

LEBESGUE SPACE ESTIMATES FOR SPHERICAL MAXIMAL FUNCTIONS ON HEISENBERG GROUPS

JORIS ROOS ANDREAS SEEGER RAJULA SRIVASTAVA

ABSTRACT. We prove $L^p \rightarrow L^q$ estimates for local maximal operators associated with dilates of codimension two spheres in Heisenberg groups; these are sharp up to two endpoints. The results can be applied to improve currently known bounds on sparse domination for global maximal operators. We also consider lacunary variants, and extensions to Métivier groups.

1. INTRODUCTION

Let $\mathbb{H}^n = \mathbb{R}^{2n} \times \mathbb{R}$ be the Heisenberg group of real Euclidean dimension $2n + 1$. Writing $x = (\underline{x}, x_{2n+1})$ with $\underline{x} \in \mathbb{R}^{2n}$, the group law is given by

$$x \cdot y = (\underline{x} + \underline{y}, x_{2n+1} + y_{2n+1} + \underline{x}^\top J \underline{y}),$$

where $\underline{x}^\top J \underline{y} = \frac{1}{2} \sum_{j=1}^n (x_{n+j} y_j - x_j y_{n+j})$. A natural dilation structure on \mathbb{H}^n is given by the parabolic dilations $\delta_t(x) = (t\underline{x}, t^2 x_{2n+1})$. These are automorphic, i.e. satisfy $\delta_t(x \cdot y) = \delta_t x \cdot \delta_t y$, and map the horizontal subspace $\mathbb{R}^{2n} \times \{0\}$ into itself.

Let μ be the normalized rotation-invariant measure on the $2n - 1$ dimensional sphere in $\mathbb{R}^{2n} \times \{0\}$, centered at the origin and let μ_t denote its t -dilate defined by $\langle \mu_t, f \rangle = \langle \mu, f \circ \delta_t \rangle$. The spherical means on the Heisenberg group,

$$f * \mu_t(x) = \int_{S^{2n-1}} f(x - t\omega, x_{2n+1} - t\underline{x}^\top J \omega) d\mu(\omega)$$

were introduced by Nevo and Thangavelu [25]. In the theory of generalized Radon transforms, they can be viewed as model operators for which the incidence relation (the support of the Schwartz kernel) has codimension two, in contrast with the classical codimension one spherical means ([32, 31, 10, 12]).

The original interest in [25] was in pointwise convergence and ergodic results and hence in L^p -estimates for the maximal function

$$\mathfrak{M}f = \sup_{t>0} |f * \mu_t|.$$

Date: August 9, 2021.

Research supported in part by NSF grant DMS-1764295.

A sharp result was proved by Müller and the second author [23] and, independently and by a different method, by Narayanan and Thangavelu [24]; namely for $n \geq 2$, the L^p boundedness of \mathfrak{M} holds if and only if $p > \frac{2n}{2n-1}$. It is conjectured that this statement holds true even when $n = 1$ but this problem is currently still open (see [4] for a recent positive result for \mathfrak{M} acting on $L^p(\mathbb{H}^1)$ functions of the form $x \mapsto f_\circ(|x|, x_3)$).

Our renewed interest is prompted by recent work of Bagchi, Hait, Roncal and Thangavelu [3], in which the authors consider (p, q') -sparse domination results for the operator \mathfrak{M} with consequences for weighted inequalities [6]. The primary ingredient in the proof of such a result is an induction argument relying on an $L^p \rightarrow L^q$ estimate for the *local* maximal function

$$Mf = \sup_{t \in I} |f * \mu_t|$$

which is also of independent interest. Here I denotes a compact subinterval of $(0, \infty)$. The objective is to find the best possible value of q in such an inequality. For the sparse bounds one also needs to establish a closely related ε -regularity property, namely an $L^p \rightarrow L_\varepsilon^q(L^\infty(I))$ estimate for the spherical means acting on compactly supported functions (*cf.* (8.2) below). If $q > p$ the operator norm in such estimates will depend on I and it is no loss of generality to assume $I = [1, 2]$. In [3] it is proved that M maps $L^p(\mathbb{H}^n)$ to $L^q(\mathbb{H}^n)$ provided that $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the triangle with corners $(0, 0)$, $(\frac{2n-1}{2n}, \frac{2n-1}{2n})$, $(\frac{3n+1}{3n+7}, \frac{6}{3n+7})$. The authors ask whether this result is essentially sharp, indeed results in [30], [31], [20] for the Euclidean analogues suggest that it is not. In the following theorem we provide $L^p \rightarrow L^q$ bounds that are sharp, possibly except for two endpoints at which we prove restricted weak type inequalities. Implications on sparse bounds for the global operator \mathfrak{M} will be discussed in §8, *cf.* (8.1).

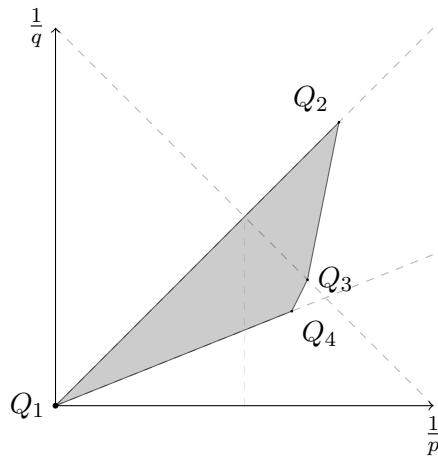


FIGURE 1. The region \mathcal{R} in Theorem 1.1, for $n = 2$.

Theorem 1.1. *Let $n \geq 2$. Let \mathcal{R} be the closed quadrilateral with corners*

$$(1.1) \quad \begin{aligned} Q_1 &= (0, 0), & Q_2 &= \left(\frac{2n-1}{2n}, \frac{2n-1}{2n}\right), \\ Q_3 &= \left(\frac{n}{n+1}, \frac{1}{n+1}\right), & Q_4 &= \left(\frac{2n^2+n}{2n^2+3n+2}, \frac{2n}{2n^2+3n+2}\right). \end{aligned}$$

Then

- (i) M is of restricted weak type (p, q) for all $(\frac{1}{p}, \frac{1}{q}) \in \mathcal{R}$.
- (ii) $M : L^p(\mathbb{H}^n) \rightarrow L^q(\mathbb{H}^n)$ is bounded if $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of \mathcal{R} , or to the open boundary segments (Q_2, Q_3) , (Q_3, Q_4) , or to the half open boundary segments $[Q_1, Q_2)$, $[Q_1, Q_4)$.
- (iii) M does not map $L^p(\mathbb{H}^n)$ to $L^q(\mathbb{H}^n)$ if $(\frac{1}{p}, \frac{1}{q}) \notin \mathcal{R}$.
- (iv) M does not map $L^p(\mathbb{H}^n)$ to $L^p(\mathbb{H}^n)$ for $(\frac{1}{p}, \frac{1}{p}) = Q_2$.

We shall reduce the proof to estimates for standard oscillatory integrals of Carleson-Sjölin-Hörmander type, in particular to a variant of Stein's theorem [33] which was formulated in [22] and which relies on the maximal possible number of nonvanishing curvatures for a cone in the fibers of the canonical relation. It came as a surprise to the authors that such a simple reduction should be possible; as far as we know this has not been observed for maximal functions associated with classes of generalized Radon transforms with incidence relations of codimension two, or higher. We shall now discuss cases with codimension greater than two.

Some extensions. We extend Theorem 1.1 in two directions, already considered in [23]. One extension deals with the situation on \mathbb{H}^n where the subspace $\mathbb{R}^{2n} \times \{0\}$ is replaced by a general subspace transversal to the center; this tilted space is then no longer invariant under the automorphic dilations. Another extension is obtained by replacing the Heisenberg group with other two step nilpotent groups with higher dimensional center; here we will consider the class of Métivier groups [21] which also includes the groups of Heisenberg type [17].

The Lie algebra \mathfrak{g} of a two step nilpotent group G splits as $\mathfrak{g} = \mathfrak{w} \oplus \mathfrak{z}$, so that $[\mathfrak{w}, \mathfrak{z}] = \{0\}$ and $[\mathfrak{w}, \mathfrak{w}] \subset \mathfrak{z}$. The Métivier groups are characterized by a nondegeneracy condition, namely that for every nontrivial linear functional ϑ on \mathfrak{z} , the bilinear form \mathcal{J}^ϑ on $\mathfrak{w} \times \mathfrak{w}$ defined by $\mathcal{J}^\vartheta = \vartheta([X, Y])$ is nondegenerate. This implies that \mathfrak{w} is of even dimension. We set $\dim(\mathfrak{w}) = 2n$, $\dim(\mathfrak{z}) = m$ and let $d = 2n + m$ denote the Euclidean dimension of G . Identifying \mathfrak{w} with \mathbb{R}^{2n} and \mathfrak{z} with \mathbb{R}^m , we use exponential coordinates $x = (\underline{x}, \bar{x}) \in \mathbb{R}^{2n} \times \mathbb{R}^m$; the group multiplication is then given by

$$(1.2) \quad x \cdot y = (\underline{x} + \underline{y}, \bar{x} + \bar{y} + \underline{x}^\top J \underline{y}).$$

Here $\underline{x}^\top J \underline{y} = \sum_{i=1}^m \underline{x}^\top J_i \underline{y} \bar{e}_i \in \mathbb{R}^m$ with $\{\bar{e}_1, \dots, \bar{e}_m\}$ being the standard basis in \mathbb{R}^m and J_1, \dots, J_m denote skew symmetric matrices acting on \mathbb{R}^{2n} . The nondegeneracy condition on \mathcal{J}^ϑ then says that for every $\theta \in \mathbb{R}^m \setminus \{0\}$, the $2n \times 2n$ matrix $J^\theta = \sum_{i=1}^m \theta_i J_i$ is invertible. In the special case of groups

of Heisenberg type we also have $(J^\theta)^2 = -|\theta|^2 I$. We note that for every m there are groups of Heisenberg type with an m -dimensional center. Kaplan [17] points out the connection with Radon-Hurwitz numbers $\rho_{\text{RH}}(k)$ defined as follows ([16, 27]): if $k = (2\ell + 1)2^{4p+q}$ with $q \in \{0, 1, 2, 3\}$ and for some $\ell \in \{0, 1, 2, 3, \dots\}$, then $\rho_{\text{RH}}(k) = 8p + 2^q$. By [27], [17] there are $(2n + m)$ -dimensional groups of Heisenberg type with an m -dimensional center if and only if $m < \rho_{\text{RH}}(2n)$. For odd n , we have $\rho_{\text{RH}}(2n) = 2$, hence $m = 1$.

The automorphic dilations on G are given by $\delta_t(\underline{x}, \bar{x}) = (t\underline{x}, t^2\bar{x})$. We now let \mathfrak{v} be a $2n$ -dimensional subspace of \mathfrak{g} which is transversal to the center, i.e. in exponential coordinates

$$V = \{(\underline{x}, \Lambda \underline{x}) : \underline{x} \in \mathbb{R}^{2n}\},$$

where Λ is an $m \times 2n$ matrix with real entries. Notice that V is invariant under the dilation group $\{\delta_t\}$ only when $\Lambda = 0$. Define a measure $\mu^\Lambda \equiv \mu_1^\Lambda$ supported on V and its automorphic dilates μ_t^Λ by

$$\langle \mu_t^\Lambda, f \rangle = \int_{S^{2n-1}} f(t\omega, t^2\Lambda\omega) d\mu(\omega).$$

We consider the convolution operator $f \mapsto f * \mu_t^\Lambda$ given explicitly by

$$f * \mu_t^\Lambda(x) = \int_{S^{2n-1}} f(\underline{x} - t\omega, \bar{x} - t^2\Lambda\omega - t\underline{x}^\top J\omega) d\mu(\omega)$$

and the associated local maximal function

$$(1.3) \quad Mf = \sup_{t \in I} |f * \mu_t^\Lambda|.$$

These integral operators can be viewed as generalized Radon transforms associated to a family of surfaces of codimension $m + 1$. We note that when $m = 1$ and $\Lambda = 0$, we recover the spherical means on the Heisenberg group considered in Theorem 1.1. Let $\|\cdot\|$ denote the operator norm of a matrix with respect to the Euclidean norm. We state an extension of Theorem 1.1 under the assumption that $\Lambda^\theta = \sum_{i=1}^m \theta_i \Lambda_i$ is sufficiently small.

Theorem 1.2. *Let $n \geq 2$, $d = 2n + m$, let M be as in (1.3), and suppose that Λ satisfies*

$$(1.4) \quad \min_{\theta \in S^{m-1}} [\|(J^\theta)^{-1}\|^{-1} - \|\Lambda^\theta\|] > 0.$$

Let \mathcal{R} be the closed quadrilateral with corners

$$(1.5) \quad \begin{aligned} Q_1 &= (0, 0), & Q_2 &= \left(\frac{d-m-1}{d-m}, \frac{d-m-1}{d-m}\right), \\ Q_3 &= \left(\frac{d-1}{d+m}, \frac{m+1}{d+m}\right), & Q_4 &= \left(\frac{d(d-1)}{d^2+(d+1)m+1}, \frac{(m+1)(d-1)}{d^2+(d+1)m+1}\right). \end{aligned}$$

Then

- (i) M is of restricted weak type (p, q) for all $(\frac{1}{p}, \frac{1}{q}) \in \mathcal{R}$.
- (ii) $M : L^p(\mathbb{H}^n) \rightarrow L^q(\mathbb{H}^n)$ is bounded if $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of \mathcal{R} , or to the open boundary segments (Q_2, Q_3) , (Q_3, Q_4) , or to the half open boundary segments $[Q_1, Q_2)$, $[Q_1, Q_4)$.

Remarks. (i) For $m = 1$ and $\Lambda = 0$, we recover the positive results in Theorem 1.1 for the spherical means on Heisenberg groups.

(ii) Under the smallness condition (1.4), we give an alternative proof of the result for maximal operators in [1] which relied on decoupling estimates to prove local $L^p \rightarrow L^p_{(2n-1)/p'}$ regularity results for the averaging operators acting on compactly supported functions on \mathbb{H}^n , in the range $1 < p < \frac{4n+2}{2n+3}$. Our approach is to use L^p space-time estimates instead. However, the L^p -Sobolev result in [1] is interesting in its own right, and is still needed for the $L^p(\mathbb{H}^n)$ boundedness of the maximal operator in the range $p > \frac{2n}{2n-1}$ if one does not impose any condition on Λ . The use of L^2 space-time estimates is implicit already in the work by Narayanan and Thangavelu [24] who use the group Fourier transform on the Heisenberg group to estimate a relevant square-function involving generalizations of spherical means. L^2 space-time estimates for the relevant Fourier integral operators have also been used in a more recent paper by Joonil Kim [18]; for the basic idea see also the work on variable coefficient Nikodym estimates in [22].

(iii) For $m+3 \leq 2n$, one can use an alternative approach to the $L^{q'} \rightarrow L^q$ estimates which does not require assumption (1.4), *cf.* Remark 5.1. One also obtains the endpoint restricted weak type estimate for the point Q_2 provided that $m+3 < 2n$.

(iv) The example in §6.5 demonstrating the sharpness of the line Q_3Q_4 in the case $m = 1$ seems to be new. It would be interesting to see whether there exists similar examples for $m \geq 2$ and to settle the problem of sharpness for those cases. It would also be interesting to analyze what happens when the size restriction (1.4) on Λ is dropped.

L^p improving estimates for spherical averages. We now discuss another problem considered in [3], concerning sparse bounds for the lacunary spherical maximal function $\sup_k |f * \mu_{2^k}|$ on the Heisenberg groups. Again, essentially sharp sparse bounds (*cf.* (8.4)) follow from essentially sharp results on the $L^p(\mathbb{H}^n) \rightarrow L^q(\mathbb{H}^n)$ boundedness for the averaging operators $f \mapsto f * \mu$ and a closely related ε -regularity property. In [3] a partial result is proved; namely the $L^p(\mathbb{H}^n) \rightarrow L^q(\mathbb{H}^n)$ boundedness holds for $n \geq 2$ if $(\frac{1}{p}, \frac{1}{q})$ belongs to the triangle with corners $(0, 0)$, $(1, 1)$ and $(\frac{3n+1}{3n+4}, \frac{3}{3n+4})$; further, the method of [3] does not seem to yield a result for $n = 1$. Here we prove sharp results for all Heisenberg groups; indeed we formulate a general result for Métivier groups of dimension $d = 2n + m$.

Theorem 1.3. (i) When $n \geq 2$ and $m < 2n - 2$, the inequality

$$(1.6) \quad \|f * \mu^\Lambda\|_{L^q(G)} \lesssim \|f\|_{L^p(G)}$$

holds for all $f \in L^p(G)$ if and only if $(\frac{1}{p}, \frac{1}{q})$ belongs to the closed triangle $\Delta(P_1P_2P_3)$ with $P_1 = (0, 0)$, $P_2 = (1, 1)$ and $P_3 = (\frac{2n+m}{2n+2m+1}, \frac{m+1}{2n+2m+1})$.

(ii) For $m = 2n - 2$, inequality (1.6) holds if $(1/p, 1/q) \in \Delta(P_1 P_2 P_3) \setminus P_3$. It fails for $(\frac{1}{p}, \frac{1}{q}) \notin \Delta(P_1 P_2 P_3)$.

(iii) For $m = 2n - 1$, inequality (1.6) holds if $(\frac{1}{p}, \frac{1}{q})$ lies in the convex hull of $\{(0, 0), (1, 1), (\frac{4m^2+3m+1}{6m^2+5m+1}, \frac{m+1}{3m+1}), (\frac{6m+1}{9m+3}, \frac{3m+2}{9m+3}), (\frac{2m}{3m+1}, \frac{2m^2+2m}{6m^2+5m+1})\}$. This result is sharp at least when $m = 1$.

In contrast to Theorem 1.2, no assumption on Λ is needed; in fact, the estimate for convolution with μ_t^Λ (with *fixed* t) is equivalent to the corresponding inequality for $\Lambda = 0$, as one can see by a change of variable argument involving shear transformations.

The above result be obtained using essentially known results on generalized Radon transforms and oscillatory integral operators with fold singularities (cf. [26], [13], [11]). For $m = 1$ (the Heisenberg case) the pentagon in (iii) reduces to a trapezoid and we get the sharp result

Corollary 1.4. *The inequality*

$$(1.7) \quad \|f * \mu^\Lambda\|_{L^q(\mathbb{H}^n)} \lesssim \|f\|_{L^p(\mathbb{H}^n)}$$

holds for all $f \in L^p(\mathbb{H}^n)$ if and only if one of the following holds:

- (i) $n \geq 2$ and $(\frac{1}{p}, \frac{1}{q})$ belongs to the closed triangle with corners $(0, 0)$, $(1, 1)$ and $(\frac{2n+1}{2n+3}, \frac{2}{2n+3})$.
- (ii) $n = 1$ and $(\frac{1}{p}, \frac{1}{q})$ belongs to the closed trapezoid with corners $(0, 0)$, $(1, 1)$, $(\frac{2}{3}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{3})$.

Remark. In view of the restriction $m < \rho_{\text{RH}}(2n)$, only cases with $m \leq 2n - 1$ occur in Theorem 1.3. Observe that $\rho_{\text{RH}}(4n + 2) = 2$, and ρ_{RH} takes the values 4, 8, 4, 8 for $2n = 4, 8, 12, 16$. Also $\rho_{\text{RH}}(2n) < 2 \log_2(2n) + 3$, hence clearly $\rho_{\text{RH}}(2n) \leq 2n - 2$ for $2n \geq 10$.

The only cases with $m = 2n - 1$ are $(m, 2n + m) = (1, 3), (7, 15)$. In these instances, the codimension $m + 1$ of our sphere in G exceeds half of the dimension of G . The first situation ($m = 1$ and $2n = 2$) corresponds to the Heisenberg group \mathbb{H}^1 , for which the region in part (iii) of Theorem 1.3 is a trapezoid. In this case, we also establish the sharpness of our result.

In the only two cases with $m = 2n - 2$, namely $(m, 2n + m) = (2, 6), (6, 14)$, we do not have a definitive answer for the endpoint $P_3 = (\frac{2n+m}{2n+2m+1}, \frac{m+1}{2n+2m+1})$. All endpoints in all the other cases are covered, since part (i) of the theorem applies.

Further directions. It would also be interesting to investigate $L^p \rightarrow L^q$ mapping properties of maximal functions with respect to arbitrary dilation sets $E \subset [1, 2]$ (see [2, 28] for the Euclidean analogue of this question). We will take up this problem in a subsequent paper [29].

Plan of the paper. The proof of Theorem 1.2 is contained in the next three sections. In §2 we describe the basic estimates and how they can be reduced

to problems about oscillatory integral operators. In §3 and §4 we show how to apply in our context two well known theorems on oscillatory integral operators acting on L^2 functions. Theorem 1.3 will be proved in §5. We establish the necessary conditions in §6 and §7. In §8 we briefly discuss the implications for sparse bounds.

Notation. Partial derivatives will often be denoted by subscripts. P denotes the $(2n-1) \times 2n$ matrix $P = (I_{2n-1} \ 0)$. By $A \lesssim B$ we mean that $A \leq C \cdot B$, where C is a constant and $A \approx B$ signifies that $A \lesssim B$ and $B \lesssim A$. For coefficient vectors $\bar{y} = (\bar{y}_1, \dots, \bar{y}_m)$, and sets of $1 \times 2n$ vectors $\{\Lambda_i\}_{i=1}^m$, or $2n \times 2n$ matrices $\{J_i\}_{i=1}^m$, we abbreviate $\Lambda^{\bar{y}} = \sum_{i=1}^m \bar{y}_i \Lambda_i$ and $J^{\bar{y}} = \sum_{i=1}^m \bar{y}_i J_i$.

2. MAIN ESTIMATES

We use the notation $*_J$ for convolution when the choice of J in (1.2) is emphasized. Let v be a nonnegative bump function on \mathbb{R}^{2n} supported in a neighborhood of e_{2n} , normalized so that $\int_{SO(2n)} v(R^{-1}e_{2n}) dR = 1$; here dR denotes the normalized Haar measure on $SO(2n)$. Then we have $\int_{SO(2n)} v(R^{-1}\omega) dR = 1$ for all $\omega \in S^{2n-1}$, and using this, Fubini's theorem and a change of variables, we can write the convolution as

$$f *_J \mu_t^\Lambda(x) = \int_{R \in SO(2n)} f_R *_R \mu_t^\Lambda [v\mu_t^\Lambda]_t(R^\top \underline{x}, \bar{x}) dR,$$

where $f_R(y) = f(R\underline{y}, \bar{y})$. Note that replacing (J_i, Λ_i) with $(R^\top J_i R, \Lambda_i R)$ does not affect condition (1.4) and therefore, by the integral Minkowski inequality, it suffices to prove our theorems with μ^Λ replaced by $v\mu^\Lambda$.

By a localization argument we may assume that the function f is supported in a small neighborhood of the origin. To see this we use the group translation to tile G . Let $Q_0 = [-\frac{1}{2}, \frac{1}{2})^{2n+m}$ and, for $\mathbf{n} \in \mathbb{Z}^{2n+m}$, let $Q_\mathbf{n} = \mathbf{n} \cdot Q_0$, i.e. $Q_\mathbf{n} = \{(\mathbf{n} + \underline{z}, \bar{\mathbf{n}} + \bar{z} + \underline{\mathbf{n}} J \underline{z}) : z \in Q_0\}$. One then verifies that $\sum_{\mathbf{n} \in \mathbb{Z}^{2n+m}} \mathbf{1}_{Q_\mathbf{n}} = 1$. Moreover, the measures μ_t are supported in $\{w \in G : |w| \leq 2, |\bar{w}| \leq 4\|\Lambda\|\}$, hence in the union of $Q_\mathbf{k}$ with $|\mathbf{k}_j| \leq 2$, $j \leq 2n$, $|\mathbf{k}_{2n+i}| \leq 2 + 4\|\Lambda\|$, $i = 1, \dots, m$. Denote this set of indices by \mathfrak{J} . Then

$$\text{supp}([f \mathbf{1}_{Q_\mathbf{n}}] * \mu_t) \subset \bigcup_{\mathbf{k} \in \mathfrak{J}} (\mathbf{n} \cdot Q_0 \cdot Q_\mathbf{k}) \subset \bigcup_{\tilde{\mathbf{n}} \in \mathfrak{J}(\mathbf{n})} Q_{\tilde{\mathbf{n}}},$$

where $\mathfrak{J}(\mathbf{n})$ is a set of indices $\tilde{\mathbf{n}}$ with $|\mathbf{n}_j - \tilde{\mathbf{n}}_j| \leq C(\Lambda, J, n)$ for $j = 1, \dots, 2n+m$. This consideration allows us to reduce to the case where f is supported in a small neighborhood of the origin.

Splitting $\underline{y} = (y', y_{2n})$ and using the parametrization $\omega = (w', g(w'))$ with $g(w') = \sqrt{1 - |w'|^2}$ near the north pole e_{2n} of the sphere, we are led to consider the generalized Radon transforms associated to the incidence relation given by the equations

$$(2.1) \quad y_{2n} = \mathfrak{S}^{2n}(x, t, y'), \quad \bar{y} = \bar{\mathfrak{S}}(x, t, \underline{y})$$

where

$$(2.2a) \quad \mathfrak{S}^{2n}(x, t, y') = x_{2n} - tg\left(\frac{x' - y'}{t}\right)$$

$$(2.2b) \quad \bar{\mathfrak{S}}(x, t, y) = \bar{x} + t\Lambda(\underline{x} - \underline{y}) + \underline{x}^\top J\underline{y},$$

where y' is small, \underline{x} is near e_{2n} on the support of v and

$$(2.3) \quad g(0) = 1, \quad \nabla g(0) = 0, \quad g''(0) = -I_{2n-1}, \quad g'''(0) = 0.$$

Using (2.1) and (2.2a) to express y_{2n} in (2.2b), we conclude that (2.1) is equivalent with

$$(2.4) \quad \begin{aligned} y_{2n} &= \mathfrak{s}^{2n}(x, t, y') := \mathfrak{S}^{2n}(x, t, y'), \\ \bar{y} &= \bar{\mathfrak{s}}(x, t, y') := \bar{\mathfrak{S}}(x, t, y', \mathfrak{S}^{2n}(x, t, y')). \end{aligned}$$

Recall that $P = (I_{2n-1} \ 0)$. We compute for $i = 1, \dots, m$,

$$\begin{aligned} \bar{\mathfrak{s}}_i(x, t, y') &= \bar{x}_i + t\Lambda_i \underline{x} - t\Lambda_i P^\top y' + \underline{x}^\top J_i P^\top y' \\ &\quad + (x_{2n} - tg\left(\frac{x' - y'}{t}\right))(\underline{x}^\top J_i e_{2n} - t\Lambda_i e_{2n}) \end{aligned}$$

and also have $\mathfrak{s}^{2n}(x, t, y') = x_{2n} - tg\left(\frac{x' - y'}{t}\right)$. We can thus write, for f with small support near 0,

$$f * (v\mu^\Lambda)_t(x) = \int \chi_1(x, t, y') f(y', \mathfrak{s}^{2n}(x, t, y'), \bar{\mathfrak{s}}(x, t, y')) dy',$$

where χ_1 is a smooth and compactly supported function so that on its support y' is small and \underline{x} is near e_{2n} . The right hand side represents an operator with Schwartz kernel

$$K(x, t, y) = \chi_1(x, t, y') \delta_0(\mathfrak{s}^{2n}(x, t, y') - y_{2n}, \bar{\mathfrak{s}}(x, t, y') - \bar{y}),$$

where δ_0 denotes the Dirac measure at the origin in \mathbb{R}^{m+1} . We express δ_0 via the Fourier transform

$$(2.5) \quad K(x, t, y) = \chi_1(x, t, y') \int_{\theta \in \mathbb{R}^{m+1}} e^{i\psi(x, t, y, \theta)} \frac{d\theta}{(2\pi)^{m+1}}$$

with

$$(2.6) \quad \psi(x, t, y, \theta) = \theta_{2n}(\mathfrak{s}^{2n}(x, t, y') - y_{2n}) + \bar{\theta} \cdot (\bar{\mathfrak{s}}(x, t, y') - \bar{y}).$$

Note that K is well defined as an oscillatory integral distribution (indeed from definition (2.4) we see that $x \mapsto K(x, t, y)$ and $y \mapsto K(x, t, y)$ are well defined as oscillatory integral distributions on \mathbb{R}^d).

We now perform a dyadic decomposition of this modified kernel. Let ζ_0 be a smooth radial function on \mathbb{R}^{m+1} with compact support in $\{|\theta| < 1\}$ such that $\zeta_0(\theta) = 1$ for $|\theta| \leq 1/2$. Setting $\zeta_1(\theta) = \zeta_0(\theta/2) - \zeta_0(\theta)$ and $\zeta_k(\theta) = \zeta_1(2^{1-k}\theta)$ for $k \geq 1$, we define

$$A_t^k f(x) = \int \chi_1(x, t, y') \int_{\theta \in \mathbb{R}^{m+1}} \zeta_k(\theta) e^{i\psi(x, t, y, \theta)} \frac{d\theta}{(2\pi)^{m+1}} f(y) dy$$

and let

$$M^k f(x) = \sup_{t \in [1, 2]} |A_t^k f(x)|.$$

The basic estimates for M^k are summarized in the following proposition.

Proposition 2.1. *Assume (1.4) holds.*

(i) *For $1 \leq p \leq \infty$,*

$$(2.7) \quad \|M^k f\|_p \lesssim 2^{\frac{k}{p}} 2^{-k(d-m-1)\min(\frac{1}{p}, \frac{1}{p'})} \|f\|_p.$$

(ii) *For $2 \leq q \leq \infty$,*

$$(2.8) \quad \|M^k f\|_{L^q(\mathbb{R}^d)} \lesssim 2^{k(m+1-\frac{d+m}{q})} \|f\|_{q'}.$$

(iii) *For $q \geq q_5 := \frac{2(d+1)}{d-1}$,*

$$(2.9) \quad \|M^k f\|_{L^q(\mathbb{R}^d)} \lesssim 2^{-k(\frac{d}{q}-\frac{m+1}{2})} \|f\|_2.$$

2.1. *Proof of Theorem 1.2, given Proposition 2.1.* It suffices to show the required bounds for $\mathcal{M}f(x) := \sum_{k \geq 0} M^k f$.

We note that for $n \geq 2, m \geq 1$ (so $d \geq 5$) and $q_5 := \frac{2(d+1)}{d-1}$ we have $\frac{d}{q_5} - \frac{m+1}{2} > 0$ and $m+1 - \frac{d+m}{2} < 0$. To deduce the required restricted weak type estimates for \mathcal{M} at Q_2, Q_3, Q_4 we recall the Bourgain interpolation argument ([7], [8]): Suppose we are given sublinear operators T_k so that for $k \geq 1$,

$$\|T_k\|_{L^{p_0,1} \rightarrow L^{q_0,\infty}} \lesssim 2^{ka_0} \text{ and } \|T_k\|_{L^{p_1,1} \rightarrow L^{q_1,\infty}} \lesssim 2^{-ka_1}$$

for some $p_0, q_0, p_1, q_1 \in [1, \infty], a_0, a_1 > 0$. Then the operator $\sum_{k \geq 1} T_k$ is of restricted weak type (p, q) , where

$$(\frac{1}{p}, \frac{1}{q}, 0) = (1 - \vartheta)(\frac{1}{p_0}, \frac{1}{q_0}, a_0) + \vartheta(\frac{1}{p_1}, \frac{1}{q_1}, -a_1)$$

and $\vartheta = \frac{a_0}{a_0 + a_1} \in (0, 1)$.

The restricted weak type estimate for \mathcal{M} at $Q_2 = (\frac{d-m-1}{d-m}, \frac{d-m-1}{d-m})$ now follows from (2.7). Similarly, the restricted weak type bound at $Q_3 = (\frac{d-1}{d+m}, \frac{m+1}{d+m})$ follows from (2.8). Finally, the restricted weak type bound at $Q_4 = (\frac{1}{p_4}, \frac{1}{q_4})$ with

$$\frac{1}{p_4} = \frac{d(d-1)}{d(d-1)+(d+1)(m+1)}, \quad \frac{1}{q_4} = \frac{(m+1)(d-1)}{d(d-1)+(d+1)(m+1)}$$

follows from interpolating (2.9) for $q = q_5$ with the case $q = \infty$ of (2.8), since for $n = d - m \geq 2$

$$(\frac{1}{p_4}, \frac{1}{q_4}, 0) = \vartheta(\frac{1}{2}, \frac{1}{q_5}, -\frac{d}{q_5} + \frac{m+1}{2}) + (1 - \vartheta)(1, 0, m+1)$$

with $\vartheta = \frac{2(d+1)(m+1)}{d(d-1)+(d+1)(m+1)} \in (0, 1)$. Since bounds for \mathcal{M} imply bounds for M , this concludes the proof of part (i) of Theorem 1.2. Part (ii) is immediate by interpolation.

2.2. *Reduction to space-time bounds.* We use the standard Sobolev inequality

$$(2.10) \quad \sup_{t \in [1,2]} |F(t)| \lesssim \|F\|_p + \|F\|_p^{1/p'} \|F'\|_p^{1/p},$$

where the L^p norms are taken on $[1, 2]$, see [34, p.499]. We apply it to $F(t) = A_t^k f(x) \equiv A^k f(x, t)$, integrate in x and then use Hölder's inequality to obtain Proposition 2.1 as a consequence of the following

Proposition 2.2. *Assume (1.4) holds.*

(i) *For $1 \leq p \leq \infty$*

$$(2.11) \quad \|A^k\|_{L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d \times [1,2])} \lesssim 2^{-k(d-m-1) \min(\frac{1}{p}, \frac{1}{p'})}$$

(ii) *For $2 \leq q \leq \infty$,*

$$(2.12) \quad \|A^k f\|_{L^{q'}(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d \times [1,2])} \lesssim 2^{k(m+1 - \frac{d+m+1}{q})}.$$

(iii) *For $q \geq \frac{2(d+1)}{d-1}$,*

$$(2.13) \quad \|A^k\|_{L^2(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d \times [1,2])} \lesssim 2^{-k(\frac{d+1}{q} - \frac{m+1}{2})} \|f\|_2.$$

(iv) *The same estimates hold for $2^{-k} \frac{d}{dt} A^k$ in place of A^k .*

For later calculations it will be convenient to introduce the nonlinear shear transformation in the x -variables (smoothly depending on t)

$$\begin{aligned} \underline{x}(x, t) &= \underline{x}, \\ \bar{x}_i(x, t) &= \bar{x}_i - t\Lambda_i \underline{x} - x_{2n}(\underline{x}^\top J_i e_{2n} - t\Lambda_i e_{2n}). \end{aligned}$$

By a change of variables it suffices to prove the above space-time inequalities for $A_t^k f(\underline{x}(x, t), t)$ and $2^{-k} \frac{d}{dt} A_t^k f(\underline{x}(x, t), t)$. Using the homogeneity we see that both terms are linear combinations of expressions of the form

$$(2.14) \quad \mathcal{A}^k f(x, t) = 2^{k(m+1)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{m+1}} e^{i2^k \Psi(x, t, y, \theta)} b(x, t, y', \theta) f(y) d\theta dy$$

where the symbol b is compactly supported in $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{d-m-1} \times \mathbb{R}^{m+1}$ with y' near zero and \underline{x} near e_{2n} on the support of b and $|\theta| \in [1/2, 2]$. The phase function Ψ is given by

$$(2.15) \quad \Psi(x, t, y, \theta) = \theta_{2n}(S^{2n}(x, t, y') - y_{2n}) + \sum_{i=1}^m \bar{\theta}_i(\bar{S}_i(x, t, y') - \bar{y}_i)$$

with $(S^{2n}, \bar{S})|_{(x, t, y')} = (\mathfrak{s}^{2n}, \bar{\mathfrak{s}})|_{(\underline{x}(x, t), t, y')}$, that is

$$(2.16) \quad \begin{aligned} S^{2n}(x, t, y') &= x_{2n} - tg(\frac{x' - y'}{t}) \\ \bar{S}_i(x, t, y') &= x_{2n+i} + (\underline{x}^\top J_i - t\Lambda_i)(P^\top y' - tg(\frac{x' - y'}{t})e_{2n}), \end{aligned}$$

with g is as in (2.3). The Schwartz kernel of \mathcal{A}^k is given by

$$(2.17) \quad \mathcal{K}^k(x, t, y) = \int_{\mathbb{R}^{m+1}} e^{i2^k \Psi(x, t, y, \theta)} b(x, t, y', \theta) d\theta$$

and integration by parts yields the estimate

$$(2.18) \quad |\mathcal{K}^k(x, t, y)| \leq C_N \frac{2^{k(m+1)}}{(1 + 2^k|y_{2n} - S^{2n}(x, t, y')| + 2^k|\bar{y} - \bar{S}(x, t, y')|)^N}.$$

This estimate (together with the specific expressions for S^{2n} , \bar{S}) yields for all $k \geq 1$ the bounds

$$(2.19) \quad \|\mathcal{A}_t^k\|_{L^1 \rightarrow L^1} + \|\mathcal{A}_t^k\|_{L^\infty \rightarrow L^\infty} \lesssim 1,$$

$$(2.20) \quad \|\mathcal{A}_t^k\|_{L^1 \rightarrow L^\infty} \lesssim 2^{k(m+1)}.$$

In view of these estimates it suffices in what follows to consider the case of large k . The bounds (2.11), (2.12) then follow by an interpolation argument using (2.19), (2.20) and the local L^2 space-time estimate

$$(2.21) \quad \|\mathcal{A}^k f\|_{L^2(\mathbb{R}^d \times [1, 2])} \lesssim 2^{-k \frac{d-m-1}{2}} \|f\|_2.$$

This gives a gain over the estimate $\|\mathcal{A}_t^k\|_{L^2 \rightarrow L^2} \lesssim 2^{-k(\frac{d-m-1}{2} - \frac{1}{6})}$ for fixed time t established in [23] via estimates for oscillatory integrals with fold singularities in [11]. As mentioned before, the papers [24] and [18] work with similar space-time estimates.

To prove (2.21) we use an oscillatory integral operator

$$T_k f(x, t) = \int_{\mathbb{R}^d} e^{i2^k \Phi(x, t, y)} b(x, t, y) f(y) dy$$

where $b \in C_c^\infty(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d)$ is as in (2.14), and

$$(2.22) \quad \Phi(x, t, y) = y_{2n} S^{2n}(x, t, y') + \sum_{i=1}^m \bar{y}_i \bar{S}_i(x, t, y').$$

Setting $F_k(y) = \int_{\mathbb{R}^{m+1}} f(y', w_{2n}, \bar{w}) e^{-i2^k(y_{2n}w_{2n} + \bar{y} \cdot \bar{w})} dw_{2n} d\bar{w}$ we have

$$\mathcal{A}^k f(x, t) = T_k F_k(x, t)$$

and by Plancherel's theorem $\|F_k\|_2 = (2^{-k} 2\pi)^{(m+1)/2} \|f\|_2$. Hence (2.21) follows from

Proposition 2.3. *Assume (1.4) holds. For all $f \in L^2(\mathbb{R}^d)$,*

$$(2.23) \quad \|T_k f\|_{L^2(\mathbb{R}^d \times [1, 2])} \lesssim 2^{-k \frac{d}{2}} \|f\|_2$$

The proof will be given in §3 using the standard Hörmander L^2 estimate ([15]). By the same argument, the $L^2 \rightarrow L^q$ bound (2.13) is reduced to the estimate

Proposition 2.4. *Assume that (1.4) holds. Then for $q \geq q_5 = \frac{2(d+1)}{d-1}$ and $f \in L^2(\mathbb{R}^d)$,*

$$(2.24) \quad \|T_k f\|_{L^q(\mathbb{R}^d \times [1, 2])} \lesssim 2^{-k \frac{d+1}{q}} \|f\|_2.$$

This will be proved in §4 using a result in [22].

3. PROOF OF PROPOSITION 2.3

By Hörmander's classical L^2 bound ([34, ch. IX.1]) applied after a partition of unity and a slicing argument with a suitable subset of d of the (x, t) -variables, it suffices to prove that the rank of the $(d + 1) \times d$ mixed Hessian matrix $\Phi''_{(x,t),y}$ is equal to d . Equivalently, for

$$(3.1) \quad \Xi(x, t, y) := \nabla_{x,t}\Phi(x, t, y) = y_{2n}\nabla_{x,t}S^{2n} + \sum_{i=1}^m \bar{y}_i \nabla_{x,t}\bar{S}_i,$$

we need to check that (using subscripts to denote partial derivatives)

$$(3.2) \quad \text{rank } (\Xi_{y_1}, \dots, \Xi_{y_d}) = d$$

for every $(x, t, y) \in \text{supp}(b)$; in particular, $|\bar{y}| \approx 1$, and $x' - y'$ is small.

Recall that P denotes the $(2n - 1) \times 2n$ matrix $P = (I_{2n-1} \ 0)$. We calculate

$$\begin{aligned} \Xi(x, t, y) &= y_{2n} \begin{pmatrix} -\nabla g\left(\frac{x'-y'}{t}\right) \\ 1 \\ \vec{0}_m \\ h\left(\frac{x'-y'}{t}\right) \end{pmatrix} \\ &+ \sum_{i=1}^m \bar{y}_i \begin{pmatrix} PJ_i P^\top y' - tg\left(\frac{x'-y'}{t}\right) PJ_i e_{2n} - (\underline{x}^\top J_i e_{2n} - t\Lambda_i e_{2n}) \nabla g\left(\frac{x'-y'}{t}\right) \\ e_{2n}^\top J_i (P^\top y' - tg\left(\frac{x'-y'}{t}\right) e_{2n}) \\ e_i^m \\ h\left(\frac{x'-y'}{t}\right) (\underline{x}^\top J_i - t\Lambda_i) e_{2n} - \Lambda_i (P^\top y' - tg\left(\frac{x'-y'}{t}\right) e_{2n}) \end{pmatrix} \end{aligned}$$

where e_i^m denotes the i -th standard basis vector in \mathbb{R}^m and

$$h(x') = \langle x', \nabla g(x') \rangle - g(x'),$$

with

$$(3.3) \quad h(0) = -1, \nabla h(0) = 0, h''(0) = -I_{2n-1}.$$

The non-degeneracy assumption on J implies that $y_{2n}I_{2n} + J_{\bar{y}}$ is invertible whenever $(y_{2n}, \bar{y}) \neq 0$. More precisely, we have the following auxiliary lemma for its operator norm (taken with respect to the standard Euclidean norm in \mathbb{R}^{2n}); this is a quantitative extension of a lemma in [23].

Lemma 3.1. *Let B be a real skew-symmetric $N \times N$ matrix and let I_N be the $N \times N$ identity matrix.*

(i) *Suppose N is even. Then $\rho I_N + B$ is invertible if and only if either $\rho \neq 0$ or B is invertible. Moreover, for the Euclidean operator norm of the inverse,*

$$(3.4) \quad \|(\rho I_N + B)^{-1}\| = \begin{cases} |\rho|^{-1} & \text{if } \det B = 0 \\ (\rho^2 + \|B^{-1}\|^{-2})^{-1/2} & \text{if } \det B \neq 0 \end{cases}$$

(ii) Suppose that N is odd. Then $\rho I_N + B$ is invertible if and only if $\rho \neq 0$ and we have $\|(\rho I_N + B)^{-1}\| = |\rho|^{-1}$. Moreover $\det(\rho I_N + B) = c(\rho, B)\rho$ where c depends smoothly on ρ, B and $c(\rho, B) \neq 0$ if $\text{rank } B = N - 1$.

Proof. We first consider the case $N = 2n$. When acting on \mathbb{C}^{2n} the skew symmetric matrix B has an orthonormal basis of eigenvectors, with purely imaginary eigenvalues. If v is a complex eigenvector with eigenvalue $i\beta$, then \bar{v} is an eigenvector with eigenvalue $-i\beta$, moreover $B(\text{Re } v) = -\beta \text{Im } v$ and $B(\text{Im } v) = \beta \text{Re } v$. There is then an orthonormal basis u_1, \dots, u_{2n} of \mathbb{R}^{2n} such that $Bu_{2k-1} = -\beta_k u_{2k}$ and $Bu_{2k} = \beta_k u_{2k-1}$. Also, $(\rho I \pm B)u_{2k-1} = \rho u_{2k-1} \mp \beta_k u_{2k}$ and $(\rho I \pm B)u_{2k} = \beta_k u_{2k-1} \pm \rho u_{2k}$. Thus $\rho I \pm B$ are invertible if and only if either $\rho \neq 0$ or $\min_k |\beta_k| \neq 0$.

We have $((\rho I + B)^{-1})^\top (\rho I + B)^{-1} = (\rho^2 I - B^2)^{-1}$, by the skew-symmetry of B . Observe that $\rho^2 I - B^2$ acts on $\mathbb{V}_k := \text{span}\{u_{2k-1}, u_{2k}\}$ as $(\rho^2 + \beta_k^2)I_{\mathbb{V}_k}$ and it follows that

$$\begin{aligned} \|(\rho I + B)^{-1}\| &= \|((\rho I + B)^{-1})^\top (\rho I + B)^{-1}\|^{1/2} \\ &= \max_{k=1, \dots, n} (\rho^2 + \beta_k^2)^{-1/2} = (\rho^2 + \min_{k=1, \dots, n} \beta_k^2)^{-1}. \end{aligned}$$

Since $\|B^{-1}\|^{-1} = \min_k |\beta_k|$ we obtain the claimed expression for the operator norm.

Next consider the case $N = 2n - 1$, $n \geq 2$ (the case $N = 1$ is trivial). The proof uses the same argument as above. We can now find an orthonormal bases u_1, \dots, u_{2n-1} such that $Bu_{2k-1} = -\beta_k u_{2k}$ and $Bu_{2k} = \beta_k u_{2k-1}$ for $k = 1, \dots, n - 1$, and $Bu_{2n-1} = 0$. Let $f(\rho, B) = \det(\rho I + B)$ then $f(0, B) = 0$ since N is odd, moreover $c(\rho, B) = f(\rho, B)/\rho$ is a polynomial in ρ and the entries of B . By the above computation $f(\rho, B) = \rho \prod_{k=1}^{n-1} (\rho^2 + \beta_k^2)^2$. If the rank of B is $N - 1$ then the β_k are nonzero and thus $c(\rho, B) \neq 0$. \square

We proceed to check (3.2). Recall the notation $J^{\bar{y}} = \sum_{i=1}^m \bar{y}_i J_i$ and $\Lambda^{\bar{y}} = \sum_{i=1}^m \bar{y}_i \Lambda_i$. We compute, for $j = 1, \dots, 2n - 1$, the partial derivatives (using $e_{2n}^\top J^{\bar{y}} e_{2n} = 0$),

$$\begin{aligned} \Xi_{y_j} &= \begin{pmatrix} t^{-1}(y_{2n} + (\underline{x}^\top J^{\bar{y}} - t\Lambda^{\bar{y}})e_{2n})\partial_j \nabla g(\frac{x' - y'}{t}) + P J^{\bar{y}}(e_j + \partial_j g(\frac{x' - y'}{t})e_{2n}) \\ e_{2n}^\top J^{\bar{y}}(e_j + \partial_j g(\frac{x' - y'}{t})e_{2n}) \\ \vec{0}_m \\ -t^{-1}(y_{2n} + (\underline{x}^\top J^{\bar{y}} - t\Lambda^{\bar{y}})e_{2n})\partial_j h(\frac{x' - y'}{t}) - \Lambda^{\bar{y}}(e_j + \partial_j g(\frac{x' - y'}{t})e_{2n}) \end{pmatrix}, \\ \Xi_{y_{2n}} &= \begin{pmatrix} -\nabla g(\frac{x' - y'}{t}) \\ 1 \\ \vec{0}_m \\ h(\frac{x' - y'}{t}) \end{pmatrix}, \end{aligned}$$

and, with $\bar{y}_i \equiv y_{2n+i}$,

$$\Xi_{y_{2n+i}} = \begin{pmatrix} PJ_i P^\top y' - tg\left(\frac{x'-y'}{t}\right) PJ_i e_{2n} - (\underline{x}^\top J_i e_{2n} - t\Lambda_i e_{2n}) \nabla g\left(\frac{x'-y'}{t}\right) \\ e_{2n}^\top J_i (P^\top y' - tg\left(\frac{x'-y'}{t}\right) e_{2n}) \\ e_i^m \\ h\left(\frac{x'-y'}{t}\right) (\underline{x}^\top J_i - t\Lambda_i) e_{2n} - \Lambda_i (P^\top y' - tg\left(\frac{x'-y'}{t}\right) e_{2n}) \end{pmatrix}.$$

Let $\Pi : \mathbb{R}^{2n+m+1} \rightarrow \mathbb{R}^{2n+m}$ be the natural projection omitting the time variable $t = x_{2n+m+1}$. Let

$$(3.5) \quad \sigma \equiv \sigma(x, t, y) = y_{2n} + (\underline{x}^\top J^{\bar{y}} - t\Lambda^{\bar{y}}) e_{2n}$$

and let $B = B(x, t, y)$ be the $(2n-1) \times (2n-1)$ matrix

$$B = PJ^{\bar{y}} e_{2n} (\nabla g\left(\frac{x'-y'}{t}\right))^\top$$

with rank at most one. We have

$$\Pi \Xi_y = \begin{pmatrix} t^{-1} \sigma g''\left(\frac{x'-y'}{t}\right) + PJ^{\bar{y}} P^\top + B & -\nabla g\left(\frac{x'-y'}{t}\right) & * \\ e_{2n}^\top J^{\bar{y}} P^\top & 1 & * \\ 0 & 0 & I_m \end{pmatrix}$$

and therefore (using elementary column operations and the skew symmetry of $J^{\bar{y}}$)

$$(3.6) \quad \det \Pi \Xi_y = \det \left(t^{-1} \sigma g''\left(\frac{x'-y'}{t}\right) + PJ^{\bar{y}} P^\top + B - B^\top \right).$$

Since $PJ^{\bar{y}} P^\top + B - B^\top$ is a skew-symmetric $(2n-1) \times (2n-1)$ matrix, we see from Lemma 3.1 that $\Pi \Xi_y$ is invertible if and only if $\sigma \neq 0$. Equivalently $\det \Phi''_{xy} \neq 0$ if and only if $\sigma \neq 0$. If $\sigma = 0$, then $\Pi \Xi_y$ is not invertible and we have to use the t -derivatives.

Remark 3.2. For later reference in §5 we include the following remarks which establish the oscillatory integral operator $f \mapsto T_k f(\cdot, t)$ as an operator with a folding canonical relation (i.e. two-sided fold singularities). We examine the one-dimensional kernel and cokernel of the matrix in (3.6) for $x' = y'$, $\sigma = 0$.

(i) Consider $b = (b', b_{2n}, \bar{b})^\top$ in the kernel. Then $\bar{b} = 0$, $b_{2n} = -e_{2n}^\top J^{\bar{y}} P^\top b'$ and $PJ^{\bar{y}} P^\top b' = 0$ with $b' \neq 0$. This also implies that $e_{2n}^\top J^{\bar{y}} P^\top b' \neq 0$ (since otherwise $P^\top b'$ would be in the kernel of the invertible matrix $J^{\bar{y}}$ and b' would be zero). Let $V_L = \sum_{j=1}^{2n-1} b_j \partial/\partial y_j + b_{2n} \partial/\partial y_{2n}$ with $b_{2n} = -e_{2n}^\top J^{\bar{y}} P^\top b' \neq 0$, then $\frac{\partial \sigma}{\partial y_{2n}} = 1$ and from part (ii) of Lemma 3.1 we get $V_L(\det \Pi \Xi_y) \neq 0$.

(ii) Let $a^\top = (a_1, \dots, a_{2n+m})$ be in the cokernel of $\Pi \Xi_y$. Then a right kernel vector field $V_R = \sum_{j=1}^{2n+m} a_j \partial/\partial x_j$ satisfies $a_{2n} = 0$ (when evaluated at $x' = y'$, $\sigma = 0$), $PJ^{\bar{y}} P^\top a' = 0$ with $a' \neq 0$ and \bar{a} is determined by a' . Note that $V_R \sigma = -e_{2n}^\top J^{\bar{y}} P^\top a' \neq 0$ which leads to $V_R(\det \Pi \Xi_y) \neq 0$.

The case of small σ . We consider \bar{y} in an ε -neighborhood of $\bar{y}_0 \neq 0$, with small $\varepsilon > 0$. We look at the $(2n+m+1) \times (2n+m)$ matrix Ξ_y for $x' = y'$ and small σ . Since $g''(0) = -I_{2n-1}$ and $h(0) = -1$, we have

$$\Xi_y \Big|_{x'=y'} = \begin{pmatrix} -t^{-1}\sigma I_{2n-1} + J^{\bar{y}}P^\top & 0 & * \\ e_{2n}^\top J^{\bar{y}}P^\top & 1 & * \\ 0 & 0 & I_m \\ -\Lambda^{\bar{y}}P^\top & -1 & * \end{pmatrix}$$

We recall that $J^{\bar{y}}$ is invertible for $\bar{y} \neq 0$ (and take \bar{y} near $\bar{y}_0 \neq 0$). This implies that for $\tilde{\Lambda}^{\bar{y}} = (\Lambda^{\bar{y}}P^\top, 0)$ the $2n \times 2n$ matrix with rows $e_1^\top J^{\bar{y}}, \dots, e_{2n-1}^\top J^{\bar{y}}, e_{2n}^\top J^{\bar{y}} - \tilde{\Lambda}^{\bar{y}}$ is invertible. To see this, let $\{c_k\}_{k=1}^{2n}$ be such that $\sum_{k=1}^{2n-1} c_k e_k^\top J^{\bar{y}} + c_{2n} (e_{2n}^\top J^{\bar{y}} - \tilde{\Lambda}^{\bar{y}}) = 0$. This gives $\sum_{k=1}^{2n} c_k e_k^\top = c_{2n} \tilde{\Lambda}^{\bar{y}} (J^{\bar{y}})^{-1}$ and thus $\sum_{k=1}^{2n-1} c_k^2 + c_{2n}^2 (1 - \|\tilde{\Lambda}^{\bar{y}}(J^{\bar{y}})^{-1}\|^2) = 0$. By (1.4), setting $\theta = \bar{y}/|\bar{y}|$,

$$\|\tilde{\Lambda}^{\bar{y}}(J^{\bar{y}})^{-1}\| \leq \|\tilde{\Lambda}^{\bar{y}}\| \|(J^{\bar{y}})^{-1}\| \leq \|\Lambda^{\bar{y}}\| \|(J^{\bar{y}})^{-1}\| < 1$$

which implies that the c_k are all zero.

The preceding consideration also yields that $2n-1$ of the truncated rows $e_1^\top J^{\bar{y}}P^\top, \dots, e_{2n-1}^\top J^{\bar{y}}P^\top, e_{2n}^\top J^{\bar{y}}P^\top - \Lambda^{\bar{y}}P^\top$ are linearly independent. For $\kappa \in \{1, \dots, 2n-1\}$, we let $P^{(\kappa)} : \mathbb{R}^{2n-1} \rightarrow \mathbb{R}^{2n-2}$ denote the map that omits the κ^{th} coordinate. We also let $\Pi^{(\kappa)}$ be the corresponding linear map from \mathbb{R}^{2n+m+1} to \mathbb{R}^{2n+m} that omits the κ^{th} coordinate.

Since the skew symmetric matrix $PJ^{\bar{y}}P^\top$ is not invertible we see that there is a $\kappa \in \{1, \dots, 2n-1\}$ (depending on \bar{y}) such that the $(2n-1) \times (2n-1)$ matrix

$$\begin{pmatrix} P^{(\kappa)}PJ^{\bar{y}}P^\top \\ (e_{2n}^\top J^{\bar{y}} - \Lambda^{\bar{y}})P^\top \end{pmatrix}$$

is invertible. By elementary row operations this implies that

$$\Pi^{(\kappa)}\Xi_y \Big|_{x'=y'} = \begin{pmatrix} -t^{-1}\sigma P^{(\kappa)}PI_{2n} + P^{(\kappa)}PJ^{\bar{y}}P^\top & 0 & * \\ e_{2n}^\top J^{\bar{y}}P^\top & 1 & * \\ 0 & 0 & I_m \\ -\Lambda^{\bar{y}}P^\top & -1 & * \end{pmatrix}$$

is invertible for $\sigma = 0$. The above calculations for $\bar{y} = \bar{y}_0$, $\sigma = 0$ and $x' = y'$ extend by continuity to small choices of $|\sigma|$, $|x' - y'|$ and $|\bar{y} - \bar{y}_0|$, and for these we obtain that $\Pi^{(\kappa)}\Xi_y$ is invertible. This concludes the verification of (3.2) and thus the proof of Proposition 2.3. \square

4. PROOF OF PROPOSITION 2.4

Let $\Xi = \nabla_{x,t}\Phi$ as in (3.1), $N \in \mathbb{R}^{d+1}$ be a unit vector, and let $\mathcal{C}^N(x, t, y)$ be the $d \times d$ curvature matrix with respect to N given by

$$(4.1) \quad \mathcal{C}_{jl}^N = \frac{\partial^2}{\partial y_j \partial y_l} \langle N, \Xi \rangle$$

We apply an oscillatory integral result in [22] according to which Proposition 2.4 holds provided that (3.2) and the additional curvature condition

$$(4.2) \quad \langle N, \Xi_{y_j} \rangle = 0, \quad j = 1, \dots, d \quad \implies \quad \text{rank } \mathcal{C}^N = d - 1$$

is satisfied; i.e. the conic surface $\Sigma_{x,t}$ parametrized by $y \mapsto \Xi(x, t, y)$ has the maximal number $d - 1$ of nonvanishing principal curvatures. It remains to verify (4.2); here we shall use our size assumption (1.4) on Λ .

Let σ be as in (3.5). For $x' = y'$, using the properties of g, h in (2.3), (3.3) we get

$$\begin{aligned} \Xi_{y_j} \Big|_{x'=y'} &= -t^{-1} \sigma e_j + \begin{pmatrix} J^{\bar{y}} e_j \\ \vec{0}_m \\ -\Lambda^{\bar{y}} e_j \end{pmatrix}, \\ \Xi_{y_{2n}} \Big|_{x'=y'} &= \begin{pmatrix} \vec{0}_{2n-1} \\ 1 \\ \vec{0}_m \\ -1 \end{pmatrix}, \quad \Xi_{y_{2n+i}} \Big|_{x'=y'} = \begin{pmatrix} PJ_i P^\top y' - tPJ_i e_{2n} \\ e_{2n}^\top J_i P^\top y' \\ e_i^m \\ (t\Lambda_i - x^\top J_i)e_{2n} - \Lambda_i(P^\top y' - te_{2n}) \end{pmatrix}. \end{aligned}$$

We now consider a unit vector

$$N \Big|_{x'=y'} = (\underline{\alpha}, \bar{\alpha}, \alpha_{d+1})^\top = (\alpha', \alpha_{2n}, \bar{\alpha}, \alpha_{d+1})^\top \in \mathbb{R}^{d+1}$$

perpendicular to Ξ_{y_i} , $\Xi_{y_{2n}}$, $\Xi_{\bar{y}_i}$. Evaluating for $x' = y'$, we get

(4.3a)

$$0 = \langle N, \Xi_{y_j} \rangle \Big|_{x'=y'} = -t^{-1} \sigma \alpha_j + \underline{\alpha}^\top J^{\bar{y}} e_j - \alpha_{d+1} \Lambda^{\bar{y}} e_j, \quad j \leq 2n - 1.$$

(4.3b)

$$0 = \langle N, \Xi_{y_{2n}} \rangle \Big|_{x'=y'} = \alpha_{2n} - \alpha_{d+1},$$

(4.3c)

$$\begin{aligned} 0 = \langle N, \Xi_{\bar{y}_i} \rangle \Big|_{x'=y'} &= \alpha'^\top (PJ_i P^\top y' - tPJ_i e_{2n}) + \alpha_{2n} e_{2n}^\top J_i P^\top y' + \bar{\alpha}_i \\ &\quad + \alpha_{d+1} ((t\Lambda_i - x^\top J_i)e_{2n} - \Lambda_i(P^\top y' - te_{2n})), \quad i = 1, \dots, m. \end{aligned}$$

Equation (4.3c) above expresses $\bar{\alpha}_i$ in terms of $\underline{\alpha}$ and α_{d+1} and turns out to be not really relevant to our calculations. Normalizing $|N| = 1$ we have $|\underline{\alpha}| \approx 1$.

Remark. It is instructive to see that when $\Lambda = 0$ and for the special case of the Heisenberg type group, i.e. when $(J^{\bar{y}})^2 = -|\bar{y}|^2 I$, the projection of the normal vector N to \mathbb{R}^{2n} is tangential to the sphere for $\sigma = 0$, indeed in that case (as we evaluate at the northpole of the sphere with normal vector e_{2n}) we see from (4.3a) that $\underline{\alpha}$ is perpendicular to $\text{span}\{J^{\bar{y}} e_1, \dots, J^{\bar{y}} e_{2n-1}\}$ which contains e_{2n} .

The second derivative vectors are given by

$$\Xi_{y_j y_k} = \begin{pmatrix} -t^{-2} \sigma \partial_{jk} \nabla g\left(\frac{x'-y'}{t}\right) - t^{-1} (PJ^{\bar{y}} e_{2n}) \partial_{jk}^2 g\left(\frac{x'-y'}{t}\right) \\ 0 \\ \vec{0}_m \\ t^{-2} \sigma \partial_{jk}^2 h\left(\frac{x'-y'}{t}\right) + t^{-1} \Lambda^{\bar{y}} e_{2n} \partial_{jk} g\left(\frac{x'-y'}{t}\right) \end{pmatrix},$$

for $1 \leq j, k \leq 2n-1$, and

$$\Xi_{y_j y_k} = 0, \text{ if } 2n \leq j, k \leq 2n+m.$$

Moreover,

$$\Xi_{y_j y_{2n}} = \begin{pmatrix} t^{-1} \partial_j \nabla g\left(\frac{x'-y'}{t}\right) \\ 0 \\ \vec{0}_m \\ -t^{-1} \partial_j h\left(\frac{x'-y'}{t}\right) \end{pmatrix}, \quad 1 \leq j \leq 2n-1,$$

and, for $i = 1, \dots, m$ and $j = 1, \dots, 2n-1$,

$$\Xi_{y_j y_{2n+i}} = \begin{pmatrix} PJ_i e_j + PJ_i e_{2n} \partial_j g\left(\frac{x'-y'}{t}\right) + t^{-1} (\underline{x}^{\top} J_i - t \Lambda_i) e_{2n} \partial_j \nabla g\left(\frac{x'-y'}{t}\right) \\ e_{2n}^{\top} J_i e_j \\ \vec{0} \\ -t^{-1} (\underline{x}^{\top} J_i - t \Lambda_i) e_{2n} \partial_j h\left(\frac{x'-y'}{t}\right) - \Lambda_i (e_j + \partial_j g\left(\frac{x'-y'}{t}\right) e_{2n}) \end{pmatrix},$$

We evaluate at $x' = y'$, using $g''(0) = h''(0) = -I_{2n-1}$, $g'''(0) = 0$, and see that the components of the curvature matrix \mathcal{C}^N at $x' = y'$ are given by

$$\begin{aligned} \langle N, \Xi_{y_j y_j} \rangle \Big|_{x'=y'} &= (\alpha')^{\top} t^{-1} PJ^{\bar{y}} e_{2n} - \alpha_{d+1} t^{-2} \sigma + \alpha_{d+1} (-t \Lambda^{\bar{y}} e_{2n}), \\ &= (\underline{\alpha})^{\top} t^{-1} J^{\bar{y}} e_{2n} - \alpha_{d+1} (t^{-2} \sigma + t^{-1} \Lambda^{\bar{y}} e_{2n}), \\ \langle N, \Xi_{y_j y_k} \rangle \Big|_{x'=y'} &= 0, \quad \text{if } j \neq k, \end{aligned}$$

for $1 \leq j, k \leq 2n-1$. Moreover for $1 \leq j \leq 2n-1$,

$$\begin{aligned} \langle N, \Xi_{y_j y_{2n}} \rangle \Big|_{x'=y'} &= -\alpha_j t^{-1}, \\ \langle N, \Xi_{y_j y_{2n+i}} \rangle \Big|_{x'=y'} &= \underline{\alpha}^{\top} J_i e_j - \alpha_j t^{-1} ((\underline{x}^{\top} J_i - t \Lambda_i) e_{2n}) - \alpha_{d+1} \Lambda_i e_j, \quad 1 \leq i \leq m, \end{aligned}$$

and

$$\langle N, \Xi_{y_j y_k} \rangle \Big|_{x'=y'} = 0, \quad 2n \leq j, k \leq d = 0.$$

Thus we get for the $d \times d$ curvature matrix \mathcal{C}^N ,

$$\mathcal{C}^N \Big|_{x'=y'} = \begin{pmatrix} cI_{2n-1} & PA \\ A^{\top} P^{\top} & 0 \end{pmatrix}$$

where $c = c(t, x, y)$ is given by

$$(4.4) \quad c = t^{-1} \underline{\alpha}^{\top} J^{\bar{y}} e_{2n} - t^{-2} \alpha_{d+1} \sigma - t^{-1} \alpha_{d+1} \Lambda^{\bar{y}} e_{2n}$$

and $A^\top P^\top$ is the $(m+1) \times (2n-1)$ matrix obtained from the following $(m+1) \times 2n$ matrix A^\top by deleting the last column; here

$$A^\top = \begin{pmatrix} -t^{-1}(\underline{\alpha})^\top \\ \underline{\alpha}^\top J_1 - t^{-1}((\underline{x}^\top J_1 - t\Lambda_1)e_{2n})\underline{\alpha}^\top - \alpha_{d+1}\Lambda_1 \\ \vdots \\ \underline{\alpha}^\top J_m - t^{-1}((\underline{x}^\top J_m - t\Lambda_m)e_{2n})\underline{\alpha}^\top - \alpha_{d+1}\Lambda_m \end{pmatrix}.$$

We combine (4.4), (4.3a) and (4.3b) to get

$$(4.5) \quad (-t^{-1}\sigma I + J^{\bar{y}})\underline{\alpha} - \alpha_{2n}(\Lambda^{\bar{y}})^\top = cte_{2n}.$$

Therefore, by Lemma 3.1, and writing $\vartheta = \bar{y}/|\bar{y}|$,

$$\begin{aligned} |c| &= t^{-1}\|(\frac{\sigma}{t}I - J^{\bar{y}})\underline{\alpha} + \alpha_{2n}\Lambda^{\bar{y}}\| = t^{-1}|\bar{y}|\|(\frac{\sigma}{|\bar{y}|t}I - J^{\bar{y}})\underline{\alpha} + \alpha_{2n}\Lambda^{\bar{y}}\| \\ &\geq t^{-1}|\bar{y}|\|\underline{\alpha}\|((\frac{\sigma^2}{|\bar{y}|^2t^2} + \|(J^{\bar{y}})^{-1}\|^{-2})^{1/2} - \|\Lambda^{\bar{y}}\|) \end{aligned}$$

and thus

$$(4.6) \quad |c| \geq t^{-1}|\bar{y}|\|\underline{\alpha}\|(\|(J^{\bar{y}})^{-1}\|^{-1} - \|\Lambda^{\bar{y}}\|)$$

which is bounded away from zero by assumption (1.4).

We finish by verifying that \mathcal{C}^N has rank $d-1 = 2n+m-1$. We have the factorization

$$(4.7) \quad \begin{pmatrix} cI_{2n-1} & PA \\ 0_{m+1,2n-1} & -c^{-1}A^\top P^\top PA \end{pmatrix} = \begin{pmatrix} I_{2n-1} & 0_{2n-1,m+1} \\ -c^{-1}A^\top P^\top & I_{m+1} \end{pmatrix} \begin{pmatrix} cI_{2n-1} & PA \\ A^\top P^\top & 0_{m+1} \end{pmatrix}$$

where PA is an $(2n-1) \times (m+1)$ matrix, I_{2n-1} is the $(2n-1) \times (2n-1)$ identity matrix, I_{m+1} is the $(m+1) \times (m+1)$ identity matrix, 0_{m+1} is the $(m+1) \times (m+1)$ zero matrix and $0_{2n-1,m+1}$ is the $(2n-1) \times (m+1)$ zero matrix.

Thus, the rank of the curvature matrix is $2n-1 + \text{rank}(PA)$ and the rank of PA the same as the rank of the $(2n-1) \times (m+1)$ matrix

$$(4.8) \quad (-t^{-1}P\underline{\alpha} \quad P\underline{v}^{(1)} \quad \dots \quad P\underline{v}^{(m)}) \text{ with } \underline{v}^{(k)} = J_k^\top \underline{\alpha} - \Lambda_k^\top \alpha_{d+1}.$$

We observe that the extended columns $-t^{-1}\underline{\alpha}$, $\underline{v}^{(1)}$, ..., $\underline{v}^{(m)}$ are linearly independent vectors in \mathbb{R}^{2n} . To see this let $(w_{2n}, \bar{w}) \in \mathbb{R}^{1+m}$ be such that $-t^{-1}\underline{\alpha}w_{2n} + \sum_{k=1}^m \underline{v}^{(k)}w_k = 0$. This is equivalent with $-t^{-1}\underline{\alpha}w_{2n} + \sum_{k=1}^m J_k^\top \underline{\alpha}w_k = \sum_{k=1}^m \Lambda_k^\top w_k$. If $\bar{w} = 0$ then we must also have $w_{2n} = 0$ since $\underline{\alpha} \neq 0$. We thus need to show that $\bar{w} \neq 0$ leads to a contradiction. Let $\bar{\omega} = \bar{w}/\|\bar{w}\|$. Since $\alpha_{d+1} = \alpha_{2n}$ we get $\underline{\alpha} = \alpha_{2n}(-t^{-1}\omega_{2n}I - J^{\bar{\omega}})^{-1}(\Lambda^{\bar{\omega}})^\top$ and thus by Lemma 3.1

$$|\underline{\alpha}| \leq (\frac{\omega_{2n}^2}{t^2} + \|(J^{\bar{\omega}})^{-1}\|^{-2})^{-1/2} \|\Lambda^{\bar{\omega}}\| |\alpha_{2n}| \leq \|(J^{\bar{\omega}})^{-1}\| \|\Lambda^{\bar{\omega}}\| |\underline{\alpha}|$$

Since by assumption $\|\Lambda^{\bar{\omega}}\| < \|(J^{\bar{\omega}})^{-1}\|^{-1}$ for $|\bar{\omega}| = 1$ we get $\underline{\alpha} = 0$, a contradiction.

We have thus verified that the $m + 1$ vectors $-t^{-1}\underline{\alpha}$, $\underline{v}^{(1)}$, ..., $\underline{v}^{(m)}$ are linearly independent and hence the rank of the matrix (4.8) is at least m . This proves (4.2) and finishes the proof of Proposition 2.4.

5. PROOF OF THEOREM 1.3

Let $\sigma(x, t, y', \theta_{2n}, \bar{\theta}) = \theta_{2n} + (\underline{x}^\top J^{\bar{\theta}} - t\Lambda^{\bar{\theta}})e_{2n}$ which is comparable to the ‘rotational curvature’ of the fixed time operator. We use the oscillatory representation of the kernel in (2.14) and split $\mathcal{A}_t^k = \sum_{\ell=0}^{[k/3]-1} \mathcal{A}_t^{k,\ell} + \tilde{\mathcal{A}}_t^{k,[k/3]}$ where

(5.1)

$$\mathcal{A}_t^{k,\ell} f(x) = 2^{k(m+1)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{m+1}} e^{i2^k \Psi(x, t, y', \theta_{2n}, \bar{\theta})} b_\ell(x, t, y, \theta) d\theta_{2n} d\bar{\theta} f(y) dy,$$

and b_ℓ is supported where $|\theta_{2n}| \approx 2^{-\ell}$ when $\ell \leq [k/3] - 1$ and supported where $|\theta_{2n}| \lesssim 2^{-k/3}$ if $\ell = [k/3]$.

The operators $\mathcal{A}_t^{k,\ell}$ are bounded on L^1 and L^∞ uniformly in k and ℓ . A trivial kernel estimate yields

$$(5.2) \quad \|\mathcal{A}_t^{k,\ell} f\|_\infty \lesssim 2^{k(m+1)} 2^{-\ell} \|f\|_1$$

We also have the L^2 estimates

$$(5.3) \quad \|\mathcal{A}_t^{k,\ell} f\|_2 \lesssim 2^{-k\frac{d-m-1}{2}} 2^{\ell/2} \|f\|_2$$

for $\ell \leq [k/3]$. These follow, after an application of Plancherel’s theorem, from corresponding bounds for the oscillatory integral operators with phase function Φ as in (2.22)

$$T_{k,\ell} f(x, t) = 2^{k(m+1)} \int_{\mathbb{R}^d} e^{i2^k \Phi(x, t, y)} b_\ell(x, t, y) f(y) dy,$$

namely

$$(5.4) \quad \|T_{k,\ell} f(\cdot, t)\|_2 \lesssim 2^{\ell/2} 2^{-kd/2} \|f\|_2.$$

The estimate (5.4) follows from bounds in [11] (cf. Remark 3.2). Interpolation of the trivial L^1 estimate and (5.3) and summing in ℓ yields an $L^p \rightarrow L^p$ estimate $\|\mathcal{A}_t^k\|_{L^p \rightarrow L^p} = O(2^{-\epsilon(p)k})$ with $\epsilon(p) > 0$ for $1 < p < \infty$.

We may interpolate between (5.2) and (5.3) and obtain

$$(5.5) \quad \|\mathcal{A}_t^{k,\ell} f\|_q \lesssim 2^{k(m+1-\frac{d+m+1}{q})} 2^{\ell(\frac{3}{q}-1)} \|f\|_{q'}, \quad 2 \leq q \leq \infty$$

which implies

$$(5.6) \quad \|\mathcal{A}_t^k f\|_q \leq C_q \begin{cases} 2^{k(m+1-\frac{d+m+1}{q})} \|f\|_{q'}, & 3 < q \leq \infty \\ k 2^{k(m+1-\frac{d+m+1}{q})} \|f\|_{q'}, & q = 3 \\ 2^{k(m+\frac{2}{3}-\frac{d+m}{q})} \|f\|_{q'}, & 2 \leq q < 3 \end{cases}$$

For the case $q = 3$, the Bourgain interpolation trick (as discussed in §2.1) also yields

$$(5.7) \quad \|\mathcal{A}_t^k f\|_{L^{3,\infty}} \lesssim 2^{k(m+1-\frac{d+m+1}{3})} \|f\|_{L^{\frac{3}{2},1}}.$$

In the case $m < 2n - 2$, we have $\frac{d+m+1}{m+1} > 3$ and thus get a uniform estimate for \mathcal{A}_t^k when $q = \frac{d+m+1}{m+1} = \frac{2n+2m+1}{m+1}$. For $m = 2n - 2$, we have $\frac{d+m+1}{m+1} = 3$ and obtain the restricted weak type (q', q) estimate for \mathcal{A}_t^k uniformly in k . For $m = 2n - 1$, we get a uniform $L^{q'} \rightarrow L^q$ -bound when $q = \frac{3(d+m)}{3m+2} = \frac{9m+3}{3m+2}$.

To combine the \mathcal{A}_t^k , we use standard applications of Littlewood-Paley theory, writing $\mathcal{A}_t^k = L_k \mathcal{A}_t^k L_k + E_k$ where the L_k satisfy Littlewood-Paley inequalities

$$\left\| \left(\sum_{k \geq 0} |L_k f|^2 \right)^{1/2} \right\|_r \lesssim \|f\|_r, \quad \left\| \sum_{k \geq 0} L_k f_k \right\|_r \lesssim \left\| \left(\sum_{k \geq 0} |L_k f_k|^2 \right)^{1/2} \right\|_r$$

for $1 < r < \infty$ and the error term E_k has $L^p \rightarrow L^q$ operator norm $O(2^{-k})$ for all $1 \leq p, q \leq \infty$. Since $q' \leq 2 \leq q$, a standard application of Littlewood-Paley inequalities in conjunction with Minkowski's inequalities allows us to deduce the endpoint estimate for P_3 when $m < 2n - 2$ and the $L^{q'} \rightarrow L^q$ bound for $q = \frac{9m+3}{3m+2}$ for the case $m = 2n - 1$. The inequalities for $(1/p, 1/q)$ on the interior parts of the edges $P_1 P_3$ and $P_2 P_3$ follow by interpolation.

When $q = 3$ and $m = 2n - 2$, we still get uniform bounds for \mathcal{A}_t^k on the interiors of $P_1 P_3$ and $P_2 P_3$, and interpolating (5.7) with $L^1 \rightarrow L^1$ and $L^\infty \rightarrow L^\infty$ bounds gives us sharp $L^p \rightarrow L^q$ estimates for \mathcal{A}_t^k on these edges. The above Littlewood-Paley trick still works for those (p^{-1}, q^{-1}) on the open edges which satisfy $p \leq 2 \leq q$ and thus for those (p^{-1}, q^{-1}) we get the $L^p \rightarrow L^q$ boundedness for the averages. A further interpolation finishes the argument.

Remark 5.1. For the case $m + 3 < 2n$ the above estimates (5.5) also give a sharp result for the $L^{q'} \rightarrow L^q$ estimate for the full maximal operator in Theorem 1.2, without imposing the condition (1.4) on Λ . By applying (2.10) for q in place of p we get

$$(5.8) \quad \left\| \sup_{t \in [1,2]} |\mathcal{A}_t^{k,\ell} f| \right\|_q \lesssim 2^{k(m+1-\frac{d+m}{q})} 2^{\ell(\frac{3}{q}-1)} \|f\|_{q'}, \quad 2 \leq q \leq \infty,$$

which implies $\left\| \sup_{t \in [1,2]} |\mathcal{A}_t^k f| \right\|_q \lesssim 2^{k(m+1-\frac{d+m}{q})} \|f\|_{q'}$ for $q > 3$ and hence the $L^{q_0,1} \rightarrow L^{q_0,\infty}$ bound for the maximal operator M for $q_0 = \frac{d+m}{m+1} = \frac{2n+m}{m+1}$, provided that $\frac{d+m}{m+1} > 3$, i.e. $m + 3 < 2n$. Moreover one obtains the $L^{q'} \rightarrow L^q$ bound for M in the range $2 \leq q < \frac{2(n+m)}{m+1}$ if $m + 3 \leq 2n$.

Finally we consider $L^p \rightarrow L^q$ bounds for $p \neq q'$ in the case $m = 2n - 1$. To this end, we will now give a further estimate based on $L^2 \rightarrow L^q$ estimates for oscillatory integral operators in [13].

Proposition 5.2. *For $1 \leq p \leq 2$, $\frac{1}{q} = \frac{d-1}{d}(1 - \frac{1}{p})$,*

$$(5.9) \quad \|\mathcal{A}_t^k f\|_q \lesssim 2^{-k(d-1-\frac{d+m}{p})} \|f\|_p.$$

Proof. This follows by an interpolation between the trivial $L^1 \rightarrow L^\infty$ estimate with operator norm $O(2^{k(m+1)})$ and the $L^2 \rightarrow L^{q_0}$ estimate with $q_0 = \frac{2d}{d-1}$ and operator norm $\lesssim 2^{-k(d/q_0-(m+1)/2)} = 2^{-k(d-m-2)/2}$. The $L^2 \rightarrow L^{q_0}$ bound follows via Plancherel's theorem from the estimate

$$(5.10) \quad \|T_k f(\cdot, t)\|_{q_0} \lesssim 2^{-kd/q_0} \|f\|_2, \quad q_0 = \frac{2d}{d-1}.$$

This in turn is a consequence of [13, Thm. 2.2] once we show that the $d - 1$ dimensional conic variety $\Sigma_{x,t}^{\text{fold}} = \{\nabla_x \Phi(x, t, y) : \sigma(x, t, y) = 0\}$ is a $d - 1$ dimensional cone with $d - 2$ nonvanishing principal curvatures everywhere (with $d = 2n + m$).

Let Ξ be as in (3.1) and let $\Pi\Xi \in \mathbb{R}^d$ be the spatial component of Ξ (omitting the last component from Ξ). Let

$$y_{2n} = \mathfrak{y}_{2n}(\bar{y}) := (t\Lambda^{\bar{y}} - x^\top J^{\bar{y}})e_{2n}$$

denote the solution of the equation $\sigma(x, t, y) = 0$. We define

$$\begin{aligned} \xi(x, t, y', \bar{y}) &= \Pi\Xi(x, t, y', \mathfrak{y}_{2n}(\bar{y}), \bar{y})) \\ &= \begin{pmatrix} PJ^{\bar{y}}P^\top y' - tg(\frac{x'-y'}{t})PJ^{\bar{y}}e_{2n} \\ (t\Lambda^{\bar{y}} - x^\top J^{\bar{y}})e_{2n} + e_{2n}^\top J^{\bar{y}}(P^\top y' - tg(\frac{x'-y'}{t})e_{2n}) \end{pmatrix} \end{aligned}$$

From (3.2) we see that $\xi_{y_1}, \dots, \xi_{y_{2n-1}}, \xi_{y_{2n+1}}, \dots, \xi_{y_{2n+m}}$ are linearly independent, which establishes $\Sigma_{x,t}^{\text{fold}}$ as a manifold of dimension $2n - 1 + m$.

We compute for $j, k \in \{1, \dots, 2n\}$ and $i, l \in \{1, \dots, m\}$,

$$\xi_{y_j y_k} = \begin{pmatrix} -t^{-1} \partial_{jk} g(\frac{x'-y'}{t}) PJ^{\bar{y}} e_{2n} \\ 0 \\ 0 \end{pmatrix}, \quad \xi_{y_j \bar{y}_i} = \begin{pmatrix} PJ_i e_j + \partial_j g(\frac{x'-y'}{t}) PJ_i e_{2n} \\ e_{2n}^\top J_i (e_j + \partial_j g(\frac{x'-y'}{t}) e_{2n}) \\ 0 \end{pmatrix}$$

and $\xi_{\bar{y}_i \bar{y}_l} = 0$.

Define the normal vector ν for $x' = y'$ by $\nu^\top = (\underline{\alpha}^\top, \bar{\alpha}^\top)$, and let $\alpha' = P\underline{\alpha}$; so that $\nu^\top \xi_{y_j} |_{x'=y'} = 0$ for $j = 1, \dots, 2n - 1$ and hence $\underline{\alpha}^\top J^{\bar{y}} e_j = 0$. Since $J^{\bar{y}}$ is invertible this implies that either $\underline{\alpha}^\top J^{\bar{y}} e_{2n} \neq 0$ or $\underline{\alpha} = 0$. But the latter possibility would also imply $\bar{\alpha} = 0$ from the conditions $\langle \nu, \xi_{\bar{y}_i} \rangle = 0$. Hence we have $\gamma := \underline{\alpha}^\top J^{\bar{y}} e_{2n} \neq 0$.

Let \mathcal{C} denote the $(2n-1+m) \times (2n-1+m)$ curvature matrix with respect to the normal ν , with entries $\langle \nu, \xi_{y_j y_k} \rangle$ where $j, k \in \{1, \dots, 2n+m\} \setminus \{2n\}$.

When $x' = y'$ it is given by

$$\mathcal{C}|_{x'=y'} = \langle \nu, \xi''_{yy} \rangle = \begin{pmatrix} -t^{-1}\gamma I_{2n-1} & PM \\ M^\top P^\top & 0 \end{pmatrix} \quad \text{with } \gamma = \underline{\alpha}^\top J^{\bar{y}} e_{2n} \neq 0,$$

where M is the $2n \times m$ matrix with m columns $\sum_{j=1}^{2n} (\alpha^\top J_i e_j) e_j = -J_i \underline{\alpha}$ and hence PM is the $(2n-1) \times m$ matrix with columns $-P J_i \underline{\alpha}$, $i = 1, \dots, m$. Using a lower dimensional version of the factorization (4.7) we see that the rank of \mathcal{C} at $x' = y'$ is equal to the rank of

$$\begin{pmatrix} \gamma I_{2n-1} & PM \\ 0_{m,2n-1} & -\gamma^{-1} M^\top P^\top PM \end{pmatrix}, \quad \gamma = \underline{\alpha}^\top J^{\bar{y}} e_{2n}$$

that is, $\text{rank } \mathcal{C}|_{x'=y'} = 2n-1+m-1 = d-2$. \square

Conclusion of the proof of Theorem 1.3. It remains to finish the argument for the ‘off-diagonal’ estimates in part (iii) of this theorem. Note that an $L^p \rightarrow L^q$ estimate implies an $L^{q'} \rightarrow L^{p'}$ estimate since the dual operator is similar with J replaced by $-J$.

Let $p_1 = \frac{d+m}{d-1}$, and $q_1 = \frac{d(d+m)}{(d-1)(m+1)}$. Since $d = 2n+m \geq m+2$, we have $p_1 \leq 2$ and $q_1 \geq \frac{2d}{d-1}$. Proposition 5.2 yields the $L^{p_1} \rightarrow L^{q_1}$ boundedness of the operators \mathcal{A}_t^k with norm uniform in k . The Littlewood-Paley arguments above also allow us to deduce the $L^{p_1} \rightarrow L^{q_1}$ boundedness of \mathcal{A}_t , since $p_1 \leq 2 \leq q_1$. For $m = 2n-1$, we have $d = 2n+m = 2m+1$, and in this case, $(\frac{1}{p_1}, \frac{1}{q_1}) = (\frac{2m}{3m+1}, \frac{2m^2+2m}{6m^2+5m+1})$ and $(1 - \frac{1}{q_1}, 1 - \frac{1}{p_1}) = (\frac{4m^2+3m+1}{6m^2+5m+1}, \frac{m+1}{3m+1})$. \square

6. NECESSARY CONDITIONS FOR MAXIMAL OPERATORS

We provide five counter-examples, corresponding to each edge of the quadrilateral \mathcal{R} for the Heisenberg group \mathbb{H}^n (in particular, $m = 1$), and one for the point Q_2 . These show the necessity of all the conditions in Theorem 1.1 and of some of the conditions in Theorem 1.2. The first four are suitable modifications of those in [31] for the Euclidean case, which were in turn adapted from standard examples for spherical means and maximal functions. These examples will be presented for all Métivier groups. The fifth example seems to be new; it replaces the Knapp type example in the Euclidean case.

6.1. The line connecting Q_1 and Q_2 . This is the necessary condition $p \leq q$ imposed by translation invariance and noncompactness of the group G (see [14] for the analogous argument in the Euclidean case).

6.2. The line connecting Q_2 and Q_3 . Let B_δ be the ball of radius δ centered at the origin. Let f_δ be the characteristic function of $B_{10\delta}$. Then

$$\|f_\delta\|_p \approx \delta^{(2n+m)/p}.$$

Let $C_\circ := 10(1 + \|\Lambda\| + \max_i \|J_i\|)$. For $1 \leq t \leq 2$ we consider the sets

$$R_{\delta,t} := \{(\underline{x}, \bar{x}) : |\underline{x}| - t \leq \delta/C_\circ, |\bar{x} - t\Lambda \underline{x}| \leq \delta/C_\circ\}.$$

Then $|R_{\delta,t}| \gtrsim \delta^{m+1}$. Let $\Sigma_{x,t} = \{\omega \in S^{2n-1} : |\underline{x} - t\omega| \leq \delta/4\}$ which has spherical measure $\approx \delta^{2n-1}$.

If $x \in R_{\delta,t}$ and $\omega \in \Sigma_{x,t}$ then $|\underline{x} - t\omega| \leq \delta$ and

$$|\bar{x} - t\underline{x}^\top J\omega - t^2\Lambda\omega| \leq |\bar{x} - t\Lambda\underline{x}| + |\underline{x}^\top J(t\omega - \underline{x})| + t|\Lambda(t\omega - \underline{x})| \leq 3\delta$$

(here we have used the skew symmetry of the J_i). We get

$$f_\delta * \sigma_t(\underline{x}, \bar{x}) = \int_{S^{2n-1}} f_\delta(\underline{x} - t\omega, \bar{x} - t\underline{x}^\top J\omega - t^2\Lambda\omega) d\sigma(\omega) \gtrsim \delta^{2n-1}$$

for $x \in R_{\delta,t}$. Passing to the maximal operator we consider $|\underline{x}| \in [1, 2]$ and put $t(x) = |\underline{x}|$. Then setting

$$R_\delta = \{x : 1 \leq |\underline{x}| \leq 2, |\bar{x} - |\underline{x}|\Lambda\underline{x}| \leq \delta/C_\circ\}$$

we have $|R_\delta| \gtrsim \delta^m$ and $|f_\delta * \sigma_{t(x)}(x)| \geq \delta^{2n-1}$ for $x \in R_\delta$.

This yields the inequality

$$\delta^{2n-1}\delta^{m/q} \lesssim \delta^{(2n+m)/p},$$

and consequently, the necessary condition

$$(6.1) \quad \frac{m}{q} + 2n - 1 \geq \frac{2n + m}{p},$$

that is, $(1/p, 1/q)$ lies on or above the line connecting Q_2 and Q_3 .

6.3. The point Q_2 . For $p = p_2 := \frac{2n}{2n-1} = \frac{d-m}{d-m-1}$ the $L^p \rightarrow L^p$ bound fails. Here one uses a modification of Stein's example [32] for the Euclidean spherical maximal function. One considers the function f_α defined by $f_\alpha(\underline{v}, v_{2n+1}) = |\underline{v}|^{-\frac{2n}{p_2}} |\log |v||^{-\alpha}$ for $|\underline{v}| \leq 1/2$, $|v_{2n+1}| \leq 1$ which belongs to L^{p_2} for $\alpha > 1/p_2$. One finds that if $t(\underline{x}) = |\underline{x}|$ then for $\alpha < 1$ the integrals $f * \sigma_{t(x)}(x)$ are ∞ on a set of positive measure. If one choose α close to 1 this also shows that M does not map any of the Lorentz spaces $L^{p_2, q}$ for $q < \infty$ to $L^{p_2, \infty}$.

6.4. The line connecting Q_1 and Q_4 . For this line we just use the counterexample for the individual averaging operators, bounding the maximal function from below by an averaging operator. Given $t \in [1, 2]$, let $g_{\delta,t}$ be the characteristic function of the set $\{(\underline{y}, \bar{y}) : ||\underline{y}|| - t \leq C_0\delta, |\bar{y} - t\Lambda\underline{y}| \leq C_0\delta\}$ with $C_0 = 10 \sum_{i=1}^m \|J_i\|$. Thus $\|g_{\delta,t}\|_p \lesssim \delta^{(m+1)/p}$.

Let $x = (\underline{x}, \bar{x})$ be such that $|\underline{x}| \leq \delta$ and $|\bar{x} - t\Lambda\underline{x}| \leq \delta$. For any $\omega \in S^{2n-1}$, we have that $t|\underline{x}^\top J\omega| \lesssim 2\delta$. Thus

$$\begin{aligned} & ||\underline{x} - t\omega| - t| \leq 2\delta \\ & |\bar{x} - t\underline{x}^\top J\omega - t^2\Lambda\omega - t\Lambda(\underline{x} - t\omega)| \leq |\bar{x} - t\Lambda\underline{x}| + t|\underline{x}^\top J\omega| \leq C_0\delta \end{aligned}$$

implying that $|g_{\delta,t} * \sigma_t(x)| \gtrsim 1$. This yields the inequality $\delta^{(2n+m)/q} \leq \delta^{(m+1)/p}$ which leads to the necessary condition

$$(6.2) \quad \frac{1}{q} \geq \frac{m+1}{2n+m} \frac{1}{p},$$

that is, $(1/p, 1/q)$ lies on or above the line connecting Q_1 and Q_4 .

6.5. *The line connecting Q_3 and Q_4 , $m = 1$.* We now consider the case $m = 1$; after a change of variables we may assume that the skew symmetric matrix J satisfies the Heisenberg condition $J^2 = -I$. Pick a unit vector $u \in \mathbb{R}^{2n}$ so that $\Lambda^\top \in \mathbb{R}u$, and set $v = Ju/\|Ju\|$ (thus $\langle u, v \rangle = 0$). Let $V = \text{span}\{u, v\}$ and let V^\perp denote the orthogonal complement of V in \mathbb{R}^{2n} . Finally, let π, π_\perp be the orthogonal projection to V, V^\perp respectively. Note that J maps V into itself, since $J^2 = -I$, and since J is skew-symmetric it also maps V^\perp into itself.

For sufficiently large C_1 (say, $C_1 = 10(2 + \|\Lambda\|)$) and small $\delta \ll C_1^{-1}$ let

$$Q_\delta = \{(\underline{y}, y_d) : |\pi_\perp(\underline{y})| \leq C_1\delta^{1/2}, |\pi(\underline{y})| \leq C_1\delta, |y_d| \leq C_1\delta\}.$$

Let $f_\delta = \mathbb{1}_{Q_\delta}$, so that $\|f_\delta\|_p^p \lesssim \delta^{(2n-2)/2+3}$, i.e. $\|f_\delta\|_p \lesssim \delta^{(n+2)/p}$.

For $1 \leq t \leq 2$, let

$$R_\delta^t = \{(\underline{x}, x_d) : |\pi_\perp(\underline{x})| \leq \delta^{1/2}, |\pi(\underline{x})| - t| \leq \delta, |x_d - t\Lambda\underline{x}| \leq \delta, \\ 1/4 < \langle \underline{x}, u \rangle, \langle \underline{x}, v \rangle < 3/4\}$$

and let $R_\delta = \cup_{9/8 \leq t \leq 15/8} R_\delta^t$. Then $|R_\delta| \approx \delta^{(2n-2)/2+2}\delta^{-1} = \delta^n$.

For $x \in R_\delta$, we derive a lower bound for $\sigma_{t(x)} * f_\delta(x)$, setting $t(x) := |\pi(\underline{x})|$. Let

$$S_x = \{\omega \in S^{2n-1} : |\pi_\perp(\omega)| \leq \delta^{1/2}, \left| \langle \omega, u \rangle - \frac{\langle \underline{x}, u \rangle}{|\pi(\underline{x})|} \right| \leq \delta, \langle \omega, v \rangle > 0\},$$

which has spherical measure $\gtrsim \delta^{(2n-2)/2+1} = \delta^n$.

For $x \in R_\delta$, $\omega \in S_x$, $t(x) = |\pi(\underline{x})|$ we have

$$(6.3) \quad |\pi_\perp(\underline{x} - t(x)\omega)| \leq 3\delta^{1/2}$$

and

$$|\langle \underline{x} - t(x)\omega, u \rangle| \leq \delta.$$

Since $\omega \in S^{2n-1}$, $\langle \omega, v \rangle > 0$ and $\langle \underline{x}, u \rangle, \langle \underline{x}, v \rangle \in [1/4, 3/4]$, we also get

$$|\langle \underline{x} - t(x)\omega, v \rangle| \leq 2 \left| \frac{\langle \underline{x}, v \rangle}{|\pi(\underline{x})|} - \langle \omega, v \rangle \right| \leq 8 \left| \frac{\langle \underline{x}, v \rangle^2}{|\pi(\underline{x})|^2} - \langle \omega, v \rangle^2 \right| \\ = 8 \left| 1 - \frac{\langle \underline{x}, u \rangle^2}{|\pi(\underline{x})|^2} - 1 + \langle \omega, u \rangle^2 + |\pi_\perp(\omega)|^2 \right| \leq 8 \left| \langle \omega, u \rangle - \frac{\langle \underline{x}, u \rangle}{|\pi(\underline{x})|} \right| + 8\delta^2 \leq 9\delta.$$

Hence

$$(6.4) \quad |\pi(x - t(x)\omega)| \leq 10\delta, \quad \omega \in S_x.$$

Now, since J acts on V and V^\perp ,

$$x^\top J\omega = (x - t(x)\omega)^\top J\omega = (\pi(x - t(x)\omega))^\top J\omega + (\pi_\perp(x - t(x)\omega))^\top J(\pi_\perp\omega)$$

and thus from (6.3) and (6.4).

$$|x^\top J\omega| \leq \delta + 3\delta^{1/2}\delta^{1/2} = 4\delta, \quad \omega \in S_x.$$

From this we finally we obtain, writing $x_d - t^2 \Lambda \omega = x_d - t\Lambda(\underline{x}) + t\Lambda(\underline{x} - t\omega)$ and using that $\Lambda^\top \in V$,

$$|x_d - t(x)\underline{x}^\top J\omega - t(x)^2 \Lambda \omega| \leq |x_d - t(x)\Lambda \underline{x}| + t(x)\|\Lambda\| |\pi(\underline{x} - t(x)\omega)| + 4\delta \leq C\delta.$$

These inequalities imply

$$\begin{aligned} Mf_\delta(x) &\geq f_\delta * \sigma_{t(x)}(x) = \int_{S_x} f_\delta(\underline{x} - t(x)\omega, x_d - t(x)\underline{x}^\top J\omega - t(x)^2 \Lambda \omega) d\sigma(\omega) \\ &\geq |S_x| \gtrsim \delta^n \text{ for } x \in R_\delta. \end{aligned}$$

Hence we get

$$\|Mf_\delta\|_q / \|f_\delta\|_p \gtrsim |R_\delta|^{1/q} \delta^n \delta^{-(n+2)/p} \gtrsim \delta^{n/q + n - (n+2)/p}$$

and letting $\delta \rightarrow 0$, we obtain the necessary condition

$$(6.5) \quad \frac{n}{q} + n \geq \frac{n+2}{p},$$

that is, the necessary condition for $m = 1$ is that $(1/p, 1/q)$ lies on or above the line connecting Q_3 and Q_4 .

7. NECESSARY CONDITIONS FOR AVERAGING OPERATORS

We now prove the necessity of the conditions in Corollary 1.4, and of some of the conditions in Theorem 1.3.

7.1. Necessary condition for $n \geq 2$ and $m \leq 2n-2$. For $n \geq 2$ the sharpness of Theorem 1.3 follows from the considerations in §6. Concerning the line Q_2Q_3 we use the example in §6.2 to get

$$\|f_\delta * \sigma_t\|_q \geq \delta^{2n-1+(m+1)/q-(2n+m)/p} \|f_\delta\|_p$$

which gives the necessary condition $\frac{2n+m}{p} - \frac{m+1}{q} \leq 2n-1$. The calculation in §6.4 only involves the averaging operator and yields the necessary condition $\frac{1}{q} \geq \frac{m+1}{2n+m} \frac{1}{p}$.

7.2. Sharpness for $n = 1$. Here we can assume by a change of variables that $\Lambda = 0$ and that $x^\top J y = x_2 y_1 - x_1 y_2$. We now consider the circular means on $G = \mathbb{H}^1$ given by

$$Af(x) = \int f(x_1 - \cos s, x_2 - \sin s, x_3 - x_2 \cos s + x_1 \sin s) ds.$$

We need to prove the necessary condition

$$(7.1) \quad 6(1/p - 1/q) \leq 1,$$

i.e. $(1/p, 1/q)$ cannot lie below the line connecting the points $(1/2, 1/3)$ and $(2/3, 1/2)$. This is in analogy with the situation for integrals along the moment curve (s, s^2, s^3) in the Euclidean situation of \mathbb{R}^3 ; there the operator is tested on indicator functions of $(\delta, \delta^2, \delta^3)$ -boxes. We show how to modify that example in our situation.

Let f_δ be the indicator function of the parallelepiped

$$P_\delta = \{(y : |y_1| \leq (2\delta)^2, |y_2| \leq 2\delta, |y_3 + y_2| \leq (2\delta)^3\}$$

and

$$V_\delta = \{x : |x_1 - 1| \leq \delta^2, |x_2| \leq \delta, |x_3| \leq \delta^3\}.$$

For $|s| \leq \delta$, and $x \in V_\delta$ we have

$$\begin{aligned} |x_1 - \cos s| &\leq |x_1 - 1| + \frac{s^2}{2} + s^3 \leq 3\delta^2 \\ |x_2 - \sin s| &\leq |x_2| + |\sin s| \leq 2\delta \end{aligned}$$

and

$$\begin{aligned} &|x_3 - x_2 \cos s + x_1 \sin s| + (x_2 - \sin s)| \\ &\leq |x_3| + |x_2||1 - \cos s| + |\sin s||x_1 - 1| \leq 3\delta^3 \end{aligned}$$

Thus if $y = (x_1 - \cos s, x_2 - \sin s, x_3 - x_2 \cos s + x_1 \sin s)$ for $0 \leq s \leq \delta$ and $x \in V_\delta$, then $y \in P_\delta$ and thus $Af_\delta(x) \geq \delta$. Hence

$$\|Af_\delta\|_q \geq |V_\delta|^{1/q} \delta = \delta^{1+6/q}$$

and since $\|f_\delta\|_p = |P_\delta|^{1/p} \lesssim \delta^{6/p}$, we obtain the necessary condition (7.1).

8. IMPLICATIONS FOR SPARSE BOUNDS

As mentioned in the introduction one principal goal of [3] was to derive for the global maximal operator \mathfrak{M} inequalities of the form

$$(8.1) \quad \int_{\mathbb{H}^n} \mathfrak{M}f(x)w(x)dx \leq C \sup \left\{ \Lambda_{\mathcal{S},p_1,p_2}(f, w) : \mathcal{S} \text{ sparse} \right\},$$

where the supremum is taken over *sparse families* of nonisotropic Heisenberg cubes (see [3] for precise definitions and constructions) and the sparse forms $\Lambda_{\mathcal{S},p_1,p_2}$ are given by

$$\Lambda_{\mathcal{S},p_1,p_2}(f, w) = \sum_{S \in \mathcal{S}} |S| \left(\frac{1}{|S|} \int |f|^{p_1} \right)^{1/p_1} \left(\frac{1}{|S|} \int_S |f|^{p_2} \right)^{1/p_2}.$$

Relying entirely on arguments in [3] and using our $L^p \rightarrow L^q$ bounds we can show that the sparse bound (8.1) holds if $(1/p_1, 1 - 1/p_2)$ lies in the interior of the quadrilateral $Q_1Q_2Q_3Q_4$ in (1.1) (or on the open line segment Q_1Q_2), a result which is sharp up to the boundary.

For the proof of sparse bounds for the global maximal operator the relevance of $L^p \rightarrow L^q$ results of localized maximal functions was recognized by Lacey [19] in his work on the Euclidean spherical maximal function. Here we mention that the recent paper [5] gives very general results about this correspondence for the Euclidean geometry; Theorem 1.4 of that paper is of particular relevance here (see also [9] for some results in spaces of homogeneous type). Moreover we refer to [5] for general results about necessary conditions.

For the proof of (8.1) we use the argument in [3]. One needs to supplement the $L^p \rightarrow L^q$ bounds for the local maximal operator by a mild regularity result, namely

$$(8.2) \quad \sup_{|h| \leq 1} |h|^{-\varepsilon} \left\| \sup_{t \in [1,2]} |(\tau_h f - f) *_{\mathcal{J}} \mu_t| \right\|_q \lesssim \|f\|_p$$

for some $\varepsilon > 0$; here τ_h is the right translation operator, i.e. $\tau_h f(y) = f(y - h, \bar{y} - \bar{h} - \underline{y}^\top J_h)$. One also needs to verify a dual condition which in our case is implied by (8.2) and the symmetry of the sphere. If $(1/p, 1/q)$ belongs to the interior of the boundedness region in Theorem 1.2 then our approach yields (8.2) with an $\varepsilon(p, q) > 0$. To prove this one needs to show, by the localization argument in the beginning of §2 and the subsequent dyadic decomposition, that the operator \mathcal{A}^k in (2.14) satisfies

$$(8.3) \quad \|\mathcal{A}^k(\tau_h f - f)\|_{L^q(\mathbb{R}^d \times [1,2])} \lesssim 2^{-k(\frac{1}{q} + a(p,q))} (2^k |h|)^\varepsilon \|f\|_p$$

for $|h| \ll 1$ and functions f supported near the origin, with $a(p, q) > 0$ in the interior of the boundedness region. By taking means it suffices to prove this for $\varepsilon = 0$ and $\varepsilon = 1$. The case for $\varepsilon = 0$ is immediate from the already proven results. For the case $\varepsilon = 1$ we use a change of variables, followed by the fundamental theorem of calculus, and a change of variable again, with the fact that $(\tau_{sh}\underline{y})^\top J_i h = \underline{y}^\top J_i h$ to write

$$\begin{aligned} & 2^{-k(m+1)} \mathcal{A}^k[\tau_h f - f](x, t) \\ &= \int_0^1 \int f(\tau_{sh}\underline{y}) \int e^{i2^k \Psi(x, t, y, \theta)} (2^k \beta_1 + \beta_2) \Big|_{(h, x, t, y, \theta)} d\theta dy ds \end{aligned}$$

with

$$\begin{aligned} \beta_1(h, x, t, y, \theta) &= ib(x, t, y', \theta) [h^\top \nabla_{\underline{y}} \Psi + \bar{h}^\top \nabla_{\bar{y}} \Psi + \underline{y}^\top J^{\nabla_{\bar{y}} \Psi} h]_{(x, t, y, \theta)}, \\ \beta_2(h, x, t, y, \theta) &= (h')^\top \nabla_{y'} b|_{(x, t, y', \theta)}. \end{aligned}$$

Thus, taking into account the explicit form of the phase function (2.15), one can reduce the case for $\varepsilon = 1$ in (8.3) to estimates for operators of the form (2.14) already handled (note that here $\nabla_{\bar{y}} \Psi = -\theta$).

Finally, by similar arguments one gets the regularity result for fixed t ,

$$\|\mathcal{A}_t^k(\tau_h f - f)\|_{L^q(\mathbb{R}^d)} \lesssim 2^{-kb(p,q)} (2^k |h|)^\varepsilon \|f\|_p$$

where $b(p, q) > 0$ in the interior of the boundedness region in Corollary 1.4. Again, using the reasoning in [3] this yields an improved sparse bound for the lacunary maximal function, namely

$$(8.4) \quad \int_{\mathbb{H}^n} \sup_{k \in \mathbb{Z}} |f * \mu_{2^k}(x)| w(x) dx \leq C \sup \{ \Lambda_{\mathcal{S}, p_1, p_2}(f, w) : \mathcal{S} \text{ sparse} \}$$

whenever $(1/p_1, 1 - 1/p_2)$ belongs to the interior of the boundedness region in Corollary 1.4.

Remark. The reader may wonder whether it is necessary to use the sparse bounds as in [3] for the proof of $L^p(\mathbb{H}^n) \rightarrow L^p(\mathbb{H}^n)$ bounds for the lacunary spherical maximal function, for $1 < p \leq \infty$ and $n \geq 1$. We are grateful to both Luz Roncal and an anonymous referee for raising this question. Indeed a more direct proof can be given; one can for example modify the arguments in [23]; alternatively one can rely on a straightforward modification of the Calderón-Zygmund arguments in [1, §6].

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JORIS ROOS: DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF MASSACHUSETTS LOWELL, LOWELL, MA 01854, USA, & SCHOOL OF MATHEMATICS, THE UNIVERSITY OF EDINBURGH, EDINBURGH EH9 3FD, UK

E-mail address: jroos.math@gmail.com

ANDREAS SEEGER: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, 480 LINCOLN DRIVE, MADISON, WI, 53706, USA.

E-mail address: seeger@math.wisc.edu

RAJULA SRIVASTAVA: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, 480 LINCOLN DRIVE, MADISON, WI, 53706, USA.

E-mail address: rsrivastava9@wisc.edu