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Convexified Open-Loop Stochastic Optimal Control for Linear Systems with Log-Concave Disturbances

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***Abstract*—We consider open-loop solutions to the stochastic optimal control of a linear dynamical system with an additive non-Gaussian, log-concave disturbance. We propose a novel, sampling-free approach, based on characteristic functions and convex optimization, to cast the stochastic optimal control problem as a difference-of-convex program. Our method invokes higher moments, resulting in less conservatism compared to moment-based approaches. We employ piecewise affine approximations and the convex-concave procedure for efficient solution via standard conic solvers. We demonstrate that the proposed solution is competitive with sampling and moment based approaches, without compromising probabilistic constraints.**

I. Introduction

Stochastic optimal control requires enforcement of chance constraints, which permit violation of the state constraints with a probability below a specified threshold [1], [2]. However, in the presence of non-Gaussian disturbances, such constraints are hard to implement in a computationally tractable manner analytically. Existing approaches to accommodate non-Gaussian disturbances involve sampling or moment-based methods [1]. Sampling approaches often result in trade-offs between accuracy, feasibility, and computational complexity [3], [4], even with sample reduction techniques [5], [6]. Moment-based approaches [7]–[9] can induce conservatism that significantly reduces the solution space, as well as non-convexities associated with simultaneous risk allocation and controller synthesis [9]–[11].

In this paper, we propose a method for stochastic optimal control of linear systems with log-concave disturbances, that results in a scalable solution, avoiding both moment bounds and sampling. Our approach uses risk allocation, which is employed in moment based methods. Risk allocation uses Boole's inequality to decompose joint chance constraints into simpler, individual chance constraints [9]–[11]. The creation of new decision variables needed to allocate risk across individual chance constraints yields a non-convex problem. Such problems are typically solved via iterative, coordinate descent methods [9], [11], yielding suboptimal solutions with additional conservatism.

We propose an approach for risk allocation that results in a convex formulation, and enables simultaneous (as opposed to iterative) risk allocation and controller synthesis. The key to this is 1) the use of characteristic functions (the Fourier transform of the probability density function) to enforce chance constraints, and 2) a reformulation of risk allocation as a difference-of-convex program. The former enables straightforward calculation of chance constraints, via simple one-dimensional integrals. The latter enables local solutions via convex optimization with clear convergence guarantees [12], when the disturbance is log-concave. Lastly, since convex constraints that arise may be non-conic, we employ piecewise affine approximations, so that standard conic solvers may be used to solve the stochastic optimal control problem.

The primary limitations of our approach are that a) the difference-of-convex reformulation is tractable only for controllers that are limited to open-loop policies, which are more conservative than affine feedback policies, particularly over long time horizons [13, Ch. 2, Sec. 4], and b) open-loop controllers are notoriously resistant to stability guarantees [13], [14]. Although, reference tracking can prove difficult with open-loop control in the presence of uncertainty, it can be facilitated by adding an extra term in the cost function [15], [16] or employing a reference governor [17]. We addess this limitation by presuming systems that are Schur stable, by using an LQR based pre-stabilizing controller that can satisfy input constraints, similarly to [18]. Other approaches to ensure stochastic stability involve augmenting the state vector [19], although extension to non-Gaussian disturbances is unclear [14], [20].

Further, open-loop control has advantages over feedback-based approaches: 1) it admits enforcement of hard input constraints without approximations or additional conservatism [18], [21], [22], 2) it can be used in problems where sensory feedback may be unavailable, such as hypersonic vehicles [23], [24], and 3) it is computationally less expensive than constrained, feedback-based control. Thus, open-loop control synthesis is commonplace in stochastic model predictive control [1] for many of these reasons. In addition, while alternative approaches such as robust control or saturated affine disturbance feedback may yield tractable, convex methodologies for affine controller synthesis, they often require artificially bounding disturbances, or ignoring available information about

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the stochasticity [9], [18], [21], [25]–[27]. These approaches are particularly ill-suited to systems with long-tailed or heavy distributions which would truncate or saturate the true nature of the uncertainty.

The main contribution of this paper is a convex solution for stochastic, open-loop optimal control of linear dynamical systems with log-concave disturbances, that can be solved via conic solvers. Our approach is based on our preliminary work [28], in which we propose a mixed-integer program to solve constrained, stochastic, open-loop optimal control of linear Gaussian systems. This paper extends the approach in [28] to disturbances with log-concave probability distribution functions (pdfs), by employing characteristic functions to evaluate chance constraints. The log-concavity property is critical for efficient computation, because it assures convexity of the chance constraints. Further, we show that we can construct a difference-of-convex reformulation of the risk allocation constraint, which, in combination with piecewise approximation, results in a conic program. This approach is superior to [28], [29], as it does not incur additional conservatism in the risk allocation caused by convex restriction.

The organization of the paper is as follows: We present the problem formulation in Section II. Section III describes the reformulation of the stochastic optimal control problem using risk allocation, piecewise affine approximation, and difference-of-convex programming. We demonstrate our approach on two motion planning examples, and compare performance to state-of-the-art moment based and sampling approaches in Section IV, and summarize our contribution in Section V.

II. Problem statement

We use the following notation in the paper: The discrete-time interval $\mathbb{N}_{[a,b]}$ enumerates all natural numbers from integers a to b. Random vectors have a bold case $\boldsymbol{v}$ and the trace operator is $\mathrm{tr}(\cdot)$.

Consider a stochastic, linear, time-varying system,

$$\boldsymbol{x}(k+1) = A(k)\boldsymbol{x}(k) + B(k)u(k) + D(k)\boldsymbol{w}(k) \tag{1}$$

with state $\boldsymbol{x}(k) \in \mathbb{R}^n$, input $u(k) \in \mathcal{U} \subset \mathbb{R}^m$, and disturbance $\boldsymbol{w}(k) \in \mathbb{R}^p$. We presume that the system matrix $A(k)$ is Schur stable and the set $\mathcal{U}$ is a convex and compact polytope [18]. We consider a random initial condition $\boldsymbol{x}(0) \sim \psi_{\boldsymbol{x}(0)}$ from a known probability density function (pdf) and a time horizon of $N \in \mathbb{N}$. The concatenated disturbance random vector $\boldsymbol{W} = [\boldsymbol{w}(0)^\top \ \boldsymbol{w}(1)^\top \ \cdots \ \boldsymbol{w}(N-1)^\top]^\top \in \mathbb{R}^{pN}$ has a pdf $\psi_{\boldsymbol{W}}$, such that for an independent but not necessarily identical random disturbance process $\boldsymbol{w}(k) \sim \psi_{\boldsymbol{w}}$ with $k \in \mathbb{N}_{[0,N-1]}$, $\psi_{\boldsymbol{W}} = \prod_{k=0}^{N-1} \psi_{\boldsymbol{w}}$. A function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is log-concave, if $\log(f)$ is concave [30, Sec. 3.5.1.]. We follow the convention that $\log(0) \triangleq -\infty$.

Assumption 1. *$\psi_{\boldsymbol{x}(0)}$ and $\psi_{\boldsymbol{W}}$ are log-concave [31, Sec. 2.3].*

Log-concave probability densities include Gaussian and exponential disturbances as well as uniform disturbances over convex sets [32]. Since log-concavity is preserved under products, log-concave ψ_{w_k} yields log-concave ψ_W.

We define the concatenated state vector and concatenated input vector associated with the dynamics (1) as follows:

$$\boldsymbol{X} = \left[\boldsymbol{x}(0)^\top \ \ldots \ \boldsymbol{x}(N)^\top\right]^\top \in \mathbb{R}^{nN}, \tag{2a}$$

$$U = \left[u(0)^\top \ \ldots \ u(N-1)^\top\right]^\top \in \mathcal{U}^N \subset \mathbb{R}^{m(N-1)}. \tag{2b}$$

From (1) and (2), we have

$$\boldsymbol{X} = \bar{A}\boldsymbol{x}(0) + \bar{B}U + \bar{D}\boldsymbol{W} \tag{3}$$

where the matrices $\bar{A} \in \mathbb{R}^{nN\times n}$, $\bar{B} \in \mathbb{R}^{nN\times m(N-1)}$, and $\bar{D} \in \mathbb{R}^{nN\times p(N-1)}$ are obtained from (1) via concatenation [33]. The affine transformation of random vectors from log-concave distributions is log-concave [31, Lemma 2.1]. Due to the linearity of (3), the mean and the covariance vector of $\boldsymbol{X}$ admit closed-form expressions,

$$\mu_{\boldsymbol{X},U} = \bar{A}\mu_{\boldsymbol{x}(0)} + \bar{B}U + \bar{D}\mu_{\boldsymbol{W}} \tag{4a}$$

$$C_{\boldsymbol{X},U} = \bar{A}C_{\boldsymbol{x}(0)}\bar{A}^\top + \bar{D}C_{\boldsymbol{W}}\bar{D}^\top. \tag{4b}$$

We are interested in solving the quadratic tracking problem

$$\begin{aligned} \underset{U}{\text{minimize}} \quad & \mathbb{E}_{\boldsymbol{X}}[(\boldsymbol{X}-X_{\mathrm{d}})^\top Q(\boldsymbol{X}-X_{\mathrm{d}}) + U^\top R U] && \text{(5a)} \\ \text{subject to} \quad & (4), && \\ & U \in \mathcal{U}^N, && \text{(5b)} \\ & \mathbb{P}\{\mathbf{X} \in \mathscr{S}\} \geq 1-\Delta && \text{(5c)} \end{aligned}$$

with positive semi-definite matrices $Q \in \mathbb{R}^{(nN)\times(nN)}$ and $R \in \mathbb{R}^{(m(N-1))\times(m(N-1))}$, a desired trajectory $X_{\mathrm{d}} \in \mathscr{S}$, and polytopic input and state constraints. We define the set $\mathscr{S} = \{X \in \mathbb{R}^{nN} : PX \leq q\}$ with $P = \left[p_1^\top \ \ldots \ p_L^\top\right]^\top \in \mathbb{R}^{L\times nN}$ and $q = \left[q_1 \ \ldots \ q_L\right]^\top \in \mathbb{R}^L$, for $L \in \mathbb{N}$ hyperplanes in the polytope. We presume a probabilistic constraint violation threshold $\Delta \in [0,1)$. The key difference between this problem and those in [10], [11], [28], [34] is that we consider non-Gaussian, log-concave disturbances $\psi_{\mathbf{W}}$.

For a Gaussian disturbance, risk allocation is an established approach to assure (5c). However, under Assumption 1, evaluation of the resulting chance constraints is not straightforward. We propose an approach based in characteristic functions, that is sample and moment-bound free, to solve (5). In contrast to moment based approaches, which employ lower order moments, our approach uses *all* moments of the distribution, and does not require sampling.

We propose to solve two problems:

Problem 1. *Extend risk allocation to log-concave disturbances without moment-based bounds or sampling.*

Problem 2. *Solve* (5) *under Assumption 1 using a convex reformulation that employs the risk allocation technique from Problem 1, in a manner amenable to conic solvers.*

We address problem 1 by using characteristic functions to enforce chance constraints. We address problem 2 through piecewise affine approximations of a reverse convex constraint.

III. Convexification of non-Gaussian joint chance constraints

A. Risk-allocation for log-concave disturbances

The standard risk-allocation approach [10], [11], [28], [34], transforms the joint chance constraints (5c) into a set of individual chance constraints via Boole's inequality. That is, given $\boldsymbol{Z} = \bar{A}\boldsymbol{x}(0) + \bar{D}\boldsymbol{W}$,

$$\begin{aligned} &\mathbb{P}\{P\boldsymbol{X} \le q\} \ge 1-\Delta \qquad (6)\\ \Leftrightarrow &\mathbb{P}\left\{\cap_{i=1}^{L}\left\{p_i^\top \boldsymbol{Z} \le q_i - p_i^\top \bar{B}U\right\}\right\} \ge 1-\Delta\\ \Leftrightarrow &\mathbb{P}\left\{\cup_{i=1}^{L}\left\{p_i^\top \boldsymbol{Z} > q_i - p_i^\top \bar{B}U\right\}\right\} \le \Delta\\ \Leftarrow &\sum_{i=1}^{L}\mathbb{P}\left\{p_i^\top \boldsymbol{Z} > q_i - p_i^\top \bar{B}U\right\} \le \Delta\\ \Leftrightarrow &\begin{cases} \mathbb{P}\left\{p_i^\top \boldsymbol{Z} \le q_i - p_i^\top \bar{B}U\right\} \ge 1-\delta_i,\ \forall i \in \mathbb{N}_{[1,L]},\\ \sum_{i=1}^{L}, \delta_i \le \Delta,\ \delta_i \in [0,\Delta],\ \forall i \in \mathbb{N}_{[1,L]}. \end{cases} \qquad (7)\end{aligned}$$

The risk of violating the constraint $p_i^\top X \le q_i$, $i \in \mathbb{N}_{[1,L]}$ is represented by the decision variable $\delta_i \in [0,1)$. We have $\delta_i \le \Delta$ since $\sum_{i=1}^{L}\delta_i \le \Delta$ and δ_i are non-negative.

Let $\Phi_{p_i^\top \boldsymbol{Z}} : \mathbb{R} \to [0,1]$ denote the cumulative distribution function of the random variable $p_i^\top \boldsymbol{Z}$,

$$\Phi_{p_i^\top \boldsymbol{Z}}(q') = \mathbb{P}\left\{p_i^\top \boldsymbol{Z} \le q'\right\}, \qquad (8)$$

for any scalar $q' \in \mathbb{R}$. We use $\Phi_{p_i^\top \boldsymbol{Z}}$ to rewrite the constraints (7) as

$$\Phi_{p_i^\top \boldsymbol{Z}}\left(q_i - p_i^\top \bar{B}U\right) \ge 1-\delta_i \qquad \forall i \in \mathbb{N}_{[1,L]}, \quad (9a)$$

$$\sum_{i=1}^{L}\delta_i \le \Delta,\ \delta_i \in [0,\Delta], \quad \forall i \in \mathbb{N}_{[1,L]}, \quad (9b)$$

Any feasible controller $U \in \mathcal{U}^N$ with a feasible risk allocation $\delta \triangleq [\delta_1 \ \cdots \ \delta_L] \in [0,1]^L$ that satisfies (9) also satisfies (5c).

B. Enforcing chance constraints using characteristic functions

The characteristic function of the random vector $\boldsymbol{Z}$ with pdf $\psi_{\boldsymbol{Z}}(z)$ is defined as

$$\Psi_{\boldsymbol{Z}}(\bar{\beta}) \triangleq \mathbb{E}_{\boldsymbol{Z}}\left[\exp\left(j\bar{\beta}^\top \boldsymbol{Z}\right)\right] = \int_{\mathbb{R}^p}\exp(j\bar{\beta}^\top z)\psi_{\boldsymbol{Z}}(z)dz \qquad (10)$$

which is the Fourier transform of $\psi_{\boldsymbol{Z}}(z)$ and $\bar{\beta} \in \mathbb{R}^{nN}$. Furthermore, from [35, Eq. 22.6.3], the characteristic function of the random variable $p_i^\top \boldsymbol{Z}$ is given by

$$\Psi_{p_i^\top \boldsymbol{Z}}(\beta) = \Psi_{\boldsymbol{x}(0)}(\bar{A}^\top p_i\beta)\Psi_{\boldsymbol{W}}(\bar{D}^\top p_i\beta) \qquad (11)$$

for some $\beta \in \mathbb{R}$.

The main insight we use in this paper is that the evaluation of the cumulative distribution function in (9a) can be evaluated by a one-dimensional integration, i.e., for any $s \in \mathbb{R}$,

$$\Phi_{p_i^\top \boldsymbol{Z}}(s) = \frac{1}{2} - \frac{1}{2\pi}\int_0^\infty \mathrm{Im}\left(\frac{\exp(j\beta s)\Psi_{\boldsymbol{Z}}(\beta)}{j\beta}\right)d\beta, \qquad (12)$$

where $\mathrm{Im}(z)$ denotes the imaginary component of a complex number z. Equation (12) enables enforcing the chance constraint in (9a) using only $\Psi_{\boldsymbol{Z}}$ as given by (11). Equation (12) follows from the inversion of the characteristic function [36]–[38]. We implement (12) using quadrature techniques which have well defined error bounds [39].

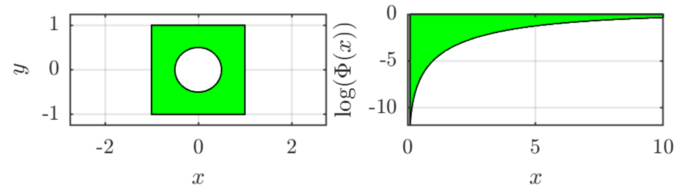

Fig. 1. Left: $f(x,y) = x^2 + y^2 \ge r^2$ within a unit box. Right: The epigraph of $f(x) = \log(\Phi(x))$ of a log-concave cumulative distribution function. Both functions are reverse convex, meaning that the complements of the inequalities, i.e. $x^2 + y^2 \le r^2$ and $\log(\Phi(x)) \ge t$, respectively, are convex.

Lemma 1 ([40, Thm. 4.2.1]). *If $\psi_{p_i^\top \boldsymbol{Z}}$ is log-concave, then $\Phi_{p_i^\top \boldsymbol{Z}}$ is log-concave over $\mathbb{R}$.*

Using (9) and Lemma 1, we approximate (5) as follows,

$$\begin{aligned} \underset{U,t}{\text{minimize}}\ \ & (\mu_{\boldsymbol{X},U} - X_{\mathrm{d}})^\top Q(\mu_{\boldsymbol{X},U} - X_{\mathrm{d}}) + U^\top RU\\ & + \mathrm{tr}(QC_{\boldsymbol{X},U}) \qquad (13a)\\ \text{subject to}\ \ & (5b),\\ \forall i \in \mathbb{N}_{[1,L]},\ & p_i^\top \bar{B}U + \Phi^{-1}_{p_i^\top \boldsymbol{Z}}(\epsilon) \le q_i \qquad (13b)\\ \forall i \in \mathbb{N}_{[1,L]},\ & \log\left(\Phi_{p_i^\top \boldsymbol{Z}}(q_i - p_i^\top \bar{B}U)\right) \ge t_i \qquad (13c)\\ \forall i \in \mathbb{N}_{[1,L]},\ & t_i \in [\log(1-\Delta), 0] \qquad (13d)\\ \forall i \in \mathbb{N}_{[1,L]},\ & \log\left(\sum_{i=1}^{L}\exp(t_i)\right) \ge \log(L-\Delta). \qquad (13e)\end{aligned}$$

for a small scalar $\epsilon > 0$ and a change of variables

$$t_i \triangleq \log(1-\delta_i),\ \forall i \in \mathbb{N}_{[1,L]} \qquad (14)$$

with $t = [t_1 \cdots t_L]^\top \in \mathbb{R}^L$.

We now establish the relationship between (5) and (13), and show that (13) is a non-convex program with a reverse convex constraint. Recall that reverse-convex constraints are optimization constraints of the form $f(\cdot) \ge 0$, where $f(\cdot)$ is a convex function, as shown in Figure 1.

Theorem 1. *Under Assumption 1, the following statements hold for any $\Delta \in [0,1)$ and any $\epsilon > 0$:*

1) *Every feasible solution of* (13) *is feasible for* (5)*, and*
2) *The cost and the constraints* (13b)–(13c) *are convex. However,* (13e) *is a reverse convex constraint.*

Proof: 1) We need to show that satisfaction of (13b)–(13e) satisfies (5c). Recall that the collection of constraints (9) tighten (5c). Therefore, it is sufficient to show that the satisfaction of constraints (13b)–(13e) guarantee satisfaction of (9).

The constraint (13b) ensures that the constraint (13c) is well-defined, since the satisfaction of (13b) ensures that $\Phi_{p_i^\top \boldsymbol{Z}}(q_i - p_i^\top \bar{B}U)$ is positive. Under (14), satisfaction of (13c) and (13d) implies satisfaction of (9a) and $\delta_i \in [0,\Delta]$, respec-

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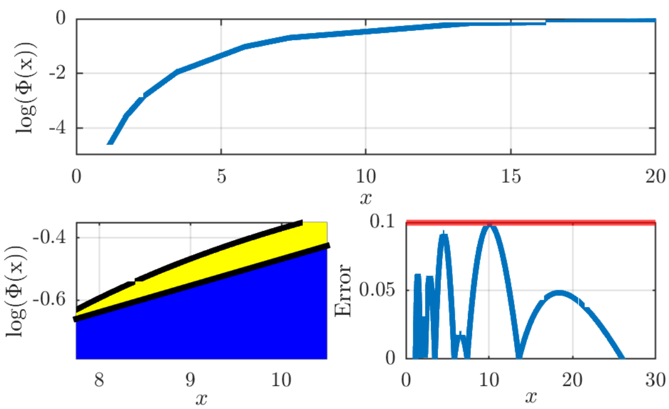

Fig. 2. Top: Log of the cumulative distribution function of an affine transformation of a random vector $a^\top \boldsymbol{w}_t$, with $\boldsymbol{w}_t = [\boldsymbol{w}_1\ \boldsymbol{w}_2\ \boldsymbol{w}_3]^\top \in \mathbb{R}^3$ and scale parameters $\overline{\lambda}_{\boldsymbol{w}} = [0.5\ 0.25\ 0.1667]^\top$. Bottom Left: A piecewise affine underapproximation (blue) of the log of the cumulative distribution function (yellow). Bottom Right: The difference $f(x) - \ell_f(x)$ as in (17), with $\eta = 0.1$.

tively. Finally, we show that (13e) and (9b) are equivalent via simple algebraic manipulations,

$$\sum_{i=1}^{L} \delta_i \leq \Delta \Leftrightarrow L - \sum_{i=1}^{L}(1-\delta_i) \leq \Delta \tag{15a}$$

$$\Leftrightarrow \log\left(\sum_{i=1}^{L} \exp(t_i)\right) \geq \log\left(L - \Delta\right) \tag{15b}$$

In other words, every feasible solution (U, t) of (13) maps to a feasible solution to (9) with $U \in \mathcal{U}^N$, and thereby is feasible for (5).

Proof of 2) The cost (13a) is a convex quadratic function of U. By construction, the constraints (13b) and (13d) are linear constraints in U and t. The convexity of (13c) follows from Lemma 1 and the definition of log-concavity. Recall that $\log\left(\sum_{i=1}^{L}\exp(t_i)\right)$ is a convex function in t [30, Sec. 3.1.5], hence (13e) is a reverse-convex constraint. ∎

C. Conic reformulation of (13c) *via piecewise affine approximation*

We now focus on enforcing the convex constraint (13c). Although convex, the constraint (13c) is not conic, which prevents the use of standard conic solvers. We present tight conic reformulation of (13c) via piecewise affine approximations.

Given a concave function $f : \mathcal{D} \to \mathcal{R}$ for bounded intervals $\mathcal{D}, \mathcal{R} \subset \mathbb{R}$, we define its piecewise affine underapproximation as $\ell_f : \mathbb{R} \to \mathbb{R}$ for some $m_j, c_j \in \mathbb{R}$ for $j \in \mathbb{N}_{[1,N_f]}$ and $N_f \in \mathbb{N}$ distinct affine elements,

$$\ell_f(x) \triangleq \min_{j \in \mathbb{N}_{[1,N_f]}} (m_j x + c_j). \tag{16}$$

For a user specified approximation error $\eta > 0$, we can find a ℓ_f for a concave f such that

$$\ell_f(x) \leq f(x) \leq \ell_f(x) + \eta, \tag{17}$$

with the sandwich algorithm [41]. The sandwich algorithm has convergence guarantees that can be balanced between the user-defined error and the number of affine pieces [28], [29], [41].

In (13), we use the piecewise affine underapproximation of the concave functions $f_i = \log\left(\Phi_{p_i^\top \boldsymbol{Z}}\right)$ with $N_i \in \mathbb{N}$ distinct pieces for every $i \in \mathbb{N}_{[1,L]}$ (as shown in Figure 2) to enforce (13c). The functions f_i have bounded domain and range in $\mathbb{R}$ due to (13b). We evaluate f_i using the one-dimensional numerical integration of characteristic functions, as in (12). We obtain the following optimization problem,

$$\begin{aligned} &\underset{U,t}{\text{minimize}}\ \ (13a) \\ &\text{subject to}\ \ (5b),\ (13b),\ (13d),\ (13e) \\ &\underset{\forall j \in \mathbb{N}_{[1,N_i]}}{\forall i \in \mathbb{N}_{[1,L]}},\ \ m_{i,j}\left(q_i - p_i^\top \bar{B} U\right) + c_{i,j} \geq t_i \end{aligned} \tag{18a}$$

By Theorem 1 and the use of piecewise affine underapproximations of $\log(\Phi_{p_i^\top \boldsymbol{Z}})$ in (13c), every feasible solution of (18) is feasible for (13), and thereby (5).

D. Solving (18) *via difference of convex programming*

The optimization problem (18) has a quadratic cost (13a), and linear constraints (5b), (13b), and (18a) in the decision variables U and t, and a reverse-convex constraint (13e). We now discuss a tractable solution to (18) using difference of convex programming [12].

Difference of convex programs are non-convex optimization problems of the form,

$$\begin{aligned} &\underset{x \in \mathbb{R}^n}{\text{minimize}} && f(x) - g(x) \\ &\text{subject to} && f_i(x) - g_i(x) \leq 0, \quad \forall i \in \mathbb{N}_{[1,M]} \end{aligned} \tag{19}$$

where f, g, f_i, and g_i are convex for $i \in \mathbb{N}_{[1,M]}$, $M \in \mathbb{N}$. The penalty based convex-concave procedure [12] solves (19) via sequential convex optimization, and is agnostic to the feasibility of the initial condition [12], [42].

Given the current estimate for the risk allocation $r = [r_1\ \cdots\ r_L]^\top \in [0,1]^L$ (i.e., the initialization for t), the penalty based convex-concave procedure solves the following convex approximation of (18) at every iteration,

$$\underset{U,t,s}{\text{minimize}}\ \ (13a) + \tau_k s \tag{20a}$$

$$\text{subject to}\ \ (5b),\ (13b),\ (13d),\ (18a)$$

$$s \geq 0 \tag{20b}$$

$$\begin{aligned} &\log\left(\textstyle\sum_{i=1}^{L}\exp(r_i)\right) \\ &+ \tfrac{1}{\sum_{i=1}^{L}\exp(r_i)} \textstyle\sum_{i=1}^{L}\exp(r_i)(t_i - r_i) \\ &\qquad + s \geq \log(L - \Delta) \end{aligned} \tag{20c}$$

where $\tau_k \geq 0$ for $k \in \mathbb{N}$ are optimization hyperparameters. The constraint (20c) corresponds to the first-order approximation of the reverse-convex constraint (13e), which is relaxed by a scalar slack variable s. We penalize the slack variable s in the objective (20a). The problem (20) is convex, since (20c) is a linear constraint in t and s, and all other constraints and the objective are convex (Theorem 1.b).

Starting with an arbitrary risk allocation $\delta_0 \in [0,1]^L$, we iteratively solve (20) with monotonically increasing values of τ_k to promote feasibility. In the numerical experiments, we

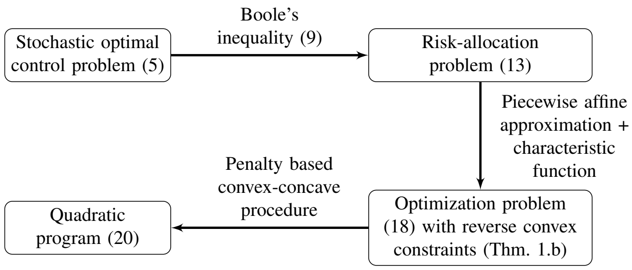

Fig. 3. Manipulations that result in a convexified reformulation of (5) that is amenable to conic solvers.

chose a uniform risk allocation $\delta_0 = \frac{\Delta}{L} I_L$, where I_L is a L-dimensional vector of ones. The corresponding initialization of $\overline{r}$ is therefore $\overline{r}_0 = \log(1-\delta_0)$. The convex-concave procedure converges to a local fixed-point when a pre-specified violation tolerance η_{viol} is met and the difference in cost between iterations k is less than a pre-specified tolerance η_{dc} [12]. However, the convergence to a local minima remains an open problem [43]. In addition, the procedure may terminate prematurely if τ_k reaches a user specified maximum number of iterations, τ_{max}.

In summary, we have transformed the original stochastic optimal control problem presented in (5) into a convex quadratic problem, via the steps shown in Figure 3. We first employed risk allocation (13), then converted the non-conic convex constraints in (13) into conic convex constraints using piecewise affine approximations, as well as the characteristic function. Finally, we utilize difference-of-convex programming to address the remaining reverse convex constraint (13e). Thus, our approach solves a convex (quadratic) program (20) iteratively to compute a local optimum of (5).

IV. Numerical Examples

We apply the proposed approach to two examples: 1) a double integrator, and 2) a quadrotor flying in an environment with a crosswind. We compare the performance of the controller produced by our approach to other open-loop methods: 1) a scenario approach [5], 2) a particle based approach [4], and 3) a moment based approach [7], [8]. We presume that the system is pre-stabilized via LQR using MATLAB's `dlqr(·)` function [44], [45]. When feasible, we also compare performance with the empirical characteristic function (ECF) approach in [29]. The number of samples for the scenario approach is determined by first specifying Δ, as well as a confidence bound $\xi = 1 \times 10^{-16}$ of not achieving Δ [5]. To ensure a fair comparison, we use the same number of samples for the particle and ECF approach. We measure the performance of the controllers based on the computed cost, the probability of constraint satisfaction, and the computation time. For methods which explicitly use samples in the constraints, performance is determined from the average of three runs. For the double integrator, $N_s = 91$ samples were used, and for the quadrotor, $N_s = 143$ samples were used. We used Monte-Carlo simulation with 10^5 samples for validation.

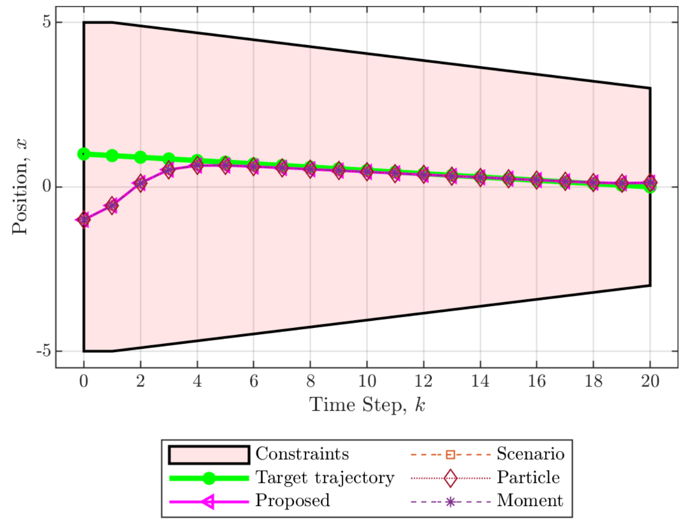

Fig. 4. Mean trajectories from the proposed approach, scenario approach [3], [5], particle based approach [4], and moment based approach [7], [8]. For all approaches, we presume a constraint violation threshold of $\Delta = 0.1$. Note that all approaches track the reference trajectory.

All computations are done with MATLAB on an Intel Core i9-10900K CPU with 3.70GHz and 128GB RAM. We implemented our algorithm, the scenario approach, the particle approach, and the empirical characteristic function approach in YALMIP [46] with MOSEK [47]. We used `fmincon` for the moment approach. For implementation of the proposed approach, we used $\tau_{max} = 10000$, $\tau_0 = 0.1$, and $\eta_{viol} = 1.2$. For the stopping criteria, we used an error tolerance of $\eta_{dc} = 0.1$. For the sandwich algorithm that generates the piecewise affine approximation for our approach and the ECF approach, we chose an absolute error of $\eta = 0.01$ for both examples.

A. Constrained control of a stochastic double integrator

We first consider a double integrator system,

$$\boldsymbol{x}(k+1) = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} \frac{T_s^2}{2} \\ T_s \end{bmatrix} u(k) + \boldsymbol{w}(k) \tag{21}$$

with state $\boldsymbol{x}(k) \in \mathbb{R}^2$, input set $\mathcal{U} = [-4, 4]$, exponential disturbance $\boldsymbol{w}(k)$ with scale $\overline{\lambda}_{\boldsymbol{w}}(k) \in \mathbb{R}^2_+$, sampling time $T_s = 0.25$s, and initial position $x(0) = [-1 \ 0]^\top$.

We seek to solve a constrained optimal control problem subject to dynamics (21), with quadratic cost (5a) that encodes our desire to track $X_\mathrm{d} \in \mathbb{R}^{nN}$, penalize high velocities, and minimize control effort. Specifically, we choose $Q = \mathrm{diag}([100 \ \ 5]) \otimes I_{(nN)\times(nN)}$, $R = 0.5 I_{(mN)\times(mN)}$, $(X_\mathrm{d})_t = [m_r t + c_r \ 0]^\top$, $\forall t \in \mathbb{N}_{[0,N]}$, and problem parameters $m_1, m_2, m_r, c_1, c_2, c_r$ as $0.1, -0.1, -0.05, -5, 5$, and 1 respectively. We define the time varying state constraints as

$$\mathscr{T} = \left\{(t, x) \in \mathbb{N}_{[0,N]} \times \mathbb{R}^2 : m_1 t + c_1 \leq x_1 \leq m_2 t + c_2\right\}$$

and maintain a constraint satisfaction of 95%, i.e. $\Delta = 0.05$.

We compute optimal control trajectories using our approach, the scenario approach, the particle filter approach, and the

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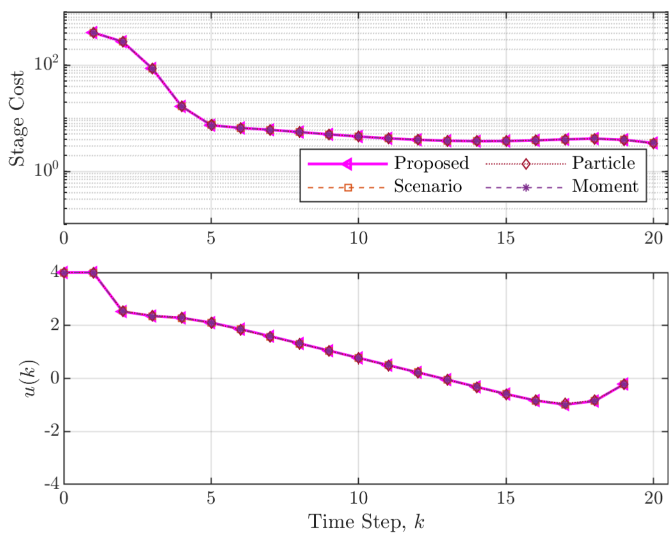

Fig. 5. Top: Stage cost (the cost incurred at each time step) and control effort over time, for the double integrator. The stage cost of all approaches are similar to highlight that reference tracking is possible for all approaches. Bottom: The optimal input for each approach.

TABLE I

DOUBLE INTEGRATOR EXAMPLE: COST AND CONSTRAINT SATISFACTION $(1-\Delta)$ FOR COMPUTED VALUES (COMP) AND MONTE CARLO (MC) SIMULATION (10^5 SAMPLES) FOR ALL BUT THE ECF METHOD (SEE FIGURE 6). WE LIST OFFLINE AND ONLINE COMPUTATION FOR ALL METHODS WHERE REASONABLE. SAMPLING/PARTICLE APPROACHES USE $N_s = 91$ SAMPLES.

Method	Cost		$1-\Delta$		Time (s)	
	Comp	MC	Comp	MC	Online	Offline
Proposed	863.2	863.2	0.95	1	0.95	56.47
Scenario	862.4	863.2	0.95	1	0.30	N/A
Particle	865.2	863.2	0.95	1	3.50	N/A
Moment	863.2	863.2	0.95	1	3.98	N/A
ECF	N/A	N/A	N/A	N/A	N/A	N/A

moment based approach, over a time horizon $N = 20$ and with scale parameter $\lambda_{\boldsymbol{w}} = [10\ 100]^\top$. The empirical characteristic function approach cannot be used (Figure 6), since the approach requires a concave region of the cumulative distribution function to exist [29, Sec. 3.B.]. Figure 4 shows the optimal trajectories for all approaches, where each track the reference trajectory closely while ensuring constraint satisfaction in the presence of uncertainty. Figure 5 shows that the stage cost (the cost at each time step) is similar amongst all approaches except the moment approach. Note that we cannot use the ECF approach here due to the absence of concavity in cumulative distribution function (Figure 6).

All methods generate similar trajectories, which track the reference. They also have similar costs and inputs, as shown in Table II, which compares the cost and probability of satisfaction to Monte Carlo estimates for 10^5 simulated trajectories. This example shows that under nominal conditions with non-Gaussian stochasticity, i.e. minimal risk allocation, all methods track the reference trajectory closely.

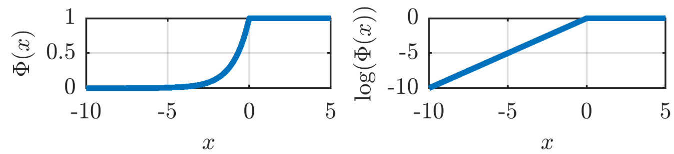

Fig. 6. The cumulative distribution function (left) and the log of the cumulative distribution function (right) for a negative affine transformation of an exponential random variable with scale parameter $\lambda = 1$. Because the empirical characteristic function approach requires a concave region of the cumulative distribution function to exist [29, Sec. III.B.], it cannot be used to solve the double integrator problem. In contrast, our approach is feasible, since the log of the cumulative distribution function is log-linear.

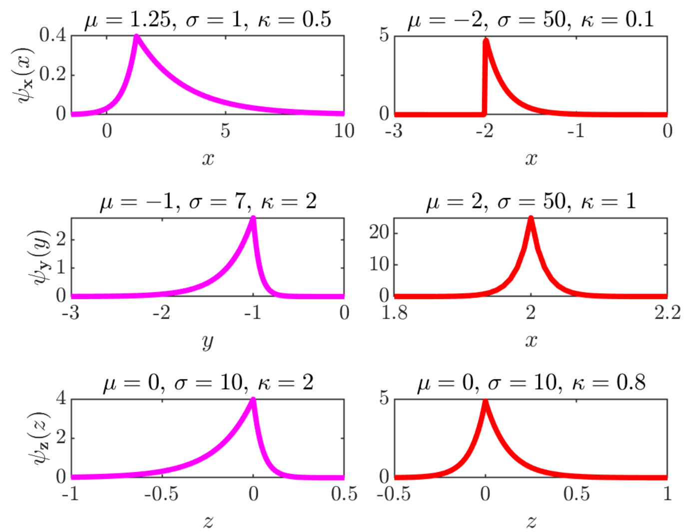

Fig. 7. The asymmetric Laplace distribution that affects the states representing quadcopter position in x, y, z. The disturbances follow the magenta distributions (left) for the first half of the time horizon, and then follow the red distributions (right) for the second half of the time horizon. The parameters of the distribution are noted above each plot.

B. Quadrotor in the crosswinds of a harsh environment

We consider a rigid-body quadcopter model [48]. The state is defined as a 12-dimensional vector, $x = [\phi\ \theta\ \psi\ \dot\phi\ \dot\theta\ \dot\psi\ \dot p_x\ \dot p_y\ \dot p_z\ p_x\ p_y\ p_z]^\top$, that captures orientation, angular rotation, speed, and position. The net thrust is described by u_1, and the moments around the p_x, p_y, and p_z axes created by the difference in the motor speeds are described by u_2, u_3, and u_4. We presume the mass is $m = 0.478$ kg and the moments of inertia are $I_{xx} = I_{yy} = 0.0117$ kg m^2 and $I_{zz} = 0.00234$ kg m^2 [49]. We linearize the nonlinear dynamics about a hovering point, and time discretize the dynamics via a zero-order hold with sampling time $T_s = 0.25$.

We incorporate the effect of wind into the quadcopter model as an additive stochastic disturbance, that takes the form of an asymmetric Laplace distribution with characteristic function,

$$\Psi_{\boldsymbol{w}}(t; \mu, \sigma, \kappa) = \frac{\exp(j\mu t)}{(1 + \frac{jt\kappa}{\sigma})(1 - \frac{jt}{\kappa\sigma})}, \tag{22}$$

whose location, shape, and asymmetry suddenly changes halfway through the time horizon, as shown in Figure 7, with distribution parameters [50], [51]. That is, the distribution is non-stationary, but independent. We presume the wind directly influences the translational motion, i.e. p_x, p_y, and p_z.

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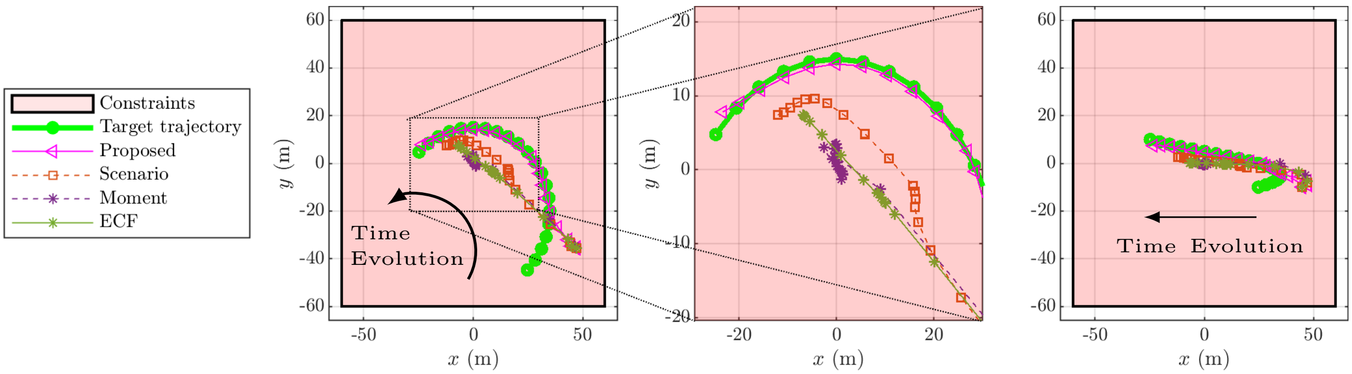

Fig. 8. Mean trajectories for the quadcopter example. Our approach has the lowest cost, and a probabilistic constraint satisfaction, with a reasonable overall solve time, that is closest (but still above) the desired threshold (Table II). This can be seen in the fact that the trajectory for our approach is close to the reference trajectory (middle plot) compared to the scenario approach which overshoots before recovering to track the reference trajectory. In essence, our approach enables a better trajectory because it can effectively account for the risk of violating the constraint satisfaction in the control optimization process.

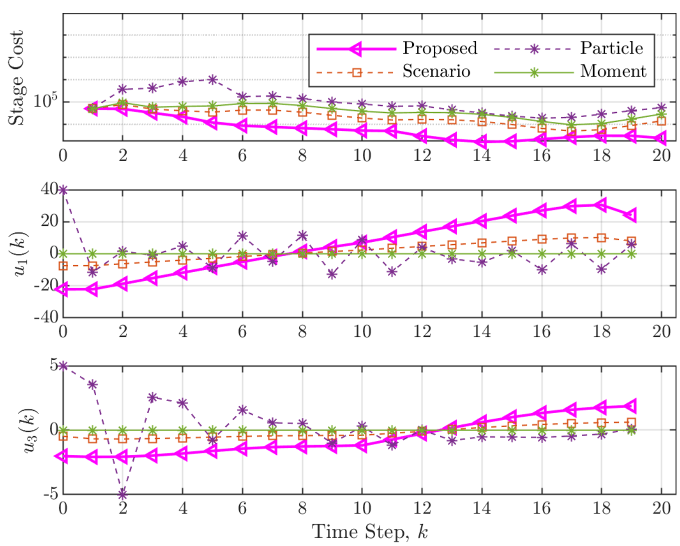

Fig. 9. The stage cost and input at each time step for all approaches compared in the quadcopter example. Our proposed, moment, and ECF methods have comparable inputs. However, note the scenario approach has differing inputs for u_1 and u_3, and correspondingly higher cost (Table II). The input u_2 and u_4 are not shown because they are quite similar.

TABLE II

QUADCOPTER EXAMPLE: COST AND CONSTRAINT SATISFACTION $(1-\Delta)$ FOR COMPUTED (COMP) AND MONTE-CARLO (MC) SIMULATION WITH 10^5 SAMPLES, FOR ALL BUT THE PARTICLE CONTROL METHOD. SAMPLING AND ECF APPROACH USE $N_s = 141$ SAMPLES.

Method	Cost		$1-\Delta$		Time (s)	
	Comp	MC	Comp	MC	Online	Offline
Proposed	228.4	228.3	0.95	1	0.86	856.8
Scenario	698.4	550.6	0.95	1	1.54	N/A
Particle	N/A	N/A	N/A	N/A	N/A	N/A
Moment	381.9	382.0	0.95	1	266.2	N/A
ECF	932.0	933.1	1	1	0.36	1949.3

We solve the stochastic optimal control problem (5) for a time horizon of $N = 20$ with cost weights $Q = \mathrm{diag}([10I_{1\times 9N} \;\; 100I_{1\times 3N}]) \otimes I_{N\times N}$ and $R = I_{4N\times 4N}$. The desired reference trajectory X_d is defined for the state variables p_x and p_y by generating waypoints via the following functions,

$$X_{\mathrm{d},x}(k) = r\sin(\theta(k)), \tag{23a}$$
$$X_{\mathrm{d},y}(k) = r\cos(\theta(k)) - 20, \tag{23b}$$

with $r = 35$, and $\theta(k) \in \mathbb{R}$ decreases from $3\pi/4$ to $-\pi/4$. The vertical desired position, $X_{d,x}(k) \in \mathbb{R}$ is defined by linearly spaced waypoints from -10 to 10. The input set is $\mathcal{U} = [-40, 40] \times [-5, 5]^3$ and the constraint set $\mathscr{S}$,

$$\mathscr{S} = \left\{x \in \mathbb{R}^{12} : |p_x| \leq 60,\ |p_y| \leq 60,\ |p_z| \leq 60\right\}$$

imposes restrictions on the translational motion. The initial condition is $x(0) = [0 \ \cdots \ 0 \ 44.75 \ -34.75 \ -10\ ]^\top$.

We choose a constraint satisfaction probability of 95% ($\Delta = 0.05$). Figure 8 shows the computed trajectories our approach and those we compare it to. All approaches except for the particle approach find an optimal, open-loop controller, but with varying conservatism (Table II). The particle approach exceeds our cutoff time of an hour.

In contrast to the first example, only our proposed approach is the only approach which tracks the reference trajectory closely with probabilistic constraint satisfaction (Figure 8). As shown in Figure 9, our stage cost is the lowest amongst all approaches. Although our approach requires an additional offline calculation for the piecewise underapproximation, the overall cost to solve time is the best out of all approaches relative to online solve time, as shown in Table II.

The piecewise affine approximation for the ECF approach takes longer offline time due to it using a sum of characteristic functions via data to construct the cumulative distribution function. In contrast, since we are given the characteristic function in our approach, the offline time for the piecewise affine approximation is slightly lower. Nonetheless, our approach has a lower cost due to exploiting the log-concavity properties of the distribution. Whereas the ECF approach relies on a conservative concave restriction about the inflection point of the cumulative distribution function (Table II).

V. CONCLUSION

We presented a convex optimization based approach for the constrained, optimal control of a linear dynamical system with additive, log-concave disturbance. Our formulation utilizes a characteristic function based risk allocation technique to assure probabilistic safety for a log-concave disturbance. Our approach solves a tractable difference-of-convex program to synthesize the desired controller. Our reforumulation is amenable to standard conic solvers via the use of piecewise affine approximations that provide tight bounds. Numerical experiments show the efficacy of our approach in comparison to scenario, particle control, and moment based approaches.

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