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On classification of nonunital amenable simple C^*-algebras III: The range and the reduction

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We prove that every stably projectionless separable simple amenable C^*-algebra in the UCT class has rationally generalized tracial rank one. Following Elliott's earlier work, we show that the Elliott invariant of any finite separable simple C^*-algebra with finite nuclear dimension can always be described as a scaled simple ordered group pairing together with a countable abelian group (which unifies the unital and nonunital, as well as stably projectionless cases). We also show that, for any given such invariant set, there is a finite separable simple C^*-algebra whose Elliott invariant is the given set, a refinement of the range theorem of Elliott. In the stably projectionless case, modified model C^*-algebras are constructed in such a way that they are of generalized tracial rank one and have other technical features.

1. Introduction

This paper is a part of the Elliott program of classification of simple separable C^*-algebras with finite nuclear dimension (or, equivalently, simple separable amenable $\mathcal{Z}$-stable C^*-algebras by [Castillejos et al. 2021; Castillejos and Evington 2020; Winter 2012; Tikuisis 2014]). In fact it is the first part of the research results which give a unified classification of separable finite simple C^*-algebras of finite nuclear dimension which satisfy the universal coefficient theorem (UCT).

Briefly, a full classification theorem for a class of C^*-algebras, say $\mathcal{A}$, consists of three parts. The first part is a description of the Elliott invariant for the C^*-algebras in $\mathcal{A}$; see [Elliott 1995]. The second part is the range (or model) theorem, i.e., for any given Elliott invariant set as described in the first part, there is a model C^*-algebra (with certain desired properties) in $\mathcal{A}$ such that its Elliott invariant set is the given one [Elliott 1996]. The third part is the isomorphism theorem which asserts that any two C^*-algebras in the class $\mathcal{A}$ are isomorphic if and only if they have the same Elliott invariant.

Let $\mathcal{A}$ be the class of separable simple amenable $\mathcal{Z}$-stable C^*-algebras in the UCT class. This paper contains the first two parts of the classification of C^*-algebras

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in $\mathcal{A}$, together with a reduction theorem. X. Jiang and H. Su [1999] constructed a unital infinite-dimensional separable simple amenable C^*-algebra $\mathcal{Z}$ which has exactly the same Elliott invariant as the complex field $\mathbb{C}$. It is known (see [Elliott et al. 2015, Corollary 4.11]) that the Jiang–Su algebra $\mathcal{Z}$ is the unique unital infinite-dimensional separable simple C^*-algebra with finite nuclear dimension in the UCT class to have this property. Let A be any unital separable simple C^*-algebra with weakly unperforated $K_0(A)$. Then $A \otimes \mathcal{Z}$ and A have exactly the same Elliott invariant (see [Gong et al. 2000, Theorem 1(b)]). The invariant set may be described as a six-tuple $(K_0(A), K_0(A)_+, [1_A], T(A), \rho_A, K_1(A))$, where $T(A)$ is the tracial state space of A and $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$ (the space of all real affine continuous functions on $T(A)$) is a pairing. Therefore, currently we study the classification of simple separable $\mathcal{Z}$-stable C^*-algebras. The classification results for unital separable simple amenable $\mathcal{Z}$-stable C^*-algebras in the UCT class can be found in [Phillips 2000; Kirchberg and Phillips 2000; Gong et al. 2020a; 2020b; Elliott et al. 2015; Tikuisis et al. 2017]. For separable simple C^*-algebras A with $K_0(A)_+ \neq \{0\}$, the classification can be easily reduced to the unital case.

This paper mainly studies the case that $K_0(A)_+ = \{0\}$ and A is finite, i.e., the stably projectionless case. Note that a separable simple C^*-algebra A with $K_0(A)_+ = \{0\}$ could still have interesting $K_0(A)$. Moreover, one could also have a nontrivial pairing ρ_A.

Let A be a nonunital separable simple C^*-algebra and $\widetilde{T}(A)$ the set of densely defined lower semicontinuous traces. Let $\widetilde{A}$ be the unitization of A and $\pi : \widetilde{A} \to \mathbb{C}$ the quotient map. Suppose that A is an algebraically simple C^*-algebra and $p \in M_n(\widetilde{A})$ is a nonzero projection and m is the rank of $\pi(p)$. One may define

$$\rho_A([p] - [1_m])(\tau) = \tau(p) - m\|\tau\|$$

(for $\tau \in \widetilde{T}(A)$) which gives a pairing $\rho_A : K_0(A) \to \mathrm{Aff}(\widetilde{T}(A))$ (the set of all real continuous affine functions on $\widetilde{T}(A)$). On the other hand, if A is stable, then a naive extension of the above pairing may not make sense as $\tau(p) = \infty$ (when A is stably projectionless) and $\|\tau\| = \infty$. If one chooses a hereditary C^*-subalgebra B of A which is algebraically simple, then one may define a pairing $\rho_B : K_0(B) \to \mathrm{Aff}(\widetilde{T}(B))$. However, one needs to define a pairing which is independent of B, so a somewhat more careful pairing is deployed. As the first part of the classification, following Elliott's earlier work [1995] and including [Elliott et al. 2020a] and [Gong et al. 2000], we combine several previous results to state that for any separable amenable simple $\mathcal{Z}$-stable C^*-algebra $(A \cong A \otimes \mathcal{Z})$, its Elliott invariant may always be described by a scaled simple ordered group pairing together with a countable abelian group (see Definition 2.15 and Theorem 5.2). For the second part of the classification, we modify the range theorem proved in [Elliott 1996] (see also [Li 2020] and the beginning of Section 4 of this paper).

To be more specific, in the stably projectionless case, modifying Elliott's original construction, we show that any Elliott invariant (including the scale) mentioned above can be realized by a model simple C^*-algebra of generalized tracial rank one. Moreover, for technical purposes, we show that these model simple C^*-algebras have other technical properties (see Theorem 4.118, Remark 4.135 and Theorem 4.137).

Furthermore, following [Elliott et al. 2020a, Corollary A.7], we found that, in the stably projectionless case, a pairing mentioned above actually gives a previously unexpected stronger feature of weak unperforation (see Corollary 3.6), a feature that plays an important technical role in the later part of [Gong and Lin 2020b] (see, for example, Theorem 3.7 in the current paper).

As mentioned above, the isomorphism theorem in [Gong and Lin 2020b] is first established for those separable simple C^*-algebras of finite nuclear dimension which have rationally generalized tracial rank one. In this paper, with the model theorem mentioned above, we also show that every separable stably projectionless simple C^*-algebra with finite nuclear dimension in the UCT class actually has rationally generalized tracial rank one (Theorem 6.169) which leads to the final classification of stably projectionless simple C^*-algebras with finite nuclear dimension in the UCT class [Gong and Lin 2020b].

The paper is organized as follows. Section 2 serves as preliminaries. Section 3 discusses the existence of $\mathcal{W}$-traces. In Section 4, we first construct a class of simple C^*-algebras which are inductive limits of 1-dimensional noncommutative CW complexes with arbitrary simple pairings. Then, together with the construction in [Gong and Lin 2020a], we construct simple C^*-algebras with arbitrary simple pairings and arbitrary K_1-groups. These are simple C^*-algebras which are locally approximated by subhomogeneous C^*-algebras whose spectra have dimension no more than 3. In Section 5, we discuss the range of the invariant sets of the stably finite separable simple $\mathcal{Z}$-stable C^*-algebras, and construct models which exhaust all the possible values of Elliott invariant for those separable simple C^*-algebras. In Section 6, we show that, in the UCT class, all separable finite simple C^*-algebras with finite nuclear dimension have rationally generalized tracial rank at most one (Theorem 6.169).

2. Preliminaries

Definition 2.1. Let A be a C^*-algebra. Denote by $A^{\mathbf{1}}$ the unit ball of A. For $a \in A_+$, denote by $\mathrm{Her}(a)$ the hereditary C^*-subalgebra $\overline{aAa}$. If $a, b \in A_+$, we write $a \lesssim b$ (a is Cuntz smaller than b), if there exists a sequence $x_n \in A$ such that $a = \lim_{n\to\infty} x_n^* x_n$ and $x_n x_n^* \in \mathrm{Her}(b)$. If both $a \lesssim b$ and $b \lesssim a$, then we say a is Cuntz equivalent to b. The Cuntz equivalence class represented by a is denoted by $\langle a \rangle$. A projection $p \in M_n(A)$ defines an element $[p] \in K_0(A)_+$. We also use $[p]$ to denote the Cuntz equivalence class represented by p.

Definition 2.2. Let A be a C^*-algebra. Denote by $T(A)$ the tracial state space of A (which could be the empty set). Let $\mathrm{Aff}(T(A))$ be the space of all real-valued affine continuous functions on $T(A)$. Let $\widetilde{T}(A)$ be the cone of densely defined, positive lower semicontinuous traces on A equipped with the topology of pointwise convergence on elements of the Pedersen ideal $\mathrm{Ped}(A)$ of A. So $\widetilde{T}(A)$ may be viewed as the cone in the dual space of the vector space $\mathrm{Ped}(A)$.

Let B be another C^*-algebra with $\widetilde{T}(B) \neq \{0\}$ and let $\varphi : A \to B$ be a homomorphism. Since $\mathrm{Ped}(A)$ is the minimal dense ideal of A, $\varphi(\mathrm{Ped}(A)) \subset \mathrm{Ped}(B)$. *In what follows we also write φ for $\varphi \otimes \mathrm{id}_{M_k} : M_k(A) \to M_k(B)$ whenever it is convenient.*

We write $\varphi_T : \widetilde{T}(B) \to \widetilde{T}(A)$ for the induced continuous affine map. Denote by $\widetilde{T}^b(A)$ the subset of $\widetilde{T}(A)$ which is bounded on A. Of course $T(A) \subset \widetilde{T}^b(A)$. Set $T_0(A) := \{t \in \widetilde{T}(A) : \|\tau\| \leq 1\}$. It is a compact convex subset of $\widetilde{T}(A)$.

Let $r \geq 1$ be an integer and $\tau \in \widetilde{T}(A)$. We continue to write τ on $A \otimes M_r$ for $\tau \otimes \mathrm{Tr}$, where Tr is the standard (unnormalized) trace on M_r. Let S be a convex subset (of a convex topological cone). We assume that a convex cone contains 0, but a convex set S may or may not contain 0. Denote by $\mathrm{Aff}(S)$ the set of affine continuous functions on S with the property that, if $0 \in S$, then $f(0) = 0$ for all $f \in \mathrm{Aff}(S)$. As in [Robert 2012], define

$$\mathrm{Aff}_+(S) = \{f : \mathrm{Aff}(S) : f(\tau) > 0 \text{ for } \tau \neq 0\} \cup \{0\}, \tag{2.3}$$

$$\mathrm{LAff}_{f,+}(S) = \{f : S \to [0, \infty) : \exists\, \{f_n\},\ f_n \nearrow f,\ f_n \in \mathrm{Aff}_+(S)\}, \tag{2.4}$$

$$\mathrm{LAff}_+(S) = \{f : S \to [0, \infty] : \exists\, \{f_n\}, f_n \nearrow f,\ f_n \in \mathrm{Aff}_+(S)\}, \tag{2.5}$$

$$\mathrm{LAff}^{\sim}(S) = \{f_1 - f_2 : f_1 \in \mathrm{LAff}_+(S) \text{ and } f_2 \in \mathrm{Aff}_+(S)\}. \tag{2.6}$$

For most of this paper, $S = \widetilde{T}(A)$, $S = T(A)$, or $S \subset T_0(A)$ in the above definition are used. Moreover, for $S \subset T_0(A)$, $\mathrm{LAff}_{b,+}(S)$ is the subset of those bounded functions in $\mathrm{LAff}_{f,+}(S)$. Recall that $0 \in \widetilde{T}(A)$ and if $f \in \mathrm{LAff}(\widetilde{T}(A))$, then $f(0) = 0$.

Definition 2.7. A convex topological cone T is a subset of a topological vector space such that for any $\alpha, \beta \in \mathbb{R}_+$ and $x, y \in T$, $\alpha x + \beta y \in T$, where $\mathbb{R}_+$ is the set of nonnegative real numbers. A subset $\Delta \subset T$ is called a base of T, if Δ is convex and for any $x \in T \setminus \{0\}$, there is a unique pair $(\alpha_x, \tau_x) \in (\mathbb{R}_+ \setminus \{0\}) \times \Delta$ such that $x = \alpha_x \tau_x$. In this article, *all convex topological cones are those with a metrizable Choquet simplex Δ as its base.* Note also that the function from $T \setminus \{0\}$ to $(\mathbb{R}_+ \setminus \{0\}) \times \Delta$ sending x to (α_x, τ_x) is continuous.

A simple ordered group pairing is a triple (G, T, ρ), where G is a countable abelian group, T is a convex topological cone with a Choquet simplex as its base and $\rho : G \to \mathrm{Aff}(T)$ is a homomorphism. *In what follows, for a pair of functions f and g on T, we write $f > g$ if $f(\tau) > g(\tau)$ for all $\tau \in T \setminus \{0\}$.* Define $G_+ = \{g \in G : \rho(g) > 0\} \cup \{0\}$. If $G_+ \neq \{0\}$, then (G, G_+) is an ordered group. It has the property that if $ng > 0$ for some integer $n > 0$, then $g > 0$. In other words, (G, G_+) is weakly unperforated. Moreover, if $G_+ \neq \{0\}$, (G, G_+) is a simple ordered group, i.e., every element $g \in G_+ \setminus \{0\}$ is an order unit (recall that the compact set Δ is a base for T). In general, we allow the case $G_+ = \{0\}$ (but ρ may not be 0).

A scaled simple ordered group pairing is a quintuple $(G, \Sigma(G), T, s, \rho)$ such that (G, T, ρ) is a simple ordered group pairing, where $s \in \mathrm{LAff}_+(T) \setminus \{0\}$ and

$$\Sigma(G) := \{g \in G_+ : \rho(g) < s\} \quad \text{or} \quad \Sigma(G) := \{g \in G_+ : \rho(g) < s\} \cup \{u\}, \tag{2.8}$$

where $u \in G_+$ and $\rho(u) = s$. We allow $\Sigma(G) = \{0\}$. Note also that $s(\tau)$ could be infinite for some $\tau \in T$. It is called a unital scaled simple ordered group pairing, if $\Sigma(G) = \{g \in G_+ : \rho(g) < s\} \cup \{u\}$ with $\rho(u) = s$, in which case u is called the unit of $\Sigma(G)$. Note that, in this case, u is the maximum element of $\Sigma(G)$, and one may write (G, u, T, ρ) for $(G, \Sigma(G), T, s, \rho)$. If $\Sigma(G)$ has no unit (which includes the case that $\Sigma(G)$ has a maximum element x but $\rho(x) < s$), then $\Sigma(G)$ is determined by s. One may write (G, T, s, ρ) for $(G, \Sigma(G), T, s, \rho)$ (see Theorem 5.2 below). (Note that $\Sigma(G) = \{0\}$ corresponds to the projectionless case and $G_+ = \{0\}$ to the stably projectionless case).

Let $(G_i, \Sigma(G_i), T_i, s_i, \rho_i)$ be scaled simple ordered group pairings, $i = 1, 2$. A map

$$\Gamma_0 = (\kappa_0, \kappa_T) : (G_1, \Sigma(G_1), T_1, s_1, \rho_1) \to (G_2, \Sigma(G_2), T_2, s_2, \rho_2)$$

is said to be a homomorphism if there is a group homomorphism $\kappa_0 : G_1 \to G_2$ and a continuous cone map $\kappa_T : T_2 \to T_1$ (preserving 0) such that

$$\rho_2(\kappa_0(g))(t) = \rho_1(g)(\kappa_T(t)) \quad \text{for all } g \in G_1 \text{ and } t \in T_2, \tag{2.9}$$

$$\kappa_0(\Sigma(G_1)) \subset \Sigma(G_2) \text{ and } s_1(\kappa_T(t)) \le s_2(t) \quad \text{for all } t \in T_2. \tag{2.10}$$

We say that a homomorphism Γ_0 is an isomorphism if κ_0 is an isomorphism, $\kappa_0(\Sigma(G_1)) = \Sigma(G_2)$, κ_T is a cone homeomorphism, and $s_1(\kappa_T(t)) = s_2(t)$ for all $t \in T_2$.

Definition 2.11. For any $\varepsilon > 0$, define $f_\varepsilon \in C([0, \infty))_+$ by letting $f_\varepsilon(t) = 0$ if $t \in [0, \varepsilon/2]$, $f_\varepsilon(t) = 1$ if $t \in [\varepsilon, \infty)$ and $f_\varepsilon(t)$ be linear in $(\varepsilon/2, \varepsilon)$.

Let A be a C^*-algebra and τ be a quasitrace. For each $a \in A_+$ define

$$d_\tau(a) = \lim_{\varepsilon \to 0} \tau(f_\varepsilon(a)).$$

Note that $f_\varepsilon(a) \in \mathrm{Ped}(A)$ for all $a \in A_+$.

Let S be a convex subset of $\widetilde{T}(A)$ and $a \in M_n(A)_+$. The function $\hat{a}(s) = s(a)$ (for $s \in S$) is an affine function from S to $[0, \infty]$. Define

$$\widehat{\langle a \rangle}(s) = d_s(a) = \lim_{\varepsilon \to 0} s(f_\varepsilon(a))$$

(for $s \in S$), which is a lower semicontinuous function. If $a \in \mathrm{Ped}(A)_+$, then the map $\tau \mapsto \hat{a}(\tau)$ is in $\mathrm{Aff}_+(S)$ and $\widehat{\langle a \rangle} \in \mathrm{LAff}_+(S)$ (see Definition 2.2), in general. Note that $\hat{a}$ is different from $\widehat{\langle a \rangle}$. In most cases, S is $\widetilde{T}(A)$, $T_0(A)$, or $T(A)$. Note also that there is a canonical map from $\mathrm{Cu}(A)$ to $\mathrm{LAff}_+(\widetilde{T}(A))$ sending $\langle a \rangle$ to $\widehat{\langle a \rangle}$.

2.12. If A is a unital C^*-algebra and $T(A) \neq \varnothing$, then there is a canonical homomorphism $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$. Now consider the case that A is not unital. Let $\pi_{\mathbb{C}}^A : \widetilde{A} \to \mathbb{C}$ be the quotient map. Suppose that $T(A) \neq \varnothing$. Let $\tau_{\mathbb{C}} := \tau_{\mathbb{C}}^A : \widetilde{A} \to \mathbb{C}$ be the tracial state which factors through $\pi_{\mathbb{C}}^A$. Then

$$T(\widetilde{A}) = \{t\tau_{\mathbb{C}}^A + (1-t)\tau : t \in [0, 1],\ \tau \in T(A)\}. \tag{2.13}$$

The map $T(A) \hookrightarrow T(\widetilde{A})$ induces a map $\mathrm{Aff}(T(\widetilde{A})) \to \mathrm{Aff}(T(A))$. Then the map $\rho_{\widetilde{A}} : K_0(\widetilde{A}) \to \mathrm{Aff}(T(\widetilde{A}))$ induces a homomorphism $\rho' : K_0(A) \to \mathrm{Aff}(T(A))$ by

$$\rho' : K_0(A) \to K_0(\widetilde{A}) \xrightarrow{\rho_{\widetilde{A}}} \mathrm{Aff}(T(\widetilde{A})) \to \mathrm{Aff}(T(A)). \tag{2.14}$$

However, in the case that $A \neq \mathrm{Ped}(A)$, we do not use ρ' in general, as it is possible that $T(A) = \varnothing$ but $\widetilde{T}(A)$ is rich (consider the case $A \cong A \otimes \mathcal{K}$).

Definition 2.15. Let A be a C^*-algebra with $\widetilde{T}(A) \neq \{0\}$. If $\tau \in \widetilde{T}(A)$ is bounded on A, then τ can be extended naturally to a trace on $\widetilde{A}$. Recall that $\widetilde{T}^b(A)$ is the set of bounded traces on A. Denote by $\rho_A^b : K_0(A) \to \mathrm{Aff}(\widetilde{T}^b(A))$ the homomorphism defined by

$$\rho_A^b([p] - [q]) = \tau(p) - \tau(q)$$

for all $\tau \in \widetilde{T}^b(A)$ and for projections $p, q \in M_n(\widetilde{A})$ (for some integer $n \geq 1$) with $\pi_{\mathbb{C}}^A(p) = \pi_{\mathbb{C}}^A(q)$. Note that $p - q \in M_n(A)$. Therefore $\rho_A^b([p] - [q])$ is continuous on $\widetilde{T}^b(A)$. In the case that $\widetilde{T}^b(A) = \widetilde{T}(A)$ (for example, $A = \mathrm{Ped}(A)$), we write $\rho_A := \rho_A^b$.

Let A be a σ-unital C^*-algebra with a strictly positive element $0 \leq e \leq 1$. Put $e_n := f_{1/2^n}(e)$. Then $\{e_n\}$ forms an approximate identity for A. Note that $e_n \in \mathrm{Ped}(A)$ for all n. Set $A_n = \mathrm{Her}(e_n) := \overline{e_n A e_n}$. Denote by $\iota_n : A_n \to A_{n+1}$ and $j_n : A_n \to A$ the embeddings. They extend to $\iota_n^\sim : \widetilde{A}_n \to \widetilde{A}_{n+1}$ and $j_n^\sim : \widetilde{A}_n \to \widetilde{A}$ unitally. Note that $e_n \in \mathrm{Ped}(A_{n+1})$. Thus ι_n and j_n induce continuous cone maps

$$\iota_{nT}^b : \widetilde{T}^b(A_{n+1}) \to \widetilde{T}^b(A_n) \quad \text{and} \quad j_{nT} : \widetilde{T}(A) \to \widetilde{T}^b(A_n)$$

defined by $\iota_{nT}^b(\tau)(a) = \tau(\iota_n(a))$ for $\tau \in \widetilde{T}^b(A_{n+1})$, and $j_{nT}(\tau)(a) = \tau(j_n(a))$ for all $\tau \in \widetilde{T}(A)$ and all $a \in A_n$. Denote by

$$\iota_n^\sharp : \mathrm{Aff}(\widetilde{T}^b(A_n)) \to \mathrm{Aff}(\widetilde{T}^b(A_{n+1})) \quad \text{and} \quad j_n^\sharp : \mathrm{Aff}(\widetilde{T}^b(A_n)) \to \mathrm{Aff}(\widetilde{T}(A))$$

the induced continuous linear maps. Recall that $\bigcup_{n=1}^\infty A_n$ is dense in $\mathrm{Ped}(A)$. A direct computation shows that one may obtain the following inverse limit of convex topological cones (with continuous cone maps):

$$\widetilde{T}^b(A_1) \xleftarrow{\iota_{1T}^{\ b}} \widetilde{T}^b(A_2) \xleftarrow{\iota_{2T}^{\ b}} \widetilde{T}^b(A_3) \leftarrow \cdots \leftarrow \widetilde{T}(A). \tag{2.16}$$

To justify (2.16), set $T' = \lim_{\leftarrow} \widetilde{T}^b(A_n) \subset \prod_{n=1}^{\infty} \widetilde{T}^b(A_n)$. Define $\Gamma : \widetilde{T}(A) \to T'$ by $\Gamma(\tau) = \{\tau_n\}$, where $\tau_n = \tau|_{A_n}$. Let I_n be the (closed two-sided) ideal of A generated by A_n. Let $\sim_T$ be the equivalence relation $\sim$ defined in [Cuntz and Pedersen 1979, §2]. Recall that $\{x \in (I_n)_+ : \exists y \in A_{n+}$ such that $x \sim_T y\}$ is a positive part of an ideal J_n containing A_n (see the remark after Proposition 4.7 of [Cuntz and Pedersen 1979]). Therefore, if $t \in \widetilde{T}(A)$ and $t|_{A_n} = \tau|_{A_n}$, then $t|_{J_n} = \tau|_{J_n}$. Put $J = \bigcup_{n=1}^{\infty} J_n$. It follows that $t|_J = \tau|_J$. Since J is a dense ideal, it contains $\mathrm{Ped}(A)$. Hence t and τ are the same element in $\widetilde{T}(A)$. This implies that the map Γ is injective. It is also clear that Γ is a continuous cone map. To see that Γ is surjective, let $\{\tau_n\} \in T'$. Recall that $\tau_n \in \widetilde{T}^b(A_n)$ and $\tau_{n+1}|_{A_n} = \tau_n$. Let $\tilde{\tau}_n$ be a trace in $\widetilde{T}(A)$ which extends τ_n (see Lemma 4.6 of [Cuntz and Pedersen 1979]). Then $\tilde{\tau}_{n+1}|_{J_n} = \tilde{\tau}_n|_{J_n}$, as argued above. Define $\tilde{\tau}$ on J by $\tilde{\tau}|_{J_n} = \tilde{\tau}_n|_{J_n}$. Since $\tilde{\tau}$ is finite on $\bigcup_{n=1}^{\infty} A_n$, which is in $\mathrm{Ped}(A)$ and dense in A, $\tilde{\tau}$ is a lower semicontinuous densely defined trace. Then one may view $\tilde{\tau} \in \widetilde{T}(A)$, and note also $\Gamma(\tilde{\tau}) = \{\tau_n\}$. This shows that Γ is surjective.

To see Γ is open, consider $O_{a,>\beta} = \{\tau \in \widetilde{T}(A) : \tau(a) > \beta\}$, where $a \in \mathrm{Ped}(A)_+ \setminus \{0\}$ and $\beta \in \mathbb{R}$. Then $O_{a,>\beta} = \bigcup_k \{\tau : \tau(e_k a e_k) > \beta\}$. Thus

$$\Gamma(O_{a,>\beta}) = \bigcup_k \{\{\tau_n\} \in T' : \tau_k(e_k a e_k) > \beta\},$$

which is open in the product topology. Now consider $O_{a,<\beta} = \{\tau \in \widetilde{T}(A) : \tau(a) < \beta\}$. Since $a \in \mathrm{Ped}(A) \subset J$, we may assume that $a \in J_N$ for some $N \geq 1$. There is an element $b \in (A_N)_+$ such that $b \sim_T a$. Note that $\{\tau \in \widetilde{T}^b(A_N) : \tau(b) < \beta\}$ is open. It follows that $O := \big\{\{\tau_k\} \in \prod_{k=1}^{\infty} \widetilde{T}^b(A_k) : \tau_N(b) < \beta\big\}$ is open in $\prod_{k=1}^{\infty} \widetilde{T}^b(A_n)$. Since $\Gamma(O_{a,<\beta}) = O \cap T'$, it is also open in T'. This implies that Γ is open.

The continuous cone map j_{nT} is the same as the cone map $\iota_{\infty,nT} : \widetilde{T}(A) \to \widetilde{T}^b(A_n)$ given by the inverse limit. One also obtains the induced inductive limit

$$\mathrm{Aff}(\widetilde{T}^b(A_1)) \xrightarrow{\iota_1^\sharp} \mathrm{Aff}(\widetilde{T}^b(A_2)) \xrightarrow{\iota_2^\sharp} \mathrm{Aff}(\widetilde{T}^b(A_3)) \to \cdots \to \mathrm{Aff}(\widetilde{T}(A)). \tag{2.17}$$

Hence one also has the following commutative diagram:

$$\begin{tikzcd}
K_0(A_1) \arrow[r, "\iota_{1*o}"] \arrow[d, "\rho_{A_1}"'] & K_0(A_2) \arrow[r, "\iota_{2*o}"] \arrow[d, "\rho_{A_2}"'] & K_0(A_3) \arrow[r] \arrow[d, "\rho_{A_3}"'] & \cdots \arrow[r] & K_0(A) \\
\mathrm{Aff}(\widetilde{T}^b(A_1)) \arrow[r, "\iota_{1,2}^\sharp"] & \mathrm{Aff}(\widetilde{T}^b(A_2)) \arrow[r, "\iota_2^\sharp"] & \mathrm{Aff}(\widetilde{T}^b(A_3)) \arrow[r] & \cdots \arrow[r] & \mathrm{Aff}(\widetilde{T}(A))
\end{tikzcd}$$

Thus one obtains a homomorphism $\rho : K_0(A) \to \mathrm{Aff}(\widetilde{T}(A))$. If A is assumed to be simple, then each A_n is simple and e_n is full in A_n. Therefore, since $e_n \in \mathrm{Ped}(A)$

and $A_n = \mathrm{Ped}(A_n)$ (see Theorem 2.1 of [Tikuisis 2014]). It follows that, when A is simple, $\widetilde{T}^b(A_n) = \widetilde{T}(A_n)$ for all n. But we do not assume that A is simple in general.

Note that, if $\widetilde{T}(A) = \widetilde{T}^b(A)$, for any $n \geq 1$, one also has the following commutative diagram:

$$\begin{tikzcd}
K_0(A_n) \arrow[r, "\iota_{n*0}"] \arrow[d, "\rho_{A_n}"'] & K_0(A_{n+1}) \arrow[r, "j_{n+1,*0}"] \arrow[d, "\rho_{A_{n+1}}"'] & K_0(A) \arrow[d, "\rho_A"'] \\
\mathrm{Aff}(\widetilde{T}^b(A_n)) \arrow[r, "{\iota_n}^{\sharp}"] & \mathrm{Aff}(\widetilde{T}^b(A_{n+1})) \arrow[r, "{j_{n+1}}^{\sharp}"] & \mathrm{Aff}(\widetilde{T}(A))
\end{tikzcd}$$

It follows that $\rho = \rho_A$ in the case that $\widetilde{T}^b(A) = \widetilde{T}(A)$.

Let us briefly point out that the definition above does not depend on the choice of e. Suppose that $0 \leq e' \leq 1$ is another strictly positive element. We similarly define e'_n. Put $A'_n = \mathrm{Her}(e'_n)$. Note that, for any m, there is $k(m) \geq n$ such that $\|e - e'_{k(m)} e e'_{k(m)}\| < 1/2^m$. By applying a result of Rørdam (see, for example, Lemma 3.3 of [Elliott et al. 2020b] and its proof), one has, for each $n \geq 1$, an integer $k(n) > k(n-1) \geq 1$ and a partial isometry $w_n \in A^{**}$ such that $w_n w_n^* e_n = e_n w_n w_n^*$, $w_n^* c w_n \in A'_{k(n)}$ for $c \in A_n$ and $\|w_n e_n - e_n\| < 1/2^n$. Define $\varphi_n : A_n \to A'_{k(n)}$ by $\varphi_n(c) = w_{n+1}^* c w_{n+1}$ for $c \in A_n$. For each m, let $E_{n,m} = e_n \otimes 1_{M_m}$ and $W_{n,m} = w_n \otimes 1_{M_m}$. Then $\|W_{n,m} E_{n,m} - E_{n,m}\| < 1/2^n$. It follows that

$$\|(\varphi_n \otimes \mathrm{id}_{M_m})(c) - c\| < \left(\frac{1}{2^n}\right)\|c\|$$

for all $c \in M_m(A_n)$ and $m \in \mathbb{N}$. Moreover, for any $\tau \in \widetilde{T}(A)$, $\tau(\varphi_n(c)) = \tau(c)$ for all $c \in A_n$. Symmetrically, one has monomorphisms $\psi_k : A'_k \to A_{N(k)}$ such that $\|(\psi_k \otimes \mathrm{id}_{M_m})(a) - a\| < (1/2^k)\|a\|$ and $\tau(\psi_k(a)) = \tau(a)$ for all $a \in M_m(A'_k)$, $\tau \in \widetilde{T}(A)$ and $m \in \mathbb{N}$. Thus, by passing to subsequences, one obtains the following commutative diagram:

$$\begin{tikzcd}
K_0(A_1) \arrow[r, "\iota_{1*0}"] \arrow[dd, "\varphi_{1*0}"] \arrow[drr, "\rho^b_{A_1}"'] & K_0(A_2) \arrow[r, "\iota_{2*0}"] \arrow[drr, "\rho^b_{A_2}"] & \cdots \arrow[rrr] & & & K_0(A) \arrow[dr, "\rho"] \arrow[dd, leftrightarrow, "\mathrm{id}_{K_0(A)}"] & \\
& & \mathrm{Aff}(\widetilde{T}^b(A_1)) \arrow[r, "\iota_1^{\sharp}"'] \arrow[dd, "\varphi_1^{\sharp}"] & \mathrm{Aff}(\widetilde{T}^b(A_2)) \arrow[r, "\iota_2^{\sharp}"'] & \cdots \arrow[rr] & & \mathrm{Aff}(\widetilde{T}(A)) \arrow[dd, leftrightarrow, "\mathrm{id}_{\mathrm{Aff}(\widetilde{T}(A))}"'] \\
K_0(A'_1) \arrow[r, "\iota'_{1*0}"] \arrow[drr, "\rho^b_{A'_1}"'] & K_0(A'_2) \arrow[r, "\iota'_{2*0}"] \arrow[uu, "\psi_{2*0}"] \arrow[drr, "\rho^b_{A'_2}"] & \cdots \arrow[rrr] & & & K_0(A) \arrow[dr, "\rho'"] & \\
& & \mathrm{Aff}(\widetilde{T}^b(A'_1)) \arrow[r, "{\iota'_1}^{\sharp}"'] & \mathrm{Aff}(\widetilde{T}^b(A'_2)) \arrow[r, "{\iota'_2}^{\sharp}"'] \arrow[uu, "\psi^{\sharp}"] & \cdots \arrow[rr] & & \mathrm{Aff}(\widetilde{T}(A))
\end{tikzcd}$$

This implies that $\rho' = \rho$, where ρ' is induced by choosing e' instead of e.

Throughout, we define $\rho := \rho_A$. Moreover, this definition of pairing is consistent with the conventional definition of ρ_A in the case that A is unital, or the case $A = \mathrm{Ped}(A)$ (see 2.12).

We write $\pi_{\mathrm{aff}}^{\rho,A} : \mathrm{Aff}(\widetilde{T}(A)) \to \mathrm{Aff}(\widetilde{T}(A))/\overline{\rho_A(K_0(A))}$ for the quotient map. This may be simplified to $\pi_{\mathrm{aff}}^{\rho}$ if A is clear. When $T(A) \neq \varnothing$, we use the same notation for the quotient map $\mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$. In this case, we also write $\rho_A^{\sim} : K_0(\widetilde{A}) \to \mathrm{Aff}(T(A))$ for the map defined by $\rho_A^{\sim}([p])(\tau) = \tau(p)$ for projections $p \in M_l(\widetilde{A})$ (for all integers l) and for $\tau \in T(A)$.

Suppose that $\Phi : A \to B$ is a homomorphism. Then $\Phi(\mathrm{Ped}(A)) \subset \mathrm{Ped}(B)$. Let $0 \le e_A \le 1$ and $0 \le e_B \le 1$ be strictly positive elements of A and B, respectively. Let $e_n^A = f_{1/2^n}(e_A)$ and $e_n^B = f_{1/2^n}(e_B)$ be as defined above. Define $A_n = \mathrm{Her}(e_n^A)$ and $B_n = \mathrm{Her}(e_n^B)$. Then

$$\lim_{n\to\infty} \|\Phi(e_A) - e_n^B \Phi(e_A) e_n^B\| = 0.$$

By passing to a subsequence, as above (applying Lemma 3.3 of [Elliott et al. 2020b] repeatedly), one has a sequence of homomorphisms $\Psi_n : A_n \to B_{k(n)}$ such that $\|\iota_{B_{k(n)}} \circ \Psi_n(a) - \Phi \circ \iota_{A_n}(a)\| < (1/n)\|a\|$ and $\tau(\iota_{B_{k(n)}} \circ \Psi_n(a)) = \tau(\Phi \circ \iota_{A_n}(a))$ for all $a \in M_m(A_n)$ (for every $m \in \mathbb{N}$) and $\tau \in \widetilde{T}(B)$ (recall that we write H for $H \otimes \mathrm{id}_{M_k}$). Drawing a similar diagram as above, one obtains the following commutative diagram:

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\rho_A} & \mathrm{Aff}(\widetilde{T}(A)) \\ \Big\downarrow{\scriptstyle \Phi_{*0}} & & \Big\downarrow{\scriptstyle \Phi^\sharp} \\ K_0(B) & \xrightarrow{\rho_B} & \mathrm{Aff}(\widetilde{T}(B)) \end{array} \tag{2.18}$$

At least in the simple case, the construction above was pointed out by Elliott (see part (iv) of [Elliott 1995, §7] as well as Proposition 5.1 below for more detail). The above also works when we do not assume that quasitraces are traces (but quasitraces would be used). We omit a more general definition here to avoid longer discussion.

Definition 2.19. We now describe the Elliott invariant for separable simple C^*-algebras (see [Elliott 1995; 1996]). Let us consider the case $\widetilde{T} \neq \{0\}$. In this case the Elliott invariant is the sextuple

$$\mathrm{Ell}(A) := (K_0(A), \Sigma(K_0(A)), \widetilde{T}(A), \Sigma_A, \rho_A, K_1(A)),$$

where $\Sigma(K_0(A)) = \{x \in K_0(A) : x = [p] \text{ for some projection } p \in A\}$, and Σ_A is a function in $\mathrm{LAff}_+(\widetilde{T}(A))$ defined by

$$\Sigma_A(\tau) = \sup\{\tau(a) : a \in \mathrm{Ped}(A)_+,\ \|a\| \le 1\} \tag{2.20}$$

(see Theorem 5.2). Let $e_A \in A$ be a strictly positive element. Then $\Sigma_A(\tau) = \lim_{\varepsilon\to 0}\tau(f_\varepsilon(e_A))$ for all $\tau \in \widetilde{T}(A)$, which is independent of the choice of e_A.

Let B be another separable C^*-algebra. A map $\Gamma : \mathrm{Ell}(A) \to \mathrm{Ell}(B)$ is a homomorphism if Γ gives group homomorphisms $\kappa_i : K_i(A) \to K_i(B)$ $(i = 0, 1)$ and a continuous cone map $\gamma : T(B) \xrightarrow{\sim} T(A)$ such that $\rho_B(\kappa_0(x))(\tau) = \rho_A(x)(\gamma(\tau))$ for all $x \in K_0(A)$ and $\tau \in \widetilde{T}(A)$, $\kappa_0(\Sigma(K_0(A))) \subset \Sigma(K_0(B))$ and $\Sigma_A(\gamma(\tau)) \le \Sigma_B(\tau)$ for all $\tau \in \widetilde{T}(B)$.

We say that Γ is an isomorphism if Γ is a homomorphism, κ_i is a group isomorphism $(i = 0, 1)$, γ is a cone homeomorphism, $\kappa_0(\Sigma(K_0(A))) = \Sigma(K_0(B))$, and $\Sigma_A(\gamma(\tau)) = \Sigma_B(\tau)$ for all $\tau \in \widetilde{T}(B)$.

In the case that $\rho_A(K_0(A)) \cap \mathrm{LAff}_+(\widetilde{T}(A)) = \{0\}$, we often consider the (special) reduced case that $T(A)$ is compact, which gives a base for $\widetilde{T}(A)$. Then, we may write $\mathrm{Ell}(A) = (K_0(A), T(A), \rho_A, K_1(A))$. Note that, in this situation, $\Sigma(K_0(A)) = \{0\}$, $\widetilde{T}(A)$ is determined by $T(A)$ and $\Sigma_A(\tau) = 1$ for all $\tau \in T(A)$.

Definition 2.21 [Robert 2016]. Let A be a C^*-algebra. We say A has almost stable rank one, if A has the property that the set of invertible elements of the unitization $\widetilde{B}$ of every nonzero hereditary C^*-subalgebra B of A is dense in B. A is said to stably have almost stable rank one, if $M_n(A)$ has almost stable rank one for all integers $n \ge 1$.

Definition 2.22. Let A be a C^*-algebra with $T(A) \ne \varnothing$. Suppose that A has a strictly positive element $e_A \in \mathrm{Ped}(A)_+$ with $\|e_A\| = 1$. Then $0 \notin \overline{T(A)}^w$, the closure of $T(A)$ in $\widetilde{T}(A)$ [Elliott et al. 2020b, Theorem 4.7]. Define

$$\lambda_s(A) = \inf\{d_\tau(e_A) : \tau \in \overline{T(A)}^w\} = \lim_{n\to\infty}(\inf\{\tau(f_{1/n}(e_A)) : \tau \in T(A)\}) > 0.$$

Let A be a C^*-algebra with $T(A) \ne \{0\}$. There is an affine map

$$r_{\mathrm{aff}} : A_{\mathrm{s.a.}} \to \mathrm{Aff}(T_0(A)), \quad r_{\mathrm{aff}}(a) : \tau \mapsto \hat{a}(\tau) = \tau(a) \quad \text{for all } \tau \in T_0(A), a \in A_{\mathrm{s.a.}}.$$

Denote by $A^q_{\mathrm{s.a.}}$ the space $r_{\mathrm{aff}}(A_{\mathrm{s.a.}})$ and $A^q_+ = r_{\mathrm{aff}}(A_+)$.

Definition 2.23. Let A and B be two C^*-algebras. A sequence of linear maps $L_n : A \to B$ is said to be approximately multiplicative if

$$\lim_{n\to\infty}\|L_n(a)L_n(b) - L_n(ab)\| = 0 \quad \text{for all } a, b \in A.$$

Let $\varphi, \psi : A \to B$ be homomorphisms. We say φ and ψ are asymptotically unitarily equivalent if there is a continuous path of unitaries $\{u(t) : t \in [1, \infty)\}$ in B (if B is not unital, $u(t) \in \widetilde{B}$) such that

$$\lim_{t\to\infty} u^*(t)\varphi(a)u(t) = \psi(a) \quad \text{for all } a \in A.$$

We say φ and ψ are strongly asymptotically unitarily equivalent if $u(1) \in U_0(B)$ (or in $U_0(\widetilde{B})$).

Definition 2.24. Let A and B be C^*-algebras, and let $T : A_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ be defined by $a \mapsto (N(a), M(a))$, where $N(a) \in \mathbb{N}$ and $M(a) \in \mathbb{R}_+ \setminus \{0\}$. Let $\mathcal{H} \subset A_+ \setminus \{0\}$. A map $L : A \to B$ is said to be T-$\mathcal{H}$-full, if, for any $a \in \mathcal{H}$ and any $b \in B_+$ with $\|b\| \le 1$, any $\varepsilon > 0$, there are $x_1, x_2, \dots, x_N \in B$ with $N \le N(a)$ and $\|x_i\| \le M(a)$ such that

$$\left\| \sum_{j=1}^{N} x_j^* L(a) x_j - b \right\| \le \varepsilon. \tag{2.25}$$

L is said to be exactly T-$\mathcal{H}$-full, if $\varepsilon = 0$ in the above formula.

Definition 2.26. Let A and B be C^*-algebras and $\varphi_0, \varphi_1 : A \to B$ be homomorphisms. By the mapping torus M_{φ_0,φ_1}, we mean the following C^*-algebra:

$$M_{\varphi_0,\varphi_1} = \{(f, a) \in C([0, 1], B) \oplus A : f(0) = \varphi_0(a) \text{ and } f(1) = \varphi_1(a)\}. \tag{2.27}$$

One has the short exact sequence

$$0 \to SB \xrightarrow{\iota} M_{\varphi,\psi} \xrightarrow{\pi_e} A \to 0,$$

where $\iota : SB \to M_{\varphi,\psi}$ is the embedding and π_e is the quotient map from $M_{\varphi,\psi}$ to A. Denote by $\pi_t : M_{\varphi,\psi} \to B$ the point evaluation at $t \in [0, 1]$.

Let F_1 and F_2 be two finite-dimensional C^*-algebras. Suppose that there are (not necessary unital) homomorphisms $\varphi_0, \varphi_1 : F_1 \to F_2$. Denote the mapping torus M_{φ_1,φ_2} by

$$\begin{aligned} A &= A(F_1, F_2, \varphi_0, \varphi_1) \\ &= \{(f, g) \in C([0, 1], F_2) \oplus F_1 : f(0) = \varphi_0(g) \text{ and } f(1) = \varphi_1(g)\}. \end{aligned}$$

Denote by $\mathcal{C}$ the class of all C^*-algebras of the form $A = A(F_1, F_2, \varphi_0, \varphi_1)$. These C^*-algebras are called Elliott–Thomsen building blocks as well as one-dimensional noncommutative CW complexes; see [Elliott and Thomsen 1994; Elliott 1996].

Recall that $\mathcal{C}_0$ is the class of all $A \in \mathcal{C}$ with $K_0(A)_+ = \{0\}$ such that $K_1(A) = 0$ and $\lambda_s(A) > 0$, and $\mathcal{C}_0^{(0)}$ the class of all $A \in \mathcal{C}_0$ such that $K_0(A) = 0$. Denote by $\mathcal{C}'$, $\mathcal{C}_0'$ and $\mathcal{C}_0^{0'}$ the class of all full hereditary C^*-subalgebras of C^*-algebras in $\mathcal{C}$, $\mathcal{C}_0$ and $\mathcal{C}_0^{(0)}$, respectively.

Definition 2.28 (cf. [Elliott et al. 2020b, Definition 8.1 and Proposition 8.2]). Recall the definition of class $\mathcal{D}$ and $\mathcal{D}_0$.

Let A be a nonunital simple C^*-algebra with a strictly positive element $a \in A$ with $\|a\| = 1$. Suppose that there exists $1 > \mathfrak{f}_a > 0$, for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $b \in A_+ \setminus \{0\}$, there are $\mathcal{F}$-ε-multiplicative completely positive contractive linear maps $\varphi : A \to A$ and $\psi : A \to D$ for some C^*-subalgebra $D \subset A$

with $D \in \mathcal{C}_0'$ (or $\mathcal{C}_0^{0'}$) such that $D \perp \varphi(A)$, and

$$\|x - (\varphi(x) + \psi(x))\| < \varepsilon \qquad \text{for all } x \in \mathcal{F} \cup \{a\}, \tag{2.29}$$

$$c \lesssim b, \tag{2.30}$$

$$t(f_{1/4}(\psi(a))) \geq \mathfrak{f}_a \qquad \text{for all } t \in T(D), \tag{2.31}$$

where c is a strictly positive element of $\overline{\varphi(A)A\varphi(A)}$. Then we say $A \in \mathcal{D}$ (or $\mathcal{D}_0$).

Note that, by Remark 8.11 of [Elliott et al. 2020b], D can *always* be chosen to be in $\mathcal{C}_0$ (or $\mathcal{C}_0^{(0)}$).

When $A \in \mathcal{D}$ and is separable, then $A = \mathrm{Ped}(A)$ (see Corollary 11.3 of [Elliott et al. 2020b]). Let $a \in A_+$ with $\|a\| = 1$ be a strict positive element. Put

$$d = \inf\{\tau(f_{1/4}(a)) : \tau \in T(A)\} > 0. \tag{2.32}$$

Then, for any $0 < \eta < d$, $\mathfrak{f}_a$ can be chosen to be $d - \eta$ (see Remark 9.2 of [Elliott et al. 2020b]). One may also assume that $f_{1/4}(\psi(a))$ is full in D. Furthermore, there exists a map: $T : A_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ which is independent of $\mathcal{F}$ and ε such that, for any finite subset $\mathcal{H} \subset A_+ \setminus \{0\}$, we can further require that ψ is exactly T-$\mathcal{H}$-full (see Theorem 8.3 and Remark 9.2 of [Elliott et al. 2020b]). For any $n \geq 1$, one can choose a strictly positive element $b \in A$ with $\|b\| = 1$ such that $f_{1/4}(b) \geq f_{1/n}(a)$. Therefore, if A has continuous scale, d can be chosen to be 1 if the strictly positive element is chosen accordingly.

In [Elliott et al. 2020b], it is proved that if A a separable simple C^*-algebra in $\mathcal{D}$, then A is stably projectionless, has stable rank one and $\mathrm{Cu}(A) = \mathrm{LAff}_+(\widetilde{T}(A))$, and every 2-quasitrace on A is a trace (see Propositions 9.3 and 11.11 of [Elliott et al. 2020b]).

Let A be a nonzero separable stably projectionless simple C^*-algebra. Recall that A has generalized tracial rank one, written $\mathrm{gTR}(A) = 1$, if there exists $e \in \mathrm{Ped}(A)_+$ with $\|e\| = 1$ such that $\overline{eAe} \in \mathcal{D}$ (see Definition 11.6 of [Elliott et al. 2020b]). It should be noted that, in the definition of D above, if we assume that A is unital, and replace $\mathcal{C}_0$ by $\mathcal{C}$, then $\mathrm{gTR}(A) \leq 1$ (see Definitions 9.1 and 9.2 and Remark 9.3 of [Gong et al. 2020a]). But the condition (2.31) and constant $\mathfrak{f}_a$ are not needed. In the case $K_0(A)_+ \neq \{0\}$ but A is not unital, we may define $\mathrm{gTR}(A) \leq 1$, if for some nonzero projection $e \in M_k(A)$, $\mathrm{gTR}(eM_k(A)e) \leq 1$ (see [Gong et al. 2020a]). A C^*-algebra A is said to have rationally generalized tracial rank at most one, if $A \otimes U$ has generalized tracial rank at most one for some infinite-dimensional UHF-algebras U.

Definition 2.33. Let $A \in \mathcal{D}$ be as defined in Definition 2.28. If, in addition, for any integer n, we can choose D and $\psi : A \to D$ to satisfy the following condition:

$D = M_n(D_1)$ for some $D_1 \in \mathcal{C}_0$ such that

$$\psi(x) = \mathrm{diag}(\overbrace{\psi_1(x), \psi_1(x), \ldots, \psi_1(x)}^{n}) \quad \text{for all } x \in \mathcal{F}, \tag{2.34}$$

where $\psi_1 : A \to D_1$ is an $\mathcal{F}$-ε-multiplicative completely positive contractive linear map, then we say $A \in \mathcal{D}^d$.

Note that here, as in [Elliott et al. 2020b, Theorem 8.3 and Remark 9.2], the map T mentioned in Definition 2.28 is also assumed to exist and $\mathfrak{f}_a$ can be also chosen as $d - \eta$ for any $\eta > 0$ with d as in (2.32) for a certain strictly positive element a.

Remark 2.35. It follows from Theorems 10.4 and 10.7 of [Elliott et al. 2020b] that if $A \in \mathcal{D}_0$, then $A \in \mathcal{D}^d$. Moreover, D_1 can be chosen in $\mathcal{C}_0^{(0)}$, and if $A \in \mathcal{D}$, then D_1 can be chosen in $\mathcal{C}_0$. If A is a separable simple C^*-algebra in $\mathcal{D}$ and A has an approximate divisible property defined in [Elliott et al. 2020b, Definition 10.1], then $A \in \mathcal{D}^d$.

Definition 2.36. Throughout the paper, $\mathcal{W}$ is the separable simple C^*-algebra which is an inductive limit of C^*-algebras in $\mathcal{C}_0^{(0)}$ with a unique tracial state, which is first constructed in [Razak 2002]. It is proved in [Elliott et al. 2020a] that $\mathcal{W}$ is the unique separable simple C^*-algebra with finite nuclear dimension which is KK-contractible and with a unique tracial state. Denote by $\tau_{\mathcal{W}}$ the unique tracial state of $\mathcal{W}$.

Let A be a C^*-algebra and let τ be a nonzero trace of A. We say that τ is a $\mathcal{W}$-trace, if there exists a sequence of approximately multiplicative completely positive contractive linear maps $\varphi_n : A \to \mathcal{W}$ such that

$$\lim_{n\to\infty} \tau_{\mathcal{W}} \circ \varphi_n(a) = \tau(a) \quad \text{for all } a \in A. \tag{2.37}$$

Throughout, Q is the UHF-algebra with $K_0(Q) = \mathbb{Q}$ and $[1_Q] = 1$ and with the unique tracial state tr. Recall that $T_{\mathrm{qd}}(A)$ is the set of those $\tau \in T(A)$ such that there exists a sequence of approximately multiplicative completely positive contractive linear maps $\varphi_n : A \to Q$ such that

$$\lim_{n\to\infty} \mathrm{tr} \circ \varphi_n(a) = \tau(a) \quad \text{for all } a \in A. \tag{2.38}$$

Definition 2.39 [Gong and Lin 2020a, Definition 9.3]. Let A be a separable C^*-algebra. We say that A has property (W), if there is a map $T : A_+^{\mathbf{1}} \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ and a sequence of approximately multiplicative completely positive contractive linear maps $\varphi_n : A \to \mathcal{W}$ such that, for any finite subset $\mathcal{H} \subset A_+^{\mathbf{1}} \setminus \{0\}$, there exists an integer $n_0 \geq 1$ such that φ_n is exactly T-$\mathcal{H}$-full (see Definition 5.5 and Theorem 5.7 of [Elliott et al. 2020b]) for all $n \geq n_0$.

3. $\mathcal{W}$-traces

Theorem 3.1. *Let Δ be a compact convex set and let G be a countable abelian subgroup of* $\mathrm{Aff}(\Delta)$. *Suppose that $G \cap \mathrm{Aff}_+(\Delta) = \{0\}$. Then there exists $t \in \Delta$ such that $g(t) = 0$ for all $g \in G$.*

Proof. Let us assume that $0 \notin \Delta$; otherwise we can choose $t = 0$. Let

$$S_+ = \{f \in \mathrm{Aff}(\Delta) : f(x) > 0 \text{ for all } x \in \Delta\} = \mathrm{Aff}_+(\Delta) \setminus \{0\}.$$

It is an open convex subset of $\mathrm{Aff}(\Delta)$. Let G_1 be the convex hull of G. Note that if $g_1, g_2 \in G$ and $r \in \mathbb{Q}$ with $0 < r < 1$, then $rg_1 + (1-r)g_2 \notin S_+$. To see this, we note that there is an integer $m \geq 1$ such that mr and $m(1-r)$ are both integers. In other words, $m(rg_1 + (1-r)g_2) \in G$. Therefore $m(rg_1 + (1-r)g_2) \notin S_+$. Hence $rg_1 + (1-r)g_2 \notin S_+$. Since S_+ is open, this implies that $G_1 \cap S_+ = \varnothing$.

By the Hahn–Banach separating theorem, there is a real continuous linear functional f on $\mathrm{Aff}(\Delta)$ and $r_0 \in \mathbb{R}$ such that

$$f(s) < r_0 \leq f(g) \quad \text{for all } s \in S_+ \text{ and } g \in G_1. \tag{3.2}$$

If $f(g) \geq r_0$, then $f(-g) \leq -r_0$. Note that if $g \in G$, then $mg \in G$ for any $m \in \mathbb{Z}$. Hence $mf(g) \geq r_0$ for all $m \in \mathbb{Z}$. It follows that $f(g) = 0$ for all $g \in G$ and $r_0 \leq 0$.

By (3.2),

$$-f(s) > -r_0 \geq 0 \quad \text{for all } s \in S_+. \tag{3.3}$$

Let $f' = -f$. Since $\mathrm{Aff}_+(\Delta) = S_+ \cup \{0\} \subset \overline{S_+}$,

$$f'(s) \geq 0 \quad \text{for all } s \in \mathrm{Aff}(\Delta)_+ = \{s \in \mathrm{Aff}(\Delta) : s(\tau) \geq 0\}. \tag{3.4}$$

In other words, f' is a positive linear functional on $\mathrm{Aff}(\Delta)$. Let $f_1 = f'/\|f'\|$. Then $f_1(1_\Delta) = 1$. Consider $\mathrm{Aff}(\Delta) \subset C(\Delta)$. By the Hahn–Banach extension theorem there is a linear functional $\bar{f_1}$ on $C(\Delta)$ such that $(\bar{f_1})|_{\mathrm{Aff}(\Delta)} = f_1$, and $\|\bar{f_1}\| = \|f_1\| = 1$. Since $\|\bar{f_1}\| = \bar{f_1}(1_\Delta) = 1$, $\bar{f_1}$ is a positive functional (see Proposition 3.1.4 of [Pedersen 1979]), and therefore it is a state of $C(\Delta)$. Let $S(C(\Delta))$ be the state space. Then it is compact and convex. By the Krein–Milman theorem, $\bar{f_1}$ is the limit of $\{\mu_n\}$, where

$$\mu_n = \sum_{i=1}^{m(n)} \alpha_{n,i} \rho_{n,i},$$

$0 \leq \alpha_{n,i} \leq 1$ are positive numbers with $\sum_{i=1}^{m(n)} \alpha_{n,i} = 1$, and $\rho_{n,i}$ are pure states of $C(\Delta)$. Note that, for each i and n, there is $t_{i,n} \in \Delta$ such that $\rho_{n,i}(a) = a(t_{i,n})$ for all $a \in C(\Delta)$. Since Δ is convex,

$$\tau_n = \sum_{i=1}^{m(n)} \alpha_{i,n} t_{i,n} \in \Delta.$$

Then $a(\tau_n) = a(\mu_n)$ for all $a \in \mathrm{Aff}(\Delta)$, by the computation

$$a(\mu_n) = \sum_{i=1}^{m(n)} \alpha_{n,i} a(\rho_{n,i}) = \sum_{i=1}^{m(n)} \alpha_{n,i} a(t_{i,n})$$
$$= a\left(\sum_{i=1}^{m(n)} \alpha_{n,i} t_{i,n}\right) = a(\tau_n) \quad \text{for all } a \in \mathrm{Aff}(\Delta).$$

Since Δ is compact, we conclude that there is $\tau \in \Delta$ such that

$$\bar{f}_1(a) = a(\tau) \quad \text{for all } a \in \mathrm{Aff}(\Delta). \tag{3.5}$$

We have just shown that

$$g(\tau) = 0 \quad \text{for all } g \in G. \qquad \square$$

Corollary 3.6 (cf. [Elliott et al. 2020a, Corollary A.7]). *Let $A \in \mathcal{D}$ be a separable simple C^*-algebra with continuous scale. Then there exists $t_o \in T(A)$ such that $\rho_A(x)(t_o) = 0$ for all $x \in K_0(A)$.*

Proof. By Theorem 11.5 and Proposition 11.8 of [Elliott et al. 2020b], A has stable rank one and $\mathrm{Cu}(A) = \mathrm{LAff}_+(T(A))$ (see also [Elliott et al. 2020b, Proposition 9.1]). It follows from Corollary A.7 of [Elliott et al. 2020a] that

$$\rho_A(K_0(A)) \cap \mathrm{Aff}_+(T(A)) = \{0\}.$$

By Theorem 3.1, there is $t_o \in T(A)$ such that $\rho_A(x)(t_o) = 0$ for all $x \in K_0(A)$. $\square$

Theorem 3.7. *Let $A \in \mathcal{D}$ be a separable simple C^*-algebra with continuous scale. Then A has at least one $\mathcal{W}$-tracial state. Moreover, A has property* (W), *i.e., there is a map $T : A_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ and a sequence of approximately multiplicative completely positive contractive linear maps $\varphi_n : A \to \mathcal{W}$ such that, for any finite subset $\mathcal{H} \subset A_+ \setminus \{0\}$, there exists $n_0 \geq 1$ such that φ_n is exactly T-$\mathcal{H}$-full* (see Definition 2.24) *for all $n \geq n_0$, and there exists a $\tau \in T(A)$ such that*

$$\tau(a) = \lim_{n\to\infty} \tau_{\mathcal{W}} \circ \varphi_n(a) \quad \textit{for all } a \in A.$$

Proof. Fix a strictly positive element $e \in A_+$. It follows from Remark 9.2 of [Elliott et al. 2020b] that, for any $\frac{1}{2} > \eta > 0$, we may choose $\mathfrak{f}_e > 1-\eta$ in Definition 2.28. By Definition 2.28, we obtain two sequences of mutually orthogonal C^*-subalgebras $B_n, C_n \subset A$, $B_n = \overline{a_n A a_n}$ for some positive elements $a_n \in A$ with $\|a_n\| = 1$, $C_n \in \mathcal{C}_0$. We also have two sequences of completely positive contractive linear maps $\varphi_{n,0} : A \to B_n$ and $\varphi_{n,1} : A \to C_n$ such that $\varphi_{n,0}(A) \perp C_n$ and the following hold:

$$\lim_{n\to\infty} \|x - (\varphi_{n,0}(x) + \varphi_{n,1}(x))\| = 0 \qquad \text{for all } x \in A, \tag{3.8}$$

$$\lim_{n\to\infty} \|\varphi_{n,i}(xy) - \varphi_{n,i}(x)\varphi_{n,i}(y)\| = 0 \qquad \text{for all } x, y \in A,\ i = 0, 1, \tag{3.9}$$

$$\lim_{n\to\infty} \sup\{d_\tau(a_n) : \tau \in T(A)\} = 0, \tag{3.10}$$

$$\tau(f_{1/2}(\varphi_{n,1}(e))) > 1 - \eta \qquad \text{for all } \tau \in T(C_n). \tag{3.11}$$

Let $b_n = f_{1/2}(\varphi_{n,1}(e)) \in C_n$. Then $0 \le b_n \le 1$. The inequality (3.11) implies that

$$\inf\{\tau(b_n) : \tau \in T(C_n)\} > 1 - \eta. \tag{3.12}$$

Hence $\lambda_s(C_n) > 1 - \eta$ (see Definition 2.22).

As noted in the middle of Definition 2.28, A is stably projectionless, and has stable rank one, $\mathrm{Ped}(A) = A$ (see [Elliott et al. 2020b, Corollary 11.3]), $QT(A) = T(A)$ and $\mathrm{Cu}(A) = \mathrm{LAff}_+(\widetilde{T}(A))$. By Theorem 7.3 of [Elliott et al. 2020b] (see also [Robert 2012, Theorem 6.2.3] of [Robert and Santiago 2021, Theorem 6.11]),

$$\mathrm{Cu}^\sim(A) = K_0(A) \sqcup \mathrm{LAff}_+^\sim(\widetilde{T}(A)).$$

By Corollary 3.6, there is $t_D \in T(A)$ such that $\rho_A(x)(t_D) = 0$ for all $x \in K_0(A)$. Recall that $\mathrm{Cu}^\sim(\mathcal{W}) = \{0\} \sqcup (\mathbb{R} \cup \{\infty\})$; see [Robert 2012, Theorem 6.2.3]. Define $\gamma : \mathrm{Cu}^\sim(A) \to \mathrm{Cu}^\sim(\mathcal{W})$ by $\gamma|_{K_0(A)} = 0$ and $\gamma(f)(\tau_{\mathcal{W}}) = f(t_D)$ for $f \in \mathrm{LAff}_+^\sim(\widetilde{T}(A))$, where $\tau_{\mathcal{W}}$ is the unique tracial state of $\mathcal{W}$. We can now see that γ is a morphism in **Cu**. Note that $\gamma(\langle c\rangle) \ne 0$ for any $c \in A_+ \setminus \{0\}$ as A is simple. Let $\iota_n : C_n \to A$ be the embedding. By Theorem 1.0.1 of [Robert 2012], there exists, for each n, a homomorphism $\psi_n : C_n \to \mathcal{W}$ such that $\mathrm{Cu}^\sim(\psi_n) = \gamma \circ \mathrm{Cu}^\sim(\iota_n)$. In particular, by (3.12), $d_{\tau_{\mathcal{W}}}(\psi_n(e_{C_n})) > 1 - \eta$, where e_{C_n} is a strictly positive element of C_n. Since ι_n is injective, $\mathrm{Cu}^\sim(\psi_n)(\langle c\rangle) \ne 0$ for any $c \in C_{n+} \setminus \{0\}$. It follows that ψ_n is injective. Define $\Phi_n = \psi_n \circ \varphi_{n,1} : A \to \mathcal{W}$. Then Φ_n is a sequence of completely positive contractive linear maps satisfying the following:

$$\lim_{m\to\infty} \|\Phi_m(xy) - \Phi_m(x)\Phi_m(y)\| = 0 \qquad \text{for all } x, y \in A, \tag{3.13}$$

$$\lim_{m\to\infty} \|\Phi_m(x)\| = \|x\| \qquad \text{for all } x \in A, \tag{3.14}$$

$$\lim_{m\to\infty} \tau_{\mathcal{W}}(f_{1/2}(\Phi_m(e))) \ge (1 - \eta). \tag{3.15}$$

Note that this holds for each $\frac{1}{2} > \eta > 0$. By choosing $\eta_n \to 0$, we may further assume that there exists an increasing sequence $\{e_n\}$ of positive elements with $0 \le e_n \le 1$ such that, in the above, we have

$$\lim_{n\to\infty} \tau_{\mathcal{W}}(\Phi_m(e_m)) = 1. \tag{3.16}$$

Then, by passing to a subsequence, we may assume that there exists $\tau \in T(A)$ such that

$$\lim_{n\to\infty} \tau_{\mathcal{W}}(\Phi_m(a)) = \tau(a) \quad \text{for all } a \in A. \tag{3.17}$$

The last part of the statement follows from the first part and Theorem 5.7 of [Elliott et al. 2020b]. □

Theorem 3.18. *Let C be a separable amenable C^*-algebra with the property* (W), *let A be a separable simple C^*-algebra with continuous scale which satisfies the UCT and has the property* (W), *and let $\kappa \in KL(C, A)$. Then there exists a sequence of completely positive contractive linear maps $\{\varphi_n\}$ from C to $A \otimes M_{m(n)}$ (for some integers $m(n)$) such that*

$$\lim_{n\to\infty} \|\varphi_n(a)\varphi_n(b) - \varphi_n(ab)\| = 0 \quad \textit{for all } a, b \in C \textit{ and } [\{\varphi_n\}] = \kappa. \tag{3.19}$$

Proof. Note that C satisfies the conditions in Definition 9.3 of [Gong and Lin 2020a]. Therefore, the theorem follows from the combination of Theorem 10.8 and Lemma 12.5 of [Gong and Lin 2020a], i.e., one first applies Theorem 10.8 to obtain maps from C to $A \otimes \mathcal{Z}_0 \otimes M_{k(n)}$, and then applies Lemma 12.5 to obtain maps from $A \otimes \mathcal{Z}_0 \otimes M_{k(n)}$ to $A \otimes M_{l(n)}$ (for some integers $k(n)$ and $l(n)$). □

4. Range and Models

This section is a refinement of Elliott's construction of model simple C^*-algebras. The main results of this section are stated as Theorem 4.118 and Theorem 4.137. Both are restatements of Elliott's original theorem [Elliott 1996] with some technical additions. These refinements are needed for our purposes. Some subtle details described in the Elliott construction are also dealt with (see, for example, the first line of page 88 of [Elliott 1996]). To avoid some technical difficulties in the construction described on page 88 of [Elliott 1996] in the general setting, we do not restrict ourselves to those building blocks described in [Elliott 1996]; see also [Li 2020, Remark 1.4]. The refined construction in this section is similar to that of the unital case (see also Sections 2.1 and 2.2 of [Li 2020]) in spirit. However, for the later purpose of the isomorphism theorem, we also require that the nontorsion infinitesimal elements of K_0-groups be stored at the building blocks. So the construction of the model C^*-algebras in this section is somewhat different from what is described in [Elliott 1996] and [Li 2020]. Other features in Theorem 4.118 and Theorem 4.137 are also needed. We also make this section self-contained as much as possible. Some previously omitted computations are presented.

4.1. Let Δ be any compact metrizable Choquet simplex and let G be any countable abelian group. Let $\rho : G \to \mathrm{Aff}(\Delta)$ be any homomorphism satisfying the condition

$$\text{for any } g \in G, \text{ there is a } \tau \in \Delta \text{ such that } \rho(g)(\tau) \le 0. \tag{$*$}$$

In other words, $\rho(G) \cap \mathrm{Aff}_+(\Delta) \setminus \{0\} = \varnothing$ (recall that $\mathrm{Aff}_+(\Delta)$ denotes the set of all continuous affine functions $f : \Delta \to \mathbb{R}$ such that $f(\tau) > 0$ for any $\tau \in \Delta$ and the zero function). Note that we include the case that $\rho(G) = \{0\}$.

In the first part of this section, we assume that G is torsion free, and construct a stably projectionless simple C^*-algebra A with continuous scale such that $K_0(A) = G$, $K_1(A) = \{0\}$, $T(A) = \Delta$ and $\rho_{_A} : K_0(A) \to \mathrm{Aff}(T(A))$ is the map $\rho : G \to \mathrm{Aff}(\Delta)$, when one identifies $K_0(A)$ with G, and $T(A)$ with Δ. The C^*-algebra A is an inductive limit of C^*-algebras $A_n \in \mathcal{C}_0$ (Elliott–Thomsen building blocks) of the form

$$\begin{aligned} A_n &= A(F_n, E_n, \beta_{n,0}, \beta_{n,1}) \\ &:= \bigl\{(f, a) \in C([0, 1], E_n) \oplus F_n \mid \beta_{n,0}(a) = f(0), \beta_{n,1}(a) = f(1)\bigr\}, \end{aligned}$$

where F_n, E_n are finite-dimensional C^*-algebras, and $\beta_{n,0}, \beta_{n,1} : F_n \to E_n$ are (not necessarily unital) homomorphisms.

Note that if $\beta_{n,0} \oplus \beta_{n,1} : F_n \to E_n \oplus E_n$ is injective, then the element a is completely determined by f, so we can simply write (f, a) as f. In our construction, we will always be in this situation.

Since the limit algebra A to be constructed is stably projectionless, in each step, the algebra A_n will also be stably projectionless. The construction presented here is a refinement of Elliott's construction [1996] and is similar to [Gong et al. 2020a, §13] (which is for the unital case).

4.2. Let us keep the notation in Notation 13.1 and 13.2 of [Gong et al. 2020a]. In particular,

$$x^{\sim k} := \{\overbrace{x, x, \dots, x}^{k}\}.$$

Let A be an Elliott–Thomsen building block. Denoted by $\mathrm{Sp}(A)$ the set of the equivalence classes of all irreducible representations of A, and $RF(A)$ the set of finite-dimensional representations of A. As in [Gong et al. 2020a], each element of $RF(A)$ can be regarded as a subset of $\mathrm{Sp}(A)$ with multiplicities. For any homomorphism $\varphi : A \to M_k(\mathbb{C})$, let $\mathrm{Sp}(\varphi) = \{x \in \mathrm{Sp}(A); \ker(\varphi) \supset \ker(x)\}$.

Suppose that φ is (unitarily equivalent to) a direct sum of k_1 copies of x_1, k_2 copies of x_2, ..., and k_i copies of x_i, where $x_1, x_2, \dots, x_i \in \mathrm{Sp}(A)$. Then we write $\mathrm{SP}(\varphi) := \{x_1^{\sim k_1}, x_2^{\sim k_2}, \dots, x_i^{\sim k_i}\}$. Note that if $\varphi, \psi : A \to M_k$ are two homomorphisms then φ and ψ are unitarily equivalent if and only if $\mathrm{SP}(\varphi) = \mathrm{SP}(\psi)$.

Let C be a vector space and $x = (x_1, x_2, \dots, x_n)$, where $x_i \in C$, $i = 1, 2, \dots, n$. For each integer $k \geq 1$, consider k-tuple $S = (i_1, i_2, \dots, i_k)$, where $i_s \in \{1, 2, \dots, n\}$. We write

$$\mathrm{diag}_{j\in S}(x_j) := \mathrm{diag}(x_{i_1}, x_{i_2}, \dots, x_{i_k})$$

for a diagonal element in $M_k(C)$. In particular, we use $\mathrm{diag}_{1\leq j\leq n}(x_j)$ to denote $\mathrm{diag}(x_1, x_2, \dots, x_n)$.

We adopt the following convention:

$$\operatorname{diag}(\operatorname{diag}(a,b),\operatorname{diag}(c,d,e))=\operatorname{diag}(a,b,c,d,e),$$
$$\operatorname{diag}(\operatorname{diag}_{1\le i\le 3}(a_i),\operatorname{diag}_{1\le i\le 2}(b_i))=\operatorname{diag}(a_1,a_2,a_3,b_1,b_2),$$
$$\operatorname{diag}\big(\operatorname{diag}(a_1^{\sim 2},a_2^{\sim 3},a_3),\operatorname{diag}(b_1,b_2^{\sim 3})\big)=\operatorname{diag}(a_1,a_1,a_2,a_2,a_2,a_3,b_1,b_2,b_2,b_2).$$

As in Notation 13.1 and 13.2 of [Gong et al. 2020a], for any two subhomogeneous algebras A and B, and a homomorphism $\varphi : A \to B$, if $\theta \in \operatorname{Sp}(B)$ is represented by $\theta : B \to M_k(\mathbb{C})$, then we use $\varphi|_\theta$ to denote $\theta \circ \varphi : A \to M_k(\mathbb{C})$.

4.3. Let us fix some notation for this section. Let

$$F_n=\bigoplus_{i=1}^{p_n} M_{[n,i]}(\mathbb{C}) \quad\text{and}\quad E_n=\bigoplus_{i=1}^{l_n} M_{\{n,i\}}(\mathbb{C}),$$

where $p_n, l_n, [n,i], \{n,i\}$ are positive integers which will be constructed later. Let $\beta_{n,0}, \beta_{n,1} : F_n \to E_n$ be two (not necessarily unital) homomorphisms. Put

$$\begin{aligned} A_n &= A(F_n,E_n,\beta_{n,0},\beta_{n,1})\\ &= \big\{(f,a)\in C([0,1],E_n)\oplus F_n \mid \beta_{n,0}(a)=f(0),\ \beta_{n,1}(a)=f(1)\big\}. \end{aligned}$$

Write $(f,a)=(f_1,f_2,\dots,f_{l_n};a_1,a_2,\dots,a_{p_n})\in A_n$, where $f_i\in C([0,1],M_{\{n,i\}}(\mathbb{C}))$ and $a_j\in M_{[n,j]}(\mathbb{C})$. Let $\eta_{n,i}(t),\theta_{n,j}\in\operatorname{Sp}(A_n)$, for $0<t<1$, $i=1,2,\dots,l_n$, $j=1,2,\dots,p_n$, be defined as

$$\eta_{n,i}(t)(f,a)=f_i(t)\in M_{\{n,i\}}(\mathbb{C})\subset E_n \quad\text{and}\quad \theta_{n,j}(f,a)=a_j\in M_{[n,j]}(\mathbb{C})\subset F_n.$$

We also use the notation $\eta_{n,i}(0)$ and $\eta_{n,i}(1)$ with

$$\eta_{n,i}(0)(f,a)=f_i(0) \quad\text{and}\quad \eta_{n,i}(1)(f,a)=f_i(1).$$

But $\eta_{n,i}(0),\eta_{n,i}(1)\in RF(A_n)$ (rather than $\operatorname{Sp}(A_n)$). Sometimes we use $(f,a)(\eta_{n,i}(t))$ and $(f,a)(\theta_{n,j})$ to denote $\eta_{n,i}(t)(f,a)$ and $\theta_{n,j}(f,a)$, respectively, or simply use $f(\eta_{n,i}(t))$, $f(\theta_{n,j})$ without a, as in this paper, a is completely determined by f. Moreover, we may write

$$\operatorname{Sp}(A_n)=\left(\bigsqcup_{j=1}^{l_n}\{\eta_{n,j}(t)\mid t\in(0,1)\}\right)\sqcup\{\theta_{n,1},\theta_{n,2},\dots,\theta_{n,p_n}\}. \tag{4.4}$$

For $\delta>0$, let

$$S_j=\{\eta_{n,j}(t):t\in T_j\subset(0,1)\} \quad\text{and}\quad S=\left(\bigsqcup_{j=1}^{l_n}S_j\right)\cup\{\theta_{n,1},\theta_{n,2},\dots,\theta_{n,p_n}\},$$

where T_j is a δ-dense subset of $(0,1)$. Then we say that S is δ-dense in $\operatorname{Sp}(A_n)$. Let $\mathcal{F}\subset A_n\setminus\{0\}$ be a finite subset. Then, for all sufficiently small δ, if S is δ-dense in A_n, then for each $f\in\mathcal{F}$, there exists $s\in S$ such that $f(s)\neq 0$.

Write $(\beta_{n,0})_*, (\beta_{n,1})_* : K_0(F_n)=\mathbb{Z}^{p_n} \to K_0(E_n)=\mathbb{Z}^{l_n}$ as $\mathfrak{b}_0^n=(b_{0,ij}^n)$, $\mathfrak{b}_1^n=(b_{1,ij}^n)$, with $b_{0,ij}^n \in \mathbb{N}$, $b_{1,ij}^n \in \mathbb{N}$. (If there is no confusion, we may omit n from $\mathfrak{b}_0^n$, $b_{1,ij}^n$, etc. in the notation above.) Then

$$\begin{aligned} \eta_{n,i}(0) &= \{\theta_{n,1}^{\sim b_{0,i1}}, \theta_{n,2}^{\sim b_{0,i2}}, \dots, \theta_{n,p_n}^{\sim b_{0,ip_n}}\}, \\ \eta_{n,i}(1) &= \{\theta_{n,1}^{\sim b_{1,i1}}, \theta_{n,2}^{\sim b_{1,i2}}, \dots, \theta_{n,p_n}^{\sim b_{1,ip_n}}\}. \end{aligned} \tag{4.5}$$

By Proposition 3.6 of [Gong et al. 2020a] (note that the unital condition is not used in the proof), if $\alpha_0, \alpha_1 : F_n \to E_n$ satisfy

$$(\alpha_0)_* = (\beta_{n,0})_*, (\alpha_1)_* = (\beta_{n,1})_* : K_0(F_n) \to K_0(E_n),$$

then $A(F_n, E_n, \alpha_0, \alpha_1) = A(F_n, E_n, \beta_{n,0}, \beta_{n,1}) = A_n$.

Let us introduce the following notation. For $1 \le j \le l_n$, let $\pi_{E_n^j}$ be the canonical projection from $E_n = \bigoplus_{k=1}^{l_n} M_{\{n,k\}}(\mathbb{C})$ to $E_n^j = M_{\{n,j\}}(\mathbb{C})$ and let $\beta_{n,0}^j = \pi_{E_n^j} \circ \beta_{n,0}$ and $\beta_{n,1}^j = \pi_{E_n^j} \circ \beta_{n,1}$.

If $(\beta_{n,0}^j)_*([1_{F_n}]) \le (\beta_{n,1}^j)_*([1_{F_n}])$, then there is a unitary $u_j \in E_n^j$ such that $\beta_{n,0}^j(1_{F_n}) \le \operatorname{Ad} u_j \circ \beta_{n,1}^j(1_{F_n})$, and if $(\beta_{n,0}^j)_*([1_{F_n}]) \ge (\beta_{n,1}^j)_*([1_{F_n}])$, then there is a unitary $u_j \in E_n^j$ such that $\beta_{n,0}^j(1_{F_n}) \ge \operatorname{Ad} u_j \circ \beta_{n,1}^j(1_{F_n})$. Thus, for convenience, replacing $\beta_{n,1}^j$ by $\operatorname{Ad} u_j \circ \beta_{n,1}^j$, if necessary, we may always assume:

If $(\beta_{n,0}^j)_*([1_{F_n}]) \le (\beta_{n,1}^j)_*([1_{F_n}])$, then $\beta_{n,0}^j(1_{F_n}) \le \beta_{n,1}^j(1_{F_n})$; and if $(\beta_{n,0}^j)_*([1_{F_n}]) \ge (\beta_{n,1}^j)_*([1_{F_n}])$, then $\beta_{n,0}^j(1_{F_n}) \ge \beta_{n,1}^j(1_{F_n})$. $(**)$

If $(**)$ holds, $\max(\beta_{n,0}^j(1_{F_n}), \beta_{n,1}^j(1_{F_n}))$ makes sense. Let

$$P_{n,j} = 1_{E_n^j} - \max(\beta_{n,0}^j(1_{F_n}), \beta_{n,1}^j(1_{F_n})).$$

4.6. Let $A = A(F, E, \beta_0, \beta_1)$, where $F = \bigoplus_{i=1}^p M_{R_i}(\mathbb{C})$, $E = \bigoplus_{i=1}^l M_{r_i}(\mathbb{C})$ and $(\beta_0)_*, (\beta_1)_* : K_0(F) = \mathbb{Z}^p \to K_0(E) = \mathbb{Z}^l$ represented by matrices $(b_{0,ij})$, $(b_{1,ij})$. From [Gong and Lin 2020a, 3.4] (see also Definition 2.22), we have

$$\lambda_s(A) = \min_i \left\{ \frac{\sum_{j=1}^p b_{0,ij} R_j}{r_i}, \frac{\sum_{j=1}^p b_{1,ij} R_j}{r_i} \right\}.$$

4.7. For C^*-algebra $A_n = A(F_n, E_n, \beta_{n,0}, \beta_{n,1})$ as in 4.3, we fix a strictly positive element $e_n^A \in A_n$ defined by $e_n^A = (f_1, f_2, \dots, f_{l_n}, a_1, a_2, \dots, a_{p_n})$ with $a_i = 1_{F_n^i} \in F_n^i$ and

$$f_j(t) = (1-t)\beta_{n,0}^j(1_{F_n}) + t\beta_{n,1}^j(1_{F_n}) + t(1-t)P_{n,j}.$$

From the definition of $P_{n,j}$, we know that for $0 < t < 1$,

$$\operatorname{rank}(f_j(t)) = \operatorname{rank}(\max(\beta_{n,0}^j(1_{F_n}), \beta_{n,1}^j(1_{F_n}))) + \operatorname{rank}(P_{n,j}) = \operatorname{rank}(E_n^j) = \{n, j\}.$$

Hence e_n^A is strictly positive. Let $a_n^A := (f, g) \in A_n$ be defined by

$$g = 1_{F_n} \quad \text{and} \quad f(t) = (1-t)\beta_{n,0}(g) + t\beta_{n,1}(g). \tag{4.8}$$

Then $a_n^A \le e_n^A$. Note that a_n^A may not be a strictly positive element.

4.9 (order unit and M large). Suppose (G, Δ, ρ) is as in 4.1 with condition $(*)$. Let us assume that G is torsion free. Choose a countable dense subgroup $G^1 \subset \mathrm{Aff}(\Delta)$ with $1_\Delta \in G^1$. Put $H = G \oplus G^1$ and define $\tilde{\rho} : H \to \mathrm{Aff}(\Delta)$ by $\tilde{\rho}((g, f))(\tau) = \rho(g)(\tau) + f(\tau)$ for all $(g, f) \in G \oplus G^1$ and $\tau \in \Delta$. Define $H_+ \ni 0$ such that $H_+ \setminus \{0\}$ is the set of elements $(g, f) \in G \oplus G^1$ with $\tilde{\rho}((g, f))(\tau) > 0$ for all $\tau \in \Delta$. Then $(H, H_+, 1_\Delta)$ is a simple ordered group with Riesz interpolation property (see [Gong et al. 2020a, 13.9]). Following [Gong et al. 2020a, 13.9–13.12], write H as an inductive limit of

$$H_1' \xrightarrow{\gamma_{12}'} H_2' \xrightarrow{\gamma_{23}'} \cdots \to H \tag{4.10}$$

of direct sum of finite copies of ordered group $(\mathbb{Z}, \mathbb{Z}_+)$ with the property

$$\gamma_{n,\infty}'(x) \in H_+ \setminus \{0\} \quad \text{for any } x \in (H_n')_+ \setminus \{0\}. \tag{4.11}$$

Let $H_n' = \left(\mathbb{Z}^{p_n'}, \mathbb{Z}_+^{p_n'}\right)$ be with a p_n'-tuple $\tilde{u}_n'$ of positive integers as the order unit. Furthermore, $\gamma_{n,n+1}'(\tilde{u}_n') = \tilde{u}_{n+1}'$ and $\gamma_{n,\infty}'(\tilde{u}_n') = 1_\Delta$.

We modify the order unit $\tilde{u}_n$ to a smaller one for future use. Since (H, H_+) is a simple Riesz group, we may assume that all maps $\gamma_{n,n+1}'$ have multiplicity at least 3 (i.e., $a_{ij} \ge 3$ for all i, j) if $\gamma_{n,n+1}'$ is represented by the matrix $(a_{ij})_{p_{n+1}' \times p_n'}$, then the condition that for all $\gamma_{n,n+1}'$, $a_{ij} > 0$ for all i, j implies that (4.11) holds. Consequently, the order unit $\tilde{u}_n = (a_1, a_2, \ldots, a_{p_n'}) \in \mathbb{Z}^{p_n'}$ satisfies $a_i \ge 3$ for all i, provided $n \ge 2$. We assume that this is true for all n, since if it is not true for $n = 1$, then we simply drop the first term from the limit procedure.

Suppose $\tilde{u}_n' = (a_1, a_2, \ldots, a_{p_n'}) \in \mathbb{Z}^{p_n'}$. Choose $u_n' = (a_1 - 2, a_2, \ldots, a_{p_n'}) \in \mathbb{Z}^{p_n'}$. The assumption that $\gamma_{n,n+1}'$ has multiplicities at least 3 implies $\gamma_{n,n+1}'(u_n') < u_{n+1}'$. Now set

$$H_n' = (\mathbb{Z}^{p_n'}, \mathbb{Z}_+^{p_n'}, u_n')$$

with the new order unit u_n'. Note that $\gamma_{n,\infty}'(u_n') < 1_\Delta$, and for any element $x \in H_+$ with $x < 1_\Delta$, there is an integer n such that $\gamma_{n,\infty}'(u_n') > x$.

The inductive limit $\lim(H_n', u_n', \gamma_{n,m}')$ of the scaled ordered groups has the following property. For any n and $M > 0$, there is N such that if $m > N$, then $\gamma_{n,m}'$ is M-large in the following sense: Suppose that the matrix $(a_{ij})_{p_m' \times p_n'}$ represents $\gamma_{n,m}'$ and $u_n' = (x_1, x_2, \ldots, x_{p_n'})$, $u_m' = (y_1, y_2, \ldots, y_{p_m'})$. Then

$$a_{ij} \ge M, \quad y_i - \sum_{k=1}^{p_n'} a_{ik} x_k \ge M \quad \text{for } 1 \le i \le p_m',\ 1 \le j \le p_n'. \tag{4.12}$$

We also use the inductive limit system $\lim(H_n', \tilde{u}_n', \gamma_{n,m}')$, which preserves the units. Compare two inductive limit systems, writing $\lim((H_n', (H_n')_+, u_n'), \gamma_{n,m}') = (H, H_+, \Sigma H)$ and $\lim((H_n', (H_n')_+, \tilde{u}_n'), \gamma_{n,m}') = (H, H_+, \Sigma_1 H)$. We have

$$\Sigma H = \{h \in H_+ : h < 1_\Delta\} \quad \text{and} \quad \Sigma_1 H = \{h \in H_+ : h \le 1_\Delta\}. \tag{4.13}$$

Furthermore, $\gamma_{n,\infty}'(u_n') < 1_\Delta$ and $\gamma_{n,\infty}'(\tilde{u}_n') = 1_\Delta$.

For any fixed positive integer M, there is a positive integer N such that if $n \ge N$, and if we write $u_n' = (b_1, b_2, \dots, b_{p_n'})$ and $\tilde{u}_n' = (b_1+2, b_2, \dots, b_{p_n'})$, then $b_1 \ge 2M$ and

$$\tilde{\rho}(\gamma_{n,\infty}'(u_n')) \ge \frac{b_1}{b_1+2}\tilde{\rho}(\gamma_{n,\infty}'(\tilde{u}_n')) = \frac{b_1}{b_1+2} \cdot 1_\Delta \ge \Big(1 - \frac{1}{M}\Big) \cdot 1_\Delta. \tag{4.14}$$

Let $G_n' = (\gamma_{n,\infty}')^{-1}(\gamma_{n,\infty}'(H_n') \cap G)$. Then $G_n' \subset H_n'$. Since $G \cap H_+ = \{0\}$, by (4.11), we have

$$G_n' \cap (H_n')_+ = \{0\}. \tag{4.15}$$

Furthermore, G is the inductive limit of

$$G_1' \xrightarrow{\gamma_{1,2}'|_{G_1'}} G_2' \xrightarrow{\gamma_{2,3}'|_{G_2'}} \cdots \to G_n' \to \cdots \to G. \tag{4.16}$$

Recall that a subgroup $G \subset H$ is said to be relatively divisible if for any $g \in G$, $m \in \mathbb{N} \setminus \{0\}$, and $h \in H$ with $g = mh$, there is $g' \in G$ such that $g = mg'$. As in [Gong et al. 2020a, 13.12], G_n is a relatively divisible subgroup of H_n, and H_n/G_n is a torsion-free finitely generated abelian group. Write $H_n/G_n = \mathbb{Z}^{l_n'}$. We have the following commutative diagram (see [Gong et al. 2020a, 13.12]):

$$\begin{tikzcd}
G_1' \arrow[r, "\gamma_{12}'|_{G_1}"] \arrow[d, hook] & G_2' \arrow[r] \arrow[d, hook] & \cdots \arrow[r] & G \arrow[d, hook] \\
H_1' \arrow[r, "\gamma_{12}'"] \arrow[d] & H_2' \arrow[r] \arrow[d] & \cdots \arrow[r] & H \arrow[d] \\
H_1'/G_1' \arrow[r, "\tilde{\gamma}_{12}'"] & H_2'/G_2' \arrow[r] & \cdots \arrow[r] & H/G
\end{tikzcd}$$

In the process of the construction above, we use subsequences $(\{k(n)\})$ and obtain the following diagram:

$$\begin{tikzcd}
G_{k(1)}' \arrow[r, "\gamma_{k(1),k(2)}'|_{G_{k(1)}}"] \arrow[d, hook] & G_{k(2)}' \arrow[r] \arrow[d, hook] & \cdots \arrow[r] & G \arrow[d, hook] \\
H_{k(1)}' \arrow[r, "\gamma_{k(1),k(2)}'"] \arrow[d] & H_{k(2)}' \arrow[r] \arrow[d] & \cdots \arrow[r] & H \arrow[d] \\
H_{k(1)}'/G_{k(1)}' \arrow[r, "\tilde{\gamma}_{k(1),k(2)}'"] & H_{k(2)}'/G_{k(2)}' \arrow[r] & \cdots \arrow[r] & H/G
\end{tikzcd}$$

Rewrite the sequence above as

$$\begin{tikzcd} G_1 \arrow[r, "\gamma_{12}|_{G_1}"] \arrow[d, hook] & G_2 \arrow[r] \arrow[d, hook] & \cdots \arrow[r] & G \arrow[d, hook] \\ H_1 \arrow[r, "\gamma_{12}"] \arrow[d] & H_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & H \arrow[d] \\ H_1/G_1 \arrow[r, "\tilde{\gamma}_{12}"] & H_2/G_2 \arrow[r] & \cdots \arrow[r] & H/G \end{tikzcd} \tag{4.17}$$

Write $H_n = \mathbb{Z}^{p_n}$ ($p_n = p'_{k(n)}$ as $H_n = H'_{k(n)}$) and $H_n/G_n = \mathbb{Z}^{l_n}$ ($l_n = l'_{k(n)}$), and write

$$\gamma_{n,n+1} = \gamma'_{k(n),k(n+1)} : \mathbb{Z}^{p_n} \to \mathbb{Z}^{p_{n+1}} \quad \text{as} \quad \mathfrak{c}^{n,n+1} = (c^{n,n+1}_{ij})_{1\le i\le p_{n+1},\, 1\le j\le p_n},$$
$$\tilde{\gamma}_{n,n+1} : \mathbb{Z}^{l_n} \to \mathbb{Z}^{l_{n+1}} \quad \text{as} \quad \mathfrak{d}^{n,n+1} = (d^{n,n+1}_{ij})_{1\le i\le l_{n+1},\, 1\le j\le l_n}.$$

In case of no confusion, we write $\mathfrak{c}^{n,n+1}$ as $\mathfrak{c} = (c_{ij})$ and $\mathfrak{d}^{n,n+1}$ as $\mathfrak{d} = (d_{ij})$ (omitting $n, n+1$). Since $G_n \subsetneqq H_n$, we know that $l_n \ge 1$.

Write

$$\begin{tikzcd} H_n = \mathbb{Z}^{p_n} \arrow[r, "\gamma_{n,n+1}"] \arrow[d, "\pi_n"] & H_{n+1} = \mathbb{Z}^{p_{n+1}} \arrow[d, "\pi_{n+1}"] \\ H_n/G_n = \mathbb{Z}^{l_n} \arrow[r, "\tilde{\gamma}_{n,n+1}"] & H_{n+1}/G_{n+1} = \mathbb{Z}^{l_{n+1}} \end{tikzcd} \tag{4.18}$$

where π_n is the (surjective) quotient map. We have

$$\pi_{n+1} \cdot \mathfrak{c}^{n,n+1} = \mathfrak{d}^{n,n+1} \cdot \pi_n. \tag{4.19}$$

4.20 (construction of A_1). Choose $k(1) \ge 1$ such that

$$u'_{k(1)} := ([1, 1], [1, 2], \ldots, [1, p'_{k(1)}]) \in \mathbb{N}^{p'_{k(1)}}$$

satisfies $[1, 1] \ge 2 \cdot 8 = 16$. Let

$$H_1 := H'_{k(1)}, \quad u_1 := u'_{k(1)} := ([1, 1], [1, 2], \ldots, [1, p_1]) \in H_1,$$
$$p_1 := p'_{k(1)}, \quad \tilde{u}_1 := ([1, 1] + 2, [1, 2], \ldots, [1, p_1]) \in H_1.$$

Then, by (4.14),

$$\tilde{\rho}(\gamma'_{k(1),\infty}(u_1)) \ge \left(1 - \tfrac{1}{8}\right) \cdot 1_\Delta$$

(that is, $\tilde{\rho}(\gamma_{1,\infty}(u_1)) \ge \left(1 - \frac{1}{8}\right) \cdot 1_\Delta$, after the diagram (4.17) is obtained). Recall that we also have $\gamma'_{k(1),\infty}(\tilde{u}_1) = 1_\Delta$. Let $G_1 = G'_{k(1)}$ and $\pi_1 : H_1 = \mathbb{Z}^{p_1} \to H_1/G_1 = \mathbb{Z}^{l_1}$ be the quotient map.

For a homomorphism $\mathbb{Z}^k \to \mathbb{Z}^l$ represented by a matrix $B = (b_{ij})_{1\le i\le l,\, 1\le j\le k}$, we define $|||B||| := \max_{i,j} |b_{ij}| \cdot k \cdot l$.

Let $\mathcal{M}_1 = 2^4 |||\pi_1|||$. The map π_1 can be written as the difference $\mathfrak{b}^1_1 - \mathfrak{b}^1_0$ of two maps

$$\mathfrak{b}^1_0, \mathfrak{b}^1_1 : \mathbb{Z}^{p_1} \to \mathbb{Z}^{l_1},$$

corresponding to two $l_1 \times p_1$ matrices

$$\mathfrak{b}_0^1 = (b_{0,ij}^1)_{1\le i\le l_1,\, 1\le j\le p_1} \quad \text{and} \quad \mathfrak{b}_1^1 = (b_{1,ij}^1)_{1\le i\le l_1,\, 1\le j\le p_1} \tag{4.21}$$

such that

$$b_{0,ij}^1 \ge \mathcal{M}_1 \quad \text{and} \quad b_{1,ij}^1 \ge \mathcal{M}_1. \tag{4.22}$$

(Later on we will define $\mathcal{M}_{n+1}$ when we construct A_{n+1}.) Namely, we construct $\mathfrak{b}_0^1$ and $\mathfrak{b}_1^1$ as below. Assume $\pi_1 : \mathbb{Z}^{p_1} \to \mathbb{Z}^{l_1}$ is given by the matrix $(p_{ij}^1)_{1\le i\le l_1,\, 1\le j\le p_1}$. Choose $\mathfrak{b}_0^1 = (b_{0,ij}^1)$ with $b_{0,ij}^1 = |p_{ij}| + \mathcal{M}_1$ and $b_{1,ij}^1 = b_{0,ij}^1 + p_{ij}$. Then $\mathfrak{b}_0^1$ and $\mathfrak{b}_1^1$ satisfy (4.22) and $\mathfrak{b}_1^1 - \mathfrak{b}_0^1 = \pi_1 : \mathbb{Z}^{p_1} \to \mathbb{Z}^{l_1}$.

Recall that the unit $u_1' \in H_1' = H_1$ can be written as $([1,1],[1,2],\dots,[1,p_1]) \in \mathbb{N}^{p_1}$. Since $b_{0,ij}^1 \ge \mathcal{M}_1$, $b_{1,ij}^1 \ge \mathcal{M}_1$, we have

$$|b_{1,ij}^1 - b_{0,ij}^1| \le \frac{1}{p_1 l_1} |||\pi_1||| \le \frac{1}{16}\cdot\frac{1}{2}\mathcal{M}_1 \le \frac{1}{32}\min(b_{0,ij}^1, b_{1,ij}^1).$$

Consequently, $\max(b_{0,ij}^1, b_{1,ij}^1) \le \bigl(1+\frac{1}{32}\bigr)\min(b_{0,ij}^1, b_{1,ij}^1)$. Hence

$$\begin{aligned}\Bigl(1+\frac{1}{8}\Bigr)\sum_{j=1}^{p_1}&\max(b_{0,ij}^1, b_{1,ij}^1)[1,j]\\ &\le \Bigl(1+\frac{1}{8}\Bigr)\Bigl(1+\frac{1}{32}\Bigr)\sum_{j=1}^{p_1}\min(b_{0,ij}^1, b_{1,ij}^1)[1,j]\\ &\le \biggl(\Bigl(1+\frac{1}{4}\Bigr)\sum_{j=1}^{p_1}\min(b_{0,ij}^1, b_{1,ij}^1)[1,j]\biggr) - 1.\end{aligned} \tag{4.23}$$

By (4.23), we may choose an integer $\{1,i\}$ such that

$$\begin{aligned}\Bigl(1+\frac{1}{8}\Bigr)\sum_{j=1}^{p_1}\max(b_{0,ij}^1, b_{1,ij}^1)[1,j] &\le \{1,i\}\\ &\le \Bigl(1+\frac{1}{4}\Bigr)\sum_{j=1}^{p_1}\min(b_{0,ij}^1, b_{1,ij}^1)[1,j].\end{aligned} \tag{4.24}$$

Let $F_1 = \bigoplus_{i=1}^{p_1} M_{[1,i]}(\mathbb{C})$, $E_1 = \bigoplus_{i=1}^{l_1} M_{\{1,i\}}(\mathbb{C})$, and $\beta_{1,0}, \beta_{1,1} : F_1 \to E_1$ be defined by

$$\beta_{1,0}(a_1 \oplus a_2 \oplus \cdots \oplus a_{p_1}) = \bigoplus_{i=1}^{l_1} \mathrm{diag}\bigl(a_1^{\sim b_{0,i1}^1}, a_2^{\sim b_{0,i2}^1}, \dots, a_{p_1}^{\sim b_{0,ip_1}^1}\bigr),$$

$$\beta_{1,1}(a_1 \oplus a_2 \oplus \cdots \oplus a_{p_1}) = \bigoplus_{i=1}^{l_1} \mathrm{diag}\bigl(a_1^{\sim b_{1,i1}^1}, a_2^{\sim b_{1,i2}^1}, \dots, a_{p_1}^{\sim b_{1,ip_1}^1}\bigr).$$

Evidently, $\beta_{1,0} \oplus \beta_{1,1} : F_1 \to E_1 \oplus E_1$ is injective. Then

$$(\beta_{0,1})_* = \mathfrak{b}_0^1 \quad \text{and} \quad (\beta_{1,1})_* = \mathfrak{b}_1^1 : K_0(F_1) = \mathbb{Z}^{p_1} \to K_0(E_1) = \mathbb{Z}^{l_1}.$$

Define

$$\begin{aligned} A_1 &= A(F_1, E_1, \beta_{1,0}, \beta_{1,1}) \\ &= \{(f, a) \in C([0,1], E_1) \oplus F_1 \mid \beta_{1,0}(a) = f(0), \beta_{1,1}(a) = f(1)\}. \end{aligned}$$

Then $K_0(A_1) = G_1$ and $K_0(F_1) = H_1$, and the quotient map $\pi_1^A : A_1 \to F_1$ defined by $\pi_1^A(f, a) = a$ induces $(\pi_1^A)_* : K_0(A_1) = G_1 \to K_0(F_1) = H_1$, which is the inclusion map. Since $\mathfrak{b}_1^1 - \mathfrak{b}_0^1 = \pi_1$ is surjective (onto $\mathbb{Z}^{l_1}$), $K_1(A_1) = 0$, by Proposition 3.5 of [Gong et al. 2020a]. Furthermore by (4.15), $K_0(A_1)_+ = G_1 \cap (H_1)_+ = \{0\}$ (see again Proposition 3.5 of [Gong et al. 2020a]) and thus $A_1 \in \mathcal{C}_0$.

By (4.24), we have

$$\lambda_s(A_1) = \min_i \left\{ \frac{\sum_{j=1}^{p_1} b_{0,ij}^1 \cdot [1, j]}{\{1, i\}}, \frac{\sum_{j=1}^{p_1} b_{1,ij}^1 \cdot [1, j]}{\{1, i\}} \right\} \geq \frac{1}{1 + \frac{1}{4}} \geq \frac{3}{4}.$$

4.25 (inductive assumption for A_n). Suppose that we have constructed $A_n = A(F_n, E_n, \beta_{n,0}, \beta_{n,1}) \in \mathcal{C}_0$ with

$$F_n = \bigoplus_{j=1}^{p_n} M_{[n,j]}(\mathbb{C}), \qquad E_n = \bigoplus_{i=1}^{l_n} M_{\{n,i\}}(\mathbb{C}), \qquad \beta_{n,0}, \beta_{n,1} : F_n \to E_n$$

such that the following conditions hold:

(a) $K_0(F_n) = H_n = H'_{k(n)}$, $K_0(A_n) = G_n = G'_{k(n)}$ (for some $k(n)$) and the quotient map $\pi_n^A : A_n \to F_n$ induces the inclusion map $(\pi_n^A)_* : K_0(A_n) = G_n \to K_0(F_n) = H_n$, $G_n \cap (H_n)_+ = \{0\}$, and $K_1(A_n) = \{0\}$.

(b) $(\beta_{n,0})_* = \mathfrak{b}_0^n = (b_{0,ij}^n)_{1 \leq i \leq l_n,\, 1 \leq j \leq p_n}$ and $(\beta_{n,1})_* = \mathfrak{b}_1^n = (b_{1,ij}^n)_{1 \leq i \leq l_n,\, 1 \leq j \leq p_n}$ with $b_{0,ij}^n \geq \mathcal{M}_n$, $b_{1,ij}^n \geq \mathcal{M}_n$ for a pregiven positive number $\mathcal{M}_n$. (Note that $(\mathfrak{b}_1^n) - (\mathfrak{b}_0^n) = \pi_n : K_0(F_n) = H_n = \mathbb{Z}^{p_n} \to K_0(E_n) = H_n/G_n = \mathbb{Z}^{l_n}$, which is surjective.)

(c) In $K_0(F_n)$, we have

$$[1_{F_n}] = u_n = u'_{k(n)} := ([n, 1], [n, 2], \ldots, [n, p_n]) \in \mathbb{N}^{p_n}$$

with $[n, 1] \geq 2 \cdot 8^n$. As a consequence (see (4.14)), we have

$$\tilde{\rho}(\gamma'_{k(n),\infty}(u_n)) \geq \left(1 - \frac{1}{8^n}\right) \cdot 1_\Delta.$$

(Note that $\tilde{u}_n = ([n, 1] + 2, [n, 2], \ldots, [n, p_n]) \in H_n$ satisfies $\gamma'_{k(n),\infty}(\tilde{u}_n) = 1_\Delta$.)

(d) $\beta_{n,0}$ and $\beta_{n,1}$ satisfy the property $(**)$ in the last paragraph of 4.3.

(e) We have

$$\Big(1+\frac{1}{8^n}\Big)\cdot\sum_{j=1}^{p_n}\max\{b^n_{0,ij},\,b^n_{n,ij}\}[n,\,j]\le\{n,i\}$$
$$\le\Big(1+\frac{1}{4^n}\Big)\cdot\sum_{j=1}^{p_n}\min\{b^n_{0,ij},\;b^n_{n,ij}\}[n,\,j].$$

Consequently, $\lambda_s(A_n)>1-1/4^n$.

Note that the homomorphism $\pi^A_n:A_n\to F_n$ induces the commutative diagram

$$\begin{array}{ccc} K_0(A_n) & \xrightarrow{\rho_{A_n}} & \mathrm{Aff}(T_0(A_n)) \\ {\scriptstyle(\pi^A_n)_{*0}}\big\downarrow & & \big\downarrow{\scriptstyle\pi^{A\sharp}_n} \\ K_0(F_n) & \xrightarrow{\rho_{F_n}} & \mathrm{Aff}(T_0(F_n)) \end{array}$$

From (a), when we identify $K_0(A_n)=G_n$ and $K_0(F_n)=H_n$, the map $(\pi^A_n)_{*0}$ is identified with the inclusion from G_n to H_n, and we have

$$\begin{array}{ccc} G_n & \xrightarrow{\rho_{A_n}} & \mathrm{Aff}(T_0(A_n)) \\ {\scriptstyle(\pi^A_n)_{*0}}\big\downarrow & & \big\downarrow{\scriptstyle\pi^{A\sharp}_n} \\ H_n & \xrightarrow{\rho_{F_n}} & \mathrm{Aff}(T_0(F_n)) \end{array}\tag{4.26}$$

From $b^n_{0,ij}\ge\mathcal{M}_n>0$ in part (b), we know $\beta_{n,0}$ is injective, and hence so is $\beta_{n,0}\oplus\beta_{n,1}:F_n\to E_n\oplus E_n$.

We construct $A_{n+1}=A(F_{n+1},E_{n+1},\beta_{n+1,0},\beta_{n+1,1})$ and a positive integer $\mathcal{M}_{n+1}$, and two homomorphisms $\varphi^o_{n,n+1},\varphi_{n,n+1}:A_n\to A_{n+1}$ as below.

4.27 (the definition of A_{n+1}). Let

$$\mathcal{L}_{n+1}=2^{4(n+1)}\cdot\max\{\{n,i\}:1\le i\le l_n\}\cdot nl_n.\tag{4.28}$$

Recall that $H_n=H'_{k(n)}$ (in the inductive limit system in (4.10)) and H is a simple Riesz group. There is $k(n+1)>k(n)$ such that the map $\gamma'_{k(n),k(n+1)}$ is $\mathcal{L}_{n+1}$ large in the sense of (4.12) in 4.9. Write $H_{n+1}=H'_{k(n+1)}=\mathbb{Z}^{p_{n+1}}$ and represent the map $\gamma_{n,n+1}=\gamma'_{k(n),k(n+1)}:\mathbb{Z}^{p_n}\to\mathbb{Z}^{p_{n+1}}$ by the matrix $\mathfrak{c}^{n,n+1}=(c^{n,n+1}_{ij})_{p_{n+1}\times p_n}$. We further require that the unit

$$u_{n+1}=u'_{k(n+1)}:=([n+1,1],[n+1,2],\ldots,[n+1,p_{n+1}])\in\mathbb{N}^{p_{n+1}}$$

satisfies the condition $[n+1,1]\ge 2\cdot 8^{n+1}$ (see (c) in 4.25). Then we have

$$c^{n,n+1}_{ij}\ge\mathcal{L}_{n+1}\quad\text{and}\quad[n+1,i]-\sum_{k=1}^{p_n}c^{n,n+1}_{ik}[n,k]\ge\mathcal{L}_{n+1}.\tag{4.29}$$

It follows from

$$\begin{aligned}\gamma'_{k(n),k(n+1)}(\tilde{u}'_{k(n)}) &= \tilde{u}'_{k(n+1)},\\ \tilde{u}_n &= \tilde{u}'_{k(n)} = ([n,1]+2,[n,2],\dots,[n,p_n]),\\ \tilde{u}_{n+1} &= \tilde{u}'_{k(n+1)} = ([n+1,1]+2,[n+1,2],\dots,[n+1,p_{n+1}])\end{aligned}$$

and (c) of 4.25 that

$$\sum_{j=1}^{p_n} c_{ij}^{n,n+1}[n,j] \ge \Big(1-\frac{1}{8^n}\Big)\cdot[n+1,i]. \tag{4.30}$$

Write $G_{n+1}=G'_{k(n+1)}\subset H_{n+1}$ and write $H_{n+1}/G_{n+1}=\mathbb{Z}^{l_{n+1}}$. Suppose that the map $\gamma_{n,n+1}:H_n\to H_{n+1}$ induces

$$\widetilde{\gamma_{n,n+1}}:H_n/G_n=\mathbb{Z}^{l_n}\to H_{n+1}/G_{n+1}=\mathbb{Z}^{l_{n+1}}.$$

Write $\widetilde{\gamma_{n,n+1}}=\mathfrak{d}^{n,n+1}=(d_{ij}^{n,n+1})_{l_{n+1}\times l_n}$, where $d_{ij}^{n,n+1}\in\mathbb{Z}$. Set

$$\mathcal{M}_{n+1}=2^{4(n+1)}l_n\cdot n\cdot(|||\mathfrak{d}^{n,n+1}|||+2)\cdot\max\{\{n,k\}:1\le k\le l_n\}\cdot|||\pi_{n+1}|||, \tag{4.31}$$

where $\pi_{n+1}:H_{n+1}=\mathbb{Z}^{p_{n+1}}\to H_{n+1}/G_{n+1}=\mathbb{Z}^{l_{n+1}}$ is the quotient map and $|||\cdot|||$ is defined in the end of 4.9.

We construct the algebra A_{n+1} as below (see 4.20 for similar construction of A_1). Write the map $\pi_{n+1}:H_{n+1}=\mathbb{Z}^{p_{n+1}}\to H_{n+1}/G_{n+1}=\mathbb{Z}^{l_{n+1}}$ as a difference $\pi_{n+1}=\mathfrak{b}_1^{n+1}-\mathfrak{b}_0^{n+1}$ of two matrices $\mathfrak{b}_0^{n+1}=(b_{0,ij}^{n+1})_{1\le i\le l_{n+1},\,1\le j\le p_{n+1}}$ and $\mathfrak{b}_1^{n+1}=(b_{1,ij}^{n+1})_{1\le i\le l_{n+1},\,1\le j\le p_{n+1}}$ satisfying

$$b_{0,ij}^{n+1}\ge\mathcal{M}_{n+1}\quad\text{and}\quad b_{1,ij}^{n+1}\ge\mathcal{M}_{n+1}\quad\text{for all } i,j. \tag{4.32}$$

Consequently, we obtain

$$|b_{1,ij}^{n+1}-b_{0,ij}^{n+1}|\le\frac{1}{2^{4(n+1)}}\min(b_{1,ij}^{n+1},b_{0,ij}^{n+1}). \tag{4.33}$$

As in the calculation in (4.23), we have

$$\begin{aligned}\Big(1+\frac{1}{8^{n+1}}\Big)&\sum_{j=1}^{p_{n+1}}\max(b_{0,ij}^{n+1},b_{1,ij}^{n+1})[n+1,j]\\ &\le\bigg(\Big(1+\frac{1}{4^{n+1}}\Big)\sum_{j=1}^{p_{n+1}}\min(b_{0,ij}^{n+1},b_{1,ij}^{n+1})[n+1,j]\bigg)-1.\end{aligned}$$

One may then choose an integer $\{n+1,i\}$ which satisfies

$$\Big(1+\frac{1}{8^{n+1}}\Big)\sum_{j=1}^{p_{n+1}}\max(b_{0,ij}^{n+1},b_{1,ij}^{n+1})[n+1,j]\le\{n+1,i\} \tag{4.34}$$

$$\le\Big(1+\frac{1}{4^{n+1}}\Big)\sum_{j=1}^{p_{n+1}}\min(b_{0,ij}^{n+1},b_{1,ij}^{n+1})[n+1,j]. \tag{4.35}$$

From (4.19), we have

$$(\mathfrak{b}_1^{n+1} - \mathfrak{b}_0^{n+1}) \cdot \mathfrak{c}^{n,n+1} = \mathfrak{d}^{n,n+1} \cdot (\mathfrak{b}_1^n - \mathfrak{b}_0^n). \tag{4.36}$$

Let $F_{n+1} = \bigoplus_{j=1}^{p_{n+1}} M_{[n+1,j]}(\mathbb{C})$ and $E_{n+1} = \bigoplus_{i=1}^{l_{n+1}} M_{\{n+1,j\}}(\mathbb{C})$, and define $\beta_{n+1,0}, \beta_{n+1,1} : F_{n+1} \to E_{n+1}$ by

$$\beta_{n+1,0}(a_1 \oplus a_2 \oplus \cdots \oplus a_{p_{n+1}}) = \bigoplus_{i=1}^{l_{n+1}} \operatorname{diag}\big(a_1^{\sim b_{0,i1}^{n+1}}, a_2^{\sim b_{0,i2}^{n+1}}, \dots, a_{p_{n+1}}^{\sim b_{0,ip_{n+1}}^{n+1}}, 0^{\sim k_0}\big),$$

$$\beta_{n+1,1}(a_1 \oplus a_2 \oplus \cdots \oplus a_{p_{n+1}}) = \bigoplus_{i=1}^{l_{n+1}} \operatorname{diag}\big(a_1^{\sim b_{1,i1}^{n+1}}, a_2^{\sim b_{1,i2}^{n+1}}, \dots, a_{p_{n+1}}^{\sim b_{1,ip_{n+1}}^{n+1}}, 0^{\sim k_1}\big),$$

where $k_0 = \{n+1, i\} - \sum_{j=1}^{p_{n+1}} b_{0,ij}^{n+1}[n+1, j]$ and $k_1 = \{n+1, i\} - \sum_{j=1}^{p_{n+1}} b_{1,ij}^{n+1}[n+1, j]$. Define $A_{n+1} = A(F_{n+1}, E_{n+1}, \beta_{n+1,0}, \beta_{n+1,1})$. Moreover,

$$(\beta_{n+1,0})_* = \mathfrak{b}_0^{n+1},\ (\beta_{n+1,1})_* = \mathfrak{b}_1^{n+1} : K_0(F_{n+1}) = \mathbb{Z}^{p_{n+1}} \to K_0(E_{n+1}) = \mathbb{Z}^{l_{n+1}}.$$

We have

$$K_0(A_{n+1}) = G_{n+1} \subseteq K_0(F_{n+1}) = H_{n+1}.$$

Since π_{n+1} is surjective, by [Gong et al. 2020a, Proposition 3.5], $K_1(A_{n+1}) = 0$. Also $K_0(A_{n+1})_+ = G_{n+1} \cap (H_{n+1})_+ = \{0\}$, which implies $A_{n+1} \in \mathcal{C}_0$. Thus conditions (a), (b), (c), (d) and (e) (see (4.35)) in 4.25 for $n+1$ follow. This ends the construction of A_{n+1}.

With A_{n+1} constructed above we construct two homomorphisms

$$\varphi_{n,n+1}^o, \varphi_{n,n+1} : A_n \to A_{n+1}$$

in the next few sections, namely from 4.37 to 4.76.

4.37 (definition of ψ and ψ^o). As notation introduced in 4.3, we have

$$\operatorname{Sp}(A_n) = \{\theta_{n,1}, \dots, \theta_{n,p_n}\} \cup \{\eta_{n,1}(t), \dots, \eta_{n,l_n}(t) \mid 0 < t < 1\}.$$

To simplify the notation, let us use $\mathfrak{c} = (c_{ij})$ to denote $\mathfrak{c}^{n,n+1} = (c_{ij}^{n,n+1})$ and $\mathfrak{d} = (d_{ij})$ to denote $\mathfrak{d}^{n,n+1} = (d_{ij}^{n,n+1})$. Let $\psi_{n,n+1} : F_n \to F_{n+1}$ be a (nonunital) homomorphism defined by

$$\psi_{n,n+1}(a_1 \oplus a_2 \oplus \cdots \oplus a_{p_n}) = \bigoplus_{i=1}^{p_{n+1}} \operatorname{diag}\big(a_1^{\sim c_{i1}}, a_2^{\sim c_{i2}}, \dots, a_{p_n}^{\sim c_{ip_n}}, 0^{\sim\sim}\big), \tag{4.38}$$

where $0^{\sim\sim}$ denotes some suitable number of copies of 0 — to avoid introducing too much notation, sometimes we do not indicate how many copies it has (but it is usually easy to calculate, for example it is $[n+1, i] - \sum_{k=1}^{p_n} c_{ik}[n, k]$ copies here). Then $(\psi_{n,n+1})_* = \tilde{\gamma}_{n,n+1} = \mathfrak{c} = (c_{ij}) : K_0(F_n) \to K_0(F_{n+1})$.

Let $\psi^o : A_n \to F_{n+1}$ be defined by $\psi^o = \psi_{n,n+1} \circ \pi_n^A$. Let

$$c'_{1j} = c_{1j} - (n-1)\sum_{i=1}^{l_n} b^n_{0,ij} \quad \text{for } 1 \le j \le p_n. \tag{4.39}$$

By 4.25, (4.28), and (4.29), we estimate

$$\begin{aligned} c_{1,j} > c'_{1j} = c_{1,j} - (n-1)\sum_{i=1}^{l_n} b^n_{0,ij} &\ge c_{1,j} - (n-1)l_n \max\{\{n,i\} : 1 \le i \le p_n\} \\ &> c_{1,j} - \frac{\mathcal{L}_{n,n+1}}{2^{4(n+1)}} > \Big(1 - \frac{1}{2^{4(n+1)}}\Big)c_{1j}. \end{aligned} \tag{4.40}$$

Define $\psi : A_n \to F_{n+1}$ by sending

$$f = (f_1, f_2, \ldots, f_{l_n}, a_1, a_2, \ldots, a_{p_n}) \mapsto \psi(f) = (b_1, b_2, \ldots, b_{p_{n+1}}),$$

where

$$\begin{aligned} b_1 = \operatorname{diag}\Big(&f_1\Big(\frac{1}{n}\Big), f_1\Big(\frac{2}{n}\Big), \ldots, f_1\Big(\frac{n-1}{n}\Big), f_2\Big(\frac{1}{n}\Big), f_2\Big(\frac{2}{n}\Big), \ldots, f_2\Big(\frac{n-1}{n}\Big), \\ &\ldots, f_{l_n}\Big(\frac{1}{n}\Big), f_{l_n}\Big(\frac{2}{n}\Big), \ldots, f_{l_n}\Big(\frac{n-1}{n}\Big), a_1^{\sim c'_{11}}, a_2^{\sim c'_{12}}, \ldots, a_{p_n}^{\sim c'_{1p_n}}, 0^{\sim\sim}\Big) \end{aligned} \tag{4.41}$$

(from (4.29) and (4.28), $\sum_{i=1}^{l_n}(n-1)\{n,i\} + \sum_{j=1}^{p_n} c'_{1j}[n,j] < [n+1,1]$), and for $i \ge 2$,

$$b_i = \operatorname{diag}\big(a_1^{\sim c_{i1}}, a_2^{\sim c_{i2}}, \ldots, a_{p_n}^{\sim c_{ip_n}}, 0^{\sim\sim}\big). \tag{4.42}$$

Note that $\pi_{n+1,i}(\psi^o(f)) = \pi_{n+1,i}(\psi(f))$ for $i \ge 2$, where $\pi_{n+1,i} : F_{n+1} \to F^i_{n+1}$ is the projection map.

For each $0 \le t \le 1$, define $\psi_t : A_n \to F_{n+1}$ by sending

$$f = (f_1, f_2, \ldots, f_{l_n}, a_1, a_2, \ldots, a_{p_n}) \mapsto \psi_t(f) = (b'_1, b'_2, \ldots, b'_{p_{n+1}})$$

with $b'_i = b_i$ for $i \ge 2$ (see (4.42)) and

$$\begin{aligned} b'_1 = \operatorname{diag}\Big(&f_1\Big(\frac{1}{n}(1-t)\Big), \ldots, f_1\Big(\frac{n-1}{n}(1-t)\Big), f_2\Big(\frac{1}{n}(1-t)\Big), \\ &\ldots, f_2\Big(\frac{n-1}{n}(1-t)\Big), \ldots, f_{l_n}\Big(\frac{1}{n}(1-t)\Big), \\ &\ldots, f_{l_n}\Big(\frac{n-1}{n}(1-t)\Big), a_1^{\sim c'_{11}}, a_2^{\sim c'_{12}}, \ldots, a_{p_n}^{\sim c'_{1p_n}}\Big). \end{aligned} \tag{4.43}$$

When $t = 0$, we have $b'_1 = b_1$ (see (4.41)), and therefore $\psi = \psi_0$. For $t = 1$,

$$b'_1 = \operatorname{diag}\big(f_1(0)^{\sim(n-1)}, f_2(0)^{\sim(n-1)}, \ldots, f_{l_n}(0)^{\sim(n-1)},\ a_1^{\sim c'_{11}}, a_2^{\sim c'_{12}}, \ldots, a_{p_n}^{\sim c'_{1p_n}}\big).$$

By (4.5) and the definition of $c'_{1,j}$, we have $\mathrm{SP}(\psi_1) = \mathrm{SP}(\psi^o)$. Hence

$$KK(\psi) = KK(\psi_0) = KK(\psi_1) = KK(\psi^o) \in KK(A_n, F_{n+1}). \tag{4.44}$$

4.45 (definition of χ and μ). Recall that $\mathfrak{d} = (d_{ij})_{l_{n+1}\times l_n}$ is the matrix which represents $\tilde{\gamma}_{n,n+1} : H_n/G_n = Z^{l_n} \to H_{n+1}/G_{n+1} = Z^{l_{n+1}}$. For $1 \le j \le l_{n+1}$, $1 \le k \le l_n$, define

$$\tilde{d}_{jk} = \begin{cases} |d_{jk}| & \text{if } d_{jk} \neq 0, \\ 2 & \text{if } d_{jk} = 0. \end{cases} \tag{4.46}$$

For $1 \le j \le l_{n+1}$, let $L_j = \sum_{k=1}^{l_n} \tilde{d}_{jk}\{n,k\}$. Define $\chi : A_n \to \bigoplus_{j=1}^{l_{n+1}} M_{L_j}(C[0,1])$ by sending

$$f = (f_1, f_2, \ldots, f_{l_n}, a_1, a_2, \ldots, a_{p_n}) \mapsto \chi(f) = \bigoplus_{j=1}^{l_{n+1}} (F_{j,1}(t), F_{j,2}(t), \ldots, F_{j,l_n}(t)),$$

where

$$F_{jk}(t) = \begin{cases} \{f_k(t)^{\sim |d_{jk}|}\} & \text{if } d_{jk} > 0, \\ \{f_k(1-t)^{\sim |d_{jk}|}\} & \text{if } d_{jk} < 0, \\ \{f_k(t), f_k(1-t)\} & \text{if } d_{jk} = 0. \end{cases} \tag{4.47}$$

For any $1 \le j \le l_{n+1}$, let $\pi_j^E : \bigoplus_{k=1}^{l_{n+1}} M_{L_k}(C[0,1]) \to M_{L_j}(C[0,1])$ be the projection. Then, for any $0 < t < 1$,

$$\{\eta_{n,i}(t), \eta_{n,i}(1-t) : 1 \le i \le l_n\} \subset \mathrm{Sp}((\pi_j^E \circ \chi)|_t) \cup \mathrm{Sp}((\pi_j^E \circ \chi)|_{1-t}). \tag{4.48}$$

This implies that $\mathrm{Sp}(A_n) = \bigcup_{0 \le t \le 1} \mathrm{Sp}(\chi|_t)$. Hence χ is injective.

Define two subsets J_0, J_1 of the index set $\{1, 2, \ldots, l_{n+1}\}$ by $j \in J_0$ if and only if $b_{0,j1}^{n+1} > b_{1,j1}^{n+1}$; and $j \in J_1$ if and only if $b_{1,j1}^{n+1} > b_{0,j1}^{n+1}$. (Note that if $b_{0,j1}^{n+1} = b_{1,j1}^{n+1}$, then j is neither in J_0 nor in J_1.)

Let $B_j = |b_{0,j1}^{n+1} - b_{1,j1}^{n+1}|$ and $K_j = (n-1)B_j \cdot \sum_{i=1}^{l_n} \{n, i\}$.

Define $\mu : A_n \to \bigoplus_{j \in J_0 \cup J_1} M_{K_j}(C[0,1])$ by sending

$$f = (f_1, f_2, \ldots, f_{l_n}, a_1, a_2, \ldots, a_{p_n}) \mapsto \mu(f) = \bigoplus_{j \in J_0 \cup J_1} (G_{j,1}(t), G_{j,2}(t), \ldots, G_{j,l_n}(t)),$$

where

$$G_{jk}(t) = \begin{cases} \begin{array}{l} \mathrm{diag}\Big(f_k\big(\tfrac{1}{n}(1-t)\big)^{\sim B_j}, f_k\big(\tfrac{2}{n}(1-t)\big)^{\sim B_j}, \\ \qquad\qquad \ldots, f_k\big(\tfrac{n-1}{n}(1-t)\big)^{\sim B_j}\Big) \end{array} & \text{if } j \in J_0, \\ \mathrm{diag}\Big(f_k\big(\tfrac{1}{n}t\big)^{\sim B_j}, f_k\big(\tfrac{2}{n}t\big)^{\sim B_j}, \ldots, f_k\big(\tfrac{n-1}{n}t\big)^{\sim B_j}\Big) & \text{if } j \in J_1. \end{cases} \tag{4.49}$$

Note that, for $j \in J_0$,

$$\begin{aligned} G_{jk}(0) &= \mathrm{diag}\Big(f_k\Big(\frac{1}{n}\Big)^{\sim B_j}, f_k\Big(\frac{2}{n}\Big)^{\sim B_j}, \ldots, f_k\Big(\frac{n-1}{n}\Big)^{\sim B_j}\Big), \\ G_{jk}(1) &= \mathrm{diag}(f_k(0)^{\sim (n-1)B_j}), \end{aligned} \tag{4.50}$$

and, for $j \in J_1$,

$$\begin{aligned} G_{jk}(0) &= \operatorname{diag}(f_k(0)^{\sim(n-1)B_j}), \\ G_{jk}(1) &= \operatorname{diag}\Big(f_k\Big(\frac{1}{n}\Big)^{\sim B_j}, f_k\Big(\frac{2}{n}\Big)^{\sim B_j}, \dots, f_k\Big(\frac{n-1}{n}\Big)^{\sim B_j}\Big). \end{aligned} \tag{4.51}$$

4.52 (notation: $\kappa_0^j(\theta_{n,i})$, $\kappa_1^j(\theta_{n,i})$, $\bar\kappa_0^j(\theta_{n,i})$, $\bar\kappa_1^j(\theta_{n,i})$ — the multiplicities of $\theta_{n,i}$ for certain homomorphisms).

Let $\xi_0^o, \xi_1^o, \xi_0, \xi_1 : A_n \to E_{n+1} = \bigoplus_{i=1}^{l_{n+1}} M_{\{n+1,i\}}(\mathbb{C})$ be defined by $\xi_0^o = \beta_{n+1,0}\circ\psi^o$, $\xi_1^o = \beta_{n+1,1}\circ\psi^o$, $\xi_0 = \beta_{n+1,0}\circ\psi$ and $\xi_1 = \beta_{n+1,1}\circ\psi$.

It is also convenient to introduce the following notation: for a homomorphism $\varphi : A \to M_l(C[0,1])$, we use $\varphi|_0, \varphi|_1 : A \to M_l(\mathbb{C})$ to denote the map given by $\varphi|_0(a) := \varphi(a)(0)$ and $\varphi|_1(a) := \varphi(a)(1)$.

For fixed $1 \le i \le l_{n+1}$, let π_i^E be the projection from $E_{n+1} = \bigoplus_{j=1}^{l_{n+1}} M_{\{n+1,j\}}(\mathbb{C})$ to the i-th summand $E_{n+1}^i = M_{\{n+1,i\}}(\mathbb{C})$, from $\bigoplus_{j=1}^{l_{n+1}} C([0,1], M_{L_j}(\mathbb{C}))$ to the i-th summand $C([0,1], M_{L_i}(\mathbb{C}))$, or from $\bigoplus_{j=1}^{l_{n+1}} C([0,1], M_{L_j+K_j}(\mathbb{C}))$ to the i-th summand $C([0,1], M_{L_i+K_i}(\mathbb{C}))$.

In the next two lemmas, we compare $\pi_j^E\circ\xi_0^o$ and $\pi_j^E\circ\xi_1^o$ with $(\pi_j^E\circ\chi)|_0$ and $(\pi_j^E\circ\chi)|_1$, and compare $\pi_j^E\circ\xi_0$ and $\pi_j^E\circ\xi_1$ with $(\pi_j^E\circ(\chi\oplus\mu))|_0$ and $\pi_j^E\circ(\chi\oplus\mu))|_1$.

Let $f = (f_1, f_2, \dots, f_{l_n}, a_1, a_2, \dots, a_{p_n})$. Up to conjugating unitaries, we write

$$\begin{aligned} (\pi_j^E\circ\chi)|_0(f) &= \operatorname{diag}\big(a_1^{\sim\kappa_0^j(\theta_{n,1})}, a_2^{\sim\kappa_0^j(\theta_{n,2})}, \dots, a_{p_n}^{\sim\kappa_0^j(\theta_{n,p_n})}, 0^{\sim\sim}\big), \\ (\pi_j^E\circ\chi)|_1(f) &= \operatorname{diag}\big(a_1^{\sim\kappa_1^j(\theta_{n,1})}, a_2^{\sim\kappa_1^j(\theta_{n,2})}, \dots, a_{p_n}^{\sim\kappa_1^j(\theta_{n,p_n})}, 0^{\sim\sim}\big), \\ \pi_j^E\circ\xi_0^o(f) &= \operatorname{diag}\big(a_1^{\sim\bar\kappa_0^j(\theta_{n,1})}, a_2^{\sim\bar\kappa_0^j(\theta_{n,2})}, \dots, a_{p_n}^{\sim\bar\kappa_0^j(\theta_{n,p_n})}, 0^{\sim\sim}\big), \\ \pi_j^E\circ\xi_1^o(f) &= \operatorname{diag}\big(a_1^{\sim\bar\kappa_1^j(\theta_{n,1})}, a_2^{\sim\bar\kappa_1^j(\theta_{n,2})}, \dots, a_{p_n}^{\sim\bar\kappa_1^j(\theta_{n,p_n})}, 0^{\sim\sim}\big). \end{aligned}$$

(Here again we use $0^{\sim\sim}$ to denote some 0's without specifying how many copies there are.) The motivation of the above notation is that $\theta_{n,j}$ is the representation of A_n by sending f to a_j.

Lemma 4.53. *For any* $1 \le i \le p_n$, *we have*

$$\bar\kappa_0^j(\theta_{n,i}) - \kappa_0^j(\theta_{n,i}) = \bar\kappa_1^j(\theta_{n,i}) - \kappa_1^j(\theta_{n,i}) \ge \Big(1 - \frac{1}{2^{4(n+1)}}\Big)\bar\kappa_0^j(\theta_{n,i}). \tag{4.54}$$

Proof. The equality follows from (4.36), i.e., $(\mathfrak{b}_1^{n+1} - \mathfrak{b}_0^{n+1})\cdot\mathfrak{c}^{n,n+1} = \mathfrak{d}^{n,n+1}\cdot(\mathfrak{b}_1^n - \mathfrak{b}_0^n)$. To see this, we first note that

$$\begin{aligned} (\pi_j^E\circ\chi)|_0(f) = \operatorname{diag}\big(&\operatorname{diag}_{\{k:d_{jk}>0\}}(f_k(0)^{\sim d_{jk}}), \\ &\operatorname{diag}_{\{k:d_{jk}<0\}}(f_k(1)^{\sim|d_{jk}|}), \operatorname{diag}_{\{k:d_{jk}=0\}}(f_k(0), f_k(1))\big), \\ (\pi_j^E\circ\chi)|_1(f) = \operatorname{diag}\big(&\operatorname{diag}_{\{k:d_{jk}>0\}}(f_k(1)^{\sim d_{jk}}), \\ &\operatorname{diag}_{\{k:d_{jk}<0\}}(f_k(0)^{\sim|d_{jk}|}), \operatorname{diag}_{\{k:d_{jk}=0\}}(f_k(1), f_k(0))\big). \end{aligned}$$

Note that $\kappa_0^j(\theta_{n,i})$ is the multiplicity of $\theta_{n,i}$ in $\mathrm{SP}((\pi_j^E \circ \chi)|_0)$, and $\kappa_1^j(\theta_{n,i})$ the multiplicity of $\theta_{n,i}$ in $\mathrm{SP}((\pi_j^E \circ \chi)|_1)$. Also note that the homomorphism $(f_1, f_2, \ldots, f_{l_n}, a_1, a_2, \ldots, a_{p_n}) \to f_k(0)$ defines $\eta_{n,k}(0) \in \mathrm{SP}(A_n)$ (see 4.3). Hence

$$\begin{aligned}&\mathrm{SP}((\pi_j^E \circ \chi)|_0)\\ &\quad = \{\eta_{n,k}(0)^{\sim d_{jk}} : d_{jk} > 0\} \cup \{\eta_{n,k}(1)^{\sim |d_{jk}|} : d_{jk} < 0\} \cup \{\eta_{n,k}(0), \eta_{n,k}(1) : d_{jk} = 0\}.\end{aligned}$$

By (4.5), we have

$$\kappa_0^j(\theta_{n,i}) = \sum_{\{k : d_{jk} > 0\}} b_{0,ki}^n d_{jk} + \sum_{\{k : d_{jk} < 0\}} b_{1,ki}^n |d_{jk}| + \sum_{\{k : d_{jk} = 0\}} (b_{0,ki}^n + b_{1,ki}^n).$$

Similarly, we have

$$\kappa_1^j(\theta_{n,i}) = \sum_{\{k : d_{jk} > 0\}} b_{1,ki}^n d_{jk} + \sum_{\{k : d_{jk} < 0\}} b_{0,ki}^n |d_{jk}| + \sum_{\{k : d_{jk} = 0\}} (b_{1,ki}^n + b_{0,ki}^n).$$

Hence $\kappa_1^j(\theta_{n,i}) - \kappa_0^j(\theta_{n,i}) = \sum_{k=1}^{l_n} (b_{1,ki}^n - b_{0,ki}^n) d_{jk}$. On the other hand,

$$\bar{\kappa}_0^j(\theta_{n,i}) = \sum_{k=1}^{p_{n+1}} b_{0,jk}^{n+1} c_{ki} \quad \text{and} \quad \bar{\kappa}_1^j(\theta_{n,i}) = \sum_{k=1}^{p_{n+1}} b_{1,jk}^{n+1} c_{ki}. \tag{4.55}$$

Hence

$$\bar{\kappa}_1^j(\theta_{n,i}) - \bar{\kappa}_0^j(\theta_{n,i}) = \sum_{k=1}^{p_{n+1}} (b_{1,jk}^{n+1} - b_{0,jk}^{n+1}) c_{ki}.$$

That is, $\kappa_1^j(\theta_{n,i}) - \kappa_0^j(\theta_{n,i})$ is the ji-th entry of the matrix $\mathfrak{d}^{n,n+1}(\mathfrak{b}_1^n - \mathfrak{b}_0^n)$; and $\bar{\kappa}_1^j(\theta_{n,i}) - \bar{\kappa}_0^j(\theta_{n,i})$ is the ji-th entry of the matrix $(\mathfrak{b}_1^{n+1} - \mathfrak{b}_0^{n+1})\mathfrak{c}^{n,n+1}$. By (4.36), we have

$$\kappa_1^j(\theta_{n,i}) - \kappa_0^j(\theta_{n,i}) = \bar{\kappa}_1^j(\theta_{n,i}) - \bar{\kappa}_0^j(\theta_{n,i}),$$

as desired.

Furthermore, it follows from $b_{0,jk}^{n+1} \geq \mathcal{M}_{n+1}$ that

$$\begin{aligned}\kappa_0^j(\theta_{n,i}) &< \sum_{k=1}^{l_n} (|d_{jk}| + 2)\{n, k\} \leq \frac{1}{2^{4(n+1)}} \mathcal{M}_{n+1}\\ &\leq \frac{1}{2^{4(n+1)}} b_{0,j1}^{n+1} \leq \frac{1}{2^{4(n+1)}} \bar{\kappa}_0^j(\theta_{n,i}).\end{aligned} \tag{4.56}$$

Hence the inequality in (4.54) also follows. □

4.57 (notation: $\sigma_k^j(\eta_{n,i})$, $\bar{\sigma}_k^j(\eta_{n,i})$, $\lambda_k^j(\theta_{n,i})$, $\bar{\lambda}_k^j(\theta_{n,i})$ for $k = 0, 1$ — the multiplicities of $\eta_{n,i}(l/n)$ and $\theta_{n,i}$ for certain homomorphisms).

Let $f=(f_1, f_2, \dots, f_{l_n}, a_1, a_2, \dots, a_{p_n})$. Then, for each fixed j, one may write

$$
\begin{aligned}
&(\pi_j^E\circ(\chi\oplus\mu))|_0(f)\\
&\quad=\mathrm{diag}\Big(f_1\Big(\frac{1}{n}\Big)^{\sim\sigma_0^j(\eta_{n,1})}, f_1\Big(\frac{2}{n}\Big)^{\sim\sigma_0^j(\eta_{n,1})}, \dots, f_1\Big(\frac{n-1}{n}\Big)^{\sim\sigma_0^j(\eta_{n,1})},\\
&\qquad f_2\Big(\frac{1}{n}\Big)^{\sim\sigma_0^j(\eta_{n,2})}, \dots, f_2\Big(\frac{n-1}{n}\Big)^{\sim\sigma_0^j(\eta_{n,2})}, \dots,\\
&\qquad f_{l_n}\Big(\frac{1}{n}\Big)^{\sim\sigma_0^j(\eta_{n,l_n})}, \dots, f_{l_n}\Big(\frac{n-1}{n}\Big)^{\sim\sigma_0^j(\eta_{n,l_n})},\\
&\qquad a_1^{\sim\lambda_0^j(\theta_{n,1})}, a_2^{\sim\lambda_0^j(\theta_{n,2})}, \dots, a_{p_n}^{\sim\lambda_0^j(\theta_{n,p_n})}, 0^{\sim\sim}\Big),\\
&(\pi_j^E\circ(\chi\oplus\mu))|_1(f)\\
&\quad=\mathrm{diag}\Big(f_1\Big(\frac{1}{n}\Big)^{\sim\sigma_1^j(\eta_{n,1})}, f_1\Big(\frac{2}{n}\Big)^{\sim\sigma_1^j(\eta_{n,1})}, \dots, f_1\Big(\frac{n-1}{n}\Big)^{\sim\sigma_1^j(\eta_{n,1})},\\
&\qquad f_2\Big(\frac{1}{n}\Big)^{\sim\sigma_1^j(\eta_{n,2})}, \dots, f_2\Big(\frac{n-1}{n}\Big)^{\sim\sigma_1^j(\eta_{n,2})}, \dots,\\
&\qquad f_{l_n}\Big(\frac{1}{n}\Big)^{\sim\sigma_1^j(\eta_{n,l_n})}, \dots, f_{l_n}\Big(\frac{n-1}{n}\Big)^{\sim\sigma_1^j(\eta_{n,l_n})},\\
&\qquad a_1^{\sim\lambda_1^j(\theta_{n,1})}, a_2^{\sim\lambda_1^j(\theta_{n,2})}, \dots, a_{p_n}^{\sim\lambda_1^j(\theta_{n,p_n})}, 0^{\sim\sim}\Big),\\
&(\pi_j^E\circ\xi_0)(f)\\
&\quad=\mathrm{diag}\Big(f_1\Big(\frac{1}{n}\Big)^{\sim\bar\sigma_0^j(\eta_{n,1})}, f_1\Big(\frac{2}{n}\Big)^{\sim\bar\sigma_0^j(\eta_{n,1})}, \dots, f_1\Big(\frac{n-1}{n}\Big)^{\sim\bar\sigma_0^j(\eta_{n,1})},\\
&\qquad f_2\Big(\frac{1}{n}\Big)^{\sim\bar\sigma_0^j(\eta_{n,2})}, \dots, f_2\Big(\frac{n-1}{n}\Big)^{\sim\bar\sigma_0^j(\eta_{n,2})}, \dots,\\
&\qquad f_{l_n}\Big(\frac{1}{n}\Big)^{\sim\bar\sigma_0^j(\eta_{n,l_n})}, \dots, f_{l_n}\Big(\frac{n-1}{n}\Big)^{\sim\bar\sigma_0^j(\eta_{n,l_n})},\\
&\qquad a_1^{\sim\bar\lambda_0^j(\theta_{n,1})}, a_2^{\sim\bar\lambda_0^j(\theta_{n,2})}, \dots, a_{p_n}^{\sim\bar\lambda_0^j(\theta_{n,p_n})}, 0^{\sim\sim}\Big),\\
&(\pi_j^E\circ\xi_1)(f)\\
&\quad=\mathrm{diag}\Big(f_1\Big(\frac{1}{n}\Big)^{\sim\bar\sigma_1^j(\eta_{n,1})}, f_1\Big(\frac{2}{n}\Big)^{\sim\bar\sigma_1^j(\eta_{n,1})}, \dots, f_1\Big(\frac{n-1}{n}\Big)^{\sim\bar\sigma_1^j(\eta_{n,1})},\\
&\qquad f_2\Big(\frac{1}{n}\Big)^{\sim\bar\sigma_1^j(\eta_{n,2})}, \dots, f_2\Big(\frac{n-1}{n}\Big)^{\sim\bar\sigma_1^j(\eta_{n,2})}, \dots,\\
&\qquad f_{l_n}\Big(\frac{1}{n}\Big)^{\sim\bar\sigma_1^j(\eta_{n,l_n})}, \dots, f_{l_n}\Big(\frac{n-1}{n}\Big)^{\sim\bar\sigma_1^j(\eta_{n,l_n})},\\
&\qquad a_1^{\sim\bar\lambda_1^j(\theta_{n,1})}, a_2^{\sim\bar\lambda_1^j(\theta_{n,2})}, \dots, a_{p_n}^{\sim\bar\lambda_1^j(\theta_{n,p_n})}, 0^{\sim\sim}\Big).
\end{aligned}
$$

In particular, the multiplicities of $f_i(1/n), f_i(2/n), \dots, f_i((n-1)/n)$, in each of the four homomorphisms above, are the same — this is why we use the notation $\sigma_0^j(\eta_{n,i})$ instead of $\sigma_0^j(\eta_{n,i}(l/n))$.

Lemma 4.58. *For all* $1 \le i \le l_n$,

$$\bar{\sigma}_0^j(\eta_{n,i}) - \sigma_0^j(\eta_{n,i}) = \bar{\sigma}_1^j(\eta_{n,i}) - \sigma_1^j(\eta_{n,i}) \le \min(b_{0,j1}^{n+1}, b_{1,j1}^{n+1}), \tag{4.59}$$

and for all $1 \le i \le p_n$,

$$\bar{\lambda}_0^j(\theta_{n,i}) - \lambda_0^j(\theta_{n,i}) = \bar{\lambda}_1^j(\theta_{n,i}) - \lambda_1^j(\theta_{n,i}) \ge \Big(1 - \frac{2}{2^{4(n+1)}}\Big)\bar{\kappa}_0^j(\theta_{n,i}).$$

Proof. By (e) of 4.25 (and $[n, i] > 2$), (4.28) and (4.29) as well as (4.55), for any $1 \le i \le p_n$, we obtain

$$\begin{aligned}(n-1)\max(b_{0,j1}^{n+1}, b_{0,j1}^{n+1})\Big(\sum_{k=1}^{l_n} b_{0,ki}^n\Big) &\le \max(b_{0,j1}^{n+1}, b_{0,j1}^{n+1})\Big(\frac{1}{2}(n-1)\sum_{k=1}^{l_n}\{n, k\}\Big)\\ &\le \min(b_{0,j1}^{n+1}, b_{0,j1}^{n+1}) \cdot c_{11} \cdot \frac{1}{2^{4(n+1)}}\\ &\le \frac{1}{2^{4(n+1)}}\min(\bar{\kappa}_0^j(\theta_{n,i}), \bar{\kappa}_1^j(\theta_{n,i})). \end{aligned}\tag{4.60}$$

From the definition of ξ_0, ξ_1 (see the first paragraph of 4.52 and (4.41) and (4.42)), we have

$$\begin{aligned}&(\pi_j^E \circ \xi_0)(f)\\ &\quad= \mathrm{diag}\Big(f_1\Big(\frac{1}{n}\Big)^{\sim b_{0,j1}^{n+1}}, f_1\Big(\frac{2}{n}\Big)^{\sim b_{0,j1}^{n+1}}, \dots, f_1\Big(\frac{n-1}{n}\Big)^{\sim b_{0,j1}^{n+1}}, f_2\Big(\frac{1}{n}\Big)^{\sim b_{0,j1}^{n+1}},\\ &\qquad\qquad \dots, f_2\Big(\frac{n-1}{n}\Big)^{\sim b_{0,j1}^{n+1}}, \dots, f_{l_n}\Big(\frac{1}{n}\Big)^{\sim b_{0,j1}^{n+1}}, \dots, f_{l_n}\Big(\frac{n-1}{n}\Big)^{\sim b_{0,j1}^{n+1}},\\ &\qquad\qquad a_1^{\sim\left(c'_{11} b_{0,j1}^{n+1} + \sum_{k=2}^{p_{n+1}} c_{k1} b_{0,jk}^{n+1}\right)}, a_2^{\sim\left(c'_{12} b_{0,j1}^{n+1} + \sum_{k=2}^{p_{n+1}} c_{k2} b_{0,jk}^{n+1}\right)}, \dots,\\ &\qquad\qquad\qquad a_{p_n}^{\sim\left(c'_{1p_n} b_{0,j1}^{n+1} + \sum_{k=2}^{p_{n+1}} c_{kp_n} b_{0,jk}^{n+1}\right)}\Big).\end{aligned}\tag{4.61}$$

Hence we always have $\bar{\sigma}_0^j(\eta_{n,i}) = b_{0,j1}^{n+1}$. Similarly, we have $\bar{\sigma}_1^j(\eta_{n,i}) = b_{1,j1}^{n+1}$. It follows that

$$\bar{\sigma}_0^j(\eta_{n,i}) - \sigma_0^j(\eta_{n,i}) \le b_{0,j1}^{n+1} \quad \text{and} \quad \bar{\sigma}_1^j(\eta_{n,i}) - \sigma_1^j(\eta_{n,i}) \le b_{1,j1}^{n+1}.$$

So, for (4.59), it remains to show that $\bar{\sigma}_0^j(\eta_{n,i}) - \sigma_0^j(\eta_{n,i}) = \bar{\sigma}_1^j(\eta_{n,i}) - \sigma_1^j(\eta_{n,i})$.

Combining (4.61), (4.39), and (4.55), we calculate (see also 4.57)

$$\begin{aligned}\bar{\lambda}_0^j(\theta_{n,i}) &= c'_{1i} b_{0,j1}^{n+1} + \sum_{k=2}^{p_{n+1}} c_{ki} b_{0,jk}^{n+1} = \Big(c_{1i} - \sum_{k=1}^{l_n} b_{0,ki}^n\Big) b_{0,j1}^{n+1} + \sum_{k=2}^{p_{n+1}} c_{ki} b_{0,jk}^{n+1}\\ &= \sum_{k=1}^{p_{n+1}} c_{ki} b_{0,jk}^{n+1} - (n-1) b_{0,j1}^{n+1}\Big(\sum_{k=1}^{l_n} b_{0,ki}^n\Big)\\ &= \bar{\kappa}_0^j(\theta_{n,i}) - (n-1) b_{0,j1}^{n+1}\Big(\sum_{k=1}^{l_n} b_{0,ki}^n\Big).\end{aligned}\tag{4.62}$$

Similarly,

$$\bar{\lambda}_1^j(\theta_{n,i}) = c'_{1i} b_{1,j1}^{n+1} + \sum_{k=2}^{p_{n+1}} c_{ki} b_{1,jk}^{n+1} = \bar{\kappa}_1^j(\theta_{n,i}) - (n-1) b_{1,j1}^{n+1} \left(\sum_{k=1}^{l_n} b_{0,ki}^n \right). \tag{4.63}$$

We divide the proof into three cases: $j \notin J_0 \cup J_1$, $j \in J_0$ and $j \in J_1$.

Case 1: $j \notin J_0 \cup J_1$. In this case, $b_{0,j1}^{n+1} = b_{1,j1}^{n+1}$, $K_j = 0$ and $\pi_j^E \circ (\chi \oplus \mu) = \pi_j^E \circ \chi$. Consequently,

$$\sigma_0^j(\eta_{n,i}) = 0 = \sigma_1^j(\eta_{n,i}), \quad \lambda_0^j(\theta_{n,i}) = \kappa_0^j(\theta_{n,i}) \quad \text{and} \quad \lambda_1^j(\theta_{n,i}) = \kappa_1^j(\theta_{n,i}). \tag{4.64}$$

Hence $\bar{\sigma}_0^j(\eta_{n,i}) - \sigma_0^j(\eta_{n,i}) = b_{0,j1}^{n+1} = b_{1,j1}^{n+1} = \bar{\sigma}_1^j(\eta_{n,i}) - \sigma_1^j(\eta_{n,i})$.

It follows from (4.64), $b_{0,j1}^{n+1} = b_{1,j1}^{n+1}$, (4.62), (4.63), Lemma 4.53 and (4.60) that

$$\begin{aligned} \bar{\lambda}_1^j(\theta_{n,i}) - \lambda_1^j(\theta_{n,i}) &= \bar{\kappa}_0^j(\theta_{n,i}) - (n-1) b_{0,j1}^{n+1} \left(\sum_{k=1}^{l_n} b_{0,ki}^n \right) - \kappa_1^j(\theta_{n,i}) \\ &= \bar{\lambda}_0^j(\theta_{n,i}) - \lambda_0^j(\theta_{n,i}) \\ &= \bar{\kappa}_0^j(\theta_{n,i}) - \kappa_0^j(\theta_{n,i}) - (n-1) b_{0,j1}^{n+1} \left(\sum_{k=1}^{l_n} b_{0,ki}^n \right) \\ &\geq \left(1 - \frac{2}{2^{4(n+1)}} \right) \bar{\kappa}_0^j(\theta_{n,i}). \end{aligned} \tag{4.65}$$

Case 2: $j \in J_0$. Let $i \in \{1, 2, \dots, l_n\}$. By (4.50),

$$\sigma_1^j(\eta_{n,i}) = 0 \quad \text{and} \quad \sigma_0^j(\eta_{n,i}) = B_j = b_{0,j1}^{n+1} - b_{1,j1}^{n+1}. \tag{4.66}$$

Recall that we have computed above that $\bar{\sigma}_0^j(\eta_{n,i}) = b_{0,j1}^{n+1}$ and $\bar{\sigma}_1^j(\eta_{n,i}) = b_{1,j1}^{n+1}$. Thus (by (4.66))

$$\bar{\sigma}_0^j(\eta_{n,i}) - \sigma_0^j(\eta_{n,i}) = b_{0,j1}^{n+1} - (b_{0,j1}^{n+1} - b_{1,j1}^{n+1}) = b_{1,j1}^{n+1} = \bar{\sigma}_1^j(\eta_{n,i}) - \sigma_1^j(\eta_{n,i}).$$

Now let $i \in \{1, 2, \dots, p_n\}$. We calculate $\lambda_0^j(\theta_{n,i})$, $\lambda_0^j(\theta_{n,i})$, $\bar{\lambda}_0^j(\theta_{n,i})$ and $\bar{\lambda}_1^j(\theta_{n,i})$. From (4.50) (and (4.5)), we have that

$$\mathrm{SP}((\pi_j^E \circ \mu)|_1) = \bigcup \{\eta_{n,k}(0)^{\sim(n-1)B_j} : 1 \leq k \leq l_n\} = \left\{ \theta_{n,i}^{\sim(n-1)B_j \sum_{k=1}^{l_n} b_{0,ki}^n} : 1 \leq i \leq p_n \right\},$$

and that $\mathrm{SP}((\pi_j^E \circ \mu)|_0)$ does not contain any $\theta_{n,i}$.

Recall that $\lambda_1^j(\theta_{n,i})$ is the multiplicity of $\theta_{n,i}$ of the spectrum of $(\pi_j^E \circ (\chi \oplus \mu))|_1$. Hence

$$\lambda_1^j(\theta_{n,i}) = \kappa_1^j(\theta_{n,i}) + (n-1) B_j \sum_{k=1}^{l_n} b_{0,ki}^n \quad \text{and} \quad \lambda_0^j(\theta_{n,i}) = \kappa_0^j(\theta_{n,i}). \tag{4.67}$$

By (4.62), (4.63) and (4.67) as well as 4.53, we have

$$\begin{aligned}\bar{\lambda}_0^j(\theta_{n,i}) - \lambda_0^j(\theta_{n,i}) &= \bar{\kappa}_0^j(\theta_{n,i}) - \kappa_0^j(\theta_{n,i}) - (n-1)b_{0,j1}^{n+1}\Big(\sum_{k=1}^{l_n} b_{0,ki}^n\Big)\\ &= \bar{\kappa}_1^j(\theta_{n,i}) - \kappa_1^j(\theta_{n,i}) - (n-1)b_{0,j1}^{n+1}\Big(\sum_{k=1}^{l_n} b_{0,ki}^n\Big)\\ &= \bar{\kappa}_1^j(\theta_{n,i}) - \kappa_1^j(\theta_{n,i}) - (n-1)(B_j + b_{1,j1}^{n+1})\Big(\sum_{k=1}^{l_n} b_{0,ki}^n\Big)\\ &= \bar{\lambda}_1^j(\theta_{n,i}) - \lambda_1^j(\theta_{n,i}).\end{aligned}$$

Also, by (4.60) and Lemma 4.53, we estimate

$$\begin{aligned}\bar{\lambda}_0^j(\theta_{n,i}) - \lambda_0^j(\theta_{n,i}) &= \bar{\kappa}_0^j(\theta_{n,i}) - \kappa_0^j(\theta_{n,i}) - (n-1)b_{0,j1}^{n+1}\Big(\sum_{k=1}^{l_n} b_{0,ki}^n\Big)\\ &\geq \Big(1 - \frac{2}{2^{4(n+1)}}\Big)\bar{\kappa}_0^j(\theta_{n,i}).\end{aligned}$$

<u>Case 3</u>: $j \in J_1$. This case is proved exactly the same as Case 2, but replacing (4.50) with (4.51). □

4.68 (definition of φ^o, φ). Set

$$\begin{aligned}\kappa^j(i) &= \bar{\kappa}_0^j(\theta_{n,i}) - \kappa_0^j(\theta_{n,i}) = \bar{\kappa}_1^j(\theta_{n,i}) - \kappa_1^j(\theta_{n,i}),\\ \sigma^j(i) &= \bar{\sigma}_0^j(\eta_{n,i}) - \sigma_0^j(\eta_{n,i}) = \bar{\sigma}_1^j(\eta_{n,i}) - \sigma_1^j(\eta_{n,i}),\\ \lambda^j(i) &= \bar{\lambda}_0^j(\theta_{n,i}) - \lambda_0^j(\theta_{n,i}) = \bar{\lambda}_1^j(\theta_{n,i}) - \lambda_1^j(\theta_{n,i}).\end{aligned}$$

Let $\varphi^o : A_n \to C\big([0,1], \bigoplus_{j=1}^{l_{n+1}} M_{o(j)}(\mathbb{C})\big)$ be the homomorphism defined by sending $f = (f_1, f_2, \ldots, f_{l_n}, a_1, a_2, \ldots, a_{p_n})$ to

$$\varphi^o(f) = \bigoplus_{j=1}^{l_{n+1}} \operatorname{diag}\big(\pi_j^E \circ \chi(f), a_1^{\sim\kappa^j(1)}, a_2^{\sim\kappa^j(2)}, \ldots, a_{p_n}^{\sim\kappa^j(p_n)}\big), \tag{4.69}$$

where $o(j) = L_j + \sum_{i=1}^{p_n} \kappa^j(i)[n,i]$. Let $\varphi : A_n \to C\big([0,1], \bigoplus_{j=1}^{l_{n+1}} M_{o'(j)}(\mathbb{C})\big)$ be the homomorphism defined by sending $f = (f_1, f_2, \ldots, f_{l_n}, a_1, a_2, \ldots, a_{p_n})$ to

$$\begin{aligned}\varphi(f) = \bigoplus_{j=1}^{l_{n+1}} \operatorname{diag}\Big(&(\pi_j^E \circ (\chi \oplus \mu))(f), f_1\Big(\frac{1}{n}\Big)^{\sim\sigma^j(1)},\\ &f_1\Big(\frac{2}{n}\Big)^{\sim\sigma^j(1)}, \ldots, f_1\Big(\frac{n-1}{n}\Big)^{\sim\sigma^j(1)}, f_2\Big(\frac{1}{n}\Big)^{\sim\sigma^j(2)}, \ldots,\\ &f_2\Big(\frac{n-1}{n}\Big)^{\sim\sigma^j(2)}, \ldots, f_{l_n}\Big(\frac{1}{n}\Big)^{\sim\sigma^j(l_n)}, \ldots, f_{l_n}\Big(\frac{n-1}{n}\Big)^{\sim\sigma^j(l_n)},\\ &a_1^{\sim\lambda^j(1)}, a_2^{\sim\lambda^j(2)}, \ldots, a_{p_n}^{\sim\lambda^j(p_n)}\Big),\end{aligned} \tag{4.70}$$

where

$$o'(j) = L_j + K_j + (n-1)\sum_{k=1}^{l_n} \sigma^j(k)\{n,k\} + \sum_{i=1}^{p_n} \lambda^j(i)[n,i].$$

With the following lemma, both maps φ^o and φ can be regarded as maps from A_n to $C([0,1], E_{n+1})$ by adding suitably many copies of 0's in the equations (4.69) and (4.70).

Lemma 4.71. *We have the inequalities* $o(j) \le \{n+1, j\}$ *and* $o'(j) \le \{n+1, j\}$ (see 4.68). *Furthermore, we have*

$$\begin{aligned} \lambda^j(i) &\ge \Big(1 - \frac{2}{2^{4(n+1)}}\Big)\sum_{k=1}^{p_{n+1}} b_{0,jk}^{n+1} c_{ki}, \\ \sum_{i=1}^{p_n} \lambda^j(i)[n,i] &> \Big(1 - \frac{2}{4^{n+1}}\Big)\{n+1, j\}. \end{aligned} \tag{4.72}$$

Proof. Recall (see 4.68) that $\kappa^j(i) \le \bar{\kappa}_0^j(\theta_{n,i})$, $\lambda^j(i) \le \bar{\lambda}_0^j(\theta_{n,i}) \le \bar{\kappa}_0^j(\theta_{n,i})$. Also from Lemma 4.58, we have $\sigma^j(k) \le \min(b_{0,j1}^{n+1}, b_{1,j1}^{n+1})$. By (4.28) and (4.29), we have

$$\begin{aligned} n\sum_{k=1}^{l_n} \sigma^j(k)\{n,k\} &\le n\sum_{k=1}^{l_n} \min(b_{0,j1}^{n+1}, b_{1,j1}^{n+1}) \\ &\le \frac{1}{2^{4(n+1)}}\mathcal{L}_{n+1} \min(b_{0,j1}^{n+1}, b_{1,j1}^{n+1}) \\ &\le \frac{1}{2^{4(n+1)}} c_{11} \min(b_{0,j1}^{n+1}, b_{1,j1}^{n+1}) \\ &\le \frac{1}{2^{4(n+1)}} [n+1, 1] \min(b_{0,j1}^{n+1}, b_{1,j1}^{n+1}). \end{aligned} \tag{4.73}$$

From the definition of $\mathcal{M}_{n+1}$ (see (4.31)), we know that

$$\sum_{j=1}^{l_{n+1}} (L_j + K_j) \le \frac{1}{2^{4(n+1)}}\mathcal{M}_{n+1}. \tag{4.74}$$

Combining with (4.55), (4.31), (4.32) and (4.33), (and recall $L_j = \sum_{k=1}^{l_n} \tilde{d}_{jk}\{n,k\}$), we have

$$\begin{aligned} o(j) &= L_j + \sum_{i=1}^{p_n} \kappa^j(i)[n,i] \le L_j + \sum_{i=1}^{p_n} \bar{\kappa}_0^j(\theta_{n,i})[n,i] \\ &= L_j + \sum_{i=1}^{p_n}\sum_{k=1}^{p_{n+1}} b_{0,jk}^{n+1} c_{ki}[n,i] = \sum_{k=1}^{l_n} \tilde{d}_{jk}\{n,k\} + \sum_{k=1}^{p_{n+1}} [n+1,k] b_{0,jk}^{n+1} \\ &\le \frac{\mathcal{M}_{n+1}}{2^{4(n+1)}} + \sum_{k=1}^{p_{n+1}} [n+1,k] b_{0,jk}^{n+1} \le \Big(1 + \frac{2}{2^{4(n+1)}}\Big)\sum_{k=1}^{p_{n+1}} \min(b_{0,jk}^{n+1}, b_{1,jk}^{n+1})[n+1,k]. \end{aligned}$$

Combining (4.74), (4.31) and (4.73), we obtain

$$\begin{aligned} o'(j) &= L_j + K_j + (n-1)\sum_{k=1}^{l_n} \sigma^j(k)\{n,k\} + \sum_{i=1}^{p_n} \lambda^j(i)[n,i] \\ &\le \Big(1+\frac{2}{2^{4(n+1)}}\Big)\sum_{k=1}^{p_{n+1}} \min(b^{n+1}_{0,jk}, b^{n+1}_{1,jk})[n+1,k] + (n-1)\sum_{k=1}^{l_n}\sigma^j(k)\{n,k\} \\ &\le \Big(1+\frac{3}{2^{4(n+1)}}\Big)\sum_{k=1}^{p_{n+1}} \min(b^{n+1}_{0,jk}, b^{n+1}_{1,jk})[n+1,k]. \end{aligned}$$

In summary, we conclude from (4.35) that

$$\max(o(j), o'(j)) \le \Big(1+\frac{3}{2^{4(n+1)}}\Big)\sum_{k=1}^{p_{n+1}} \min(b^{n+1}_{0,jk}, b^{n+1}_{1,jk})[n+1,k] \le \{n+1, j\}.$$

The first inequality of (4.72) follows from Lemma 4.58 and (4.55). Using (4.30) and (e) in 4.25 (with $n+1$ in place of n), we calculate

$$\begin{aligned} \sum_{i=1}^{p_n} \lambda^j(i)[n,i] &\ge \Big(1-\frac{2}{2^{4(n+1)}}\Big)\sum_{k=1}^{p_{n+1}}\sum_{i=1}^{p_n} b^{n+1}_{0,jk} c_{ki}[n,i] \\ &\ge \Big(1-\frac{2}{2^{4(n+1)}}\Big)\Big(1-\frac{1}{8^n}\Big)\sum_{k=1}^{p_{n+1}} b^{n+1}_{0,jk}[n+1,k] \\ &\ge \Big(1-\frac{2}{2^{4(n+1)}}\Big)\Big(1-\frac{1}{8^n}\Big)\Big(1-\frac{1}{4^{n+1}}\Big)\{n+1,j\} \\ &> \Big(1-\frac{2}{4^{n+1}}\Big)\{n+1,j\}. \end{aligned}$$ □

From the definition of φ^o and φ, we get the following lemma.

Lemma 4.75. *We have* $\mathrm{SP}(\varphi^o|_0) = \mathrm{SP}(\xi^o_0)$, $\mathrm{SP}(\varphi^o|_1) = \mathrm{SP}(\xi^o_1)$, $\mathrm{SP}(\varphi|_0) = \mathrm{SP}(\xi_0)$ *and* $\mathrm{SP}(\varphi|_1) = \mathrm{SP}(\xi_1)$.

Proof. Note that $\mathrm{SP}(\pi^E_j \circ \varphi^o|_0) = \big\{\mathrm{SP}((\pi^E_j\circ\chi)|_0), \theta_{n,1}^{\sim\kappa^j(1)}, \theta_{n,2}^{\sim\kappa^j(2)}, \dots, \theta_{n,p_n}^{\sim\kappa^j(p_n)}\big\} = \mathrm{SP}(\pi^E_j \circ \xi^o_0)$ since $\kappa^j(i) = \bar\kappa^j_0(\theta_{n,i}) - \kappa^j_0(\theta_{n,i})$. The proofs of the other three parts are similar. □

4.76 (definition of $\varphi^o_{n,n+1}$ and $\varphi_{n,n+1}$). By the lemma above, there are unitaries $V_0, V_1, U_0, U_1 \in E_{n+1}$ such that $\mathrm{Ad}\, V_i \circ (\varphi^o|_i) = \xi^o_i$ and $\mathrm{Ad}\, U_i \circ (\varphi|_i) = \xi_i$ for $i = 0, 1$. Since the unitary group of E_{n+1} is path connected, there are two continuous paths of unitaries $V := V(t)$, $U := U(t)$, $0 \le t \le 1$ such that $V(0) = V_0$, $V(1) = V_1$, $U(0) = U_0$ and $U(1) = U_1$.

Now we define two homomorphisms $\varphi^o_{n,n+1}, \varphi_{n,n+1} : A_n \to A_{n+1}$ by

$$\varphi^o_{n,n+1}(f) = (\operatorname{Ad} V \circ \varphi^o)(f) \oplus \psi^o(f) \in A_{n+1} \subset C([0,1], E_{n+1}) \oplus F_{n+1}, \quad (4.77)$$

$$\varphi_{n,n+1}(f) = (\operatorname{Ad} U \circ \varphi)(f) \oplus \psi(f) \in A_{n+1} \subset C([0,1], E_{n+1}) \oplus F_{n+1}. \quad (4.78)$$

Note that

$$(\operatorname{Ad} V \circ \varphi^o)(f)(0) = (\operatorname{Ad} V_0 \circ (\varphi^o|_0))(f) = \xi^o_0(f) = \beta_{n+1,0}(\psi^o(f)),$$
$$(\operatorname{Ad} V \circ \varphi^o)(f)(1) = (\operatorname{Ad} V_1 \circ (\varphi^o|_1))(f) = \xi^o_1(f) = \beta_{n+1,1}(\psi^o(f)).$$

In other words, the element $\varphi^o_{n,n+1}(f)$ is in A_{n+1} rather than $C([0,1], E_{n+1}) \oplus F_{n+1}$. Similarly, $\varphi_{n,n+1}$ is also a homomorphism from A_n to A_{n+1}.

So the construction is completed. We obtain two inductive systems $A^o = \lim(A_n, \varphi^o_{n,n+1})$ and $A = \lim(A_n, \varphi_{n,n+1})$. We summarize the properties of $\varphi^o_{n,n+1}$ and $\varphi_{n,n+1}$ in 4.80, 4.101 and Theorem 4.107 below, and use these properties to prove that A is a simple C^*-algebra which satisfies the desired properties in 4.1.

Before we present other properties of $\varphi^o_{n,n+1}$ and $\varphi_{n,n+1}$, let us point out that both of them are injective, since χ is injective. Also let us rewrite the part of property (c) in 4.25 as

(c′) $\tilde{\rho}(\gamma_{n,\infty}(u_n)) \geq (1 - 1/8^n) \cdot 1_\Delta$ (since $\gamma_{n,\infty} = \gamma'_{k(n),\infty}$).

4.79 (K-theory of $\varphi_{n,n+1}$). Let

$$I_n = C_0((0,1), E_n) \subset A_n \quad \text{and} \quad I_{n+1} = C_0((0,1), E_{n+1}) \subset A_{n+1}.$$

From the definition of $\varphi^o_{n,n+1}$, we have the commutative diagram

$$\begin{array}{ccc} A_n & \xrightarrow{\varphi^o_{n,n+1}} & A_{n+1} \\ \downarrow{\scriptstyle \pi^A_n} & & \downarrow{\scriptstyle \pi^A_{n+1}} \\ A_n/I_n = F_n & \xrightarrow{\psi_{n,n+1}} & A_{n+1}/I_{n+1} = F_{n+1} \end{array}$$

Note that $K_0(A_n) = G_n$, $K_0(A_{n+1}) = G_{n+1}$, $K_0(F_n) = H_n$, $K_0(F_{n+1}) = H_{n+1}$, and the maps

$$(\pi^A_n)_* : K_0(A_n) \to K_0(F_n) = H_n \quad \text{and} \quad (\pi^A_{n+1})_* : K_0(A_{n+1}) \to K_0(F_{n+1}) = H_{n+1}$$

are inclusions. Also, recall that $(\psi_{n,n+1})_* = \gamma_{n,n+1} : H_n \to H_{n+1}$. Consequently, $(\varphi^o_{n,n+1})_* = \gamma_{n,n+1}|_{G_n} : K_0(A_n) \to K_0(A_{n+1})$. The commutative diagram

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\varphi^o_{12}} & A_2 & \xrightarrow{\varphi^o_{23}} & \cdots & \longrightarrow & A^o \\ \downarrow & & \downarrow & & & & \downarrow \\ F_1 & \xrightarrow{\psi_{12}} & F_2 & \xrightarrow{\psi_{23}} & \cdots & \longrightarrow & F \end{array}$$

induces the commutative diagram

$$\begin{array}{ccccccc}
K_0(A_1)=G_1 & \xrightarrow{\gamma_{12}|_{G_1}} & K_0(A_2)=G_2 & \xrightarrow{\gamma_{23}|_{G_2}} & \cdots & \longrightarrow & K_0(A^o)=G \\
\downarrow & & \downarrow & & & & \downarrow \\
K_0(F_1)=H_1 & \xrightarrow{\gamma_{12}} & K_0(F_2)=H_2 & \xrightarrow{\gamma_{23}} & \cdots & \longrightarrow & K_0(F)=H
\end{array}$$

Hence $K_0(A^o)=G$ as a subgroup of $K_0(F)=H$.

On the other hand, $\pi^A_{n+1}\circ\varphi^o_{n,n+1}=\psi^o$ is homotopy equivalent to $\pi^A_{n+1}\circ\varphi_{n,n+1}=\psi$ (see (4.44)). Thus we know that

$$(\pi^A_{n+1}\circ\varphi_{n,n+1})_*=(\pi^A_{n+1}\circ\varphi^o_{n,n+1})_*:K_0(A_n)=G_n\to K_0(F_{n+1})=H_{n+1}.$$

Consequently, $(\pi^A_{n+1})_*\circ(\varphi_{n,n+1})_*=(\pi^A_{n+1})_*\circ\gamma_{n,n+1}|_{G_n}$. Since $(\pi^A_{n+1})_*$ is an inclusion, we have $(\varphi_{n,n+1})_*=\gamma_{n,n+1}|_{G_n}$. Therefore, the inductive limit

$$A_1\xrightarrow{\varphi_{12}}A_2\xrightarrow{\varphi_{23}}A_3\to\cdots\to A$$

also induces the K-theory maps

$$K_0(A_1)=G_1\xrightarrow{\gamma_{12}|_{G_1}}K_0(A_2)\xrightarrow{\gamma_{23}|_{G_2}}K_0(A_3)\to\cdots\to K_0(A)\quad(\text{or } K_0(A^o)).$$

Hence $K_0(A)=G=K_0(A^o)$. Since $A_n\in\mathcal{C}_0$, $K_1(A)=\{0\}$.

Lemma 4.80. *A is a simple C^*-algebra.*

Proof. Note that, by 4.58 and 4.68, $\lambda^j(i)>0$. From (4.78) and (4.70), it follows that

$$\{\theta_{n,i}:1\le i\le p_n\}\subset \mathrm{Sp}(\varphi_{n,n+1}|_{\eta_{n+1,j}(t)})\quad\text{for any } 1\le j\le l_{n+1},\ 0<t<1. \tag{4.81}$$

Note that from (4.78), for any $\theta\in\mathrm{Sp}(F_{n+1})$, $\mathrm{SP}(\varphi_{n,n+1}|_\theta)=\mathrm{SP}(\psi|_\theta)$ (see 4.2). It follows from the definition of ψ in 4.37 (see (4.41) and (4.42)) that we have

$$\{\theta_{n,i}:1\le i\le p_n\}\subset\mathrm{Sp}(\varphi_{n,n+1}|_{\theta_{n+1,j}})\quad\text{for all } 1\le j\le p_{n+1}, \tag{4.82}$$

$$\{\eta_{n,j}(k/n),\theta_{n,i}:1\le i\le p_n,1\le j\le l_n,1\le k\le n-1\}\subset\mathrm{Sp}(\varphi_{n,n+1}|_{\theta_{n+1,1}}) \tag{4.83}$$

(see also (4.41)).

From equations (4.70) (which tells us χ is a part of φ), (4.78) (which tells us, for any $\eta_{n+1,j}(t)\in(0,1)_j\subset\mathrm{Sp}(C([0,1],M_{\{n+1,j\}}(\mathbb{C})))$, that $\mathrm{SP}((\varphi_{n,n+1})_{\eta_{n+1,j}(t)})=\mathrm{SP}(\varphi|_{\eta_{n+1,j}(t)})$), and (4.48), we know that for all $1\le j\le l_{n+1}$,

$$\{\eta_{n,i}(t),\eta_{n,i}(1-t):1\le i\le l_n\}\subset\mathrm{Sp}(\varphi_{n,n+1}|_{\eta_{n+1,j}(t)})\cup\mathrm{Sp}(\varphi_{n,n+1}|_{\eta_{n+1,j}(1-t)}). \tag{4.84}$$

For the composition of two homomorphisms $\varphi:A\to B$ and $\psi:B\to C$ among three subhomogeneous algebras, it is well known and easy to see that, for any $x\in\mathrm{Sp}(C)$, one has $\mathrm{Sp}(\psi\circ\varphi)|_x=\bigcup_{y\in\mathrm{Sp}(\psi|_x)}\mathrm{Sp}(\varphi|_y)$. We will repeatedly use this fact.

From (4.82), we have

$$\{\theta_{n,i} \mid 1 \le i \le p_n\} \subset \mathrm{Sp}(\varphi_{n,m}|_{\theta_{m,j}}) \quad \text{for all } 1 \le j \le p_m. \tag{4.85}$$

From (4.84), we know that, for any $m \ge n+1$ and $1 \le j \le l_m$,

$$\{\eta_{n,i}(t), \eta_{n,i}(1-t) : 1 \le i \le l_n\} \subset \mathrm{Sp}(\varphi_{n,m}|_{\eta_{m,j}(t)}) \cup \mathrm{Sp}(\varphi_{n,m}|_{\eta_{m,j}(1-t)}). \tag{4.86}$$

Fix $m+2 > m \ge n$. Let $Z := \{\eta_{m,j}(k/m), \theta_{m,i} : 1 \le i \le p_m, 1 \le j \le l_m, 1 \le k \le m-1\}$. It follows from (4.86) and (4.85) that

$$\{\eta_{n,j}(k/m), \theta_{n,i} : 1 \le i \le p_n, 1 \le j \le l_n, 1 \le k \le m-1\} \subset \bigcup_{z \in Z} \mathrm{Sp}(\varphi_{n,m}|_z).$$

(When $n = m$, we use the convention $\varphi_{n,n} = \mathrm{id}$.) Applying (4.83) with m in place of n, we know that $Z \subset \mathrm{Sp}(\varphi_{m,m+1}|_{\theta_{m+1,1}})$. For any $x \in \mathrm{Sp}(A_{m+2})$, from (4.83) with $m+1$ in place of n, we have $\theta_{m+1,1} \subset \mathrm{Sp}(\varphi_{m+1,m+2}|_x)$. Hence

$$\mathrm{Sp}(\varphi_{n,m+2}|_x) \supset \left\{\eta_{n,i}\left(\frac{k}{m}\right) : 1 \le k \le m-1,\ 1 \le i \le l_n\right\} \cup \{\theta_{n,i} : 1 \le i \le p_n\}. \tag{4.87}$$

The latter is $1/m$-dense in $\mathrm{Sp}(A_n)$ (see (4.4) and lines below that).

It is standard to show that A is simple (see [Dădărlat et al. 1992]). To see this, let $a, b \in A_+^{\mathbf{1}} \setminus \{0\}$. It suffices to show that b is in the (closed) ideal generated by a. Let $\frac{1}{2} > \varepsilon > 0$. There are $n \ge 1$ and $a_0 \in (A_n)_+^{\mathbf{1}}$ such that $\|\varphi_{n,\infty}(a_0) - a\| < \varepsilon/4$. It follows from Lemma 3.1 of [Elliott et al. 2020b] that there is $r_0 \in A$ such that

$$0 \ne (\varphi_{n,\infty}(a_0) - \varepsilon/4)_+ = r_0^* a r_0. \tag{4.88}$$

Put $a_1 := (a_0 - \varepsilon/4)_+ \in A_n$. Choose an integer $m' > n$ and $b_1 \in A_{m'+}$ such that $\|\varphi_{m',\infty}(b_1) - b\| < \varepsilon/4$. By (4.4) and lines below that, we may assume that, for some $\delta > 0$ and for any δ-dense subset S of $\mathrm{Sp}(A_n)$, there is $s \in S$ such that $a_1(s) > 0$. Choose $m > n$ such that $1/m < \delta$. Then by what has been proved above (see (4.87)), we have

$$\varphi_{n,m+2}(a_1)(x) > 0 \quad \text{for all } x \in \mathrm{Sp}(A_{m+2}). \tag{4.89}$$

By choosing a large m, we may assume that $m > m'$. It follows from Proposition 6.3 of [Elliott et al. 2020b] that $\varphi_{n,m+2}(a_1)$ is full in A_{m+2}. Therefore, there are $x_1, x_2, \dots, x_K \in A_{m+2}$ such that

$$\left\| \sum_{i=1}^{K} x_i^* \varphi_{n,m+2}(a_1) x_i - \varphi_{m',m+2}(b_1) \right\| < \frac{\varepsilon}{4}. \tag{4.90}$$

This implies (see (4.88)) that

$$\left\| \sum_{i=1}^{K} \varphi_{m+2,\infty}(x_i)^* r_0^* a r_0 \varphi_{m+2,\infty}(x_i) - b \right\| < \varepsilon. \tag{4.91}$$

This shows that b is in the closed ideal generated by a, whence A is simple. $\square$

4.92. Let $\partial_e(T(A))$ denote the set of extremal tracial states of A. It is well known (see Lemma 2.2 of [Thomsen 1998]) that there is a one-to-one correspondence between $\mathrm{Sp}(A_n)$ and $\partial_e(T(A_n))$ given by sending the irreducible representation $\theta : A_n \to M_l(\mathbb{C})$ to the extremal trace τ_θ defined by $\tau_\theta(a) = \mathrm{tr}(\theta(a))$, where tr is the normalized trace on $M_l(\mathbb{C})$. Using the calculation in [Gong et al. 2020a, 3.8] (see [Thomsen 1998] also), we know that $\mathrm{Aff}(T_0(A_n))$ (for the definition of $T_0(A_n)$, see Definition 2.2) consists of elements

$$(g_1, g_2, \ldots, g_{l_n}, x_1, x_2, \ldots, x_n) \in C([0,1], \mathbb{R}^{l_n}) \oplus \mathbb{R}^{p_n}$$

with the conditions

$$g_j(0) = \frac{1}{\{n,i\}} \sum_{i=1}^{p_n} b^n_{0,ji} x_i[n,i] \quad \text{and} \quad g_j(1) = \frac{1}{\{n,i\}} \sum_{i=1}^{p_n} b^n_{1,ji} x_i[n,i]. \tag{4.93}$$

Note that the norm on $\mathrm{Aff}(T_0(A_n))$ is given by

$$\|(g_1, g_2, \ldots, g_{l_n}, x_1, x_2, \ldots, x_n)\| = \max\{\sup_{1\le t\le 1} |g_j(t)|, |x_i| : 1 \le j \le l_n, 1 \le i \le p_n\}.$$

Let $\psi^T_{n,n+1} : T_0(F_{n+1}) \to T_0(F_n)$ and $\psi^\sharp_{n,n+1} : \mathrm{Aff}(T_0(F_n)) \to \mathrm{Aff}(T_0(F_{n+1}))$ be the affine maps induced by $\psi_{n,n+1} : F_n \to F_{n+1}$, and $\varphi^T_{n,n+1} : T_0(A_{n+1}) \to T_0(A_n)$ and $\varphi^\sharp_{n,n+1} : \mathrm{Aff}(T_0(A_n)) \to \mathrm{Aff}(T_0(A_{n+1}))$ be induced by $\varphi_{n,n+1} : A_n \to A_{n+1}$. Note that F_n is unital. There is a unique element in $\mathrm{Aff}(T_0(F_n))$, denoted by $1_{T(F_n)}$, such that $1_{T(F_n)}(\tau) = 1$ for all $\tau \in T(F_n)$. Even though $\mathrm{Aff}(T_0(F_n))$ and $\mathrm{Aff}(T_0(F_{n+1}))$ have units $1_{T(F_n)}$ and $1_{T(F_{n+1})}$, respectively, $\psi^\sharp_{n,n+1}$ does not preserve the units, since $\psi_{n,n+1}$ is not unital.

Lemma 4.94. (a) *We have* $\psi^\sharp_{n,n+1}(1_{T(F_n)}) \ge (1 - 1/8^n) \cdot 1_{T(F_{n+1})}$. (*Equivalently, for any* $\tau \in T(F_{n+1})$, $\|\psi^T_{n,n+1}(\tau)\| \ge (1 - 1/8^n)$.) *Consequently,*

$$\psi^\sharp_{n,m}(1_{T(F_n)}) \ge \left(\prod_{i=n}^{m-1} \left(1 - \frac{1}{8^i}\right)\right) \cdot 1_{T(F_m)}$$

for any $m > n$.

(b) *Suppose that* $f \in \mathrm{Aff}(T_0(A_n))$ *with* $\|f\| \le 1$ *satisfying*

$$\pi^{A\,\sharp}_n(f) \ge \alpha \cdot 1_{T(F_n)} \in \mathrm{Aff}(T_0(F_n))$$

for some $\alpha \in (0,1]$, *where* $\pi^{A\,\sharp}_n : \mathrm{Aff}(T_0(A_n)) \to \mathrm{Aff}(T_0(F_n))$ *is induced by* $\pi^A_n : A_n \to F_n$. *Then, for any* $\tau \in T(A_{n+1})$, $\varphi^\sharp_{n,n+1}(f)(\tau) \ge (1 - 2/4^{n+1})\alpha$.

Consequently, for $a_n^A, e_n^A \in (A_n)_+$ (see 4.6), *we have*

$$\begin{aligned}\varphi^\sharp_{n,n+1}(\widehat{e_n^A})(\tau) &\geq \varphi^\sharp_{n,n+1}(\widehat{a_n^A})(\tau) \geq 1-\frac{2}{4^{n+1}} \quad \textit{for any } \tau\in T(A_{n+1}),\\ \varphi^\sharp_{n,m}(\widehat{e_n^A})(\tau) &\geq \varphi^\sharp_{n,m}(\widehat{a_n^A})(\tau)\\ &\geq \left(\prod_{i=n}^{m-2}\Big(1-\frac{1}{8^i}\Big)\right)\Big(1-\frac{2}{4^m}\Big) \quad \textit{for any } m>n+1,\end{aligned} \tag{4.95}$$

where $\widehat{e_n^A}$ *and* $\widehat{a_n^A}$ *are the elements in* $\mathrm{Aff}(T_0(A_n))$ *corresponding to* e_n^A *and* a_n^A, *respectively.* (*Also, for any* $\tau\in\widetilde{T}(A_{n+1})$, $\|\varphi^T_{n,n+1}(\tau)\|\geq(1-2/4^{n+1})$.) *Furthermore, for any* $\tau\in T(A)$,

$$\varphi^\sharp_{n,\infty}(\widehat{e_n^A})(\tau)\geq\varphi^\sharp_{n,\infty}(\widehat{a_n^A})(\tau)\geq\prod_{i=n}^{\infty}\Big(1-\frac{1}{8^i}\Big)\geq\Big(1-\frac{1}{4^n}\Big). \tag{4.96}$$

Proof. (a) We only need to show that

$$\psi^\sharp_{n,n+1}(1_{T(F_n)})(\tau)=\psi^T_{n,n+1}(\tau)(1_{F_n})\geq 1-\frac{1}{8^n}$$

for $\tau\in\partial_e(T(F_{n+1}))$. Let $\tau=\tau_{\theta_{n+1,i}}$ be defined by

$$\tau(a_1,a_2,\dots,a_{p_{n+1}})=\mathrm{tr}(a_i),$$

where tr is the normalized trace of $F^i_{n+1}=M_{[n+1,i]}(\mathbb{C})$. Let tr_j denote the normalized trace on $F_n^j=M_{[n,j]}(\mathbb{C})$. Then, for $b=(b_1,b_2,\dots,b_{p_n})$, we have

$$\psi^T_{n,n+1}(\tau)(b)=\frac{1}{[n+1,i]}\sum_{j=1}^{p_n}\mathrm{tr}_j(b_j)c_{ij}[n,j]$$

(recalling $c_{ij}=c_{ij}^{n,n+1}$). In particular, if $b=1_{F_n}$, then by (4.30) (noting $\hat{1}_{F_n}=1_{T(F_n)}$), we have

$$\psi^T_{n,n+1}(\tau)(1_{F_n})=\frac{1}{[n+1,i]}\sum_{j=1}^{p_n}\mathrm{tr}_j(b_j)c_{ij}[n,j]\geq\Big(1-\frac{1}{8^n}\Big).$$

(b) Keep the notation from the proof of part (a). Again we only need to calculate $\varphi^\sharp_{n,n+1}(f)(\tau)$ for $\tau\in\partial_e(T(F_{n+1}))$. First suppose that $\tau=\tau_{\theta_{n+1,i}}$ defined by $\tau(f_1,f_2,\dots,f_{l_n},a_1,a_2,\dots,a_{p_{n+1}})=\mathrm{tr}(a_i)$. Writing

$$f=(g_1,g_2,\dots,g_{l_n},b_1,b_2,\dots,b_{p_n})\in\mathrm{Aff}(T_0(A_n)),$$

we have $b_i \geq \alpha \in \mathbb{R}$ for all i. Then, by (4.78), (4.41), (4.42) and (4.40), we have

$$\begin{aligned}(\varphi^{\sharp}_{n,n+1}(f))(\tau_{\theta_{n+1,i}}) &= (\psi^{\sharp}(f))(\tau_{\theta_{n+1,i}})\\ &\geq \frac{1}{[n+1,i]}\Big(\mathrm{tr}_1(b_1)c'_{i1}[n,1]+\sum_{j=2}^{p_n}\mathrm{tr}_j(b_j)c_{ij}[n,j]\Big)\\ &\geq \frac{1}{[n+1,i]}\Big(1-\frac{1}{2^{4(n+1)}}\Big)\sum_{j=1}^{p_n}\mathrm{tr}_j(b_j)c_{ij}[n,j]\\ &= \Big(1-\frac{1}{2^{4(n+1)}}\Big)\frac{1}{[n+1,i]}\sum_{j=1}^{p_n}\alpha c_{ij}[n,j]\\ &\geq \Big(1-\frac{2}{8^n}\Big)\alpha.\end{aligned}$$

Now suppose that $\tau = \tau_{\eta_{n+1,j}(t)}$ (for $0 < t < 1$) is defined by

$$\tau(f_1, f_2, \dots, f_{l_n}, a_1, a_2, \dots, a_{p_{n+1}}) = \mathrm{tr}(f_j(t)),$$

where tr is the normalized trace on $E^j_{n+1} = M_{\{n+1,j\}}(\mathbb{C})$. By (4.78), (4.70) and (4.72), we have

$$\begin{aligned}(\varphi^{\sharp}_{n,n+1}(f))(\tau_{\eta_{n+1,j}(t)}) = (\varphi^{\sharp}(f))(\tau_{\eta_{n+1,j}(t)}) &\geq \frac{1}{\{n+1,j\}}\Big(\sum_{i=1}^{p_n}\mathrm{tr}_i(b_i)\lambda^j(i)[n,i]\Big)\\ &= \frac{1}{\{n+1,j\}}\Big(\sum_{i=1}^{p_n}\alpha\lambda^j(i)[n,i]\Big) \geq \Big(1-\frac{2}{4^{n+1}}\Big)\alpha.\end{aligned}$$

Other parts of (b) follow from $\pi^A_n(e^A) = \pi^A_n(a^A) = 1_{F_n}$ and $e^A \geq a^A$. ☐

4.97. Note that the map $\pi^{A\,\sharp}_n : \mathrm{Aff}(T_0(A_n)) \to \mathrm{Aff}(T_0(F_n))$ induced by $\pi^A_n : A_n \to F_n$ is given by

$$\pi^{A\,\sharp}_n(g_1, g_2, \dots, g_{l_n}, x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_{p_n}).$$

Define $\xi_n : \mathrm{Aff}(T_0(F_n)) \to \mathrm{Aff}(T_0(A_n))$ by

$$\xi_n(x_1, x_2, \dots, x_{p_n}) = (g_1, g_2, \dots, g_{l_n}, x_1, x_2, \dots, x_{p_n})$$

$$\text{where}\quad \begin{pmatrix}\{n,1\}g_1(t)\\ \{n,2\}g_2(t)\\ \vdots\\ \{n,l_n\}g_{l_n}(t)\end{pmatrix} = (\mathfrak{b}^n_0 + t(\mathfrak{b}^n_1 - \mathfrak{b}^n_0))\begin{pmatrix}[n,1]x_1\\ [n,2]x_2\\ \vdots\\ [n,p_n]x_{p_n}\end{pmatrix}. \tag{4.98}$$

Then $\pi^{A\,\sharp}_n \circ \xi_n = \mathrm{id}|_{\mathrm{Aff}(T(F_n))}$. Define $\xi_{n,n+1} : \mathrm{Aff}(T(A_n)) \to \mathrm{Aff}(T(A_{n+1}))$ by $\xi_{n,n+1} = \xi_{n+1} \circ \psi^{\sharp}_{n,n+1} \circ \pi^{A\,\sharp}_n$. Note that if $\psi^{\sharp}_{n,n+1}(x_1, x_2, \dots, x_{p_n}) = (y_1, y_2, \dots, y_{l_n})$, then

$$\begin{pmatrix} [n+1,1]y_1 \\ [n+1,2]y_2 \\ \vdots \\ [n+1,p_{n+1}]y_{p_{n+1}} \end{pmatrix} = \mathfrak{c}^{n,n+1} \begin{pmatrix} [n,1]x_1 \\ [n,2]x_2 \\ \vdots \\ [n,p_n]x_{p_n} \end{pmatrix}. \tag{4.99}$$

If $\xi_{n,n+1}(g_1, g_2, \ldots, g_{l_n}, x_1, x_2, \ldots, x_{p_n}) = (h_1, h_2, \ldots, h_{l_{n+1}}, y_1, y_2, \ldots, y_{p_{n+1}})$, then (4.99) holds and

$$\begin{pmatrix} \{n+1,1\}h_1(t) \\ \{n+1,2\}h_2(t) \\ \vdots \\ \{n+1,l_{n+1}\}h_{l_{n+1}}(t) \end{pmatrix} = (\mathfrak{b}_0^{n+1} + t(\mathfrak{b}_1^{n+1} - \mathfrak{b}_0^{n+1})) \begin{pmatrix} [n+1,1]y_1 \\ [n+1,2]y_2 \\ \vdots \\ [n+1,p_{n+1}]y_{p_{n+1}} \end{pmatrix}. \tag{4.100}$$

Lemma 4.101. *The following estimate holds*:

$$\|\varphi_{n,n+1}^{\sharp} - \xi_{n,n+1}\| < \frac{1}{2^{n+1}}. \tag{4.102}$$

Proof. Let $g = (g_1, g_2, \ldots, g_{l_n}, x_1, x_2, \ldots, x_{p_n}) \in \mathrm{Aff}(T(A_n))$ with $\|g\| \le 1$. Without lose of generality, we assume that $0 \le g_j(t) \le 1$ and $0 \le x_i \le 1$ for all i, j, t. Write

$$\begin{aligned} {\varphi_{n,n+1}^{o}}^{\sharp}(g) &= (h_1', h_2', \ldots, h_{l_{n+1}}', y_1', y_2', \ldots, y_{p_{n+1}}') \in \mathrm{Aff}(T(A_{n+1})), \\ \varphi_{n,n+1}^{\sharp}(g) &= (h_1, h_2, \ldots, h_{l_{n+1}}, y_1, y_2, \ldots, y_{p_{n+1}}) \in \mathrm{Aff}(T(A_{n+1})), \\ \xi_{n,n+1}(g) &= (f_1, f_2, \ldots, f_{l_{n+1}}, z_1, z_2, \ldots, z_{p_{n+1}}) \in \mathrm{Aff}(T(A_{n+1})). \end{aligned}$$

Recall that $\pi_{n+1}^A \circ \varphi_{n,n+1}^o = \psi^o = \psi_{n,n+1} \circ \pi_n^A : A_n \to F_{n+1}$ and $\pi_{n+1}^A \circ \varphi_{n,n+1} = \psi : A_n \to F_{n+1}$.

Note that

$$\pi_{n+1}^{A\sharp} \circ \xi_{n,n+1} = \pi_{n+1}^{\sharp} \circ \xi_{n+1} \circ (\psi_{n,n+1} \circ \pi_n^A)^{\sharp} = (\psi_{n,n+1} \circ \pi_n^A)^{\sharp} = \pi_{n+1}^{\sharp} \circ {\varphi_{n,n+1}^{o}}^{\sharp}.$$

Hence we have $z_i = y_i'$ for all $1 \le i \le p_{n+1}$.

Using (4.38), (4.42) and (4.41), we calculate that

$$\begin{aligned} y_i &= y_i' = z_i \quad \text{for } i \ge 2, \\ y_1' &= \frac{1}{[n+1,1]} \sum_{i=1}^{p_n} c_{1,i} x_i [n,i], \\ y_1 &= \frac{1}{[n+1,1]} \Big(\sum_{i=1}^{l_n} \sum_{k=1}^{n-1} g_i\Big(\frac{k}{n}\Big)\{n,i\} + \sum_{i=1}^{p_n} c_{1,i}' x_i [n,i] \Big). \end{aligned}$$

Since $\|{\varphi^o_{n,n+1}}^\sharp\| \le 1$, $|y_1'| \le 1$. By (4.28) and (4.29), we have

$$\begin{aligned}\left|\sum_{i=1}^{l_n}\sum_{k=1}^{n-1} g_i\Big(\frac{k}{n}\Big)\{n,i\}\right| &\le (n-1)l_n \max\{\{n,i\} : 1 \le i \le p_n\}\\ &\le \frac{1}{2^{4(n+1)}}\mathcal{L}_{n+1} \le \frac{1}{2^{4(n+1)}}c_{11} < \frac{1}{2^{4(n+1)}}[n+1,1]. \qquad (4.103)\end{aligned}$$

Hence $(1/[n+1,1])\big|\sum_{i=1}^{l_n}\sum_{k=1}^{n-1} g_i(k/n)\{n,i\}\big| \le 1/2^{4(n+1)}$. Combining with (4.40) (recall $c_{ij} = c_{ij}^{n,n+1}$), we obtain

$$\begin{aligned}|y_1 - y_1'| &\le \frac{1}{[n+1,1]}\sum_{i=1}^{p_n}(c_{1i} - c'_{1,i})x_i[n,i] + \frac{1}{2^{4(n+1)}}\\ &\le \frac{1}{2^{4(n+1)}}\left(\frac{1}{[n+1,1]}\sum_{i=1}^{p_n} c_{1i}x_i[n,i]\right) + \frac{1}{2^{4(n+1)}}\\ &= \frac{1}{2^{4(n+1)}}y_1' + \frac{1}{2^{4(n+1)}} \le \frac{2}{2^{4(n+1)}}.\end{aligned}$$

By (4.33), (4.35) and (4.100) (note that $z_k \in [0,1]$), we know that

$$\begin{aligned}|f_i(t) - f_i(0)| &\le \frac{1}{\{n+1,i\}}\sum_{k=1}^{p_{n+1}}\big|b_{1,ik}^{n+1} - b_{0,ik}^{n+1}\big|[n+1,k]z_k\\ &\le \frac{1}{\{n+1,i\}}\frac{1}{2^{4(n+1)}}\sum_{k=1}^{p_{n+1}}\max\big\{b_{1,ik}^{n+1}, b_{0,ik}^{n+1}\big\}[n+1,k] \le \frac{1}{2^{4(n+1)}}\end{aligned}$$

for any $1 \le i \le l_{n+1}$ and $0 \le t \le 1$.

Note that, by (4.74) and (4.32) as well as (4.34),

$$L_j + K_j \le \frac{1}{2^{4(n+1)}}\mathcal{M}_{n+1} \le \frac{1}{2^{4(n+1)}}\{n+1,j\}. \qquad (4.104)$$

It is easy to see from the definition of $\varphi_{n,n+1}$ (see (4.78) and (4.70)) that all the functions $h_j'(t)$ and $h_j(t)$ are approximately constant within $1/2^{4(n+1)}$. To be more precise, we may regard $\varphi_{n,n+1}$ as a homomorphism from A_n to $C([0,1], E_{n+1})$ which is unitarily equivalent to φ (see (4.78)). Hence

$$(h_1, h_2, \ldots, h_{l_{n+1}}) = \varphi^\sharp(g) \in \mathrm{Aff}(T(C([0,1], E_{n+1}))).$$

On the other hand, by (4.70), $\varphi = \bigoplus_{j=1}^{l_{n+1}}\varphi_j$ can be written as $(\chi\oplus\mu)\oplus\varphi' = \bigoplus_{j=1}^{l_{n+1}}(\chi\oplus\mu)_j \oplus \varphi_j'$, where

$$(\chi\oplus\mu) = \bigoplus_{j=1}^{l_{n+1}}(\chi\oplus\mu)_j : A_n \to \bigoplus_{j=1}^{l_{n+1}} M_{L_j+K_j}(C[0,1])$$

is defined in 4.45 and

$$\varphi' = \bigoplus_{j=1}^{l_{n+1}} \varphi'_j : A_n \to \bigoplus_{j=1}^{l_{n+1}} M_{\{n+1,j\}-(L_j+K_j)}(C[0,1])$$

sends $f = (f_1, f_2, \dots, f_{l_n}, a_1, a_2, \dots, a_{p_n})$ to

$$\begin{aligned}\varphi'(f) = \bigoplus_{j=1}^{l_{n+1}} \operatorname{diag}\Big(& f_1\Big(\frac{1}{n}\Big)^{\sim\sigma^j(1)}, f_1\Big(\frac{2}{n}\Big)^{\sim\sigma^j(1)}, \dots, f_1\Big(\frac{n-1}{n}\Big)^{\sim\sigma^j(1)}, f_2\Big(\frac{1}{n}\Big)^{\sim\sigma^j(2)}, \\ & \dots, f_2\Big(\frac{n-1}{n}\Big)^{\sim\sigma^j(2)}, \dots, f_{l_n}\Big(\frac{1}{n}\Big)^{\sim\sigma^j(l_n)}, \dots, f_{l_n}\Big(\frac{n-1}{n}\Big)^{\sim\sigma^j(l_n)}, \\ & a_1^{\sim\lambda^j(1)}, a_2^{\sim\lambda^j(2)}, \dots, a_{p_n}^{\sim\lambda^j(p_n)}, 0^{\sim\sim}\Big). \end{aligned} \tag{4.105}$$

In particular, $(\varphi')^\sharp(g)$ is constant (that is, $(\varphi'_i)^\sharp(g)(t) = (\varphi'_i)^\sharp(g)(0)$ for $t \in [0,1]$ and $i \in \{1, 2, \dots, l_{n+1}\}$). Consequently, for any $1 \le i \le l_{n+1}$ and $0 \le t \le 1$, we obtain

$$\begin{aligned}|h_i(t) - h_i(0)| &= |\varphi_i^\sharp(g)(t) - \varphi_i^\sharp(g)(0)| \\ &= \frac{1}{\{n+1,j\}}\Big|(K_j + L_j)\big((\chi\oplus\mu)_j^\sharp(g)(t) - (\chi\oplus\mu)_j^\sharp(g)(0)\big) \\ &\qquad + (\{n+1,j\} - K_j - L_j)\big((\varphi'_i)^\sharp(g)(t) - (\varphi'_i)^\sharp(g)(0)\big)\Big| \\ &\le \frac{K_j + L_j}{\{n+1,j\}} \le \frac{1}{2^{4(n+1)}}.\end{aligned}$$

Note that $y_i = y'_i = z_i$ for all $i \ge 2$, $y'_1 = z_1$ and $|y'_1 - y_1| \le 1/2^{4(n+1)}$. By the formulae (4.93), we have $h'_i(0) = f_i(0)$ and $|h_i(0) - f_i(0)| < 1/2^{4(n+1)}$ (as $\beta_{n+1,0}$ is a homomorphism). Consequently,

$$|h_i(t) - f_i(t)| \le \frac{2}{2^{4(n+1)}} < \frac{1}{2^{n+1}}. \qquad \square$$

4.106. For a separable C^*-algebra A, one has a standard metric on $T_0(A)$ (see Definition 2.2), i.e., $d(t_1, t_2) := \sum_{n=1}^\infty (1/2^{n+1})|t_1(a_n) - t_2(a_n)|$ for all $t_1, t_2 \in T_0(A)$, where $\{a_n\}$ is a fixed dense sequence of $A^{\mathbf{1}}_{\text{s.a.}}$. In the following proof, we will use this metric.

Theorem 4.107. *The C^*-algebra $A = \lim(A_n, \varphi_{n,m})$ satisfies the following conditions*:

(a) *A is simple*,

(b) *$K_0(A) = G$, $K_1(A) = 0$, $T(A) = \Delta$ and $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$ is the map ρ from G to* Aff Δ *identifying $K_0(A)$ with G and $T(A)$ with Δ.*

Moreover, $A \in \mathcal{D}$ (see Definition 2.28) *and has continuous scale.*

Proof. From 4.79 and Lemma 4.80, A is simple, and $K_0(A) = G$. Consider the projective limit

$$T_0(A_1) \xleftarrow{\varphi^T_{1,2}} T_0(A_2) \xleftarrow{\varphi^T_{1,2}} T_0(A_3) \leftarrow \cdots \leftarrow T_0(A), \tag{4.108}$$

where $\varphi^T_{i,i+1} : T_0(A_{i+1}) \to T_0(A_i)$ is the affine continuous map induced by $\varphi_{i,i+1}$. Suppose that $\tau \in \overline{T(A)}^w$ is written as $\tau = \lim_{k\to\infty} \tau_k$, where $\tau_k \in T(A)$. Then, for any fixed n, by (4.96),

$$\tau(\varphi_{n,\infty}(e^A_n)) = \lim_{k\to\infty} \tau_k(\varphi_{n,\infty}(e^A_n)) = \lim_{k\to\infty} \big(\varphi^\sharp_{n,\infty}(\widehat{e^A_n})\big)(\tau_k) \geq \Big(1 - \frac{1}{4^n}\Big).$$

Hence $\|\tau\| \geq (1-1/4^n)$. Since n is arbitrary, $\|\tau\| = 1$ and $\tau \in T(A)$. That is, $T(A)$ is compact. Note that F is a nonunital simple AF algebra with

$$(K_0(F), K_0(F)_+, \Sigma(F)) = (H, H_+, \{x \in H_+ : \tilde\rho(x)(\tau) < 1 \text{ for all } \tau \in \Delta\}).$$

Therefore $T(F) = \Delta$, identifying $\rho_F : K_0(F) \to \mathrm{Aff}(T(F))$ and $\tilde\rho : H \to \mathrm{Aff}(\Delta)$.

On the other hand, by Lemma 4.101, we have the approximately commuting diagram

$$\begin{tikzcd}
\mathrm{Aff}(T_0(A_1)) \arrow[r, "\varphi^\sharp_{1,2}"] \arrow[d, "\pi_1^{A\sharp}"'] & \mathrm{Aff}(T_0(A_2)) \arrow[r, "\varphi^\sharp_{2,3}"] \arrow[d, "\pi_2^{A\sharp}"'] & \mathrm{Aff}(T_0(A_3)) \arrow[r] \arrow[d, "\pi_3^{A\sharp}"'] & \cdots \arrow[r] & \mathrm{Aff}(T_0(A)) \\
\mathrm{Aff}(T_0(F_1)) \arrow[r, "\psi^\sharp_{1,2}"'] \arrow[ur, "\xi_2\circ\psi^\sharp_{1,2}"] & \mathrm{Aff}(T_0(F_2)) \arrow[r, "\psi^\sharp_{2,3}"'] \arrow[ur, "\xi_2\circ\psi^\sharp_{2,3}"] & \mathrm{Aff}(T_0(F_3)) \arrow[r, "\psi^\sharp_{3,4}"'] \arrow[ur, "\xi_3\circ\psi^\sharp_{3,4}"] & \cdots \arrow[r] & \mathrm{Aff}(T_0(F))
\end{tikzcd}$$

of real Banach spaces (recall $\xi_{n,n+1} = \xi_{n+1}\circ\psi^\sharp_{n,n+1}\circ\pi_n^{A\sharp}$ and $\pi_n^{A\sharp}\circ\xi_n = \mathrm{id}|_{\mathrm{Aff}(T(F_n))}$). Let $\Pi^\sharp : \mathrm{Aff}(T_0(A)) \to \mathrm{Aff}(T_0(F))$ be the continuous linear isomorphism induced by the above approximately commutative diagram. Note that we also have the projective limit

$$T_0(F_1) \xleftarrow{\psi^T_{1,2}} T_0(F_2) \xleftarrow{\psi^T_{1,2}} T_0(F_3) \leftarrow \cdots \leftarrow T_0(F). \tag{4.109}$$

Together with (4.108), by applying the above approximately commutative diagram, we obtain the approximately commutative diagram

$$\begin{tikzcd}
T_0(A_1) & T(A_2) \arrow[l, "\varphi^T_{1,2}"'] & T_0(A_3) \arrow[l, "\varphi^T_{2,3}"'] & \cdots T_0(A) \arrow[l] \\
T_0(F_1) \arrow[u, "\pi_1^{AT}"'] & T_0(F_2) \arrow[l, "\psi^T_{1,2}"'] \arrow[u, "\pi_2^{AT}"'] & T_0(F_3) \arrow[l, "\psi^T_{2,3}"'] \arrow[u, "\pi_3^{AT}"'] & \cdots T_0(F) \arrow[l]
\end{tikzcd}$$

as compact convex sets with the metric mentioned in 4.106, which gives an affine continuous map $\Pi^T : T_0(F) \to T_0(A)$. Combining the two approximately commutative diagrams above, we have $\Pi^\sharp(f)(t) = f(\Pi^T(t))$ for all $t \in T(F)$. Consider the functions $g_n := \varphi^\sharp_{n,\infty}(e^A_n) \in \mathrm{Aff}(T_0(A))$. By (4.96), on $T(A) \subset T_0(A)$, g_n

converges uniformly to the affine function g_A with $g_A(\tau) = 1$ for all $\tau \in T(A)$ ($g_A(0) = 0$). Since $g_n(r\tau) = r\tau$ for all $\tau \in T(A)$ and $0 \le r \le 1$, and $T(A)$ is compact, g_n converges to g_A uniformly on $T_0(A)$. Note that $\Pi^\sharp(g_n) = \psi^\sharp_{n,\infty}(\pi_n^A(e_n^A))$. It follows from (4.95) that $\Pi^\sharp(g_n)$ converges to 1 uniformly on $T(F)$. Then, by the first approximately commutative diagram above, $\Pi^\sharp(g_A) = 1$ on $T(F)$. Since $\Pi^\sharp(f)(t) = f(\Pi^T(t))$ for all $t \in T(F)$, Π^T maps $T(F)$ to the compact set $\{t \in T_0(A) : g_A(t) = 1\}$. The fact that $\Pi^\sharp$ is an isomorphism implies that Π^T is an affine homeomorphism. Since $T(A) = \{t \in T_0(A) : g_A(t) = 1\}$, this implies that Π^T maps $\Delta = T(F)$ onto $T(A)$.

Recall from (4.26) that we have, for each n, the commutative diagram

$$\begin{tikzcd}
G_n \arrow[r, "\rho_{A_n}"] \arrow[d, "(\pi_n^A)_{*0}"'] & \mathrm{Aff}(T_0(A_n)) \arrow[d, "{\pi_n^A}^\sharp"] \\
H_n \arrow[r, "\rho_{F_n}"] & \mathrm{Aff}(T_0(F_n))
\end{tikzcd}$$

where $\rho_{A_n} : G_n = K_0(A_n) \to \mathrm{Aff}(T(A_n))$ is induced by $\rho_{\widetilde{A}_n} : K_0(\widetilde{A}_n) \to \mathrm{Aff}(T(\widetilde{A}_n))$ (see (2.14)). Let $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$ be the map given as in 2.12 by $\rho_{\widetilde{A}} : K_0(\widetilde{A}) \to \mathrm{Aff}(T(\widetilde{A}))$. Then $\rho_A = \lim_{n\to\infty} \rho_{A_n}$. We obtain the approximately commutative diagram

$$\begin{tikzcd}
G_n \arrow[r, "\varphi_{n,n+1,*0}"] \arrow[drr, "\rho_{A_n}"'] \arrow[dd, "\pi^A_{n*0}"'] & G_{n+1} \arrow[r, "\varphi_{n+1,n+2,*0}"] \arrow[drr, "\rho_{A_{n+1}}"'] \arrow[dd, "\pi^A_{n+1,*0}"'] & \cdots \arrow[r] & \cdots \arrow[r] & G \arrow[drr, "\rho_A"] \arrow[dd, "\iota"] & & \\
& & \mathrm{Aff}(T_0(A_n)) \arrow[r, "\varphi^\sharp_{n,n+1}"'] \arrow[dd, "{\pi_n^A}^\sharp"'] & \mathrm{Aff}(T_0(A_{n+1})) \arrow[r, "\varphi^\sharp_{n+1,n+2}"'] \arrow[dd, "{\pi_{n+1}^A}^\sharp"'] & \cdots \arrow[r] & \cdots \arrow[r] & \mathrm{Aff}(T_0(A)) \arrow[dd, "\Pi^\sharp"] \\
H_n \arrow[r, "\psi_{n,n+1*0}"] \arrow[drr, "\rho_{F_n}"'] & H_{n+1} \arrow[r, "\psi_{n+1,n+2*0}"] \arrow[drr, "\rho_{F_{n+1}}"'] & \cdots \arrow[r] & \cdots \arrow[r] & H \arrow[drr, "\rho_F"] & & \\
& & \mathrm{Aff}(T_0(F_n)) \arrow[r, "\psi^\sharp_{n,n+1}"'] & \mathrm{Aff}(T_0(F_{n+1})) \arrow[r, "\psi^\sharp_{n+1,n+2}"'] & \cdots \arrow[r] & \cdots \arrow[r] & \mathrm{Aff}(T_0(F))
\end{tikzcd}$$

where the top, bottom and the back diagrams are commutative, and the front plane is approximately commutative. Thus, we obtain the commutative diagram

$$\begin{tikzcd}
G \arrow[r, "\rho_A"] \arrow[d, "\iota"'] & \mathrm{Aff}(T_0(A)) \arrow[d, "\Pi^\sharp"] \\
H \arrow[r, "\rho_F"] & \mathrm{Aff}(T_0(F))
\end{tikzcd}$$

where the map from G to H is given by 4.79. Since $\rho_F = \tilde{\rho}$, we obtain $\rho_A = \rho : G \to \mathrm{Aff}(\Delta)$ as desired.

To see $A \in \mathcal{D}$, choose $n_0 \geq 1$ such that $\lambda_s(A_n) \geq \frac{63}{64}$ for all $n \geq n_0$ (see (e) of 4.25). Let $e \in A_{n_0}$ be a strictly positive element. By (4.95), we may assume that

$$\begin{aligned} \tau(\varphi_{n_0,n}(e)) &\geq \tfrac{63}{64} \quad \text{for all } \tau \in T(A_n), \\ t(\varphi_{n_0,\infty}(e)) &\geq \tfrac{63}{64} \quad \text{for all } t \in T(A). \end{aligned} \tag{4.110}$$

In particular, $\varphi_{n_0,n}(e)$ is full in A_n for all $n \geq 1$. Choose $\delta > 0$ such that

$$\begin{aligned} \tau(\varphi_{n_0,n}((e-\delta)_+)) &\geq \tfrac{31}{32} \quad \text{for all } \tau \in T(A_n), \\ t(\varphi_{n_0,\infty}(e-\delta)_+) &\geq \tfrac{31}{32} \quad \text{for all } t \in T(A). \end{aligned}$$

Choose $k \geq 1$ such that

$$f_{1/4}(e^{1/k}) \geq (e-\delta)_+. \tag{4.111}$$

Let $B = \overline{\varphi_{n_0,\infty}(e)A\varphi_{n_0,\infty}(e)}$. Then B is a hereditary C^*-subalgebra of A. Let us first show $B \in \mathcal{D}$. Put $a = \varphi_{n_0,\infty}(e^{1/k})$ and choose $\mathfrak{f}_a = \frac{1}{2}$. Let $B_n = \overline{\varphi_{n_0,n}(e)A_n\varphi_{n_0,n}(e)}$. Note that $B_n \in \mathcal{C}_0'$ (recall Definition 2.26).

Now fix a finite subset $\mathcal{F} \subset B$ and $0 < \varepsilon < \frac{1}{16}$. We may assume that $\mathcal{F} \subset \varphi_{n,\infty}(B_n)$ for some $n > n_0$. Choose $0 < \eta < \varepsilon$ such that, if $a_1, a_2 \in B_+$ with $0 \leq a_1, a_2 \leq 1$ and $\|a_1 - a_2\| < \eta$, then $\|f_{1/4}(a_1) - f_{1/4}(a_2)\| < \varepsilon/8$.

Choose a completely positive contractive linear map $\psi : B \to B_n \cong \varphi_{n,\infty}(B_n)$ (see 2.3.13 of [Lin 2001], for example) such that

$$\|\psi(b) - b\| < \tfrac{\varepsilon}{2} \quad \text{for all } b \in \mathcal{F} \cup \{a\}. \tag{4.112}$$

Then

$$\|f_{1/4}(\psi(a)) - f_{1/4}(a)\| < \tfrac{\varepsilon}{8}. \tag{4.113}$$

It follows, for all $\tau \in T(B_n)$, by identifying $\varphi_{n,\infty}(B_n)$ with B_n, that

$$\begin{aligned} \tau(f_{1/4}(\psi(a))) &\geq \tau(f_{1/4}(\varphi_{n_0,n}(e^{1/k}))) - \tfrac{\varepsilon}{8} \\ &\geq \tau((\varphi_{n_0,n}(e-\delta)_+)) \geq \tfrac{31}{32} - \tfrac{\varepsilon}{8} > \mathfrak{f}_a. \end{aligned} \tag{4.114}$$

Define $\varphi = 0$. By (4.112), (4.114) and Definition 2.28, $B \in \mathcal{D}$. By Corollary 11.3 of [Elliott et al. 2020b], $B = \mathrm{Ped}(B)$, and by Theorem 9.4 of [Elliott et al. 2020b], B has strict comparison for positive elements. It follows from [Brown 1977] that $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$. Therefore A is isomorphic to a hereditary C^*-subalgebra of $B \otimes \mathcal{K}$. It follows that A has strict comparison for positive elements. Let e_A be a strictly positive element. Since $T(A)$ is compact, $d_\tau(e_A) = 1$ for all $\tau \in T(A)$. It follows that $\widehat{\langle e_A \rangle}$ is continuous on $\widetilde{T}(A)$. By Theorem 5.4 of [Elliott et al. 2020b], A has continuous scale. Therefore $A = \mathrm{Ped}(A)$ (see Theorem 3.3 of [Lin 1991]). Hence $a \in \mathrm{Ped}(A)$. It follows from Proposition 11.7 of [Elliott et al. 2020b] (see also Definition 1.6 of [Elliott et al. 2020b]) that $A \in \mathcal{D}$. $\square$

4.115. Now let G and K be any countable abelian groups, Δ a compact Choquet simplex and $\rho : G \to \operatorname{Aff}\Delta$ a homomorphism with the property that for any $g \in G$, there is a $\tau \in \Delta$ such that $\rho(g)(\tau) \le 0$ (see condition $(*)$ in 4.1).

We construct a simple stably projectionless, stably finite C^*-algebra A such that $K_0(A) \cong G$, $K_1(A) \cong K$, $T(A) \cong \Delta$, and the map $\rho_A : K_0(A) \to \operatorname{Aff}(T(A))$ is the map ρ when one identifies $K_0(A)$ with G and $T(A)$ with Δ.

Note that if $K = 0$ and G is torsion free, then the algebra satisfying the condition is already constructed (see Theorem 4.107 above).

Note $\operatorname{Tor}(G) \subset \ker\rho$. Write $G_T = G/\ker\rho$, which may be viewed as a subgroup of $\operatorname{Aff}\Delta$, and write $G_f = \ker\rho/\operatorname{Tor}(G)$, which is a torsion free group.

Lemma 4.116. *G can be written as an inductive limit of finitely generated subgroups ($G_n = G_{n,T} \oplus G_{n,f} \oplus G_{n,\mathrm{tor}}, \gamma_{n,m}$) with* $\operatorname{Tor}(G_n) = G_{n,\mathrm{tor}}$ *such that the following hold*:

(a) *According to the decompositions $G_n = G_{n,T} \oplus G_{n,f} \oplus G_{n,\mathrm{tor}}$ and $G_{n+1} = G_{n+1,T} \oplus G_{n+1,f} \oplus G_{n+1,\mathrm{tor}}$, the map $\gamma_{n,n+1}$ may be written as*

$$\begin{pmatrix} \gamma_{T,T}^{n,n+1} & 0 & 0 \\ \gamma_{T,f}^{n,n+1} & \gamma_{f,f}^{n,n+1} & 0 \\ \gamma_{T,\mathrm{tor}}^{n,n+1} & \gamma_{f,\mathrm{tor}}^{n,n+1} & \gamma_{\mathrm{tor},\mathrm{tor}}^{n,n+1} \end{pmatrix},$$

that is, the components of $\gamma_{n,n+1}$ from $G_{n,\mathrm{tor}}$ to $G_{n+1,T} \oplus G_{n+1,f}$ and from $G_{n,f}$ to $G_{n,T}$ are zero maps. In particular,

$$\gamma_{n,n+1}(G_{n,\mathrm{tor}}) \subset G_{n+1,\mathrm{tor}} \quad \textit{and} \quad \gamma_{n,n+1}(G_{n,\mathrm{Inf}}) \subset G_{n+1,\mathrm{Inf}},$$

where $G_{n,\mathrm{Inf}} = G_{n,f} \oplus G_{n,\mathrm{tor}}$.

(b) $\ker\rho = \lim(G_{n,\mathrm{Inf}}, \gamma_{n,m}|_{G_{n,\mathrm{Inf}}})$ *and* $\operatorname{Tor}(G) = \lim(G_{n,\mathrm{tor}}, \gamma_{n,m}|_{G_{n,\mathrm{tor}}})$.

(c) *Let*

$$\tilde\gamma_{n,n+1} = \gamma_{T,T}^{n,n+1} : G_{n,T} = G_n/G_{n,\mathrm{Inf}} \to G_{n+1,T} = G_{n+1}/G_{n+1,\mathrm{Inf}},$$

$$\tilde{\tilde\gamma}_{n,n+1} = \begin{pmatrix} \gamma_{T,T}^{n,n+1} & 0 \\ \gamma_{T,f}^{n,n+1} & \gamma_{f,f}^{n,n+1} \end{pmatrix} : G_{n,T} \oplus G_{n,f} = G_n/G_{n,\mathrm{tor}} \to G_{n+1,T} \oplus G_{n+1,f} = G_{n+1}/G_{n+1,\mathrm{tor}}$$

be the quotient maps induced by $\gamma_{n,n+1}$. Then $G_T = G/\ker\rho = \lim(G_{n,T}, \tilde\gamma_{n,m})$ and $G/\operatorname{tor}(G) = \lim(G_{n,T} \oplus G_{n,f}, \tilde{\tilde\gamma}_{n,m})$.

(d) *$\gamma_{T,T}^{n,n+1}$, $\gamma_{f,f}^{n,n+1}$ and $\gamma_{\mathrm{tor},\mathrm{tor}}^{n,n+1}$ are injective. Consequently, $\gamma_{n,n+1}, \tilde\gamma_{n,n+1}$ and $\tilde{\tilde\gamma}_{n,n+1}$ are injective.*

Proof. Let $G_f = \ker\rho/\operatorname{tor}(G)$. Write $G_T = \bigcup_{n=1}^\infty G_{n,T}$ with $G_{1,T} \subset G_{2,T} \subset \cdots \subset G_{n,T} \subset \cdots \subset G_T$, $G_f = \bigcup_{n=1}^\infty G_{n,f}$ with $G_{1,f} \subset G_{2,f} \subset \cdots \subset G_{n,f} \subset \cdots \subset G_f$, and $\mathrm{Tor}(G)=\bigcup_{n=1}^{\infty} G_{n,\mathrm{tor}}$ with $G_{1,\mathrm{tor}}\subset G_{2,\mathrm{tor}}\subset\cdots\subset G_{n,\mathrm{tor}}\subset\cdots\subset\mathrm{Tor}(G)$, where each $G_{n,T}$, $G_{n,f}$ and $G_{n,\mathrm{tor}}$ are finitely generated. Denote by $\iota_{G_{n,f}}: G_{n,f}\to G_f$ the embedding.

Since G_f is torsion free, the extension $0\to\mathrm{Tor}(G)\to\ker\rho\to G_f\to 0$ is pure, i.e., every finitely generated subgroup lifts. Thus, for each n, there is a homomorphism $\xi_n: G_{n,f}\to\ker\rho$ such that the diagram

$$\begin{tikzcd} & G_{n,f} \arrow[dl, dashed, "\xi_n"'] \arrow[d, "\iota_{G_{n,f}}"] \\ \ker\rho \arrow[r, "\pi"] & G_f \end{tikzcd}$$

commutes ($\pi\circ\xi_n=\iota_{G_{n,f}}$, $n=1,2,\dots$). Define $\gamma'_{n,f}: G_{n,f}\to\mathrm{Tor}(G)$ by $\gamma'_{n,f}(h)=\xi_n(h)-\xi_{n+1}(h)\in\mathrm{Tor}(G)$. Since $G_{n,f}$ is finitely generated, there is an integer $m>n$ such that $\gamma'_{n,f}(G_{n,f})\subset G_{m,\mathrm{tor}}$. By passing to a subsequence, we may assume $\gamma'_{n,f}(G_{n,f})\subset G_{n+1,\mathrm{tor}}$. Define $\gamma_{f,\mathrm{tor}}^{n,n+1}=\gamma'_{n,f}$ and

$$\chi_{n,n+1}: G_{n,\mathrm{Inf}}=G_{n,f}\oplus G_{n,\mathrm{tor}}\to G_{n+1,\mathrm{Inf}}=G_{n+1,f}\oplus G_{n+1,\mathrm{tor}}$$

by

$$\chi_{n,n+1}=\begin{pmatrix}\gamma_{f,f}^{n,n+1} & 0\\ \gamma_{f,\mathrm{tor}}^{n,n+1} & \gamma_{\mathrm{tor},\mathrm{tor}}^{n,n+1}\end{pmatrix},$$

where $\gamma_{f,f}^{n,n+1}: G_{n,f}\to G_{n+1,f}$ and $\gamma_{\mathrm{tor},\mathrm{tor}}^{n,n+1}: G_{n,\mathrm{tor}}\to G_{n+1,\mathrm{tor}}$ are the inclusion maps. Then $\ker\rho=\lim(G_{n,f}\oplus G_{n,\mathrm{tor}},\chi_{n,n+1})$ and $\chi_{n,n+1}$ are injective. Repeating this procedure with G_T in place of G_f and $\ker\rho$ in place of $\mathrm{Tor}(G)$, and of course passing to a subsequence again, we obtain the other parts of the map $\gamma_{n,n+1}$ as desired. □

4.117. Let us recall Theorem 7.11 of [Gong and Lin 2020a]. Let G_0, G_1 be any countable abelian groups and T be any compact metrizable Choquet simplex. There is a simple $\mathcal{Z}$-stable C^*-algebra $B_T\in\mathcal{D}_0$ with continuous scale such that $K_0(B_T)=\ker(\rho_{B_T})=G_0$, $K_1(B_T)=G_1$ and $T(B_T)=T$. Moreover B_T is locally approximated by subhomogeneous C^*-algebras with spectrum having dimension no more than 3 (see Proposition 7.7 of [Gong and Lin 2020a]). More precisely, $B_T=\lim_{k\to\infty}(E_{n(k)}\oplus W_k,\Phi_{k,k+1})$, where $E_n=M_{(n!)^2}(A(W,\alpha_n))$ and W_k is in $\mathcal{C}_0$ with $K_0(W_k)=\{0\}$, and $\Phi_{k,k+1}$ is injective (see the constructions in Definition 7.2 of [Gong and Lin 2020a]; also the notation in [Gong and Lin 2020a, 11.3] and the discussions in [Gong and Lin 2020a, 7.3–7.10]). Note that, by [Gong and Lin 2020a, Proposition 7.7], B_T is locally approximated by subhomogeneous C^*-algebras with spectrum having dimension no more than 3.

If Δ is a compact metrizable Choquet simplex, then $\mathrm{Aff}(\Delta)$ can be regarded as a subset of $\mathrm{LAff}_+(\tilde{\Delta})$ (here $\tilde{\Delta}$ is the cone generated by Δ and 0) by regarding

$f : \Delta \to \mathbb{R}$ as $\tilde{f} : \tilde{\Delta} \to \mathbb{R}$ defined by $\tilde{f}(\lambda\tau) = \lambda f(\tau)$ for $\lambda \in [0, \infty)$ and $\tau \in \Delta$. In particular $1_\Delta \in \mathrm{LAff}_+(\tilde{\Delta})$. For $\alpha > 0$, denote by $\alpha\Delta$ the subset $\{\alpha\tau : \tau \in \Delta\}$ of $\tilde{\Delta}$. Note that when Δ is compact, we may identify $\mathrm{Aff}(\Delta)$ with $\mathrm{Aff}(\tilde{\Delta})$ (recall $f(0) = 0$ and see the end of Definition 2.2) — that is, for $f \in \mathrm{Aff}(\Delta)$, we assume f is extended to $f \in \mathrm{Aff}(\tilde{\Delta})$ defined by $f(\alpha\tau) = \alpha f(\tau)$ for any $\alpha \in \mathbb{R}_+$ and $\tau \in \Delta$.

Theorem 4.118. *Let Δ be a metrizable Choquet simplex, G_0 a countable abelian group, $\rho : G_0 \to \mathrm{Aff}(\Delta)$ a homomorphism such that $\rho(G_0) \cap \mathrm{Aff}_+(\Delta) = \{0\}$, and G_1 a countable abelian group. Then there is a simple C^*-algebra*

$$A = \lim_{n\to\infty} (B_n \oplus C_n \oplus D_n, \varphi_{n,m}),$$

where C_n and D_n are simple C^*-algebras in Theorem 4.107 and B_n is in 4.117 (as B_T — see [Gong and Lin 2020a, Theorem 7.11 (and 7.2)]) *such that φ_n maps strictly positive elements to strictly positive elements,*

$$((K_0(A), \Sigma(K_0(A)), T(A), \rho_A, \Sigma_A), K_1(A)) = ((G_0, \{0\}, \Delta, \rho, 1_\Delta), G_1), \quad (4.119)$$

$\varphi_{n,\infty}(K_0(C_n)) \cap \ker\rho_A = \{0\}$, $\ker\rho_{C_n} = \{0\}$, $\ker\rho_{D_n} = K_0(D_n)$, and $K_0(B_n)$ is torsion. Moreover, A has continuous scale, $A \in \mathcal{D}$ and

$$\lim_{n\to\infty} \inf\{d_\tau(\varphi_{n,\infty}(x_n)) : \tau \in A\} = 0,$$

where $x_n \in B_n \oplus D_n$ is any strictly positive element.

Moreover, one may require that $\varphi_{n*i}|_{K_i(B_n)}$, $\varphi_{n*i}|_{K_i(C_n)}$ and $\varphi_{n*i}|_{K_i(D_n)}$ are all injective, and $K_i(B_n)$, $K_i(C_n)$ and $K_i(D_n)$ are finitely generated.

Proof. For convenience, we write $G_0 = G$ and $G_1 = K$. Choose finitely generated subgroups $K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots \subset K$ such that $K = \bigcup_{n=1}^\infty K_n$. Write $G = \lim(G_{n,T} \oplus G_{n,f} \oplus G_{n,\mathrm{tor}}, \gamma_{n,m})$ as in Lemma 4.116. We adopt the notation $\gamma_{n,n+1}$ and $\gamma_{a,b}^{n,n+1}$, where $a, b = T$, f and tor from Lemma 4.116.

It follows from Theorem 7.11 and Proposition 7.8 of [Gong and Lin 2020a] that there is a simple C^*-algebra $B_n \in \mathcal{B}_T$ such that

$$((K_0(B_n), \Sigma(K_0(B_n)), T(B_n), \rho_{B_n}), K_1(B_n)) = ((G_{n,\mathrm{tor}}, \{0\}, \Delta_0, 0), K_n),$$

where Δ_0 is a single point. Here we mean $\rho_{B_n} = 0$. Note that $B_n \in \mathcal{D}_0 \subset \mathcal{D}$ is a $\mathcal{Z}$-stable simple C^*-algebra with continuous scale (see [Gong and Lin 2020a, 7.7]).

By Theorem 4.107, there is also a simple C^*-algebra C_n' with continuous scale and with the form in Theorem 4.107 such that $(K_0(C_n'), T(C_n'), \rho_{C_n'}) = (G_{n,T}, \Delta, \xi_{T,n,\infty})$, where $\xi_{T,n,\infty}$ is the homomorphism of the form

$$G_{n,T} \xrightarrow{\tilde{\gamma}_{n,\infty}} G/\ker\rho = G_T \subset \mathrm{Aff}(\Delta)$$

induced by the inductive limit $G_T = \lim_{n\to\infty}(G_{n,T}, \tilde{\gamma}_{n,m})$. Note that, by [Robert 2012, Theorem 6.2.3],

$$\mathrm{Cu}^{\sim}(C_n') = K_0(C_n') \sqcup \mathrm{LAff}_+(\tilde{\Delta}), \tag{4.120}$$

where $\tilde{\Delta}$ is the cone generated by Δ and $\{0\}$. Note that $\tilde{\gamma}_{n,n+1}$ and id_Δ induce a morphism $\xi^{\mathrm{cu}}_{T,n} : \mathrm{Cu}^{\sim}(C_n') \to \mathrm{Cu}^{\sim}(C_{n+1}')$. By [Robert 2012, Theorem 1.0.1], there is a homomorphism $\psi_n : C_n' \to C_{n+1}'$ which sends strictly positive elements to strictly positive elements and $\mathrm{Cu}^{\sim}(\psi_n) = \xi^{\mathrm{cu}}_{T,n}$. In particular, $\psi_{n*0} = \xi^{\mathrm{cu}}_{T,n}|_{K_0(C_n')} = \tilde{\gamma}_{n,n+1}$. We continue to write ψ_n for the extension $\psi_n \otimes \mathrm{id}_{M_3} : M_3(C_n') \to M_3(C_{n+1}')$. Note that, since $C_1' \in \mathcal{D}$, by Proposition 11.8 of [Elliott et al. 2020b], one may choose $c_1 \in (C_1')_+$ such that $d_\tau(c_1) = \frac{1}{2}$ for all $\tau \in T(C_1')$. Let $C_1 = \overline{c_1 C_1' c_1}$. Let $n \geq 2$. Choose $c_{n,0}, c_{n,b}, c_{n,c}, c_{n,d} \in (C_n')_+$ such that $d_\tau(c_{n,0}) = 1/2^{n+1}$, and $d_\tau(c_{n,b}) = d_\tau(c_{n,d}) = 1/2^{n+2}$ (and $d_\tau(c_{n,c}) = 1 - 1/2^n$) for all $\tau \in T(C_n')$ and $n = 2, 3, \ldots$. Put $c_n := c_{n,b} \oplus c_{n,c} \oplus c_{n,d} \in M_3(C_n')$ and $C_n = \overline{c_n M_3(C_n') c_n}$. Note that $c_n \in M_3(C_n')$ satisfies that $\tau(c_n) = (1 - 1/2^n) + 1/2^{n+2} + 1/2^{n+2} = 1 - 1/2^{n+1}$ and defines $\widehat{\langle c_n \rangle} = (1 - 1/2^{n+1}) \cdot 1_\Delta \in \mathrm{LAff}_+(\tilde{\Delta}) \subset \mathrm{Cu}^{\sim}(C_n')$. Similarly, $c_{n+1,c}$ also defines $\widehat{\langle c_{n+1,c} \rangle} = (1 - 1/2^{n+1}) \cdot 1_\Delta \in \mathrm{LAff}_+(\tilde{\Delta}) \subset \mathrm{Cu}^{\sim}(C_{n+1}')$. Consequently, $\langle c_{n+1,c} \rangle = \langle \psi_n(c_n) \rangle$ in $\mathrm{Cu}(C_{n+1}')$. By [Robert 2016, Theorem 1.2], one has $\mathrm{Her}(\psi_n(C_n)) \cong \overline{c_{n+1,c} C_{n+1}' c_{n+1,c}}$. Therefore, there is a homomorphism $\varphi_{n,c,c} : C_n \to C_{n+1}$ such that $\mathrm{Cu}^{\sim}(\varphi_{n,c,c}) = \mathrm{Cu}^{\sim}(\psi_n|_{C_n})$ and $\langle \varphi_{n,c,c}(c_n) \rangle = \langle c_{n+1,c} \rangle$. Note this fact, which will be used later: when we identify both $T(C_n)$ and $T(C_{n+1})$ with Δ, the map $\varphi^\sharp_{n,c,c} : \mathrm{Aff}(T(C_n)) = \mathrm{Aff}(\Delta) \to \mathrm{Aff}(T(C_{n+1})) = \mathrm{Aff}(\Delta)$ is given by

$$\varphi^\sharp_{n,c,c}(f) = \frac{1 - 1/2^{n+1}}{1 - 1/2^{n+2}} f \quad \text{for all } f \in \mathrm{Aff}(T(C_n)). \tag{4.121}$$

Denote by $C_{n,\mathfrak{p}} = \overline{c_{n,\mathfrak{p}} C_n c_{n,\mathfrak{p}}}$, where $\mathfrak{p} = b, c, d$.

By Theorem 4.107, there is a simple C^*-algebra D_n of the form in Theorem 4.107 such that $K_0(D_n) = G_{n,f}$ and $\ker \rho_{D_n} = K_0(D_n)$, and $T(D_n) = \Delta_0$, the single point. As in the previous case, there is $\psi_{d,n} : D_n \to D_{n+1}$ such that $\psi_{d,n*0} = \gamma^{n,n+1}_{f,f} : G_{n,f} \to G_{n+1,f}$ (see Lemma 4.116), and $\psi_{n,d}$ sends strictly positive elements to strictly positive elements (using again [Robert 2012, Theorem 1.0.1]). Choose $d_{n,b}, d_{n,c}, d_{n,d} \in (D_n)_+$ such that $d_\tau(d_{n,b}) = d_\tau(d_{n,d}) = 1/2^{n+2}$, and $d_\tau(d_{n,c}) = 1 - 1/2^{n+1}$ for all $\tau \in T(D_n)$. Define

$$D_n' := \overline{(d_{n,b} \oplus d_{n,c} \oplus d_{b,d}) M_3(D_n) (d_{n,b} \oplus d_{n,c} \oplus d_{n,d})}.$$

Then, by [Robert 2016], $D_n' \cong D_n$. Without loss of generality, we may assume that $D_n = D_n'$ and $d_{n,b}, d_{n,c}$ and $d_{n,d}$ are mutually orthogonal in D_n. Put $D_{n,\mathfrak{p}} = \mathrm{Her}(d_{n,\mathfrak{p}})$, $\mathfrak{p} = b, c, d$.

Choose $b_{n,b}, b_{n,c}, b_{n,d} \in (B_n)_+$ such that $d_\tau(b_{n,b}) = d_\tau(b_{n,d}) = 1/2^{n+2}$ and

$d_\tau(b_{n,c}) = 1 - 1/2^{n+1}$ for all $\tau \in T(B_n)$. Define

$$B_n' := \overline{(b_{n,b} \oplus b_{n,c} \oplus b_{n,d}) M_2(B)(b_{n,b} \oplus b_{n,c} \oplus b_{n,d})}.$$

Since B_n is stably projectionless and $\mathcal{Z}$-stable [Gong and Lin 2020a], by [Robert 2016, Theorem 1.2], $B_n' \cong B_n$. We may assume that $b_{n,b}$, $b_{n,c}$ and $b_{n,d}$ are mutually orthogonal in B_n and $B_n' = B_n$. Define $B_{n,\mathfrak{p}} = \overline{b_{n,\mathfrak{p}} B b_{n,\mathfrak{p}}}$, where $\mathfrak{p} = b, c, d$.

Denote by $\iota_{n,\mathfrak{p}}^b : B_{n,\mathfrak{p}} \to B_n$ the embedding ($\mathfrak{p} = b, c, d$). Note that for $\mathfrak{p} = b, c, d$,

$$\begin{aligned}((K_0(B_{n,\mathfrak{p}}), \widetilde{T}(B_{n,\mathfrak{p}}), 0), K_1(B_{n,\mathfrak{p}})) &\cong ((K_0(B_n), \widetilde{T}(B_n), 0), K_1(B_n)) \\ &= ((G_{n,\mathrm{tor}}, \tilde{\Delta}_0, 0), K_n).\end{aligned} \tag{4.122}$$

By Theorem 12.8 of [Gong and Lin 2020a], there is a homomorphism $\varphi'_{n,b,b} : B_n \to B_{n+1,b} \subset B_{n+1}$ which sends strictly positive elements to strictly positive elements such that $\varphi_{n,b,b*0} = \gamma^{n,n+1}_{\mathrm{tor},\mathrm{tor}} : K_0(B_n) = G_{n,\mathrm{tor}} \to K_0(B_{n+1}) = G_{n+1,\mathrm{tor}}$ and $\varphi_{n,b,b*1} = \iota : K_1(B_n) = K_n \hookrightarrow K_1(B_{n+1}) = K_{n+1}$. Let $\varphi_{n,b,b} := \iota^b_{n+1,b} \circ \varphi'_{n,b,b}$. Let W_n be a simple C^*-algebra which is an inductive limit of C^*-algebras in $\mathcal{C}_0$ such that $K_0(W_n) = K_1(W_n) = 0$ and $T(W_n) = \Delta_0$, $n = 1, 2, \ldots$. It follows from Theorem 12.8 of [Gong and Lin 2020a] again that there is $h_{n,b,w} : B_n \to W_n$ which sends strictly positive elements to strictly positive elements and $h_{n,b,w T}$ gives the identity on Δ_0. Note that, for $\mathfrak{p} = b, c, d$,

$$\begin{aligned}&((K_0(C_{n,\mathfrak{p}}), \Sigma(K_0(C_{n,\mathfrak{p}})), \widetilde{T}(C_{n,\mathfrak{p}}), \rho_{C_{n,\mathfrak{p}}}), K_1(C_{n,\mathfrak{p}})) \\ &\qquad\qquad \cong ((K_0(C_n'), \{0\}, \tilde{\Delta}, \rho_{C_n'}), \{0\}).\end{aligned} \tag{4.123}$$

By [Robert 2012, Theorem 1.0.1], there is a homomorphism $h_{n,w,c} : W_n \to C_{n+1,b}$ mapping strictly positive elements to strictly positive elements. Define $\varphi_{n,b,c} : B_n \to C_{n+1,b} \subset C_{n+1}$ by $\varphi_{n,b,c} := h_{n,w,c} \circ h_{n,b,w}$. Similarly, one obtains a homomorphism $\varphi_{n,b,d} : B_n \to D_{n+1,b} \subset D_{n+1}$ which factors through W_n and sends strictly positive elements to strictly positive elements.

Note that, by [Elliott et al. 2020b, Theorem 11.5], B_{n+1} has stable rank one, and by [Robert 2012, Theorem 6.2.3] (see also Theorem 7.3, Corollary 11.3 and Proposition 11.8 of [Elliott et al. 2020b]),

$$\mathrm{Cu}^\sim(B_{n+1,\mathfrak{p}}) = K_0(B_{n+1}) \sqcup \mathrm{LAff}^\sim(T(B_{n+1})), \quad \mathfrak{p} = c, b, d. \tag{4.124}$$

Let $\tau_{w,0} \in T(C_n)$ be such that $\rho_{C_n}(x)(\tau_{w,0}) = 0$ for all $x \in K_0(C_n)$ given by Theorem 3.1. Define $\xi_{c,b} : \mathrm{Cu}^\sim(C_n) \to \mathrm{Cu}^\sim(B_{n+1,c})$ by

$$\xi_{c,b}|_{K_0(C_n)} := \gamma^{n,n+1}_{T,\mathrm{tor}} : K_0(C_n) = G_{n,T} \to K_0(B_{n+1,c}) = K_0(B_{n+1}) = G_{n+1,\mathrm{tor}}$$

and $\xi_{c,b}|_{\mathrm{LAff}^\sim(T(C_n))}$ by $\xi_{c,b}(f)(t) = f(\tau_{w,0})$ for $f \in \mathrm{LAff}^\sim(T(C_n))$ and $t \in T(B_{n+1,c})$. Note that $\rho_{B_n} = 0$. One then checks that $\xi_{c,b}$ is a morphism in **Cu**. Since $B_{n+1,c}$

has stable rank one, by applying [Robert 2012, Theorem 1.0.1] again, one obtains a homomorphism $\varphi'_{n,c,b} : C_n \to B_{n+1,c}$ such that $\mathrm{Cu}(\varphi'_{n,c,b}) = \xi_{c,b}$ which sends strictly positive elements to strictly positive elements. Define $\varphi_{n,c,b} := \iota^b_{n+1,c} \circ \varphi'_{n,c,b}$, where $\iota^b_{n+1,c} : B_{n+1,c} \to B_{n+1}$ is the embedding.

Using $\tau_{w,0}$ and [Robert 2012, Theorem 1.0.1] again, one obtains a homomorphism $\varphi_{n,c,d} : C_n \to D_{n+1,c} \subset D_{n+1}$ such that $\varphi_{n,c,d*0} = \gamma^{n,n+1}_{T,f} : G_{n,T} \to G_{n+1,f}$, which sends strictly positive elements of C_n to strictly positive elements of $D_{n+1,c}$.

Denote by $\iota^d_{n,\mathfrak{p}} : D_{n,\mathfrak{p}} \to D_n$ the embedding ($\mathfrak{p} = b, c, d$). As above, applying [Robert 2012, Theorem 1.0.1], as B_{n+1} has stable rank one, one obtains a homomorphism $\varphi_{n,d,b} : D_n \to B_{n+1,d} \subset B_{n+1}$ such that $\varphi_{n,b,d*0} = \gamma^{n,n+1}_{f,\mathrm{tor}} : K_0(D_n) = G_{n,f} \to K_0(B_{n+1,d}) = G_{n+1,\mathrm{tor}}$. By factoring through W_n again, one obtains a homomorphism $\varphi_{n,d,c} : D_n \to C_{n+1,d} \subset C_{n+1}$, which sends strictly positive elements of D_n to strictly positive elements of $C_{n+1,d}$. By applying Theorem 1.0.1 of [Robert 2012] again (recall $\rho_{D_n} = 0$ for all n), one also has a homomorphism $\varphi_{n,d,d} : D_n \to D_{n+1,d} \subset D_{n+1}$ such that $\varphi_{n,d,d*0} = \gamma^{n,n+1}_{f,f} : K_0(D_n) = G_{n,f} \to K_0(D_{n+1,d}) = G_{n+1,f}$, which sends strictly positive elements to strictly positive elements.

Now define $\varphi_{n,n+1} : B_n \oplus C_n \oplus D_n \to B_{n+1} \oplus C_{n+1} \oplus D_{n+1}$ by

$$\begin{aligned}
\varphi_{n,n+1}|_{B_n} &= \varphi_{n,b,b} \oplus \varphi_{n,b,c} \oplus \varphi_{n,b,d},\\
\varphi_{n,n+1}|_{C_n} &= \varphi_{n,c,b} \oplus \varphi_{n,c,c} \oplus \varphi_{n,c,d},\\
\varphi_{n,n+1}|_{D_n} &= \varphi_{n,d,b} \oplus \varphi_{n,d,c} \oplus \varphi_{n,d,d}.
\end{aligned}$$

Put $A_n = B_n \oplus C_n \oplus D_n$. Then $K_0(A_n) = G_n = G_{T,n} \oplus G_{n,f} \oplus G_{n,\mathrm{tor}}$ and $K_1(A_n) = K_n$. It is clear that $\varphi_{n,n+1}$ sends strictly positive elements to strictly positive elements as constructed above. Moreover,

$$\varphi_{n,n+1*0} := \gamma_{n,n+1} : G_n \to G_{n+1} \quad \text{and} \quad \varphi_{n,n+1*1} = \iota : K_n \hookrightarrow K_{n+1}. \tag{4.125}$$

Denote by $\iota_n : C_n \to C'_n$ the embedding and by $\iota_{nT} : T(C'_n) \to T(C_n)$ the induced affine homeomorphism defined by

$$\iota_{nT}(\tau)(c) = \frac{1}{1-1/2^{n+1}} \tau(\iota_n(c))$$

for all $c \in C_n$ (recall $T(C'_n) = \Delta$). Then, for any

$$(f, g_f, g_t) \in G_{n,T} \oplus G_{n,f} \oplus G_{n,\mathrm{tor}} = G_n,$$

we have

$$\rho_{A_n}(f \oplus g_f \oplus g_t)(\tau) = \rho(\gamma_{n,\infty}(f))(\iota_{nT}^{-1}(\tau)) \quad \text{for all } \tau \in T(C_n), \tag{4.126}$$

$$\rho_{A_n}(f \oplus g_f \oplus g_t)(\tau) = 0 \quad \text{for all } \tau \in T(B_n), \tag{4.127}$$

$$\rho_{A_n}(f \oplus g_f \oplus g_t)(\tau) = 0 \quad \text{for all } \tau \in T(D_n) \tag{4.128}$$

(recall $\rho : G \to \mathrm{Aff}(\Delta) = \mathrm{Aff}(\tilde{\Delta})$). Define $A := \lim_{n\to\infty}(A_n, \varphi_{n,n+1})$. Then $K_0(A) = G = G_0$ and $K_1(A) = K = G_1$. Since B_n, C_n and D_n are simple, and all maps $\varphi_{n,\mathfrak{p},\mathfrak{q}}$ (both $\mathfrak{p}$ and $\mathfrak{q}$ are among b, c, and d) are nonzero, $\varphi_{n,n+1}$ maps any nonzero element of $B_n \oplus C_n \oplus D_n$ to a full element in $B_{n+1} \oplus C_{n+1} \oplus D_{n+1}$. It follows that A is simple. Note that, for any $b \in B_n \oplus D_n$ and any $\tau \in T(A_{n+1})$,

$$|\tau(\varphi_{n,n+1}(b))| < \frac{1}{2^n}\|b\|. \tag{4.129}$$

Let $q_{n,a,b} : A_n \to B_n$, $q_{n,a,c} : A_n \to C_n$ and $q_{n,a,d} : A_n \to D_n$ be the projection maps. Denote by $j_{n,c,a} : C_n \to A_n$ the homomorphism defined by $j_{n,c,a}(c) = 0\oplus c\oplus 0$ for all $c \in C_n$. Identify $T(C_n) = \Delta$ as above. Let $\lambda_n : \mathrm{Aff}(T(C_n)) \to \mathrm{Aff}(T(C_{n+1}))$ be defined as $\mathrm{id} : \mathrm{Aff}(\Delta) \to \mathrm{Aff}(\Delta)$, where we identify both $T(C_n)$ and $T(C_{n+1})$ with Δ. Then, by (4.121),

$$\|\lambda_n(f) - \varphi^\sharp_{n,c,c}(f)\| = \left\|\left(1 - \frac{1 - 1/2^{n+1}}{1 - 1/2^{n+2}}\right) f\right\| < \frac{1}{2^n}\|f\|. \tag{4.130}$$

For each $f \in \mathrm{Aff}(T(A_n))$, we may write $f = f_b \oplus f_c \oplus f_d$, where $f_b = q_{n,a,b}(f)$, $f_c = q_{n,a,c}(f)$ and $f_d = q_{n,a,d}(f)$. By (4.129) and (4.130), we have

$$\begin{aligned}\|j^\sharp_{n+1,c,a} &\circ \lambda_n \circ q^\sharp_{n,a,c}(f) - \varphi^\sharp_{n,n+1}(f)\| \\ &\le \|\lambda_n(f_c) - \varphi^\sharp_{n,c,c}(f_c)\| + \|\varphi^\sharp_{n,n+1}(f_b \oplus 0 \oplus f_d)\| < \frac{2}{2^n}\|f\|.\end{aligned} \tag{4.131}$$

Let $\tilde{\lambda}_n = \lambda_n \circ q^\sharp_{n,a,c}$. Then we have

$$\|j^\sharp_{n+1,c,a} \circ \tilde{\lambda}_n(f) - \varphi^\sharp_{n,n+1}(f)\| < \frac{2}{2^n}\|f\| \quad \text{and} \quad \tilde{\lambda}_n \circ j^\sharp_{n,c,a} = \lambda_n. \tag{4.132}$$

Recall that $T(A_n)$ and $T(C_n) = \Delta$ are all compact as constructed above. Moreover, by (4.132), we obtain the following approximately commutative diagram:

$$\begin{tikzcd} \mathrm{Aff}(T(A_1)) \arrow[r, "\varphi^\sharp_{1,2}"] \arrow[dr, "\tilde{\lambda}_1"] & \mathrm{Aff}(T(A_2)) \arrow[r, "\varphi^\sharp_{2,3}"] \arrow[dr, "\tilde{\lambda}_2"] & \mathrm{Aff}(T(A_3)) \arrow[r] & \cdots \arrow[r] & \mathrm{Aff}(T(A)) \\ \mathrm{Aff}(T(C_1)) \arrow[u, "j^\sharp_{1,c,a}"] \arrow[r, "\lambda_1"] & \mathrm{Aff}(T(C_2)) \arrow[u, "j^\sharp_{2,c,a}"] \arrow[r, "\lambda_2"] & \mathrm{Aff}(T(C_3)) \arrow[u, "j^\sharp_{3,c,a}"] \arrow[r] & \cdots \arrow[r] & \mathrm{Aff}(\Delta) \end{tikzcd} \tag{4.133}$$

Note that, since $\varphi_{n,n+1}$ maps strictly positive elements to strictly positive elements, $\varphi^\sharp_{n,n+1} : \mathrm{Aff}(T(A_n)) \to \mathrm{Aff}(T(A_{n+1}))$ and $\lambda_n : \mathrm{Aff}(T(C_n)) \to \mathrm{Aff}(T(C_{n+1}))$ are identities when we identify all $T(C_n)$ with Δ. Therefore there is an affine homeomorphism $\Lambda : \Delta \to T(A)$ which induces the diagram above. In other words, $T(A) = \Delta$. Moreover, by (4.126), identifying $K_0(A_n)$ with G_n, we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Aff}(T(A_1)) \arrow[r, "\varphi_{1,2}^{\sharp}"] & \mathrm{Aff}(T(A_2)) \arrow[r, "\varphi_{2,3}^{\sharp}"] & \mathrm{Aff}(T(A_3)) \arrow[r] & \cdots \arrow[r] & \mathrm{Aff}(T(A)) \\
G_1 \arrow[u, "\rho_{A_1}"] \arrow[r, "\varphi_{1,2*0}"] & G_2 \arrow[u, "\rho_{A_2}"] \arrow[r, "\varphi_{2,3*0}"] & G_3 \arrow[u, "\rho_{A_3}"] \arrow[r] & \cdots \arrow[r] & G \arrow[u, "\rho_A"]
\end{tikzcd}$$

Combining with (4.133), we obtain

$$((K_0(A), T(A), \rho_A), K_1(A)) = ((G_0, \Delta, \rho), G_1). \tag{4.134}$$

Since $\gamma_{T,T}^{n,n+1}$ is injective, we have $\varphi_{n,n+1*0}(C_n) \cap \ker \rho_A = \{0\}$. By (4.129), $\lim_{n\to\infty} \inf\{d_\tau(\varphi_{n,\infty}(x_n)) : \tau \in A\} = 0$, where $x_n \in B_n \oplus D_n$ is any strictly positive element. It follows from Lemma 4.116 that $\varphi_{n*i}|_{K_i(B_n)}$, $\varphi_{n*i}|_{K_i(C_n)}$ and $\varphi_{n*i}|_{K_i(D_n)}$ are all injective, and $K_i(B_n)$, $K_i(C_n)$ and $K_i(D_n)$ are finitely generated. It remains to show that $A \in \mathcal{D}$. However, this follows from the fact that B_n, C_n and D_n are in $\mathcal{D}$ and have continuous scales (and A is simple). By Theorem 9.4 of [Gong and Lin 2020a], A has strict comparison. Since $T(A) = \Delta$ is compact, by Theorem 5.3 of [Gong and Lin 2020a], A has continuous scale. $\square$

Remark 4.135. With the last part of Lemma 4.116 in mind and with some obvious modification, one may also have the following forms of inductive limit:

(1) $A = \lim_{n\to\infty}(B_n \oplus C_n, \varphi_{n,n+1})$, in which $B_n \in \mathcal{B}_T$, $K_0(B_n) = \mathrm{Tor}(K_0(A))$, $K_1(B_n) = K_1(A)$, $T(B_n) = \Delta$, and C_n is a simple C^*-algebra as in Theorem 4.107, $K_0(C_n) = G_{n,T} \oplus G_{n,f}$ and $T(C_n) = T(A)$.

(2) $A = \lim_{n\to\infty}(B_n \oplus C_n, \varphi_{n,n+1})$, in which $B_n \in \mathcal{B}_T$, $K_0(B_n) = \mathrm{Inf}(K_0(A))$, $K_1(B_n) = K_1(A)$, $T(B_n) = \Delta$, and C_n is a simple C^*-algebra as in Theorem 4.107, $K_0(C_n) = G_{n,T}$ and $T(C_n) = T(A)$, $\ker \rho_{C_n} = 0$.

In particular, in this modified construction, $B_n = B_1$ for all $n \geq 1$. However, while $K_0(C_n)$ is finitely generated, $K_i(B_n)$ is not. Thus, one also has the following form:

(3) $A = \lim_{n\to\infty}(B_n \oplus C_n \oplus D_n)$ as in Theorem 4.118 except that $B_n = B_1$ for all $n \geq 1$ (thus $K_i(B_n)$ may not be finitely generated).

Definition 4.136. Let $\mathcal{M}_1$ denote the class of stably projectionless simple C^*-algebras with continuous scale constructed in Theorem 4.118, or in Remark 4.135.

By, Theorem 4.107, there is a simple C^*-algebra A which is an inductive limit of C^*-algebras in $\mathcal{C}_0$ such that A has a unique tracial state, $K_0(A) = \mathbb{Z}$, $\ker \rho_A = K_0(A) = \mathbb{Z}$, and $K_1(A) = \{0\}$. By Corollary 15.7 of [Gong and Lin 2020a], $A \cong \mathcal{Z}_0$, the unique stably finite separable simple C^*-algebra with finite nuclear dimension in the UCT class.

For future usage, let us state the following version of Elliott's theorem [1996] (see also Theorem 1.3 of [Li 2020]).

Theorem 4.137. *Let Δ be a metrizable Choquet simplex, G_0 and G_1 countable abelian groups, and $\rho : G_0 \to \mathrm{Aff}(\Delta)$ a homomorphism with $\rho(G_0) \cap \mathrm{Aff}_+(\Delta) = \{0\}$. There is a simple C^*-algebra $A \in \mathcal{M}_1$ with continuous scale such that*

$$((K_0(A), T(A), \rho_A), K_1(A)) = ((G_0, \Delta, \rho), G_1). \tag{4.138}$$

Moreover, if A is as constructed in Theorem 4.118 or as (3) *in Remark 4.135, then it also satisfies the following conditions: for any finitely generated group $G_{0,T} \oplus G_{f,\mathrm{inf}} \oplus G_{0,\mathrm{tor}} \oplus G_r \subset \underline{K}(A)$, where $G_r \cap K_0(A) = \{0\}$, $G_{0,T} \subset K_0(A)$ is a free subgroup with $G_{0,T} \cap \ker \rho_A = \{0\}$, $G_{f,\mathrm{inf}} \subset \ker \rho_A$ is a free subgroup, $G_{0,\mathrm{tor}} \subset \mathrm{Tor}(K_0(A))$, any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$, and any $\sigma > 0$, there are mutually orthogonal C^*-subalgebras E_n as defined in* [Gong and Lin 2020a, Definition 7.2 and Proposition 7.7] (see 4.117) *with strictly positive element $a_{e,n}$ and C_n, $D_n \in \mathcal{C}_0$ with strictly positive elements $a_{c,n}$ and $a_{d,n}$, respectively, satisfying*

$$G_{0,T} \oplus G_{0,\mathrm{tor}} \oplus G_{0,\mathrm{inf}} \oplus G_r \subset [\iota_n](E_n \oplus C_n \oplus D_n), \quad K_0(E_n) = \mathrm{Tor}(K_0(E_n)), \tag{4.139}$$

$$G_{0,T} \subset \iota_{n*0}(K_0(C_n)), \quad G_{0,\mathrm{inf}} \subset \iota_{n*0}(K_0(D_n)), \quad G_{0,\mathrm{tor}} \subset \iota_{n*0}(E_n), \tag{4.140}$$

$$\iota_{n*0}(K_0(C_n)) \cap \ker \rho_A = \{0\}, \quad \iota_{n*0}(K_0(D_n)) \cap \mathrm{Tor}(K_0(A)) = \{0\}, \tag{4.141}$$

$$\iota_{n*0}(K_0(D_n)) \subset \ker \rho_A, \tag{4.142}$$

where $\iota_n : E_n \oplus C_n \oplus D_n \to A$ is the embedding,

$$a \approx_\varepsilon \varphi_{e,n}(a) \oplus \varphi_{c,n}(a) \oplus \varphi_{d,n}(a) \quad \textit{for all } a \in \mathcal{F}, \tag{4.143}$$

where $\varphi_{e,n} : A \to E_n$, $\varphi_{c,n} : A \to C_n$ and $\varphi_{d,n} : A \to D_n$ are completely positive contractive linear maps which are $\mathcal{F}$-$\varepsilon/2$-multiplicative,

$$d_\tau(a_{e,n}) + d_\tau(a_{d,n}) < \sigma \quad \textit{and} \quad d_\tau(a_{c,n}) > 1 - \sigma \quad \textit{for all } \tau \in T(A), \tag{4.144}$$

$$\lambda_s(C_n) > 1 - \sigma \quad \textit{and} \quad \lambda_s(D_n) > 1 - \sigma. \tag{4.145}$$

Proof. Let $A = \lim(A_n = B_n \oplus C_n' \oplus D_n', \varphi_{n,m})$ be as in Theorem 4.118 such that (4.138) holds (or A is as in (3) of Remark 4.135). Note that the $\varphi_{n,n+1}$ are injective, so we can regard A_n as a subalgebra of A. Choose m with the strictly positive elements $b \in B_m$, $c \in C_m'$ and $d \in D_m'$ such that

(i) $\mathcal{F} \subset_{\varepsilon/10} A_m = B_m \oplus C_m' \oplus D_m'$,

(ii) $G_{0,T} \oplus G_{0,\mathrm{tor}} \oplus G_{0,\mathrm{inf}} \oplus G_r \subset [\iota](A_m)$, and

$$G_{0,T} \subset \iota_{*0}(K_0(C_m')), \quad G_{0,\mathrm{inf}} \subset \iota_{*0}(K_0(D_m')), \quad G_{0,\mathrm{tor}} \subset \iota_{*0}(B_m),$$

(iii) $d_\tau(b + d) < \sigma/2$ and $d_\tau(c) > 1 - \sigma/4$ for all $\tau \in T(A)$.

Note that D'_m and C'_m can be written, respectively, as inductive limits of $D_{m,n}$ and $C_{m,n}$ with $D_{m,n}, C_{m,n} \in \mathcal{C}_0$. Also from 4.117, $B_m = \lim_{n\to\infty}(E_{k(n)} \oplus W_n, \Phi_{n,n+1})$, where $E_k = M_{(k!)^2}(A(W, \alpha_k))$ and W_n is in $\mathcal{C}_0$ with $K_0(W_k) = \{0\}$. Since B_m has continuous scale (see 4.117), for n large enough, we have $\lambda_s(E_{k(n)}) > 1-\sigma$ and $\lambda_s(W_n) > 1-\sigma$ [Elliott et al. 2020b]. One can choose n large enough so that

(iv) $\mathcal{F} \subset_{\varepsilon/5} E_{k(n)} \oplus W_n \oplus C_{m,n} \oplus D_{m,n}$,

(v) $G_{0,T} \oplus G_{0,\mathrm{tor}} \oplus G_{0,\mathrm{inf}} \oplus G_r \subset [\iota](E_{k(n)} \oplus W_n \oplus C_{m,n} \oplus D_{m,n})$, and also $G_{0,T} \subset \iota_{*0}(K_0(C_{m,n}))$, $G_{0,\mathrm{inf}} \subset \iota_{*0}(K_0(D_{m,n}))$,

(vi) for the strictly positive element $e_n^A \in C_{m,n}$, $d_\tau(e_n^A) > 1-\sigma/4$ for all $\tau \in T(\widetilde{C}_m)$,

(vii) $\lambda_s(C_{m,n}) > 1-\sigma$, $\lambda_s(D_{m,n}) > 1-\sigma$ and $\lambda_s(W_n) > 1-\sigma$.

Let $C_n = C_{m,n}$, $D_n = D_{m,n} \oplus W_n$, $E_n = E_{k(n)}$ and let $a_{c,n} = e_n^A \in C_n$, $a_{e,n} \in E_n$, $a_{d,n} \in D_n$ be strictly positive elements. Then (4.144) follows from (iii) and (vi). It is standard to construct completely positive contractive linear maps $\varphi_{e,n} : A \to E_n$, $\varphi_{c,n} : A \to C_n$ and $\varphi_{d,n} : A \to D_n$ to finish the proof. $\square$

Remark 4.146. In the statement of Theorem 4.137 we may replace B_n by a simple C^*-algebra of the form B_T, as in Theorem 7.11 of [Gong and Lin 2020a], with continuous scale, and C_n and D_n are simple C^*-algebras which are constructed in Theorem 4.107 with continuous scale and retain (4.139) and (4.140). Moreover, we may assume that $a_{e,n} + a_{c,n} + a_{d,n}$ is a strictly positive element. Note that all $K_i(B_n)$, $K_i(C_n)$ and $K_i(D_n)$ are finitely generated ($i = 0, 1$).

The technical conditions (4.139), (4.140), (4.141) and (4.142) will be used later in the isomorphism theorem in [Gong and Lin 2020b].

5. Range of the Elliott invariant

The following statement implies that, in the case that A is simple, the pairing does not depend on where the unitization occurs.

Proposition 5.1. *Let A be a σ-unital simple C^*-algebra with $a \in \mathrm{Ped}(A)_+ \setminus \{0\}$. Suppose that $\iota : B := \mathrm{Her}(a) \to A$ is the embedding. Then ι_{*0} is an isomorphism and $\rho_B(x)(\iota_T(\tau)) = \rho_A(\iota_{*0}(x))(\tau)$ for all $x \in K_0(B)$ and for all $\tau \in \widetilde{T}(A)$, where $\iota_T : \widetilde{T}(A) \to \widetilde{T}(B)$ is the map induced by ι.*

Proof. We claim that in general, without assuming A is simple, if B is a σ-unital full hereditary C^*-subalgebra of A, then $\iota_{*i} : K_i(B) \to K_i(A)$ is an isomorphism ($i = 0, 1$). This is a reconstruction of the argument in [Brown 1977].

If B is a full corner of A, i.e., $B = pAp$ for some projection $p \in M(A)$, then the claim follows from Corollary 2.6 of [Brown 1977]. In general, let C be the C^*-subalgebra of $A \otimes M_2$ consisting of the sum $\sum a_{ij} \otimes e_{ij}$ such that $a_{11} \in B$, $a_{12} \in \overline{BA}$, $a_{21} \in \overline{AB}$ and $a_{22} \in A$, where $\{e_{ij}\}_{1\le i,j\le 2}$ is a system of matrix units. We identify B with the full corner $B \otimes e_{11}$ and A with the full corner $A \otimes e_{22}$ of C; see the proof of Theorem 2.8 of [Brown 1977]. Let $j_A : A \to C$ and $j_B : B \to C$ be embeddings. Then $(j_A)_{*i}$ and $(j_B)_{*i}$ are isomorphisms ($i = 0, 1$).

On the other hand, $j_B(B)$ and $j_A \circ \iota(B)$ may be identified with $B \otimes e_{11}$ and $B \otimes e_{22}$ in $M_2(B) \subset C$. Denote by $j : M_2(B) \to C$ the embedding. Then $j \circ j_B = j_B$ and $j \circ j_A|_{B\otimes e_{22}} = j_A|_{B\otimes e_{22}}$. Since $j_{*i} = (j \circ j_B)_{*i} = (j \circ j_A|_{B\otimes e_{22}})_{*i} : K_i(B) \to K_i(C)$, one has that $(j_A)_{*i} \circ \iota_{*i} = (j_A \circ \iota)_* = (j \circ j_B)_{*i} = (j_B)_{*i}$ is also an isomorphism. Since $(j_A)_{*i}$ is an isomorphism, so is ι_{*i} ($i = 0, 1$). This proves the claim.

The lemma follows from the claim and the commutative diagram (2.18) immediately. □

Let us state the following result (see Definition 2.19), which also holds if we replace the condition that A has finite nuclear dimension by that A is $\mathcal{Z}$-stable (i.e., $A \cong A \otimes \mathcal{Z}$) and all 2-quasitraces are traces.

Theorem 5.2. *Let A be a separable finite simple C^*-algebra with finite nuclear dimension. Then $(K_0(A), \Sigma(K_0(A)), \widetilde{T}(A), \widehat{\langle e_A \rangle}, \rho_A)$ is a scaled simple ordered group pairing* (see Definitions 2.7 and 2.15).

Proof. It follows from [Winter 2012; Tikuisis 2014] that A is $\mathcal{Z}$-stable. By [Rørdam 2004, Corollary 5.1] (see also [Kirchberg 1997]), $\widetilde{T}(A) \neq \{0\}$. Then, if A is unital, $(K_0(A), K_0(A)_+, [1_A])$ is a weakly unperforated simple ordered group (see [Gong et al. 2000]) with the scale determined by the order unit $[1_A]$, and $g \in K_0(A)_+ \setminus \{0\}$ if and only if $\rho_A(g)(\tau) > 0$ for all $\tau \in T(A)$. So the unital case follows. Suppose that A is not unital and $K_0(A)_+ \neq \{0\}$. Let $x \in K_0(A)_+ \setminus \{0\}$. Then there is a nonzero projection $p \in M_r(A)$ for some integer $r \geq 1$ such that $[p] = x$. If follows that $A_1 := pAp$ is a unital simple C^*-algebra with finite nuclear dimension (see Corollary 2.8 of [Winter and Zacharias 2010]). Therefore $(K_0(A_1), K_0(A_1)_+)$ is a weakly unperforated simple ordered group such that $g \in K_0(A_1)_+ \setminus \{0\}$ if and only if $\rho_{A_1}(g)(\tau) > 0$ for all $\tau \in T(A_1)$. Note that $A \otimes \mathcal{K} \cong A_1 \otimes \mathcal{K}$. It follows that $(K_0(A), \widetilde{T}, \rho)$ is a simple ordered group pairing.

Let $e_A \in A$ be a strictly positive element with $\|e_A\| = 1$ and let $s(\tau) = d_\tau(e_A)$ for all $\tau \in \widetilde{T}(A)$. If $p \in A$ is a projection, then, $f_{1/n}(e_A)p \approx_{1/2} p$ for some integer $n \geq 1$. It follows that there is a projection $q \in \mathrm{Her}(f_{1/n}(e_A))$ such that $[q] = [p]$. It follows that

$$f_{1/2n}(e_A)q = q \quad \text{and} \quad (1-q)f_{1/2n}(e_A)(1-q) \neq 0.$$

Since A is simple, this implies that $\tau(p) = \tau(q) < d_\tau(e_A)$. Then

$$\Sigma(K_0(A)) = \{g \in G_+ : g = [p] \text{ for some projection } p \in A\} = \{g \in G_+ : \rho(g) < s\}.$$

It follows that $(K_0(A), \Sigma(K_0(A)), \widetilde{T}(A), s, \rho_A)$ is a scaled simple ordered group pairing. (Even though $K_0(A)_+ \neq \{0\}$, it is still possible that $\Sigma(K_0(A)) = \{0\}$.)

Now assume that $K_0(A)_+ = \{0\}$. In other words, A is stably projectionless. It follows from Remark 5.2 of [Elliott et al. 2020b], for example, that one may choose $a \in A_+ \setminus \{0\}$ such that $A_1 := \overline{aAa}$ has continuous scale. It follows from (1) of Theorem 5.3 of [Elliott et al. 2020b] that $T(A_1)$ is nonempty and compact (see also [Lin 1991, Theorem 3.3; Rørdam 2004, Corollary 5.1]). By [Pedersen 1979, 5.2.2], every tracial state $\tau \in T(A_1)$ extends to a lower semicontinuous trace on A, which is finite on $\mathrm{Ped}(A)$ as A is simple. Again, since A is simple, the extension is unique. It follows that $T(A_1)$ is a basis for the cone $\widetilde{T}(A)$. Since $\mathrm{Aff}(\widetilde{T}(A))$ is a lattice (see [Pedersen 1966, Corollary 3.3; 1969, Theorem 3.1]), $\widetilde{T}(A)$ is a convex topological cone with a Choquet simplex as its base. For any $x \in K_0(A_1)$, by Corollary A.7 of [Elliott et al. 2020a], $\rho_{A_1}(x)(\tau) = 0$ for some $\tau \in T(A_1)$. Since $A \otimes \mathcal{K} \cong A_1 \otimes \mathcal{K}$, this implies that $(K_0(A), \{0\}, \widetilde{T}(A), \widehat{\langle e_A \rangle}, \rho_A)$ is a scaled simple ordered group pairing (see Proposition 5.1). $\square$

The following range theorem was given in [Elliott 1996] (see also [Li 2020]).

Theorem 5.3. *Let $(G_0, \Sigma(G_0), T, s, \rho)$ be a scaled simple ordered group pairing and G_1 be a countable abelian group. Then there is a simple separable amenable C^*-algebra A which satisfies the UCT such that*

$$((K_0(A), \Sigma(K_0(A)), \widetilde{T}(A), \widehat{\langle e_A \rangle}, \rho_A), K_1(A)) = ((G_0, \Sigma(G_0), T, s, \rho), G_1). \quad (5.4)$$

A is unital if and only if $\Sigma(G_0)$ has a unit u. (This means that u is the nonzero maximum in $\Sigma(G_0)$ and $\rho(u) = s$. See Definition 2.7.) If $\rho(G_0) \cap \mathrm{Aff}_+(T) \neq \{0\}$, then A can be chosen to have rationally generalized tracial rank at most one (see Definition 2.28) and be an inductive limit of subhomogeneous C^*-algebras of spectra with dimension no more than 3. If $\rho(G_0) \cap \mathrm{Aff}_+(T) = \{0\}$, then A is stably projectionless and A can be chosen to have generalized tracial rank one (see Definition 2.28) and be locally approximated by subhomogeneous C^*-algebras with the spectra having dimension no more than 3.

If $G_1 = \{0\}$ and G_0 is torsion free, then A can be chosen to be an inductive limit of 1-dimensional NCCW complexes in $\mathcal{C}_0$.

Proof. Let us first consider the case that $\Sigma(G_0)$ has a unit u. Let

$$\Delta := \{t \in T : \rho(u)(t) = 1\}.$$

Then Δ is a base for the cone T. Recall that $\rho(G) \subset \mathrm{Aff}(T)$. Therefore Δ is a compact convex subset. Moreover, it is a base for T. Since T has a metrizable Choquet simplex as a base, $\mathrm{Aff}(T) = \mathrm{Aff}(\Delta)$ is a lattice. Therefore Δ is a Choquet simplex. Let $G_{0+} = \{g \in G : \rho(g) > 0\} \cup \{0\}$. It follows from Theorem 13.50 of [Gong et al. 2020a] that there is a unital simple C^*-algebra A which has rationally generalized tracial rank at most one and is an inductive limit of subhomogeneous

C^*-algebras of spectra with dimension no more than 3 such that

$$((K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), \rho_A), K_1(A)) = ((G_0, G_{0+}, u, G_1, \Delta, \rho), G_1).$$

Put $\widetilde{T}(A) := \{r\tau : r \in \mathbb{R}_+, \tau \in T(A)\}$. Then $\widetilde{T}(A) = T$. Also, $\Sigma(K_0(A)) = \Sigma(G_0)$. This proves the case that $\Sigma(G_0)$ has a unit.

Consider the case $\rho(G_0) \cap \mathrm{Aff}_+(T) \neq \{0\}$ and $\Sigma(G_0)$ has no unit. Choose $v \in G_0$ such that $\rho(v) \in \mathrm{Aff}_+(T) \setminus \{0\}$. Put $\Sigma_1(G) = \{g \in G_+ : \rho(g) < v\} \cup \{v\}$. Since $\rho(v) \in \mathrm{Aff}_+(T)$, as above, $\Delta := \{t \in T : \rho(v)(t) = 1\}$ is a Choquet simplex. Then, by what has been shown, there is a C^*-algebra A_1 which has rationally tracial rank at most one such that

$$\begin{aligned}((K_0(A_1), \Sigma(K_0(A)), T(A_1), \rho_A([1_{A_1}]), \rho_{A_1}), K_1(A_1)) \\ = ((G_0, \Sigma_1(G_0), T_1, \rho'), G_1), \quad (5.5)\end{aligned}$$

where $T_1 := \{r\xi : r \in \mathbb{R}_+, \xi \in \Delta\} = T$ and $\rho' : G \to \mathrm{Aff}(T_1)$ is defined to be the same as ρ, when we identify T_1 with T. Choose an element $e_A \in A_1 \otimes \mathcal{K}$ such that $\widehat{\langle e_A \rangle}(\tau) = s(\tau)$ for all $\tau \in \widetilde{T}(A_1) = T_1 = T$ (see Theorem 5.5 of [Brown et al. 2008]). Define $A = \overline{e_A(A_1 \otimes \mathcal{K})e_A}$. One then checks that

$$((K_0(A), \Sigma(K_0(A)), \widetilde{T}(A), \widehat{\langle e_A \rangle}, \rho_A), K_1(A)) \cong ((G_0, \Sigma(G_0), T, s, \rho), G_1). \quad (5.6)$$

Now we consider the case that $\rho(G) \cap \mathrm{Aff}_+(T) = \{0\}$. Let Δ be a base of T which is a Choquet simplex. Define $\rho' : G \to \mathrm{Aff}(\Delta)$ to be the same map as ρ by restricting a function in $\mathrm{Aff}(T)$ to $\mathrm{Aff}(\Delta)$. By Theorem 4.118, there is a simple C^*-algebra $A_1 \in \mathcal{D}$ with continuous scale which is an inductive limit of $B_n \oplus C_n \oplus D_n$, where B_n is locally approximated by subhomogeneous C^*-algebras with spectra having dimension no more than 3 (see 4.117), and $C_n \oplus D_n$ is an inductive limit of C^*-algebras in $\mathcal{C}_0$ such that

$$(K_0(A_1), T(A_1), \rho_{A_1}, K_1(A_1)) = (G_0, G_1, \Delta, \rho', G_1). \quad (5.7)$$

Choose $e_A \in (A_1 \otimes \mathcal{K})_+ \setminus \{0\}$ such that $\widehat{\langle e_A \rangle}(t) = \lim_{n\to\infty} t(e_A^{1/n}) = s(t)$ for all $t \in \Delta$. Define $A := \overline{e_A(A_1 \otimes \mathcal{K})e_A}$. Then A has generalized tracial rank one (see Definition 2.28). One then checks that

$$((K_0(A), \Sigma(K_0(A)), \widetilde{T}(A), \widehat{\langle e_A \rangle}, \rho_A), K_1(A)) = ((G_0, \{0\}, T, s, \rho), G_1). \quad (5.8)$$

The last part of the lemma follows immediately from Theorem 4.107. □

Corollary 5.9. *Let A_1 be a simple separable C^*-algebra in $\mathcal{D}$ with continuous scale, U be an infinite-dimensional UHF algebra and $A = A_1 \otimes U$.*

There exists an inductive limit algebra B as constructed in Theorem 4.118 such that $A = A_1 \otimes U$ and B have the same Elliott invariant. Moreover, the C^*-algebra B has the following properties:

Let G_0 be a finitely generated subgroup of $K_0(B)$ with decomposition $G_0 = G_{00} \oplus G_{01}$, where G_{00} vanishes under all states of $K_0(B)$. Suppose $\mathcal{P} \subset \underline{K}(B)$ is a finite subset which generates a subgroup G such that $G_0 \subset G \cap K_0(B)$.

Then, for any $\epsilon > 0$, any finite subset $\mathcal{F} \subset B$, any $1 > r > 0$, and any positive integer K, there is an $\mathcal{F}$-ϵ-multiplicative map $L : B \to B$ such that:

(1) *$[L]|_{\mathcal{P}}$ is well defined.*

(2) *$[L]$ induces the identity maps on G_{00}, $G \cap K_1(B)$, $G \cap K_0(B, \mathbb{Z}/k\mathbb{Z})$ and $G \cap K_1(B, \mathbb{Z}/k\mathbb{Z})$ for $k = 1, 2, \ldots,$ and $i = 0, 1$.*

(3) *$\|\rho_B \circ [L](g)\| \le r\|\rho_B(g)\|$ for all $g \in G \cap K_0(B)$, where ρ_B is the canonical positive homomorphism from $K_0(B)$ to $\mathrm{Aff}(T(B))$.*

(4) *For any element $g \in G_{01}$, we have $g - [L](g) = Kf$ for some $f \in K_0(B)$.*

(5) *$d_\tau(e_0) < r$ for all $\tau \in T(B)$, where $e_0 \in \overline{L(B)BL(B)}$ is strictly positive.*

Proof. Consider $\mathrm{Ell}(A_1)$. By Theorem 4.118, there is an inductive system $B_1 = \varinjlim(T_i \oplus C_i, \psi_{i,i+1})$ (where $T_i := B_i \oplus D_i$ in Theorem 4.118) such that

(i) $\rho_{T_i} = 0 : K_0(T_i) \to \mathrm{Aff}(T(T_i))$ and $C_i \in \mathcal{C}_0$ with $K_1(C_i) = \{0\}$,

(ii) for the strictly positive element $e_{T_i} \in T_i$, $\lim \tau(\varphi_{i,\infty}(e_{T_i})) = 0$ uniformly on $\tau \in T(B_1)$,

(iii) $\ker(\rho_{B_1}) = \bigcup_{i=1}^{\infty}(\psi_{i,\infty})_{*0}(K_0(T_i))$, and

(iv) $\mathrm{Ell}(B_1) = \mathrm{Ell}(A_1)$.

Put $B = B_1 \otimes U$. Then $\mathrm{Ell}(A) = \mathrm{Ell}(B)$. Let $\mathcal{P} \subset \underline{K}(B)$ be a finite subset, and let G be the subgroup generated by $\mathcal{P}$, which we may assume to contain G_0. Then there is a positive integer M' such that $G \cap K_*(B, \mathbb{Z}/k\mathbb{Z}) = \{0\}$ if $k > M'$. Put $M = M'!$. Then $Mg = 0$ for any $g \in G \cap K_*(B, \mathbb{Z}/k\mathbb{Z})$, $k = 1, 2, \ldots$.

Let $\varepsilon > 0$, a finite subset $\mathcal{F} \subset B$, and $0 < r < 1$ be given. Choose a finite subset $\mathcal{G} \subset B$ and $0 < \varepsilon' < \varepsilon$ such that $\mathcal{F} \subset \mathcal{G}$ and for any $\mathcal{G}$-ϵ'-multiplicative map $L : B \to B$, the map $[L]_{\mathcal{P}}$ is well defined, and $[L]$ is a homomorphism on G.

Since $B = B_1 \otimes U$, we may write $U = \varinjlim(M_{m(n)}, \iota_{n,n+1})$, where $m(n) \mid m(n+1)$ and $\iota_{n,n+1} : M_{m(n)} \to M_{m(n+1)}$ is defined by $a \mapsto a \otimes 1_{m(n+1)}$. Choosing a sufficiently large i_0 and n_0, we may assume that $[\psi_{i_0,\infty}](\underline{K}((T_{i_0} \oplus C_{i_0}) \otimes M_{m(n_0)})) \supset G$. In particular, we may assume, by (i) and (iii) above, that $\rho_{T_i \otimes M_{m(n_0)}} = 0$ and $G \cap \ker \rho_{B_1 \otimes M_{m(n_0)}} \subset (\psi_{i_0,\infty})_{*0}(K_0(T_i) \otimes M_{m(n_0)})$. Let $G' \subset \underline{K}((T_{i_0} \oplus C_{i_0}) \otimes M_{m(n_0)})$ be such that $[\psi_{i_0,\infty}](G') \supset G$.

One may assume that, for each $f \in \mathcal{G}$, there exists $i > i_0, n_0$ such that

$$f = (f_0 \oplus f_1) \otimes 1_m \in (T_i' \oplus C_i') \otimes M_m \tag{5.10}$$

for some $f_0 \in T_i'$, $f_1 \in C_i'$, and $m > 2MK/r$, where $m = m(n)/m(i)$, $T_i' = \psi_{i,\infty}'(T_i \otimes M_{m(i)})$, $C_i' = \psi_{i,\infty}'(C_i \otimes M_{m(i)})$, and where $\psi_{i,\infty}' = \psi_{i,\infty} \otimes \iota_{i,\infty}$. Moreover, one may assume that $\tau(1_{T_i'}) < r/2$ for all $\tau \in T(A_1)$.

Choose a large n such that $m = M_0 + l$ with M_0 divisible by KM and $0 \le l < KM$. Then define a map $L : (T_i' \oplus C_i') \otimes M_m \to (T_i' \oplus C_i') \otimes M_m$ to be

$$L((f_{i,j} \oplus g_{i,j})_{m\times m}) = (f_{i,j})_{m\times m} \oplus E_l (g_{i,j})_{m\times m} E_l,$$

where

$$E_l = \mathrm{diag}(\underbrace{1_{(C_i')^\sim}, 1_{(C_i')^\sim}, \dots, 1_{(C_i')^\sim}}_{l}, \underbrace{0, 0, \dots, 0}_{M_0}).$$

Note that L is also a contractive completely positive linear map from $(T_i' \oplus C_i') \otimes M_m$ to B, where we identify B with $B \otimes M_m$. (Note also that $E_l \notin C_i' \otimes M_m$ but $E_l (g_{ij}) E_l \in C_i' \otimes M_m$.) We then extend L to a completely positive linear map from B to B. Also define $R : (T_i' \oplus C_i') \otimes M_m \to T_i' \oplus C_i'$ to be

$$R\begin{pmatrix} f_{1,1} \oplus g_{1,1} & f_{1,2} \oplus g_{1,2} & \cdots & f_{1,m} \oplus g_{1,m} \\ f_{2,1} \oplus g_{2,1} & f_{2,2} \oplus g_{2,2} & \cdots & f_{2,m} \oplus g_{2,m} \\ & & \ddots & \\ f_{m,1} \oplus g_{m,1} & f_{m,2} \oplus g_{m,2} & \cdots & f_{m,m} \oplus g_{m,m} \end{pmatrix} = g_{1,1}, \tag{5.11}$$

where $f_{j,k} \in T_i'$ and $g_{j,k} \in C_i'$, and extend it to a contractive completely positive linear map $B \to B$, where $T_i' \oplus C_i'$ is regarded as a corner of $(T_i' \oplus C_i') \otimes M_m \subset B$. Then L and R are $\mathcal{G}$-ϵ'-multiplicative. Hence $[L]|_{\mathcal{P}}$ is well defined. Moreover,

$$d_\tau(e_0) = d_\tau(L(e_B)) < d_\tau(e_{T_i'}) + \frac{l}{m} < \frac{r}{2} + \frac{MK}{2MK/r} = r \quad \text{for all } \tau \in T(B),$$

where e_B and $e_{T_i'}$ are strictly positive elements in B and T_i', respectively.

Note that, for any f in the form (5.10), if f is written in the form $(f_{jk} \oplus g_{jk})_{m\times m}$, then $g_{jj} = g_{11}$ and $g_{jk} = 0$ for $j \neq k$. Hence one has

$$f = L(f) + \bar{R}(f),$$

where $\bar{R}(f)$ may be written as

$$\bar{R}(f) = \mathrm{diag}\{\underbrace{0, 0, \dots, 0}_{l}, \underbrace{(0 \oplus g_{1,1}), \dots, (0 \oplus g_{1,1})}_{M_0}\}.$$

Hence for any $g \in G$,

$$g = [L](g) + M_0[R](g).$$

Then, if $g \in (G_{0,1})_+ \subset (G_0)_+$, one has

$$g - [L](g) = M_0[R](g) = K\left(\left(\frac{M_0}{K}\right)[R](g)\right).$$

Also if $g \in G \cap K_i(B, \mathbb{Z}/k\mathbb{Z})$ $(i = 0, 1)$, one also has

$$g - [L](g) = M_0[R](g).$$

Since $Mg = 0$ and $M \mid M_0$, one has $g - [L](g) = 0$.

Since L is the identity on $\psi'_{i,\infty}(T_i \otimes M_{m(i)})$ and $i > i_0$, by (iii), $[L]$ is the identity map on $G \cap \ker \rho_B$. Since $K_1(S_i) = 0$ for all i, L induces the identity map on $G \cap K_1(B)$. It follows that L is the desired map. $\square$

Lemma 5.12. *Let $C = A(F_1, F_2, \beta_0, \beta_1) \in \mathcal{C}_0$ and let $N_0 \geq 1$ be an integer. There exists $\sigma > 0$ satisfying the following condition: For any order preserving homomorphism $\kappa : K_0(\widetilde{C}) \to \mathbb{R}$ such that, for any $x \in K_0(\widetilde{C})_+ \setminus \{0\}$ with $N_0\kappa(x) > 1$ and $\kappa([1_{\widetilde{C}}]) = 1$, there exists $t \in T(C)$ such that*

$$t(h) \geq \sigma \int_{s\in[0,1]} T(\lambda(h)(s))\, d\mu(s) \quad \textit{for all } h \in C_+, \tag{5.13}$$

$$\kappa(x) = \rho_C(x)(t) \quad \textit{for all } x \in K_0(\widetilde{C}), \tag{5.14}$$

where $\lambda : C \to C([0, 1], F_2)$ is the natural embedding and $T(b) = \sum_{j=1}^{k} \mathrm{tr}_j(\psi_j(b))$ for all $b \in F_2$, where $F_2 = \bigoplus_{j=1}^{k} M_{r(j)}$, $\psi_j : F_2 \to M_{r(j)}$ is the projection map, tr_j is the normalized trace on $M_{r(j)}$, and μ is Lebesgue measure.

Proof. Let us write $F_1 = \bigoplus_{i=1}^{l} M_{R(i)}$. Denote by $q_i : F_1 \to M_{R(i)}$ the projection map and $\bar{t}_i$ the tracial state of $M_{R(i)}$, $i = 1, 2, \ldots, l$. Let $\pi_e : C \to F_1$ be the quotient map and $T : K_0(F_2) \to \mathbb{R}$ be defined by $T(x) = \sum_{j=1}^{k} \rho_{M_{r(j)}} \circ \psi_{j*0}(x)(\mathrm{tr}_j)$ for all $x \in K_0(F_2)$.

Define homomorphisms $\beta'_0 : \mathbb{C} \to F_2$ by $\beta'_0(1_{\mathbb{C}}) := 1_{F_2} - \beta_0(1_{F_1})$ and $\beta'_1(1_{\mathbb{C}}) := 1_{F_2} - \beta_1(1_{F_1})$ (at least one of them is not zero as C is not unital). We may write $\widetilde{C} = A(F_1^{\sim}, F_2, \beta_0^{\sim}, \beta_1^{\sim})$, where $F_1^{\sim} = F_1 \oplus \mathbb{C}$ and $\beta_i^{\sim} = \beta_i \oplus \beta'_i$, $i = 0, 1$. Denote by $q_{l+1} : F_1^{\sim} \to \mathbb{C}$ the projection map and $\pi_e^{\sim} : \widetilde{C} \to F_1^{\sim}$ the quotient map which extends π_e. Recall that $T([\beta_0^{\sim}(\pi_e^{\sim}(1_{\widetilde{C}}))]) = k$ and for any projection $p \in M_m(\widetilde{C})$, $T(p) \leq mT([\beta_0^{\sim}(\pi_e^{\sim}(1_{\widetilde{C}}))]) = mk$.

Let $p_1, p_2, \ldots, p_s \in M_m(\widetilde{C})$ be a set of minimal projections for some integer $m \geq 1$ such that they generate $K_0(\widetilde{C})_+$ (see Theorem 3.15 of [Gong et al. 2020a]). There is $\sigma_0 > 0$ such that $\sigma_0 mk < \frac{1}{2}$. Choose $\sigma := \sigma_0/2N_0$. Since $N_0\kappa(x) > 1$ for all $x \in K_0(\widetilde{C})_+ \setminus \{0\}$, for any $p \in \{p_1, p_2, \ldots, p_s\}$, we have

$$N_0\kappa([p]) - \sigma_0 T \circ (\beta_0 \circ \pi_e)(p) > 0. \tag{5.15}$$

Define $\Gamma : K_0(\widetilde{C}) \to \mathbb{R}$ by

$$\Gamma(x) = \kappa(x) - \sigma_0 T \circ (\beta_0^{\sim} \circ \pi_e)_{*0}(x) \quad \text{for all } x \in K_0(\widetilde{C}). \tag{5.16}$$

By Proposition 3.5 of [Gong et al. 2020a], $\pi_{e*0} : K_0(\widetilde{C}) \to K_0(F_1^{\sim})$ is an order embedding. It follows from Theorem 3.2 of [Goodearl and Handelman 1976] that there is an order preserving homomorphism $\Gamma^\sim : K_0(F_1^\sim) \to \mathbb{R}$ such that $\Gamma^\sim \circ \pi_{e*0} = \Gamma$. Put $\alpha_j := \Gamma^\sim([e_j]) \geq 0$, where $e_j := q_j \circ \pi_e^\sim(1_{\widetilde{C}})$, $j = 1, 2, \ldots, l+1$. Define, for $(f, b) \in C^\sim = \{C([0, 1], F_2) \oplus F_1^\sim : f(0) = \beta_0^\sim(b) \text{ and } f(1) = \beta_1^\sim(b)\}$,

$$t((f, b)) = \sigma \sum_{j=1}^{k} \int_{s\in(0,1)} \mathrm{tr}_j(f)(s)\, dm(s) + \sum_{i=1}^{l+1} \alpha_i \bar{t}_i(q_i(\pi_e^\sim(b))). \tag{5.17}$$

For any projection $p = (p, \pi_e^\sim(p)) \in M_N(\widetilde{C})$ (for some $N \geq 1$), we have

$$\begin{aligned}\rho_C([p])(t) &= t(p, \pi_e^\sim(p)) \\ &= \sigma \sum_{j=1}^{k} \int_{s\in(0,1)} \mathrm{tr}_j(p)(s)\, dm(s) + \sum_{i=1}^{l+1} \alpha_i \bar{t}_i(q_i(\pi_e^\sim(p))) \\ &= \sigma_0 T \circ (\beta_0^\sim \circ \pi_e)_{*0}([p]) + \Gamma^\sim([\pi_e^\sim(p)]) = \kappa([p]).\end{aligned} \tag{5.18}$$

Moreover, if $h = (f, b) \in C_+$, we have

$$t(h) \geq \sigma \sum_{j=1}^{k} \int_{s\in(0,1)} \mathrm{tr}_j(f)(s)\, dm(s) = \sigma \int_{s\in[0,1]} T(\lambda(h))\, d\mu(s). \qquad \square$$

Lemma 5.19. *Let $C \in \mathcal{C}_0$ and let $G_1 \subset \mathrm{Aff}(\Delta)$ be a countable subgroup such that $G_1 \cap \mathrm{Aff}_+(\Delta) = \{0\}$, where Δ is a metrizable Choquet simplex. Let $\eta : K_0(\widetilde{C}) \to G_1 + \mathbb{Z}1_\Delta$ be an order preserving homomorphism, where $1_\Delta \in \mathrm{Aff}(\Delta)$ is the constant function with value 1 (with $\mathrm{Aff}_+(\Delta)$ as the (strictly) positive cone of $\mathrm{Aff}(\Delta)$ — see Definition 2.2). Then there is a morphism $\eta^\sim : \mathrm{Cu}^\sim(\widetilde{C}) \to G_1 \sqcup \mathrm{LAff}_+^\sim(\Delta)$ in* **Cu** *such that $\eta^\sim|_{K_0(\widetilde{C})} = \eta$.*

Proof. Write $C = A(\beta_0, \beta_1, F_1, F_2)$, where $F_1 = \bigoplus_{i=1}^k M_{R(i)}$, $F_2 = \bigoplus_{j=1}^l M_{r(i)}$, and $\beta_i : F_1 \to F_2$ are homomorphisms. Recall that $\mathrm{Aff}(T(C))$ is identified as a subspace of $C([0, 1], \mathbb{R}^l) \oplus \mathbb{R}^k$ and $\mathrm{Cu}^\sim(\widetilde{C})$ is identified with a subgroup of

$$K_0(\widetilde{C}) \sqcup \mathrm{LSC}([0, 1], \mathbb{Z}^l) \oplus \mathbb{Z}^{k+1} \tag{5.20}$$

(see Proposition 3.6 of [Gong and Lin 2020a]). Since $K_0(\widetilde{C})_+$ is finitely generated (see Theorem 3.15 of [Gong et al. 2020a]) by $g_1, g_2, \ldots, g_s$, there exists an integer $N_0 \geq 1$ such that

$$N_0 \eta(x) > 1_\Delta \quad \text{for all } x \in K_0(\widetilde{C})_+ \setminus \{0\}. \tag{5.21}$$

Let $\sigma > 0$ be given by Lemma 5.12 for $2N_0$.

Recall that the map $\rho : K_0(\widetilde{C}) \to \mathrm{Aff}(T(\widetilde{C}))$ is injective (see Proposition 3.5 of [Gong et al. 2020a] for example). By [Tsang 2005, Lemma 5.1], we may write $\mathrm{Aff}(\Delta) = \lim_{n\to\infty}(\mathbb{R}^{a(n)}, \lambda_n)$, where $a(n) \geq 1$ are integers and $\lambda_n : \mathbb{R}^{a(n)} \to \mathbb{R}^{a(n+1)}$ is an order preserving map which also preserves the canonical order unit. Let

$G_2 := \eta(K_0(C))$. Note that G_2 is a finitely generated subgroup of G_1. Therefore there is a sequence of subgroups $G_{n,2} \subset \mathbb{R}^{a(n)}$ such that $G_2 = \lim_{n\to\infty}(G_{n,2}, \lambda_n)$ and $\lambda_n|_{G_{n,2}}$ is injective. Let $G_{n,2}^{\sim} = G_{n,2} + \mathbb{Z}\cdot 1$ and $G_2^{\sim} := G_2 + \mathbb{Z}\cdot 1_\Delta$.

Since $K_0(\widetilde{C})_+$ is finitely generated, one obtains (for all large n) an order preserving map $\eta_n : K_0(\widetilde{C}) \to \mathbb{R}^{a(n)}$ such that $\lambda_{n,\infty} \circ \eta_n = \eta$, where $\lambda_{n,\infty}$ is induced by the inductive limit system. There exists $n_1 \geq 1$ such that, for all $n \geq n_1$,

$$2N_0\eta_n(x) > 1 \quad \text{for all } x \in K_0(\widetilde{C})_+ \setminus \{0\}. \tag{5.22}$$

Write $\mathbb{R}^{a(n)} = \bigoplus_{i=1}^{a(n)} \mathbb{R}_i$. Define $q_i : \mathbb{R}^{a(n)} \to \mathbb{R}_i$ to be the projection. Consider the order preserving map $q_i \circ \eta_n : K_0(\widetilde{C}) \to \mathbb{R}$ which preserves the order unit. Since ρ is injective, we may view $K_0(\widetilde{C})$ as an order subgroup of $C([0,1], \mathbb{R}^l) \oplus \mathbb{R}^{k+1}$. By Lemma 5.12, there is an order preserving map $\gamma_{n,i} : C([0,1], \mathbb{R}^l) \oplus \mathbb{R}^{k+1} \to \mathbb{R}_i$ such that $\gamma_{n,i}|_{K_0(\widetilde{C})} = q_i \circ \eta_n$ and

$$\gamma_{n,i}(\hat{h}) \geq \frac{\sigma}{2} \int_{[0,1]} T(\hat{h})(t)\, d\mu, \tag{5.23}$$

where $h \in \widetilde{C}_+ \setminus \{0\}$, which we identify with the corresponding element in $C([0,1], F_2)$, and $\hat{h}$ is the associated affine function (for all large $n \geq n_1$).

Define

$$\gamma_n : C([0,1], \mathbb{R}^l) \oplus \mathbb{R}^{k+1} \to \mathbb{R}^{a(n)}, \quad g \mapsto (\gamma_{n,1}(g), \gamma_{n,2}(g), \ldots, \gamma_{n,a(n)}(g))$$

for all g. Note that $\gamma_n|_{K_0(\widetilde{C})} = \eta_n$ for all n. Define $\gamma : C([0,1], \mathbb{R}^l) \oplus \mathbb{R}^{k+1} \to \mathrm{Aff}(\Delta)$ by $\gamma(g) := \lambda_{n,\infty} \circ \gamma_n$. Then γ is an order preserving map which preserves the order unit. It is linear and therefore continuous (with the supremum norm). The condition $\lambda_{n,\infty} \circ \eta_n = \eta$ implies that $\gamma|_{K_0(\widetilde{C})} = \eta$. Note also that the condition (5.23) implies that, for each $f \in (C([0,1], \mathbb{R}^l) \oplus \mathbb{R}^{k+1})_+ \setminus \{0\}$,

$$\gamma(f)(t) > 0 \quad \text{for all } t \in \Delta. \tag{5.24}$$

Then $\gamma^{\sim}$ extends to an order preserving affine map from $\mathrm{LSC}([0,1], \mathbb{Z}^l) \oplus \mathbb{Z}^{k+1}$ to $\mathrm{LAff}_+^{\sim}(\Delta)$. Define $\eta^{\sim} : \mathrm{Cu}^{\sim}(\widetilde{C}) \to G_1 \sqcup \mathrm{LAff}_+^{\sim}(\Delta)$ by $\eta^{\sim}|_{K_0(\widetilde{C})} = \eta$ and $\eta^{\sim}|_{\mathrm{LSC}([0,1],\mathbb{Z}^l)\oplus\mathbb{Z}^{k+1}} = \gamma^{\sim}|_{\mathrm{LSC}([0,1],\mathbb{Z}^l)\oplus\mathbb{Z}^{k+1}}$. It is then straightforward to verify that $\eta^{\sim}$ is a map in **Cu**. □

Theorem 5.25. *Let $A \in \mathcal{D}$ be with continuous scale and $C \in \mathcal{C}_0$. Suppose that there is a strictly positive homomorphism $\alpha : K_0(\widetilde{C}) \to K_0(\widetilde{A})$ such that $\alpha([1_{\widetilde{C}}]) = [1_{\widetilde{A}}]$ and $\alpha(K_0(C)) \subset K_0(A)$. Then there exists a homomorphism $h : C \to A$ such that $h_{*0} = \alpha$.*

Proof. Since A is a stably projectionless $\mathcal{Z}$-stable simple C^*-algebra with stable rank one (i.e., all C^*-algebras in $\mathcal{D}$ have stable rank one [Elliott et al. 2020b, Proposition 11.11]), by [Robert 2012, Theorem 6.2.3] (see also [Elliott et al. 2020b, Theorem 7.3; Robert and Santiago 2021, Theorem 6.11],

$$\mathrm{Cu}^{\sim}(A) = K_0(A) \sqcup \mathrm{LAff}^{\sim}_{+}(T(A)). \tag{5.26}$$

By Lemma 5.19, α extends to a morphism $\alpha^{\sim} : \mathrm{Cu}^{\sim}(\widetilde{C}) \to \mathrm{Cu}^{\sim}(\widetilde{A})$ in **Cu**. Let $e_C \in C$ be a strictly positive element with $\|e_C\| = 1$. Then $\alpha^{\sim}(\langle e_C\rangle) \le [1_{\widetilde{C}}]$. Let $e_A \in A$ be a strictly positive element. Since A has continuous scale, $d_\tau(e_A) = 1$ for all $\tau \in T(A)$. Therefore, by Theorem 7.3 of [Elliott et al. 2020b] (see also Lemma 6.10 of [Robert and Santiago 2021]), $\alpha^{\sim}(\langle e_C\rangle) \le \langle e_A\rangle$ in $\mathrm{Cu}^{\sim}(A)$. Since A has stable rank one, by [Robert 2012, Theorem 1.0.1], there is a homomorphism $h : C \to A$ such that $\mathrm{Cu}^{\sim}(h) = \alpha^{\sim}|_{\mathrm{Cu}^{\sim}(C)}$. In particular, $h_{*0} = \alpha$. □

6. Reduction

This section is a nonunital version of the corresponding results in [Elliott et al. 2015]. Most of the results are taken from [Elliott et al. 2015] with some modification.

Lemma 6.1 [Elliott et al. 2015, Lemma 3.1]. *Let A be a nonunital simple separable amenable quasidiagonal C^*-algebra satisfying the UCT. Assume that $A \cong A \otimes Q$. Let a finite subset $\mathcal{G}$ of $\widetilde{A} \otimes Q$ and $\varepsilon_1, \varepsilon_2 > 0$ be given. Let $p_1, p_2, \dots, p_s \in M_m(\widetilde{A} \otimes Q)$ (for an integer $m \ge 1$) be projections such that $[1], [p_1], [p_2], \dots, [p_s] \in K_0(\widetilde{A} \otimes Q)$ are $\mathbb{Q}$-linearly independent. (Recall that $K_0(\widetilde{A} \otimes Q) \cong K_0((\widetilde{A} \otimes Q) \otimes Q) \cong K_0(\widetilde{A} \otimes Q) \otimes \mathbb{Q}$.) There are a $\mathcal{G}$-ε_1-multiplicative completely positive linear map $\sigma : \widetilde{A} \otimes Q \to Q$ with $\sigma(1)$ a projection satisfying*

$$\mathrm{tr}(\sigma(1)) < \varepsilon_2$$

(where tr denotes the unique tracial state on Q), and $\delta > 0$ such that for any $r_1, r_2, \dots, r_s \in \mathbb{Q}$ with

$$|r_i| < \delta, \quad i = 1, 2, \dots, s,$$

there is a $\mathcal{G}$-ε_1-multiplicative completely positive linear map $\mu : \widetilde{A} \otimes Q \to Q$ with $\mu(1) = \sigma(1)$ such that

$$[\sigma(p_i)] - [\mu(p_i)] = r_i, \quad i = 1, 2, \dots, s.$$

Proof. Let us first consider the case $m = 1$. The proof in this case is exactly the same as that of Lemma 3.1 of [Elliott et al. 2015]. The only place in that proof mentioning simplicity is the lines shortly after equation (3.1), where one claims that the algebra is the closure of an increasing sequence of unital amenable RFD C^*-algebras. We replace this part as follows: Since A is simple, separable, amenable and quasidiagonal, by Corollary 5.5 of [Blackadar and Kirchberg 2001], A is a strong NF-algebra. Since Q is a strong NF-algebra, by Corollary 7.1.6 of [Blackadar and Kirchberg 1997], $\widetilde{A} \otimes Q$ is a strong NF-algebra. It follows from

Corollary 6.16 of [Blackadar and Kirchberg 1997] that, indeed, $\widetilde{A}\otimes Q$ is the closure of an increasing sequence of unital amenable RFD C^*-algebras. The rest of the proof of this case remains the same.

If $m > 1$, one notes that there are $p_i' \in \widetilde{A}\otimes Q$ such that $[p_i'] = (1/m)[p_i]$, $i = 1, 2, \ldots, s$. Then $[1], [p_1'], \ldots, [p_s']$ are $\mathbb{Q}$-linearly independent. Replace r_i by r_i/m for $i = 1, 2, \ldots, s$. If $\sigma([p_i']) - \mu([p_i']) = r_i/m$ for $i = 1, 2, \ldots, s$, then $\sigma([p_i]) - \mu([p_i]) = r_i$ for $i = 1, 2, \ldots, s$. □

Corollary 6.2 [Elliott et al. 2015, Lemma 3.1]. *Let A be a nonunital simple separable amenable quasidiagonal C^*-algebra satisfying the UCT. Assume that $A \cong A\otimes Q$. Let a finite subset $\mathcal{G}$ of $\widetilde{A}$ and $\varepsilon_1, \varepsilon_2 > 0$ be given. Let $p_1, p_2, \ldots, p_s \in M_m(\widetilde{A})$ (for some integer $m \geq 1$) be projections such that $[1], [p_1], [p_2], \ldots, [p_s] \in K_0(\widetilde{A})$ are $\mathbb{Q}$-linearly independent. There are a $\mathcal{G}$-ε_1-multiplicative completely positive linear map $\sigma : \widetilde{A} \to Q$ with $\sigma(1)$ a projection satisfying*

$$\operatorname{tr}(\sigma(1)) < \varepsilon_2$$

(where tr *denotes the unique tracial state on Q), and $\delta > 0$ such that for any $r_1, r_2, \ldots, r_s \in \mathbb{Q}$ with*

$$|r_i| < \delta, \quad i = 1, 2, \ldots, s,$$

there is a $\mathcal{G}$-ε_1-multiplicative completely positive linear map $\mu : \widetilde{A} \to Q$, with $\mu(1) = \sigma(1)$, such that

$$[\sigma(p_i)] - [\mu(p_i)] = r_i, \quad i = 1, 2, \ldots, s.$$

Proof. Let $B = \widetilde{A}\otimes Q$. One has the split short exact sequence

$$0 \to A\otimes Q \to \widetilde{A}\otimes Q \to Q \to 0.$$

This gives the split short exact sequence

$$0 \to K_0(A\otimes Q) \to K_0(\widetilde{A}\otimes Q) \to \mathbb{Q} \to 0.$$

Since $A \cong A\otimes Q$, $K_0(A) = K_0(A\otimes Q)$ and $K_0(\widetilde{A}) = K_0(A\otimes Q)\oplus\mathbb{Z}$ is a subgroup of $K_0(\widetilde{A}\otimes Q)$. Apply Lemma 6.1 to B, and then choose $\sigma|_{\widetilde{A}}$ and $\mu|_{\widetilde{A}}$. □

Remark 6.3. Let $A \cong A\otimes Q$ be a separable stably projectionless simple C^*-algebra. Suppose that $K_0(A) \neq \{0\}$. Then there exists $x = [p] - k[1_{\widetilde{A}}] \in K_0(A) \setminus \{0\}$, where $k \in \mathbb{N}$, and $p \in M_n(\widetilde{A})$. If $[p] = r[1_{\widetilde{A}}]$ for some rational number $r \in \mathbb{Q}$, then $x = (r-k)[1_{\widetilde{A}}]$. Since $x \neq 0$, $r \neq k$. But then either $x = (r-k)[1_{\widetilde{A}}]$ or $-x := (k-r)[1_{\widetilde{A}}]$ is a nonzero positive element in $K_0(A)$. This contradicts the fact that $K_0(A)_+ = \{0\}$. In other words, $[p]$ and $[1_{\widetilde{A}}]$ are $\mathbb{Q}$-linearly independent. Put $p_1 := p$. Choose $r_1 \neq 0$ and let $[\sigma(p_1)] - [\mu(p_1)] = r_1$. Then $[\mu](x) \neq [\sigma](x)$. In other words, at least one of the maps $\mu|_A$ and $\sigma|_A$ is not zero.

Lemma 6.4 [Elliott et al. 2015, Lemma 3.3]. *Let A be a nonunital simple separable amenable quasidiagonal C^*-algebra satisfying the UCT. Assume that $A \cong A \otimes Q$. Let $\mathcal{G}$ be a finite subset of A, let $\varepsilon_1, \varepsilon_2 > 0$, and let $p_1, p_2, \dots, p_s \in M_m(\widetilde{A})$ (for some integer m) be projections such that $[1_A], [p_1], [p_2], \dots, [p_s] \in K_0(\widetilde{A})$ are $\mathbb{Q}$-linearly independent. There exists $\delta > 0$ satisfying the following condition*:

Let $\psi_k : Q^l \to Q^r$, $k = 0, 1$, be unital homomorphisms, where $l, r \in \{1, 2, \dots\}$. Set

$$\mathbb{D} = \{x \in \mathbb{Q}^l : (\psi_0)_{*0}(x) = (\psi_1)_{*0}(x)\} \subset \mathbb{Q}^l.$$

There exists a $\mathcal{G}$-ε_1-multiplicative completely positive linear map $\Sigma : \widetilde{A} \to Q^l$ such that $\Sigma(1_{\widetilde{A}})$ is a projection, with the properties

$$\tau(\Sigma(1_{\widetilde{A}})) < \varepsilon_2, \quad \tau \in T(Q^l),$$
$$[\Sigma(1_{\widetilde{A}})],\ [\Sigma(p_j)] \in \mathbb{D}, \quad j = 1, 2, \dots, s,$$

and, for any $r_1, r_2, \dots, r_s \in \mathbb{Q}^r$ satisfying

$$|r_{i,j}| < \delta,$$

where $r_i = (r_{i,1}, r_{i,2}, \dots, r_{i,r})$, $i = 1, 2, \dots, s$, there is a $\mathcal{G}$-ε_1-multiplicative completely positive linear map $\mu : \widetilde{A} \to Q^r$, with $\mu(1_{\widetilde{A}})$ a projection, such that

$$[\psi_0 \circ \Sigma(p_i)] - [\mu(p_i)] = r_i, \quad i = 1, 2, \dots, s[\mu(1_{\widetilde{A}})] = [\psi_0 \circ \Sigma(1_{\widetilde{A}})].$$

Proof. The proof is exactly the same as that of Lemma 3.3 of [Elliott et al. 2015] but using Lemma 6.1 (and Corollary 6.2) above instead of [Elliott et al. 2015, Lemma 3.1]. □

Lemma 6.5 [Elliott et al. 2015, Lemma 3.4]. *Let A be a nonunital simple separable amenable quasidiagonal C^*-algebra satisfying the UCT. Assume that $A \cong A \otimes Q$. Let $\mathcal{G} \subset A$ be a finite subset, let $\varepsilon_1, \varepsilon_2 > 0$ and let $p_1, p_2, \dots, p_s \in M_m(\widetilde{A})$ be projections such that $[1_{\widetilde{A}}], [p_1], [p_2], \dots, [p_s] \in K_0(\widetilde{A})$ are $\mathbb{Q}$-linearly independent. Then there exists $\delta > 0$ satisfying the following condition*:

Let $\psi_k : Q^l \to Q^r$, $k = 0, 1$, be unital homomorphisms, where $l, r \in \{1, 2, \dots\}$. Set

$$\mathbb{D} = \{x \in \mathbb{Q}^l : (\psi_0)_{*0}(x) = (\psi_1)_{*0}(x)\} \subset \mathbb{Q}^l.$$

There exists a $\mathcal{G}$-ε_1-multiplicative completely positive linear map $\Sigma : \widetilde{A} \to Q^l$ such that $\Sigma(1_{\widetilde{A}})$ is a projection, with the properties

$$\tau(\Sigma(1_{\widetilde{A}})) < \varepsilon_2, \quad \tau \in T(Q^l),$$
$$[\Sigma(1_{\widetilde{A}})], [\Sigma(p_i)] \in \mathbb{D}, \quad i = 1, 2, \dots, s,$$

and, for any $r_1, r_2, \dots, r_s \in \mathbb{Q}^l$ satisfying

$$|r_{i,j}| < \delta,$$

where $r_i = (r_{i,1}, r_{i,2}, \dots, r_{i,l})$, $i = 1, 2, \dots, s$, *there is a* $\mathcal{G}$-ε_1-*multiplicative completely positive linear map* $\mu : \widetilde{A} \to Q^l$, *with* $\mu(1_{\widetilde{A}}) = \Sigma(1_{\widetilde{A}})$, *such that*

$$[\Sigma(p_i)] - [\mu(p_i)] = r_i, \quad i = 1, 2, \dots, s.$$

Proof. The proof is exactly the same as that of Lemma 3.4 of [Elliott et al. 2015], also using Lemma 6.1 (and Corollary 6.2) instead of [Elliott et al. 2015, Lemma 3.1]. □

Lemma 6.6. *Let* A *be a nonunital simple separable amenable* C^*-*algebra with* $T(A) = T_{\mathrm{qd}}(A) \neq \varnothing$ *which satisfies the UCT. Assume that* $A \otimes Q \cong A$ *and* A *have continuous scale. For any* $\sigma > 0$, $\varepsilon > 0$, *and any finite subset* $\mathcal{F}$ *of* A, *there exist a finite set of* $\mathcal{P} \subset K_0(A)$ *and* $\delta > 0$ *with the following property*:

Denote by $G \subseteq K_0(A)$ *the subgroup generated by* $\mathcal{P}$. *Let* $\kappa : G \to K_0(C)$ *be a homomorphism which extends to a positive homomorphism* $\kappa^{\sim} : G + \mathbb{Z} \cdot [1_{\widetilde{A}}] \to K_0(C)$ *such that* $\kappa^{\sim}([1_{\widetilde{A}}]) = [1_C]$, *where* $C = C([0, 1], Q)$, *and let* $\lambda : T(C) \to T(A)$ *be a continuous affine map such that*

$$|\tau(\kappa(x)) - \rho_A(\lambda(\tau))(x)| < \delta, \quad x \in \mathcal{P}, \tau \in T(C). \tag{6.7}$$

Then there is an $\mathcal{F}$-ε-*multiplicative completely positive linear map* $L : A \to C$ *such that*

$$|\tau \circ L(a) - \lambda(\tau)(a)| < \sigma, \quad a \in \mathcal{F}, \tau \in T(C). \tag{6.8}$$

Proof. Let ε, σ and $\mathcal{F}$ be given. We may assume that every element of $\mathcal{F}$ has norm at most one. Fix a strictly positive element $e \in A$ with $\|e\| = 1$. Choose

$$0 < d < \inf\{\tau(f_{1/2}(e)) : \tau \in T(A)\}.$$

This is possible since $T(A)$ is compact.

Let δ_1 (in place of δ), $\mathcal{G}$, and $\mathcal{P}_1$ (in place of $\mathcal{P}$) be as assured by Lemma 7.2 of [Elliott et al. 2020a] for $\mathcal{F}$ and ε, as well as d. We may assume that $\mathcal{F} \subset \mathcal{G}$ and $f_{1/2}(e)$, $f_{1/4}(e) \in \mathcal{G}$.

We may assume $\mathcal{P}_1 = \{x_1, x_2, \dots, x_{s'}\}$, where $x_i = [q_i] - [\bar{q}_i]$, where $q_i \in M_m(\widetilde{A})$ is a projection and $\bar{q}_i \in M_m(\mathbb{C} \cdot 1_{\widetilde{A}})$ is a scalar matrix. Take $\mathcal{P}^{\sim} = \{1_{\widetilde{A}}, p_1, p_2, \dots, p_s\}$ such that x_i is in the subgroup G_0 generated by $\{[1_{\widetilde{A}}], [p_1], [p_2], \dots, [p_s]\}$ (in $K_0(\widetilde{A})$). Deleting one or more of $p_1, p_2, \dots, p_s$ (but not $1_{\widetilde{A}}$), we may assume that the set $\{[1_A], [p_1], \dots, [p_s]\}$ is $\mathbb{Q}$-linearly independent. Define

$$\mathcal{P} = \{[p_i] - [\bar{p}_i] : 1 \le i \le s\},$$

where $\bar{p}_i \in M_m(\mathbb{C} \cdot 1_{\widetilde{A}})$ is a scalar matrix such that $\pi^A_{\mathbb{C}}(p_i) = \bar{p}_i$ $(1 \le i \le s)$, where $\pi^A_{\mathbb{C}} : \widetilde{A} \to \mathbb{C}$ is the quotient map.

Note that $p_i = \bar{p}_i + y_i$, where $\bar{p}_i \in M_m(\mathbb{C} \cdot 1_{\widetilde{A}})$ is a scalar projection and $y_i \in A$, $i = 1, 2, \dots, s$. Let c_i be the rank of $\bar{p}_i$, $i = 1, 2, \dots, s$. We may assume that $G_0 \cap K_0(A)$ is generated by $\mathcal{P}$ (by enlarging $\mathcal{P}$ if necessary). Then G_0 is the subgroup of $K_0(\widetilde{A})$ generated by $[1_{\widetilde{A}}]$ and G. Therefore we may assume that $[\bar{p}_i] = c_i[1_{\widetilde{A}}]$ for some integer $1 \le c_i \le m$, $i = 1, 2, \ldots, s$. Choose $M = 1 + m$ and choose $0 < \sigma_1 < \min\{\sigma, d/32\}$ such that $(1 - \sigma_1)/(1 + \sigma_1) > 127/128$.

Let $0 < \delta_2 < 1$ (in place of δ) be as given by Lemma 6.1 for $\varepsilon_1 = \delta_1$, $\varepsilon_2 = \sigma_1/4$, $\mathcal{G}$ and $\{p_1, p_2, \ldots, p_s\}$. Write $y_i = (y_{j,k}^{(i)})_{m\times m}$, where $y_{j,k}^{(i)} \in A$. Choose $\mathcal{G}_1 = \{y_{j,k}^{(i)} : j, k, i\}$ and $\mathcal{G}_2 = \mathcal{G} \cup \mathcal{G}_1$.

Put $\delta_3 = \min\{\delta_1/M, \delta_2/32M, d/128\}$. We choose $0 < \delta < \delta_3$ and a finite subset $\mathcal{G}_3 \supset \mathcal{G}_2$ such that for any $\mathcal{G}_3$-δ-multiplicative contractive completely positive linear map $L : A \to B$ (any unital C^*-algebra B), $[L(p_i)]$ is well defined and

$$\|[L(p_i)] - L(p_i)\| < \delta_3, \quad i = 1, 2, \ldots, s. \tag{6.9}$$

Let us show that $\mathcal{P}$ and δ are as desired.

Let κ and λ be given satisfying (6.7). We write κ for $\kappa^\sim$ for convenience. Then recall $\kappa([1_{\widetilde{A}}]) = [1_C]$. Note that

$$\lambda(\tau)(\bar{p}_i) = c_i = \tau(\kappa(\bar{p}_i)), \quad i = 1, 2, \ldots. \tag{6.10}$$

So, as κ is positive, we may identify $\kappa([p_i])$ with a projection in $M_m(C)$ as $M_m(C)$ has stable rank one. Hence, by (6.7) and (6.10), for all $\tau \in T(C)$,

$$|\tau(\kappa([p_i])) - \rho_A(\lambda(\tau))([p_i])| < \delta, \quad i = 1, 2, \ldots, s. \tag{6.11}$$

Let $\lambda_* : \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(C))$ be defined by

$$\lambda_*(f)(\tau) = f(\lambda(\tau))$$

for all $f \in \mathrm{Aff}(T(A))$ and $\tau \in T(C)$. Identify $\partial_e(T(C))$ with $[0, 1]$, and put $\eta = \min\{\delta, \sigma_1/12\}$. Choose a partition

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$$

of the interval $[0, 1]$ such that

$$|\lambda_*(\hat{g})(t_j) - \lambda_*(\hat{g})(t_{j-1})| < \frac{\eta}{m^2}, \quad g \in \mathcal{G}_3, \ j = 1, 2, \ldots, n. \tag{6.12}$$

Since $T(A) = T_{\mathrm{qd}}(A)$, there are unital $\mathcal{G}_3$-δ-multiplicative completely positive linear maps $\Psi_j : A \to Q$, $j = 0, 1, 2, \ldots, n$, such that

$$|\mathrm{tr} \circ \Psi_j(g) - \lambda_*(\hat{g})(t_j)| < \frac{\eta}{m^2}, \quad g \in \mathcal{G}_3. \tag{6.13}$$

Denote by $\Psi_j^\sim : \widetilde{A} \to Q$ the unitization (which maps $1_{\widetilde{A}}$ to 1_Q). Recall that $[p_i] = m_i[1_{\widetilde{A}}] + x_i$, $i = 1, 2, \ldots, s$. It then follows from (6.9), (6.13) and (6.11), that, for each $i = 1, 2, \dots, s$ and each $j = 1, 2, \dots, n$,

$$\begin{aligned}
&|\mathrm{tr}([\Psi_j^{\sim}(p_i)]) - \mathrm{tr}([\Psi_0^{\sim}(p_i)])| \\
&\qquad < |\mathrm{tr}(\Psi_j^{\sim}(p_i)) - \mathrm{tr}(\Psi_0^{\sim}(p_i))| + 2\delta_3 \\
&\qquad < 2\delta_3 + 2\eta + |\lambda_*(\widehat{p_i})(t_j) - \lambda_*(\widehat{p_i})(t_0)| \\
&\qquad < 2\delta_3 + 2\eta + 2\delta + |\mathrm{tr} \circ \pi_{t_j}(\kappa([p_i])) - \mathrm{tr} \circ \pi_0(\kappa([p_i]))| \\
&\qquad = 2\delta_3 + 2\eta + 2\delta \le 6\delta_3 \le \delta_2. \qquad (6.14)
\end{aligned}$$

(Here, as before, π_t is the point evaluation at $t \in [0, 1]$.) We also have, by (6.12) and (6.13), that

$$|\mathrm{tr}(\Psi_j(g)) - \mathrm{tr}(\Psi_{j+1}(g))| < 3\eta, \quad g \in \mathcal{G}_3,\ j = 1, 2, \dots, n. \qquad (6.15)$$

Recall that $\lambda(\tau)(\widehat{f_{1/2}(e)}) \ge d$ for all $\tau \in T(C)$. So we also have, by (6.12),

$$\mathrm{tr}(\Psi_j(f_{1/2}(e))) > \tfrac{31}{32} d. \qquad (6.16)$$

Consider the differences

$$r_{i,j} := \mathrm{tr}([\Psi_j^{\sim}(p_i)]) - \mathrm{tr}([\Psi_0^{\sim}(p_i)]), \quad i = 1, 2, \dots, s,\ j = 1, 2, \dots, n. \quad (6.17)$$

Recall that $[\Psi_j^{\sim}(p_i)]$ and $[\Psi_0^{\sim}(p_i)]$ are in $K_0(Q)$. By (6.14), $|r_{i,j}| < \delta_2$. Applying Lemma 6.1, we obtain a projection $e \in Q$ with $\mathrm{tr}(e) < \sigma_1/4$ and $\mathcal{G}$-δ_1-multiplicative unital completely positive linear maps $\psi_0^{\sim}, \psi_j^{\sim} : \widetilde{A} \to eQe$, $j = 1, 2, \dots, n$, such that

$$[\psi_0^{\sim}(p_i)] - [\psi_j^{\sim}(p_i)] = r_{i,j}, \quad i = 1, 2, \dots, s,\ j = 1, 2, \dots, n. \qquad (6.18)$$

Consider the direct sum maps

$$\Phi_j'^{\sim} := \psi_j^{\sim} \oplus \Psi_j^{\sim} : \widetilde{A} \to (1 \oplus e) M_2(Q)(1 \oplus e), \quad j = 0, 1, 2, \dots, n.$$

Since $\delta \le \delta_1$, these maps are $\mathcal{G}$-δ_1-multiplicative. By (6.17) and (6.18),

$$[\Phi_j'^{\sim}(p_i)] = [\Phi_0'^{\sim}(p_i)], \quad i = 1, 2, \dots, s,\ j = 1, 2, \dots, n. \qquad (6.19)$$

Define $s : \mathbb{Q} \to \mathbb{Q}$ by $s(x) = x/(1 + \mathrm{tr}(e))$, $x \in \mathbb{Q}$. Choose a (unital) isomorphism

$$S : (1 \oplus e) M_2(Q)(1 \oplus e) \to Q$$

such that $S_{*0} = s$.

Consider the composed maps (still $\mathcal{G}$-δ_1-multiplicative and now unital)

$$\Phi_j := S \circ \Phi_j' : A \to Q, \quad j = 0, 1, 2, \dots, n.$$

By (6.19),

$$[\Phi_j]|_{\mathcal{P}} = [\Phi_{j-1}]|_{\mathcal{P}}, \quad j = 1, 2, \dots, n,$$

and by (6.15) and the fact that $\mathrm{tr}(e) < \sigma_1/4$,

$$|\mathrm{tr}\circ\Phi_j(a) - \mathrm{tr}\circ\Phi_{j-1}(a)| < 3\eta + \frac{\sigma_1}{4} \le \frac{\sigma}{2}, \quad a \in \mathcal{F},\ j = 1, 2, \ldots, n. \tag{6.20}$$

Moreover, by (6.16) and by the choice of σ_1,

$$\mathrm{tr}(\Psi_j(f_{1/2}(e))) \ge \frac{d}{2}, \quad j = 1, 2, \ldots, n. \tag{6.21}$$

Applying Lemma 7.2 of [Elliott et al. 2020a] successively for $j = 1, 2, \ldots, n$ (to the pairs (Φ_0, Φ_1), $(\mathrm{Ad}\, u_1 \circ \Phi_1, \mathrm{Ad}\, u_1 \circ \Phi_2), \ldots, (\mathrm{Ad}\, u_{n-1} \circ \cdots \circ \mathrm{Ad}\, u_1 \circ \Phi_{n-1}$, $\mathrm{Ad}\, u_{n-1} \circ \cdots \circ \mathrm{Ad}\, u_1 \circ \Phi_n))$, one obtains, for each j, a unitary $u_j \in Q$ and a unital $\mathcal{F}$-ε-multiplicative completely positive linear map $L_j : A \to \mathrm{C}([t_{j-1}, t_j], Q)$ such that

$$\pi_0 \circ L_1 = \Phi_0, \quad \pi_{t_1} \circ L_1 = \mathrm{Ad}\, u_1 \circ \Phi_1, \tag{6.22}$$

and

$$\pi_{t_{j-1}} \circ L_j = \pi_{t_{j-1}} \circ L_{j-1},\ \pi_{t_j} \circ L_j = \mathrm{Ad}\, u_j \circ \cdots \circ \mathrm{Ad}\, u_1 \circ \Phi_j,\ j = 2, 3, \ldots, n. \tag{6.23}$$

Furthermore, in view of (6.20), we may choose the maps L_j such that

$$|\mathrm{tr}\circ\pi_t \circ L_j(a) - \lambda(\mathrm{tr}\circ\pi_t)(a)| < \sigma, \quad t \in [t_{j-1}, t_j],\ a \in \mathcal{F},\ j = 1, 2, \ldots, n. \tag{6.24}$$

Define $L : A \to \mathrm{C}([0, 1], Q)$ by $\pi_t \circ L = \pi_t \circ L_j$, $t \in [t_{j-1}, t_j]$, $j = 1, 2, \ldots, n$. Since L_j, $j = 1, 2, \ldots, n$, are $\mathcal{F}$-ε-multiplicative (use (6.22) and (6.23)), we have that L is a $\mathcal{F}$-ε-multiplicative completely positive linear map $A \to \mathrm{C}([0, 1], Q)$. It follows from (6.24) that L satisfies (6.8), as desired. □

6.25. Let A be a separable stably projectionless simple C^*-algebra with continuous scale. Recall that $\tau_{\mathbb{C}}^A$ is the tracial state of $\widetilde{A}$ which vanishes on A and $T(\widetilde{A}) = \{s \cdot t_{\mathbb{C}}^A + (1-s)\cdot\tau : \tau \in T(A),\ 0 \le s \le 1\}$ (see 2.12). For each projection $p \in M_m(\widetilde{A})$, one may write $p = \bar{p} + a$, where $\bar{p}$ is a scalar matrix in $M_m(\mathbb{C}\cdot 1_{\widetilde{A}})$ and $a \in M_m(A)_{\mathrm{s.a.}}$. Let $\bar{p}$ have rank $k(p)$. For each $\tau \in T(A)$, define $r_{\widetilde{A}}(\tau)([p]) = k(p) + \tau(a)$ and $r_{\widetilde{A}}(\tau_{\mathbb{C}}^A)([p]) = k(p)$. This gives a map $r_{\widetilde{A}} : T(\widetilde{A}) \to \mathrm{Hom}(K_0(\widetilde{A}), \mathbb{R})$. Let $r_A : T(A) \to \mathrm{Hom}(K_0(A), \mathbb{R})$ be defined by $r_A(\tau) = r_{\widetilde{A}}(\tau)|_{K_0(A)}$ for any $\tau \in T(A)$ and $r_A^{\sim} : T(A) \to \mathrm{Hom}(K_0(\widetilde{A}), \mathbb{R})$ be defined by $r_A^{\sim} = r_{\widetilde{A}}|_{T(A)}$.

Suppose that C is another separable stably projectionless simple C^*-algebra with continuous scale and suppose that there is an isomorphism

$$\Gamma : (K_0(A), T(A), \rho_A) \cong (K_0(C), T(C), \rho_C). \tag{6.26}$$

Recall that this means $\Gamma|_{K_0(A)}$ is a group isomorphism, $\Gamma|_{T(A)}$ is an affine homeomorphism, and $\rho_A(\Gamma^{-1}(\tau))(x) = \rho_C(\tau)(\Gamma(x))$ for $x \in K_0(A)$ and $\tau \in T(C)$.

Then Γ extends to an order isomorphism

$$\Gamma^{\sim} : (K_0(\widetilde{A}), K_0(\widetilde{A})_+, [1_{\widetilde{A}}], T(\widetilde{A}), r_{\widetilde{A}}) \to (K_0(\widetilde{C}), K_0(\widetilde{C})_+, [1_{\widetilde{C}}], T(\widetilde{C}), r_{\widetilde{C}})$$

by defining $\Gamma^{\sim}([1_{\widetilde{A}}]) = [1_{\widetilde{C}}]$ and $\Gamma^{\sim}(\tau_{\mathbb{C}}^{A}) = \tau_{\mathbb{C}}^{C}$. To see this, we note that $\Gamma^{\sim}|_{K_0(\widetilde{A})}$ is an isomorphism and $\Gamma^{\sim}_{T(\widetilde{A})}$ is an affine homeomorphism. If $y := m \cdot [1_{\widetilde{A}}] + x \geq 0$ for some positive integer m and $x \in K_0(A)$, then there is a projection $p \in M_K(A)$ for some integer $K \geq 1$ such that $[p] = y$. Assume that $y \neq 0$. Then $p \neq 0$. Choose $a \in (pAp)_+^{\mathbf{1}} \setminus \{0\}$. Then $a \leq p$. It follows that $\tau(a) > 0$ for all $\tau \in T(A)$. Therefore $\tau(p) > 0$ for all $\tau \in T(A)$. This also means that $m + r_A(\tau)(x) > 0$ for all $\tau \in T(A)$.

On the other hand, $\pi_{\mathbb{C}}^{A}(p) \neq 0$, where $\pi_{\mathbb{C}}^{A} : \widetilde{A} \to \mathbb{C}$ is the quotient map. It follows that $\tau_{\mathbb{C}}^{A}(p) > 0$. This implies that $t(p) > 0$ for all $t \in T(\widetilde{A})$. One checks that, for $\tau \in T(C)$,

$$r_{\widetilde{A}}((\Gamma^{\sim})^{-1}(\tau))(y) = r_{\widetilde{A}}(\Gamma^{-1}(\tau)(m + x)) = m + r_A(\Gamma^{-1}(\tau))(x) > 0. \tag{6.27}$$

Also

$$r_{\widetilde{A}}((\Gamma^{\sim})^{-1}(\tau_{\mathbb{C}}^{C}))(y) = r_A(\tau_{\mathbb{C}}^{A})(y) = m > 0. \tag{6.28}$$

This implies that

$$r_{\widetilde{C}}(t)(\Gamma^{\sim}(y)) = r_{\widetilde{A}}((\Gamma^{\sim})^{-1})(y) > 0. \tag{6.29}$$

Therefore $\Gamma^{\sim}$ is an order isomorphism.

Lemma 6.30. *Let A be a nonunital but σ-unital simple C^*-algebra with strict comparison for positive elements which has almost stable rank one. Suppose that $QT(A) = \mathrm{T}(A)$, $A = \mathrm{Ped}(A)$ and the canonical map $\iota : W(A) \to \mathrm{LAff}_{b,+}(\overline{T(A)}^{\mathrm{w}})$ is surjective. Fix a strictly positive element $a \in A$. Then A has an approximate identity $\{e_n\}$ such that $\overline{e_n A e_n}$ has continuous scale of each n. Moreover, $e_n a e_n \sim e_n$ for all n.*

Proof. Let $a \in A_+$ with $\|a\| = 1$ be a strictly positive element. We may assume that a is not Cuntz equivalent to a projection as A is not unital. For $\varepsilon_1 = \frac{1}{2}$, by Lemma 7.2 of [Elliott et al. 2020b], there are $0 < \varepsilon_2 < \frac{1}{4}$ and $e_1 \in A$ such that $0 \leq f_{\varepsilon_1}(a) \leq e_1 \leq f_{\varepsilon_2}(a)$ and $\overline{e_1 A e_1}$ has continuous scale (see also Lemma 5.3 of [Elliott et al. 2020b], for example). By an induction and repeatedly applying Lemma 7.2 of [Elliott et al. 2020b], one obtains a sequence $\{e_n\} \subset A_+^1$ such that

$$0 \leq f_{\varepsilon_n}(a) \leq e_n \leq f_{\varepsilon_{n+1}}(a), \tag{6.31}$$

$\overline{e_n A e_n}$ has continuous scale and $0 < \varepsilon_n < 1/2^n$, $n = 1, 2, \ldots$. Since $\{f_{\varepsilon_n}(a)\}$ forms an approximate identity, one then verifies that $\{e_n\}$ also forms an approximate identity for A.

To see the last part of the lemma, we note that

$$e_n \sim e_n e_n e_n \leq e_n f_{\varepsilon_{n+1}}(a) e_n \lesssim e_n a e_n \leq e_n. \tag{6.32}$$

It follows that $e_n \sim e_n a e_n$ for all n. $\square$

The following is a nonunital version of Theorem 2.2 of [Winter 2016].

In the next statement, as in Definition 5.3 of [Elliott et al. 2020a], $\mathcal{S}$ is a fixed class of nonunital separable amenable C^*-algebras C such that $T(C) \neq \varnothing$ and $0 \notin \overline{T(C)}^w$. A simple C^*-algebra A is said to be in the class $\mathcal{R}$ if A is separable, has continuous scale and $T(A) \neq \varnothing$.

Theorem 6.33 (cf. [Winter 2016, Theorem 2.2; Elliott et al. 2020a, Theorem 5.4]). *Let A be a stably projectionless separable simple C^*-algebra with continuous scale and with* $\dim_{\mathrm{nuc}} A = m < \infty$.

Fix a positive element $e \in A_+$ with $0 \le e \le 1$ such that $\tau(e), \tau(f_{1/2}(e)) \ge r_0 > 0$ for all $\tau \in T(A)$. Let

$$C = \overline{\bigcup_{n=1}^{\infty} C_n}$$

be a nonunital simple C^*-algebra with continuous scale, where $C_n \subseteq C_{n+1}$ and $C_n \in \mathcal{S}$, which also satisfies condition (1) *in* [Elliott et al. 2020a, Definition 5.3]. *Suppose that there is an affine homeomorphism $\Gamma : T(C) \to T(A)$ and suppose that there is a sequence of completely positive contractive linear maps $\sigma_n : A \to C$ with* $\mathrm{im}(\sigma_n) \subset C_n$ *and a sequence of injective homomorphisms $\rho_n : C_n \to A$ such that*

$$\lim_{n\to\infty} \|\sigma_n(ab) - \sigma_n(a)\sigma_n(b)\| = 0, \quad a, b \in A, \tag{6.34}$$

$$\lim_{n\to\infty} \sup\{|t \circ \sigma_n(a) - \Gamma(t)(a)| : t \in T(C)\} = 0, \qquad a \in A, \tag{6.35}$$

$$\lim_{n\to\infty} \sup\{|\tau(\rho_n \circ \sigma_n(a)) - \tau(a)| : \tau \in T(A)\} = 0, \qquad a \in A. \tag{6.36}$$

Then A has the following property: *For any finite set $\mathcal{F} \subseteq A$ and any $\varepsilon > 0$, there are a projection $p \in M_{4(m+2)}(\widetilde{A})$, a C^*-subalgebra $S \subseteq pM_{4(m+2)}(A)p$ with $S \in \mathcal{S}$ and an $\mathcal{F}$-ε-multiplicative completely positive contractive linear map $L : A \to S$ such that*

(1) $\|[p, 1_{4(m+2)} \otimes a]\| < \varepsilon$ *for all* $a \in \mathcal{F}$,

(2) $p(1_{4(m+2)} \otimes a)p \in_{\varepsilon} S$ *for all* $a \in \mathcal{F}$,

(3) $\|L(a) - p(1_{4(m+2)} \otimes a)p\| < \varepsilon$ *for all* $a \in \mathcal{F}$,

(4) $p \sim e_{11}$ *in* $M_{4(m+2)}(\widetilde{A})$,

(5) $\tau(L(e)),\ \tau(f_{1/2}(L(e))) > \dfrac{7r_0}{32(m+2)}$ *for all* $\tau \in T(M_{4(m+2)}(A))$,

(6) $(1_{4(m+2)} - p)M_{4(m+2)}(A)(1_{4(m+2)} - p) \in \mathcal{R}$ *and*

(7) $t(f_{1/4}(L(e))) \ge \dfrac{r_0}{8}\lambda_s(C_1)$ *for all* $t \in T(S)$ (*see Definition 2.22 for* λ_s).

Proof. This is a slight modification of Theorem 5.4 of [Elliott et al. 2020a], which is a variation of Theorem 2.2 of [Winter 2016]. The proof is almost the same as that of Theorem 5.4 of [Elliott et al. 2020a], which itself is a repetition of the original proof of Theorem 2.2 of [Winter 2016].

Since A has finite nuclear dimension, one has that $A \cong A \otimes \mathcal{Z}$ (see [Winter 2012] for the unital case and [Tikuisis 2014] for the nonunital case). Therefore, A has strict comparison for positive elements (see Corollary 4.7 of [Rørdam 2004]).

The proof is essentially the same as that of Theorem 2.2 of [Winter 2016]. We give the proof in the present very much analogous situation for the convenience of the reader. Let $e \in A_+$ with $\|e\| = 1$, $\tau(e) > r_0$ and $\tau(f_{1/2}(e)) > r_0$ for all $\tau \in T(A)$.

Let (e_n) be an (increasing) approximate unit for A. Since $A \in \mathcal{R}$, and A is also assumed to be projectionless, one may assume that $\mathrm{sp}(e_n) = [0, 1]$. Since $\dim_{\mathrm{nuc}}(A) \le m$, by Lemma 5.2 of [Elliott et al. 2020a], there is a system of $(m+1)$-decomposable completely positive approximations

$$\widetilde{A} \xrightarrow{\psi_j} F_j^{(0)} \oplus F_j^{(1)} \oplus \cdots \oplus F_j^{(m)} \oplus \mathbb{C} \xrightarrow{\varphi_j} \widetilde{A}, \quad j = 1, 2, \ldots$$

such that

$$\varphi_j(F_j^{(l)}) \subseteq A, \quad l = 0, 1, \ldots, m, \tag{6.37}$$

$$\varphi_j|_{\mathbb{C}}(1_{\mathbb{C}}) = 1_{\widetilde{A}} - e_j, \tag{6.38}$$

where e_j is an element of (e_n).

Write

$$\varphi_j^{(l)} = \varphi_j|_{F_j^{(l)}} \quad \text{and} \quad \varphi_j^{(m+1)} = \varphi_j|_{\mathbb{C}}, \quad l = 0, 1, \ldots, m.$$

Set $F_j^{(m+1)} = \mathbb{C}$. Let $\psi_j^{(l)} = \pi_l \circ \psi_j$ for $j = 0, 1, 2, \ldots, m+1$, where

$$\pi_l : \bigoplus_{k=0}^{m+1} F_j^{(k)} \to F_j^{(l)} VSP2$$

is the projection. As in Lemma 5.2 of [Elliott et al. 2020a], one may assume that

$$\lim_{j\to\infty} \|\varphi_j^{(l)}\psi_j^{(l)}(1_{\widetilde{A}})a - \varphi_j^{(l)}\psi_j^{(l)}(a)\| = 0, \quad l = 0, 1, \ldots, m, \ a \in A. \tag{6.39}$$

Note that $\varphi_j^{(l)} : F_j^{(l)} \to A$ is of order zero, and the relation for an order zero map is weakly stable (see $(\mathcal{P})$ and $(\mathcal{P}1)$ of [Kirchberg and Winter 2004, Proposition 2.5]). On the other hand, if i is large enough, then $\sigma_i \circ \varphi_j^{(l)}$ satisfies the relation for order zero to within an arbitrarily small tolerance, since σ_i will be sufficiently multiplicative. It follows that there are order zero maps

$$\tilde{\varphi}_{j,i}^{(l)} : F_j^{(l)} \to C_i$$

such that

$$\lim_{i\to\infty} \|\tilde{\varphi}_{j,i}^{(l)}(c) - \sigma_i(\varphi_j^{(l)}(c))\| = 0, \quad c \in F_j^{(l)}.$$

We identify C_i with $S_i = \rho_i(C_i) \subseteq A$, $\sigma_i : A \to C_i$ with $\rho_i \circ \sigma_i : A \to S_i \subseteq A$ and

$\tilde{\varphi}^{(l)}_{j,i}$ with $\rho_i \circ \tilde{\varphi}^{(l)}_{j,i}$. There is a positive linear map (automatically order zero)

$$\tilde{\varphi}^{(m+1)}_{j,i} : \mathbb{C} \ni 1 \mapsto 1_{\widetilde{A}} - \sigma_i(e_j) \in \widetilde{S_i} = \mathrm{C}^*(S_i, 1_{\widetilde{A}}) \subseteq \widetilde{A}, \quad i \in \mathbb{N}.$$

Note that

$$\tilde{\varphi}^{(m+1)}_{j,i}(\lambda) = \sigma_i(\varphi^{(m+1)}_j(\lambda)), \quad \lambda \in F^{(m+1)}_j = \mathbb{C}, \tag{6.40}$$

where one still uses σ_i to denote the induced map $\widetilde{A} \to \widetilde{S_i}$.

Note that for each $l = 0, 1, \dots, m$,

$$\lim_{i\to\infty} \|f(\tilde{\varphi}^{(l)}_{j,i})(c) - \sigma_i(f(\varphi^{(l)}_j)(c))\| = 0, \quad c \in (F^{(l)}_j)_+, \; f \in \mathrm{C}_0((0,1])_+,$$

(see [Winter and Zacharias 2009, 4.2] for the definition of $f(\psi)$, where ψ is an order zero map) and hence, from (6.36),

$$\lim_{i\to\infty} \sup_{\tau\in T(A)} |\tau(f(\tilde{\varphi}^{(l)}_{j,i})(c) - f(\varphi^{(l)}_j)(c))| = 0, \quad c \in (F^{(l)}_j)_+, \; f \in \mathrm{C}_0((0,1])_+.$$

Also note that

$$\limsup_{i\to\infty} \|f(\tilde{\varphi}^{(l)}_{j,i})(c)\| \le \|f(\varphi^{(l)}_j)(c)\|, \quad c \in (F^{(l)}_j)_+, \; f \in \mathrm{C}_0((0,1])_+.$$

Applying Lemma 5.1 of [Elliott et al. 2020a] to $(\tilde{\varphi}^{(l)}_{j,i})_{i\in\mathbb{N}}$ and $\varphi^{(l)}_j$ for each $l = 0, 1, \dots, m$, we obtain contractions

$$s^{(l)}_{j,i} \in M_4(A) \subseteq M_4(\widetilde{A}), \quad i \in \mathbb{N},$$

such that, for all $c \in F^{(l)}_j$,

$$\lim_{i\to\infty} \|s^{(l)}_{j,i}(1_4 \otimes \varphi^{(l)}_j(c)) - (e_{1,1} \otimes \tilde{\varphi}^{(l)}_{j,i}(c))s^{(l)}_{j,i}\| = 0, \tag{6.41}$$

$$\lim_{i\to\infty} \|(e_{1,1} \otimes \tilde{\varphi}^{(l)}_{j,i}(c))s^{(l)}_{j,i}(s^{(l)}_{j,i})^* - e_{1,1} \otimes \tilde{\varphi}^{(l)}_{j,i}(c)\| = 0. \tag{6.42}$$

Note that $\mathrm{sp}(e_j) = [0,1]$. Put $C_0 = C_0((0,1])$. Define

$$\Delta_j(\hat{f}) = \inf\{\tau(f(e_j)) : \tau \in T(A)\} \quad \text{for all } f \in (C_0)_+ \setminus \{0\}. \tag{6.43}$$

Since A is assumed to have continuous scale, $T(A)$ is compact and $\Delta_j(\hat{f}) > 0$ for all $f \in (C_0)_+ \setminus \{0\}$. For $l = m+1$, since $\mathrm{sp}(e_j) = [0,1]$, by considering Δ_j for each j, since i is chosen after j is fixed, by applying Corollary A.16 of [Elliott et al. 2020a], one obtains unitaries

$$s^{(m+1)}_{j,i} \in \widetilde{A}, \quad i \in \mathbb{N},$$

such that

$$\lim_{i\to\infty} \|s^{(m+1)}_{j,i} e_j - \sigma_i(e_j)s^{(m+1)}_{j,i}\| = 0,$$

and hence

$$\lim_{i\to\infty} \|s^{(m+1)}_{j,i}(1_{\widetilde{A}} - e_j) - (1_{\widetilde{A}} - \sigma_i(e_j))s^{(m+1)}_{j,i}\| = 0.$$

By (6.38) and (6.40), one has

$$\lim_{i\to\infty} \|s_{j,i}^{(m+1)}\varphi_j^{(m+1)}(c) - \tilde{\varphi}_{j,i}^{(m+1)}(c)s_{j,i}^{(m+1)}\| = 0, \quad c \in F_j^{(m+1)} = \mathbb{C}.$$

Considering the element $e_{1,1} \otimes s_{j,i}^{(m+1)} \in M_4 \otimes \widetilde{A}$, and still denoting it by $s_{j,i}^{(m+1)}$, one has

$$\lim_{i\to\infty} \|s_{j,i}^{(m+1)}(1_4 \otimes \varphi_j^{(m+1)}(c)) - (e_{1,1} \otimes \tilde{\varphi}_{j,i}^{(m+1)}(c))s_{j,i}^{(m+1)}\| = 0, \quad c \in F_j^{(m+1)} = \mathbb{C}$$

and

$$(e_{1,1} \otimes \tilde{\varphi}_{j,i}^{(m+1)}(c))s_{j,i}^{(m+1)}(s_{j,i}^{(m+1)})^* = e_{1,1} \otimes \tilde{\varphi}_{j,i}^{(m+1)}(c).$$

Therefore,

$$\lim_{i\to\infty} \|s_{j,i}^{(l)}(1_4 \otimes \varphi_j^{(l)}(c)) - (e_{1,1} \otimes \tilde{\varphi}_{j,i}^{(l)}(c))s_{j,i}^{(l)}\| = 0, \tag{6.44}$$

$$\lim_{i\to\infty} \|(e_{1,1} \otimes \tilde{\varphi}_{j,i}(c))s_{j,i}^{(l)}(s_{j,i}^{(l)})^* - \tilde{\varphi}_{j,i}(c)\| = 0, \tag{6.45}$$

for $c \in F_j^{(l)}$, $l = 0, 1, \dots, m+1$. Let $\tilde{\sigma}_i : \widetilde{A} \to \widetilde{C}_i$ and $\tilde{\rho}_i : \widetilde{C}_i \to \widetilde{A}$ denote the unital maps induced by $\sigma_i : A \to C_i$ and $\rho_i : C_i \to A$, respectively.

Consider the contractions

$$s_j^{(l)} := (s_{j,i}^{(l)})_{i\in\mathbb{N}} \in (M_4 \otimes \widetilde{A})_\infty, \quad l = 0, 1, \dots, m+1, \ j = 1, 2, \dots.$$

By (6.44) and (6.45), these satisfy

$$s_j^{(l)}(1_4 \otimes \iota(\varphi_j^{(l)}(c))) = (e_{1,1} \otimes \bar{\rho}\bar{\sigma}(\varphi_j^{(l)}(c)))s_j^{(l)},$$
$$(e_{1,1} \otimes \bar{\rho} \circ \bar{\sigma}(\varphi_j^{(l)}(c)))s_j^{(l)}(s_j^{(l)})^* = (e_{1,1} \otimes \bar{\rho} \circ \bar{\sigma}(\varphi_j^{(l)}(c))),$$

where

$$\bar{\sigma} : \widetilde{A} \to \prod_{n=1}^{\infty} \widetilde{C_n} \Big/ \bigoplus_{n=1}^{\infty} \widetilde{C_n} \quad \text{and} \quad \bar{\rho} : \prod_{n}^{\infty} \widetilde{C_n} \Big/ \bigoplus_{n}^{\infty} \widetilde{C_n} \to (\widetilde{A})_\infty$$

are the homomorphisms induced by $\tilde{\sigma}_i$ and $\tilde{\rho}_i$, and the map $\iota : \widetilde{A} \to (\widetilde{A})_\infty$ is the canonical embedding. Let

$$\bar{\iota} : (\widetilde{A})_\infty \to ((\widetilde{A})_\infty)_\infty$$

be the embedding induced by the canonical embedding ι and let

$$\bar{\gamma} : (\widetilde{A})_\infty \to ((\widetilde{A})_\infty)_\infty$$

denote the homomorphism induced by the composed map

$$\bar{\rho}\bar{\sigma} : \widetilde{A} \to (\widetilde{A})_\infty.$$

For each $l = 0, 1, \dots, m+1$, let

$$\bar{\varphi}^{(l)} : \prod_j F_j^{(l)} \Big/ \bigoplus_j F_j^{(l)} \to A_\infty \quad \text{and} \quad \bar{\psi}^{(l)} : A \to \prod_j F_j^{(l)} \Big/ \bigoplus_j F_j^{(l)} \tag{6.46}$$

denote the maps induced by $\varphi_j^{(l)}$ and $\psi_j^{(l)}$.

Consider the contraction

$$\bar{s}^{(l)} = (s_j^{(l)}) \in (M_4 \otimes \widetilde{A}_\infty)_\infty.$$

Then

$$\begin{aligned} \bar{s}^{(l)}(1_4 \otimes \bar{\iota}\bar{\varphi}^{(l)}\bar{\psi}^{(l)}(a)) &= (e_{1,1} \otimes \bar{\gamma}\bar{\varphi}^{(l)}\bar{\psi}^{(l)}(a))\bar{s}^{(l)}, \quad a \in \widetilde{A}, \\ (e_{1,1} \otimes \bar{\gamma}\bar{\varphi}^{(l)}\bar{\psi}^{(l)}(a))\bar{s}^{(l)}(\bar{s}^{(l)})^* &= (e_{1,1} \otimes \bar{\gamma}\bar{\varphi}^{(l)}\bar{\psi}^{(l)}(a)). \end{aligned}$$

By (6.39), one has

$$\bar{\varphi}^{(l)}\bar{\psi}^{(l)}(1_{\widetilde{A}})\iota(a) = \bar{\varphi}^{(l)}\bar{\psi}^{(l)}(a), \quad a \in A.$$

Hence $[\bar{\varphi}^{(l)}\bar{\psi}^{(l)}(1_{\widetilde{A}}), \iota(b)] = 0$ and $(\bar{\varphi}^{(l)}\bar{\psi}^{(l)}(1_{\widetilde{A}})\iota(b))^{1/2} = (\bar{\varphi}^{(l)}\bar{\psi}^{(l)}(1_{\widetilde{A}}))^{1/2}\iota(b^{1/2})$ for any $b \in A_+$. It follows that

$$(\bar{\varphi}^{(l)}\bar{\psi}^{(l)}(1_{\widetilde{A}}))^{1/2}\iota(a) \in \mathrm{C}^*(\bar{\varphi}^{(l)}\bar{\psi}^{(l)}(A)) \quad \text{for } a \in A,$$

and hence

$$\begin{aligned} &\bar{s}^{(l)}(1_4 \otimes (\bar{\iota}\bar{\varphi}^{(l)}\bar{\psi}^{(l)}(1_{\widetilde{A}}))^{1/2})(1_4 \otimes \bar{\iota}\iota(a)) \\ &\qquad = \bar{s}^{(l)}(1_4 \otimes \bar{\iota}\bar{\varphi}^{(l)}\bar{\psi}^{(l)}(1_{\widetilde{A}})^{1/2}\iota(a)) \\ &\qquad = (e_{1,1} \otimes \bar{\gamma}(\bar{\varphi}^{(l)}\bar{\psi}^{(l)}(1_{\widetilde{A}}))^{1/2}\iota(a))\bar{s}^{(l)} \\ &\qquad = (e_{1,1} \otimes \bar{\gamma}\iota(a)(\bar{\varphi}^{(l)}\bar{\psi}^{(l)}(1_{\widetilde{A}}))^{1/2})\bar{s}^{(l)} \\ &\qquad = (e_{1,1} \otimes \bar{\gamma}(\iota(a)))(e_{1,1} \otimes \bar{\gamma}(\bar{\varphi}^{(l)}\bar{\psi}^{(l)}(1_{\widetilde{A}}))^{1/2})\bar{s}^{(l)}. \end{aligned} \tag{6.47}$$

Set

$$\begin{aligned} \bar{v} &= \sum_{l=0}^{m+1} e_{1,l} \otimes ((e_{1,1} \otimes \bar{\gamma}\bar{\varphi}^{l}\bar{\psi}^{(l)}(1_{\widetilde{A}}))^{1/2}\bar{s}^{(l)}) \\ &= \sum_{l=0}^{m+1} e_{1,l} \otimes (\bar{s}^{(l)}(1_4 \otimes \bar{\iota}\bar{\varphi}^{l}\bar{\psi}^{(l)}(1_{\widetilde{A}}))^{1/2}) \in M_{m+2}(\mathbb{C}) \otimes M_4(\mathbb{C}) \otimes (\widetilde{A}_\infty)_\infty. \end{aligned}$$

Then

$$\bar{v}\bar{v}^* = \sum_{l=0}^{m+1} e_{1,1} \otimes (e_{1,1} \otimes \bar{\gamma}\bar{\varphi}^{l}\bar{\psi}^{(l)}(1_{\widetilde{A}})) = e_{1,1} \otimes e_{1,1} \otimes \bar{\gamma}(1_{\widetilde{A}}).$$

Thus, $\bar{v}$ is a partial isometry. Moreover, for any $a \in \widetilde{A}$,

$$\begin{aligned}
&\bar{v}(1_{(m+2)} \otimes 1_4 \otimes \bar{\iota}\iota(a)) \\
&\qquad = \sum_{l=0}^{m+1} e_{1,l} \otimes (\bar{s}^{(l)}(1_4 \otimes \bar{\iota}\bar{\varphi}^l \bar{\psi}^{(l)}(1_{\widetilde{A}})^{1/2})(1_4 \otimes \bar{\iota}\iota(a))) \\
&\qquad = \sum_{l=0}^{m+1} e_{1,l} \otimes (e_{1,1} \otimes \bar{\gamma}(\iota(a)))(e_{1,1} \otimes \bar{\gamma}(\bar{\varphi}^{(l)}\bar{\psi}^{(l)}(1_{\widetilde{A}})^{1/2}\bar{s}^{(l)}) \quad \text{(by (6.47))} \\
&\qquad = (e_{1,1} \otimes e_{1,1} \otimes \bar{\gamma}(\iota(a))) \sum_{l=0}^{m+1} e_{1,l} \otimes e_{1,1} \otimes \bar{\gamma}(\bar{\varphi}^{(l)}\bar{\psi}^{(l)}(1_{\widetilde{A}})^{1/2}\bar{s}^{(l)}) \\
&\qquad = (e_{1,1} \otimes e_{1,1} \otimes \bar{\gamma}(\iota(a)))\bar{v}.
\end{aligned}$$

Hence, we obtain

$$\bar{v}^*\bar{v}(1_{m+2}\otimes 1_4\otimes\bar{\iota}\iota(a)) = \bar{v}^*(e_{1,1}\otimes e_{1,1}\otimes\bar{\gamma}\iota(a))\bar{v} = (1_{m+2}\otimes 1_4\otimes\bar{\iota}\iota(a))\bar{v}^*\bar{v}, \quad a \in \widetilde{A}.$$

Then for any finite set $\mathcal{G} \subseteq \widetilde{A}$ and $\delta > 0$, there are $i \in \mathbb{N}$ and $v_i \in M_{m+2}(\mathbb{C}) \otimes M_4(\mathbb{C}) \otimes \widetilde{A}$ such that

$$v_i v_i^* = e_{1,1} \otimes e_{1,1} \otimes \tilde{\rho}_i(1_{\widetilde{S}_i}) = e_{1,1} \otimes e_{1,1} \otimes 1_{\widetilde{A}}, \tag{6.48}$$

$$\|[v_i^* v_i, 1_{m+2} \otimes 1_4 \otimes a]\| < \delta \quad \text{for all } a \in \mathcal{G}, \tag{6.49}$$

$$\|v_i^* v_i(1_{m+2} \otimes 1_4 \otimes a) - v_i^*(e_{1,1} \otimes e_{1,1} \otimes \tilde{\rho}_i\tilde{\sigma}_i(a))v_i\| < \delta \quad \text{for all } a \in \mathcal{G}, \tag{6.50}$$

$$\tau(\rho_i \circ \sigma_i(e)),\ \tau(f_{1/2}(\rho_i \circ \sigma_i(e))) \ge \frac{15r_0}{16} \quad \text{for all } \tau \in T(A). \tag{6.51}$$

Define $\kappa_i : \widetilde{S}_i \to M_{m+2} \otimes M_4 \otimes \widetilde{A}$ by

$$\kappa_i(s) = v_i^*(e_{1,1} \otimes e_{1,1} \otimes \rho_i(s))v_i.$$

Note that

$$\kappa_i(S_i) \subseteq M_{m+2} \otimes M_4 \otimes A.$$

Then κ_i is an embedding; and on setting $p_i = 1_{\kappa_i(\widetilde{S}_i)} = v_i^* v_i$, we have

(i) $p_i \sim e_{1,1} \otimes e_{1,1} \otimes 1_{\widetilde{A}}$,

(ii) $\|[p_i, 1_{m+2} \otimes 1_4 \otimes a]\| < \delta$, $a \in \mathcal{G}$,

(iii) $p_i(1_{m+2} \otimes 1_4 \otimes a)p_i \in_\delta \kappa_i(\widetilde{S}_i)$, $a \in \mathcal{G}$.

Note that A is $\mathcal{Z}$-stable (by [Winter 2012]) and hence has strict comparison (by [Robert 2016]). Let $e' \in (1_{4(m+2)} - p_i)M_{4(m+2)}(A)(1_{4(m+2)} - p_i)$ be a strictly positive element. By (i), $\mathrm{d}_\tau(e') = \tau(1_{4(m+2)} - p_i) = \tau(1_{4(m+2)} - e_{1,1} \otimes e_{1,1} \otimes 1_{\widetilde{A}})$ for all $\tau \in T(A)$, where τ is naturally extended to $\widetilde{A}$. Since A and $M_{4(m+2)}(A)$ have continuous scale, $\tau \mapsto \mathrm{d}_\tau$ is continuous on $T(A)$. Hence

$$(1_{4(m+2)} - p_i)M_{4(m+2)}(A)(1_{4(m+2)} - p_i)$$

also has continuous scale (see Proposition 5.4 of [Elliott et al. 2020b]) and is still in the reduction class $\mathcal{R}$ (so condition (6) holds).

Define $L_i : A \to \kappa_i(S_i)$ by $L_i(a) = v_i^*(e_{1,1} \otimes e_{1,1} \otimes \rho_i(\sigma_i(a)))v_i$ for all $a \in A$. Then

(iv) $\|L_i(a) - p_i(1_{4(m+2)} \otimes a)p_i\| < \delta$ for all $a \in \mathcal{G}$ and

(v) $\tau(L_i(e)),\ \tau(f_{1/2}(L_i(e))) \geq \dfrac{15r_0}{64(m+2)}$ for all $\tau \in T(M_{4(m+2)}(A))$.

Let $\tau_i \in T(\kappa_i(S_i))$. Then $\tau_i \circ L_i$ is a positive linear functional. Let $\bar{t}$ be a weak $*$-limit of $\{\tau_i \circ L_i\}$. Note that, for any $\frac{1}{2} > \varepsilon > 0$, since A has continuous scale, there is $e_A \in A$ with $\|e_A\| = 1$ such that $\tau(e_A) > 1 - \varepsilon/2$ for all $\tau \in T(A)$. By (6.36) (see also (6.48)), we may assume that $\tau_i \circ L_i(e_A) > 1 - \varepsilon$ for all large i. It follows that $\bar{t}(e_A) \geq 1 - \varepsilon$. Hence $\|\bar{t}\| \geq 1 - \varepsilon$ for any $\frac{1}{2} > \varepsilon > 0$. It follows that $\bar{t}$ is a state of A. Then, by (6.34) and (6.36), $\bar{t}$ is a tracial state of A. Therefore, with sufficiently small δ and large $\mathcal{G}$ (and sufficiently large i), also by (6.51), we may assume that

$$t(f_{1/4}(L_i(e))) \geq \frac{7r_0}{8} \quad \text{for all } t \in T(\kappa_i(S_i)). \tag{6.52}$$

Recall that $\kappa_i(S_i) \cong C_i$. Fix (a large i) above and set $S = \kappa_i(S_i)$ and $L = L_i$. Recall also that $\lambda_s(C_1) \leq 1$. So (6.52) also implies that

$$t(f_{1/4}(L(e))) \geq \frac{3r_0}{8}\lambda_s(C_1) \quad \text{for all } t \in T(S). \tag{6.53}$$

The conclusion of the theorem follows from (i),(ii), (iii), (iv), (v), and (6.53). □

Theorem 6.54 (cf. [Elliott et al. 2020a, Theorem 5.7]). *Let A be a stably projectionless separable simple C^*-algebra with continuous scale and with* $\dim_{\mathrm{nuc}} A = m < \infty$.

Suppose that every hereditary C^*-subalgebra B of A with continuous scale has the following properties: Let $e_B \in B$ be a strictly positive element with $\|e_B\| = 1$ and $\tau(e_B) > 1 - \frac{1}{64}$ for all $\tau \in T(B)$. Let C be a nonunital simple C^*-algebra which is an inductive limit $C = \overline{\bigcup_{n=1}^{\infty} C_n}$, where $C_n \subset C_{n+1}$ and $C_n \in \mathcal{C}_0$, with continuous scale such that $T(C) \cong T(B)$. For each affine homeomorphism $\gamma : T(B) \to T(C)$, there exist sequences of completely positive contractive linear maps $\sigma_n : B \to C$ with $\mathrm{im}(\sigma_n) \subset C_n$ *and injective homomorphisms $\rho_n : C_n \to B$ such that*

$$\lim_{n\to\infty} \|\sigma_n(xy) - \sigma_n(x)\sigma_n(y)\| = 0 \quad \textit{for all } x, y \in B, \tag{6.55}$$

$$\lim_{n\to\infty} \sup\{|t \circ \sigma_n(b) - \gamma^{-1}(t)(b)| : t \in T(C)\} = 0 \quad \textit{for all } b \in B, \tag{6.56}$$

$$\lim_{n\to\infty} \sup\{|\tau(\rho_n \circ \sigma_n(b)) - \tau(b)| : \tau \in T(B)\} = 0 \quad \textit{for all } b \in B. \tag{6.57}$$

Suppose also that every hereditary C^*-subalgebra A is tracially approximately divisible. Then $A \in \mathcal{D}$.

Proof. The proof is exactly the same as that of Theorem 5.7 of [Elliott et al. 2020a]. By [Tikuisis 2014] (see also [Winter 2012]), $A'\otimes\mathcal{Z}\cong A'$ for every hereditary C^*-subalgebra A' of A. It follows from [Robert 2016] that A has almost stable rank one. Let B be a hereditary C^*-subalgebra with continuous scale. Then B has finite nuclear dimension (see [Winter and Zacharias 2010]). By [Tikuisis 2014] again, B is $\mathcal{Z}$-stable. It follows from Theorem 6.6 of [Elliott et al. 2011] that the map from $\mathrm{Cu}(B)$ to $\mathrm{LAff}_+(\widetilde{T}(B))$ is surjective. Note that the map from $W(B)$ to $\mathrm{LAff}_{b,+}(T(B))$ is also surjective. We apply Theorem 6.33 above and Lemma 5.5 of [Elliott et al. 2020a].

Since B has continuous scale, we may choose a strictly positive element $e\in B$ with $\|e\|=1$ and $e'\in B_+$ with $\|e'\|\le 1$ such that $f_{1/2}(e)e'=e'f_{1/2}(e)=e'$ and $d_\tau(f_{1/2}(e'))>1-1/(128(m+2))$ for all $\tau\in T(B)$. Let $1>\varepsilon>0$, $\mathcal{F}\subset B$ be a finite subset and $b\in B_+\setminus\{0\}$. Choose $b_0\in B_+\setminus\{0\}$ such that $64(m+2)\langle b_0\rangle\le\langle b\rangle$ in $\mathrm{Cu}(A)$. Since we assume that B is tracially approximately divisible (see Definitions 10.1 of [Elliott et al. 2020b] or 5.6 of [Elliott et al. 2020a]), there are $e_0\in B_+$ and a hereditary C^*-subalgebra A_0 of B such that $e_0\perp M_{4(m+2)}(A_0)$, $e_0\lesssim b_0$ and

$$\mathrm{dist}(x,\,B_{1,d})<\frac{\varepsilon}{64(m+2)}\quad\text{for all } x\in\mathcal{F}\cup\{e\},$$

where $B_{1,d}\subseteq B_s:=\overline{e_0Be_0}\oplus M_{4(m+2)}(A_0)\subseteq B$ and

$$B_{1,d}=\{x_0\oplus(\overbrace{x_1\oplus x_1\oplus\cdots\oplus x_1}^{4(m+2)}):x_0\in\overline{e_0Be_0},\,x_1\in A_0\}.\tag{6.58}$$

Without loss of generality, we may further assume that $\mathcal{F}\cup\{e'\}\subseteq B_{1,d}$. Let $P:B_s\to M_{4(m+2)}(A_0)$ be a projection map and $P^{(1)}:M_{4(m+2)}(A_0)\to A_0=A_0\otimes e_{11}$ be defined by $P^{(1)}(a)=(1_{\widetilde{A}_0}\otimes e_{11})a(1_{\widetilde{A}_0}\otimes e_{11})$ for $a\in M_{4(m+2)}(A_0)$, where $\{e_{ij}\}_{4(m+2)\times 4(m+2)}$ is a system of matrix unit. Put $\mathcal{F}_0=\{P(x):x\in\mathcal{F}\}$. Therefore, we may assume, without loss of generality, that $\|e_0x-xe_0\|<\varepsilon/(64(m+2))$, and there is $e_1\in M_{4(m+2)}(A_0)$ with $0\le e_1\le 1$ such that $\|e_1x-xe_1\|<\varepsilon/(64(m+2))$ and $\|e_1P(x)-P(x)\|<\varepsilon/(64(m+2))$ for all $x\in\mathcal{F}\cup\{e,\,e',\,f_{1/2}(e),\,f_{1/4}(e),\,f_{1/2}(e')\}$. Moreover, as the map from $W(A)_+$ to $\mathrm{LAff}_{b,+}(T(A))$ is surjective by Lemma 6.30, without loss of generality, we may assume that A_0 has continuous scale.

Without loss of generality, we may further assume that $\mathcal{F}\cup\{e\}\subseteq B_{1,d}$. Write

$$x=x_0+\overbrace{x_1\oplus x_1\oplus\cdots\oplus x_1}^{4(m+1)}.$$

Let $\mathcal{F}_1=\{x_1:x\in\mathcal{F}\cup\{e\}\}\subset A_0$. Note that we may write $\overbrace{x_1\oplus x_1\oplus\cdots\oplus x_1}^{4(m+2)}=x_1\otimes 1_{4(m+2)}$. Note that $\dim_{\mathrm{nuc}}A_0=m$ (see [Winter and Zacharias 2010]). Also, A_0 is a nonunital separable simple C^*-algebra which has continuous scale. We may then apply Theorem 6.33 to A_0 with $\mathcal{S}=\mathcal{C}_0$. By Theorem 4.107, $C=\overline{\bigcup_{n=1}^\infty C_n}$, where $C_n \in \mathcal{C}_0$, $C_n \subset C_{n+1}$, and C_n satisfies condition (1) in Definition 5.3 of [Elliott et al. 2020a]. Moreover, $\lambda_s(C_n) \geq \frac{1}{2}$ for all n. Put $r_0 = 1 - 1/(64(m+2))$. Choose $\eta_0 = 7/(32(m+2))$ and $\lambda = \frac{3}{16}$. Thus, by applying Theorem 6.33, we have the below estimates, with $\varphi_1(b) = (E-p)b(E-p)$ for all $b \in M_{4(m+2)}(A_0)$, where $E = 1_{M_{4(m+2)}(\widetilde{A_0})}$, and $p \in M_{4(m+2)}(\widetilde{A_0})$ is a projection given by Theorem 6.33, and $L : A_0 \to D_1 \subset pM_{4(m+2)}(A_0)p$ is an $\mathcal{F}_1$-ε-multiplicative completely positive contractive linear map:

$$\|x \otimes 1_{4(m+2)} - (\varphi_1(x \otimes 1_{4(m+2)}) + L(x))\| < \frac{\varepsilon}{4} \quad \text{for all } x \in \mathcal{F}_1, \tag{6.59}$$

$$d_\tau(\varphi_1(P(e))) \leq 1 - \frac{1}{4(m+2)} \quad \text{for all } \tau \in T(M_{4(m+2)}(A_0)), \tag{6.60}$$

$$\tau'(\varphi_1(P(e))), \tau'\big(f_{1/2}(\varphi_1(P(e)))\big) \geq r_0 - \frac{\varepsilon}{4} \quad \text{for all } \tau' \in T((1-p)M_{4(m+1)}(A_0)(1-p)), \tag{6.61}$$

$$D_1 \in \mathcal{C}_0,\ D_1 \subseteq pM_{4(m+2)}(A_0)p, \tag{6.62}$$

$$\tau(L(P^{(1)}(e))) \geq r_0\eta_0 \quad \text{for all } \tau \in T(M_{4(m+2)}(A_0)), \tag{6.63}$$

$$t\big(f_{1/4}(L(P^{(1)}(e)))\big) \geq r_0\lambda \quad \text{for all } t \in T(D_1). \tag{6.64}$$

Let $B_1 = (1-p)M_{4(m+1)}(A_0)(1-p) \oplus \overline{e_0Be_0}$ and $\varphi : B \to B_1$ be defined by $\varphi(a) = \varphi_1(e_1ae_1) + e_0ae_0$ for $a \in A$. Define $L_1 : B \to D_1$ by $L_1(b) = L(P^{(1)}(e_1^{1/2}be_1^{1/2}))$. Then both φ and L_1 are $\mathcal{F}$-ε-multiplicative. Put

$$\eta = \frac{\eta_0}{2} < \frac{\eta_0}{1 + \varepsilon/(64(m+2))}.$$

Then, in addition to (6.64) and (6.63),

$$\begin{aligned} \|x - (\varphi(x) + L_1(x))\| &< \varepsilon && \text{for all } x \in \mathcal{F}, \\ d_\tau(\varphi(e)) &\leq 1 - \eta && \text{for all } \tau \in T(B), \\ \tau'(\varphi(e)), \tau'(f_{1/2}(\varphi(e))) &\geq r - \varepsilon && \text{for all } \tau' \in T(B_1), \\ \tau(L_1(e)) &\geq r_0\eta && \text{for all } \tau \in T(B). \end{aligned}$$

Note that these hold for every such B. Thus, the hypotheses of Theorem 5.5 of [Elliott et al. 2020a] are satisfied. We then apply Theorem 5.5 of [Elliott et al. 2020a]. □

Theorem 6.65 (cf. [Elliott et al. 2015, Theorem 4.4]). *Let A be a separable stably projectionless simple C^*-algebra of finite nuclear dimension which satisfies the UCT. Assume that A has continuous scale*, $T(A) = T_{\mathrm{qd}}(A) \neq \varnothing$, $\mathrm{Cu}(A) = \mathrm{LAff}_+(T(A))$.

Suppose that there exists a simple C^*-algebra C with continuous scale such that $C=\lim_{n\to\infty}(C_n,\iota_n)$, where each $C_n=C_n'\otimes Q$ and $C_n'\in\mathcal{C}_0$, and such that

$$(K_0(A\otimes Q), T(A\otimes Q), r_{A\otimes Q})\cong (K_0(C), T(C), r_C). \tag{6.66}$$

Then, $A\otimes Q\in\mathcal{D}$.

Proof. Since A is simple, the assumption $T(A)=T_{\mathrm{qd}}(A)\neq\varnothing$ immediately implies that A is both stably finite and quasidiagonal. We may assume that $A\otimes Q\cong A$. Let $B\subset A$ be a hereditary C^*-subalgebra with continuous scale. We may write $B=\mathrm{Her}(b)$ for some $b\in A_+$. Since $A\cong A\otimes Q$, there is $a_0\in A_+$ such that $d_\tau(a_0\otimes 1_U)=d_\tau(b)$ for all $\tau\in T(A)$. Since we assume that $\mathrm{Cu}(A)=\mathrm{LAff}_+(T(A))$, $\langle a_0\otimes 1_U\rangle=\langle b\rangle$. By Theorem 1.2 of [Robert 2016], $B\cong\mathrm{Her}(a_0\otimes 1_U)$. However, $\mathrm{Her}(a_0\otimes 1_U)=\overline{a_0Aa_0}\otimes Q$. It follows that $B\cong\overline{a_0Aa_0}\otimes U\cong B\otimes Q$.

By Theorem 4.107, together with the assumption $A\cong A\otimes Q$, there is a simple C^*-algebra $C=\lim_{n\to\infty}(C_n,\iota_n)$ with continuous scale, where each C_n is the tensor product of a C^*-algebra in $\mathcal{C}_0$ with Q and ι_n is injective, such that

$$(K_0(A), T(A), r_A)\cong (K_0(C), T(C), r_C)$$

is given by Γ. It follows that there is an order isomorphism

$$\Gamma^{\sim}:(K_0(\widetilde{A}), K_0(\widetilde{A})_+, [1_{\widetilde{A}}], T(\widetilde{A}), r_{\widetilde{A}})\cong (K_0(\widetilde{C}), K_0(\widetilde{C})_+, [1_{\widetilde{C}}], T(\widetilde{C}), r_{\widetilde{C}}).$$

We continue to write ι_n and $\iota_{n,\infty}$ for the inclusions of $\iota_n:\widetilde{C}_n\to\widetilde{C}_{n+1}$ and $\iota_{n,\infty}:\widetilde{C}_n\to\widetilde{C}$, $n=1,2,\dots$. Let us write Γ_{Aff} and $\Gamma^{\sim}_{\mathrm{Aff}}$ for the corresponding maps from $\mathrm{Aff}(T(A))$ to $\mathrm{Aff}(T(C))$ and from $\mathrm{Aff}(T(\widetilde{A}))$ to $\mathrm{Aff}(T(\widetilde{C}))$. Since A and C have continuous scale, $T(A)$ and $T(C)$ are compact.

Let a finite subset $\mathcal{F}$ of A and $\varepsilon>0$ be given. Fix a strictly positive element $e\in A$ with $\|e\|=1$. Choose

$$0<d<\inf\{\tau(f_{1/2}(e)):\tau\in T(A)\}. \tag{6.67}$$

This is possible since $T(A)$ is compact. Let $0<\sigma<\min\{d,\varepsilon\}/2^{10}$. Since A has continuous scale, one may choose $e_1=f_{\varepsilon'}(e)$ for some $\frac{1}{4}>\varepsilon'>0$ such that

$$\tau(e_1)>1-\tfrac{\sigma}{128}\quad\text{for all }\tau\in T(A). \tag{6.68}$$

Without loss of generality, we may assume that e, $f_{1/2}(e)$, $f_{1/4}(e)$ and $e_1\in\mathcal{F}$.

Let the finite set $\mathcal{P}$ of $K_0(A)$, the finite subset $\mathcal{G}_1$ (in place of $\mathcal{G}$) of A, and $\delta_0>0$ (in place of δ) be as assured by Lemma 7.2 of [Elliott et al. 2020a] for $\mathcal{F}$ and $\varepsilon/4$. We may also assume that, for any $\mathcal{G}_1$-δ_0-multiplicative contractive completely positive linear map L from A,

$$\|f_{1/2}(L(e))-L(f_{1/2}(e))\|<\frac{d}{2^{10}}\quad\text{and}\quad\|f_{\varepsilon'}(L(e))-L(f_{\varepsilon'}(e))\|<\frac{d}{2^{10}}. \tag{6.69}$$

Choose a finite subset $\mathcal{P}_0$ of projections in $M_m(\widetilde{A})$ (for some integer $m \geq 1$) such that $\mathcal{P} \subset \{[p]-[q] : p, q \in \mathcal{P}_0\}$. We may assume that $1_{\widetilde{A}} \in \mathcal{P}$. Write $\mathcal{P}_0 = \{1_{\widetilde{A}}, p_1, p_2, \ldots, p_s\}$. Deleting some elements (but not 1_A), we may assume that the set

$$\mathcal{P}_0 = \{[1_{\widetilde{A}}], [p_1], [p_2], \ldots, [p_s]\} \subset K_0(\widetilde{A})$$

is $\mathbb{Q}$-linearly independent.

Choose $\delta_0' = \delta_0/4m^2$. We may also assume, without loss of generality, that $L^\sim$ is $\mathcal{G}_1 \cup \mathcal{P}_0$-$\delta_0$-multiplicative, if L is a $\mathcal{G}_1$-δ_0'-multiplicative contractive completely positive linear map from A to a C^*-algebra B, and $L^\sim : M_m(\widetilde{A}) \to M_m(\widetilde{B})$ is the usual extension of the unitization of L.

Put $\mathcal{G} = \mathcal{F} \cup \mathcal{G}_0$ and $\delta = \min\{\varepsilon/8, \delta_0'/2, d/2^{10}\}$. We may further assume, without loss of generality, that every element of $\mathcal{G}$ has norm at most one.

Let $\delta_1 > 0$ (in place of δ) be as assured by Lemma 6.4 for $\mathcal{G}$, δ (in place of ε_1), and $\sigma/64$ (in place of ε_2) and for $\mathcal{P}_0$. We may assume that $\delta_1 \leq \delta$.

Let $\delta_3 > 0$ (in place of δ) be as assured by Lemma 6.5 for $\mathcal{G}$, $\delta_1/8$ (in place of ε_1) and $\min\{\delta_1/32, \sigma/256\}$ (in place of ε_2) (and for $\mathcal{P}_0$).

Let $\mathcal{P}_1$ (in place of $\mathcal{P}$) and $\delta_2 > 0$ (in place of δ) be as assured by Lemma 6.6 for $\delta_1/8$ (in place of ε), $\min\{\delta_1/32, \sigma/256\}$ (in place of σ), and $\mathcal{G}$ (in place of $\mathcal{F}$). Replacing $\mathcal{P}$ and $\mathcal{P}_1$ by their union, we may assume that $\mathcal{P} = \mathcal{P}_1$. Note that we still use the notation $\mathcal{P}_0$ for the related set of projections.

By Lemmas 2.8 and 2.9 of [Elliott and Niu 2016], there are unital positive linear maps

$$\gamma : \mathrm{Aff}(T(\widetilde{A})) \to \mathrm{Aff}(T(\widetilde{C}_{n_1}))$$

for some $n_1 \geq 1$ such that

$$\|(\imath_{n_1,\infty})_{\mathrm{Aff}} \circ \gamma(\hat{f}) - \Gamma^\sim_{\mathrm{Aff}}(\hat{f})\| < \min\left\{\frac{\sigma}{128}, \delta_2, \frac{\delta_3}{2}\right\}, \quad f \in \mathcal{F} \cup \mathcal{P}_0 \tag{6.70}$$

(recall that $\hat{f}$ is the element in $\mathrm{Aff}(T(\widetilde{A}))$ corresponding to $f \in A_+ \subset \widetilde{A}_+$).

We may assume, without loss of generality, that for $i = 1, 2, \ldots, s$, there are projections $p_i' \in M_m(\widetilde{C_{n_1}})$ such that $\Gamma^\sim([p_i]) = \imath_{n_1,\infty}([p_i'])$. To simplify notation, assume that $n_1 = 1$. Let $\overline{G}_0$ denote the subgroup of $K_0(\widetilde{A})$ generated by $\mathcal{P}$, and $G_0 = K_0(A) \cap \overline{G}_0$. Since $\overline{G}_0$ is free abelian, there is a homomorphism $\Gamma' : \overline{G}_0 \to K_0(\widetilde{C}_1)$ such that

$$(\imath_{1,\infty})_{*0} \circ \Gamma'|_{G_0} = \Gamma|_{G_0} \quad \text{and} \quad (\imath_{1,\infty})_{*0} \circ \Gamma' = \Gamma^\sim|_{\overline{G}_0}.$$

We may assume (since $\mathcal{P}$ is a basis for $\overline{G}_0$) that

$$\Gamma'([p_i]) = [p_i'], \quad i = 1, 2, \ldots, s. \tag{6.71}$$

Since the pair $(\Gamma^{\sim}_{\mathrm{Aff}}, \Gamma^{\sim}|_{K_0(\widetilde{A})})$ is compatible, as a consequence of (6.70) and (6.71) we have

$$\|\widehat{p_i'} - \gamma(\widehat{p_i})\|_\infty < \min\Big\{\delta_2, \frac{\delta_3}{2}\Big\}, \quad i = 1, 2, \dots, s. \tag{6.72}$$

Write

$$\begin{aligned} C_1 &= (\psi_0, \psi_1, Q^r, Q^l) \\ &= \{(f, a) \in \mathrm{C}([0,1], Q^r) \oplus Q^l : f(0) = \psi_0(a) \text{ and } f(1) = \psi_1(a)\}, \end{aligned}$$

where $\psi_0, \psi_1 : Q^l \to Q^r$ are homomorphisms. Note that since we assume that $K_0(C_1)_+ = \{0\}$, C_1 is stably projectionless.

Set $1^{\bar{r}} := \mathrm{id}_{Q^r}$ and $1^{\bar{l}} := \mathrm{id}_{Q_l}$. Define $\psi_0^{\sim} : Q^l \oplus \mathbb{C} \to Q^r$ by $\psi_0^{\sim}(a, c) = \psi_0(a) + (1^{\bar{r}} - \psi_0(1^{\bar{l}}))c$ for all $(a, c) \in Q^l \oplus \mathbb{C}$ ($a \in Q^l$ and $c \in \mathbb{C}$), and define $\psi_1^{\sim} : Q^l \oplus \mathbb{C} \to Q^r$ by $\psi_1^{\sim}(a, c) = \psi_1(a) + (1^{\bar{r}} - \psi_1(1^{\bar{r}}))c$ for all $(a, c) \in Q^l \oplus \mathbb{C}$. It is understood that if ψ_0 is unital, $\psi_0^{\sim} = \psi_0$, and if ψ_1 is unital, $\psi_1^{\sim} = \psi_1$. We then identify

$$\begin{aligned} \widetilde{C}_1 &= (\psi_0^{\sim}, \psi_1^{\sim}, Q^r, Q^l) \\ &= \{(f, b) \in \mathrm{C}([0,1], Q^r) \oplus (Q^l \oplus \mathbb{C}) : f(0) = \psi_0^{\sim}(b) \text{ and } f(1) = \psi_1^{\sim}(b)\}. \end{aligned}$$

Denote by

$$\pi_{\mathrm{e}} : C_1 \to Q^l,\ (f, a) \mapsto a, \quad \text{and} \quad \pi_{\mathrm{e}}^{\sim} : \widetilde{C}_1 \to Q^l \oplus \mathbb{C},\ (f, b) \mapsto b$$

the canonical quotient map, and by $j : C_1 \to \mathrm{C}([0,1], Q^r)$ the canonical maps

$$j((f, a)) = f, \quad (f, a) \in C_1.$$

For convenience, in what follows, we also consider $\widetilde{C}_1 \otimes Q$. We continue to use $\psi_0^{\sim}$ for the extension $\pi_0^{\sim}(a, x) := \psi_0(a) + (1^{\bar{r}} - \psi_0(1^{\bar{l}}))x$ for all $(a, x) \in Q^l \oplus Q$ (where $a \in Q^l$ and $x \in Q$), and define

$$\psi_1^{\sim} : Q^{l+1} \to Q^r, \quad (a, x) \mapsto \psi_1(a) + (1^{\bar{r}} - \psi_1(1^{\bar{r}}))x$$

for all $(a, x) \in Q^{l+1}$. We also identify

$$\widetilde{C_1} \otimes Q = \{(f, b) \in \mathrm{C}([0,1], Q^r) \oplus Q^{l+1} : f(0) = \psi_0^{\sim}(b) \text{ and } f(1) = \psi_1^{\sim}(b)\}.$$

We also continue to write $\pi_e^{\sim}$ for the extension $\pi_e^{\sim} : \widetilde{C}_1 \otimes Q \to Q^{l+1}$. Denote by $\gamma^* : T(\widetilde{C}_1) \to T(\widetilde{A})$ the continuous affine map dual to γ. Let $\pi_{\mathbb{C}}^{C_1} : \widetilde{C}_1 \to \mathbb{C}$ be the quotient map and $\tau_{\mathbb{C}}^{C_1}$ be the trace of $\widetilde{C}_1$, which factors through $\mathbb{C}$. We also write $\pi_{\mathbb{C}}^{C_1}$ for the extension $\pi_{\mathbb{C}}^{C_1} : \widetilde{C}_1 \otimes Q \to Q$ and $\tau_{\mathbb{C}}^{C_1}$ for the tracial state of $\widetilde{C}_1 \otimes Q$ vanishing on C_1.

It follows from (6.70) that

$$|\gamma^*(\tau_{\mathbb{C}}^{C_1})(a)| < \min\Big\{\frac{\sigma}{128}, \delta_2, \frac{\delta_3}{2}\Big\} \quad \text{for all } a \in \mathcal{F}. \tag{6.73}$$

Note that, for any $\tau \in T(\widetilde{C_1})$, $\tau = s\tau_{\mathbb{C}} + (1-s)\tau'$, where $\tau' \in T(C_1)$ and $0 \le s \le 1$.

Denote by $\theta_1, \theta_2, \ldots, \theta_l$ the extreme tracial states of C_1 factoring through $\pi_{\mathrm{e}} : C_1 \to Q^l$. Put $\theta_{l+1} = \tau_{\mathbb{C}}$.

For any finite subset $\mathcal{G}' \supset \mathcal{G}$, and $0 < \eta < \min\{\delta_1/8, \delta_3/8\}$, by the assumption $T(A) = T_{\mathrm{qd}}(A)$, there is a unital $\mathcal{G}'$-η-multiplicative completely positive linear map $\Phi : \widetilde{A} \to Q^{l+1}$ (in fact, we can define each component $\pi_j \circ \Phi : \widetilde{A} \to Q$ of Φ separately) such that

$$|\mathrm{tr}_j \circ\Phi(a) - \gamma^*(\theta_j)(a)| < \min\left\{\frac{13\delta_1}{32}, \frac{\delta_3}{4}, \frac{\sigma}{32}\right\}, \qquad a \in \mathcal{G} \cup \mathcal{P}_0,\ j = 1, 2, \ldots, l, l+1, \tag{6.74}$$

where tr_j is the tracial state supported on the j-th direct summand of Q^l for $j = 1, 2, \ldots, l$, and tr_{l+1} is the tracial state of $\mathbb{C}$ (recall that we also write $\mathrm{tr} \circ\Phi$ for $\mathrm{tr} \otimes \mathrm{Tr}_m \circ(\Phi \otimes \mathrm{id}_m)$, where Tr_m is the nonnormalized trace on M_m). We also assume that $\Phi|_A$ maps A to Q^l (namely, $\pi_{l+1} \circ \Phi : \widetilde{A} \to Q$ can be defined to be the homomorphism taking A to $0 \in Q$ and $1_{\widetilde{A}}$ to $1 \in Q$) and $\Phi(1_{\widetilde{A}}) = (\underbrace{1, 1, \ldots, 1}_{l+1})$. We may also assume that

$$\mathrm{tr}_j(f_{1/2}(\Phi(e))) \geq \frac{63d}{64}, \qquad j = 1, 2, \ldots, l. \tag{6.75}$$

Moreover, we may also assume that

$$|\mathrm{tr}_j([\Phi(p_i)]) - \mathrm{tr}_j(\Phi(p_i))| < \frac{\delta_3}{4}, \qquad i = 1, 2, \ldots, s,\ j = 1, 2, \ldots, l+1. \tag{6.76}$$

Set

$$\begin{aligned} \mathbb{D}_0 &:= (\pi_{\mathrm{e}})_{*0}(K_0(C_1)) = \ker((\psi_0)_{*0} - (\psi_1)_{*0}) \subseteq \mathbb{Q}^l, \\ \mathbb{D} &:= (\pi_{\mathrm{e}}^{\sim})_{*0}(K_0(\widetilde{C}_1 \otimes Q)) = \ker((\psi_0^{\sim})_{*0} - (\psi_1^{\sim})_{*0}) \subseteq \mathbb{Q}^{l+1}. \end{aligned} \tag{6.77}$$

It follows from (6.74) that

$$|\tau(\Phi(a)) - (\pi_{\mathrm{e}}^{\sim})_{\mathrm{Aff}}(\gamma(\hat{a}))(\tau)| < \min\left\{\frac{13\delta_1}{32}, \frac{\delta_3}{4}, \frac{\sigma}{32}\right\}, \quad a \in \mathcal{G},\ \tau \in T(Q^{l+1}), \tag{6.78}$$

where $(\pi_{\mathrm{e}}^{\sim})_{\mathrm{Aff}} : \mathrm{Aff}(T(\widetilde{C_1} \otimes Q)) \to \mathrm{Aff}(T(Q^{l+1}))$ is the map induced by $\pi_{\mathrm{e}}^{\sim}$. By (6.78) for $a \in \mathcal{P}_0$, together with (6.76) and (6.72),

$$|\tau([\Phi(p_i)]) - \tau \circ (\pi_{\mathrm{e}})_{*0} \circ \Gamma'([p_i])| < \delta_3, \qquad \tau \in T(Q^{l+1}),\ i = 1, 2, \ldots, s.$$

Therefore, applying Lemma 6.5, with

$$r_i = [\Phi(p_i)] - (\pi_{\mathrm{e}})_{*0} \circ \Gamma'([p_i]) \in \mathbb{Q}^{l+1}$$

(note that $|r_{ij}| < \delta_3$ for $j = 1, 2, \ldots, l+1$ and $i = 1, 2, \ldots, s$), we obtain $\mathcal{G}$-$\delta_1/8$-multiplicative completely positive linear maps $\Sigma_1, \mu_1 : \widetilde{A} \to Q^{l+1}$, with

$\Sigma_1(1_{\widetilde{A}}) = \mu_1(1_{\widetilde{A}})$ a projection, such that

$$\tau(\Sigma_1(1_{\widetilde{A}})) < \min\left\{\frac{\delta_1}{32}, \frac{\sigma}{256}\right\}, \quad \tau \in T(Q^{l+1}), \tag{6.79}$$

$$[\Sigma_1(\mathcal{P}_0)] \subseteq \mathbb{D}, \quad \text{and} \tag{6.80}$$

$$[\Sigma_1(p_i)] - [\mu_1(p_i)] = r_i = [\Phi(p_i)] - (\pi_{\mathrm{e}})_{*0} \circ \Gamma'([p_i]), \quad i = 1, 2, \ldots, s. \tag{6.81}$$

Consider the (unital) direct sum map

$$\Phi' := \Phi \oplus \mu_1 : \widetilde{A} \to (1 \oplus \Sigma_1(1_{\widetilde{A}}))M_2(Q^{l+1})(1 \oplus \Sigma_1(1_{\widetilde{A}})). \tag{6.82}$$

Note that Φ', like μ_1 and Φ, is $\mathcal{G}$-$\delta_1/8$-multiplicative. It follows from (6.81) that

$$\begin{aligned} [\psi_0(\Phi'(p_i))] &= (\psi_0^{\sim})_{*0}([\mu_1(p_i)] + [\Phi(p_i)]) \\ &= (\psi_0^{\sim})_{*0}([\Sigma_1(p_i)] + (\pi_{\mathrm{e}}^{\sim})_{*0} \circ \Gamma'([p_i])), \end{aligned} \tag{6.83}$$

$$\begin{aligned} [\psi_1(\Phi'(p_i))] &= (\psi_1^{\sim})_{*0}([\mu_1(p_i)] + [\Phi(p_i)]) \\ &= (\psi_1^{\sim})_{*0}([\Sigma_1(p_i)] + (\pi_{\mathrm{e}}^{\sim})_{*0} \circ \Gamma'([p_i])) \end{aligned} \tag{6.84}$$

for $i = 1, 2, \ldots, s$. It follows from (6.83) and (6.84), in view of (6.80) and the fact (using (6.71)) that $(\pi_{\mathrm{e}})_{*0} \circ \Gamma'([p_i]) \in (\pi_{\mathrm{e}}^{\sim})_{*0}(K_0(\widetilde{C_1} \otimes Q)) = \mathbb{D}$, that $[\Phi'(p_i)] \in \mathbb{D}$, $i = 1, 2, \ldots, s$, i.e.,

$$[\psi_0(\Phi'(p_i))] = [\psi_1(\Phi'(p_i))], \quad i = 1, 2, \ldots, s. \tag{6.85}$$

Set $B = \mathrm{C}([0, 1], Q^r)$, and (as before) write $\pi_t : B \to Q^r$ for the point evaluation at $t \in [0, 1]$. Since $1_{\widetilde{A}} \in \mathcal{P}_0$, by (6.80), $[\Sigma_1(1_{\widetilde{A}})] \in \mathbb{D}$. Hence there is a projection $e_0 \in B$ such that $\pi_0(e_0) = \psi_0^{\sim}(\Sigma_1(1_{\widetilde{A}}))$ and $\pi_1(e_0) = \psi_1^{\sim}(\Sigma_1(1_{\widetilde{A}}))$. It then follows from (6.79) (applied just for τ factoring through $\psi_0^{\sim}$ — alternatively, for τ factoring through $\psi_1^{\sim}$) that

$$\tau(e_0) < \min\left\{\frac{\delta_1}{32}, \frac{\sigma}{256}\right\}, \quad \tau \in T(B). \tag{6.86}$$

Let $j^* : T(B) \to T(\widetilde{C}_1)$ denote the continuous affine map dual to the canonical unital map $j : \widetilde{C}_1 \to B$. Let $\gamma_1 : T(B) \to T(\widetilde{A})$ be defined by $\gamma_1 := \gamma^* \circ j^*$, and let $\kappa : \overline{G}_0 \to K_0(B)$ be defined by $\kappa := j_{*0} \circ \Gamma'$. Then, by (6.71) and (6.72), for all $\tau \in T(B)$,

$$\begin{aligned} |\tau(\kappa([p_i])) - \gamma_1(\tau)(p_i)| &= |\tau(j_{*0}(\Gamma'([p_i]))) - (\gamma^* \circ j^*)(\tau)(p_i)| \\ &= |j^*(\tau)([p_i']) - \gamma(\widehat{p_i})(j^*(\tau))| < \delta_2, \quad 1 \le i \le k. \end{aligned} \tag{6.87}$$

The estimate (6.87) ensures that we can apply Lemma 6.6 with κ and γ_1 (note that $\Gamma'([1_{\widetilde{A}}]) = [1_{\widetilde{C_1}}]$ and hence $\kappa([1_{\widetilde{A}}]) = [1_B]$) to obtain a $\mathcal{G}$-$\delta_1/8$-multiplicative completely positive linear map $\Psi' : A \to B$ such that

$$|\tau \circ \Psi'(a) - \gamma_1(\tau)(a)| < \min\left\{\frac{\delta_1}{32}, \frac{\sigma}{256}\right\}, \quad a \in \mathcal{G}, \ \tau \in T(B). \tag{6.88}$$

Let $\Psi'^{\sim} : \widetilde{A} \to B$ be the unitization of Ψ'. Amplifying $\Psi'^{\sim}$ slightly (by first identifying Q^r with $Q^r \otimes Q$ and then considering $H_0(f)(t) = f(t) \otimes (1 + e_0(t))$ for $t \in [0, 1]$), we obtain a unital $\mathcal{G}$-$\delta_1/8$-multiplicative completely positive linear map $\Psi : \widetilde{A} \to (1 \oplus e_0)M_2(B)(1 \oplus e_0)$ such that (by (6.88) and (6.86)), for all $a \in \mathcal{G}$ and $\tau \in T(B)$,

$$|\tau \circ \Psi(a) - \gamma_1(\tau)(a)| < 2 \min\left\{\frac{dt_1}{32}, \frac{\sigma}{256}\right\} = \min\left\{\frac{\delta_1}{16}, \frac{\sigma}{128}\right\}. \tag{6.89}$$

By (6.67), as $\gamma_1(\tau) \in T(A)$ and $\sigma < d/2^{10}$,

$$\tau(\Psi(f_{1/2}(e))) \geq \frac{d(2^9 - 1)}{2^9} \quad \text{for all } \tau \in T(B). \tag{6.90}$$

Note that, for any element $a \in C_1$,

$$\tau(\psi_i(\pi_{\mathrm{e}}(a))) = \tau(\pi_i(j(a))), \quad \tau \in T(Q^r),\ i = 0, 1. \tag{6.91}$$

(Recall that $j : \widetilde{C}_1 \to B$ is the canonical map.) Therefore (by (6.91)), for any $a \in \mathcal{G}$,

$$\begin{aligned} |\tau(\psi_0(\Phi(a))) - \gamma(\hat{a})(\tau \circ \pi_0 \circ j)| &= |\tau(\psi_0(\Phi(a))) - \gamma(\hat{a})(\tau \circ \psi_0 \circ \pi_{\mathrm{e}})| \\ &= |\tau \circ \psi_0(\Phi(a)) - (\pi_{\mathrm{e}})_{\mathrm{Aff}}(\gamma(\hat{a}))(\tau \circ \psi_0)| \\ &< \min\left\{\frac{13\delta_1}{32}, \frac{\sigma}{32}\right\} \quad \text{(by (6.78))} \end{aligned} \tag{6.92}$$

for all $\tau \in T(Q^r)$. The same argument shows that

$$|\tau(\psi_1(\Phi(a))) - \gamma(\hat{a})(\tau \circ \pi_1 \circ j)| < \min\left\{\frac{13\delta_1}{32}, \frac{\sigma}{32}\right\}, \quad a \in \mathcal{G},\ \tau \in T(Q^r). \tag{6.93}$$

Then, for any $\tau \in T(Q^r)$ and any $a \in \mathcal{G}$, we have

$$\begin{aligned} &|\tau \circ \psi_0 \circ \Phi'(a) - \tau \circ \pi_0 \circ \Psi(a)| \\ &\quad = |\tau \circ \psi_0(\Phi(a) \oplus \mu_1(a)) - \tau \circ \pi_0 \circ \Psi(a)| \\ &\quad < |\tau \circ \psi_0(\Phi(a) \oplus \mu_1(a)) - \gamma_1(\tau \circ \pi_0)(a)| + \min\left\{\frac{\delta_1}{16}, \frac{\sigma}{128}\right\} \quad \text{(by (6.89))} \\ &\quad < |\tau \circ \psi_0(\Phi(a)) - \gamma_1(\tau \circ \pi_0)(a)| + \min\left\{\frac{3\delta_1}{32}, \frac{3\sigma}{256}\right\} \quad \text{(by (6.79))} \\ &\quad = |\tau \circ \psi_0(\Phi(a)) - \gamma(\hat{a})(\tau \circ \pi_0 \circ j)| + \min\left\{\frac{3\delta_1}{32}, \frac{3\sigma}{256}\right\} \\ &\quad < \min\left\{\frac{13\delta_1}{32}, \frac{\sigma}{32}\right\} + \min\left\{\frac{3\delta_1}{32}, \frac{3\sigma}{256}\right\} \quad \text{(by (6.92))} \\ &\quad \leq \frac{13\delta_1}{32} + \frac{3\delta_1}{32} = \frac{\delta_1}{2}. \end{aligned} \tag{6.94}$$

The same argument, using (6.93) instead of (6.92), shows that

$$|\tau\circ\psi_1\circ\Phi'(a)-\tau\circ\pi_1\circ\Psi(a)|<\min\left\{\frac{13\delta_1}{32},\frac{\sigma}{32}\right\}+\min\left\{\frac{3\delta_1}{32},\frac{3\sigma}{256}\right\} \\ \le\frac{\delta_1}{2}\quad\text{for all }\tau\in T(Q^r)\text{ and }a\in\mathcal{G}. \tag{6.95}$$

(The σ estimates (6.94) and (6.95) will be used later to verify (6.106) and (6.107)).

Noting that Ψ and Φ' are $\delta_1/8$-multiplicative on $\{1_{\widetilde{A}}, p_1, p_2, \ldots, p_s\}$, we may assume

$$|\tau([\Psi(p_i)])-\tau(\Psi(p_i))|<\frac{\delta_1}{4}\quad\text{and}\quad|\tau([\Phi'(p_i)])-\tau(\Phi'(p_i))|<\frac{\delta_1}{4}$$

for all $\tau\in T(Q^r)$ and $1\le i\le s$. Combining these inequalities with (6.94) and (6.95), we have

$$|\tau([\pi_0\circ\Psi(p_i)])-\tau([\psi_0\circ\Phi'(p_i)])|<\delta_1,\quad i=1,2,\ldots,s,\ \ \tau\in T(Q^r). \tag{6.96}$$

Therefore (in view of (6.96)), applying Lemma 6.4 with

$$r_i'=[\pi_0\circ\Psi(p_i)]-[\psi_0\circ\Phi'(p_i)]\in\mathbb{Q}^r,$$

we obtain unital $\mathcal{G}$-δ-multiplicative completely positive linear maps $\Sigma_2:\widetilde{A}\to Q^{l+1}$ and $\mu_2:\widetilde{A}\to Q^r$, taking $1_{\widetilde{A}}$ into projections, such that

$$[\psi_k\circ\Sigma_2(1_{\widetilde{A}})]=[\mu_2(1_{\widetilde{A}})],\quad k=0,1, \tag{6.97}$$

$$[\Sigma_2(\mathcal{P})]\subseteq(\pi_{\mathrm{e}})_{*0}(K_0(\widetilde{C}_1))=\mathbb{D},\quad i=1,2,\ldots,s, \tag{6.98}$$

$$\tau(\Sigma_2(1_{\widetilde{A}}))<\frac{\sigma}{64},\quad \tau\in T(Q^{l+1}), \tag{6.99}$$

and, taking (6.98) into account,

$$[\psi_0\circ\Sigma_2(p_i)]-[\mu_2(p_i)]=[\psi_1\circ\Sigma_2(p_i)]-[\mu_2(p_i)] \\ =r_i'=[\pi_0\circ\Psi(p_i)]-[\psi_0\circ\Phi'(p_i)], \tag{6.100}$$

where $i=1,2,\ldots,s$. It should be also noted that, since $\tau\circ\psi_k\in T(Q^{l+1})$ for $k=0,1$,

$$\tau(\mu_2(1_{\widetilde{A}}))<\frac{\sigma}{64}\quad\text{for all }\tau\in T(Q^r). \tag{6.101}$$

Consider the four $\mathcal{G}$-δ-multiplicative direct sum maps (note that Φ' and Ψ are $\mathcal{G}$-$\delta_1/8$-multiplicative, and $\delta_1\le 8\delta$), from $\widetilde{A}$ to $M_3(Q^r)$,

$$\Phi_0:=(\psi_0\circ\Phi')\oplus(\psi_0\circ\Sigma_2),\quad \Phi_1:=(\psi_1\circ\Phi')\oplus(\psi_1\circ\Sigma_2), \tag{6.102}$$

$$\Psi_0:=(\pi_0\circ\Psi)\oplus\mu_2,\qquad \Psi_1:=(\pi_1\circ\Psi)\oplus\mu_2. \tag{6.103}$$

We then have for each $i = 1, 2, \dots, s$ that

$$\begin{aligned}[t][\Psi_0(p_i)] - [\Phi_0(p_i)] &= ([(\pi_0 \circ \Psi)(p_i)] + [\mu_2(p_i)]) \\ &\qquad - ([(\psi_0 \circ \Phi')(p_i)] + [(\psi_0 \circ \Sigma_2)(p_i)]) \\ &= ([(\pi_0 \circ \Psi)(p_i)] - [(\psi_0 \circ \Phi')(p_i)]) \\ &\qquad - ([(\psi_0 \circ \Sigma_2)(p_i)] - [\mu_2(p_i)]) \\ &= 0 \quad \text{(by (6.100))},\end{aligned}$$

and

$$\begin{aligned}[t]&[\Psi_1(p_i)] - [\Phi_1(p_i)] \\ &\quad = ([(\pi_1 \circ \Psi)(p_i)] + [\mu_2(p_i)]) \\ &\qquad\quad - ([(\psi_1 \circ \Phi')(p_i)] + [(\psi_1 \circ \Sigma_2)(p_i)]) \\ &\quad = ([(\pi_1 \circ \Psi)(p_i)] - [(\psi_1 \circ \Phi')(p_i)]) \\ &\qquad\quad - ([(\psi_1 \circ \Sigma_2)(p_i)] - [\mu_2(p_i)]) \\ &\quad = ([(\pi_0 \circ \Psi)(p_i)] - [(\psi_1 \circ \Phi')(p_i)]) \\ &\qquad\quad - ([(\psi_1 \circ \Sigma_2)(p_i)] - [\mu_2(p_i)]) \quad (\pi_0 \text{ and } \pi_1 \text{ are homotopic}) \\ &\quad = ([(\pi_0 \circ \Psi)(p_i)] - [(\psi_0 \circ \Phi')(p_i)]) \\ &\qquad\quad - ([(\psi_1 \circ \Sigma_2)(p_i)] - [\mu_2(p_i)]) = 0 \quad \text{(by (6.85))}.\end{aligned}$$

Note also that, by construction,

$$\Psi_i(1_{\tilde{A}}) = \Phi_i(1_{\tilde{A}}) = 1 \oplus \pi_i(e_0), \quad i = 0, 1. \tag{6.104}$$

Summarizing the calculations in the preceding paragraph, we have

$$[\Phi_i]|_{\mathcal{P}} = [\Psi_i]|_{\mathcal{P}}, \quad i = 0, 1. \tag{6.105}$$

On the other hand, for any $a \in \mathcal{F} \subseteq \mathcal{G}$ and any $\tau \in T(Q^r)$, we have

$$\begin{aligned}[t]&|\tau(\Phi_0(a)) - \tau(\Psi_0(a))| \\ &\quad = |\tau((\psi_0 \circ \Phi')(a) \oplus (\psi_0 \circ \Sigma_2)(a)) - \tau((\pi_0 \circ \Psi)(a) \oplus \mu_2(a))| \\ &\quad < |\tau((\psi_0 \circ \Phi')(a)) - \tau((\pi_0 \circ \Psi)(a))| + \frac{\sigma}{32} \quad \text{(by (6.99))} \\ &\quad < \min\left\{\frac{13\delta_1}{16}, \frac{\sigma}{32}\right\} + \min\left\{\frac{3\delta_1}{16}, \frac{3\sigma}{256}\right\} + \frac{\sigma}{32} \quad \text{(by (6.94))} \\ &\quad \le \frac{5\sigma}{64}.\end{aligned} \tag{6.106}$$

The same argument, using (6.95) instead of (6.94), also shows that

$$|\tau(\Phi_1(a)) - \tau(\Psi_1(a))| < \frac{5\sigma}{64}, \quad a \in \mathcal{F},\ \tau \in T(Q^r). \tag{6.107}$$

Since $1_{\widetilde{A}} \in \mathcal{P}$, by (6.98), $[\Sigma_2(1_{\widetilde{A}})] \in \mathbb{D}$, and so there is a projection $e_1 \in B$ such that $\pi_0(e_1) = \psi_0(\Sigma_2(1_{\widetilde{A}}))$ and $\pi_1(e_1) = \psi_1(\Sigma_2(1_{\widetilde{A}}))$. It then follows from (6.99) (applied just for τ factoring through ψ_0 — alternatively, for τ factoring through ψ_1) that

$$\tau(e_1) < \frac{\sigma}{64}, \quad \tau \in T(B). \tag{6.108}$$

Set $E_0' = 1 \oplus \pi_0(e_0) \oplus \pi_0(e_1)$, $E_1' = 1 \oplus \pi_1(e_0) \oplus \pi_1(e_1)$, and $D_0 = E_0' M_2(Q^r) E_0'$, $D_1 = E_1' M_2(Q^r) E_1'$. We estimate that, using (6.90), (6.101), (6.103) and (6.101),

$$\tau_0(\Psi_0(f_{1/2}(e))) \geq \frac{63d}{64} \quad \text{for } \tau_0 \in T(D_0), \tag{6.109}$$

$$\tau_1(\Psi_1(f_{1/2}(e))) \geq \frac{63d}{64} \quad \text{for } \tau_1 \in T(D_1). \tag{6.110}$$

Then, by (6.106) and (6.107),

$$\tau_0(\Phi_0(f_{1/2}(e))) \geq \frac{62d}{64} \quad \text{for } \tau_0 \in T(D_0), \tag{6.111}$$

$$\tau_1(\Phi_1(f_{1/2}(e))) \geq \frac{62d}{64} \quad \text{for } \tau \in T(D_1). \tag{6.112}$$

By the choice of $\mathcal{G}_1$, δ_0, and (6.69), we have

$$\tau_0(f_{1/2}(\Phi_0(e))) \geq \frac{3d}{4}, \quad \tau_0(f_{1/2}(\Psi_0(e))) \geq \frac{3d}{4} \quad \text{for } \tau_0 \in T(D_0), \tag{6.113}$$

$$\tau_1(f_{1/2}(\Phi_1(e))) \geq \frac{3d}{4}, \quad \tau_1(f_{1/2}(\Psi_1(e))) \geq \frac{3d}{4} \quad \text{for } \tau \in T(D_1). \tag{6.114}$$

Pick a sufficiently small $r' \in \left(0, \frac{1}{4}\right)$ such that

$$\|\Psi(a)((1+2r')t - r') - \Psi(a)(t)\| < \frac{\sigma}{64}, \quad a \in \mathcal{G},\ t \in \left[\frac{r'}{1+2r'}, \frac{1+r'}{1+2r'}\right]. \tag{6.115}$$

It follows from [Elliott et al. 2020a, Lemma 7.2] (with (6.105), (6.113), (6.114), (6.106), (6.107) and (6.104)) that there exist unitaries $u_0 \in D_0$ and $u_1 \in D_1$, and unital $\mathcal{F}$-$\varepsilon/4$-multiplicative completely positive linear maps $L_0 : \widetilde{A} \to C([-r', 0], D_0)$ and $L_1 : \widetilde{A} \to C([1, 1+r'], D_1)$, such that

$$\pi_{-r'} \circ L_0 = \Phi_0, \quad \pi_0 \circ L_0 = \operatorname{Ad} u_0 \circ \Psi_0, \tag{6.116}$$

$$\pi_{1+r'} \circ L_1 = \Phi_1, \quad \pi_1 \circ L_1 = \operatorname{Ad} u_1 \circ \Psi_1, \tag{6.117}$$

$$|\tau \circ \pi_t \circ L_0(a) - \tau \circ \pi_0 \circ L_0(a)| < \frac{5\sigma}{32}, \quad t \in [-r', 0], \tag{6.118}$$

$$|\tau \circ \pi_t \circ L_1(a) - \tau \circ \pi_1 \circ L_1(a)| < \frac{5\sigma}{32}, \quad t \in [1, 1+r,'] \tag{6.119}$$

where $a \in \mathcal{F}$, $\tau \in T(Q^r)$ and (as before) π_t is the point evaluation at $t \in [-r', 1+r']$.

Write $E_3 = 1 \oplus e_0 \oplus e_1 \in M_3(C([0, 1], Q^r))$ and $B_1 = E_3(M_3(C([0, 1], Q^r)))E_3$. There exists a unitary $u \in B_1$ such that $u(0) = u_0$ and $u(1) = u_1$. Consider the projection $E_4 \in M_3(\mathrm{C}([-r', 1+r'], Q^r))$ defined by $E_4|_{[-r,0]} = E_0'$, $E_4|_{[0,1]} = E_3$ and $E_4|_{[1,1+r]} = E_1'$. Set

$$B_2 = E_4\big(M_3(\mathrm{C}([-r', 1+r'], Q^r))\big)E_4.$$

Define a unital $\mathcal{F}$-$\varepsilon/4$-multiplicative (note that $\mathcal{F} \subset \mathcal{G}$ and $\delta \le \varepsilon/8$) completely positive linear map $L' : \widetilde{A} \to B_2$ by

$$L'(a)(t) = \begin{cases} L_0(a)(t), & t \in [-r', 0), \\ \mathrm{Ad}\, u(t) \circ (\pi_t \circ \Psi \oplus \mu_2)(a), & t \in [0, 1], \\ L_1(a)(t), & t \in (1, 1+r']. \end{cases} \tag{6.120}$$

Note that for any $a \in \mathcal{G}$, and any $\tau \in T(Q^r)$, by (6.120), if $t \in [0, 1]$, then

$$\begin{aligned} &|\tau(\pi_t(L'(a))) - \gamma_1(\pi_t^*(\tau))(a)| \\ &\quad = |\tau(\mathrm{Ad}\, u(t) \circ (\pi_t \circ \Psi \oplus \mu_2)(a)) - \gamma_1(\pi_t^*(\tau))(a)| \\ &\quad = |\tau(\pi_t(\Psi(a))) + \tau(\mu_2(a)) - \gamma_1(\pi_t^*(\tau))(a)| \\ &\quad < |(\pi_t^*(\tau))(\Psi(a)) - \gamma_1(\pi_t^*(\tau))(a)| + \frac{\sigma}{64} \qquad \text{(by (6.97) and (6.99))} \\ &\quad < \min\left\{\frac{\delta_1}{16}, \frac{\sigma}{128}\right\} + \frac{\sigma}{64} \le \frac{3\sigma}{128} \qquad \text{(by (6.89))}, \end{aligned} \tag{6.121}$$

where $\pi_t^* : T(Q^r) \to T(B)$ is the dual of $\pi_t : B \to Q^r$. Furthermore, if $t \in [-r', 0]$, then for any $a \in \mathcal{F}$ and any $\tau \in T(Q^r)$,

$$\begin{aligned} &|\tau(\pi_t(L'(a))) - \gamma_1(\pi_0^*(\tau))(a)| \\ &\quad = |\tau(L_0(a)(t)) - \gamma_1(\pi_0^*(\tau))(a)| \\ &\quad < |\tau(L_0(a)(0)) - \gamma_1(\pi_0^*(\tau))(a)| + \frac{5\sigma}{32} \qquad \text{(by (6.118))} \\ &\quad = |\tau(\Psi_0(a)) - \gamma_1(\pi_0^*(\tau))(a)| + \frac{5\sigma}{32} \qquad \text{(by (6.116))} \\ &\quad = |\tau((\pi_0 \circ \Psi)(a) \oplus \mu_2(a)) - \gamma_1(\pi_0^*(\tau))(a)| + \frac{5\sigma}{32} \\ &\quad < |\tau(\pi_0 \circ \Psi)(a) - \gamma_1(\pi_0^*(\tau))(a)| + \frac{\sigma}{64} + \frac{5\sigma}{32} \qquad \text{(by (6.97) and (6.99))} \\ &\quad < \min\left\{\frac{\delta_1}{16}, \frac{\sigma}{128}\right\} + \frac{\sigma}{64} + \frac{5\sigma}{32} < \frac{23\sigma}{128} \qquad \text{(by (6.89))}. \end{aligned} \tag{6.122}$$

Again, if $t \in [1, 1+r']$, then the same argument shows that for any $a \in \mathcal{F}$ and any $\tau \in T(Q^r)$,

$$|\tau(\pi_t(L'(a))) - \gamma_1(\pi_1^*(\tau))(a)| < \frac{23\sigma}{128}. \tag{6.123}$$

Let us modify L' to a unital map from $\widetilde{A}$ to B. First, let us renormalize L'. Consider the isomorphism $\eta : K_0(Q^r) = \mathbb{Q}^r \to K_0(Q^r) = \mathbb{Q}^r$ defined by

$$\eta(x_1, x_2, \dots, x_r) = \left(\frac{1}{\mathrm{tr}_1(E_3)}x_1, \frac{1}{\mathrm{tr}_2(E_3)}x_2, \dots, \frac{1}{\mathrm{tr}_r(E_3)}x_r\right),$$

for all $(x_1, x_2, \dots, x_r) \in \mathbb{Q}^r$, where (as before) tr_k is the tracial state supported on the k-th direct summand of Q^r. Then there is a (unital) isomorphism

$$\varphi : B_2 \to \mathrm{C}([-r', 1+r'], Q^r)$$

such that $\varphi_{*0} = \eta$. Let us replace the map L' with the map $\varphi \circ L'$, and still denote it by L'. Note that it follows from (6.121), (6.86) and (6.108) that for any $t \in [0, 1]$, any $a \in \mathcal{F}$ and any $\tau \in T(Q^r)$,

$$\begin{aligned} |\tau(\pi_t(L'(a))) - \gamma_1(\pi_t^*(\tau))(a)| &< \frac{\sigma}{16} + \tau(e_0) + \tau(e_1) \\ &< \frac{3\sigma}{128} + \min\left\{\frac{\delta_1}{16}, \frac{\sigma}{64}\right\} + \frac{\sigma}{64} \le \frac{7\sigma}{128}. \end{aligned} \tag{6.124}$$

The same argument, using (6.122) and (6.123) instead of (6.121), shows that for any $a \in \mathcal{F}$,

$$|\tau(\pi_t(L'(a))) - \gamma_1(\pi_0^*(\tau))(a)| < \frac{27\sigma}{128}, \quad t \in [-r', 0], \tag{6.125}$$

$$|\tau(\pi_t(L'(a))) - \gamma_1(\pi_1^*(\tau))(a)| < \frac{27\sigma}{128}, \quad t \in [1, 1+r']. \tag{6.126}$$

Now, put

$$L''(a)(t) = L'(a)((1+2r')t - r'), \quad t \in [0, 1]. \tag{6.127}$$

This perturbation does not change the trace very much, as for any $a \in \mathcal{F}$ and any $\tau \in T(Q^r)$, if $t \in [0, r'/(1+2r')]$, then

$$\begin{aligned} &|\tau(L''(a)(t)) - \tau(L'(a)(t))| \\ &\quad = |\tau(L'(a)((1+2r')t - r')) - \tau(L'(a)(t))| && \text{(by (6.127))} \\ &\quad = |\tau(L_0(a)((1+2r')t - r')) - (\tau(\Psi(a)(t)) + \mu_2(a))| \\ &\quad < |\tau(L_0(a)((1+2r')t - r')) - \tau(\Psi(a)(t))| + \frac{\sigma}{64} && \text{(by (6.97) and (6.99))} \\ &\quad < |\tau(L_0(a)(0)) - \tau(\Psi(a)(t))| + \frac{5\sigma}{32} + \frac{\sigma}{64} && \text{(by (6.118))} \\ &\quad = |\tau(\Psi_0(a)(0)) - \tau(\Psi(a)(t))| + \frac{11\sigma}{64} && \text{(by (6.116))} \\ &\quad < \frac{\sigma}{64} + \frac{11\sigma}{64} = \frac{3\sigma}{16} && \text{(by (6.97) and (6.99)).} \end{aligned}$$

Furthermore, the same argument, now using (6.119) and (6.117), shows that for any $a \in \mathcal{F}$, $\tau \in T(Q^r)$ and $t \in [(1+r')/(1+2r'), 1]$,

$$|\tau(L_0(a)((1+2r')t-r')) - (\tau(\Psi(a)(t)) + \mu_2(a))| < \frac{3\sigma}{16},$$

and, if $t \in [r'/(1+2r'), (1+r')/(1+2r')]$, then

$$\begin{aligned}|\tau(L'(a)((1+2r')t-r')) - \tau(L'(a)(t))| &= |\tau(\Psi(a)((1+2r')t-r') - \Psi(a)(t))| \\ &< \frac{\sigma}{64} \quad \text{(by (6.115)).}\end{aligned}$$

Thus

$$|\tau(L''(a)(t)) - \tau(L'(a)(t))| < \frac{3\sigma}{16}, \quad a \in \mathcal{F},\ \tau \in T(Q^r),\ t \in [0,1]. \tag{6.128}$$

Hence, by (6.128), (6.124), (6.125) and (6.126),

$$\begin{aligned}&|\tau(\pi_t(L''(a))) - \gamma_1(\pi_t^*(\tau))(a)| \\ &\qquad \le |\tau(L''(a)(t)) - \tau(L'(a)(t))| + |\tau(L'(a)(t)) - \gamma_1(\pi_t^*(\tau))(a)| \\ &\qquad < \frac{3\sigma}{16} + \frac{27\sigma}{128} = \frac{51\sigma}{128}.\end{aligned} \tag{6.129}$$

Note that L'' is a unital map from $\widetilde{A}$ to B. It is also $\mathcal{F}$-ε-multiplicative, since L' is. Consider the order isomorphism $\eta' : K_0(Q^{l+1}) = \mathbb{Q}^{l+1} \to K_0(Q^{l+1}) = \mathbb{Q}^{l+1}$ defined by

$$\eta'(y_1, y_2, \ldots, y_l) = (a_1y_1, a_2y_2, \ldots, a_ly_l) \quad \text{for } (y_1, y_2, \ldots, y_l) \in \mathbb{Q}^{l+1},$$

where

$$a_j = \frac{1}{\mathrm{tr}_j(1 \oplus \Sigma_1(1_{\widetilde{A}}) \oplus \Sigma_2(1_{\widetilde{A}}))}, \quad j = 1, 2, \ldots, l+1, \tag{6.130}$$

and (as before) tr_j is the tracial state supported on the j-th direct summand of Q^{l+1}. There exists a unital homomorphism

$$\tilde{\varphi} : (1 \oplus \Sigma_1(1_{\widetilde{A}}) \oplus \Sigma_2(1_{\widetilde{A}}))M_3(Q^{l+1})(1 \oplus \Sigma_1(1_{\widetilde{A}}) \oplus \Sigma_2(1_{\widetilde{A}})) \to Q^{l+1}$$

such that

$$\tilde{\varphi}_{*0} = \eta'.$$

Therefore, by the constructions of L'', L', L_0 and L_1 (see (6.127), (6.120), (6.116) and (6.117)), we may assume that

$$\psi_0 \circ \tilde{\varphi} \circ (\Phi' \oplus \Sigma_2) = \pi_0 \circ L'' \quad \text{and} \quad \psi_1 \circ \tilde{\varphi} \circ (\Phi' \oplus \Sigma_2) = \pi_1 \circ L'', \tag{6.131}$$

replacing L'' with $\mathrm{Ad}\, w \circ L''$ for a suitable unitary w if necessary.

Define $L : \widetilde{A} \to \widetilde{C}_1$ by $L(a) = (L''(a), \tilde{\varphi}(\Phi'(a) \oplus \Sigma_2(a)))$, an element of $\widetilde{C}_1$ by (6.131). Since L'' and $\tilde{\varphi} \circ (\Phi' \oplus \Sigma_2)$ are unital and $\mathcal{F}$-$\varepsilon/4$-multiplicative (since Φ' and Σ_2 are $\mathcal{G}$-δ-multiplicative, $\mathcal{F} \subset \mathcal{G}$, and $\delta \le \varepsilon/8$), so too is L.

Moreover, for any $a \in \mathcal{F}$, any $\tau \in T(Q^r)$, and any $t \in (0,1)$, it follows from (6.129) that

$$|\tau(\pi_t(L(a))) - \gamma_1(\pi_t^*(\tau))(a)| < \frac{51\sigma}{128}. \tag{6.132}$$

If $\tau \in T(Q^{l+1})$, then for any $a \in \mathcal{F}$,

$$\begin{aligned}
&|\tau(\pi_{\mathrm{e}}(L(a))) - \gamma^*(\pi_{\mathrm{e}}^*(\tau))(a)| \\
&\quad = |\tau(\tilde{\varphi}(\Phi'(a) \oplus \Sigma_2(a))) - \gamma^*(\pi_{\mathrm{e}}^*(\tau))(a)| \\
&\quad < |\tau(\Phi'(a) \oplus \Sigma_2(a)) - \gamma^*(\pi_{\mathrm{e}}^*(\tau))(a)| + \frac{\sigma}{32} && \text{(by (6.130), (6.86) and (6.99))} \\
&\quad < |\tau(\Phi'(a)) - \gamma^*(\pi_{\mathrm{e}}^*(\tau))(a)| + \frac{3\sigma}{64} && \text{(by (6.99))} \\
&\quad < |\tau(\Phi(a)) - \gamma^*(\pi_{\mathrm{e}}^*(\tau))(a)| + \frac{\sigma}{8} && \text{(by (6.79))} \\
&\quad < \frac{\sigma}{32} + \frac{\sigma}{8} = \frac{5\sigma}{32}. && \text{(by (6.78))}.
\end{aligned}$$

Since each extreme trace of $\widetilde{C}_1$ factors through either the evaluation map π_t or the canonical quotient map π_{e}, by (6.132),

$$|\tau(L(a)) - \gamma^*(\tau)(a)| < \frac{51\sigma}{128}, \quad \tau \in T(\widetilde{C}_1), \quad a \in \mathcal{F}. \tag{6.133}$$

From (6.73), we have $|\gamma^*(\tau_{\mathbb{C}}^{C_1})(a)| < \sigma/128$ for all $f \in \mathcal{F}$. Combing with (6.133), we have

$$|\tau_{\mathbb{C}}^{C_1}(L(a))| < \frac{52\sigma}{128} \quad \text{for all } a \in \mathcal{F}. \tag{6.134}$$

That is, $\|\pi_{\mathbb{C}}^{C_1}(L(a))\| < 52\sigma/128$. For each $a \in \mathcal{F}$, put $a' = L(a) - \lambda 1_{\widetilde{C}_1}$, where $\lambda = \pi_{\mathbb{C}}^{C_1}(L(a)) \in \mathbb{C}$. Choose an element $e_{C_1} \in C_1$ with $0 \le e_{C_1} \le 1$ such that

$$\|e_{C_1} a' e_{C_1} - a'\| < \frac{\sigma}{128} \quad \text{for all } a \in \mathcal{F}. \tag{6.135}$$

Then

$$\|e_{C_1} L(a) e_{C_1} - L(a)\| < \frac{\sigma}{128} + \frac{52\sigma}{128} = \frac{53\sigma}{128} \quad \text{for all } a \in \mathcal{F}. \tag{6.136}$$

Define $\bar{L} : A \to C_1$ by $\bar{L}(a) = e_{C_1} L(a) e_{C_1}$ for all $a \in A$. (Note that $\bar{L}$ is $\mathcal{F}$-$\varepsilon/2$-multiplicative.) Therefore, for any $a \in \mathcal{F}$ and $\tau \in T(C)$, we have

$$\begin{aligned}
&|\tau(\imath_{1,\infty}(\bar{L}(a))) - \Gamma_{\mathrm{Aff}}(\hat{a})(\tau)| \\
&\quad < |\tau(\imath_{1,\infty}(\bar{L}(a))) - \gamma^*(\imath_{1,\infty}(\tau))(a)| + |\gamma_*(\imath_{1,\infty}(\tau))(a) - \Gamma_{\mathrm{Aff}}(\hat{a})(\tau)| \\
&\quad = |\tau(\imath_{1,\infty}(\bar{L}(a))) - \gamma^*(\imath_{1,\infty}(\tau))(a)| + |(\imath_{1,\infty})_{\mathrm{Aff}}(\hat{a})(\tau) - \Gamma_{\mathrm{Aff}}(\hat{a})(\tau)| \\
&\quad < \frac{51\sigma}{128} + \frac{53\sigma}{128} + \frac{\sigma}{128} = \frac{105\sigma}{128}. && \text{(by (6.133) and (6.70))}
\end{aligned}$$

Recalling $f_{\varepsilon'}(e) \in \mathcal{F}$, combining the above inequality with (6.68), one has

$$\tau\big(\imath_{1,\infty}(\bar{L}(f_{\varepsilon'}(e)))\big) \geq \Big(1 - \frac{\sigma}{128}\Big) - \frac{105\sigma}{128} > 1 - \frac{106\sigma}{128} \quad \text{for all } \tau \in T(C). \tag{6.137}$$

Choose a strictly positive $e_A \in A$ such that $e_A \geq f_{\varepsilon'}(e)$ and put $c_1 = \imath_{1,\infty}(L(e_A))$. Then, by (6.137),

$$\tau(c_1) > 1 - \frac{106\sigma}{128} \quad \text{for all } \tau \in T(C). \tag{6.138}$$

Let $H_1 : A \to C$ be defined by $H_1 = \imath_{1,\infty} \circ \bar{L}$. Then

$$|\tau(H_1(a)) - \Gamma_{\mathrm{Aff}}(\hat{a})(\tau)| < \frac{105\sigma}{128} \quad \text{for all } a \in \mathcal{F},\ \tau \in T(C_1). \tag{6.139}$$

Since $\mathcal{F}$, ε, and σ are arbitrary, in this way we obtain a sequence of completely positive linear maps $H_n : A \to C$ such that

$$\lim_{n\to\infty} \|H_n(ab) - H_n(a)H_n(b)\| = 0 \quad \text{for all } a, b \in A, \tag{6.140}$$

$$\lim_{n\to\infty} \sup\{|\tau \circ H_n(a) - \Gamma_{\mathrm{Aff}}(\hat{a})(\tau)| : \tau \in T(C)\} = 0 \quad \text{for all } a \in A, \tag{6.141}$$

and $\tau(H_n(e_A)) \to 1$ uniformly on τ in $T(C)$.

Recall that $C_n = C_{0,n} \otimes Q$, where $C_{0,n} \in \mathcal{C}_0$. Write $Q = \overline{\bigcup_{n=1}^\infty M_{n!}}$ and each $1_{M_{n!}}$ is the identity of Q. Choose a subsequence $\{m(n)\} \subset \{n!\}$ such that

$$C = \overline{\textstyle\bigcup_{n=1}^\infty \bar{C}_n},$$

where $\bar{C}_n := C_{0,n} \otimes M_{m(n)}$, $n = 1, 2, \ldots$, and $\lambda_s(\bar{C}_n) \nearrow 1$ (see Definition 5.3 of [Elliott et al. 2020a]). Without loss of generality, we may assume that, in (6.140) and (6.140), $H_n : A \to \bar{C}_n \to C$, $n = 1, 2, \ldots$.

On the other hand, $\Gamma^{-1} : (K_0(C), T(C), r_C) \cong (K_0(A), T(A), r_A)$ gives an affine homeomorphism $\lambda_T : T(A) \to T(C)$ such that $\Gamma_{\mathrm{Aff}}(\hat{a})(\lambda_T(\tau)) = \tau(a)$ for all $a \in A_{\mathrm{s.a.}}$ and $\tau \in T(A)$. Since $K_1(C) = \{0\}$, by Corollary 7.8 of [Lin 2022], there is a sequence of injective homomorphisms $h'_k : C_k \to A$ such that, for any $c \in \bar{C}_m$,

$$\lim_{k\to\infty} \sup_{\tau\in T(A)} \{|\tau(h'_k \circ \iota_{m,k}(c)) - \lambda_T(\tau)(\iota_{m,\infty}(c))|\} = 0. \tag{6.142}$$

It follows that, by an appropriate choice of a subsequence $\{k(n)\}$ and defining $h_n := h'_{k(n)} \circ \iota_{n,k(n)}$, one obtains that

$$\lim_{n\to\infty} \sup\{|\tau \circ h_n \circ H_n(a) - \tau(a)| : \tau \in T(A)\} = 0 \quad \text{for all } a \in A. \tag{6.143}$$

Note that, as shown at the beginning of the proof, for any nonzero hereditary C^*-subalgebra B of A with continuous scale, $B \otimes Q \cong B$. So all the above work holds for any such B. Also, $B \cong B \otimes Q$ is tracially approximately divisible. By (6.140), (6.141) and (6.143), applying Theorem 6.54 to each hereditary C^*-subalgebra B of A with continuous scale, described in Theorem 6.54, we conclude $A \otimes Q \in \mathcal{D}$. $\square$

Theorem 6.144. *Let A be a separable simple C^*-algebra with continuous scale. Suppose that $A \otimes U \in \mathcal{D}$ for some infinite-dimensional UHF-algebra U. Then $A \otimes B \in \mathcal{D}$ for any infinite-dimensional UHF-algebra B.*

Proof. Suppose that $A \otimes U \in \mathcal{D}$. Then $A \otimes U$ has at least one tracial state. Since the map $a \mapsto a \otimes 1_U$ maps A into $A \otimes U$, A must be stably finite. Moreover, if τ is a 2-quasitrace for A, then $\tau \otimes t_U$ is a trace since $A \otimes U \in \mathcal{D}$ (see Proposition 9.1 of [Elliott et al. 2020b]), where t_U is the unique tracial state on U. It follows that τ is a trace. In other words, $QT(A) = T(A)$. Let B be a unital infinite-dimensional UHF-algebra. Choose a strictly positive element $e_A \in A$ with $\|e_A\| = 1$. We may assume that, as $A \otimes U$ has continuous scale,

$$d = \inf\{\tau(e_A \otimes 1_U) : \tau \in T(A \otimes U)\} > \tfrac{1}{2}. \tag{6.145}$$

Fix $\varepsilon > 0$, a finite subset $\mathcal{F} \subset A \otimes B$ and $a \in (A \otimes B)_+ \setminus \{0\}$. Note that $A \otimes B$ is finite and $\mathcal{Z}$-stable, and has strict comparison for positive elements (see Corollary 4.6 of [Rørdam 2004]). There is a nonzero element $a_0 = a_{00} \otimes b_0 \in (A \otimes B)_+$ for some $a_{00} \in A_+$ and $b_0 \in B_+$ such that $a_0 \lesssim a$ in $A \otimes B$. We may also assume that $b_0 = b_{0,1,1} \oplus b_{0,1,2} \oplus b_{0,2} \in B_+$, where $b_{0,1,1}$, $b_{0,1,2}$ and $b_{0,2}$ are mutually orthogonal nonzero positive elements. Put $b_{0,1} = b_{0,1,1} \oplus b_{0,1,2}$.

As $A \otimes B$ is simple, there is an integer $N_0 \geq 1$ such that

$$e_A \otimes 1_B \lesssim N_0 \langle a_{00} \otimes b_{0,2} \rangle. \tag{6.146}$$

We write $B = \lim_{n\to\infty}(B_n, \psi_n)$, where $B_n = M_{R(n)}$ and $\psi_n : B_n \to B_{n+1}$ is a unital embedding. If $n > m$, put $\psi_{m,n} = \psi_{n-1} \circ \cdots \circ \psi_m : B_m \to B_n$. Denote by $\psi_{n,\infty} : B_n \to B$ the unital embedding induced by the inductive limit. By Proposition 2.2 and Lemma 2.3(b) of [Rørdam 1992], to simplify notation, without loss of generality, replacing b_0 by a smaller (in Cuntz relation) element, we may assume that $b_{0,1,1}, b_{0,1,2}, b_{0,2} \in B_n$ for some large n. Since B is simple, we may assume that $R(n) > 4N_0$ for all n. It follows from (6.146) that we may assume that the range projection of b_0 has rank at least two (as a matrix).

By changing notation, without loss of generality, we may further assume that $\mathcal{F} \subset A \otimes B_1$ and $b_{0,1,1}, b_{0,1,2}, b_{0,2} \in B_1$.

Since $A_1 := A \otimes M_{R(1)} \otimes U \in \mathcal{D}$ has continuous scale, there are $\mathcal{F}$-$\varepsilon/128$-multiplicative completely positive contractive linear maps $\varphi : A_1 \to A_1$ and $\psi : A_1 \to D_0$ for some C^*-subalgebra $D_0 \subset A_1$ with $D_0 \in \mathcal{C}_0$, $D_0 \perp \varphi(A_1)$, and

$$\|x \otimes 1_U - (\varphi(x \otimes 1_U) + \psi(x \otimes 1_U))\| < \frac{\varepsilon}{128} \quad \text{for all } x \in \mathcal{F} \cup \{e_A \otimes 1\}, \tag{6.147}$$

$$\varphi(e_A \otimes 1_U) \lesssim a_{0,1} := a_{00} \otimes b_{0,1,1} \otimes 1_U, \tag{6.148}$$

$$t(f_{1/4}(\psi(e_A \otimes 1))) \geq \frac{3}{4} \qquad \text{for all } t \in T(D_0) \tag{6.149}$$

(see also Proposition 2.10 of [Elliott et al. 2020b]). Replacing φ by the map defined by $\varphi'(x) = f_\eta(\varphi(e_A \otimes 1_U))\varphi(x) f_\eta(\varphi(e_A \otimes 1_U))$ for all $x \in A_1$ for some sufficiently small η, we may assume that there is $e_0 \in A_1$ such that $e_0\varphi(x) = \varphi(x)e_0 = \varphi(x)$ for all $x \in A_1$, $e_0 \perp D_0$ and $e_0 \lesssim a_{00} \otimes b_{0,1,1} \otimes 1_U$.

Let $\mathcal{G} \subset D_0$ be a finite subset such that, for every $x \in \mathcal{F}$, there exists $x' \in \mathcal{G}$ such that $\|\psi(x \otimes 1_U) - x'\| < \varepsilon/128$. We may also assume that $\mathcal{G}$ contains a strictly positive element e_D of D_0 with $\|e_D\| = 1$.

Write $U = \overline{\bigcup_{n=1}^\infty M_{r(n)}}$, where $\lim_{n\to\infty} r(n) = \infty$ and $M_{r(n)} \subset M_{r(n+1)}$ unitally. For each n, there are $a_n, b_n \in A \otimes M_{R(1)} \otimes M_{r(n)}$, $0 \le a_n, b_n \le 1$ such that

$$\begin{aligned} a_n \perp b_n, \quad \lim_{n\to\infty} \|a_n d a_n - d\| = 0 \quad \text{for all } d \in D_0, \\ \lim_{n\to\infty} \|b_n - e_0\| = 0, \quad \lim_{n\to\infty} \|b_n\varphi(a)b_n - \varphi(a)\| = 0 \quad \text{for all } a \in A_1. \end{aligned} \tag{6.150}$$

Since $e_0 \lesssim a_{00} \otimes b_{0,1} \otimes 1_U$, by Proposition 2.2 of [Rørdam 1992], for each n, there exists $k(n)$ such that $f_{1/n}(b_{k(n)}) \lesssim a_{00} \otimes b_{0,1} \otimes 1_U$. Therefore, without loss of generality, replacing b_n by $f_{1/n}(b_{k(n)})$, we may assume (in A_1)

$$b_n \lesssim a_{00} \otimes b_{0,1,1} \otimes 1_U. \tag{6.151}$$

Put $C_n := \overline{a_n(A \otimes M_{R(1)} \otimes M_{r(n)})a_n}$ and $C_n' := \overline{b_n(A \otimes M_{R(1)} \otimes M_{r(n)})b_n}$. Note that $C_n \perp C_n'$.

Since each D_0 is weakly semiprojective, we can choose n_0 large enough such that there exists a unital homomorphism $h : D_0 \to A \otimes M_{R(1)} \otimes M_{r(n_0)}$ $(\subset A \otimes M_{R(1)} \otimes U)$ satisfying

$$\|h(x') - x'\| < \frac{\varepsilon}{64} \quad \text{for all } x' \in \mathcal{G}, \tag{6.152}$$

$$t\big(f_{1/4}(h(\psi(e_A \otimes 1)))\big) \ge \frac{1}{2} \quad \text{for all } t \in T(h(D_0)). \tag{6.153}$$

Consider $\Phi' : A \otimes M_{R(1)} \to A_1$ defined by $\Phi'(a) = \varphi(a \otimes 1_U)$ for $a \in A \otimes M_{R(1)}$. Let $s_0 : M_{R(1)} \otimes U \to M_{R(1)} \otimes M_{r(n_0)}$ be a completely positive contractive linear map such that $s_0|_{M_{R(1)}\otimes M_{r(n_0)}} = \mathrm{id}_{M_{R(1)}\otimes M_{r(n_0)}}$. Define $J := \mathrm{id}_A \otimes s_0 : A \otimes M_{R(1)} \otimes U \to A \otimes M_{R(1)} \otimes M_{r(n_0)}$ and define $\Phi_0 : A \otimes M_{R(1)} \to C'_{n_0} \subset A \otimes M_{R(1)} \otimes M_{r(n_0)}$ by $\Phi_0(a) = b_n(J \circ \Phi'(a))b_n$ for all $a \in A \otimes M_{R(1)}$. Choosing a larger n_0 if necessary, we may also assume that Φ_0 is an $\mathcal{F}$-$\varepsilon/64$-multiplicative completely positive contractive linear map such that

$$\|\Phi_0(x) - \varphi(x \otimes 1_U)\| < \frac{\varepsilon}{64} \quad \text{for all } x \in \mathcal{F}. \tag{6.154}$$

We also assume that (viewing x as an element in $A \otimes M_{R(1)}$)

$$\|x \otimes 1_{M_{r(n_0)}} - (\Phi_0(x) + h \circ \psi(x \otimes 1_U))\| < \frac{\varepsilon}{32} \quad \text{for all } x \in \mathcal{F}. \tag{6.155}$$

Note that $\Phi_0(A_1) \perp h(D_0)$. Moreover, by (6.151), we have, for all $\tau \in T(A_1)$,

$$d_\tau(\Phi_0(e_A)) \le d_\tau(a_{00} \otimes b_{0,1,1} \otimes 1_{M_{r(n_0)}}) < d_\tau(a_{00} \otimes b_{0,1} \otimes 1_{M_{r(n_0)}}). \tag{6.156}$$

Recall that $\Phi_0(e_A)$, $a_{00} \otimes b_{0,1} \otimes 1_{M_{r(n_0)}} \in A \otimes M_{R(1)} \otimes M_{r(n_0)}$. Hence (6.156) holds for any trace with the form $t \otimes \mathrm{Tr}$, where $t \in T(A \otimes M_{R(1)})$ and Tr is the normalized tracial state on $M_{r(n_0)}$.

We may assume that $\psi_{1,n_1} : B_1 \to B_{n_1}$ has multiplicities at least $N \ge 1$ such that

$$\frac{2r(n_0)R(1)}{N} < 1. \tag{6.157}$$

Write

$$R(n_1) = NR(1) = lr(n_0)R(1) + m, \tag{6.158}$$

where $l \ge 1$ and $r(n_0)R(1) > m \ge 0$ are integers. It follows that

$$\frac{m}{R(n_1)} < \frac{r(n_0)R(1)}{R(n_1)} < \frac{r(n_0)}{N} < \frac{1}{2R(1)}. \tag{6.159}$$

Since $R(1) \mid R(n_1)$, we may write $m = m^{(r)}R(1)$. Define $\rho : M_{R(1)} \to M_m$ by $x \to x \otimes 1_{M_{m^{(r)}}}$, if $m > 0$. If $m = 0$, then we omit ρ, or view $\rho = 0$.

It follows from (6.159) that, if $m \ne 0$, by viewing $M_m = M_m \oplus 0_{R(n_1)-m} \subset M_{R(n_1)} \subset B$,

$$d_\tau(e_A \otimes \rho(1_{M_{R(1)}})) < \frac{1}{2R(1)} \quad \text{for all } \tau \in T(A \otimes B). \tag{6.160}$$

Therefore, by (6.146) and the fact that $R(1) > 4N_0$,

$$\iota(e_A \otimes \rho(1_{M_{R(1)}})) \lesssim a_{00} \otimes b_{0,2}. \tag{6.161}$$

Let $\iota_1 : M_{r(n_0)R(1)} \to M_{lr(n_0)R(1)}$ be the embedding defined by $a \mapsto a \otimes 1_{M_l}$. Let $\iota_2 : M_{lr(n_0)R(1)} \to M_{R(n_1)}$ be defined by the embedding which sends rank one projections to rank one projections. Put $\iota_3 = \iota_2 \circ \iota_1$. Define $\iota_4 : A \otimes M_{R(1)} \otimes M_{r(n_0)} \to A \otimes M_{R(n_1)}$ by $\iota_4(a \otimes b) = a \otimes \iota_3(b)$ for all $a \in A$ and $b \in M_{r(n_0)R(1)}$. Note that, for all $a \in B_1 = M_{R(1)}$, we may write

$$\psi_{1,n_1}(a) = \iota_3 \circ \iota_0(a) \oplus \rho(a)$$

(modulo an inner automorphism, and if $m = 0$, $\rho = 0$). Define $\iota : A \otimes B_{n_1} \to A \otimes B$ to be the map given by $a \otimes b \mapsto a \otimes \psi_{1,\infty}(b)$.

Put $E_1 = \iota \circ \iota_4(h(D_0))$. Then $E_1 \in \mathcal{C}_0$. Let $s : B \to B_1$ be a completely positive contractive linear map such that $s|_{B_1} = \mathrm{id}_{B_1}$. Let $j := \mathrm{id}_A \otimes s : A \otimes B \to A \otimes B_1$. Define $\Phi_1' : A \otimes B \to A \otimes B$ by $\Phi_1' := \iota \circ \iota_4 \circ \Phi_0 \circ j$. Define $\Phi_1 : A \otimes B \to A \otimes B$ by $\Phi_1(a) = \Phi_1'(a) \oplus \iota((\mathrm{id}_A \otimes \rho)(j(b)))$ for all $a \in A$ and $b \in B$. Note that Φ_1 is a $\mathcal{F}$-$\varepsilon/2$-multiplicative map and $\Phi_1(A \otimes B) \perp E_1$. Put $\psi' : A \otimes B \to D_0$ by $\psi'(a \otimes b) = \psi(a \otimes s(b) \otimes 1_U)$ for $a \in A$ and $b \in B$. Define $\Psi : A \otimes B \to E_1$

by $\Psi := \iota \circ \iota_4 \circ h \circ \psi'$. Then Ψ is an $\mathcal{F}$-$\varepsilon/2$-multiplicative completely positive contractive linear map. Note that $\iota(x \otimes 1_{M_{r(n_0)}}) = \iota \circ \iota_4(x)$ for all $x \in \mathcal{F}$. Recall also $M_{R(n_1)} = M_{R(1)} \otimes M_N$. Hence, by (6.155), for all $x \in \mathcal{F}$ and (viewing $x \in A \otimes M_{R(1)}$), we estimate that

$$x = \iota(x \otimes 1_{M_N}) = \iota(x \otimes 1_{M_{lr(n_0)}}) \oplus \iota(x \otimes 1_{m^{(r)}}) \tag{6.162}$$

$$= \iota \circ \iota_4(\iota_0(x)) \oplus \iota(\mathrm{id}_A \otimes \rho)(x) \tag{6.163}$$

$$\approx_{\varepsilon/32} (\Phi_1'(x) + \Psi(x)) \oplus \iota(\mathrm{id}_A \otimes \rho)(x) = \Phi_1(x) + \Psi(x). \tag{6.164}$$

By (6.153), we also have

$$t(f_{1/4}(\Psi(e_A \otimes 1))) \geq \tfrac{1}{2} \quad \text{for all } t \in T(E_1). \tag{6.165}$$

By [Rørdam 2004], $A \otimes B$ has strict comparison. By the lines right below (6.156) and (6.161) (viewing $a_{00} \otimes b_{0,1} \otimes 1_{M_{r(n_0)}}$ as an element in $A \otimes M_{R(1)} \otimes M_{r(n_0)} \subset A \otimes B$), we conclude that

$$\Phi_1(e_A \otimes 1_B) = \Phi_1'(e_A \otimes 1_B) \oplus \iota((\mathrm{id}_A \otimes \rho)(e_A \otimes j(1_B))) \tag{6.166}$$

$$\lesssim \iota(a_{00} \otimes b_{0,1} \otimes 1_{M_{r(n_0)}}) \oplus \iota(e_A \otimes \rho(1_{B_1})) \tag{6.167}$$

$$\lesssim (a_{00} \otimes b_{0,1}) \oplus (a_{00} \otimes b_{0,2}) \lesssim a_{00} \otimes b_0 \lesssim a_0 \lesssim a. \tag{6.168}$$

Combining the last relation with (6.164) and (6.165), we obtain $A \otimes B \in \mathcal{D}$. □

Theorem 6.169. *Let A be a separable simple stably projectionless amenable C^*-algebra with continuous scale such that $T(A) \neq \{0\}$ and satisfying the UCT. Then $A \otimes U \in \mathcal{D}$ for any infinite-dimensional UHF-algebra U.*

Proof. Note that $U \cong U_1 \otimes U_2$ for some infinite-dimensional UHF-algebras U_1 and U_2. Consider $A_1 = A \otimes U_1$. Since U_1 is $\mathcal{Z}$-stable, so is A_1. It follows from [Castillejos and Evington 2020] that A_1 has finite nuclear dimension. Thus, to prove the theorem, we may assume that A has finite nuclear dimension. By [Tikuisis et al. 2017], every tracial state of A is quasidiagonal. Then, by Theorems 5.2, 4.107 and 6.65, $A \otimes Q \in \mathcal{D}$. Thus, by Theorem 6.144, $A \otimes U \in \mathcal{D}$ for any infinite-dimensional UHF-algebra U. □

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