$\sum_{i=1}^{r(x)} (U_i^x)^* U_i^x = \sum_{i=1}^{r(x)} U_i^x (U_i^x)^* = I.$$

Keeping the notation from the proof of Lemma 3.1, we write $\Gamma(\epsilon_{x,x} \otimes \epsilon_{y,y}) = \sum_{a,a' \in A}\sum_{b,b' \in B} \langle E_{x,a,a'}F_{y,b,b'}\xi, \xi\rangle$, where the assignments $e_{x,a,a'} \to E_{x,a,a'} \in B(H)$ and $f_{y,b,b'} \to F_{y,b,b'} \in B(H)$ define *-representations of $\mathcal{B}_{X,A}$ and $\mathcal{B}_{Y,B}$, respectively, with commuting ranges. Set $E_x = (E_{x,a,a'})_{a,a' \in A}$ and $F_y = (F_{y,b,b'})_{b,b' \in B}$. By Lemma 3.1,

$$F^t_{f(x)}(\tilde{U}_i^x)^*(e_a \otimes \xi) = (\tilde{U}_i^x)^* E_x(e_a \otimes \xi), \quad i = 1, \dots, r(x), \ a \in A.$$

Let $Q = \left((D_x \otimes I)(f_{f(x),a,b})^t_{(a,b)}(D_x^* \otimes I)\right)_{a,b}$ and write $Q = (q_{x,a,b})_{b,a}$. Set

$$h_{x,a,b} = e_{x,a,b} \otimes 1 - 1 \otimes q_{x,b,a}, \quad x \in X, \ a, b \in A.$$

We have

$$\begin{aligned} h^*_{x,a,b}h_{x,a,b} &= (e_{x,b,a} \otimes 1 - 1 \otimes q_{x,a,b})(e_{x,a,b} \otimes 1 - 1 \otimes q_{x,b,a}) \\ &= e_{x,b,b} \otimes 1 - e_{x,b,a} \otimes q_{x,b,a} - e_{x,a,b} \otimes q_{x,a,b} + 1 \otimes q_{x,a,a}. \end{aligned}$$

Let $s \in \mathcal{B}_{X,A} \otimes_{\max} \mathcal{B}_{Y,B}$ be such that $\Gamma = \Gamma_s$. As

$$\begin{aligned} s(e_{x,b,a} \otimes q_{x,b,a}) &= \langle E_{x,b,a}((D_x \otimes I)F^t_{f(x)}(D_x^* \otimes I))_{a,b}\xi, \xi\rangle \\ &= \langle ((D_x \otimes I)F^t_{f(x)}(D_x^* \otimes I))_{a,b}\xi, E_{x,a,b}\xi\rangle = \langle E_{x,a,b}\xi, E_{x,a,b}\xi\rangle = \langle E_{x,b,b}\xi, \xi\rangle, \end{aligned}$$

we get

$$s(h^*_{x,a,b}h_{x,a,b}) = 0, \quad x \in X, \ a, b \in A. \tag{12}$$

For $u, v \in \mathcal{B}_{X,A} \otimes_{\max} \mathcal{B}_{Y,B}$, write $u \sim v$ if $s(u - v) = 0$. Equations (12), combined with the Cauchy-Schwarz inequality, imply

$$uh_{x,a,b} \sim 0 \text{ and } h^*_{x,a,b}u \sim 0, \quad x \in X,\ a,b \in A,\ u \in \mathcal{B}_{X,A} \otimes_{\max} \mathcal{B}_{Y,B}.$$

Since $h^*_{x,a,b} = h_{x,b,a}$, we have

$$uh_{x,a,b} \sim 0 \text{ and } h_{x,a,b}u \sim 0, \quad x \in X,\ a,b \in A,\ u \in \mathcal{B}_{X,A} \otimes_{\max} \mathcal{B}_{Y,B}. \tag{13}$$

In particular,

$$ze_{x,a,b} \otimes 1 \sim z \otimes q_{x,b,a} \sim e_{x,a,b}z \otimes 1, \quad x \in X,\ a,b \in A,\ z \in \mathcal{B}_{X,A}. \tag{14}$$

Similarly, let V_i^y, $i = 1, \dots d(y)$, be partial isometries such that

$$\sum_{i=1}^{d(y)} V_i^y (V_i^y)^* = \sum_{i=1}^{d(y)} (V_i^y)^* V_i^y = I$$

and

$$\varphi(\epsilon_{g(y),g(y)} \otimes \epsilon_{y,y}) = \sum_{i=1}^{d(y)} \xi_{V_i^y} \xi^*_{V_i^y}.$$

Similarly to the proof of Lemma 3.1, letting $G_y = \sum_{i=1}^{d(y)} V_i^y$, we obtain that $F_{y,a,b}^{\sim} = \left((G_y \otimes I)(E_{g(y),a,b})^{t}_{a,b}(G_y^* \otimes I)\right)_{a,b}^{\sim}$.

Set $(p_{y,a,b})_{b,a} = \left((G_y \otimes I)(e_{g(y),a,b})^{t}_{a,b}(G_y^* \otimes I)\right)_{a,b}$ and note that $\{p_{y,a,b} : a, b\}$ is a matrix unit system, $y \in Y$. Letting $g_{y,b,b'} = p_{y,b,b'} \otimes 1 - 1 \otimes f_{y,b,b'}$, where $y \in Y$ and $b, b' \in B$, we obtain, similarly,

$$zp_{y,b,b'} \otimes 1 \sim z \otimes f_{y,b,b'} \sim p_{y,b,b'}z \otimes 1, \quad y \in Y,\ b,b' \in B,\ z \in \mathcal{B}_{X,A}. \tag{15}$$

Let z and w be (finite) words on the set $E := \{e_{x,a,b} : x \in X,\ a,b \in A\}$. We show by induction on the length $|w|$ of w that

$$zw \otimes 1 \sim wz \otimes 1. \tag{16}$$

In the case $|w| = 1$, the claim reduces to (14). Suppose (16) holds if $|w| \leq n - 1$. Let $|w| = n$ and write $w = w'e$, where $e \in E$. Using (14), we have

$$zw \otimes 1 = zw'e \otimes 1 \sim ezw' \otimes 1 \sim w'ez \otimes 1 = wz \otimes 1.$$

Let $\tau : \mathcal{B}_{X,A} \to \mathbb{C}$ be given by $\tau(z) = s(z \otimes 1)$; it is clear that τ is a state on $\mathcal{B}_{X,A}$. From (16) and the fact that the set of all linear combinations of words on E is dense in A, we conclude that τ is a trace on $\mathcal{B}_{X,A}$. Identity (15) implies that

$$s\left(e_{x,a,a'} \otimes f_{y,b,b'}\right) = \tau\left(e_{x,a,a'}p_{y,b,b'}\right), \quad x \in X,\ y \in Y,\ a,a',b,b' \in A.$$

Equality (11) is now immediate if we let $\pi : \mathcal{B}_{Y,B} \to \mathcal{B}_{X,A}$ be the *-homomorphism defined by letting $\pi(f_{y,b',b}) = p_{y,b,b'}$, $y \in Y$, $b, b' \in B$. $\square$

We will write $\Gamma = \Gamma_{\pi,\tau}$ if the CQNS correlation $\Gamma : \mathcal{D}_{XY} \to M_{AB}$ is given as in (11). Keeping the notation from the proof of Theorem 3.2, let $\rho : \mathcal{B}_{X,A} \to \mathcal{B}_{Y,B}$ be the *-homomorphism given by $\rho(e_{x,a,a'}) = q_{x,a,a'}$. We will need the following lemma, which can be thought of as a dilation result for semi-classical stochastic operator matrices.

Lemma 3.3. *Let X and A be finite sets and $(E_{x,a,a'})_{x,a,a'}$, where $x \in X$ and $a, a' \in A$, be a semi-classical stochastic operator matrix acting on a finite dimensional Hilbert space H. Then there exist matrix unit systems $(\tilde{E}_{x,a,a'})_{a,a'}$, $x \in X$, on a finite dimensional Hilbert space $\tilde{H}$, and an isometry $V : H \to \tilde{H}$, such that $V^* \tilde{E}_{x,a,a'} V = E_{x,a,a'}$ for all $x \in X$ and all $a, a' \in A$.*

Proof. Write $X = [k]$ and use induction on k. If $k = 1$, the result is a direct consequence of the Stinespring Theorem. Resorting to the inductive assumption, suppose that H_{k-1} is a finite dimensional Hilbert space, $V_{k-1} : H \to H_{k-1}$ is an isometry, and $(F_{x,a,a'})_{a,a'}$ is a matrix unit system on H_{k-1}, such that

$$V_{k-1}^* F_{x,a,a'} V_{k-1} = E_{x,a,a'}, \quad x \in [k-1], \ a, a' \in A.$$

Let $F^0_{k,a,a'} = V_{k-1} E_{k,a,a'} V_{k-1}^*$, $a, a' \in A$. Note that $(F^0_{k,a,a'})_{a,a'} \in (M_A \otimes \mathcal{B}(H_{k-1}))^+$ and $\sum_{a \in A} F^0_{k,a,a} = P_{k-1} := V_{k-1} V_{k-1}^*$. Fix $a_0 \in A$ and define

$$F_{k,a,a'} = \begin{cases} F^0_{k,a_0,a_0} + P^\perp_{k-1} & \text{if } a = a' = a_0 \\ F^0_{k,a,a'} & \text{otherwise.} \end{cases}$$

Note that $(F_{k,a,a'})_{a,a'}$ is a stochastic operator matrix acting on H_{k-1}. In addition,

$$V_{k-1}^* F_{k,a_0,a_0} V_{k-1} = V_{k-1}^* (F^0_{k,a_0,a_0} + P^\perp_{k-1}) V_{k-1} = E_{k,a_0,a_0},$$

and hence

$$V_{k-1}^* F_{k,a,a'} V_{k-1} = E_{k,a,a'}, \quad a, a' \in A.$$

By [30, Theorem 3.1], there exists a Hilbert space K and operators $V_a : H_{k-1} \to K$ such that the column operator $V_k := (V_a)_{a \in A} : H_{k-1} \to K \otimes \mathbb{C}^A$ is an isometry, and $(F_{k,a,a'})_{a,a'} = V_a^* V_{a'}$, $a, a' \in A$. Let $\tilde{H} = K \otimes \mathbb{C}^A$ and $\tilde{E}_{k,a,a'} = I_K \otimes \epsilon_{a,a'}$, $a, a' \in A$. Then $V_k^* \tilde{E}_{k,a,a'} V_k = V_a^* V_{a'} = F_{k,a,a'}$ and hence, letting $V = V_k V_{k-1}$, we have that $V : H \to \tilde{H}$ is an isometry such that $V^* \tilde{E}_{k,a,a'} V = E_{k,a,a'}$, $a, a' \in A$.

Let $P_k = V_k V_k^*$ and $\tilde{F}_{x,a,a'} = V_k F_{x,a,a'} V_k^*$, $x \in [k-1]$, $a, a' \in A$. Then

$$\tilde{F}_{x,a,a'} \tilde{F}_{x,b,b'} = V_k F_{x,a,a'} V_k^* V_k F_{x,b,b'} V_k^* = \delta_{a',b} \tilde{F}_{x,a,b'}, \quad a, a', b, b' \in A, \tag{17}$$

and

$$\sum_{a \in A} \tilde{F}_{x,a,a} = P_k.$$

Note that, if $x_0 \in [k-1]$, $a_0 \in A$ and $l = \operatorname{rank}(\tilde{F}_{x_0,a_0,a_0})$, then $\operatorname{rank}(P_k) = l|A|$. It follows that $l = \operatorname{rank}(\tilde{F}_{x,a,a})$ for all $x \in [k-1]$ and all $a \in A$. Thus, $P_k^\perp(K \otimes \mathbb{C}^A) = K_0 \otimes \mathbb{C}^A$ for some Hilbert space with $\dim K_0 = \dim K - l$.

Let

$$\tilde{F}'_{x,a,a'} = I_{K_0} \otimes \epsilon_{a,a'}, \quad x \in [k-1], a, a' \in A,$$

considered as an operator on $P_k^\perp(K \otimes \mathbb{C}^A)$, and

$$\tilde{E}_{x,a,a'} := \tilde{F}_{x,a,a'} + \tilde{F}'_{x,a,a'}, \quad x \in [k-1], a, a' \in A.$$

For $a,a',b,b' \in A$ and $x \in [k-1]$, using (17) we have

$$\begin{aligned}\tilde{E}_{x,a,a'}\tilde{E}_{x,b,b'} &= (\tilde{F}_{x,a,a'} + \tilde{F}'_{x,a,a'})(\tilde{F}_{x,b,b'} + \tilde{F}'_{x,b,b'})\\ &= \tilde{F}_{x,a,a'}\tilde{F}_{x,b,b'} + \tilde{F}'_{x,a,a'}\tilde{F}'_{x,b,b'}\\ &= \delta_{a',b}\tilde{F}_{x,a,b'} + \delta_{a',b}\tilde{F}'_{x,a,b'} = \delta_{a',b}\tilde{E}_{x,a,b'}.\end{aligned}$$

In addition, for $x \in [k-1]$ and $a,a' \in A$ we have

$$\begin{aligned}V^*\tilde{E}_{x,a,a'}V &= V^*\tilde{F}_{x,a,a'}V + V^*\tilde{F}'_{x,a,a'}V = V^*\tilde{F}_{x,a,a'}V\\ &= V_{k-1}^*V_k^*(V_k F_{x,a,a'}V_k^*)V_k V_{k-1}\\ &= E_{x,a,a'}.\end{aligned}$$

$\square$

Remark. In the notation of Lemma 3.3, if $E_{x,a,a'} = \delta_{a,a'}E_{x,a,a}$ for all x,a,a', the statement reduces to the simultaneous Naimark dilation of a finite family of POVM's exhibited in [23, Theorem 9.8]. We include the following consequence, which will be used later.

Corollary 3.4. Let X, Y, A and B be finite sets. A CQNS correlation $\Gamma : D_{XY} \to M_{AB}$ is quantum if and only if there exist finite dimensional Hilbert space H_X and H_Y, *-representations $\pi_X : B_{X,A} \to B(H_X)$ and $\pi_Y : B_{Y,B} \to B(H_Y)$, and a unit vector $\eta \in H_A \otimes H_B$, such that

$$\Gamma(\epsilon_{x,x} \otimes \epsilon_{y,y}) = \left(\langle(\pi_X(e_{x,a,a'}) \otimes \pi_Y(f_{y,b,b'}))\eta, \eta\rangle\right)_{a,a',b,b'}, \quad x \in X, y \in Y.$$

Proof. Let $(E_{x,a,a'})_{x,a,a'}$ (resp. $(F_{y,b,b'})_{y,b,b'}$) be a semi-classical stochastic operator matrix acting on finite dimensional Hilbert space H_A (resp. H_B) and $\xi \in H_A \otimes H_B$ be a unit vector such that

$$\Gamma(\epsilon_{x,x} \otimes \epsilon_{y,y}) = \left(\left\langle (E_{x,a,a'} \otimes F_{y,b,b'})\xi, \xi \right\rangle\right)_{a,a',b,b'}, \quad x \in X, y \in Y.$$

Let $(\tilde{E}_{x,a,a'})_{a,a'}$ and V (resp. $(\tilde{F}_{x,a,a'})_{a,a'}$ and W) be the matrix unit systems acting on a finite dimensional Hilbert space H_X (resp. H_Y) and the corresponding isometry, obtained via Lemma 3.3. By the universal property of the C*-algebraic free product, there exists a *-representation $\pi_X : B_{X,A} \to B(H_X)$ (resp. $\pi_Y : B_{Y,B} \to B(H_Y)$) such that $\pi_X(e_{x,a,a'}) = \tilde{E}_{x,a,a'}$ (resp. $\pi_Y(f_{y,b,b'}) = \tilde{F}_{y,b,b'}$), $x \in X$, $a,a' \in A$ (resp. $y \in Y$, $b,b' \in B$). Letting $\eta = (V \otimes W)\xi$, we obtain the required representation of Γ. $\square$

Theorem 3.5. Let $\varphi : P^{cl}_{XY} \to P_{AB}$ be a quantum output mirror game, τ be a tracial state on $B_{X,A}$ and $\rho : B_{Y,B} \to B_{X,A}$ be a unital *-homomorphism such that $\Gamma = \Gamma_{\rho,\tau}$ is a perfect quantum commuting CQNS strategy for φ. The following hold:

(i) $\Gamma \in CQ_{qa}$ if and only if τ can be chosen to be amenable;
(ii) $\Gamma \in CQ_q$ if and only if τ can be chosen to factor through a finite-dimensional *-representation of $B_{X,A}$.

Proof. (i) Assume that $\Gamma \in CQ_{qa}$. By the Remark after [30, Theorem 7.7], s can be chosen to be a state of $B_{X,A} \otimes_{\min} B_{Y,B}$. Let $\partial : B_{X,A} \to B^{op}_{X,A}$ be the

*-isomorphism given by $\partial(e_{x,a,a'}) = e^{op}_{x,a',a}$, whose existence is guaranteed by [30, Lemma 9.2]. Let $\phi : \mathcal{B}_{X,A} \otimes_{\min} \mathcal{B}^{op}_{X,A} \to \mathbb{C}$ be the state defined by letting

$$\phi = s \circ (\mathrm{id} \otimes \pi) \circ (\mathrm{id} \otimes \partial^{-1}).$$

Let $z \in \mathcal{B}_{X,A}$ and $w = e_{x_1,a_1,a'_1} \cdots e_{x_k,a_k,a'_k}$, for some $x_i \in X$, $a_i, a'_i \in A$, $i = 1,\dots,k$. Set $\bar{w} := \partial^{-1}(w^{op}) = e_{x_k,a'_k,a_k} \cdots e_{x_1,a'_1,a_1}$. Using (13), we have

$$\begin{aligned}\phi(z \otimes w^{op}) &= s(z \otimes \pi(\bar{w})) = s(z \otimes q_{x_k,a'_k,a_k} \cdots q_{x_1,a'_1,a_1}) = \\ &\tau(z e_{x_1,a_1,a'_1} \cdots e_{x_k,a_k,a'_k}) = \tau(zw).\end{aligned}$$

By linearity and continuity,

$$\phi(z \otimes w^{op}) = \tau(zw), \qquad z, w \in \mathcal{B}_{X,A}. \tag{18}$$

By [6, Theorem 6.2.7], τ is amenable.

Conversely, if τ is an amenable trace that implements Γ then the functional $\phi : \mathcal{B}_{X,A} \otimes_{\max} \mathcal{B}_{X,A} \to \mathbb{C}$ defined via the identity (18) factors through $\mathcal{B}_{X,A} \otimes_{\min} \mathcal{B}_{X,A}$; by the Remark after [30, Theorem 7.7], $\Gamma \in \mathcal{CQ}_{qa}$.

(ii) Let $\Gamma : D_{XY} \to M_{AB}$ be a perfect strategy in $\mathcal{CQ}_q$. By Corollary 3.4, there exist finite dimensional spaces H and K, representations $\pi' : \mathcal{B}_{X,A} \to \mathcal{B}(H)$ and $\rho' : \mathcal{B}_{Y,B} \to \mathcal{B}(K)$, and a unit vector $\xi \in H \otimes K$ such that $\Gamma = \Gamma_s$, where $s : \mathcal{B}_{X,A} \otimes_{\min} \mathcal{B}_{Y,B}$ is a state such that

$$s\left(e_{x,a,a'} \otimes f_{y,b,b'}\right) = \left\langle (\pi'(e_{x,a,a'}) \otimes \rho'(f_{y,b,b'}))\xi, \xi \right\rangle, \tag{19}$$

for all $x \in X, y \in Y, a, a' \in A, b, b' \in B$. The proof of Theorem 3.2 shows that the left marginal of s is a trace on $\mathcal{B}_{X,A}$ that factors through the finite dimensional space $H \otimes K$ and satisfies (11). The converse direction follows from [30, Proposition 9.15]. $\square$

Remark. In case the bijective projections $\varphi(\epsilon_{x,x} \otimes \epsilon_{f(x),f(x)})$ and $\varphi(\epsilon_{g(y),g(y)} \otimes \epsilon_{y,y})$ from Definition 2.3 have full rank, the corresponding quantum output mirror games reduces to a classical one and has possesses non-trivial local perfect strategies. However, if $|A| > 1$ and at least one of those projections has rank smaller than $|A|$, a local perfect strategy does not exist, since local CQNS correlations preserve separability of states.

The following is a partial converse of Theorem 3.2.

Proposition 3.6. *Let $\tau : \mathcal{B}_{X,A} \to \mathbb{C}$ be a tracial state and let $\rho : \mathcal{B}_{Y,B} \to \mathcal{B}_{X,A}$ be a *-homomorphism for which there exist bijections $f : X \to Y$, $g : Y \to X$ and unitary operators $U_x, V_y : \mathbb{C}^B \to \mathbb{C}^A$, $x \in X$, $y \in Y$, such that $(\rho(f_{y,b,b'}))_{b,b'} = (V_y^* \otimes I)(e_{g(y),a,a'})_{a,a'}(V_y \otimes I)$ and $(\rho(f_{f(x),b,b'}))_{b,b'} = (U_x^* \otimes I)(e_{x,a,a'})_{a,a'}(U_x \otimes I)$. Then $\Gamma_{\rho,\tau}$ is a perfect strategy for the game φ given by*

$$\varphi(\epsilon_{x,x} \otimes \epsilon_{y,y}) = \begin{cases} \xi_{\bar{U}_x}\xi^*_{\bar{U}_x} & \text{if } y = f(x), \\ \xi_{\bar{V}_y}\xi^*_{\bar{V}_y} & \text{if } x = g(y), \\ I_{AA} & \text{otherwise.} \end{cases}$$

Proof. We have that

$$\left(\tau(e_{x,a,a'}\pi(f_{f(x),b',b}))\right)_{a,a',b,b'} = (I\otimes U_x^*)\left(\tau(e_{x,a,a'}e_{x,b',b})\right)_{a,a',b,b'}(I\otimes U_x).$$

As τ is a trace and $\{e_{x,a,a'}\}_{a,a'\in A}$ is a family of matrix units, $\tau(e_{x,a,a'}e_{x,b',b}) = \delta_{a',b'}\delta_{a,b}\lambda_x$, where $\lambda_x = \tau(e_{x,a,a}) = \tau(e_{x,b,b})$ for all $a,a',b,b'\in A$ and $x\in X$. Hence

$$\begin{aligned}&\langle\Gamma(\epsilon_{x,x}\otimes\epsilon_{f(x),f(x)})(e_{a'}\otimes e_{b'}), e_a\otimes e_b\rangle\\ &= \langle(\tau(e_{x,a,a'}e_{x,b',b}))_{a,a',b,b'}(I\otimes U_x)(e_{a'}\otimes e_{b'}), (I\otimes U_x)(e_a\otimes e_b)\rangle =\\ &u^x_{a',b'}\overline{u^x_{a,b}}\lambda_x,\end{aligned}$$

where $U_x = (u^x_{a,b})_{a,b}$. On the other hand,

$$\langle\xi_{\bar U_x}\xi^*_{\bar U_x} e_{a'}\otimes e_{b'}, e_a\otimes e_b\rangle = \langle\xi_{\bar U_x}, e_a\otimes e_b\rangle\langle e_{a'}\otimes e_{b'}, \xi_{\bar U_x}\rangle = u^x_{a',b'}\overline{u^x_{a,b}}$$

showing that $\Gamma(\epsilon_{x,x}\otimes\epsilon_{f(x),f(x)}) = \lambda_x\xi_{\bar U_x}\xi^*_{\bar U_x}$ and hence for $P = \epsilon_{x,x}\otimes\epsilon_{f(x),f(x)}$, we obtain

$$\langle\Gamma(P), \varphi(P)^{\perp}\rangle = 0. \tag{20}$$

Similar arguments give (20) for $P = \epsilon_{g(y),g(y)}\otimes\epsilon_{y,y}$. ∎

The *classical-to-quantum concurrency game* is the game $\varphi : \mathcal{P}^{cl}_{XX}\to\mathcal{P}_{AA}$ defined as follows:

$$\varphi(\epsilon_{x,x}\otimes\epsilon_{y,y}) = \begin{cases} J_A & \text{if } x = y\\ I_{AA} & \text{if } x \neq y.\end{cases}$$

A CQNS correlation Γ will be called *concurrent* if Γ is a perfect strategy for the concurrency game.

Corollary 3.7. *Let $\Gamma : \mathcal{D}_{XX}\to M_{AA}$ be a quantum commuting CQNS correlation. The following are equivalent:*

(i) *Γ is concurrent;*
(ii) *there exists a tracial state $\tau : \mathcal{B}_{X,A}\to\mathbb{C}$ such that*

$$\Gamma(\epsilon_{x,x}\otimes\epsilon_{y,y}) = \left(\tau(e_{x,a,a'}e_{y,b',b})\right)_{a,a',b,b'},\quad x,y\in X. \tag{21}$$

Moreover,

(i') *$\Gamma\in\mathcal{CQ}_{qa}$ if and only if the trace τ can be chosen to be amenable;*
(ii') *$\Gamma\in\mathcal{CQ}_{q}$ if and only if τ can be chosen to factor through a finite dimensional *-representation of $\mathcal{B}_{X,A}$.*

Proof. (i)$\Rightarrow$(ii) The concurrency game is a quantum output mirror game with $B = A$, f and g the identity maps, and $\varphi(\epsilon_{x,x}\otimes\epsilon_{x,x}) = J_A = \xi_{I_A}\xi^*_{I_A}$ for every $x\in X$. In this case the *-homomorphism $\pi : \mathcal{B}_{X,A}\to\mathcal{B}_{X,A}$ from the proof of Theorem 3.2 is given by $\pi(f_{y,b,b'}) = e_{y,b,b'}$. The statement now follows from Theorem 3.2.

(ii)⇒(i) Fix $x \in X$ and note that, by the uniqueness of the trace on M_A, the restriction of τ to any of the free product terms in the definition of $\mathcal{B}_{X,A}$ coincides with the normalised trace tr; thus,

$$\tau(e_{x,a,a'}) = \frac{1}{|A|}\delta_{a,a'}, \quad a, a' \in A. \tag{22}$$

It follows that

$$\begin{aligned}
\Gamma(\epsilon_{x,x} \otimes \epsilon_{x,x}) &= \sum_{a,a',b,b' \in A} \tau(e_{x,a,a'}e_{x,b',b})\epsilon_{a,a'} \otimes \epsilon_{b,b'} \\
&= \sum_{a,b,b' \in A} \tau(e_{x,a,b'}e_{x,b',b})\epsilon_{a,b'} \otimes \epsilon_{b,b'} \\
&= \sum_{a,b,b' \in A} \tau(e_{x,a,b})\epsilon_{a,b'} \otimes \epsilon_{b,b'} = \frac{1}{|A|}\sum_{a,b \in A} \epsilon_{a,b} \otimes \epsilon_{a,b} = J_A.
\end{aligned}$$

Statements (i') and (ii') are immediate from statements (i) and (ii) in Theorem 3.5. □

Remark 3.8. Factorisable quantum channels were introduced in [1] and have been subsequently studied by a number of authors (see [19] and the references therein). It was shown in [19, Proposition 3.1] that a quantum channel $\Phi : M_A \to M_A$ is factorisable if and only if its Choi matrix has the form $\left(\tau(p_{a,a'}q_{b',b})\right)_{a,a',b,b'}$, for some matrix unit systems $(p_{a,a'})_{a,a'}$ and $(q_{b,b'})_{b,b'}$ in $M_A *_1 M_A$. It follows that the factorisable quantum channels on M_A can be identified with the perfect quantum commuting CQNS strategies for concurrent games with two inputs. Note, in addition, that the perfect quantum commuting strategies of quantum output mirror games with a single input form a subclass of the factorisable quantum channels.

Remark 3.9. Let $\mathcal{A}$ be a unital C*-algebra, equipped with a tracial state τ. Recall [30] that a semi-stochastic $\mathcal{A}$-matrix over (X, A) is a positive matrix $(q_{x,a,a'})_{x,a,a'} \in \mathcal{D}_X \otimes M_A \otimes \mathcal{A}$ such that $\sum_{a \in A} g_{x,a,a} = 1$ for all $x \in X$. A CQNS correlation $\Gamma : \mathcal{D}_{XX} \to M_{AA}$ is called tracial [30] if there exists a semi-stochastic $\mathcal{A}$-matrix $(g_{x,a,a'})_{x,a,a'}$ such that

$$\Gamma(\epsilon_{x,x} \otimes \epsilon_{y,y}) = \sum_{a,a',b,b'} \tau(g_{x,a,a'}g_{y,b',b})\epsilon_{a,a'} \otimes \epsilon_{b,b'}, \quad x, y \in X.$$

It follows from Corollary 3.7 that every concurrent quantum commuting CQNS correlation is tracial.

3.3. Algebras of classical-to-quantum games. Let $P \in \mathcal{P}^{cl}_{XX}$ and $Q \in \mathcal{P}_{AA}$. We define a linear map

$$\eta_{P,Q} : \mathcal{D}_{XX} \otimes M_{AA} \otimes \mathcal{B}_{X,A} \otimes \mathcal{B}_{X,A}^{op} \to \mathcal{B}_{X,A} \text{ by}$$

letting

$$\eta_{P,Q}(\omega \otimes u \otimes v^{op}) = \mathrm{Tr}(\omega(P \otimes Q))uv, \quad \omega \in \mathcal{D}_{XX} \otimes M_{AA}, u, v \in \mathcal{B}_{X,A}.$$

When, in addition, $Q \in \mathcal{P}^{cl}_{AA}$, define a corresponding map

$$\tilde{\Gamma}_{P,Q} : \mathcal{D}_{XX} \otimes \mathcal{D}_{AA} \otimes \mathcal{A}_{X,A} \otimes \mathcal{A}_{X,A}^{op} \to \mathcal{A}_{X,A}.$$

Both $\Gamma_{P,Q}$ and $\tilde{\Gamma}_{P,Q}$ will be considered as maps on the ampliations of the algebraic tensor products $\mathcal{B}_{X,A} \odot \mathcal{B}_{X,A}^{op}$ and $\mathcal{A}_{X,A} \odot \mathcal{A}_{X,A}^{op}$, with values in $\mathcal{B}_{X,A}$ and $\mathcal{A}_{X,A}$, respectively.

We use the notation $\langle S \rangle$ to refer to the *-ideal generated by a subset S of a *-algebra. If $\varphi : \mathcal{P}^{cl}_{XX} \to \mathcal{P}_{AA}$ is a classical-to-quantum game, set

$$\mathcal{I}(\varphi) = \left\langle \left\{ \Gamma_{P,\varphi(P)^\perp}(E_{cq} \otimes E_{cq}^{op}) : P \in \mathcal{P}^{cl}_{XX} \right\} \right\rangle \subseteq \mathcal{B}_{X,A},$$

and let $\overline{\mathcal{I}(\varphi)}$ be the closure of $\mathcal{I}(\varphi)$ in $\mathcal{B}_{X,A}$. Set $\mathcal{B}(\varphi) = \mathcal{B}_{X,A}/\mathcal{I}(\varphi)$ and $\overline{\mathcal{B}}(\varphi) = \mathcal{B}_{X,A}/\overline{\mathcal{I}(\varphi)}$. Define $\mathcal{A}(\varphi)$ and $\overline{\mathcal{A}}(\varphi)$ similarly, using the ideal

$$\left\langle \left\{ \tilde{\Gamma}_{P,\varphi(P)^\perp}(E_{cl} \otimes E_{cl}^{op}) : P \in \mathcal{P}^{cl}_{XX} \right\} \right\rangle$$

of $\mathcal{A}_{X,A}$.

Given a synchronous non-local game $\lambda : X \times X \times A \times A \to \{0,1\}$, its *-algebra $\mathcal{A}(\lambda)$ was defined in [12] as the unital *-algebra with generators selfadjoint idempotents $e^{0}_{x,a}$, where $x \in X$, $a \in A$, subject to the relations

$$\sum_{a \in A} e^0_{x,a} = 1 \text{ for all } x \in X, \text{ and } e^0_{y,b}e^0_{z,c} = 0 \text{ if } \lambda(y,z,b,c) = 0.$$

Proposition 3.10. Let $\lambda : X \times X \times A \times A \to \{0,1\}$ be a synchronous non-local game. Then $\mathcal{A}(\varphi_\lambda)$ (resp. $\overline{\mathcal{A}}(\varphi_\lambda)$) coincides with the *-algebra (resp. C*-algebra) of the game λ.

Proof. Let $\mathcal{A}(\lambda)$ be the *-algebra of the game λ as defined in [12], and note that $\mathcal{A}(\lambda) = \mathcal{A}_{X,A}/\mathcal{I}(\lambda)$, where

$$\mathcal{I}(\lambda) = \langle e_{x,a}e_{y,b} : \lambda(x,y,a,b) = 0 \rangle.$$

We show that

$$\mathcal{I}(\lambda) = \mathcal{I}(\varphi_\lambda). \tag{23}$$

Note that $\varphi_\lambda(\epsilon_{x,x} \otimes \epsilon_{y,y})^\perp = \sum_{(a,b):\ \lambda(x,y,a,b)=0} \epsilon_{a,a} \otimes \epsilon_{b,b}$; thus,

$$\tilde{\Gamma}_{\epsilon_{x,x}\otimes\epsilon_{y,y},\varphi_\lambda(\epsilon_{x,x}\otimes\epsilon_{y,y})^\perp}(E_{cl} \otimes E_{cl}^{op}) = \sum_{(a,b):\ \lambda(x,y,a,b)=0} e_{x,a}e_{y,b}.$$

Multipying from the left by $e_{x,a}$ and by $e_{y,b}$ from the right, we conclude that $e_{x,a}e_{y,b} \in \mathcal{I}(\varphi_\lambda)$ whenever $\lambda(x,y,a,b) = 0$; thus, $\mathcal{I}(\lambda) \subseteq \mathcal{I}(\varphi_\lambda)$.

Let $P = \sum_k \epsilon_{x_k,x_k} \otimes \epsilon_{y_k,y_k}$ as a finite sum. Then

$$\varphi_\lambda(P)^\perp = \sum_{\substack{(a,b):\\ \lambda(x_k,y_k,a,b)=0,\forall k}} \epsilon_{a,a} \otimes \epsilon_{b,b}$$

and hence

$$\tilde{\Gamma}_{P,\varphi_\lambda(P)^\perp}(E_{cl} \otimes E_{cl}^{op}) = \sum_{(a,b):\ \lambda(x_k,y_k,a,b)=0,\forall k} \ \sum_k e_{x_k,a}e_{y_k,b}.$$

This shows that $I(\varphi) \subseteq I(\Gamma)$, establishing (23). $\square$

Remark 3.11. We have $\Phi_{J_X^{cl}, J_A^{\perp}}(E_{cq} \otimes E_{cq}^{op}) = 0$.

Proof. The claim follows from the fact that

$$\Phi_{J_X^{cl}, J_A}\left(E_{cq} \otimes E_{cq}^{op}\right) = \frac{1}{|A|}\sum_{x\in X}\sum_{a,b\in A} e_{x,a,b}e_{x,b,a} = \sum_{x\in X}\sum_{a\in A} e_{x,a,a}$$
$$= \sum_{x\in X}\sum_{a,b\in A} e_{x,a,a}e_{x,b,b} = \Phi_{J_X^{cl}, I_{AA}}\left(E_{cq} \otimes E_{cq}^{op}\right).$$

$\square$

Corollary 3.12. Let $\varphi : \mathcal{P}^{cl}_{XX} \to \mathcal{P}_{AA}$ be a classical-to-quantum concurrent game. The following are equivalent for a CQNS correlation $\Gamma : \mathcal{D}_{XX} \to M_{AA}$:

(i) Γ is a perfect quantum commuting (resp. quantum) strategy for φ;

(ii) there exists trace τ (resp. a trace τ that factors through a finite dimensional *-representation) on $\mathcal{B}_{X,A}$ such that (21) holds and
$$\tau\left(\Phi_{P,\varphi(P)^{\perp}}\left(E_{cq} \otimes E_{cq}^{op}\right)\right) = 0,$$
for all $P \in \mathcal{P}^{cl}_{XX}$.

Proof. We only prove the statement in the case of quantum commuting strategies. Let τ be the trace of $\mathcal{B}_{X,A}$ that implements Γ, arising from Theorem 3.2. For any $P \in \mathcal{P}^{cl}_{XX}$, taking into account the duality relations (2), we have

$$\begin{aligned}
0 &= \langle \Gamma(P), \varphi(P)^{\perp}\rangle = \sum_{x,y\in X} \mathrm{Tr}\left((\epsilon_{x,x}\otimes\epsilon_{y,y})P\right)\langle \Gamma(\epsilon_{x,x}\otimes\epsilon_{y,y}), \varphi(P)^{\perp}\rangle \\
&= \sum_{x,y\in X} \mathrm{Tr}\left((\epsilon_{x,x}\otimes\epsilon_{y,y})P\right) \sum_{a,a',b,b'} \tau\left(e_{x,a,a'}e_{y,b',b}\right)\left\langle \epsilon_{a,a'}\otimes\epsilon_{b,b'}, \varphi(P)^{\perp}\right\rangle \\
&= \sum_{x,y\in X} \mathrm{Tr}\left((\epsilon_{x,x}\otimes\epsilon_{y,y})P\right) \sum_{a,a',b,b'} \tau\left(e_{x,a,a'}e_{y,b',b}\right)\mathrm{Tr}\left((\epsilon_{a,a'}\otimes\epsilon_{b,b'})\varphi(P)^{\perp}\right) \\
&= \tau\left(\Phi_{P,\varphi(P)^{\perp}}\left(E_{cq}\otimes E_{cq}^{op}\right)\right).
\end{aligned}$$

$\square$

Remark 3.13. Clearly, any trace τ on $\mathcal{B}(\varphi)$ gives rise to a perfect quantum commuting strategy for φ. If, in particular, τ is amenable on $\mathcal{B}(\varphi)$, by [6, Proposition 6.3.5], the induced trace $\tilde{\tau}$ on $\mathcal{B}_{X,A}$ is amenable, and hence the CQNS correlation defined via (21) is approximately quantum. We do not know if any perfect quantum commuting strategy for a non-local game φ arises from a trace of $\mathcal{B}(\varphi)$ in general. Similarly we are not aware if any the approximately quantum perfect strategies of a classical-to-quantum non-local game φ all arise from amenable traces of $\mathcal{B}(\varphi)$.

4. Concurrent quantum games

In this section, we define the *-algebra and the C*-algebra of a quantum concurrent game and provide a characterisation of the prefect strategies for this type of games.

4.1. Tracial descriptions. Let $\tau : \mathcal{C}_{X,A} \to \mathbb{C}$ be a tracial state; then the linear map $\Gamma_\tau : M_{XX} \to M_{AA}$, given by

$$\Gamma_\tau(\epsilon_{x,x'} \otimes \epsilon_{y,y'}) = \sum_{a,a',b,b'} \tau(e_{x,x',a,a'} e_{y',y,b',b}) \epsilon_{a,a'} \otimes \epsilon_{b,b'},$$

is a QNS correlations; the QNS correlations arising in this way were called tracial in [30]. The classes of quantum tracial (resp. locally tracial) QNS correlations are defined by requiring that τ factors through a finite dimensional (resp. abelian) *-representation.

Theorem 4.1. Let X and A be finite sets, $\varphi : \mathcal{P}_{XX} \to \mathcal{P}_{AA}$ be a concurrent game and $\Gamma : M_{XX} \to M_{AA}$ be a perfect quantum commuting QNS strategy for φ. Then there exists a tracial state $\tau : \mathcal{C}_{X,A} \to \mathbb{C}$ such that $\Gamma = \Gamma_\tau$. Moreover,

(i) if $\Gamma \in \mathcal{Q}_{qa}$ then τ can be chosen to be amenable;
(ii) if $\Gamma \in \mathcal{Q}_{q}$ then τ can be chosen to factor through a finite dimensional *-representation of $\mathcal{C}_{X,A}$;
(iii) if $\Gamma \in \mathcal{Q}_{loc}$ then τ can be chosen to factor through an abelian *-representation of $\mathcal{C}_{X,A}$.

Proof. Let $\Gamma \in \mathcal{Q}_{qc}$ be a perfect strategy for φ. By [30, Theorem 6.3], there exists a state $s : \mathcal{C}_{X,A} \otimes_{\max} \mathcal{C}_{X,A} \to \mathbb{C}$ such that

$$\Gamma(\epsilon_{x,x'} \otimes \epsilon_{y,y'}) = \sum_{a,a',b,b'} s(e_{x,x',a,a'} \otimes f_{y,y',b,b'}) \epsilon_{a,a'} \otimes \epsilon_{b,b'}, \tag{24}$$

for all $x, x', y, y' \in X$ and all $a, a', b, b' \in A$ (for clarity, we use $f_{y,y',b,b'}$ to denote the canonical generators of the second copy of $\mathcal{C}_{X,A}$). It follows that

$$\frac{1}{|X|} \sum_{x,y} \sum_{a,a',b,b'} s(e_{x,y,a,a'} \otimes f_{x,y,b,b'}) \epsilon_{a,a'} \otimes \epsilon_{b,b'} = J_A,$$

and hence

$$\sum_{x,y} s(e_{x,y,a,b} \otimes f_{x,y,a,b}) = \frac{|X|}{|A|}, \quad a, b \in A. \tag{25}$$

Let $V = (v_{a,x})_{a,x}$ be the isometry such that $e_{x,x',a,a'} = v_{a,x}^* v_{a',x'}$. Then

$$VV^* = \left(\sum_{x \in X} v_{a,x} v_{b,x}^* \right)_{a,b}$$

is a projection, and hence

$$\sum_{x\in X} v_{a,x}v_{a,x}^* \leq 1, \quad a\in A.$$

It follows that

$$\sum_{x\in X} e_{y,x,b,a}e_{x,y,a,b} = \sum_{x\in X} v_{b,y}^* v_{a,x}v_{a,x}^* v_{b,y} \leq v_{b,y}^* v_{b,y} = e_{y,y,b,b} \tag{26}$$

for all $y\in X$ and all $a,b\in A$. Thus,

$$\begin{aligned}\sum_{x,y\in X}\sum_{a,b\in A} e_{y,x,b,a}e_{x,y,a,b} &\leq \sum_{y\in X}\sum_{a,b\in A} e_{y,y,b,b}\\ &= |A|\sum_{y\in X}\sum_{b\in A} e_{y,y,b,b} = |X||A|1. \end{aligned}\tag{27}$$

Similarly,

$$\sum_{y\in X} f_{x,y,a,b}f_{y,x,b,a} \leq f_{x,x,a,a} \tag{28}$$

and

$$\sum_{x,y\in X}\sum_{a,b\in A} f_{x,y,a,b}f_{y,x,b,a} \leq |X||A|1. \tag{29}$$

Let

$$h_{x,y,a,b} = e_{x,y,a,b}\otimes 1 - 1\otimes f_{y,x,b,a}, \quad x,y\in X,\ a,b\in A.$$

Equation (25) and inequalities (27) and (29) imply

$$\begin{aligned}&\sum_{x,y,a,b} s(h_{x,y,a,b}^* h_{x,y,a,b})\\ = &\sum_{x,y,a,b} s\left((e_{y,x,b,a}\otimes 1 - 1\otimes f_{x,y,a,b})(e_{x,y,a,b}\otimes 1 - 1\otimes f_{y,x,b,a})\right)\\ = &\sum_{x,y,a,b} s\left(e_{y,x,b,a}e_{x,y,a,b}\otimes 1 + 1\otimes f_{x,y,a,b}f_{y,x,b,a}\right)\\ - &\sum_{x,y,a,b} s\left(e_{y,x,b,a}\otimes f_{y,x,b,a} + e_{x,y,a,b}\otimes f_{x,y,a,b}\right)\\ \leq\ & 2|X||A|1 - 2|X||A|1 = 0.\end{aligned}$$

It follows that

$$s(h_{x,y,a,b}^* h_{x,y,a,b}) = 0, \quad x,y\in X,\ a,b\in A. \tag{30}$$

As in the proof of Theorem 3.2, write $u\sim v$ if $s(u-v) = 0$ and note that, by (30),

$$uh_{x,y,a,b}\sim 0 \text{ and } h_{x,y,a,b}u\sim 0, \quad x,y\in X,\ a,b\in A,\ u\in C_{X,A}\otimes_{\max} C_{X,A}.$$

In particular,

(31) $(ze_{x,y,a,b}\otimes 1)\xi = (z\otimes f_{y,x,b,a})\xi = (e_{x,y,a,b}z\otimes 1)\xi, \quad x,y\in X,\ a,b\in A,\ z\in C_{X,A}.$

Using (31) and induction, as in the proof of Theorem 3.2, we conclude that the map $\tau : C_{X,A}\to\mathbb{C}$, given by $\tau(z) = s(z\otimes 1)$, is a trace on $C_{X,A}$. Identity (31) implies

$$s\left(e_{x,x',a,a'}\otimes f_{y,y',b,b'}\right) = \tau\left(e_{x,x',a,a'}e_{y',y,b',b}\right), \quad x,x',y,y'\in X,\ a,a',b,b'\in A.$$

Statements (i)-(ii) are proved similarly to Corollary 3.7. To see (iii), let Γ be a perfect strategy of class Q_{loc}. We have $\Gamma = \sum_{i=1}^n \lambda_i \Phi_i\otimes\Psi_i$ as a convex linear combination of quantum channels $\Phi_i,\Psi_i : M_X\to M_A$, $i=1,\dots,n$. Let $\left(\lambda^{(j)}_{x,x',a,a'}\right)_{a,a'} = \Phi_j(\epsilon_{x,x'})$, $x,x'\in X$, $\left(\mu^{(j)}_{y,y',b,b'}\right)_{b,b'} = \Psi_j(\epsilon_{y,y'})$, $y,y'\in X$, $\pi_j,\rho_j : C_{X,A}\to\mathbb{C}$ be the *-representations given by $\pi_j(e_{x,x',a,a'}) = \lambda^{(j)}_{x,x',a,a'}$ and $\rho_j(f_{y,y',b,b'}) = \mu^{(j)}_{y,y',b,b'}$, and $\pi^0,\rho^0 : C_{X,A}\to B(\mathbb{C}^n)$ be the *-representations given by

$$\pi^0(u) = \sum_{j=1}^n \pi_j(u)\epsilon_{j,j}, \quad \rho^0(v) = \sum_{j=1}^n \rho_j(v)\epsilon_{j,j}.$$

The images of π_0 and ρ_0 are abelian. Set $\xi = \sum_{i=1}^n \sqrt{\lambda_i}\, e_i\otimes e_i\in\mathbb{C}^n\otimes\mathbb{C}^n$; then

$$\Gamma(\epsilon_{x,x'}\otimes\epsilon_{y,y'}) = \left(\left\langle(\pi^0(e_{x,x',a,a'})\otimes\rho^0(f_{y,y',b,b'}))\xi,\xi\right\rangle\right)_{a,a',b,b'}$$

and the corresponding state s is given by

$$s(e_{x,x',a,a'}\otimes f_{y,y',b',b}) = \left\langle(\pi^0(e_{x,x',a,a'})\otimes\rho^0(f_{y,y',b,b'}))\xi,\xi\right\rangle.$$

It follows that the left marginal of s is a trace on $C_{X,A}$ that factors through the abelian representation π^0 of $C_{X,A}$. ∎

We now assume that $X = A$; we will see that in this case, we can obtain more precise conditions than the ones in Theorem 4.1 that are also sufficient. Let $\mathcal{B}_X$ be the universal C*-algebra (usually referred to as the Brown algebra), generated by the elements $u_{a,x}$, $x,a\in X$ such that the matrix $(u_{a,x})_{a,x\in A}$ is unitary. Consider the C*-subalgebra $\mathcal{C}_X$ of $\mathcal{B}_X$ generated by $p_{x,x',a,a'} = u^*_{a,x}u_{a',x'}$, $x,x',a,a'\in X$. Write $\mathcal{J}$ for the closed ideal of $C_{X,A}$, generated by the elements

$$\sum_{x\in X} e_{y,x,b,a}e_{x,y,a,b} - e_{y,y,b,b}, \quad y,a,b\in X.$$

Let $\mathcal{V}_{X,A}$ be the universal TRO of an isometry $(v_{a,x})_{a,x}$, as defined in [30, Section 5]. In the sequel, we will consider products $v^{\varepsilon_1}_{a_1,x_1}v^{\varepsilon_2}_{a_2,x_2}\cdots v^{\varepsilon_k}_{a_k,x_k}$, where ε_i is either the empty symbol or $*$, and $\varepsilon_i \neq \varepsilon_{i+1}$ for all i, as elements of either $\mathcal{V}_{X,A}$, $\mathcal{V}_{X,A}^*$, $C_{X,A}$ or the left C*-algebra corresponding to the TRO $\mathcal{V}_{X,A}$.

Lemma 4.2. The map $\pi : e_{x,x',a,a'}\to p_{x,x',a,a'}$, $x,x',a,a_0\in X$ extends to a surjective *-homomorphism $\pi : C_{X,A}\to\mathcal{C}_X$ with $\ker\pi = \mathcal{J}$.

Proof. Since $U = (u_{a,x})$ is unitary and hence an isometry, we have that $E = (p_{x,x',a,a'})_{x,x',a,a'}$ is a stochastic operator matrix; thus, there exists a *-homomorphism $\pi : C_{X,A} \to C_X$ such that $\pi(e_{x,x',a,a'}) = p_{x,x',a,a'}$. We have

$$\pi\left(\sum_{x\in X} e_{y,x,b,a}e_{x,y,a,b} - e_{y,y,b,b}\right) = \sum_{x\in X} u^*_{b,y}u_{a,x}u^*_{a,x}u_{b,y} - u^*_{b,y}u_{b,y} = 0,$$

showing that $J \subseteq \ker\pi$.

For the reverse inclusion, let $\rho : C_{X,A} \to B(K)$ be a unital *-representation. By [30, Lemma 5.1], there exists a block operator matrix $V = (V_{a,x})_{a,x}$ that is an isometry, such that $\rho(e_{x,x',a,a'}) = V^*_{a,x}V_{a',x'}$, $x, x', a, a' \in X$; we write $\rho = \rho_V$. Note that ρ annihilates J if and only if

$$V^*_{b,y}V_{b,y} - \sum_{x\in X} V^*_{b,y}V_{a,x}V^*_{a,x}V_{b,y} = 0, \quad a, b, y \in X.$$

Letting $D_a = 1 - \sum_{x\in X} V_{a,x}V^*_{a,x}$, we have that D_a is positive, $D_a^{1/2}V_{b,y} = 0$ and hence

$$D_aV_{b,y} = \left(1 - \sum_{x\in X} V_{a,x}V^*_{a,x}\right)V_{b,y} = 0. \tag{32}$$

Since $(V_{b,y}\otimes I)^*(I - VV^*)(V_{b,y}\otimes I) \in M_X(\rho(C_{X,A}))^+$ and has zeros on its main diagonal, it is the zero operator. In particular,

$$(I - VV^*)^{1/2}(V_{b,y}\otimes I) = (I - VV^*)(V_{b,y}\otimes I) = 0, \quad y, b \in A,$$

implying that

$$\sum_{x\in X} V_{a,x}V^*_{a',x}V_{b,y} = 0 \text{ whenever } a \neq a'. \tag{33}$$

The block operator matrix

$$U := \begin{pmatrix} V & I - VV^* \\ 0 & V^* \end{pmatrix}$$

is unitary; let

$$U_{a,x} = \begin{pmatrix} V_{a,x} & \delta_{a,x}I - \sum_{b\in X} V_{a,b}V^*_{x,b} \\ 0 & V^*_{x,a} \end{pmatrix}.$$

Since $(U_{a,x})_{a,x}$ is unitary, it gives rise to a *-representation π_U of $\mathcal{B}_X$ on the Hilbert space $K_1 \oplus K_2$, where $K_1 = K_2 = K$. Using (32) and (33), we have

$$\begin{aligned}\pi_U(p_{x,x',a,a'}) &= U^*_{a,x}U_{a',x'} = \begin{pmatrix} V^*_{a,x} & \sqrt{\delta_{a,x}I - \sum_{b\in X} V_{x,b}V^*_{a,b}} \\ 0 & V_{x,a}\end{pmatrix}\begin{pmatrix} V_{a',x'} & * \\ \sqrt{\delta_{a',x'}I - \sum_{b\in X} V_{a',b}V^*_{x',b}} & V^*_{x',a'}\end{pmatrix} \\ &= \begin{pmatrix} V^*_{a,x}V_{a',x'} & * \\ \sqrt{\left(\delta_{a,x}I - \sum_{b\in X} V_{x,b}V^*_{a,b}\right)}\,V_{a',x'} & * \end{pmatrix} \\ &= \begin{pmatrix} V^*_{a,x}V_{a',x'} & * \\ 0 & * \end{pmatrix}.\end{aligned}$$

It follows that K_1 is an invariant subspace for $\pi_U|_{\mathcal{C}_X}$, and $\phi_V(e_{x,x',a,a'}) = \pi_U(p_{x,x',a,a'})|_{K_1}$ for all $x, x', a, a' \in X$. This yields $\phi_V(T) = \pi_U(\iota(T))|_{K_1}$, $T \in \mathcal{C}_{X,A}$. Thus, for a fixed $T \in \mathcal{C}_{X,A}$, we have

$$\begin{aligned}\|T + \mathcal{J}\| &= \sup\{\|\phi_V(T)\| : V = (V_{a,x}) \text{ isometry with } \phi_V(\mathcal{J}) = \{0\}\} \\ &\leq \sup\{\|\pi_U(\iota(T))\| : U = (U_{a,x}) \text{ unitary}\} = \|\iota(T)\|.\end{aligned}$$

Therefore $T \in \ker \iota$ implies $T \in \mathcal{J}$. ∎

Note that, according to Lemma 4.2, we have $\mathcal{C}_{X,A}/\mathcal{J} \cong \mathcal{C}_X$.

Theorem 4.3. *Let X be a finite set and $\varphi : P_{XX} \to P_{XX}$ be a concurrent game. A quantum commuting QNS correlation $\Gamma : M_{XX} \to M_{XX}$ is a perfect strategy for φ if and only if there exists a tracial state $\tau : \mathcal{C}_X \to \mathbb{C}$ such that*

$$\Gamma(\epsilon_{x,x'} \otimes \epsilon_{y,y'}) = \left(\tau(p_{x,x',a,a'}p_{y',y,b',b})\right)_{a,a',b,b'}, \quad x, x', y, y' \in X.$$

Moreover,

(i) *$\Gamma \in \mathcal{Q}_q$ if and only if τ can be chosen to factor through a finite dimensional *-representation of $\mathcal{C}_X$;*

(ii) *$\Gamma \in \mathcal{Q}_{loc}$ if and only if τ can be chosen to factor through an abelian *-representation of $\mathcal{C}_X$.*

Proof. For clarity, we set $A = X$. Let $\Gamma \in \mathcal{Q}_{qc}$ be a perfect strategy for φ. Keeping the notation from the proofs of Theorem 4.1 and Lemma 4.2, we see that

$$s\left(\sum_{x\in X} e_{y,x,b,a}e_{x,y,a,b} \otimes 1\right) = s(e_{y,y,b,b} \otimes 1)$$

(for otherwise we would have $\sum_{x,y,a,b} s(h^*_{x,y,a,b}h_{x,y,a,b}) < 0$). It follows that the trace τ on $\mathcal{C}_{X,A}$ annihilates the elements

$$d_{y,a,b} := e_{y,y,b,b} - \sum_{x\in X} e_{y,x,b,a}e_{x,y,a,b}, \quad y \in X,\ a, b \in A.$$

As $d_{y,a,b} \geq 0$, we have that $\tau(d_{y,a,b}^{1/2}u) = 0$ for every $u \in \mathcal{C}_{X,A}$ and so $\tau(\mathcal{J}) = \{0\}$. By Lemma 4.2, $\tilde{\tau}(q(u)) := \tau(u)$ is a well-defined trace on $\mathcal{C}_X$. Identity (31) implies that

$$s\left(e_{x,x',a,a'} \otimes f_{y,y',b,b'}\right) = \tilde{\tau}\left(p_{x,x',a,a'}p_{y',y,b',b}\right), \quad x, x', y, y' \in X, a, a', b, b' \in A.$$

Conversely, let τ be a trace on $\mathcal{C}_X$ and $\Gamma : M_{XX} \to M_{AA}$ be the QNS correlation, given by

$$\Gamma(\epsilon_{x,x'} \otimes \epsilon_{y,y'}) = \sum_{a,a',b,b' \in X} \tau(p_{x,x',a,a'}p_{y',y,b',b})\epsilon_{a,a'} \otimes \epsilon_{b,b'}.$$

Write

$$w_{a,b} = \sum_{x \in X} u_{a,x}^* u_{b,x}, \quad a, b \in X.$$

We have that

$$\begin{aligned}
\Gamma(J_X) &= \frac{1}{|X|}\sum_{x,y \in X}\sum_{a,a',b,b' \in X} \tau(p_{x,y,a,a'}p_{y,x,b',b})\epsilon_{a,a'} \otimes \epsilon_{b,b'}\\
&= \frac{1}{|X|}\sum_{x,y \in X}\sum_{a,a',b,b' \in X} \tau(u_{a,x}^* u_{a',y} u_{b',y}^* u_{b,x})\epsilon_{a,a'} \otimes \epsilon_{b,b'}\\
&= \frac{1}{|X|}\sum_{x \in X}\sum_{a,a',b,b' \in X} \delta_{a',b'}\tau(u_{a,x}^* u_{b,x})\epsilon_{a,a'} \otimes \epsilon_{b,b'}\\
&= \frac{1}{|X|}\sum_{x \in X}\sum_{a,b,c \in X} \tau(u_{a,x}^* u_{b,x})\epsilon_{a,c} \otimes \epsilon_{b,c}\\
&= \frac{1}{|X|}\sum_{a,b,c \in X} \tau(w_{a,b})\epsilon_{a,c} \otimes \epsilon_{b,c}\\
&= \frac{1}{|X|}\left(\sum_{a,b \in X} \tau(w_{a,b})e_a \otimes e_b\right)\left(\sum_{c \in X} e_c \otimes e_c\right)^*
\end{aligned}$$

Since Γ is a quantum channel, $\Gamma(J_X)$ is a positive operator and hence

$$\sum_{a,b \in X} \tau(w_{a,b})e_a \otimes e_b = \sum_{c \in X} e_c \otimes e_c,$$

implying

$$\tau(w_{a,b}) = \delta_{a,b}, \quad a, b \in X,$$

and $\Gamma(J_X) = J_A$.

(i)-(ii) If $\mathcal{A}$ is a unital C*-algebra, equipped with a trace $\tau_{\mathcal{A}}$, and $\pi : \mathcal{C}_{X,A} \to \mathcal{A}$ is a *-homomorphism such that $\tau = \tau_{\mathcal{A}} \circ \pi$, then $\tau_{\mathcal{A}}(\pi(\mathcal{J})) = 0$. Let $\tilde{\pi} : \mathcal{C}_{X,A}/\mathcal{J} \to \mathcal{A}/\pi(\mathcal{J})$ be given by $\tilde{\pi}(u + \mathcal{J}) = \pi(u) + \pi(\mathcal{J})$. Then $\tilde{\pi}$ is a *-homomorphism and the map $\tau_{\mathcal{A}/\pi(\mathcal{J})} : \mathcal{A}/\pi(\mathcal{J}) \to \mathbb{C}$, given by $\tau_{\mathcal{A}/\pi(\mathcal{J})}(a + \pi(\mathcal{J})) := \tau_{\mathcal{A}}(a)$, is a well-defined trace on $\mathcal{A}/\pi(\mathcal{J})$. We have $\tau_{\mathcal{A}/\pi(\mathcal{J})}(\tilde{\pi}(q(u))) = \tau_{\mathcal{A}}(\pi(u))$, $u \in \mathcal{C}_{X,A}$. Clearly, if $\mathcal{A}$ is finite-dimensional (resp. abelian), so is $\mathcal{A}/\mathcal{I}(\mathcal{J})$. The statements now follow after an inspection of the proof of Theorem 4.1. $\Box$

We do not know if the approximately quantum perfect strategies for concurrent games admit a characterisation via amenable traces of C_X under the conditions of Theorem 4.3.

4.2. Algebras of quantum games. Similarly to concurrent classical-to-quantum games, concurrent quantum games give rise to *- and C*-algebras which we now describe. For $P \in \mathcal{P}_{XX}$ and $Q \in \mathcal{P}_{AA}$, define a linear map

$$\Gamma_{P,Q} : M_{XX} \otimes M_{AA} \otimes \mathcal{C}_{X,A} \otimes \mathcal{C}_{X,A}^{op} \to \mathcal{C}_{X,A}$$

by letting

$$\Gamma_{P,Q}(\omega \otimes u \otimes v^{op}) = \mathrm{Tr}(\omega(P \otimes Q))uv, \quad \omega \in M_{XX} \otimes M_{AA},\ u, v \in \mathcal{C}_{X,A}.$$

For a quantum game $\varphi : \mathcal{P}_{XX} \to \mathcal{P}_{AA}$, let

$$\mathcal{I}(\varphi) = \left\langle \Gamma_{P,\varphi(P)^\perp}(E \otimes E^{op}) : P \in \mathcal{P}_{XX} \right\rangle$$

be the *-ideal in $\mathcal{C}_{X,A}$ generated by $\Gamma_{P,\varphi(Q)^\perp}(E \otimes E^{op})$, $P \in \mathcal{P}_{XX}$, and $I(\varphi)$ be the closed ideal in $C_{X,A}$ generated by the same set. Write $\mathcal{C}(\varphi) = \mathcal{C}_{X,A}/\mathcal{I}(\varphi)$ (resp. $C(\varphi) = C_{X,A}/I(\varphi)$) for the quotient *-algebra (resp. quotient C*-algebra). Similarly, we define an ideal $I(\varphi)$ in C_X and its quotient, where we write E for $(p_{x,x',a,a'})_{x,x',a,a'} \in M_{XX}(C_X)$.

Similarly to Corollary 3.12, we obtain the following:

Corollary 4.4. *Let X be a finite set and $\varphi : \mathcal{P}_{XX} \to \mathcal{P}_{XX}$ be a concurrent quantum game. The following are equivalent for a QNS correlation $\Gamma : M_{XX} \to M_{XX}$:*

(i) *Γ is a perfect quantum commuting (resp. quantum/local) strategy for φ;*

(ii) *there exists a trace τ (resp. a trace τ that factors through a finite dimensional/abelian *-representation) of C_X such that*

$$\Gamma(\epsilon_{x,x'} \otimes \epsilon_{y,y'}) = \left(\tau(e_{x,x',a,a'}e_{y,y',b',b})\right)_{a,a',b,b'}, \quad x, x', y, y' \in X,$$

and

$$\tau\left(\Gamma_{P,\varphi(P)^\perp}(E \otimes E^{op})\right) = 0.$$

5. The quantum graph homomorphism game

In this section, we revisit the quantum graph homomorphism game as introduced in [30], and provide characterisations of its perfect QNS strategies of various classes.

5.1. Characterisations of the existence of perfect strategies. Let Z be a finite set, $H = \mathbb{C}^Z$, and recall that H^d stands for the dual (Banach) space of H. Let $\theta : H \otimes H \to L(H^d, H)$ be the linear map given by

$$\theta(\xi \otimes \eta)(\zeta^d) = \langle \xi, \zeta \rangle \eta, \quad \zeta \in H.$$

We have

$$\theta((S \otimes T)\zeta) = T\theta(\zeta)S^d, \quad \zeta \in H \otimes H, \; S, T \in L(H). \tag{34}$$

We denote by $m : H \otimes H \to \mathbb{C}$ the map, given by

$$m(\zeta) = \left\langle \zeta, \sum_{z \in Z} e_z \otimes e_z \right\rangle, \quad \zeta \in H \otimes H.$$

Let also $f : H \otimes H \to H \otimes H$ be the flip operator, given by $f(\xi \otimes \eta) = \eta \otimes \xi$.

Definition 5.1. A linear subspace $U \subseteq H \otimes H$ is called skew if $m(U) = \{0\}$ and symmetric if $f(U) = U$.

If U is a symmetric skew subspace of $H \otimes H$ and $S_U = \theta(U)$ then the subspace S_U of $L(H^d, H)$ has the following properties:

- $T \in S_U \implies d^{-1} T^{*} d^{-1} \in S_U$, and
- $T \in S_U \implies \sum_{z \in Z} \langle (T d)(e_z), e_z \rangle = 0$.

We call a subspace of $L(H^d, H)$ satisfying these properties a twisted operator anti-system, because of its resemblance to operator anti-systems (that is, selfadjoint subspaces of M_X each of whose elements has trace zero [3]). Given a twisted operator anti-system $S \subseteq L(H^d, H)$, one has that the subspace $U_S = \theta^{-1}(S)$ of $H \otimes H$ is symmetric and skew.

Given a graph G, let

$$U_G = \mathrm{span}\{e_x \otimes e_y : x \sim y\};$$

then U_G is a symmetric skew subspace of $\mathbb{C}^X \otimes \mathbb{C}^X$. We thus consider symmetric skew subspaces of $\mathbb{C}^X \otimes \mathbb{C}^X$ as a non-commutative version of graphs. We note that a couple of other non-commutative incarnations of graphs were considered in the literature, namely, operator subsystems in M_X in [9] – after noting that the subspace

$$S_G := \mathrm{span}\{\epsilon_{x,x'} : x \simeq x'\}$$

of M_X is an operator system, and operator anti-systems in [29] – after noting that the subspace

$$S_G^0 := \mathrm{span}\{\epsilon_{x,x'} : x \sim x'\}$$

of M_X is an operator anti-system. Our use of symmetric skew subspaces, instead of some of these concepts, is dictated by the nature of the definition of QNS correlations, adopted in [10].

We write P_U for the orthogonal projection from $\mathbb{C}^X \otimes \mathbb{C}^X$ onto U. Let $U^{\perp} \subseteq (\mathbb{C}^X \otimes \mathbb{C}^X)^d$ be the annihilator of U and write $P_{U^{\perp}} \in L((\mathbb{C}^X \otimes \mathbb{C}^X)^d)$

for the orthogonal projection onto $U_\perp$. Observe that $\xi^d \in U_\perp$ if and only if ξ belongs to the orthogonal complement $U^\perp$ of U in $\mathbb{C}^X \otimes \mathbb{C}^X$. In addition,

$$P_{U_\perp} = (P_U^\perp)^d.$$

Let $U \subseteq \mathbb{C}^X$ and $V \subseteq \mathbb{C}^A$ be symmetric skew spaces. The quantum graph homomorphism game $U \to V$ is the quantum non-local game $\varphi_{U \to V} : P_{XX} \to P_{AA}$ determined by

$$\varphi_{U \to V}(P) = \begin{cases} 0 & \text{if } P = 0 \\ P_V & \text{if } P \leq P_U \\ I_{AA} & \text{otherwise} \end{cases}$$

Definition 5.2. Let X and A be finite sets and $U \subseteq \mathbb{C}^X \otimes \mathbb{C}^X$, $V \subseteq \mathbb{C}^A \otimes \mathbb{C}^A$ be symmetric skew subspaces. We say that U is quantum commuting homomorphic (resp. quantum homomorphic, locally homomorphic) to V, and write $U \xrightarrow{qc} V$ (resp. $U \xrightarrow{q} V$, $U \xrightarrow{loc} V$), if $\varphi_{U \to V}$ has a perfect quantum commuting (resp. quantum, local) tracial strategy.

Given operator anti-systems $S \subseteq M_X$ and $T \subseteq M_A$, Stahlke [29] defines a non-commutative graph homomorphism from S to T to be a quantum channel $\Phi : M_X \to M_A$ whose family $\{M_i\}_{i=1}^m$ of Kraus operators satisfies the conditions

$$M_i S M_j^* \subseteq T, \quad i, j = 1, \dots, m;$$

if such Φ exists, one writes $S \to T$. We recall the suitable version of this notion for twisted operator anti-systems, described in [30].

Definition 5.3. Let X and A be finite sets, and $S \subseteq \mathcal{L}\left((\mathbb{C}^X)^d, \mathbb{C}^X\right)$ and $T \subseteq \mathcal{L}\left((\mathbb{C}^A)^d, \mathbb{C}^A\right)$ be twisted operator anti-systems. A homomorphism from S into T is a quantum channel

$$\Gamma : M_X \to M_A, \quad \Gamma(T) = \sum_{i=1}^m M_i T M_i^*,$$

such that

$$\overline{M_j} S M_i^d \subseteq T, \quad i, j = 1, \dots, m.$$

If S and T are twisted operator anti-systems, we write $S \to T$ as in [29] to denote the existence of a homomorphism from S to T. Further, if G and H are graphs, we write $G \to H$ if there exists a homomorphism from G to H. The following was shown in [30].

Proposition 5.4. Let X and A be finite sets, $U \subseteq \mathbb{C}^X \otimes \mathbb{C}^X$, $V \subseteq \mathbb{C}^A \otimes \mathbb{C}^A$ be symmetric skew spaces, and G, H be graphs. The following hold:

(i) $U \xrightarrow{loc} V$ if and only if $S_U \to S_V$;

(ii) $G \to H$ if and only if $U_G \xrightarrow{loc} U_H$.

Let $U_X : (\mathbb{C}^X)^d \to \mathbb{C}^X$ be the unitary operator given on the standard basis by $U_X e_x^d = e_x$, $x \in X$, and define $U_A : (\mathbb{C}^A)^d \to \mathbb{C}^A$ similarly. Then $S \subseteq L((\mathbb{C}^X)^d, \mathbb{C}^X)$ is a twisted operator anti-system if and only if the space SU_X^{-1} of M_X has the following properties:

- $T \in SU_X^{-1} \Rightarrow T^t \in SU_X^{-1}$;
- $T \in SU_X^{-1} \Rightarrow \mathrm{Tr}(T) = 0$.

Indeed, the first property is a direct consequence of the fact that

$$d^{-1}(TU_X)^* d^{-1} U_X^{-1} e_x = d^{-1} U_X^*(T^* e_x) = \sum_{y \in X} \overline{\langle T^* e_x, e_y \rangle}\, e_y = \sum_{y \in X} t_{x,y} e_y = T^t e_x,$$

while the second one follows directly from the definition of a twisted operator anti-system.

Recall from Section 2 that m_Z denotes the (normalised) maximally entangled vector in $\mathbb{C}^Z \otimes \mathbb{C}^Z$. For a symmetric skew space $U \subseteq \mathbb{C}^X$, set

$$U \otimes m_Z = \{\xi \otimes m_Z : \xi \in U\};$$

after applying the shuffle map, we view $U \otimes m_Z$ as a symmetric skew subspace of $(\mathbb{C}^X \otimes \mathbb{C}^Z) \otimes (\mathbb{C}^X \otimes \mathbb{C}^Z)$.

Theorem 5.5. Let X and A be finite sets and $U \subseteq \mathbb{C}^X \otimes \mathbb{C}^X$, $V \subseteq \mathbb{C}^A \otimes \mathbb{C}^A$ be symmetric skew spaces. The following are equivalent:

(i) $U \xrightarrow{q} V$;

(ii) $U \otimes m_Z \xrightarrow{\mathrm{loc}} V$ for some finite set Z.

Proof. (i)$\Rightarrow$(ii) Let $\Gamma : M_{XX} \to M_{AA}$ be a tracial quantum QNS correlation such that

$$\langle \Gamma(P_U), P_{V^\perp} \rangle = 0,$$

that is, such that

$$\langle \Gamma(\xi\xi^*)\eta, \eta \rangle = \langle \Gamma(\xi\xi^*), (\eta\eta^*)^d \rangle = 0, \quad \xi \in U, \eta \in V^\perp.$$

By definition of tracial quantum QNS correlation, there exists a finite dimensional C^*-algebra $\mathcal{A}$, a tracial state $\tau_{\mathcal{A}}$ on $\mathcal{A}$ and a *-homomorphism $\pi : \mathcal{C}_{X,A} \to \mathcal{A}$ such that

$$\Gamma(\epsilon_{x,x'} \otimes \epsilon_{y,y'}) = \left(\tau_{\mathcal{A}}(\pi(e_{x,x',a,a'} e_{y',y,b',b}))\right)_{a,a',b,b'}.$$

Writing $\xi = \sum_{x,y \in X} \lambda_{x,y} e_x \otimes e_y$ and $\eta = \sum_{a,b \in A} \eta_{a,b} e_a \otimes e_b$, we have

$$\Gamma(\xi\xi^*) = \sum_{a,a',b,b'} \sum_{x,x',y,y'} \tau_{\mathcal{A}}(\pi(e_{x,x',a,a'} e_{y',y,b',b})) \lambda_{x,y} \overline{\lambda_{x',y'}}\, \epsilon_{a,a'} \otimes \epsilon_{b,b'}.$$

Let

$$Y_\xi := \sum_{x',y' \in X} \overline{\lambda_{x',y'}}\, \epsilon_{x',y'}, \quad Y_\eta = \sum_{a',b' \in A} \eta_{a',b'} \epsilon_{a',b'}$$

and $E = (\tau(e_{x,x',a,a'}))_{x,x',a,a'}$; then E is a stochastic A-matrix. Observe that

$$\gamma(\xi)U_X^{-1} = \sum_{x,y\in X}\lambda_{x,y}\gamma(e_x\otimes e_y)U_X^{-1} = \sum_{x,y\in X}\lambda_{x,y}\epsilon_{y,x} = Y_\xi^{*}. \tag{35}$$

We have

$$\begin{aligned}
0 &= \langle \Gamma(\xi\xi^*)\beta, \beta\rangle\\
&= \sum_{a,a',b,b'}\sum_{x,x',y,y'} \phi_A\big(\tau(e_{x,x',a,a})\tau(e_{y',y,b,b})\big)\lambda_{x,y}\overline{\lambda_{x',y'}\beta_{a',b'}}\beta_{a,b}\\
&= (\mathrm{Tr}\otimes\phi_A)\left(\left(\sum_{x',y',a',b'}\tau(e_{x,x',a,a'})\overline{\lambda_{x',y'}\beta_{a',b'}}\tau(e_{y',y,b',b})\right)_{x,y,a,b}\right.\\
&\qquad\left.\times\left(\sum_{x,y,a,b}\overline{\lambda_{x,y}\beta_{a,b}}\epsilon_{x,y}\otimes\epsilon_{a,b}\otimes 1_A\right)^*\right)\\
&= (\mathrm{Tr}\otimes\phi_A)\left(E(Y_\xi\otimes Y_\beta\otimes 1_A)E(Y_\xi^*\otimes Y_\beta^*\otimes 1_A)\right).
\end{aligned}$$

After passing to a quotient, we may assume that A is faithfully represented on a Hilbert space H and ϕ_A is faithful. As E is positive, we have

$$E^{1/2}(Y_\xi\otimes Y_\beta\otimes 1_A)E(Y_\xi^*\otimes Y_\beta^*\otimes 1_A)E^{1/2} = 0.$$

It follows that $E^{1/2}(Y_\xi\otimes Y_\beta\otimes 1_A)E^{1/2} = 0$ and hence $E(Y_\xi^*\otimes Y_\beta^*\otimes 1_A)E = 0$.

Define a linear map $\Gamma : M_A \to M_X\otimes A$ by letting

$$\Gamma(\epsilon_{a,b}) = E_{a,b} := (\tau(e_{x,x',a,b}))_{x,x'};$$

by Choi's Theorem, Γ is a unital completely positive map. Let $\Gamma(\omega) = \sum_{i=1}^m M_i\omega M_i^*$ be a Kraus representation (here $M_i : \mathbb{C}^A \to \mathbb{C}^X\otimes H$, $i = 1,\dots,m$), and set

$$X_{a,b,i,j} = \sum_{a',b'\in A}\overline{\beta_{b',a'}}\epsilon_{a,a'}M_j^*(Y_\xi^*\otimes 1_A)M_i\epsilon_{b',b}, \qquad a,b\in A,\ i,j\in[m].$$

Let $\theta_{1,2} : \mathbb{C}^A\otimes\mathbb{C}^X\otimes H \to \mathbb{C}^X\otimes\mathbb{C}^A\otimes H$ be the flip operator defined on the elementary tensors by $\theta_{1,2}(\xi_1\otimes\xi_2\otimes\xi_3) = \xi_2\otimes\xi_1\otimes\xi_3$, and write $M_i^{1,3} : \mathbb{C}^A\otimes\mathbb{C}^A \to \mathbb{C}^X\otimes\mathbb{C}^A\otimes H$ for the operator $\theta_{1,2}(1\otimes M_i)$.

We have

$$\begin{aligned}
&\mathrm{Tr}(X_{a,b,i,j}X_{a,b,i,j}^*)\\
&= \sum_{a',b',a'',b''}\overline{\beta_{b',a'}}\beta_{b'',a''}\mathrm{Tr}\big(\epsilon_{a,a'}M_j^*(Y_\xi^*\otimes 1_A)M_i\epsilon_{b',b}\epsilon_{b,b''}M_i^*(Y_\xi\otimes\\
&\quad 1_A)M_j\epsilon_{a'',a}\big).
\end{aligned}$$

Letting $\xi_a = (\xi_{a,a'})_{a' \in A}$, considered as a row operator over M_A, we have

$$\sum_{i,j=1}^{m} \operatorname{Tr}(X_{a,b,i,j} X_{a,b,i,j}^*)$$

$$= \sum_{j,a',b',a'',b''} \delta_{b',a'} \delta_{b'',a''} \operatorname{Tr}\left(\xi_{a,a'} M_j^* (Y_\cdot \otimes 1_A) E_{b',b''} (Y_\cdot \otimes 1_A) M_j \xi_{a'',a}^* \right)$$

$$= \sum_{j=1}^{m} \operatorname{Tr}\left(\xi_a (M_j^{1,3})^* (Y^* \otimes Y^* \otimes 1_A) E (Y_\cdot \otimes Y \otimes 1_A) M_j^{1,3} \xi_a^* \right).$$

Write $R_{a,j} = E^{1/2}(Y_\cdot \otimes Y \otimes 1_A) M_j^{1,3} \xi_a^*$. Then

$$\sum_{j=1}^{m} (\operatorname{Tr} \otimes \tau_A)(R_{a,j} R_{a,j}^*)$$

$$= \sum_{j=1}^{m} \operatorname{Tr}\left(E^{1/2}(Y_\cdot \otimes Y \otimes 1_A) M_j^{1,3} \xi_a^* \xi_a (M_j^{1,3})^* (Y^* \otimes Y^* \otimes 1_A) E^{1/2} \right)$$

$$= (\operatorname{Tr} \otimes \tau_A)\left(E^{1/2}(Y_\cdot \otimes Y \otimes 1_A) E (Y^* \otimes Y^* \otimes 1_A) E^{1/2} \right) = 0,$$

giving $R_{a,j} = 0$, as we assume that the trace is faithful and therefore $\sum_{i,j=1}^{m} \operatorname{Tr}(X_{a,b,i,j} X_{a,b,i,j}^*) = 0$ implying

$$X_{a,b,i,j} = 0, \quad a, b \in A, \; i, j = 1, \dots, m. \tag{36}$$

We may assume that A is faithfully represented on $\mathbb{C}^Z$ and so that M_i is an operator from $\mathbb{C}^A$ into $\mathbb{C}^X \otimes \mathbb{C}^Z$. Let $R_j = \overline{M_j^*}$, $j = 1, \dots, m$. For $a \in A$ we have

$$\begin{aligned}
(U_X \otimes U_Z) R_i^d U_A^{-1}(e_a) &= (U_X \otimes U_Z) R_i^d (e_a^d) = (U_X \otimes U_Z)(R_i^* e_a)^d \\
&= (U_X \otimes U_Z)(\overline{M_i} e_a)^d = M_i e_a.
\end{aligned}$$

Taking into account (35), we obtain

$$\begin{aligned}
R_j \Gamma(\cdots \otimes m_Z) R_i^d &= R_j(\Gamma(\cdots) U_X^{-1} \otimes \Gamma(m_Z) U_Z^{-1})((U_X \otimes U_Z) R_i^d U_A^{-1}) U_A = M_j^*(Y^* \otimes 1)((U_X \otimes U_Z) R_i^d U_A^{-1}) U_A \\
&= M_j^*(Y^* \otimes 1) M_i U_A.
\end{aligned} \tag{37}$$

Since Γ is unital,

$$\sum_{j=1}^{m} R_j^* R_j = \sum_{j=1}^{m} \overline{M_j M_j^*} = I$$

and hence the map $\omega \to \sum_{j=1}^{m} R_j \omega R_j^*$ from M_{XZ} into M_A is a quantum channel. We claim that

$$\overline{R_j \Gamma(\cdots \otimes m_Z) R_i^d} \in S_V. \tag{38}$$

Indeed, fix $\zeta \in V^{\perp}$. Since $\psi(\zeta)U_A^{-1} = Y_\zeta^*$, taking (36) and (37) into account, we have

$$\begin{aligned}
\left\langle R_j^*\psi(\xi \otimes m_Z)R_i^d, \psi(\zeta)\right\rangle &= \operatorname{Tr}\left(M_j^*(Y_\xi^* \otimes 1)\, M_i U_A \psi(\zeta)^*\right)\\
&= \operatorname{Tr}\left(M_j^*(Y_\xi \otimes 1)M_i Y_\zeta\right)\\
&= \sum_{a',b' \in A} \overline{\zeta_{b',a'}} \left\langle M_j^*(Y_\xi \otimes 1)M_i e_{b'}, e_{a'}\right\rangle\\
&= \langle X_{a,b,i,j} e_b, e_a\rangle = 0;
\end{aligned}$$

(38) now follows.

(ii)⇒(i) By Proposition 5.4,

$$S_{U \otimes m_Z} \to S_V. \tag{39}$$

Let $(R_i)_{i=1}^m \subseteq L(\mathbb{C}^X \otimes \mathbb{C}^Z, \mathbb{C}^A)$ be a family of Kraus operators of the quantum channel implementing (39). Keeping the notation from the previous paragraphs and reversing the arguments therein, we see that, if $M_i = R_i^*$, $\xi \in U$ and $\zeta \in V^{\perp}$, then

$$\sum_{a',b' \in A} \overline{\zeta_{b',a'}} \left\langle M_j^*(Y_\xi^* \otimes 1_Z)M_i e_{b'}, e_{a'}\right\rangle = 0. \tag{40}$$

Thus, $X_{a,b,i,j} = 0$ for all $a, b \in A$ and all $i, j = 1, \dots, m$.

Letting $\Phi : M_A \to M_X \otimes M_Z$ be the unital completely positive map given by $\Phi(\omega) = \sum_{i=1}^m M_i \omega M_i^*$ and setting $E_{a,b} = \Phi(\epsilon_{a,b})$, we see that $E = (E_{a,b})_{a,b}$ is a stochastic operator matrix acting on $\mathbb{C}^Z$. By [30, Theorem 5.2], there exists a *-representation $\pi : \mathcal{C}_{X,A} \to \mathcal{B}(\mathbb{C}^Z)$ such that $(\pi(e_{x,x',a,a'}))_{x,x',a,a'} = E$. Let $\Gamma : M_{XX} \to M_{AA}$ be the linear map given by

$$\Gamma(\epsilon_{x,x'} \otimes \epsilon_{y,y'}) = \left(\operatorname{Tr}(\pi(e_{x,x',a,a'}e_{y',y,b',b}))\right)_{a,a',b,b'};$$

thus, Γ is a tracial quantum QNS correlation and, by (40) and the previous paragraphs,

$$\left\langle \Gamma(\xi\xi^*), (\zeta\zeta^*)^d\right\rangle = \operatorname{Tr}\left(E(Y_\xi \otimes Y_\zeta \otimes 1_A)E(Y_\xi^* \otimes Y_\zeta^* \otimes 1_Z)\right) = 0.$$

It follows that $\langle \Gamma(\xi\xi^*), P_{V^{\perp}}\rangle = 0$ for every $\xi \in U$, giving $\langle \Gamma(P_U), P_{V^{\perp}}\rangle = 0$. ∎

Remark 5.6. It was shown as part of the proof of Theorem 5.5 that, for symmetric skew spaces $U \subseteq \mathbb{C}^X \otimes \mathbb{C}^X$ and $V \subseteq \mathbb{C}^A \otimes \mathbb{C}^A$, we have that $U \to V$ if and only if there exist a finite-dimensional algebra $\mathcal{A}$, a unital completely positive map $\Phi : M_A \to M_X \otimes \mathcal{A}$ with Kraus representation $\Phi(T) = \sum_{i=1}^m M_i T M_i^*$, such that

$$M_j^*(\psi(U)U_X^{-1} \otimes 1_{\mathcal{A}})M_i \subseteq \psi(V)U_A^{-1}, \quad i, j = 1, \dots, m. \tag{41}$$

The same arguments allow us to conclude the equivalence (i)⇔(ii) in the following statement.

Theorem 5.7. Let X and A be finite sets and $\mathcal{U} \subseteq \mathbb{C}^X \otimes \mathbb{C}^X$, $\mathcal{V} \subseteq \mathbb{C}^A \otimes \mathbb{C}^A$ be symmetric skew spaces. The following are equivalent:

(i) $\mathcal{U} \xrightarrow{qc} \mathcal{V}$;

(ii) there exists a unital completely positive map $\Phi : M_A \to M_X \otimes \mathcal{C}_{X,A}$ with Kraus representation $\Phi(T) = \sum_{i=1}^m M_i T M_i^*$, for which inclusions (41) hold;

(iii) there exists a von Neumann algebra $\mathcal{N}$ with a faithful normal tracial state τ and a unital completely positive map $\Phi : M_A \to M_X \otimes \mathcal{N}$ with Kraus representation $\Phi(T) = \sum_{i=1}^m M_i T M_i^*$, for which inclusions (41) hold.

Proof. The equivalence (i)$\Leftrightarrow$(ii) was pointed out in Remark 5.6. The implication (iii)$\Rightarrow$(i) is similar to that of (ii)$\Rightarrow$(i) of Theorem 5.5. For (ii)$\Rightarrow$(iii), we take $\mathcal{N} = \pi_\tau(\mathcal{C}_{X,A})''$, where π_τ is the GNS representation of τ; if ξ is the cyclic vector of π_τ then $\langle \cdot \xi, \xi \rangle$ is a faithful normal trace on $\mathcal{N}$. $\square$

Let $\mathcal{S} \subseteq M_X$ and $\mathcal{T} \subseteq M_A$ be operator anti-systems. Stalhke writes [29] $\mathcal{S} \xrightarrow{*} \mathcal{T}$ if there exists a finite set B and a state $\sigma \in M_B^+$ such that $\mathcal{S} \otimes \sigma \to \mathcal{T}$; in this case he says that there exists an entanglement assisted homomorphism from $\mathcal{S}$ to $\mathcal{T}$.

Corollary 5.8. Let G, H be graphs. Then

$$\mathcal{U}_G \xrightarrow{q} \mathcal{U}_H \implies \mathcal{S}_G^0 \xrightarrow{*} \mathcal{S}_H^0.$$

Proof. First observe that $\mathcal{S}_G^0 = \vartheta(\mathcal{U}_G)\mathcal{U}_X^{-1}$. The statement now follows from Remark 5.6. $\square$

In the next corollary, we partially improve [30, Proposition 10.5] by providing a lower bound on the relaxed orthogonal rank $\xi_q(G)$.

Corollary 5.9. If G is a graph then $\xi_q(G) \geq \sqrt{\vartheta(\overline{G})}$.

Proof. We observe first that

$$\xi_q(G) = \min\{|A| : \mathcal{U}_G \xrightarrow{q} \langle m_A \rangle^\perp\}.$$

Moreover, $\vartheta(\langle m_A \rangle^\perp)\mathcal{U}_A^{-1} = (\mathbb{C}I_A)^\perp$, and hence $\mathcal{U}_G \xrightarrow{q} \langle m_A \rangle^\perp$ implies $\mathcal{S}_G^0 \xrightarrow{*} (\mathbb{C}I_A)^\perp$. It follows from [29, Corollary 20] that $\xi_q(G) \geq \sqrt{\vartheta(\overline{G})}$. $\square$

5.2. Quantum colourings of graphs. Let G be a (finite) simple graph with vertex set X. For $x, y \in X$, we write $x \sim y$ when $\{x, y\}$ is an edge of G, and $x \simeq y$ when $x \sim y$ or $x = y$. The classical-to-quantum colouring game $\Gamma_G^A : \mathcal{P}_{XX}^{cl} \to \mathcal{P}_{AA}$ is determined by the requirements

$$\Gamma_G^A(\epsilon_{x,x} \otimes \epsilon_{y,y}) = \begin{cases} J_A & \text{if } x = y \\ J_A^\perp & \text{if } x \sim y, \\ I_{AA} & \text{otherwise.} \end{cases}$$

In this subsection, we apply the previous results to give a description of perfect quantum commuting and perfect quantum strategies for the classical-to-quantum colouring game in terms of quantum channels whose Kraus operators respect certain containment relations. These relations define a "pushforward" of the graph G into M_A or, in the terminology of Weaver [33], into the quantum graph (S, M) with $S = M' = M_A$. Namely, for a von Neumann algebra $\mathcal{N}$, equipped with faithful tracial state τ, and a uni-tal completely positive map $\Phi : M_A \to D_X \otimes \mathcal{N}$ with Kraus representation $\Phi(T) = \sum_{i=1}^m M_i T M_i^*$, we consider the inclusion relations

$$M_i^*(D_X \otimes 1_{\mathcal{N}})M_j \subseteq \mathbb{C}I_A, \quad i, j \in [m], \tag{42}$$

and

$$M_i^*(S_G^0 \otimes 1_{\mathcal{N}})M_j \perp \mathbb{C}I_A, \quad i, j \in [m]. \tag{43}$$

Definition 5.10. Let X and A be finite sets and G be graph with vertex set X. A pair $(\mathcal{N}, \Phi)$, where $\mathcal{N}$ is a von Neumann algebra and $\Phi : M_A \to D_X \otimes \mathcal{N}$ is a unital completely positive map with Kraus representation $\Phi(T) = \sum_{i=1}^m M_i T M_i^*$, is called a quantum colouring of G if conditions (42) and (43) are satisfied.

Let $\mathcal{R}^{\mathfrak{u}}$ denote an ultrapower of the hyperfinite II_1-factor $\mathcal{R}$ by a free ultrafilter $\mathfrak{u}$ on $\mathbb{N}$ and $\mathrm{tr}_{\mathcal{R}^{\mathfrak{u}}}$ be its trace.

Theorem 5.11. Let G be a graph with vertex set X.

(1) The following are equivalent:
  - (i) the classical-to-quantum colouring game φ_G^A has a perfect quantum commuting strategy;
  - (ii) there exists a quantum colouring $(\mathcal{N}, \Phi)$ of G, with $\mathcal{N}$ possessing a faithful tracial state.

(2) The following are equivalent:
  - (i) φ_G^A has a perfect approximately quantum strategy;
  - (ii) there exist a quantum colouring of the form $(\mathcal{R}^{\mathfrak{u}}, \Phi)$.

(3) The following are equivalent:
  - (i) φ_G^A has a perfect quantum strategy;
  - (ii) there exists a quantum colouring $(\mathcal{N}, \Phi)$ of G, where $\mathcal{N}$ is finite dimensional.

Proof. (1) (i)$\Rightarrow$(ii) Let $\mathcal{E} : D_{XX} \to M_{AA}$ be a CQNS correlation, which is a perfect quantum commuting strategy for φ_G^A. Let τ be a trace on $\mathcal{B}_{X,A}$ associated with $\mathcal{E}$ via Corollary 3.12, and $\mathcal{N} := \pi_\tau(\mathcal{B}_{X,A})''$, where π_τ is the GNS representation corresponding to τ. If ξ is the cyclic vector of π_τ, then $\tau(T) := \langle T\xi, \xi\rangle$ is a faithful trace on $\mathcal{N}$. Let $\Gamma : M_{XX} \to M_{AA}$ be the canonical lift of $\mathcal{E}$ to a QNS correlation:

$$\begin{aligned}\Gamma(\epsilon_{x,x'} \otimes \epsilon_{y,y'}) &= \left(\delta_{x,x'}\delta_{y,y'}\tau(e_{x,a,b}e_{y,b',a'})\right)_{a,a',b,b'} \\ &= \left(\delta_{x,x'}\delta_{y,y'}\tau(\pi_\tau(e_{x,a,b}e_{y,b',a'}))\right)_{a,a',b,b'}.\end{aligned}$$

As $(\sigma_{x,x'}\pi_\tau(e_{x,a,a'}))_{x,x',a,a'}$ is a stochastic operator matrix, there exists a *-representation $\pi : \mathcal{C}_{X,A} \to \mathcal{N}$ such that $\pi(e_{x,x',a,a'}) = \sigma_{x,x'}\pi_\tau(e_{x,a,a'})$, $x, x' \in X$, $a, a' \in A$. Therefore Γ is a tracial QNS correlation with

$$\left\langle \Gamma(P_{U_G}), P_{V^\perp} \right\rangle = 0, \quad \text{where } V = \langle m_A \rangle^\perp.$$

As $\mathcal{L}(V^\perp)_{U_A^{-1}} = \mathbb{C}I_A$ and $\mathcal{L}(U_G)_{U_X^{-1}} = S_G^0$, Theorem 5.7 shows that the unital completely positive map $\Phi : M_A \to D_X \otimes \mathcal{N}$, given by $\Phi(\epsilon_{a,a'}) = (\pi(e_{x,x',a,a'}))_{x,x'}$, has a Kraus representation $\Phi(T) = \sum_{i=1}^m M_i T M_i^*$ satisfying (43). As $\langle \Gamma(\epsilon_{x,x} \otimes \epsilon_{x,x}), (\xi\xi^*)^d \rangle = 0$ whenever $\xi \in V$, similar arguments show that (42) is satisfied.

(ii)⇒(i) Let $E_{a,b} = \Phi(\epsilon_{a,b})$, $a, b \in A$. Then $E := (E_{a,b})_{a,b}$ is a semi-classical stochastic operator matrix; thus, there exists a *-representation $\pi : \mathcal{C}_{X,A} \to \mathcal{N}$ such that $(\pi(e_{x,x',a,a'}))_{x,x',a,a'} = E$. Let $\Gamma : M_{XX} \to M_{AA}$ be the QNS correlation given by

$$\Gamma(\epsilon_{x,x'} \otimes \epsilon_{y,y'}) = \left(\tau(\pi(e_{x,x',a,a'}e_{y',y,b',b}))\right)_{a,a',b,b'}, \quad x, x', y, y' \in X.$$

As $\pi(e_{x,x',a,a'}) = 0$ whenever $x \neq x'$, we have that $\Gamma = \Gamma \circ \Delta_{XX}$. By Theorem 5.7, $\langle \Gamma(P_{U_G}), P_{V^\perp} \rangle = 0$. It hence suffices to show that the CQNS correlation $\Gamma|_{D_{XX}}$ is concurrent.

Let $\xi = \sum_{a,b\in A} \lambda_{a,b} e_a \otimes e_b$ be orthogonal to m_A; thus, $\sum_{a\in A} \lambda_{a,a} = 0$. Let $Y_\xi = \sum_{a',b'\in A} \lambda_{a',b'} \epsilon_{a',b'}$. We have to show that $\langle \Gamma(\epsilon_{x,x} \otimes \epsilon_{x,x}), (\xi\xi^*)^d \rangle = 0$. As $M_j^*(\epsilon_{x,x} \otimes 1_N)M_i \in \mathbb{C}I$, there exists $\mu_x \in \mathbb{C}$ such that

$$\epsilon_{a,a'} M_j^* (\epsilon_{x,x} \otimes 1_N) M_i \epsilon_{b',b} = \delta_{a',b'} \mu_x \epsilon_{a,b}, \tag{44}$$

for all $a, a', b, b' \in A$. As in the proof of Theorem 5.5, let $\epsilon_a = (\epsilon_{a,a'})_{a'}$, considered as a row operator over M_A, and $M_i^{1,3} : \mathbb{C}^A \otimes \mathbb{C}^A \to \mathbb{C}^X \otimes \mathbb{C}^A \otimes H$ be the operator $\partial_{1,2}(1 \otimes M_i)$, where $\partial_{1,2} : \mathbb{C}^A \otimes \mathbb{C}^X \otimes H \to \mathbb{C}^X \otimes \mathbb{C}^A \otimes H$ is the flip operator defined on the elementary tensors by $\partial_{1,2}(\zeta_1 \otimes \zeta_2 \otimes \zeta_3) = \zeta_2 \otimes \zeta_1 \otimes \zeta_3$. Fix $a, b \in A$. We have

$$\begin{aligned}
&\sum_{i,j=1}^m \sum_{a',b',a'',b''\in A} \overline{\lambda_{b',a'}}\lambda_{b'',a''}\epsilon_{a,a'} M_j^*(\epsilon_{x,x}\otimes 1_N)M_i\epsilon_{b',b}\epsilon_{b,b''}M_i^*(\epsilon_{x,x}\otimes 1_N)M_j\epsilon_{a'',a}\\
&= \sum_{j=1}^m \sum_{a',b',a'',b''\in A} \overline{\lambda_{b',a'}}\lambda_{b'',a''}\epsilon_{a,a'} M_j^*(\epsilon_{x,x}\otimes 1_N)E_{b',b''}(\epsilon_{x,x}\otimes 1_N)M_j\epsilon_{a'',a}\\
&= \sum_{j=1}^m \epsilon_a (M_j^{1,3})^*(\epsilon_{x,x}\otimes Y_\xi^*\otimes 1_N)E(\epsilon_{x,x}\otimes Y_\xi\otimes 1_N)M_j^{1,3}\epsilon_a^*.
\end{aligned}$$

On the other hand, by (44),

$$\sum_{a',b',a'',b''\in A} \overline{\lambda_{b',a'}}\lambda_{b'',a''}\xi_{a,a'} M_j^{*}(\xi_{x,x}\otimes 1_N)M_i\xi_{b',b}\xi_{b,b''}M_i(\xi_{x,x}\otimes 1_N)M_j\xi_{a'',a}$$

$$= \sum_{a',a''\in A}\overline{\lambda_{a',a'}}\lambda_{a'',a''}\xi_{a,a'}M_j^{*}(\xi_{x,x}\otimes 1_N)M_i\xi_{a',a''}M_i^{*}(\xi_{x,x}\otimes 1_N)M_j\xi_{a'',a}$$

$$= \sum_{a'\in A}\overline{\lambda_{a',a'}}\xi_{a,a'}M_j^{*}(\xi_{x,x}\otimes 1_N)M_i\xi_{a',a} \left(\sum_{a'\in A}\overline{\lambda_{a',a'}}\xi_{a,a'}M_j^{*}(\xi_{x,x}\otimes 1_N)M_i\xi_{a',a}\right)^{*}$$

$$= \left(\sum_{a'\in A}\overline{\lambda_{a',a'}}\, x\xi_{a,a}\right)\left(\sum_{a'\in A}\overline{\lambda_{a',a'}}\, x\xi_{a,a}\right)^{*} = 0.$$

Hence

$$\sum_{j=1}^{m}\xi_a\, M_j^{1,3\,*}\left(\xi_{x,x}\otimes Y_{\beta}\otimes 1_N\right) E\left(\xi_{x,x}\otimes Y_{\beta}\otimes 1_N\right)M_j^{1,3}\xi_a^{*} = 0;$$

this implies $E^{1/2}(\xi_{x,x}\otimes Y_{\beta}\otimes 1_N)M_j^{1,3}\xi_a^{*} = 0$ and therefore

$$0 = (\mathrm{Tr}\otimes\tau)\left(\sum_{j=1}^{m} E^{1/2}(\xi_{x,x}\otimes Y_{\beta}\otimes 1_N)M_j^{1,3}\xi_a^{*}\xi_a(M_j^{1,3})^{*}(\xi_{x,x}\otimes Y_{\beta}\otimes 1_N)E^{1/2}\right)$$

$$= (\mathrm{Tr}\otimes\tau)\left(E^{1/2}(\xi_{x,x}\otimes Y_{\beta}\otimes 1_N)\,E\,\left(\xi_{x,x}\otimes Y_{\beta}^{*}\otimes 1_N\right)E^{1/2}\right)$$

$$= \langle \Gamma(\xi_{x,x}\otimes\xi_{x,x})\beta,\beta\rangle$$

showing that Γ is concurrent.

(2) (ii)⇒(i) The arguments are similar to those in part (1): we first obtain a *-representation $\pi : \mathcal{C}_{X,A} \to \mathcal{R}^{u}$ by letting $(\pi(e_{x,x',a,a'}))_{x,x'} = (\xi_{a,a'})$, and define

$$\Gamma(\xi_{x,x'}\otimes\xi_{y,y'}) = \left(\mathrm{tr}_{\mathcal{R}^u}(\pi(e_{x,x',a,a'}e_{y',y,b',b}))\right)_{a,a',b,b'}.$$

We have that $\tau := \mathrm{tr}_{\mathcal{R}^u}\circ\pi$ is an amenable trace on $\mathcal{C}_{X,A}$ ([6, Proposition 6.3.5 (1),(2)]). Hence the assignment $s_{\tau}(x\otimes y^{op}) := \tau(xy)$ determines a state on $\mathcal{C}_{X,A}\otimes_{\min}\mathcal{C}_{X,A}^{op}$ and if $\partial: \mathcal{C}_{X,A}\to\mathcal{C}_{X,A}^{op}$ is the *-isomorphism given by $\partial(e_{x,x',a,a'}) = e_{x',x,a',a}$ ([30, Theorem 7.7]) then $s := s_{\tau}\circ(\mathrm{id}\otimes\partial)$ is a state on $\mathcal{C}_{X,A}\otimes_{\min}\mathcal{C}_{X,A}$ such that $s(e_{x,x',a,a'}\otimes e_{y,y',b,b'}) = \tau(e_{x,x',a,a'}e_{y',y,b',b})$. Applying [30, Theorem 6.5], we obtain that Γ is an approximately quantum QNS correlation. As $\Gamma = \Gamma_{\mathcal{X}\mathcal{X}}$, by [30, Theorem 7.3], $\Gamma|_{\mathcal{D}_{XX}}$ is an approximately quantum CQNS correlation. The above arguments also give that Γ is concurrent and satisfies $\langle\Gamma(P_{\mathcal{U}_{\beta}}), P_{\mathcal{V}_{\beta}^{\perp}}\rangle = 0$.

(i)⇒(ii) By Corollary 3.7(i'), a perfect approximately quantum CQNS strategy $\mathcal{E}$ is determined by an amenable trace τ on $\mathcal{B}_{X,A}$. Hence there exists a *-homomorphism $\pi : \mathcal{B}_{X,A} \to \mathcal{R}^{\mathcal{U}}$ such that $\Gamma = \mathrm{tr}_{\mathcal{R}^{\mathcal{U}}} \circ \pi$ (see [15, Proposition 3.2]). The proof is completed similarly to (1)(i)$\Rightarrow$(1)(ii).

(3) This part of the statement is similar to (1) and (2), and uses the representation of quantum strategies established in Theorem 3.5. $\square$

In the next two propositions, we clarify some useful properties of quantum colourings. The first one is an automatic homomorphism result.

Proposition 5.12. Let X and A be finite sets, $\mathcal{N}$ be a von Neumann algebra and $\Phi : M_A \to \mathcal{D}_X \otimes \mathcal{N}$ be a unital completely positive map with Kraus representation $\Phi(T) = \sum_{i=1}^{m} M_i T M_i^*$. The following are equivalent:

(i) Φ is a *-homomorphism;
(ii) condition (42) holds.

Proof. (i)$\Rightarrow$(ii) For each $a, b \in A$, we write $\Phi(\epsilon_{a,b}) = \sum_{x \in X} \epsilon_{x,x} \otimes r_{x,a,b}$. For $a \neq b$ in A and $1 \leq i, j \leq m$, set

$$X_{a,b,i,j} = \epsilon_{a,a} M_i^* (\epsilon_{x,x} \otimes 1_{\mathcal{N}}) M_j \epsilon_{b,b} \in M_A. \tag{45}$$

We have

$$\begin{aligned}
\sum_{j=1}^{m} X_{a,b,i,j} X_{a,b,i,j}^* &= \sum_{j=1}^{m} \epsilon_{a,a} M_i^* (\epsilon_{x,x} \otimes 1_{\mathcal{N}}) M_j \epsilon_{b,b} M_j^* (\epsilon_{x,x} \otimes 1_{\mathcal{N}}) M_i \epsilon_{a,a} \\
&= \epsilon_{a,a} M_i^* (\epsilon_{x,x} \otimes 1_{\mathcal{N}}) \left(\sum_{y \in X} \epsilon_{y,y} \otimes r_{y,b,b} \right) (\epsilon_{x,x} \otimes 1_{\mathcal{N}}) M_i \epsilon_{a,a} \\
&= \epsilon_{a,a} M_i^* (\epsilon_{x,x} \otimes r_{x,b,b}) M_i \epsilon_{a,a} \\
&= \left(\epsilon_{a,a} M_i^* \left(\epsilon_{x,x} \otimes r_{x,b,b}^{1/2} \right) \right) \left(\epsilon_{a,a} M_i^* \left(\epsilon_{x,x} \otimes r_{x,b,b}^{1/2} \right) \right)^*,
\end{aligned} \tag{46}$$

where we have used the fact that $r_{x,b,b}$ is positive. Let

$$Y_{a,b,i} = \epsilon_{a,a} M_i^* \left(\epsilon_{x,x} \otimes r_{x,b,b}^{1/2} \right), \quad x \in X,\ a \neq b \in A,\ i = 1, \dots, m. \tag{47}$$

By (46), $\sum_{j=1}^{m} X_{a,b,i,j} X_{a,b,i,j}^* = Y_{a,b,i} Y_{a,b,i}^*$. Furthermore,

$$\begin{aligned}
\sum_{i=1}^{m} Y_{a,b,i}^* Y_{a,b,i} &= \sum_{i=1}^{m} \left(\epsilon_{x,x} \otimes r_{x,b,b}^{1/2} \right) M_i \epsilon_{a,a} M_i^* \left(\epsilon_{x,x} \otimes r_{x,b,b}^{1/2} \right) \\
&= \left(\epsilon_{x,x} \otimes r_{x,b,b}^{1/2} \right) \left(\sum_{y \in X} \epsilon_{y,y} \otimes r_{y,a,a} \right) \left(\epsilon_{x,x} \otimes r_{x,b,b}^{1/2} \right) \\
&= \epsilon_{x,x} \otimes \left(r_{x,b,b}^{1/2} r_{x,a,a}^{1/2} \right) \left(r_{x,b,b}^{1/2} r_{x,a,a}^{1/2} \right)^*.
\end{aligned} \tag{48}$$

Since Φ is a homomorphism, $r_{x,b,b}^{1/2} r_{x,a,a}^{1/2} = r_{x,b,b} r_{x,a,a} = 0$ if $a \neq b$. By (48), it follows that $Y_{a,b,i} = 0$ for all i and all $a \neq b$. Using (46), we have $X_{a,b,i,j} = 0$ for all i, j and $a \neq b$. By (45), this forces $M_i^* (\epsilon_{x,x} \otimes 1_{\mathcal{N}}) M_j \in \mathcal{D}_A$.

Next, we show that $M_i^*(\epsilon_{x,x}\otimes 1_N)M_j$ lies in $\mathbb{C}1_A$. We set

$$\lambda_{a,x,i,j} = \mathrm{Tr}(\epsilon_{a,a}M_i^*(\epsilon_{x,x}\otimes 1_N)M_j\epsilon_{a,a});$$

then $M_i^*(\epsilon_{x,x}\otimes 1_N)M_j = \sum_{a\in A}\lambda_{a,x,i,j}\epsilon_{a,a}$. To establish (iii), it suffices to show that $\lambda_{a,x,i,j} = \lambda_{b,x,i,j}$ for all $a,b\in A$.

Set

$$C_{a,b,x,i,j} = \epsilon_{a,a}M_i^*(\epsilon_{x,x}\otimes 1_N)M_j\epsilon_{a,a} - \epsilon_{a,b}M_i^*(\epsilon_{x,x}\otimes 1_N)M_j\epsilon_{b,a} \tag{49}$$

and observe that

$$C_{a,b,x,i,j} = (\lambda_{a,x,i,j}-\lambda_{b,x,i,j})\epsilon_{a,a}.$$

We note that $\epsilon_{a,a}M_i^*(\epsilon_{x,x}\otimes 1_N)M_j\epsilon_{b,a} = \epsilon_{a,b}M_i^*(\epsilon_{x,x}\otimes 1_N)M_j\epsilon_{a,a} = 0$, since $M_i^*(\epsilon_{x,x}\otimes 1_N)M_j\in D_A$. Therefore,

$$C_{a,b,x,i,j} = (\epsilon_{a,a}-\epsilon_{a,b})M_i^*(\epsilon_{x,x}\otimes 1_N)M_j(\epsilon_{a,a}+\epsilon_{b,a}). \tag{50}$$

Since

$$(\epsilon_{a,a}+\epsilon_{b,a})(\epsilon_{a,a}+\epsilon_{a,b}) = \epsilon_{a,a}+\epsilon_{a,b}+\epsilon_{b,a}+\epsilon_{b,b},$$

by summing over j and setting $d_{x,a,b} = r_{x,a,a}+r_{x,a,b}+r_{x,b,a}+r_{x,b,b}\geq 0$, we obtain

$$\sum_{j=1}^m C_{a,b,x,i,j}C_{a,b,x,i,j}^* = (\epsilon_{a,a}-\epsilon_{a,b})M_i^*(\epsilon_{x,x}\otimes d_{x,a,b})M_i(\epsilon_{a,a}-\epsilon_{b,a}). \tag{51}$$

Let $g_{x,a,b}\in M_N$ satisfy $g_{x,a,b}g_{x,a,b}^* = d_{x,a,b}$ and define

$$D_{a,b,x,i} = (\epsilon_{a,a}-\epsilon_{a,b})M_i^*(\epsilon_{x,x}\otimes g_{x,a,b}). \tag{52}$$

By (51),

$$\sum_{j=1}^m C_{a,b,x,i,j}C_{a,b,x,i,j}^* = D_{a,b,x,i}D_{a,b,x,i}^*.$$

Set $f_{x,a,b} = r_{x,a,a}-r_{x,a,b}-r_{x,b,a}+r_{x,b,b}$ and note that $f_{x,a,b}\geq 0$ and

$$\begin{aligned}
\sum_{i=1}^m D_{a,b,x,i}^*D_{a,b,x,i} &= (\epsilon_{x,x}\otimes g_{x,a,b}^*)M_i(\epsilon_{a,a}-\epsilon_{b,a})(\epsilon_{a,a}-\epsilon_{a,b})M_i^*(\epsilon_{x,x}\otimes g_{x,a,b})\\
&= \epsilon_{x,x}\otimes g_{x,a,b}^*(r_{x,a,a}-r_{x,a,b}-r_{x,b,a}+r_{x,b,b})g_{x,a,b}\\
&= \epsilon_{x,x}\otimes(g_{x,a,b}^*f_{x,a,b}g_{x,a,b}).
\end{aligned} \tag{53}$$

Since π is a *-homomorphism, the element $g_{x,a,b} = r_{x,a,a}+r_{x,a,b}$ satisfies the relation $g_{x,a,b}^*g_{x,a,b} = d_{x,a,b}$. A calculation then shows that $g_{x,a,b}^*f_{x,a,b}g_{x,a,b} = 0$. By (53), $D_{a,b,x,i} = 0$, and by (51), $C_{a,b,x,i,j} = 0$. This forces $\lambda_{x,a,i,j} = \lambda_{x,b,i,j}$ for all $a\neq b$. Hence, $M_i^*(\epsilon_{x,x}\otimes 1_N)M_j\in\mathbb{C}1_A$.

(ii)$\Rightarrow$(i) The assumption implies that $M_i^*(\epsilon_{x,x}\otimes 1_N)M_j\in D_A$, so equations (45)–(48) show that $r_{x,a,a}^{1/2}r_{x,b,b}^{1/2} = 0$, and hence $r_{x,a,a}r_{x,b,b} = 0$, whenever $a\neq b$. Since each $e_{r,a,a}\geq 0$ and $\sum_{a\in A} r_{x,a,a} = 1$, we have that $r_{x,a,a}^2 = r_{x,a,a}$ for each $x\in X$ and $a\in A$. As $\pi(\epsilon_{a,a}) = \sum_{x\in X}\epsilon_{x,x}\otimes r_{x,a,a}$, the diagonal matrix unit $\epsilon_{a,a}$ belongs to the multiplicative domain of Γ for each a. In particular,

$$\Gamma(\epsilon_{a,a})\Gamma(\epsilon_{b,c}) = \delta_{a,b}\Gamma(\epsilon_{a,c}) = \delta_{a,b}\sum_{x\in X}\epsilon_{x,x}\otimes r_{x,a,c}$$

for all $b,c\in A$.

Now, choose $g_{x,a,b}$ with $g_{x,a,b}^* g_{x,a,b} = d_{x,a,b}$, and $h_{x,a,b}$ with $h_{x,a,b}^* h_{x,a,b} = f_{x,a,b}$. Our assumption on Γ implies that $C_{a,b,x,i,j} = 0$ for all $a \neq b$, $x\in X$ and all i and j. By (49)–(53), $g_{x,a,b}^* f_{x,a,b} g_{x,a,b} = 0$, yielding $g_{x,a,b}h_{x,a,b}^* = 0$. Multiplying on the left by $g_{x,a,b}^*$ and on the right by $h_{x,a,b}$, we get $d_{x,a,b}f_{x,a,b} = 0$. Using the fact that $\epsilon_{a,a}$ and $\epsilon_{b,b}$ are in the multiplicative domain of Γ, a calculation shows that

$$0 = d_{x,a,b}f_{x,a,b} = r_{x,a,a} + r_{x,b,b} - r_{x,a,b}r_{x,b,a} - r_{x,b,a}r_{x,b,a}. \tag{54}$$

Multiplying equation (54) on both sides by $r_{x,a,a}$, we get $0 = r_{x,a,a} - r_{x,a,b}r_{x,b,a}$. Therefore, $r_{x,a,b}r_{x,b,a} = r_{x,a,a}$. Similarly, $r_{x,b,a}r_{x,a,b} = r_{x,b,b}$, so that $\epsilon_{a,b}$ belongs to the multiplicative domain of Γ. Since $a,b\in A$ were arbitrary with $a\neq b$, Γ must be a homomorphism, completing the proof. ∎

Remark. We note that an alternative proof of the implications (iii)⇒(ii) in Theorem 5.11 can be given, using Proposition 5.12. We have decided to present the given argument instead as it shows that, it order to conclude that the game φ_G^A has a perfect strategy (of the corresponding class) one does not need to necessarily resort to the fact that Γ has to be a homomorphism.

The next proposition shows the combinatorial meaning of (43).

Proposition 5.13. Let X and A be finite sets, G be a graph with vertex set X, and N be a von Neumann algebra. Let $\Gamma : M_A \to D_X \otimes N$ be a unital *-homomorphism with Kraus representation $\Gamma(T) = \sum_{i=1}^m M_i T M_i^*$ and write $\Gamma(\epsilon_{a,b}) = \sum_{x\in X}\epsilon_{x,x}\otimes r_{x,a,b}$, where $r_{x,a,b}\in N$, $x\in X$, $a,b\in A$. The following are equivalent:

(i) condition (43) holds;
(ii) if $v \sim w$ in G, then $\sum_{a,b\in A} r_{v,a,b}r_{w,b,a} = 0$.

Proof. Let $v\sim w$ in G, and define

$$R_{c,i,j} = \sum_{b\in A}\epsilon_{c,b}M_i^*(\epsilon_{v,w}\otimes 1_N)M_j\epsilon_{b,c}, \quad c\in A,\ i,j\in[m].$$

Set

$$T_{c,i} = \sum_{a\in A}\epsilon_{c,a}F_i(\epsilon_{v,v}\otimes e_{w,a,c}).$$

We have

$$
\begin{aligned}
&\sum_{c\in A}\sum_{i,j=1}^{m} R_{c,i,j}R_{c,i,j}^{*}\\
&= \sum_{a,b,c\in A}\sum_{i,j=1}^{m} \xi_{c,a}M_i^{*}(\xi_{v,w}\otimes 1_N)M_j\xi_{a,c}\xi_{c,b}M_j(\xi_{w,v}\otimes 1_N)M_i\xi_{b,c}\\
&= \sum_{a,b,c\in A}\sum_{i,j=1}^{m} \xi_{c,a}M_i^{*}(\xi_{v,w}\otimes 1_N)M_j\xi_{a,b}M_j(\xi_{w,v}\otimes 1_N)M_i\xi_{b,c}\\
&= \sum_{a,b,c\in A}\sum_{i=1}^{m} \xi_{c,a}M_i^{*}(\xi_{v,w}\otimes 1_N)\pi(\xi_{a,b})(\xi_{w,v}\otimes 1_N)M_i\xi_{b,c}\\
&= \sum_{a,b,c\in A}\sum_{i=1}^{m} \xi_{c,a}M_i^{*}(\xi_{v,v}\otimes r_{w,a,b})M_i\xi_{b,c}\\
&= \sum_{a,b,c\in A}\sum_{i=1}^{m} \xi_{c,a}M_i^{*}(\xi_{v,v}\otimes r_{w,a,c})(\xi_{v,v}\otimes r_{w,c,b})M_i\xi_{b,c}
\end{aligned}
$$

$$
= \sum_{c\in A}\sum_{i=1}^{m} T_{c,i}T_{c,i}^{*}, \tag{55}
$$

On the other hand,

$$
\begin{aligned}
\sum_{c\in A}\sum_{i=1}^{m} T_{c,i}^{*}T_{c,i} &= \sum_{a,b,c\in A}\sum_{i=1}^{m} (\xi_{v,v}\otimes r_{w,c,a})M_i\xi_{a,c}\xi_{c,b}M_i^{*}(\xi_{v,v}\otimes r_{w,b,c})\\
&= \sum_{a,b,c\in A}\sum_{i=1}^{m} (\xi_{v,v}\otimes r_{w,c,a})M_i\xi_{a,b}M_i^{*}(\xi_{v,v}\otimes r_{w,b,c})\\
&= \sum_{a,b,c\in A} (\xi_{v,v}\otimes r_{w,c,a})(\xi_{v,v}\otimes r_{v,a,b})(\xi_{v,v}\otimes r_{w,b,c})
\end{aligned}
$$

$$
= \xi_{v,v}\otimes \sum_{a,b,c\in A} r_{w,c,a}r_{v,a,b}r_{w,b,c}. \tag{56}
$$

Since π is a *-homomorphism,

$$
\sum_{a,b,c\in A} r_{w,c,a}r_{v,a,b}r_{w,b,c} = \left(\sum_{a,c\in A} r_{w,c,a}r_{v,a,c}\right)\left(\sum_{a,c\in A} r_{w,c,a}r_{v,a,c}\right)^{*}. \tag{57}
$$

Considering equations (55)-(57), it follows that condition (i) is equivalent to having $R_{c,i,j} = 0$ for all $c\in A$ and all $i,j\in[m]$. The latter condition is in turn equivalent to the condition $\mathrm{Tr}(M_i^{*}(\xi_{v,w}\otimes 1_N)M_j) = 0$ for all $i,j\in[m]$. The proof is complete. $\square$

5.3. Algebraic versions of the orthogonal rank. Recall that the orthogonal rank $\xi(G)$ of G is the smallest $k \in \mathbb{N}$ for which there exists an orthogonal representation of G in $\mathbb{C}^k$, that is, a collection $(\xi_x)_{x\in X}$ of unit vectors in $\mathbb{C}^k$ such that

$$x \sim y \implies \langle \xi_x, \xi_y \rangle = 0.$$

In this subsection, we discuss algebraic and C*-algebraic versions of the parameter $\xi(G)$. To place this into context, we define the relaxed classical-to-quantum colouring game as the game $\Gamma_G^A : \mathcal{P}^{cl}_{XX} \to \mathcal{P}_{AA}$ determined by the requirements

$$\Gamma_G^A(\epsilon_{x,x} \otimes \epsilon_{y,y}) = \begin{cases} J_A^{\perp} & \text{if } x \sim y, \\ I_{AA} & \text{otherwise.} \end{cases}$$

Let $x \in \{loc, q, qa, qc\}$. We consider the following two parameters:

$$\xi_x(G) = \min\{|A| : \text{there exists a perfect tracial x-strategy for } \Gamma_G^A\},$$

which we call the relaxed orthogonal x-rank of G, and

$$\xi'_x(G) = \min\{|A| : \text{there exists a perfect x-strategy for } \varphi_G^A\},$$

which we call the orthogonal x-rank of G (we set $\xi'_x(G) = \infty$ if there is no perfect strategy for φ_G^A for any A). These parameters were introduced in [30, Subsection 10.1] as quantum versions of the orthogonal rank. We note the following:

(i) Since φ_G^A is more restrictive than Γ_G^A, we have that $\xi_x(G) \leq \xi'_x(G)$;

(ii) By [30, Proposition 10.3], we have $\xi_{loc}(G) = \xi(G)$. On the other hand, if $|A| > 1$ then $\xi'_{loc}(G) = \infty$;

(iii) By [30, Proposition 10.5], $\xi'_q(K_{d^2}) = \xi'_{qc}(K_{d^2}) = d$, and hence $\xi'_q(G) \leq [\sqrt{|X|}] + 1$. By Corollary 5.9, $\xi_q(K_{d^2}) = d$.

Taking into account Remark 3.11, we see that the ideal $\mathcal{I}(\varphi_G^A)$ of $\mathcal{B}_{X,A}$ is given by

$$\mathcal{I}(\varphi_G^A) = \left\langle \left\{ \sum_{a,b\in A} e_{x,a,b}e_{y,b,a} : x \sim y \right\} \right\rangle,$$

that is, $\mathcal{B}(\varphi_G^A)$ is the universal *-algebra generated by matrix unit systems $(e_{x,a,a'})_{a,a'\in A}$, $x \in X$, subject to the relations $\sum_{a,b\in A} e_{x,a,b}e_{y,b,a} = 0$ whenever $x \sim y$. Similarly, $\mathcal{B}(\varphi_G^A)$ is the universal C*-algebra generated by such matrix unit systems, subject to these relations.

Corollary 3.12 implies the following characterisations:

Corollary 5.14. Let G be a graph with vertex set X. Then

(i) The quantum commuting colourings of G correspond to traces of $\mathcal{B}(\varphi_G^A)$. In particular,

$$\xi'_{qc}(G) = \min\{|A| : \mathcal{B}(\varphi_G^A) \text{ possesses a tracial state}\}, \text{ and}$$

(ii) The quantum colourings of G correspond to finite dimensional traces of $\mathcal{B}(\varphi_G^A)$. In particular,

$$\chi_q^0(G) = \min\{|A| : \mathcal{B}(\varphi_G^A) \text{ possesses a finite dim. *-representation}\}.$$

Proof. Suppose that τ is a tracial state on $C_{X,A}$ that annihilates the generators $A_{x,y} = \sum_{a,b\in A} e_{x,a,b}e_{y,b,a}$, $x \sim y$, of $\mathcal{I}(\varphi_G^A)$. Note that

$$A_{x,y}^*A_{x,y} = \sum_{a,b,c,d\in A} e_{y,a,b}e_{x,b,a}e_{x,c,d}e_{y,d,c} = \sum_{b,c,d\in A} e_{y,c,b}e_{x,b,d}e_{y,d,c};$$

it follows that

$$\tau(A_{x,y}^*A_{x,y}) = |A|\tau\left(\sum_{b,d\in A} e_{x,b,d}e_{y,d,b}\right) = 0.$$

Combining this with the Cauchy-Schwartz inequality we obtain the statements. $\square$

Definition 5.15. (i) The algebraic orthogonal rank $\xi_{alg}(G)$ is the smallest cardinality of a set A for which $\mathcal{B}(\varphi_G^A) = \{0\}$; if such A does not exist, set $\xi_{alg}(G) = \infty$;

(ii) The C*-algebraic orthogonal rank $\xi_{C^*}(G)$ is the smallest cardinality of a set A for which $\mathcal{B}(\varphi_G^A) = \{0\}$; if such A does not exist, set $\xi_{C^*}(G) = \infty$.

Proposition 5.16. Let G be a graph with vertex set X. Then $\xi_{C^*}(G) \geq \sqrt{\frac{|X|}{\alpha(G)}}$. Moreover, $\xi_{C^*}(K_{d^2}) = d$.

Proof. If $\xi_{C^*}(G) = \infty$ then the inequality is trivial; assume hence that $\mathcal{B}(\varphi_G^A) \neq \{0\}$. Since $\mathcal{B}(\varphi_G^A)$ is separable, it possesses a faithful state s. Let π be the corresponding GNS representation and ξ the corresponding cyclic vector. Set $E_{x,a,b} = \pi(e_{x,a,b})$ and $\xi_{x,a,b} = E_{x,a,b}\xi$, $x \in X$, $a, b \in A$. The proof of the inequality is now concluded in the same way as the proof of [30, Proposition 10.5].

For the equality, realise $A = \mathbb{Z}_d = \{0, 1, \dots, d-1\}$ and let $X = A \times A$. Let ζ be a primitive $|A|$-th root of unity and, for $x = (a', b')$ and $y = (a'', b'') \in X$, set

$$E_{x,z,z'} = \zeta^{(z'-z)b'} e_{z-a'}e_{z'-a'}^* \in M_A, \quad x = (a', b') \in X, z, z' \in A.$$

For $x = (a', b')$ and $y = (a'', b'')$ with $x \neq y$, we have

$$\begin{aligned}\sum_{z,z'\in A} E_{x,z,z'}E_{y,z',z} &= \sum_{z,z'\in A} \zeta^{(z'-z)b'}\zeta^{(z-z')b''}(e_{z-a'}e_{z'-a'}^*)(e_{z'-a''}e_{z-a''}^*) \\ &= \sum_{z,z'\in A} \delta_{z'-a',z'-a''}\delta_{z-a',z-a''}\zeta^{(z'-z)(b'-b'')} I = 0.\end{aligned}$$

In addition,

$$\begin{aligned} E_{x,z,z'}E_{x,z',z''} &= \zeta^{(z'-z)b'}\zeta^{(z''-z')b'}(e_{z-a'}e^*_{z'-a'})(e_{z'-a'}e^*_{z''-a'}) \\ &= \zeta^{(z''-z)b'}e_{z-a'}e^*_{z''-a'}; \end{aligned}$$

thus, $\mathcal{B}(\varphi^{A}_{K_{d^2}})$ is non-trivial. ∎

As the next proposition shows, the algebraic orthogonal rank can be strictly smaller than the C*-algebraic one.

Proposition 5.17. $\xi_{\mathrm{alg}}(K_{d^2}) = 2$ for all $d \geq 2$.

Proof. We first show that $\xi_{\mathrm{alg}}(K_{d^2}) \leq 2$. The case of $d = 2$ follows from Proposition 5.16, so we assume that $d \geq 3$. By Proposition 5.16, the algebra of the (classical) 4-colouring game for K_{d^2} is non-zero. Hence, there are self-adjoint idempotents $p_{v,w}$ in a non-zero, unital *-algebra $\mathcal{A}$, for $1 \leq v \leq d^2$ and $1 \leq w \leq 4$, such that $\sum_{w=1}^{4} p_{v,w} = 1$ for all v, $p_{v,w}p_{v,z} = 0$ if $w \neq z$, and

$$p_{u,w}p_{v,w} = 0, \quad u \neq v. \tag{58}$$

By Proposition 5.16, the algebra $\mathcal{B}(\varphi^{A}_{K_4})$ is non-zero when $|A| = 2$. Hence, there are elements $e_{x,a,b}$ in a unital $*$-algebra $\mathcal{B}$, for $1 \leq x \leq 4$ and $1 \leq a, b \leq 2$, such that $e_{x,a,b}e_{x,c,d} = \delta_{b,c}e_{x,a,d}$, $e_{x,a,b}^* = e_{x,b,a}$, $\sum_{a=1}^{2} e_{x,a,a} = 1$ and

$$\sum_{a,b=1}^{2} e_{x,a,b}e_{y,b,a} = 0, \quad x \neq y. \tag{59}$$

For $1 \leq v \leq d^2$ and $1 \leq a, b \leq 2$, define

$$f_{v,a,b} = \sum_{w=1}^{4} p_{v,w} \otimes e_{w,a,b} \in \mathcal{A} \otimes \mathcal{B}.$$

We will show that the elements $f_{v,a,b}$ satisfy the requirements of the generators for the classical-to-quantum colouring game for K_{d^2} with $|A| = 2$. Observe that

$$\begin{aligned} f_{v,a,b}f_{v,c,d} &= \sum_{w,z=1}^{4} p_{v,w}p_{v,z} \otimes e_{w,a,b}e_{z,c,d} = \sum_{w=1}^{4} p_{v,w} \otimes e_{w,a,b}e_{w,c,d} \\ &= \delta_{b,c}\sum_{w=1}^{4} p_{v,w} \otimes e_{w,a,d} = \delta_{b,c}f_{v,a,d}. \end{aligned}$$

Since $p_{v,w}^* = p_{v,w}$ and $e_{w,a,b}^* = e_{w,b,a}$, we have $f_{v,a,b}^* = f_{v,b,a}$. In addition,

$$\sum_{a=1}^{2} f_{v,a,a} = \sum_{a=1}^{2}\sum_{w=1}^{4} p_{v,w} \otimes e_{w,a,a} = \sum_{w=1}^{4} p_{v,w} \otimes 1 = 1 \otimes 1.$$

Lastly, using (58) and (59), assuming that $u \neq v$, we have

$$\sum_{a,b=1}^{2} f_{u,a,b} f_{v,b,a} = \sum_{a,b=1}^{2} \sum_{w,z=1}^{4} p_{u,w}p_{v,z} \otimes e_{w,a,b}e_{z,b,a}$$
$$= \sum_{w=1}^{4} p_{u,w}p_{v,w} \otimes 1 = 0.$$

Thus, there is a unital *-homomorphism from $\mathcal{B}(\varphi^{A}_{K_{d^2}})$ to $\mathcal{A}\otimes\mathcal{B}$ with $|A| = 2$, so $\chi_{alg}(K_{d^2}) \leq 2$.

It remains to show that $\chi_{alg}(K_{d^2}) \geq 2$. To this end, we show that $\mathcal{B}(\varphi^{A}_{K_{d^2}}) = \{0\}$ if $|A| = 1$. When $|A| = 1$, the relations defining $\mathcal{B}(\varphi^{A}_{K_{d^2}})$ reduce to having generators $q_{v,a,a}$, $a \in A$, $1 \leq v \leq d^2$, such that $\sum_{a\in A} q_{v,a,a} = 1$, $q^*_{v,a,a} = q_{v,a,a}$ and $q_{v,a,a}q_{w,a,a} = 0$ for $v \neq w$. Since $|A| = 1$, the first relation implies that $q_{v,a,a} = 1$ for all v, a. Then since $n \geq 2$, we may choose $1 \leq v, w \leq d^2$ with $v \neq w$. Since we must have $q_{v,a,a}q_{w,a,a} = 0$, it follows that $1 = 1^2 = 0$ in $\mathcal{B}(\varphi^{A}_{K_{d^2}})$. Hence, $\chi_{alg}(K_{d^2}) \geq 2$. ∎

References

[1] C. Anantharaman-Delaroche, On ergodic theorems for free group actions on noncommutative spaces, Probab. Theory Rel. Fields 135 (2006), 520-546.

[2] A. Bochniak, P. Kasprzak and P. Soltan, Quantum correlations on quantum spaces, preprint (2021), arXiv:2105.07820.

[3] G. Boreland, I. G. Todorov and A. Winter, Sandwich theorems and capacity bounds for non-commutative graphs, J. Combin. Theory Ser. A 177 (2021), 105302, 39 pp.

[4] M. Brannan, A. Chirvasitu, K. Eifler, S. Harris, V. I. Paulsen, X. Su and M. Wasilewski, Bigalois extensions and the graph isomorphism game, Comm. Math. Phys. 375 (2020), no. 3, 1777-1809.

[5] M. Brannan, P. Ganesan and S. J. Harris, The quantum-to-classical graph homomorphism game, preprint (2020), arXiv:2009.07229.

[6] N. P. Brown and N. Ozawa, C*-algebras and finite-dimensional approximations, American Mathematical Society, 2008.

[7] T. Cooney, M. Junge, C. Palazuelos and D. Pérez-García, Rank-one quantum games, Comput. Complexity 24 (2015), no. 1, 133-196.

[8] G. De las Cuevas, T. Drescher and T. Netzer, Quantum magic squares: dilations and their limitations, J. Math. Phys. 61 (2020), no. 11, 111704, 15 pp.

[9] R. Duan, S. Severini and A. Winter, Zero-error communication via quantum channels, non-commutative graphs and a quantum Lovász ϑ function, IEEE Trans. Inf. Theory 59 (2013), no. 2, 1164-1174.

[10] R. Duan and A. Winter, No-signalling assisted zero-error capacity of quantum channels and an information theoretic interpretation of the Lovász number, IEEE Trans. Inf. Theory 62 (2016), no. 2, 891-914.

[11] K. Dykema, V. I. Paulsen and J. Prakash, Non-closure of the set of quantum correlations via graphs, Comm. Math. Phys. 365 (2019), no. 3, 1125-1142.

[12] J. W. Helton, K. P. Meyer, V. I. Paulsen and M. Satriano, Algebras, synchronous games, and chromatic numbers of graphs, New York J. Math. 25 (2019), 328-361.

[13] Z. Ji, A. Natarajan, T. Vidick, J, Wright and H. Yuen, MIP* = RE, preprint (2020), arXiv:2001.04383.
[14] M. Junge, M. Navascues, C. Palazuelos, D. Perez-Garcia, V. Scholz and R. F. Werner, Connes' emnedding problem and Tsirelson's problem, J. Math. Phys. 52, 012102 (2011), 12 pages.
[15] E. Kirchberg, Discrete groups with Kazhdans property T and factorization property are residually finite, Math. Ann. 299 (1994), 35-63.
[16] L. Lovász, On the Shannon capacity of a graph, IEEE Trans. Inf. Theory 25 (1979), no. 1, 1-7.
[17] M. Lupini, L. Mančinska, V. I. Paulsen, D. E. Roberson, G. Scarpa, S. Severini, I. G. Todorov and A. Winter, Perfect strategies for non-local games, Math. Phys. Anal. Geom. 23 (2020), no. 1, Paper No. 7, 31 pp.
[18] L. Mančinska and D. E. Roberson, Graph homomorphisms for quantum players, 9th Conference on the Theory of Quantum Computation, Communication and Cryptography, LIPIcs. Leibniz Int. Proc. Inform. 27 (2014), 212-216.
[19] M. Musat and M. Rørdam, Factorizable maps and traces on the universal free product of matrix algebras, preprint (2019), arXiv:1903.10182.
[20] M. Musat and M. Rørdam, Non-closure of quantum correlation matrices and factorizable channels that require infinite dimensional ancilla. With an appendix by Narutaka Ozawa, Comm. Math. Phys. 375 (2020), no. 3, 1761-1776.
[21] B. Musto, D. Reutter and D. Verdon, A compositional approach to quantum functions, J. Math. Phys. 59 (2018), 081706, 42pp.
[22] N. Ozawa, About the Connes embedding conjecture: algebraic approaches, Jpn. J. Math. 8 (2013), no. 1, 147-183.
[23] V. I. Paulsen, Entanglement and non-locality, Lecture Notes, University of Waterloo, 2016.
[24] V. I. Paulsen, S. Severini, D. Stahlke, I. G. Todorov and A. Winter, Estimating quantum chromatic numbers, J. Funct. Anal. 270 (2016), no. 6, 2188-2222.
[25] V. I. Paulsen and M. Rahaman, Bisynchronous games and factorizable maps, preprint (2019), arXiv:1908.03842.
[26] V. I. Paulsen, I. G. Todorov and M. Tomforde, Operator system structures on ordered spaces, Proc. London Math. Soc. 102 (2011), 25-49.
[27] O. Regev and T. Vidick, Quantum XOR games, ACM Trans. Comput. Theory 7 (2015), no. 4, Art. 15, 43 pp.
[28] P. Soltan Quantum semigroups from synchronous games, J. Math. Phys. 60 (2019), no. 4, 042203, 8pp.
[29] D. Stahlke, Quantum zero-error source-channel coding and non-commutative graph theory, IEEE Trans. Inform. Theory 62 (2016), no. 1, 554-577.
[30] I. G. Todorov and L. Turowska, Quantum no-signalling correlations and non-local games, preprint (2020), arXiv:2009.07016.
[31] L. Trevisan, On Khot's unique games conjecture, Bull. Amer. Math. Soc. (N.S.) 49 (2012), no. 1, 91-111.
[32] J. Watrous, The theory of quantum information, Cambridge University Press, 2018.
[33] N. Weaver, Quantum Graphs as Quantum Relations, Jour. Geom. Anal. (2021), https://doi.org/10.1007/s12220-020-00578-w.

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