H. Lin and P. W. Ng (2023) "Extensions of C^*-Algebras by a Small Ideal,"
International Mathematics Research Notices, Vol. 2023, No. 12, pp. 10350–10438
Advance Access Publication May 30, 2022
https://doi.org/10.1093/imrn/rnac115

Extensions of C^*-Algebras by a Small Ideal

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We classify all essential extensions of the form

$$0 \to \mathcal{W} \to D \to A \to 0,$$

where $\mathcal{W}$ is the unique separable simple C*-algebra with a unique tracial state, which is KK-contractible and has finite nuclear dimension, and A is a separable amenable $\mathcal{W}$-embeddable C*-algebra, which satisfies the Universal Coefficient Theorem (UCT). We actually prove more general results. We also classify a class of amenable C^*-algebras, which have only one proper closed ideal $\mathcal{W}$.

1 Introduction

Motivated by the goal of classifying all essentially normal operators using Fredholm indices, Brown–Douglas–Fillmore (BDF) classified all extensions of the form

$$0 \to \mathcal{K} \to D \to C(X) \to 0,$$

Received August 16, 2021; Revised January 7, 2022; Accepted April 6, 2022

where X is a compact subset of the plane, and later, where X is an arbitrary compact metric space ([3, 5, 7]; see also [8]).

The C^*-algebra $\mathcal{K}$ is perhaps the simplest non-unital simple C^*-algebra. In recent developments of the classification of separable simple amenable C^*-algebras, however, some other seemingly nice non-unital simple C^*-algebras arise. One piquant example is $\mathcal{W}$, which was first studied by Razak [59], and is a non-unital separable simple C^*-algebra with a unique tracial state and $K_i(\mathcal{W}) = \{0\}$, $i = 0, 1$. It is in fact stably projectionless. It is proved in [19] that $\mathcal{W}$ is the only separable stably projectionless simple C^*-algebra with finite nuclear dimension satisfying the Universal Coefficient Theorem (UCT), which has said properties. It is also algebraically simple. Moreover, as we will later elaborate, $\mathcal{W}$ has another very nice feature shared with $\mathcal{K}$, namely that the corona algebra $\mathcal{C}(\mathcal{W}) = M(\mathcal{W})/\mathcal{W}$ is a purely infinite simple C^*-algebra. A natural question is whether one can classify essential extensions of the following form:

$$0 \to \mathcal{W} \to E \to C(X) \to 0. \tag{1.1}$$

Since $K_i(\mathcal{W}) = 0$ for $i = 0, 1$, one immediately realizes that $KK^1(C(X), \mathcal{W}) = 0$. However, as we will see soon, there are many nontrivial essential extensions of $C(X)$ by $\mathcal{W}$ and a variety of unitary equivalence classes of these essential extensions. In other words, the classification of these essential extensions will not follow from the usual stable KK theory.

Other questions also naturally emerge. For example, how many extensions have the form

$$0 \to \mathcal{W} \to E \to \mathcal{W} \to 0? \tag{1.2}$$

More generally, can one classify all the essential extensions of the form

$$0 \to \mathcal{W} \to E \to A \to 0 \tag{1.3}$$

for some general class of separable amenable C^*-algebras A?

As mentioned above, the classification will not follow from the usual stable KK-theory. As one may expect, some restrictions on A will be inevitably added. If A is a separable amenable C^*-algebra, then, by [32], A can always be embedded into O_2, the separable purely infinite simple C^*-algebra in the UCT class, which has trivial K_i-group ($i = 0, 1$). Since $M(\mathcal{W})/\mathcal{W}$ is simple purely infinite, O_2 can be embedded into $M(\mathcal{W})/\mathcal{W}$. This immediately implies that, for the aforementioned C^*-algebras A, essential extensions by $\mathcal{W}$ always exist. In order to have some nice description of a class of extensions, like the ones in (1.3), one may at least want to have some trivial essential extensions, that is, those essential extensions, in (1.3), that split. However, unlike the classical case, this is in general hopeless. Note that if (1.3) is a trivial extension then it induces a *-embedding of A into $M(\mathcal{W})$. But $M(\mathcal{W})$ has a faithful tracial state, which is the extension of the unique tracial state of $\mathcal{W}$. This implies that A has a faithful tracial state. So we will assume that A has a faithful tracial state. Moreover, one may also want to have some diagonal trivial extensions of the form (1.3). The conventional way to do this is to allow A to be embeddable into $\mathcal{W}$. We will then present a classification of these extensions (see Theorem 9.9).

Recent successes in the theory of classification of simple C^*-algebras also make it impossible to resist the attempt to classify at least some non-simple C^*-algebras. This is an ambitious and challenging task. At this stage, our experiments will be limited to the situation where the K-theory is still manageable and we will avoid the cases where the tracial information becomes non-traceable. One of the goals of this research is to classify some amenable C^*-algebras, which have only one ideal $\mathcal{W}$. So these C^*-algebras also have the form of E as in (1.3). Since we assume that $\mathcal{W}$ is the only ideal, A will be a separable simple amenable C^*-algebra. As discussed above, we will assume that A is embeddable into $\mathcal{W}$, and so A is a stably projectionless simple C^*-algebra. Let us point out that for any separable amenable C^*-algebra A, which has a faithful tracial state and satisfies the UCT, $A \otimes \mathcal{Z}_0$ is $\mathcal{W}$ embeddable, where $\mathcal{Z}_0$ is the unique separable simple C^*-algebra with a unique tracial state, which satisfies the UCT such that $K_0(\mathcal{Z}_0) = \mathbb{Z}$, $K_0(\mathcal{Z}_0)_+ = \{0\}$, $K_1(\mathcal{Z}_0) = \{0\}$ and has finite nuclear dimension (so $K_*(A) = K_*(A \otimes \mathcal{Z}_0)$ and $T(A) = T(A \otimes \mathcal{Z}_0)$).

Denote by $\mathcal{E}$ the class of C^*-algebras E, which are essential extensions of the form (1.3) such that A is any separable simple stably projectionless C^*-algebra with $K_0(A) = \mathrm{k}er\rho_A$, and, as customary, A has finite nuclear dimension and satisfies the UCT. Note that, in the definition of the class $\mathcal{E}$, we do not fix the quotient algebra A. We will show that, when E_1 and E_2 are two such C^*-algebras, then $E_1 \cong E_2$, if and only if they have isomorphic Elliott invariants (see Theorem 9.6).

For the remainder of this introduction, we elaborate on some aspects that were earlier alluded to. Perhaps one reason for the success of the BDF theory was that their multiplier algebra $\mathbb{B}(l_2)$ and corona algebra $\mathbb{B}(l_2)/\mathcal{K}$ have particularly nice and simple structure. Among other things, $\mathbb{B}(l_2)$ has real rank zero (it is in fact a von Neumann algebra) and strict comparison, and $\mathbb{B}(l_2)/\mathcal{K}$ is simple purely infinite. For example, the the BDF–Voiculescu result, which roughly says that all essential extensions are absorbing [1, 72], would not be true if the Calkin algebra $\mathbb{B}(l_2)/\mathcal{K}$ were not simple. We may further note that, even in the case that the ideal is stable, as long as the corona algebra is not simple, Kasparov's KK^1 group cannot be used to classify these essential extensions up to unitary equivalence.

Recall that a non-unital σ-unital simple C*-algebra B is said to have *continuous scale* if B has a sequential approximate unit $\{e_n\}$ such that

(a) $e_{n+1}e_n = e_n$ for all n, and
(b) for every $a \in B_+ \setminus \{0\}$, there exists an $N \geq 1$ such that for all $m > n \geq N$,

$$e_m - e_n \lesssim a,$$

where $e_m - e_n \lesssim a$ means that there exists a sequence $\{x_k\}$ in B for which $x_k a x_k^* \to e_m - e_n$.

(See [33].)

In [39] (Theorem 2.4 of [39]; see also Theorem 2.8 of [33]), it was shown that a simple non-unital non-elementary σ-unital C*-algebra B has continuous scale if and only if the corona algebra $\mathcal{C}(B)$ is simple, and, if and only if $\mathcal{C}(B)$ is simple purely infinite. Simple continuous scale C*-algebras are basic building blocks for generalizing extension theory (see, e.g., [34, 37]) and have been much studied. As alluded to earlier, aside from their basic role in the theory, the extension theory of these algebras are in themselves quite interesting. For example, unlike the case of the classical theory of absorbing extensions, there are no infinite repeats, and one needs to develop a type of nonstable absorption theory, where, among other things, the class of a trivial extension need not be the zero class. More refined considerations are required to take into account the new K theory that arises. Some results in this direction were first derived many years ago (see, e.g., [34] and [37]).

As mentioned above, in the present paper, we classify a class of extensions by the Razak algebra $\mathcal{W}$, which is a C*-algebra with continuous scale, $K_*(\mathcal{W}) = 0$ and unique trace. Unlike the previous cases, our canonical ideal $\mathcal{W}$ has no non-zero projections—in fact, $\mathcal{W}$ is stably projectionless, which is like the "opposite" of being real rank zero. We note that the property of real rank zero has been present implicitly since the beginnings of the subject (even though the term "real rank zero" was invented after the BDF papers [7, 8]). For example, the original BDF proof of the uniqueness of the neutral element (for the case of compact subset of the plane) was essentially the Weyl–von Neumann–Berg theorem, and it is well known among experts that under mild conditions on a C*-algebra B, $M(B)$ has a Weyl–von Neumann theorem for self-adjoint operators if and only if $M(B)$

has real rank zero [77]. Moreover, this phenomenon reoccurs throughout the original and subsequent papers. We believe that our present result is the first classification of a class of extensions by a simple projectionless C*-algebra (in fact, the first case of a simple algebra which has real rank greater than zero).

To further illustrate the results of this research and the difference from the usual stable results in the C*-algebra extension theory, we end this introduction by presenting one of our main results. Notations and terminologies in the statement will be explained later in the paper.

Theorem 1.1 (see Theorem 9.9). Let A be a separable amenable C^*-algebra, which is $\mathcal{W}$ embeddable and satisfies the UCT.

(1) Let $\tau_1, \tau_2 : A \to \mathcal{C}(\mathcal{W})$ be two essential extensions. Then $\tau_1 \sim^u \tau_2$ if and only if $KK(\tau_1) = KK(\tau_2)$.

(2) The map

$$\Lambda : \mathbf{Ext}^u(A, \mathcal{W}) \to KK(A, \mathcal{C}(\mathcal{W})) \cong \mathrm{Hom}(K_0(A), \mathbb{R}) \tag{1.4}$$

defined by $\Lambda([\tau]) = KK(\tau)$ is a group isomorphism.

(3) An essential extension $\tau : A \to \mathcal{C}(\mathcal{W})$ is trivial and diagonal if and only if $KK(\tau) = 0$, and all essential trivial and diagonal extensions of A by $\mathcal{W}$ are unitarily equivalent. In fact, the essential trivial diagonal extensions induce the neutral element of $\mathbf{Ext}^u(A, \mathcal{W})$.

(4) An essential extension $\tau : A \to \mathcal{C}(\mathcal{W})$ is trivial if and only if there exist $t \in T_f(A)$ (see Definition 2.2) and $r \in (0, 1]$ such that

$$\tau_{*0}(x) = r \cdot r_A(t)(x) \ \text{ for all } \ x \in K_0(A).$$

(5) Let $\mathcal{T}$ be the set of unitary equivalence classes of essential trivial extensions of A by $\mathcal{W}$. Then,

$$\Lambda(\mathcal{T}) = \{r \cdot h : r \in (0, 1], \ h \in \mathrm{Hom}(K_0(A), \mathbb{R})_{T_f(A)}\} \ \text{(see Definition 2.6)}.$$

(6) All quasidiagonal essential extensions of A by $\mathcal{W}$ are trivial and are unitarily equivalent.

(7) In the case where $\ker\rho_{f,A} = K_0(A)$, all essential trivial extensions of A by $\mathcal{W}$ are unitarily equivalent. Moreover, in this case, an essential extension $\tau : A \to \mathcal{C}(\mathcal{W})$ is trivial if and only if $KK(\tau) = \{0\}$.

(8) In the case where $\ker \rho_{f,A} \neq K_0(A)$, there are essential trivial extensions of A by $\mathcal{W}$ which are not quasidiagonal, and not all essential trivial extensions of A by $\mathcal{W}$ are unitarily equivalent (see (5) above).

2 Notation

Definition 2.1. For each C^*-algebra B, $M(B)$ denotes the multiplier algebra of B, and $\mathcal{C}(B) := M(B)/B$ denotes the corresponding corona algebra. For each C*-algebra extension

$$0 \to B \to D \to C \to 0 \tag{2.1}$$

(of C by B, (in the literature, the terminology is sometimes reversed and this is sometimes called an "extension of B by C"), we will work with the corresponding *Busby invariant* which is a homomorphism $\phi : C \to \mathcal{C}(B)$. Recall that (2.1) is essential if and only if ϕ is injective. We will mainly be working with essential extensions.

An extension is unital if the corresponding Busby invariant is a unital map. We will mainly be working with non-unital extensions.

Let $\phi, \psi : C \to \mathcal{C}(B)$ be two essential extensions. We say that ϕ and ψ are *(weakly) equivalent* and write $\phi \sim \psi$ if there is a partial isometry $v \in \mathcal{C}(B)$ such that $v^*v\phi(c) = \phi(c)v^*v = \phi(c)$ and $vv^*\psi(c) = \psi(c)vv^* = \psi(c)$ for all $c \in C$ and

$$v\phi(c)v^* = \psi(c) \text{ for all } c \in C. \tag{2.2}$$

Ext(A,B) denotes the collection of all (weak) equivalence classes of essential extensions $\phi : A \to \mathcal{C}(B)$.

Let $\pi : M(B) \to M(B)/B = \mathcal{C}(B)$ be the quotient map. Throughout this paper, unless otherwise stated, π always denotes this quotient map.

We say that ϕ and ψ are *unitarily equivalent* (and write $\phi \sim^u \psi$) if there exists a unitary $u \in M(B)$ such that

$$\phi(c) = \pi(u)\psi(c)\pi(u)^*$$

for all $c \in C$.

Ext$^u(A,B)$ denotes the collection of all unitary equivalence classes of essential extensions $\phi : A \to \mathcal{C}(B)$.

Definition 2.2. Let A be a C^*-algebra. Denote by $T(A)$ the tracial state space of A (which could be an empty set), given the weak* topology. Denote by $T_f(A)$ the set of all faithful tracial states of A. $T_f(A)$ is a convex subset of $T(A)$. Let $\tilde{T}(A)$ be the cone of densely defined, positive, (norm) lower semi-continuous traces on A, equipped with the topology of pointwise convergence on elements of the Pedersen ideal $\mathrm{Ped}(A)$ of A. Let B be another C^*-algebra with $T(B) \neq \emptyset$ and let $\phi : A \to B$ be a homomorphism which maps an approximate unit of A to an approximate unit of B. We will then use $\phi_T : T(B) \to T(A)$ for the induced continuous affine map.

Let I be a (closed two-sided) ideal of A and $\tau \in T(I)$. It is well known that τ can be uniquely extended to a tracial state of A (by taking $\tau(a) = \lim_\alpha \tau(ae_\alpha)$ for all $a \in A$, where $\{e_\alpha\}$ is an approximate identity for I). In what follows, we will continue to use τ for the extension. Also, when A is not unital and $\tau \in T(A)$, we will use τ for the extension to $\widetilde{A}$ as well as to $M(A)$, the multiplier algebra of A.

Definition 2.3. Let $r \geq 1$ be an integer and $\tau \in \tilde{T}(A)$. We will continue to use τ to denote the trace $\tau \otimes \mathrm{Tr}$ on $A \otimes M_r$, where Tr is the standard non-normalized trace on M_r. Let $S \subset \tilde{T}(A)$ be a convex subset. Denote by $\mathrm{Aff}(S)$ the space of all continuous real-valued affine functions on S. Define (see [62])

$$\mathrm{Aff}_+(S) \quad := \quad \{f : C(S, \mathbb{R})_+ : f \textit{ affine}, f(\tau) > 0 \text{ for all } \tau \neq 0\} \cup \{0\}, \tag{2.3}$$

$$\mathrm{L}\mathit{Aff}_+(S) \quad := \quad \{f : S \to [0, \infty] : \exists \{f_n\}, f_n \nearrow f,\ f_n \in \mathrm{Aff}_+(S) \text{ for all } n\}, \text{ and} \tag{2.4}$$

$$\mathrm{L}\mathit{Aff}^{\sim}(S) \quad := \quad \{f_1 - f_2 : f_1 \in \mathrm{L}\mathit{Aff}_+(S) \text{ and } f_2 \in \mathrm{Aff}_+(S)\}. \tag{2.5}$$

Note that $0 \in \mathrm{L}\mathit{Aff}_+(S)$. For most of this paper, $S = \tilde{T}(A)$ or $S = T(A)$ will be used in the above definition.

Recall that $T(A)$ is compact, and hence a compact convex set, when A is unital or A is simple separable finite and has continuous scale. Also, when A is simple, $T_f(A) = T(A)$.

Definition 2.4. For any $\delta > 0$, we let $f_\delta, : [0, \infty) \to [0, 1]$ be the unique continuous map satisfying

$$f_\delta(t) := \begin{cases} 0 & t \in [0, \delta/2] \\ 1 & t \in [\delta, \infty) \\ \text{linear on} & [\delta/2, \delta]. \end{cases}$$

Definition 2.5. Recall that every $\tau \in \tilde{T}(A)$ extends uniquely to a strictly lower semicontinuous trace on $M(A)_+$, which we also denote by τ.

For any $\tau \in \tilde{T}(A)$ and $a \in A_+$ (or $a \in M_m(A)_+$ for some integer $m \geq 1$),

$$d_\tau(a) := \lim_{n\to\infty} \tau(f_{1/n}(a)).$$

Note that $f_{1/n}(a)$ is in the Pedersen ideal of A. It follows that $d_\tau(a)$ is a lower semicontinuous, positive homogeneous, additive function on $\tilde{T}(A)$. (Recall that in the case where $a \in M_m(A)_+$, we continue to use τ for $\tau \otimes Tr$, where Tr is the standard non-normalized trace on M_m.)

Definition 2.6. Let A be a C^*-algebra. Let $\mathrm{Hom}(K_0(A), \mathbb{R})$ be the set of homomorphism s from $K_0(A)$ to $\mathbb{R}$. Denote by $\mathrm{Hom}(K_0(A), \mathbb{R})_+$ the set of all homomorphism s $f : K_0(A) \to \mathbb{R}$ such that $f(x) \geq 0$ for all $x \in K_0(A)_+$. Denote

$$\ker\rho_A := \{x \in K_0(A) : f(x) = 0 \ \text{for all} \ f \in \mathrm{Hom}(K_0(A), \mathbb{R})_+\}.$$

It is possible that $\mathrm{Hom}(K_0(A), \mathbb{R})_+ = \{0\}$. In that case, $\ker\rho_A = K_0(A)$. There is a homomorphism $r_A : T(A) \to \mathrm{Hom}(K_0(A), \mathbb{R})_+$ induced by $r_A(\tau)([p]) = \tau(p)$ for all projections $p \in M_m(\widetilde{A})$. The image of r_A is denoted by $\mathrm{Hom}(K_0(A), \mathbb{R})_{T(A)}$ (or just $\mathrm{Hom}(K_0(A), \mathbb{R})_T$). Note that for any $\tau \in T(A)$, $\tau([1_{\tilde{A}}]) = 1$. If A is unital and exact, then by Corollary 3.4 of [2],

$$\mathrm{Hom}(K_0(A), \mathbb{R})_+ = \{r \cdot s : r \in \mathbb{R}_+, s \in \mathrm{Hom}(K_0(A), \mathbb{R})_{T(A)}\}. \tag{2.6}$$

Let Y be a locally compact metric space and $A = C_0(Y)$. Then,

$$\mathrm{Hom}(K_0(A), \mathbb{R})_+ = \{r \cdot s|_{K_0(A)} : r \in \mathbb{R}_+, s \in \mathrm{Hom}(K_0(\widetilde{A}), \mathbb{R})_{T(A)}\}.$$

Let A be a separable exact simple C^*-algebra. Choose a nonzero element $e \in \mathrm{Ped}(A)_+$. Let $A_e = \mathrm{Her}(e) = \overline{eAe}$. Then, $\mathbb{R}_+ r_{A_e}(T(A_e)) = \mathrm{Hom}(K_0(A_e), \mathbb{R})_+$. By [4], the embedding $\iota : A_e \to A$ induces an isomorphism $\iota_* : K_0(A_e) \cong K_0(A)$. Then,

$$\mathrm{Hom}(K_0(A), \mathbb{R})_+ = \{r \cdot s \circ {\iota_*}^{-1} : r \in \mathbb{R}_+ \text{ and } s \in \mathrm{Hom}(K_0(A_e), \mathbb{R})_{T(A_e)}\}.$$

In particular,

$$\ker \rho_A = \{x \in K_0(A) : r_{A_e}(\tau)({\iota_*}^{-1}(x)) = 0 \text{ for all } \tau \in T(A_e)\}.$$

Denote $\mathrm{Hom}(K_0(A), \mathbb{R})_{T_f} := r_A(T_f(A))$. Define

$$\ker \rho_{f,A} := \{x \in K_0(A) : \lambda(x) = 0 \text{ for all } \lambda \in \mathrm{Hom}(K_0(A), \mathbb{R})_{T_f}\}.$$

It should be noted that $\ker \rho_A \subset \ker \rho_{f,A} \subset K_0(A)$. Recall that if A is simple, then $T_f(A) = T(A)$ (see Remark 8.2 for more comments on $\mathrm{Hom}(K_0(A), \mathbb{R})_{T_f}$).

Suppose that A is a σ-unital simple C^*-algebra such that A has continuous scale, every 2-quasi-trace of A is a trace and $\tilde{T}(A) \neq \{0\}$. Then $T(A)$ is compact and $\tilde{T}(A)$ is a cone with base $T(A)$. There is an order preserving homomorphism $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$ such that $\rho_A([p])(\tau) = \tau(p)$ for all projections $p \in M_\infty(\widetilde{A})$. For any unital stably finite C^*-algebra A, ρ_A can also be similarly defined (see Theorem 3.3 of [2]). We note that, with A as in this paragraph, the kernel of the group homomorphism ρ_A is the same as the object $ker\rho_A$, which is defined at the beginning of the present Definition 2.6 – and this is consistent with conventional notation.

Finally, when the context is clear, we often omit A and f and write ρ for ρ_A or $\rho_{f,A}$.

Definition 2.7. For a C^*-algebra D and for $a, b \in D_+$, $a \lesssim b$ means that there exists a sequence $\{x_n\}$ in D such that $x_n b x_n \to a$. We write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$. To avoid possible confusion, if both p and q are projections in D, we write $p \approx q$ to mean that p and q are Murray–von Neumann equivalent in D. For $a \in D_+$, we let $\mathrm{Her}_D(a) := \overline{aDa}$, the hereditary C*-subalgebra of D generated by a. Sometimes, for simplicity, we write $\mathrm{Her}(a)$ in place of $\mathrm{Her}_D(a)$. Similarly, for a C*-subalgebra $C \subseteq D$, we let $\mathrm{Her}_D(C)$ or $\mathrm{Her}(C)$ denote $\overline{CDC}$, the hereditary C*-subalgebra of D generated by C.

Definition 2.8. Let A and C be C^*-algebras. We say that a map $\phi : A \to C$ is *c.p.c.* if it is linear and completely positive contractive. Let $\mathcal{F} \subset A$ be a finite subset and let $\delta > 0$ be given. A c.p.c. map $\psi : A \to \mathcal{C}$ is said to be *$\mathcal{F}$-δ-multiplicative* if $\|\psi(fg) - \psi(f)\psi(g)\| < \delta$ for all $f, g \in \mathcal{F}$.

Definition 2.9. A non-unital C^*-algebra B *stably has almost stable rank one* if for any integer $m \geq 1$, and for any hereditary C*-subalgebra $D \subseteq M_m(B)$, $D \subseteq \overline{GL(\widetilde{D})}$.

We note that, in the literature, *almost stable rank one* often means only taking $m = 1$ in the above definition. (See, e.g., [63, Definition 3.1] and [19].)

Definition 2.10. Let A and B be C^*-algebras and let $\phi : A \to B$ be a homomorphism. We let $KK(\phi)$ denote the element in $KK(A, B)$ induced by ϕ, and we let $KL(\phi)$ denote the element in $KL(A, B)$ induced by ϕ.

Finally, we will be a bit loose in our terminology and use the term "extension" to refer both to an extension $0 \to B \to E \to A \to 0$ as well as the extension algebra E in the exact sequence.

3 Nonstable Absorption

Definition 3.1. Let A be a separable C^*-algebra, and let B be a non-unital but σ-unital C^*-algebra with continuous scale. Let $\phi, \psi : A \to \mathcal{C}(B)$ be two essential extensions. The *BDF sum* of ϕ and ψ is defined to be

$$(\phi \dot{+} \psi)(.) := S\phi(.)S^* + T\psi(.)T^*,$$

where $S, T \in \mathcal{C}(B)$ are any two isometries such that

$$SS^* + TT^* \leq 1.$$

The BDF sum $\phi \dot{+} \psi$ is well defined (i.e., independent of choice of S and T) up to weak equivalence. If, in addition, ϕ or ψ is non-unital then the BDF sum $\phi \dot{+} \psi$ is well-defined up to unitary equivalence (see, e.g., [49, 52]).

Definition 3.2. Let A be a separable C^*-algebra, and let B be a non-unital but σ-unital C^*-algebra with continuous scale. Recall that **Ext**(A, B) denotes the collection of all (weak) equivalence classes of essential extensions $\phi : A \to \mathcal{C}(B)$.

We make **Ext**(A, B) into an abelian semigroup with the sum induced by the BDF sum, that is, for all $[\phi], [\psi] \in$ **Ext**(A, B),

$$[\phi] + [\psi] := [\phi \dot{+} \psi].$$

We often also call the sum on **Ext**(A, B) the *BDF sum*. Similarly, when A is non-unital, with the BDF sum, **Ext**$^u(A, B)$ is also a semigroup.

The next result is well known, but we provide it for the convenience of the reader.

Lemma 3.3. Let A be a separable C^*-algebra, and let B be a σ-unital C^*-algebra. Let $\phi : A \to \mathcal{C}(B)$ be a non-unital essential extension. Then there exists a nonzero element $c \in \mathcal{C}(B)_+$ such that

$$c \perp ran(\phi).$$

Proof. If A is a unital C^*-algebra, then we can simply take

$$c := 1_{\mathcal{C}(B)} - \phi(1_A).$$

Suppose that A is non-unital. Then $D := \overline{\phi(A)\mathcal{C}(B)\phi(A)}$ is a σ-unital proper hereditary C*-subalgebra of $\mathcal{C}(B)$. Then, by Pedersen's double annihilator theorem (see Theorem 7.7 of [57]), $D^\perp$ is nonzero, and hence, we can take $c \in D^\perp_+ \setminus \{0\}$. ■

Proposition 3.4. Let A be a separable C^*-algebra and B be a σ-unital C^*-algebra such that $M(B)/B = \mathcal{C}(B)$ is purely infinite and simple. Let $\phi_1, \phi_2 : A \to \mathcal{C}(B)$ be two non-unital essential extensions.

Then $\phi_1 \sim \phi_2$ if and only if $\phi_1 \sim^u \phi_2$.

Moreover, if ϕ_1 and ϕ_2 are approximately unitarily equivalent, then there exists a sequence of unitaries $U_n \in M(B)$ such that

$$\lim_{n\to\infty} \pi(U_n)^*\phi_1(a)\pi(U_n) = \phi_2(a) \text{ for all } a \in A. \tag{3.1}$$

Proof. This is Proposition 2.1 of [52]. ■

Proposition 3.5. Let B be a σ-unital simple C^*-algebra with continuous scale. Let $\phi_1, \phi_2, \psi : A \to \mathcal{C}(B)$ be three non-unital essential extensions. Suppose that there is a unitary $U \in M_2(M(B))$ such that

$$\pi(U)^*(\phi_1(a) \oplus \phi_2(a))\pi(U) = \psi(a) \text{ for all } a \in A. \tag{3.2}$$

Then there is a unitary $V \in M(B)$ such that

$$\pi(V)^*(\phi_1(a) \dot{+} \psi_2(a))\pi(V) = \psi(a) \text{ for all } a \in A, \tag{3.3}$$

where $\phi_1 \oplus \phi_2 : A \to M_2(\mathcal{C}(B))$ is the orthogonal direct sum of ϕ_1 and ϕ_2, and $\phi_1 \dot{+} \phi_2$ is the BDF sum.

Proof. Write the BDF sum as

$$\phi_1 \dot{+} \phi_2 = S\phi_1(.)S^* + T\phi_2(.)T^*, \tag{3.4}$$

where $S, T \in \mathcal{C}(B)$ are isometries as in 3.1. Set $p_s = SS^*$ and $p_t = TT^*$. Then, $p_s \perp p_t$.

As in 3.1, there are unitaries $v_1, v_2 \in \mathcal{C}(B)$ such that

$$v_1 S\phi_1(a)S^* v_1^* = \phi_1(a) \text{ and } v_2 T\phi_2(a)T^* v_2 = \phi_2(a) \text{ for all } a \in A. \tag{3.5}$$

Put $E_1 = diag(1, 0)$ and $E_2 = diag(0, 1)$. There is a partial isometry $v_3 \in M_2(\mathcal{C}(B))$ such that $v_3^* v_3 = E_1$ and $v_3 v_3^* = E_2$. Define $w = v_1 p_s + v_3 v_2 p_t$. Then,

$$ww^* = v_1 p_s v_1^* + v_3 v_2 p_t v_2^* v_3^* \le 1_{M_2(\mathcal{C}(B))} \text{ and} \tag{3.6}$$

$$w^* w = p_s v_1^* v_1 p_s + p_t v_2^* v_3^* v_3 v_2 p_t = p_s + p_t. \tag{3.7}$$

Moreover,

$$w(S\phi_1(a)S^* + T\phi_2(a)T^*)w^* = \phi_1(a) \oplus \phi_2(a) \text{ for all } a \in A. \tag{3.8}$$

Therefore,

$$1_{\mathcal{C}(B)}\pi(U)w(S\phi_1(a)S^* + T\phi_2(a)T^*)w^*\pi(U)^* 1_{\mathcal{C}(B)} = \psi(a) \text{ for all } a \in A. \tag{3.9}$$

Since ψ is not unital, $\psi(A)^\perp \neq \{0\}$. As $\mathcal{C}(B)$ is purely infinite and simple, it has real rank zero (see [76]). Let $e_0 \in \psi(A)^\perp$ be a non-zero projection and $p = 1_{\mathcal{C}(B)} - e_0$. Let

$q = w\pi(U)p\pi(U)^*w^*$. Note that

$$q = w\pi(U)p\pi(U)^*w^* \neq w\pi(U)1_{\mathcal{C}(B)}\pi(U)^*w^* \leq p_s + p_t \leq 1_{\mathcal{C}(B)}. \tag{3.10}$$

In other words, $1_{\mathcal{C}(B)} - q \neq 0$. Note that p and q are equivalent projections in $\mathcal{C}(B)$. This implies that $1_{\mathcal{C}(B)} - q$ and $e_0 = 1_{\mathcal{C}(B)} - p$ are equivalent in $\mathcal{C}(B)$. Thus, there is a partial isometry $v_0 \in \mathcal{C}(B)$ such that $v_0^*v_0 = 1_{\mathcal{C}(B)} - p$ and $v_0v_0^* = 1_{\mathcal{C}(B)} - q$. Set $v_1 = qw\pi(U)p + v_0$. Then,

$$v_1^*v_1 = (v_0^* + p\pi(U)^*w^*q)(v_0 + qw\pi(U)p) \tag{3.11}$$
$$= v_0^*v_0 + p\pi(U)^*w^*qw\pi(U)p = 1_{\mathcal{C}(B)} - p + p = 1_{\mathcal{C}(B)} \text{ and} \tag{3.12}$$
$$v_1v_1^* = (v_0 + qw\pi(U)p)(v_0^* + p\pi(U)^*w^*q) \tag{3.13}$$
$$= v_0v_0^* + qw\pi(U)p\pi(U)^*w^*q = 1_{\mathcal{C}(B)} - q + q = 1_{\mathcal{C}(B)}. \tag{3.14}$$

So v_1 is a unitary. Moreover,

$$v_1(S\phi_1(a)S^* + T\phi_2(a)T^*)v_1^* = \psi(a) \text{ for all } a \in A. \tag{3.15}$$

Since both ψ and $S\phi_1(\cdot)S^* + T\phi_2(\cdot)T^*$ are not unital, by 3.4, there is a unitary $V \in M(B)$ such that

$$\pi(V)(S\phi_1(a)S^* + T\phi_2(a)T^*)\pi(V)^* = \psi(a) \text{ for all } a \in A. \tag{3.16}$$

This completes the proof. ■

Remark 3.6. By Proposition 3.5, from now on, we will not distinguish between the usual orthogonal sum of two non-unital essential extensions and the BDF sum of the same two non-unital essential extensions. Proposition 3.5 should of course be known. Let us point out the following fact: Suppose that $H_1, H_2 : A \to M(B)$ are two maps such that $\pi \circ H_1$ and $\pi \circ H_2$ are non-unital essential extensions. Then, in general, one may not be able to find unitaries $U, V \in M(B)$ such that $\mathrm{Ad}\,U \circ H_1 \perp \mathrm{Ad}\,V \circ H_2$ even in the case that both H_1 and H_2 are diagonal maps and $\pi \circ H_1 \perp \pi \circ H_2$.

Theorem 3.7. Let A be a separable nuclear C^*-algebra, and let B be a σ-unital simple C^*-algebra with continuous scale. Then $\mathbf{Ext}(A, B)$ is an abelian group. Moreover, if A is non-unital, then $\mathbf{Ext}^u(A, B)$ is also an abelian group.

Proof. This is Theorem 2.10 of [52]. See also Theorem 3.5 of [49]. The second part follows from 3.4. ■

Theorem 3.8. Let A be a separable nuclear C^*-algebra and let B be a σ-unital simple C^*-algebra with continuous scale. Suppose that $\phi, \psi : A \to C(B)$ are two monomorphisms with ϕ non-unital.

Then

$$\phi \sim^u \psi \oplus \psi_0 \tag{3.17}$$

for some non-unital monomorphism $\psi_0 : A \to C(B)$.

Proof. This is Proposition 2.7 of [52]. ■

4 Quasidiagonality

Definition 4.1. Let B be a non-unital but σ-unital C^*-algebra. A sequence $\{b_n\}$ of norm one elements in B_+ is said to be a *system of quasidiagonal units* if the following statements are true:

1. $b_m \perp b_n = 0$ for all $m \neq n$.
2. If $\{x_n\}$ is a bounded sequence in B such that $x_n \in \overline{b_n B b_n}$ for all n, then the sum $\sum x_n$ converges in the strict topology on $M(B)$.

Note that every σ-unital non-unital C^*-algebra has a system of quasidiagonal units (see, e.g., Lemma 2.2 of [53]).

The first result is an exercise in the strict topology.

Lemma 4.2. Let B be a separable non-unital C^*-algebra, which stably has almost stable rank one and let C be a separable C^*-algebra. Suppose that $\{b_n\}$ is a system of quasidiagonal units in B and $\phi_n : C \to \overline{b_n B b_n}$ is a sequence of c.p.c maps. For any permutation $\lambda : \mathbb{N} \to \mathbb{N}$, $\sum_{n=1}^{\infty} \phi_{\lambda(n)}(c)$ converges strictly for all $c \in C$ and there exists a unitary $U \in M(B)$ such that

$$U^*\left(\sum_{n=1}^{\infty} \phi_{\lambda(n)}(c)\right)U = \sum_{n=1}^{\infty} \phi_n(c) \ \text{ for all } \ c \in C. \tag{4.1}$$

Definition 4.3. Let B be a σ-unital non-unital C^*-algebra.

An element $x \in M(B)$ is said to be *diagonal*, if there exists a system $\{b_n\}$ of quasidiagonal units in B, and there exists a bounded sequence $\{x_n\}$ for which $x_n \in \overline{b_n B b_n}$ for all n, such that

$$x = \sum x_n.$$

An element $x \in M(B)$ is said to be *(generalized) quasidiagonal* if x is a sum of a diagonal element with an element of B.

A collection $\mathcal{S} \subseteq M(B)$ is *(generalized) quasidiagonal* if there exists a single system of quasidiagonal units with respect to which all the elements in $\mathcal{S}$ can be simultaneously (generalized) quasidiagonalized.

Definition 4.4. Let A be a separable C^*-algebra and B a σ-unital non-unital C^*-algebra.

An extension $\phi : A \to \mathcal{C}(B)$ is said to be *(generalized) quasidiagonal* if $\pi^{-1}(\phi(A))$ is a (generalized) quasidiagonal collection of operators.

For the rest of this paper, unless it is clearly false, when we write "quasidiagonal", we mean generalized quasidiagonal.

It is easy to prove the following analogue of a classical quasidiagonality result:

Proposition 4.5. Let A be a separable C^*-algebra, and B a non-unital but σ-unital C^*-algebra.

Suppose that $\phi : A \to \mathcal{C}(B)$ is a quasidiagonal extension such that ϕ can be lifted to a c.p.c. map $\Phi' : A \to M(B)$ (so $\pi \circ \Phi' = \phi$).

Then there exist a system $\{b_n\}$ of quasidiagonal units, and, for each n, a c.p.c. map $\phi_n : A \to \overline{b_n B b_n}$ such that $\phi = \pi \circ \Phi$, where $\Phi : A \to M(B)$ is the c.p.c. map defined by

$$\Phi := \sum \phi_n.$$

Moreover, $\{\phi_n\}$ is asymptotically multiplicative, that is, for all $a, b \in A$,

$$\|\phi_n(ab) - \phi_n(a)\phi_n(b)\| \to 0 \ \text{ as } \ n \to \infty.$$

In the above setting, we often write $\Phi = \bigoplus_{n=1}^{\infty} \phi_n$.

Proof. This is Proposition 2.5 of [53]. ■

Theorem 4.6. Let A be a separable C^*-algebra and B be a σ-unital non-unital simple C^*-algebra. Then the pointwise norm limit of quasidiagonal extensions of A by B is quasidiagonal.

As a consequence, let $h : A \to \mathcal{C}(B)$ be an essential quasidiagonal extension and let $\sigma : A \to \mathcal{C}(B)$ be an essential extension. If there exists a sequence $\{U_n\}$ of unitaries in $M(B)$ such that

$$\lim_{n\to\infty} \pi(U_n)h(a)\pi(U_n)^* = \sigma(a) \text{ for all } a \in A \tag{4.2}$$

then σ is a quasidiagonal extension.

If, in addition, B has continuous scale and the extension h is non-unital, then in the above statement, the unitaries can be taken to simply be in $C(B)$. In other words, for A and B as above, suppose, in addition, that B has continuous scale, $h : A \to C(B)$ is a non-unital essential quasidiagonal extension and $\sigma : A \to C(B)$ is an essential extension. If there exists a sequence $\{u_n\}$ of unitaries in $C(B)$ such that

$$\lim_{n\to\infty} u_n^* h(a) u_n = \sigma(a) \text{ for all } a \in A$$

then σ is a quasidiagonal extension.

Proof. This is Theorem 3.7 of [53] together with the present paper's Proposition 3.4. ■

Quasidiagonality was first defined by Halmos [25] in 1970. There is a long history of K-theoretical characterizations of quasidiagonality, going back to BDF's observation that an essentially normal operator is quasidiagonal if and only if it induces the zero element in *Ext* [7]. BDF were essentially the first to recognize that quasidiagonal extensions might be approached by K-theory, and another one of their fundamental results was that (in their setting) limits of trivial extensions correspond to quasidiagonal extensions [8]. Brown pursued this further in [5]. Further developments in the study of quasidiagonality can be found in [65], [73], and [74]. Schochet proved that stably quasidiagonal extensions are the same as limits of stably trivial extensions and can be characterized by $Pext(K_*(A), K_*(B))$ if A is assumed to be nuclear, quasidiagonal relative to B and satisfying the Universal Coefficient Theorem [67]. More recent developments in the general nonstable case, with additional regularity assumptions on B, and a historical summary, can be found in [40]. We will be implicitly using ideas with its origins in the above paper. Starting with the next result, we will be presenting various K theoretic conditions and characterizations for quasidiagonality in our setting. A good example of this is Proposition 7.18.

Theorem 4.7. Let A be a separable nuclear C^*-algebra, which satisfies the UCT and let B be a non-unital separable simple C^*-algebra with continuous scale. Suppose that there exists a non-unital essential quasidiagonal extension $\sigma : A \to \mathcal{C}(B)$ such that $KL(\sigma) = 0$. If $\phi : A \to \mathcal{C}(B)$ is a non-unital essential extension such that $KL(\phi) = 0$ then ϕ is quasidiagonal.

Proof. Let

$$A^+ := \begin{cases} A^\sim & \text{if } A \text{ is non-unital} \\ A \oplus \mathbb{C} & \text{otherwise.} \end{cases}$$

Let

$$\phi^+, \sigma^+ : A^+ \to \mathcal{C}(B)$$

be the unique unital monomorphisms that extend ϕ and σ respective. Then,

$$KL(\phi^+) = KL(\sigma^+).$$

Hence, by [40] Theorem 3.7, ϕ^+ and σ^+ are approximately unitarily equivalent. Consequently, ϕ and σ are approximately unitarily equivalent. It follows from Theorem 4.6 that ϕ is quasidiagonal. ■

5 Stable Uniqueness

Moving towards a non-unital stable uniqueness result, we next provide some definitions and results from [19].

Definition 5.1. ([19] Definition 3.13.)

Let $r_0, r_1 : \mathbb{N} \to \mathbb{Z}_+$ be maps, $T : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}_+$ be a map, and $s, R \geq 1$ be integers. We say that a C^*-algebra D belongs to the class $\mathbf{C}_{(r_0, r_1, T, s, R)}$ if all of the the following statements hold:

(a) For any integer $n \geq 1$ and any pair of projections $p, q \in \mathbb{M}_n(\widetilde{D})$ with

$$[p] = [q] \;\; in \;\; K_0(\widetilde{D}),$$

$$p \oplus 1_{\mathbb{M}_{r_0(n)}(\widetilde{D})} \sim^u q \oplus 1_{\mathbb{M}_{r_0(n)}(\widetilde{D})}.$$

(a′) For any integer $n \geq 1$ and any pair of projections $p, q \in \mathbb{M}_n(\widetilde{D})$, if

$$[p] - [q] \geq 0,$$

then there exists a projection $p' \in \mathbb{M}_{n+r_0(n)}(\widetilde{D})$ such that

$$p' \leq p \oplus 1_{\mathbb{M}_{r_0(n)}(\widetilde{D})} \;\; and \;\; p' \sim q \oplus 1_{\mathbb{M}_{r_0(n)}(\widetilde{D})}.$$

(b) For any integers $n, k \geq 1$ and any $x \in K_0(\widetilde{D})$ such that

$$-n[1_{\widetilde{D}}] \leq kx \leq n[1_{\widetilde{D}}],$$

$$-T(n,k)[1_{\widetilde{D}}] \leq x \leq T(n,k)[1_{\widetilde{D}}].$$

(c) The canonical map

$$U(\mathbb{M}_s(\widetilde{D}))/U_0(\mathbb{M}_s(\widetilde{D})) \to K_1(D)$$

is surjective.

(d) For any integer $n \geq 1$, if $u \in U(\mathbb{M}_n(\widetilde{D}))$ and $[u] = 0$ in $K_1(D)$, then

$$u \oplus 1_{\mathbb{M}_{r_1(n)}(\widetilde{D})} \in U_0(\mathbb{M}_{n+r_1(n)}(\widetilde{D})).$$

(e) For any integer $m \geq 1$, the exponential rank

$$cer(\mathbb{M}_m(\widetilde{D})) \leq R.$$

Proposition 5.2. Let B be a separable simple stably finite C^*-algebra with continuous scale, finite nuclear dimension, UCT, unique tracial state and $K_0(B) = ker\rho_B$. Let $T : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}_+$ be defined by $T(n,k) = n$. Then,

$$B \in \mathbf{C}_{0,0,1,T,7}.$$

Proof. It follows from Theorem 4.3 of [22] that $cer(M_n(\widetilde{B})) \leq 6 + \epsilon$.

Since B has stable rank one and $K_0(\widetilde{B})$ is weakly unperforated, it is easy to check that $B \in \mathbf{C}_{0,0,1,T,7}$. ■

Definition 5.3. Let A be a separable C^*-algebra, B be a non-unital C^*-algebra, and let $\sigma : A \to B$ be a positive map. Let $F : A_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}$, and let $\mathcal{E} \subset A_+ \setminus \{0\}$ be a finite set.

We say that σ is *F-$\mathcal{E}$ full* if for any $\epsilon > 0$, for any $b \in B_+$ with $\|b\| \leq 1$ and for any $a \in \mathcal{E}$, there are $x_1, x_2, ..., x_m \in B$ with $m \leq N(a)$ and $\|x_j\| \leq M(a)$, where $F(a) = (N(a), M(a))$, and such that

$$\|\sum_{j=1}^{m} x_j^* \sigma(a) x_j - b\| \leq \epsilon. \tag{5.1}$$

We say that σ is *exactly F-$\mathcal{E}$ full* if (5.1) holds with $\epsilon = 0$. If σ is F-$\mathcal{E}$-full for every finite subset $\mathcal{E}$ of $A_+ \setminus \{0\}$, then we say that σ is F-full.

Definition 5.4. We introduce some notation that will be used in the next result and later parts of the paper. For a linear map $\phi : A \to B$ between C*-algebras, we often let ϕ also denote the induced map $\phi \otimes id_{M_m} : M_m(A) \to M_m(B)$ for all m. If A and B are not unital, to simplify notation, we understand that $\phi(x)$ is $\phi^{\sim}(x)$ for any $x \in \widetilde{A}$, where $\phi^{\sim} : \widetilde{A} \to \widetilde{B}$ is the unitization of ϕ.

Let A be a unital C^*-algebra and let $x \in A$. Suppose that $\|xx^* - 1\| < 1$ and $\|x^*x - 1\| < 1$. Then $x|x|^{-1}$ is a unitary. Let us use $\lceil x \rceil$ to denote $x|x|^{-1}$. Now let A be any separable amenable C^*-algebra. Let $\mathcal{P} \subset \underline{K}(A)$ be a finite subset. Then there exist a finite subset $\mathcal{F}$ and $\epsilon > 0$ such that for any C^*-algebra B and any $\mathcal{F}$-ϵ-multiplicative c.p.c map $L : A \to B$, L induces a homomorphism $[L] : G(\mathcal{P}) \to \underline{K}(B)$, where $G(\mathcal{P})$ is the subgroup of $\underline{K}(A)$ generated by $\mathcal{P}$. Moreover (by choosing sufficiently small ϵ and large $\mathcal{F}$), if $L' : A \to B$ is another $\mathcal{F}$-ϵ-multiplicative c.p.c. map such that $\|L(x) - L'(x)\| < \epsilon$ for all $x \in \mathcal{F}$, then $[L']|_{G(\mathcal{P})} = [L]|_{G(\mathcal{P})}$. Such a triple $(\epsilon, \mathcal{F}, \mathcal{P})$ is sometimes called a KL-triple. In what follows, when we write $[L]|_{\mathcal{P}}$, we assume that L is at least $\mathcal{F}$-ϵ-multiplicative so that $[L]|_{G(\mathcal{P})}$ is well defined (see 1.2 of [38], 3.3 of [17], and 2.11 of [43], or 2.12 of [23]).

Theorem 5.5. Let A be a non-unital separable amenable C^*-algebra that satisfies the UCT, let $r_0, r_1 : \mathbb{N} \to \mathbb{Z}_+$ and $T : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be three maps, $s, R \geq 1$ be integers, and let $F : A_+ \setminus \{0\} \to \mathbb{N} \times (0, \infty)$ and $L : \bigcup_{m=1}^{\infty} U(M_m(\widetilde{A})) \to [0, \infty)$ be two additional maps.

For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{P} \subset \underline{K}(A)$, a finite subset $\mathcal{U} \subset \bigcup_{m=1}^{\infty} U(M_m(\widetilde{A}))$, a finite subset $\mathcal{E} \subset A_+ \setminus \{0\}$, and an integer $K \geq 1$ satisfying the following:

For any C^*-algebra $B \in \mathbf{C}_{(r_0, r_1, T, s, R)}$, for any two $\mathcal{G}$-δ-multiplicative c.p.c. maps $\phi, \psi : A \to B$, and for any F–$\mathcal{E}$ full $\mathcal{G}$-δ-multiplicative map $\sigma : A \to M_l(B)$ such that

$$cel(\lceil \phi(u) \rceil \lceil \psi(u)^* \rceil) \leq L(u)$$

for $u \in \mathcal{U}$, and

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}},$$

there exists a unitary $U \in M_{1+Kl}(\widetilde{B})$ such that

$$\| Ad(U) \circ (\phi \oplus S)(a) - (\psi \oplus S)(a) \| < \epsilon$$

for all $a \in \mathcal{F}$, where

$$S(a) := diag(\sigma(a), \sigma(a), \ldots, \sigma(a))$$

(the "$\sigma(a)$" is repeated K times in the diagonal).

Furthermore, if B stably has almost stable rank one, then one can choose $U \in \widetilde{M_{1+Kl}}(B)$.

Proof. This is Theorem 3.14 of [19]. ■

Remark 5.6. Note that the finite subset $\mathcal{U}$ in the statement of 5.5 may be assumed to be a subset of $U(M_m(\widetilde{A}))$ for some integer $m \geq 1$. Let $\lceil \mathcal{U} \rceil$ be the image of $\mathcal{U}$ in $U(M_m(\widetilde{A}))/U_0(M_m(\widetilde{A}))$ and let $G(\lceil \mathcal{U} \rceil)$ be the (finitely generated) subgroup generated by $\lceil \mathcal{U} \rceil$.

We provide some notation that will be used in Theorem 5.8, in the proof of Lemma 7.12, and in other places.

Definition 5.7. Let A be a non-unital C^*-algebra.

1. Let

$$\Pi_1 : U(M_\infty(\widetilde{A})) \to U(M_\infty(\widetilde{A}))/U_0(M_\infty(\widetilde{A})) = K_1(A),$$

$$\Pi_{cu} : U(M_\infty(\widetilde{A})) \to U(M_\infty(\widetilde{A}))/CU(M_\infty(\widetilde{A})),$$

and

$$\Pi_{1,cu} : U(M_\infty(\widetilde{A}))/CU(M_\infty(\widetilde{A})) \to K_1(A)$$

be the usual quotient maps.

2. Fix a homomorphism

$$J_A : K_1(A) \to U(M_\infty(\widetilde{A}))/CU(M_\infty(\widetilde{A}))$$

so that the following short exact sequence splits

$$0 \longrightarrow U_0(M_\infty(\widetilde{A}))/CU(M_\infty(\widetilde{A})) \longrightarrow U(M_\infty(\widetilde{A}))/CU(M_\infty(\widetilde{A})) \overset{\Pi_{1,cu}}{\underset{J_A}{\rightleftarrows}} K_1(A) \to 0 \qquad (5.2)$$

(see Cor. 3.3 of [68]). In other words, $\Pi_{1,cu} \circ J_A(x) = x$ for all $x \in K_1(A)$. *We will also use J instead of J_A for brevity. Moreover, in what follows, once A is given, we will assume that J is fixed.*

Fix a map $\Pi_{cu}^- : U(M_\infty(\widetilde{A}))/CU(M_\infty(\widetilde{A})) \to U(M_\infty(\widetilde{A}))$ such that $\Pi_{cu}(\Pi_{cu}^-(z)) = z$ for all $z \in \Pi_{cu}(U(M_\infty(\widetilde{A})))$. Then for each $u \in U(M_\infty(\widetilde{A}))$, we write

$$u = \Pi_{cu}^-(\Pi_{cu}(u))u_{cu}$$

where

$$u_{cu} = \Pi_{cu}^-(\Pi_{cu}(u))^* u \in CU(M_\infty(\widetilde{A})).$$

Note that Π_{cu}^- is just a map between sets. Once A is given, we will assume that Π_{cu}^- is fixed.

3. Let

$$J^{\sim} : K_1(A) \to U(M_\infty(\widetilde{A}))$$

be given by

$$J^{\sim} := \Pi_{cu}^{-} \circ J.$$

4. Once J and Π_{cu}^{-} are fixed, for any $u \in U(M_\infty(\widetilde{A}))$, one may uniquely write

$$u = \Pi_{cu}^{-}(J \circ \Pi_1(u))u_{0,cu}, \tag{5.3}$$

where

$$u_{0,cu} = \Pi_{cu}^{-}(J \circ \Pi_1(u))^* u \in U_0(M_\infty(\widetilde{A})). \tag{5.4}$$

Let $J_0^{\sim} : U(M_\infty(\widetilde{A})) \to U_0(M_\infty(\widetilde{A}))$ be defined by $J_0^{\sim}(u) = J^{\sim}(\Pi_1(u))^* u\, (= u_{0,cu}$ as *in* (5.4).

We repeat for emphasis: For a fixed C^*-algebra A, we will fix one splitting map J and a map Π_{cu}^{-} as above, which then determine $J^{\sim}$ and $J_0^{\sim}$.

5. Suppose that B is another C^*-algebra and $h : A \to B$ is a homomorphism. Denote by $h^{\dagger} : U(M_\infty(\widetilde{A}))/CU(M_\infty(\widetilde{A})) \to U(M_\infty(\widetilde{B}))/CU(M_\infty(\widetilde{B}))$ the induced homomorphism. Denote by $h^{\ddagger} : K_1(A) \to U(M_\infty(\widetilde{B}))/CU(M_\infty(\widetilde{B}))$ the homomorphism defined by $h^{\dagger} \circ J$, as J is fixed. Note that in the case where B has stable rank one (see [68, Cor. 3.4], and its remark), $U(M_\infty(\widetilde{B}))/CU(M_\infty(\widetilde{B})) = U(\widetilde{B})/CU(\widetilde{B})$. In this case, $h^{\dagger}$ is a homomorphism from $U(\widetilde{A})/CU(\widetilde{A})$ to $U(\widetilde{B})/CU(\widetilde{B})$ and $h^{\ddagger}$ maps $K_1(A)$ to $U(\widetilde{B})/CU(\widetilde{B})$.
6. Denote by $\Delta : U(\widetilde{B})/CU(\widetilde{B}) \to \mathrm{Aff}(T(\widetilde{B}))/\rho_B(K_0(\widetilde{B}))$ the determinant map which is an isometric group isomorphism (see Section 3 of [68] and Proposition 3.23 of [23]).

Theorem 5.8. Let A be a non-unital separable amenable C^*-algebra that satisfies the UCT, let $r_0, r_1 : \mathbb{N} \to \mathbb{Z}_+$ and $T : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be three maps, $s, R \geq 1$ be integers, and let $F : A_+ \setminus \{0\} \to \mathbb{N} \times (0, \infty)$ and $L : J^{\sim}(K_1(A)) \to [0, \infty)$ be two additional maps.

For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{P} \subset \underline{K}(A)$, a finite subset $\mathcal{U} \subset J^{\sim}(K_1(A))$, a finite subset $\mathcal{E} \subset A_+ \setminus \{0\}$, and an integer $K \geq 1$ satisfying the following:

For any C^*-algebra $B \in \mathbf{C}_{(r_0,r_1,T,s,R)}$, for any two $\mathcal{G}$-δ-multiplicative c.p.c. maps $\phi, \psi : A \to B$, and for any F–$\mathcal{E}$ full $\mathcal{G}$-δ-multiplicative map $\sigma : A \to M_l(B)$ such that

$$cel(\lceil \phi(u) \rceil \lceil \psi(u)^* \rceil) \leq L(u) \text{ for all } u \in \mathcal{U}, \text{ and}$$

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}},$$

there exists a unitary $U \in \mathbb{M}_{1+Kl}(\widetilde{B})$ such that

$$\|Ad(U) \circ (\phi \oplus S)(a) - (\psi \oplus S)(a)\| < \epsilon$$

for all $a \in \mathcal{F}$, where

$$S(a) := diag(\sigma(a), \sigma(a), \ldots, \sigma(a))$$

(the "$\sigma(a)$" is repeated K times in the diagonal).

Furthermore, if B stably has almost stable rank one, then one can choose $U \in \widetilde{\mathbb{M}_{1+Kl}(B)}$.

Proof. It suffices to show that there is a map $L_1 : U(M_\infty(\widetilde{A})) \to [0, \infty)$, which depends only on A and L, such that for any finite subset $\mathcal{U} \subset U(M_\infty(\widetilde{A}))$, if

$$cel(\lceil \phi(u) \rceil \lceil \psi(u)^* \rceil) \leq L(u) \text{ for all } u \in J^\sim \circ \Pi_1(\mathcal{U}), \tag{5.5}$$

then one always has that

$$cel(\lceil \phi(u) \rceil \lceil \psi(u)^* \rceil) \leq L_1(u) \text{ for all } u \in \mathcal{U}, \tag{5.6}$$

for any ϕ and ψ that are $\mathcal{G}_1$-δ_1-multiplicative, where δ_1 is sufficiently small and $\mathcal{G}_1$ is sufficiently large and which depend only on $\mathcal{U}$ and L, as we then apply Theorem 5.5 for this L_1 (choosing $\delta < \delta_1$, large $\mathcal{G} \supset \mathcal{G}_1$ and K and so on).

Let us provide the details for the resolution of this issue. As A is given, we fix a splitting homomorphism J and a map Π_{cu}^- as in 5.7 (so $J^\sim$ and $J_0^\sim$ are also fixed). Define $L_0 : U_0(M_\infty(\widetilde{A})) \to [0, \infty)$ as follows: For each $u \in U_0(M_\infty(\widetilde{A}))$, there is a smallest $m(u) \geq 1$ such that $u \in U_0(M_{m(u)}(\widetilde{A}))$. Define $L_0(u) = cel(u)$ in $U_0(M_{m(u)}(\widetilde{A}))$.

Suppose that a finite subset $\mathcal{U} \subset U(M_m(\widetilde{A}))$ is fixed. Without loss of generality, we may assume that $J_0^\sim(u) \in U_0(M_m(\widetilde{A}))$ for all $u \in \mathcal{U}$.

For each $u \in \mathcal{U}$, there are $h_1(u), h_2(u), ..., h_{k(u)}(u) \in M_m(\widetilde{A})_{s.a.}$ such that

$$\exp(ih_1(u))\exp(ih_2(u))\cdots\exp(ih_{k(u)}(u)) = J_0^{\sim}(u). \tag{5.7}$$

We choose a small $\delta_1 > 0$ and a large finite subset $\mathcal{G}_1$ of A such that for all $u \in \mathcal{U}$,

$$\|\phi(J_0^{\sim}(u)) - \exp(i\phi(h_1(u)))\exp(i\phi(h_2(u)))\cdots\exp(i\phi(h_{k(u)}(u)))\| < 1/16\pi \tag{5.8}$$

for any $\mathcal{G}_1$-δ_1-multiplicative c.p.c. map ϕ from $\widetilde{A}$. In particular,

$$cel(\lceil\phi(J_0^{\sim}(u))\rceil) \le cel(J_0^{\sim}(u)) + 1/4 \ \text{ for all } \ u \in \mathcal{U}. \tag{5.9}$$

We may also assume that

$$\lceil\phi(u)\rceil \approx_{1/64\pi} \lceil\phi(J^{\sim}(\Pi_1(u)))\rceil\lceil\phi(J_0^{\sim}(u))\rceil \ \text{ for all } \ u \in \mathcal{U}. \tag{5.10}$$

Define $L_1(u) = L(J^{\sim}(\Pi_1(u))) + 2L_0(J_0^{\sim}(u)) + 1$ for all $u \in U(M_\infty(\widetilde{A}))$.

Note that, as had been demonstrated, if δ_1 is small enough and $\mathcal{G}_1$ is large enough independent of ϕ or ψ (and also independent of B in the class $\mathbf{C}_{r_0,r_1,t,s,R}$), when both ϕ and ψ are $\mathcal{G}_1$-δ_1-multiplicative and satisfy (5.5), for all $u \in \mathcal{U}$,

$$cel(\lceil\phi(u)\rceil\lceil\psi(u)\rceil^*) \tag{5.11}$$

$$\le 1/16 + cel(\lceil\phi(J^{\sim}(\Pi_1(u)))\rceil\lceil\phi(J_0^{\sim}(u))\rceil\lceil\psi(J_0^{\sim}(u))^*\rceil\lceil\psi(J^{\sim}(\Pi_1(u)))^*\rceil) \tag{5.12}$$

$$\le 2(L_0(J_0^{\sim}(u)) + 1/4) + L(J^{\sim}(\Pi_1(u))) \le L_1(u). \tag{5.13}$$

In other words, (5.6) holds. The theorem then follows from Theorem 5.5. ∎

6 Existence and Exponential Length

Lemma 6.1. Let A be a separable algebraically simple C^*-algebra with finite nuclear dimension, which satisfies the UCT and has a unique tracial state τ_A. Suppose that A is non-unital and stably projectionless. Then we have the following:

1. A is $\mathcal{Z}$-stable and has stable rank one.
2. $K_0(A) = \ker\rho_A$.

Proof. By [70], A is $\mathcal{Z}$-stable. It follows from [63] that A stably has almost stable rank one. Since A has only one tracial state, it follows from Corollary A7 of [19] that $K_0(A) = \ker\rho_A$. By Theorem 15.5 of [22], A is classifiable and is in the class $\mathcal{D}$, which is defined in 3.9 of [22] (see also 8.1 of [18]). Therefore, by Theorem 11.5 of [18], A has stable rank one. ■

Definition 6.2. Let $\mathcal{W}$ be the *Razak algebra*, which is a non-unital, simple, separable, nuclear, continuous scale, stably projectionless C^*-algebra with a unique tracial state τ_W and $K_*(\mathcal{W}) = 0$. (Sometimes, we will write t_W, instead of τ_W, for the unique tracial state of $\mathcal{W}$.) $\mathcal{W}$ also has stable rank one and is $\mathcal{Z}$-stable (see [26, 59, 71]). It is proved in [19] that $\mathcal{W}$ is the only non-unital simple separable C^*-algebra with finite nuclear dimension, $K_i(\mathcal{W}) = \{0\}$ ($i = 0, 1$) and with a unique tracial state which satisfies the UCT and has continuous scale. From this, one can also conclude that $\mathcal{W}$ is *-isomorphic to any of its nonzero hereditary C*-subalgebras.

Remark 6.3. In fact, the proof of Lemma 6.1 shows that the C^*-algebra A is in the classifiable class $\mathcal{D}$ defined in [22] 3.9. We also note that it is not hard to check directly that $\mathcal{W}$ has properties (1) and (2) of Lemma 6.1. (See, e.g., [26].)

Lemma 6.4 (Theorem 1.1 of [62]). Let B be a separable infinite dimensional simple $\mathcal{Z}$-stable C^*-algebra with stable rank one, and for which every 2-quasi-trace of $\overline{bBb}$ is a trace, for any $b \in \mathrm{Ped}(B)_+ \setminus \{0\}$. Then there is an embedding $\phi_{w,b} : \mathcal{W} \to B$.

If B also has continuous scale and is stably projectionless, we may require that $\phi_{w,b}$ maps strictly positive elements to strictly positive elements.

Moreover, if $\phi_1, \phi_2 : \mathcal{W} \to B$ are two monomorphisms such that $d_\tau(\phi_1(a)) = d_\tau(\phi_2(a))$ holds for all $\tau \in T(B)$ and for one non-zero $a \in \mathcal{W}_+ \setminus \{0\}$, then there exists a sequence of unitaries $u_n \in \widetilde{B}$ such that

$$\lim_{n\to\infty} \mathrm{Ad}\, u_n \circ \phi_1(c) = \phi_2(c) \ \text{ for all } \ c \in \mathcal{W}.$$

Proof. Since $K_0(\mathcal{W}) = \{0\}$ and $\mathcal{W}$ has a unique tracial state, from Proposition 6.2.3 of [62], one computes that $\mathrm{Cu}^\sim(\mathcal{W}) = (-\infty, \infty]$. It is easy to see that for the first part of the Lemma, we may assume that B has continuous scale (see, e.g., Proposition 5.4 of [18]). Since B is simple, has stable rank one, and since every 2-quasi-trace of B is a trace, $T(B) \neq \emptyset$. Since B has continuous scale, it is well known that $T(B)$ is compact (see, e.g., Theorem 5.3 of [18]).

It follows from Theorem 7.3 of [18] (see also Theorem 6.2.3 of [62]) that when B is stably projectionless, $\mathrm{C}u^{\sim}(B) = K_0(B) \sqcup (\mathrm{L}Aff^{\sim}(T(B)) \setminus \{0\})$ (recall that $T(B)$ is compact as B has continuous scale). This also holds for the case that B is not stably projectionless (see the proof of Proposition 6.1.1 of [62]; in fact, here we can replace the trace space by $T(pM_m(B)p)$ for some nonzero projection $p \in M_m(B)$, and for some $m \geq 1$).

Fix a strictly positive element $e_W \in \mathcal{W}$ with $\|e_W\| = 1$ and a strictly positive element $e_B \in B$ with $\|e_B\| = 1$. Then e_W is represented by $1 \in (-\infty, \infty]$ in $\mathrm{C}u^{\sim}(\mathcal{W})$. Note also $d_\tau(e_B) = 1$ for all $\tau \in T(B)$. Choose $0 < a < 1$. Define a map $j : (-\infty, \infty] \to \mathrm{L}Aff^{\sim}(T(B)) \subseteq \mathrm{C}u^{\sim}(B)$ by $j(r) = ar$, where we view ar as a constant function on $T(B)$. Thus, j is a morphism from $\mathrm{C}u^{\sim}(\mathcal{W})$ to $\mathrm{C}u^{\sim}(B)$. Hence, by Theorem 1.0.1 of [62], there is a *-homomorphism $\phi_{w,b} : \mathcal{W} \to B$, such that

$$\mathrm{C}u^{\sim}(\phi_{w,b}) = j.$$

In the case that B is stably projectionless, e_B is not a projection. So if, in the previous paragraph, we choose $a = 1$, then $\phi_{w,b}(e_W)$ is Cuntz equivalent to e_B. Since B has stably rank one, $\mathrm{H}er(\phi_{w,b}(e_W))$ is isomorphic to $\mathrm{H}er(e_B) = B$. So we may also assume that $\phi_{w,b}$ maps strictly positive elements to strictly positive elements.

The second part of the Lemma follows immediately from the fact that $\mathrm{C}u^{\sim}(\mathcal{W}) = (-\infty, \infty]$ and Theorem 1.1 of [62]. ∎

Definition 6.5. Let B be a separable (non-unital) simple C^*-algebra with stable rank one and with continuous scale such that $K_0(B) = \ker \rho_B$. Then $U(M_\infty(\widetilde{B}))/CU(M_\infty(\widetilde{B})) = U(\widetilde{B})/CU(\widetilde{B})$. Fix $J : K_1(B) = K_1(\widetilde{B}) \to U(\widetilde{B})/CU(\widetilde{B})$. Then, by 5.7, one may write

$$U(\widetilde{B})/CU(\widetilde{B}) = (\mathrm{Aff}(T(\widetilde{B}))/\mathbb{Z}) \oplus J(K_1(\widetilde{B})). \tag{6.1}$$

Recall that $\mathrm{Aff}(T(\widetilde{B})) \cong \mathrm{Aff}(T(B)) \oplus \mathbb{R}$.

Suppose that D is a hereditary C^*-subalgebra of B, which also has continuous scale. If $\gamma_D : T(B) \to T(D)$ is an affine continuous map, denote by $\gamma^D : \mathrm{Aff}(T(D)) \to \mathrm{Aff}(T(B))$ the induced linear map defined by $\gamma^D(f)(\tau) = f(\gamma_D(\tau))$ for all $f \in \mathrm{Aff}(T(D))$ and $\tau \in T(B)$. Let $\overline{\gamma^D} : \mathrm{Aff}(T(\widetilde{D}))/\mathbb{Z} \to \mathrm{Aff}(T(\widetilde{B}))/\mathbb{Z}$ be the map induced by γ^D.

Let $j_D : D \to B$ be the inclusion map. Note that if $u \in \widetilde{D}$ is a unitary, then we may write $u = e^{2\pi i\theta} \cdot 1_{\widetilde{D}} + u_d$, where $\theta \in (-1, 1]$ and $u_d \in D$. Note that $j_D(u) = e^{2\pi i\theta} \cdot 1_{\widetilde{B}} + u_d$. Also, $(j_D)_{*1} : K_1(D) \to K_1(B)$ is an isomorphism. Moreover, by Proposition 4.5 of [22], $j_D^{\dagger} : U(\widetilde{D})/CU(\widetilde{D}) \to U(\widetilde{B})/CU(\widetilde{B})$ is an isomorphism.

Lemma 6.6. Let B be a non-unital separable finite simple C^*-algebra with finite nuclear dimension and with continuous scale.

(1) B is $\mathcal{Z}$-stable and every quasitrace on B is a trace. For each $t \in (0,1)$, there are elements $a_t, a_{1-t} \in B_+$ such that $a_t a_{1-t} = 0$, $d_\tau(a_t) = t$ and $d_\tau(a_{1-t}) = 1-t$ for all $\tau \in T(B)$, and $a_t + a_{1-t}$ is a strictly positive element of B.

(2) Suppose, in addition, that $K_0(B) = \ker\rho_B$ and B satisfies the UCT.

Then for any $t \in (0,1)$, for any $a_t \in B_+ \setminus \{0\}$ with $d_\tau(a_t) = t$ for all $\tau \in T(B)$, there is an isomorphism $\phi_t : B \to B_t := \mathrm{Her}(a_t)$ such that $KL(\phi_t) = KL(\mathrm{id}_B)$, $(\phi_t)_T = \gamma_{B_t}^{-1} : T(B_t) \to T(B)$, where $\gamma_{B_t} : T(B) \to T(B_t)$ is defined by $\gamma_{B_t}(\tau)(b) = \tau(b)/t$ for all $b \in B_t$ and $\tau \in T(B)$, and

$$(\phi_t)^\dagger|_{J(K_1(B))} = (j^\dagger_{B_t})^{-1}|_{J(K_1(B))} \ \text{and}\ (\phi_t)^\dagger|_{\mathrm{Aff}(T(\widetilde{B}))/\mathbb{Z}} = (\overline{\gamma^{B_t}})^{-1},$$

where $\overline{\gamma^{B_t}} : \mathrm{Aff}(\tilde{B}_t)/\mathbb{Z} \to \mathrm{Aff}(\tilde{B})/\mathbb{Z}$ is the induced map given in Definition 6.5.

Moreover, for any $u \in U_0(\widetilde{B})$,

$$\mathrm{dist}(uCU(\widetilde{B}), j^\dagger_{B_t}(\phi_t^\dagger(u))) \le (1-t)\mathrm{dist}(u, 1_{\widetilde{B}}). \tag{6.2}$$

Proof. We firstly prove part (1). It follows from [70] that B is $\mathcal{Z}$ stable. By [10], every quasitrace on B is a trace. By Theorem 6.8 of [20] (see also [9] Theorem 2.5), $\mathrm{Cu}(B) = V(B) \sqcup (\mathrm{LAff}_+(T(B)) \setminus \{0\})$. For both parts (1) and (2), we may assume that $T(B) \neq \emptyset$.

If B is not stably projectionless, then there is a projection $e \in M_m(B)$ for some $m \ge 1$. It follows from Cor. 3.1 of [69] that $eM_m(B)e$ is a unital simple $\mathcal{Z}$-stable C^*-algebra. By Theorem 6.7 of [64], $eM_m(B)e$ has stable rank one. It follows that B has stable rank one.

Fix a strictly positive element e_B of B. Note that $d_\tau(e_B) = 1$ for all $\tau \in T(B)$. For any $t \in (0,1)$, choose elements $a'_{1-t}, a'_t \in B_+$ that are not projections (as $\mathrm{Cu}(B) = V(B) \sqcup (\mathrm{LAff}_+(T(B)) \setminus \{0\})$) such that $d_\tau(a'_{1-t}) = 1-t$ and $d_\tau(a'_t) = t$ for all $\tau \in T(B)$. Let $b = a'_{1-t} \oplus a'_t \in M_2(B)_+$. Then $d_\tau(b) = 1$ for all $\tau \in T(B)$. Therefore $d_\tau(b) = d_\tau(e_B)$ and both b and e_B are not projections. If B is not stably projectionless, applying Theorem 3 of [14] (see also Theorem 3.3 of [9]), as $eM_m(B)e$ is unital and has stable rank one, and if B is stably projectionless, applying (the last part of) Theorem 1.2 of [63], one obtains an isomorphism $h : \overline{bM_2(B)b} \to B$. Let $a_t = h(a'_t)$ and $a_{1-t} = h(a'_{1-t})$. Then $a_t a_{1-t} = 0$ and $a_t + a_{1-t}$ is a strictly positive element of B.

For part (2), note that since B is finite and $\mathcal{Z}$ stable, it is stably finite. Note also that since $K_0(B) = \ker\rho_B$, B is stably projectionless. By Theorem 15.6 of [22], $B \in \mathcal{D}_0$

and $B \in B_T$ as defined there. Then, part (2) follows from Theorem 12.8 of [22]. In fact, note that $K_i(\mathrm{Her}(a_t)) = K_i(B)$, $i = 0, 1$. Define $\kappa_0 := \mathrm{id}_{K_0(B)}$ and $\kappa_1 := \mathrm{id}_{K_1(B)}$. Note that $B_t := \mathrm{Her}(a_t)$ also has continuous scale as $d_\tau(a_t)$ is continuous on $T(B)$ (see Proposition 5.4 of [18]). Fix $t \in (0, 1)$. The map defined by $\gamma_{B_t}(\tau)(b) = \tau(b)/t$ for all $b \in B_t$ and $\tau \in T(B)$ is an affine homeomorphism from $T(B)$ onto $T(B_t)$. Let $\kappa_T := (\gamma_{B_t})^{-1} : T(B_t) \to T(B)$. Note also that since $K_0(B) = \ker \rho_B$, $U(\widetilde{B})/CU(\widetilde{B}) \cong \mathrm{Aff}(T(\widetilde{B}))/\mathbb{Z} \oplus K_1(B)$. Let $\kappa_{cu} : U(\widetilde{B})/CU(\widetilde{B}) \to U(\widetilde{B_t})/CU(\widetilde{B_t})$ be the map defined by

$$\kappa_{cu}|_{J_B(K_1(B))} = (j^\dagger_{B_t})^{-1}|_{J_B(K_1(B))} \ \ and \ \ \kappa_{cu}|_{\mathrm{Aff}(T(\widetilde{B}))/\mathbb{Z}} = (\overline{\gamma^{B_t}})^{-1}. \tag{6.3}$$

Then by Theorem 12.8 of [22], there is an isomorphism $h_t : B \to B_t$ such that $KL(h_t) = KK(\mathrm{id}_B)$, $h^\dagger_t = \kappa_{cu}$, and $s(h_t(b)) = \kappa_T(s)(b)$ for all $s \in T(B_t)$ and all $b \in B$.

For the last part of the lemma, let $u \in U_0(\widetilde{B})$. Write $u = \exp(i2\pi a)w$, where $a = \alpha \cdot 1_{\widetilde{B}} + a_b$, $a_b \in B_{s.a.}$, $\alpha \in \mathbb{R}$ and $w \in CU(\widetilde{B})$. Moreover, we may assume that $\Delta(u)(\tau) = \alpha + \tau(a_b)$ for all $\tau \in T(\widetilde{B})$ (see [68] and Corollary 2.12 of [21], as well as (6) of 5.7). We compute that

$$j^\dagger_{B_t}(h^\dagger_t(u)) = \overline{\exp(i2\pi(\alpha \cdot 1_{\widetilde{B}} + h_t(a_b)))}, \tag{6.4}$$

where $\tau(h_t(a_b)) = t\tau(a_b)$ for all $\tau \in T(B)$. Therefore,

$$\overline{u(j^\dagger_{B_t}(h^\dagger_t(u)))^*} = \overline{\exp(i2\pi(a_b - h_t(a_b)))}. \tag{6.5}$$

Note that

$$\tau(a_b - h_t(a_b)) = (1 - t)\tau(a_b) \ \text{ for all } \ \tau \in T(B). \tag{6.6}$$

Then (6.2) follows from the fact that Δ is an isometric isomorphism. ∎

Lemma 6.7. Let B be an algebraically simple, σ-unital C^*-algebra , and let C be a σ-unital C^*-algebra. Suppose that $\sigma : C \to B$ is a nonzero homomorphism.

Then there exists a map:

$$F : C_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}$$

such that for every finite subset $\mathcal{E} \subset A_+ \setminus \{0\}$, σ is F-$\mathcal{E}$ full.

Proof. Let $A = \sigma(C)$. Then $A \subset B$ is a σ-unital C*-subalgebra. Then Proposition 5.6 of [18] applies. ■

Definition 6.8. Let A be a separable C^*-algebra. We say that A is $\mathcal{W}$ *embeddable* if there is a monomorphism $\phi : A \hookrightarrow \mathcal{W}$.

Since $\mathcal{W}$ is projectionless, if A is $\mathcal{W}$ embeddable, then A is non-unital. Let $e_A \in A$ be a strictly positive element. Consider $a = \phi(e_A)$. There is an isomorphism $s : \overline{a\mathcal{W}a} \to \mathcal{W}$. Then $s \circ \phi : A \to \mathcal{W}$ is an embedding which maps e_A to a strictly positive element of $\mathcal{W}$. So, if it is needed, one may assume that ϕ maps strictly positive elements to strictly positive elements.

Remark 6.9. If A is $\mathcal{W}$ embeddable, then $T_f(A) \neq \emptyset$. In particular, A is not purely infinite. Let τ_W be the unique tracial state of $\mathcal{W}$ and suppose that $\phi : A \hookrightarrow \mathcal{W}$ is an embedding. Then the normalization of $\tau_W \circ \phi$ is a faithful tracial state of A.

Recall that $\mathcal{Z}_0$ is the unique separable stably projectionless simple C^*-algebra with finite nuclear dimension, which satisfies the UCT and has a unique tracial state, and for which $K_0(\mathcal{Z}_0) = \mathbb{Z}$ and $K_1(\mathcal{Z}_0) = \{0\}$ (see Cor. 15.7 of [22]).

Theorem 6.10. Let A be a separable amenable C^*-algebra that is $\mathcal{W}$ embeddable, and let B be a separable simple stably projectionless C^*-algebra with finite nuclear dimension and with continuous scale. Suppose that $\ker\rho_B = K_0(B)$ and both A and B satisfies the UCT.

Then for any $x \in KL(A, B)$, there is a monomorphism $h : A \to B$ such that $KL(h) = x$.

Proof. By Theorem 15.6 of [22], $B \cong B \otimes \mathcal{Z}_0$. It follows from Theorem 10.8 of [22] that there exists a sequence of c.p.c. maps $\phi_n : A \to B \otimes \mathcal{K}$ such that

$$\lim_{n\to\infty} \|\phi_n(a)\phi_n(b) - \phi_n(ab)\| = 0 \text{ for all } a, b \in A \text{ and } [\{\phi_n\}] = x. \tag{6.7}$$

Without loss of generality, we may assume that $\phi_n : A \to B \otimes M_{r(n)}$ for some sequence $\{r(n)\} \subset \mathbb{N}$. Since $B \otimes M_{r(n)}$ is also a separable simple stably projectionless C^*-algebra with finite nuclear dimension, with continuous scale, and which satisfies the UCT, by part (2) of 6.6, and by replacing ϕ_n by $\phi_{\frac{1}{r(n)}} \circ \phi_n$, we may assume that $\phi_n : A \to B$.

Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \mathcal{F}_n, \ldots$ be an increasing sequence of finite sets whose union is dense in the unit ball of A. Let $\{\epsilon_n\}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \epsilon_n < 1$.

Fix an embedding $\iota_A : A \to \mathcal{W}$. For each $u \in U(M_m(\widetilde{A}))$ $(m = 1, 2, \ldots,)$, $\iota_A(u)$ (which abbreviates $(\iota_A \otimes \mathrm{id}_{M_m})(u)$) is in $U_0(M_m(\widetilde{\mathcal{W}}))$. Define $L_1 : \bigcup_{m=1}^{\infty} U(M_m(A)) \to \mathbb{R}_+$ by $L_1(u) = cel(\iota_A(u))$ for all $u \in \bigcup_{m=1}^{\infty} U(M_m(A))$.

By Lemma 6.4, there is a homomorphism $\phi_{w,b} : \mathcal{W} \to B$, which maps strictly positive elements to strictly positive elements. Put $\sigma_A := \phi_{w,b} \circ \iota_A : A \to B$. Note that

$$cel(\sigma_A(u)) \le L_1(u) \text{ for all } u \in \bigcup_{m=1}^{\infty} U(M_m(A)). \tag{6.8}$$

By Lemma 6.7, let $F : A_+ \setminus \{0\} \to \mathbb{N} \times (0, \infty)$ be such that σ_A is F-full in B.

Let $L = L_1 + 2\pi + 1$. Let $T : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}_+$ be the map defined by $T(n, k) = n$. We will apply Theorem 5.5. Note that by Proposition 5.2, $B \in \mathbf{C}_{0,0,1,T,7}$. So let $r_0 = r_1 = 0$, $s = 1$ and $R = 1$.

For the above data, and for each n, let $\delta_n > 0$ (in place of δ), $\mathcal{G}_n \subset A$ (in place of $\mathcal{G}$) be a finite subset, $\mathcal{P}_n \subset \underline{K}(A)$ (in place of $\mathcal{P}$) be a finite subset, $\mathcal{U}_n \subset U(M_{m(n)}(\widetilde{A})) \cap J^{\sim}(K_1(A))$ (in place of $\mathcal{U}$) be a finite subset (for some integer $m(n)$), $\mathcal{E}_n \subset A_+ \setminus \{0\}$ (in place of $\mathcal{E}$) be a finite subset, and an integer $K_n \ge 1$ (in place of K) be as provided by Theorem 5.8 for $\epsilon_n/2$ (in place of ϵ) and for $\mathcal{F}_n$ (in place of $\mathcal{F}$), as well as for the given L and T above. Passing to a subsequence if necessary, we may assume that ϕ_n is $\mathcal{G}_n$-δ_n-multiplicative and $\lceil \phi_n(u) \rceil$ is well defined for all $u \in \mathcal{U}_n$. We may also assume that $K_n \le K_{n+1}$, $n \in \mathbb{N}$.

For each $u \in \mathcal{U}_n$, define

$$z_n^{(1)} = J_B(\Pi_{1,cu}(\lceil \phi_n(u) \rceil)). \tag{6.9}$$

Then, for each $u \in \mathcal{U}_n$,

$$\lceil \phi(u) \rceil = z_n^{(0)} z_n^{(1)} \text{ and } z_n^{(0)} := \lceil \phi_n(u) \rceil J_B(\Pi_{1,cu}(\lceil \phi_n(u) \rceil^*)) \in U_0(M_{m(n)}(\widetilde{B})). \tag{6.10}$$

Define, for each n,

$$\lambda_n' = \max\{cel(\lceil \phi_n(u) \rceil J_B(\Pi_{1,cu}(\lceil \phi_n(u) \rceil^*))) : u \in \mathcal{U}_n\}. \tag{6.11}$$

Define, for each n,

$$\lambda_n = \max\{cel(\lceil \phi_n(u)\rceil \lceil \phi_{n+1}(u^*)\rceil) : u \in \mathcal{U}_n\}, \ \ n = 1, 2, \ldots.$$

Choose $J_n \in \mathbb{N}$ such that

$$2(\lambda_n' + \lambda_n) + 7\pi/J_n < 1 \ \ and \ \ J_{n+1} > J_n > K_n, \ \ n = 1, 2, \ldots. \tag{6.12}$$

Choose $t_n \in (0, 1) \cap \mathbb{Q}$ such that

$$t_{n+1} < t_n, \ (1 - t_n) > 2J_n(4(K_n + 1) + 1)t_n, \ \ n = 1, 2, \ldots. \tag{6.13}$$

Define, for each n,

$$N_n = 2J_n(4(K_n + 1) + 1), \ \ d_n = (1/N_n)(1 - t_n), \ \ r_{n+1} = (t_n - t_{n+1}). \tag{6.14}$$

One can check that

$$0 < t_n < d_n, \ \ r_{n+1} < 2J_n d_n + (t_n - t_{n+1}) < (2J_n + 1)d_n, \tag{6.15}$$

$$t_n + 2J_n d_n + r_{n+1} < (4J_n + 2)d_n, \ \ 8J_n(K_n + 1) > (4J_n + 2)K_n \ \ and \tag{6.16}$$

$$(1 - t_{n+1}) = r_{n+1} + (1 - t_n) = r_{n+1} + 2J_n d_n + 8J_n(K_n + 1)d_n \tag{6.17}$$

$$= r_{n+1} + 2J_n d_n + (N_n - 2J_n)d_n,$$

$$N_n - 2J_n \geq 4(K_n + 1)2J_n. \tag{6.18}$$

By Lemma 6.6, choose $a_{t_n}, a_{1-t_n} \in B$ such that $a_{t_n} a_{1-t_n} = 0$ and $d_\tau(a_{t_n}) = t_n$ and $d_\tau(a_{1-t_n}) = 1 - t_n$ for all $\tau \in T(B)$. Also, let $a_{d_n}, a_{r_n} \in \mathrm{Her}(a_{1-t_n})$ be such that $d_\tau(a_{d_n}) = d_n$ and $d_\tau(a_{r_n}) = r_n$ for all $\tau \in T(B)$, $n \in \mathbb{N}$. Let

$$s_n : B \to B_{n,1} := \mathrm{Her}(a_{d_n}) \subset \mathrm{Her}(a_{1-t_n}), \tag{6.19}$$

$$s_n^r : B \to B_{n,r} := \mathrm{Her}(a_{r_n}) \subset B, \ \ and \tag{6.20}$$

$$s_n^{(0)} : B \to B_{n,0} := \mathrm{Her}(a_{t_n}) \subset \mathrm{Her}(a_{t_{n-1}}) \subset B \tag{6.21}$$

be the isomorphisms given by part (2) of Lemma 6.6, $n \in \mathbb{N}$. In particular, $KL(s_n) = KL(s_n^r) = KL(s_n^0) = KL(\mathrm{id}_B)$ and

$$(s_n^{(0)})^{\dagger}|_{J_B(K_1(\widetilde{B}))} = (j_{B_{t_n}})^{\dagger^{-1}}|_{J_B(K_1(\widetilde{B}))} \text{ and} \tag{6.22}$$

$$\mathrm{d}ist(z, (j_{B_{t_n}})^{\dagger}((s_n^{(0)})^{\dagger}(z))) \le (1 - t_n)\mathrm{d}ist(z, \bar{1}) \text{ for all } z \in U_0(\widetilde{B})/CU(\widetilde{B}). \tag{6.23}$$

Moreover, viewing $\mathrm{H}er(a_{t_{n+1}}) \subset \mathrm{H}er(a_{t_n})$ and letting $j'_{B_{t_{n+1}}} : B_{t_{n+1}} \to B_{t_n}$ to be the inclusion map,

$$\overline{(s_n^{(0)})^{\dagger}(x)(j'_{B_{t_{n+1}}})^{\dagger}((s_{n+1}^{(0)})^{\dagger}(x^{-1}))} = \bar{1} \text{ for all } x \in J_{B_{t_n}}(K_1(\widetilde{B_{t_n}})) \tag{6.24}$$

and, as at the end of the proof of 6.6,

$$\mathrm{dist}((s_n^{(0)})^{\dagger}(z), (j'_{B_{t_{n+1}}})^{\dagger}((s_{n+1}^{(0)})^{\dagger}(z))) \le (t_n - t_{n+1})\mathrm{dist}(z, \bar{1}) \tag{6.25}$$

for all $z \in U_0(M_\infty(\widetilde{B}))/CU(M_\infty(\widetilde{B}))$.

Define $\Lambda_{n,0} := s_n^{(0)} \circ \phi_n : A \to B_{n,0}$. We may assume that

$$[\Lambda_{n,0}]|_{\mathcal{P}_n} = x|_{\mathcal{P}_n} = [\Lambda_{n+1,0}]|_{\mathcal{P}_n}, \tag{6.26}$$

$n \in \mathbb{N}$. By (2) of Lemma 6.6, $\mathrm{H}er(a_{1-t_n}) \cong M_{N_n}(B_{n,1})$. Moreover, define, for each n,

$$S_n := \bigoplus^{N_n} s_n \circ \sigma_A : A \to \mathrm{Her}(a_{1-t_n}) = M_{N_n}(B_{n,1}), \tag{6.27}$$

$$R_n := s_n^r \circ \sigma_A : A \to \mathrm{Her}(a_{r_n}), \tag{6.28}$$

$$S_n^l := \bigoplus^{2J_n} s_n \circ \sigma_A : A \to M_{2J_n}(B_{n,1}) \subset \mathrm{Her}(a_{1-t_n}), \text{ and} \tag{6.29}$$

$$S_n' := \bigoplus^{N_n - 2J_n} s_n \circ \sigma_A : A \to M_{N_n-2J_n}(B_{n,1}). \tag{6.30}$$

Note that each $s_n \circ \sigma_A$ is F-full in $B_{n,1}$. By (6.17) and the second part of Lemma 6.4, the maps $\bigoplus^{N_{n+1}} s_{n+1} \circ \phi_{w,b}$ and $(s_{n+1}^r \circ \phi_{w,b} \oplus \bigoplus^{2J_n} s_n \circ \phi_{w,b} \oplus \bigoplus^{N_n-2J_n} s_n \circ \phi_{w,b})$ induce the same map on $\mathrm{C}u(\mathcal{W}) = [0, \infty]$. Since $K_0(\mathcal{W}) = 0$, they induce the same map on $\mathrm{C}u^{\sim}(\mathcal{W})$. As $K_i(\mathcal{W}) = \{0\}$ $(i = 0, 1)$, by Theorem 1.11 of [62], there are unitaries $v_{n,k} \in \widetilde{\mathrm{Her}(a_{1-t_{n+1}})}$

such that

$$\lim_{k\to\infty} v_{n,k}^*(\bigoplus^{N_{n+1}} s_{n+1}\circ\phi_{w,b}(c))v_{n,k} = (s_{n+1}^r\circ\phi_{w,b}\oplus\bigoplus^{2J_n} s_n\circ\phi_{w,b}\oplus\bigoplus^{N_n-2J_n} s_n\circ\phi_{w,b})(c) \tag{6.31}$$

for all $c\in\mathcal{W}$. By replacing $v_{n,k}$ by $e^{\sqrt{-1}\theta}v_{n,k}$ for some $\theta\in(-\pi,\pi)$, we may assume that $v_{n,k}=1_{\widetilde{\mathrm{Her}(a_{1-t_{n+1}})}}+\bar{v}_{n,k}$ for some $\bar{v}_{n,k}\in\mathrm{Her}(a_{1-t_{n+1}})$. Set $v'_{n,k}=1_{\widetilde{B}}+\bar{v}_{n,k}$, $k=1,2,\ldots$, and $n=1,2,\ldots$. Define

$$\Psi_n := \Lambda_{n,0}\oplus S_n^l = \Lambda_{n,0}\oplus\bigoplus^{2J_n} s_n\circ\sigma_A : A\to\mathrm{Her}(B_{n,0}\oplus M_{2J_n}(B_{n,1}))\subset B_{n,0,1},$$

where $B_{n,0,1}:=\mathrm{Her}(B_{n,0}\oplus M_{2J_n}(B_{n,1}))\oplus\mathrm{Her}(a_{r_{n+1}})$, and

$$\Psi'_n := \Lambda_{n+1,0}\oplus\bigoplus^{2J_n} s_n\circ\sigma_A\oplus s_{n+1}^r\circ\sigma_A : A\to\mathrm{Her}(B_{n+1,0}\oplus M_{2J_n}(B_{n,1}))\oplus\mathrm{Her}(a_{r_{n+1}}).$$

Define $\Lambda_n:=\Lambda_{n,0}\oplus S_n=\Psi_n\oplus S'_n : A\to B$ and define $\Lambda'_{n+1}:=\Lambda_{n+1,0}\oplus R_{n+1}\oplus S_n^l\oplus S'_n$ $n=1,2,\ldots$. Note that

$$\Lambda'_{n+1}=\Psi'_n\oplus S'_n. \tag{6.32}$$

By (6.31),

$$\lim_{k\to\infty}(v'_{n,k})^*\Lambda_{n+1}(a)v'_{n,k}=\Lambda'_{n+1}(a)\ \text{ for all }\ a\in A. \tag{6.33}$$

Now consider the maps $\Psi_n,\Psi'_n : A\to B_{n,1,0}$ (recall $B_{n+1,0}\subset B_{n,0}$; see (6.21)). Then since $K_i(\mathcal{W})=\{0\}$, by (6.26),

$$[\Psi_n]|_{\mathcal{P}_n}=[\Psi'_n]|_{\mathcal{P}_n}=x|_{\mathcal{P}_n}. \tag{6.34}$$

Note that by viewing $\Lambda_{n,0}$ and $\Lambda_{n+1,0}$ as maps from A into $\mathrm{Her}(a_{t_n})$ (as $t_{n+1}\le t_n$), and by computing inside $U(M_{m(n)}(\widetilde{\mathrm{Her}(a_{t_n})}))/CU(M_{m(n)}(\widetilde{\mathrm{Her}(a_{t_n})}))$ and omitting $(j'_{B_{t_{n+1}}})^\dagger$, we have that

$$\overline{\lceil\Lambda_{n,0}(u)\rceil\lceil\Lambda_{n+1,0}(u^*)\rceil}=\overline{s_n^{(0)}(\lceil\phi_n(u)\rceil)s_{n+1}^{(0)}(\lceil\phi_{n+1}(u)\rceil^*)} \tag{6.35}$$

$$=\overline{s_n^{(0)}(\lceil\phi_n(u)\rceil)s_{n+1}^{(0)}(\lceil\phi_n(u)\rceil^*)s_{n+1}^{(0)}(\lceil\phi_n(u)\rceil\lceil\phi_{n+1}(u)\rceil^*)} \tag{6.36}$$

$$= \overline{s_n^{(0)}(z_n^{(0)})s_{n+1}^{(0)}(z_n^{(0)*})s_n^{(0)}(z_n^{(1)})s_{n+1}^{(0)}(z_n^{(1)*})s_{n+1}^{(0)}(\lceil\phi_n(u)\rceil\lceil\phi_{n+1}(u)\rceil^*)} \tag{6.37}$$

$$= \overline{s_n^{(0)}(z_n^{(0)})s_{n+1}^{(0)}(z_n^{(0)*})} \cdot \overline{1} \cdot \overline{s_{n+1}^{(0)}(\lceil\phi_n(u)\rceil\lceil\phi_{n+1}(u)\rceil^*)} \quad \text{(by (6.24))} \tag{6.38}$$

(recall the notation in (6.9) and (6.10)). Recall, by Theorem 4.4 of [22] (see also Theorem 15.5 and the end of 3.9 of [22] for notation), that

$$cel(w) \le 7\pi \text{ for all } w \in CU(M_m(\widetilde{D})) \tag{6.39}$$

for any hereditary C^*-subalgebra D of B.

By (6.38), (6.25), and (6.39) (computing inside $M_{m(n)}(\widetilde{\mathrm{Her}(a_{t_n})})$),

$$cel(\lceil\Lambda_{n,0}(u)\rceil\lceil\Lambda_{n+1,0}(u^*)\rceil) \le (t_n - t_{n+1})\lambda_n' + \lambda_n + 7\pi \text{ for all } u \in \mathcal{U}_n. \tag{6.40}$$

Then, by Lemma 4.2 of [22] and by (6.12) (computing inside $M_{(2J_n+1)m(n)}(\widetilde{\mathrm{Her}(a_{t_n})})$),

$$cel(\lceil\Psi_n(u)\rceil\lceil(\Lambda_{n+1,0} \oplus S_n^l)(u)\rceil^*) \tag{6.41}$$

$$= cel(\lceil\Lambda_{n,0}(u)\rceil\lceil\Lambda_{n+1,0}(u)^*\rceil \oplus S_n^l(uu^*)) \le (\lambda_n' + \lambda_n + 7\pi)/J_n + 2\pi. \tag{6.42}$$

It follows from the definition of R_{n+1} (computing inside $M_{m(n)}(\widetilde{\mathrm{Her}(a_{r_{n+1}})})$) that

$$cel(R_{n+1}(u^*)) \le L_1(u) \text{ for all } u \in \mathcal{U}_n. \tag{6.43}$$

Hence,

$$cel(\lceil\Psi_n(u)\rceil\lceil\Psi_n'(u)^*\rceil) \le 1 + 2\pi + L_1(u) = L(u) \text{ for all } u \in \mathcal{U}_n. \tag{6.44}$$

Recall that $N_n - 2J_n \ge 4(K_n+1)2J_n$ (see (6.18)). Using (6.44), (6.34) and the fact that $s_n \circ \sigma_A$ is F-full in $B_{n,1}$, and applying Theorem 5.5, we obtain a unitary $u_n' \in \widetilde{B}$ such that

$$(u_n')^*(\Lambda_{n+1}'(a))u_n' \approx_{\epsilon_n/2} \Lambda_n(a) \text{ for all } a \in \mathcal{F}_n. \tag{6.45}$$

It follows from (6.33) that there is a unitary $u_n \in \widetilde{B}$ such that

$$(u_n)^*\Lambda_{n+1}(a)u_n \approx_{\epsilon_n} \Lambda_n(a) \text{ for all } a \in \mathcal{F}_n. \tag{6.46}$$

Define $\Lambda_1''(a) := \Lambda_1(a)$ and $\Lambda_n''(a) := u_1^* \cdots u_{n-1}^* \Lambda_n(a) u_{n-1} \cdots u_1$ for all $a \in A$, $n = 2, 3...$ Then by (6.46),

$$\Lambda_n''(a) \approx_{\epsilon_n} \Lambda_{n+1}''(a) \text{ for all } a \in \mathcal{F}_n, \ n = 1, 2, ... \tag{6.47}$$

It follows that for any $m > n$,

$$\|\Lambda_n''(a) - \Lambda_m''(a)\| < \sum_{j=n}^{m} \epsilon_j \text{ for all } a \in \mathcal{F}_n. \tag{6.48}$$

Note that $\lim_{n\to\infty} \sum_{j=n}^{\infty} \epsilon_j = 0$. Since $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ and $\bigcup \mathcal{F}_n$ is dense in the unit ball of A, we conclude that for each $a \in A$, $\{\Lambda_n''(a)\}$ is Cauchy in B. Let $\Phi(a) = \lim_{n\to\infty} \Lambda_n''(a)$ for each $a \in A$. It is clear that Φ is a positive linear map. Since

$$\lim_{n\to\infty} \|\Lambda_n''(ab) - \Lambda_n''(a)\Lambda_n''(b)\| = 0 \text{ for all } a, b \in A, \tag{6.49}$$

$\Phi : A \to B$ is a homomorphism. Since σ_A is an embedding, Φ is injective. Finally, by (6.34), we have that

$$KL(\Phi) = x. \tag{6.50}$$

■

Theorem 6.11. Let A be a separable amenable C^*-algebra, which satisfies the UCT. Suppose that A is $\mathcal{W}$ embeddable. Let B be a separable simple C^*-algebra with finite nuclear dimension, with continuous scale, which satisfies the UCT and with $K_0(B) = \ker \rho_B$.

Let $\kappa \in KL(A, B)$, and let $\kappa_{ku} : K_1(A) \to U(\widetilde{B})/CU(\widetilde{B})$ be a homomorphism that is compatible with κ, that is,

$$\kappa(z) = \Pi_{1,cu} \circ \kappa_{ku}(z),$$

for all $z \in K_1(A)$. Then there exists a monomorphism $h : A \to B$ such that

$$KL(h) = \kappa \text{ and } h^{\ddagger} = \kappa_{ku}.$$

Proof. Note that we are fixing injective homomorphism s $J_A : K_1(A) \to U(\widetilde{A})/CU(\widetilde{A})$ and $J_B : K_1(B) \to U(\widetilde{B})/CU(\widetilde{B})$, which split the following short exact sequences:

$$0 \longrightarrow U_0(M_\infty(\widetilde{A}))/CU(M_\infty(\widetilde{A})) \longrightarrow U(M_\infty(\widetilde{A}))/CU(M_\infty(\widetilde{A})) \overset{\Pi_{1,cu}}{\rightleftarrows}_{J_A} K_1(A) \to 0 \text{ and}$$
$$0 \longrightarrow U_0(M_\infty(\widetilde{B}))/CU(M_\infty(\widetilde{B})) \longrightarrow U(M_\infty(\widetilde{B}))/CU(M_\infty(\widetilde{B})) \overset{\Pi_{1,cu}}{\rightleftarrows}_{J_B} K_1(B) \to 0$$

(see 5.7). Recall that for any homomorphism $\rho : A \to B$, we let $\rho^\ddagger : K_1(A) \to U(\widetilde{B})/CU(\widetilde{B})$ denote the induced map.

Note also that $K_i(A)$ is a countable abelian group for $i = 0, 1$. By Theorem 7.11 of [22], there is a stably projectionless simple C^*-algebra C, in the classifiable class and with continuous scale, such that $K_i(C) \cong K_i(A)$, $i = 0, 1$, and $T(C) = T(B)$. Let $\iota_i : K_i(A) \to K_i(C)$ be a group isomorphism. By Theorem 6.10, let $\phi : A \to C$ be a *-embedding such that $\phi_{*i} = \iota_i$ $(i = 0, 1)$. Fix an injective homomorphism $J_C : K_1(C) \to U(\widetilde{C})/CU(\widetilde{C})$ such that $\Pi_{1,cu} \circ J_C = \mathrm{id}_{K_1(C)}$ and consider the group homomorphism

$$\phi_D : K_1(A) \to U_0(\widetilde{C})/CU(\widetilde{C})$$

induced by ϕ, that is, $\phi_D := \phi^\dagger \circ J_A - J_C \circ \phi_{*1}$. In particular, $\phi_D(x) \in U_0(\widetilde{C})/CU(\widetilde{C})$ for all $x \in K_1(A)$.

Consider the group homomorphism $\lambda : K_1(C) \to U(\widetilde{C})/CU(\widetilde{C})$, which is defined by

$$\lambda(z) = J_C(z) - \phi_D \circ \iota_1^{-1}(z) \tag{6.51}$$

for all $z \in K_1(C)$.

Define $\lambda_1 : U(\widetilde{C})/CU(\widetilde{C}) \to U(\widetilde{C})/CU(\widetilde{C})$ by

$$\lambda_1|_{U_0(\widetilde{C})/CU(\widetilde{C})} = id_{U_0(\widetilde{C})/CU(\widetilde{C})} \text{ and } \lambda_1|_{J_C(K_1(C))} = \lambda \circ J_C^{-1},$$

where $J_C^{-1} = \Pi_{1,cu}|_{J_C(K_1(C))}$. By Lemma 12.10 of [22], there is a homomorphism $j : C \to C$ such that $KK(j) = KK(\mathrm{id}_C)$, $j_T = \mathrm{id}_{T(C)}$ and $j^\dagger = \lambda_1$.

Let $\psi : A \to C$ be defined by

$$\psi = j \circ \phi.$$

Then for all $x \in K_1(A)$,

$$\begin{aligned}
\psi^{\ddagger}(x) &= ((j\circ\phi)^{\dagger}\circ J_A)(x) = j^{\dagger}\circ\phi^{\dagger}\circ J_A(x)\\
&= j^{\dagger}(\phi_D(x) + J_C\circ\phi_{*1}(x)) = \lambda_1(\phi_D(x) + J_C\circ\phi_{*1}(x))\\
&= \phi_D(x) + \lambda\circ\phi_{*1}(x)\\
&= \phi_D(x) + J_C\circ\phi_{*1}(x) - \phi_D\circ\iota_1^{-1}(\phi_{*1}(x))\\
&= \phi_D(x) + J_C\circ\phi_{*1}(x) - \phi_D(x) = J_C\circ\iota_1(x). \qquad (6.52)
\end{aligned}$$

By the UCT, since ι_i is an isomorphism ($i = 0, 1$), it gives a KK equivalence and hence, there is a $\zeta \in KK(C, A)$ such that

$$\zeta \times KK(\psi) = KK(\mathrm{id}_C).$$

Let $\overline{\zeta}$ be the element in $KL(C, A)$ induced by ζ. By Lemma 12.10 of [22], there is a homomorphism $h_1 : C \to B$ such that $KL(h_1) = \kappa\circ\overline{\zeta}$, $(h_1)_T^{-1}$ is the identification of $T(C)$ and $T(B)$ and

$$h_1^{\dagger}|_{J_C(K_1(C))} = \kappa_{ku}\circ\iota_1^{-1}\circ J_C^{-1}.$$

It follows from (6.52) that if we define $h := h_1\circ\psi : A \to B$, then

$$h^{\ddagger} = h_1^{\dagger}\circ\psi^{\dagger}\circ J_A = (\kappa_{ku}\circ\iota_1^{-1}\circ J_C^{-1})\circ\psi^{\ddagger} = (\kappa_{ku}\circ\iota_1^{-1}\circ J_C^{-1})\circ J_C\circ\iota_1 = \kappa_{ku}.$$

Then one verifies that the map h satisfies the requirements. ■

7 Quasidisagonal Extensions by $\mathcal{W}$

The following proposition is an easy fact and known to the experts. We include a proof for the convenience of the reader.

Proposition 7.1. Let B be a σ-unital C^*-algebra and let $p \in M(B)\setminus B$ be a projection. Suppose that $pBp = \mathrm{Her}(a)$ for some $a \in B_+$.

Then $a^{1/n} \to p$ (as $n \to \infty$) in the strict topology on $M(B)$.

Moreover, p is the open projection in B^{**} corresponding to $\mathrm{Her}(a)$.

Proof. Fix any $x \in B_+$. Note that $\{a^{1/n}\}$ is an approximate unit for $\mathrm{H}er(a) = pBp$. Also, $px^2p \in pBp = \mathrm{H}er(a)$, and $a^{1/n} = a^{1/n}p = pa^{1/n}$ for all n.

Hence,

$$\begin{aligned} & \|x(p - a^{1/n})\|^2 \\ = \ & \|(p - a^{1/n})x^2(p - a^{1/n})\| \\ = \ & \|px^2p - px^2pa^{1/n} - a^{1/n}px^2p + a^{1/n}px^2pa^{1/n}\| \\ \to \ & \|px^2p - px^2p - px^2p + px^2p\| = 0. \end{aligned}$$

By a similar argument, $\|(p - a^{1/n})x\| \to 0$. Since x is an arbitrary element of B_+, $a^{1/n} \to p$ in the strict topology on $M(B)$.

We may assume that a is a contraction. Since $a^{1/n} \nearrow p$ in the strict topology on $M(B)$, $a^{1/n} \nearrow p$ in the weak* topology on B^{**}. So p is also the open projection in B^{**} corresponding to $\mathrm{H}er(a)$. ■

The next lemma should also be known.

Lemma 7.2. Let B be a separable simple C^*-algebra with continuous scale such that B and $B \otimes \mathcal{K}$ stably have almost stable rank one. Suppose that $\mathrm{C}u(B) = V(B) \sqcup (\mathrm{L}Aff_+(T(B)) \setminus \{0\})$.

Then, if $p, q \in M_m(M(B)) \setminus M_m(B)$ are two projections (for any integer $m \geq 1$) such that $\tau(p) = \tau(q)$ for all $\tau \in T(B)$, then $p \approx q$ in $M_m(M(B))$. Moreover,

$$K_0(M(B)) \cong \mathrm{Aff}(T(B)) \ \text{ and } \ K_0(M(B))_+ \cong \mathrm{Aff}_+(T(B)). \tag{7.1}$$

In fact,

$$V(M(B)) = V(B) \sqcup (\mathrm{Aff}_+(T(B)) \setminus \{0\}). \tag{7.2}$$

Proof. Let $\{e_{i,j}\}$ be a system of matrix units for $\mathcal{K}$. In what follows, we will identify B with $e_{1,1}(B \otimes \mathcal{K})e_{1,1}$. (Here, we abuse notation and identify $e_{1,1}$ with $1_{M(B)} \otimes e_{1,1}$. Similar for $e_{i,j}$ for all i, j.) We also note that in this way, we identify $M(B)$ with $e_{1,1}M(B \otimes \mathcal{K})e_{1,1}$. In what follows, in this proof, we also identify $1_m := \sum_{i=1}^m e_{i,i}$ with the unit of $M_m(M(B))$. Moreover, for each $\tau \in T(B)$, we will also use τ for the extensions of τ to $(B \otimes \mathcal{K})_+$ as well as to $M(B \otimes \mathcal{K})_+$.

Set $C = B \otimes \mathcal{K}$.

Let $p,\, q \in M(C)\setminus C$ be two projections such that $\tau(p) = \tau(q)$ for all $\tau \in T(B)$. Let $a \in pCp$ and $b \in qCq$ be strictly positive elements of pCp and qCq, respectively. Note that neither a nor b are projections, as $p, q \notin C$. Then the fact that $\tau(p) = \tau(q)$ for all $\tau \in T(B)$ implies that

$$d_\tau(a) = d_\tau(b) \text{ for all } \tau \in T(B). \tag{7.3}$$

It follows that $a \sim b$ in C. Since $C = B\otimes\mathcal{K}$ stably has almost stable rank one, there exists a partial isometry $v \in C^{**}$ such that $c := vav^*$ is a strictly positive element of $\mathrm{H}er(b)$,

$$v^*va = av^*v = a, \text{ and } va, v^*b \in C. \tag{7.4}$$

(See Proposition 3.3 of [63] and the paragraph above it.) From the above and since $b^{1/m}, c^{1/m} \to q$ and $a^{1/m} \to p$ in the weak* topology on C^{**}, $q = vpv^*$. Also, $qvp \in C^{**}$ is a partial isometry with left support q and right support p. So replacing v with qvp if necessary, we may assume that $v^*v = p$ and $vv^* = q$. By Proposition 7.1,

$$va^{1/m} \to vp \text{ in the strict topology on } M(C). \tag{7.5}$$

Therefore, $v = vp \in M(C)$. So v witnesses that $p \approx q$ in $M(C)$.

From what has been just proven, we conclude that if $p, q \in M_m(M(B))\setminus M_m(B)$ (for some integer $m \geq 1$) are two projections and $\tau(p) = \tau(q)$ for all $\tau \in T(B)$, then p and q are equivalent in $M_m(M(B))$.

Let $p \in M_m(M(B))$ be a projection. Then $\tau(p)$ may be viewed as a function in $\mathrm{L}Aff_+(T(B))$. However, $1_m - p \in M_m(M(B))$ is also a projection. Therefore, $\tau(1_m - p) \in \mathrm{L}Aff_+(T(B))$ $(\tau \in T(B))$. It follows that $\tau(p)$ is an affine function in $\mathrm{Aff}_+(T(B))$. This implies that the map

$$\rho : K_0(M(B)) \to \mathrm{Aff}(T(B)) \tag{7.6}$$

is an order preserving homomorphism, and we just proved that the map ρ is injective.

We now show that ρ is surjective. By the assumption, for any $f \in \mathrm{Aff}_+(T(B))\setminus\{0\}$, there is a nonzero positive element $a \in B$ such that $d_\tau(a) = f(\tau)$ for all $\tau \in T(B)$. It follows from Kasparov's absorption theorem (Theorem 2 of [31]) that there is a projection $p_1 \in M(B\otimes\mathcal{K})$ such that $p_1(B\otimes\mathcal{K}) \cong \overline{a(B\otimes\mathcal{K})}$, where the isomorphism is a unitary isomorphism of Hilbert $B\otimes\mathcal{K}$-modules.

Therefore, by Proposition 7.1, replacing a with a Cuntz equivalent positive element if necessary, we may assume that $a^{1/m}$ converges to p_1 in the strict topology on $M(B \otimes \mathcal{K})$. It follows that $\tau(p_1) = d_\tau(a)$ for all $\tau \in T(B)$.

There exists an integer m such that $m \geq f(\tau) + 1$ for all $\tau \in T(B)$. Let $g = m - f$. Then $g \in \mathrm{Aff}_+(T(B)) \setminus \{0\}$. From what has just been proved, we obtain a projection $q \in M(B \otimes \mathcal{K})$ such that $\tau(q) = g$. Without loss of generality, we may assume that $p_1 \perp q$. Then $p_1 + q = e$ is a projection in $M(B \otimes \mathcal{K})$ such that $\tau(e) = \tau(1_m)$ for all $\tau \in T(B)$. From the first part of the proof above, we conclude that there is $v \in M(B \otimes \mathcal{K})$ such that

$$v^*v = e \text{ and } vv^* = 1_m. \tag{7.7}$$

This implies that $p := vp_1v^* \leq 1_m$. In other words, $p \in M_m(M(B))$ (see the first part of the paragraph of this proof). Note that $\tau(p) = f(\tau)$, for all $\tau \in T(B)$. This proves that the map ρ is surjective. The rest of the proposition also follows. ■

Remark 7.3. As in the beginning of the proof of Theorem 6.6, if A is a separable simple finite C^*-algebra, which is $\mathcal{Z}$-stable and has continuous scale and for which every 2-quasi-trace is a trace, then $\mathrm{Cu}(B) = V(B) \sqcup (\mathrm{L}Aff_+(T(B)) \setminus \{0\})$. Later on, we often assume that B is a separable simple finite C^*-algebra, which is $\mathcal{Z}$-stable and has continuous scale.

Theorem 7.4. Let B be a σ-unital, stably projectionless, finite, simple, $\mathcal{Z}$-stable, amenable C^*-algebra with a unique tracial state τ_B.

Then $K_0(M(B)) = \mathbb{R}$, $K_1(M(B)) = \{0\}$, $K_0(\mathcal{C}(B)) = \mathbb{R} \oplus K_1(B)$, and $K_1(\mathcal{C}(B)) = \ker \rho_B = K_0(B)$. In particular, $K_0(M(\mathcal{W})) = \mathbb{R}$, $K_1(M(\mathcal{W})) = \{0\}$, $K_0(\mathcal{C}(\mathcal{W})) = \mathbb{R}$ and $K_1(\mathcal{C}(\mathcal{W})) = \{0\}$.

Moreover, if $p, q \in M_m(M(B)) \setminus M_m(B)$ are two projections (for some integer $m \geq 1$) and $\tau_B(p) = \tau_B(q)$, then there exists $v \in M_m(M(B))$ such that $v^*v = p$ and $vv^* = q$.

Proof. By [51], $K_1(M(B)) = \{0\}$. It follows from Lemma 7.2 that $K_0(M(B)) = \mathbb{R}$. Thus, the six-term exact sequence

$$\begin{array}{ccccc} K_0(B) & \rightarrow & K_0(M(B)) & \rightarrow & K_0(\mathcal{C}(B)) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{C}(B)) & \leftarrow & K_1(M(B)) & \leftarrow & K_1(B) \end{array}$$

becomes

$$\begin{array}{ccccc} K_0(B) & \to & \mathbb{R} & \to & K_0(\mathcal{C}(B)) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{C}(B)) & \leftarrow & 0 & \leftarrow & K_1(B) \end{array}$$

Note that the map from $K_0(B)$ into $\mathbb{R} = K_0(M(B))$ is induced by the map $\rho_B : K_0(B) \to \mathrm{Aff}(T(B)) = \mathbb{R}$. However, by Corollary A7 of [19], since B has a unique tracial state, $K_0(B) = \ker\rho_B$. In other words, $\rho_B(K_0(B)) = \{0\}$.

It follows from the exact sequence that

$$K_1(\mathcal{C}(B)) = \ker\rho_B = K_0(B) \text{ and}$$

$$0 \to \mathbb{R} \to K_0(\mathcal{C}(B)) \to K_1(B) \to 0.$$

Since $\mathbb{R}$ is divisible, we may write $K_0(\mathcal{C}(B)) = \mathbb{R} \oplus K_1(B)$. Note that B is stably projectionless and $\mathcal{Z}$-stable. Therefore, the last statement follows from 7.2 (see also Remark 7.3). ■

Definition 7.5. Let A be a separable C^*-algebra, and let B be a non-unital and σ-unital C^*-algebra.

A trivial extension $\phi : A \to M(B)$ is said to be *diagonal* if ϕ is quasidiagonal as in Definition 4.4 and Proposition 4.5, with the additional property that the maps ϕ_n in Proposition 4.5 can be taken to be homomorphism s.

In the above setting, we often write $\phi = \bigoplus_{n=1}^{\infty} \phi_n$ (where the sum converges in the pointwise-strict topology).

Definition 7.6. $\mathcal{T}_d$ extensions Let B be a separable simple non-unital C^*-algebra with continuous scale, and let C be a separable C^*-algebra. A monomorphism $\sigma : C \to M(B)$ is called a *$\mathcal{T}_d$ extension with model* σ_0 if $\pi \circ \sigma$ is non-unital and if σ is a diagonal essential extension of the form

$$\sigma = \bigoplus_{n=1}^{\infty} \bigoplus^{n} \phi_n \circ \sigma_0 = \bigoplus_{n=1}^{\infty} (\overbrace{\phi_n \circ \sigma_0 \oplus \ldots \oplus \phi_n \circ \sigma_0}^{n}).$$

Here, $\sigma_0 : C \to B$ is a fixed injective *-homomorphism such that $\sigma_0(e_C)$ is a strictly positive element of B, where $e_C \in C$ is a strictly positive element of C.

More precisely, this means the following:

1. There exists a system $\{b_n\}$ of quasidiagonal units for B.
2. There exists a nonzero positive element $b_{n,1} \in \overline{b_n B b_n}$ such that

$$\mathrm{Her}(b_{n,1}) \otimes M_n \cong M_n(\overline{b_{n,1} B b_{n,1}}) = \overline{b_n B b_n}$$

for all $n \geq 1$. Moreover, we may write

$$b_n = \sum_{j=1}^{n} b_{n,j}, \tag{7.8}$$

where $b_{n,j} := b_{n,1} \otimes e_{j,j}$, and $\{e_{i,j}\} \subset M_n$ is a system of matrix units.
3. $\phi_n : B \to \overline{b_{n,1} B b_{n,1}}$ is an isomorphism such that $\phi_n \circ \sigma_0(e_C) = b_{n,1}$, and $(\overbrace{\phi_n \circ \sigma_0 \oplus \ldots \oplus \phi_n \circ \sigma_0}^{n}) : C \to M_n(\mathrm{Her}(b_{n,1}))=\mathrm{Her}(b_n)$ is the diagonal map, for all $n \geq 1$.

Remark 7.7. With notation as in Definition 7.6, let us suppose that $F : C_+ \setminus \{0\} \to \mathbb{N} \times (0, \infty)$ is a map such that σ_0 is F-full. (F exists by Lemma 6.7.) Then each $\phi_n \circ \sigma_0$ is also F-full.

Note also that $\oplus_{n=1}^{m} \bigoplus^{n} \phi_n \circ \sigma_0(c)$ converges strictly to $\sigma(c)$ for all $c \in C$ (as $m \to \infty$). Note that $\sigma = \phi \circ \sigma_0$, where $\phi = \bigoplus_{n=1}^{\infty} \bigoplus^{n} \phi_n : B \to M(B)$.

Finally, note that our definition of $\mathcal{T}_d$ extension requires that $\pi \circ \sigma$ be a non-unital essential extension.

Remark 7.8. With notation as in Definition 7.6, note that if $KK(\sigma_0) = 0$ then, since $\sigma = \phi \circ \sigma_0$ and ϕ is a *-homomorphism, $KK(\pi \circ \sigma) = 0$.

Also, when C is amenable and satisfies the UCT, and when B is stably projectionless, $\mathcal{Z}$-stable, and has a unique tracial state, since $K_*(M(B)) \cong (\mathbb{R}, 0)$ is divisible, a sufficient condition for the above is that

$$K_0(\sigma_0) = 0.$$

Proposition 7.9. Let A be a separable amenable C^*-algebra, which is $\mathcal{W}$ embeddable. Then there exists a $\mathcal{T}_d$ extension $\tau : A \to \mathcal{C}(\mathcal{W})$. Moreover,

$$KK(\tau) = 0.$$

Proof. Fix a *-embedding $\sigma_A : A \to \mathcal{W}$, which maps strictly positive elements to strictly positive elements. Denote by τ_W the unique tracial state of $\mathcal{W}$. Fix a system of quasidiagonal units $\{b_k\}$ for $\mathcal{W}$ as in 4.1. Passing to a subsequence if necessary, we may assume that

$$\sum_{k=n+1}^{\infty} d_{\tau_W}(b_k) < \frac{1}{n} d_{\tau_W}(b_n) \text{ for all } n. \tag{7.9}$$

Let $t_n = \frac{1}{n+1} d_{\tau_W}(b_n)$, $n \in \mathbb{N}$. There is an element $a_n \in Her(b_n)_+ \setminus \{0\}$ with $d_{\tau_W}(a_n) \leq t_n$ such that $M_n(\mathrm{Her}(a_n)) \subseteq \mathrm{Her}(b_n)$ (by Theorem 6.6 of [20] and by strict comparison). There is, for each n, an isomorphism $\phi_n : \mathcal{W} \to Her(a_n)$. Define $\sigma : A \to M(\mathcal{W})$ by $\sigma(a) = \sum_{n=1}^{\infty} (\bigoplus^n \phi_n \circ \sigma_A)(a)$ for all $a \in A$. Note that since $\{b_n\}$ is a system of quasidiagonal units, the sum above converges in the strict topology on $M(\mathcal{W})$ for each $a \in A$. One then checks, from Definition 7.6, that $\pi \circ \sigma$ is a $\mathcal{T}_d$ extension with model σ_A.

That $KK(\pi \circ \sigma) = 0$ follows from Remark 7.8. ■

Proposition 7.10. Let C be a separable amenable C^*-algebra, which is $\mathcal{W}$ embeddable and satisfies the UCT, and let $\phi : C \to M(\mathcal{W})$ be a monomorphism. Then $\phi_{*0}(\ker \rho_{f,C}) = \{0\}$ and $\phi_{*1} = 0$.

If X is a connected and locally connected compact metric space, and $C := C_0(X \setminus \{x_0\})$ for some $x_0 \in X$, then $KK(\phi) = 0$ and $KK(\pi \circ \phi) = 0$.

Proof. Recall that $K_0(M(\mathcal{W})) = \mathbb{R}$ and $K_1(M(\mathcal{W})) = \{0\}$. The first part follows from the fact that if $p, q \in M_n(M(\mathcal{W}))$ (for some integer n) are two projections and $\tau_W(p) = \tau_W(q)$, then there exists a $v \in M_n(M(\mathcal{W}))$ such that $v^*v = p$ and $vv^* = q$. (See Theorem 7.4.)

In the case that $C = C_0(X \setminus \{x_0\})$, since X is connected, $K_0(C) = \ker \rho_C$. It follows that $\phi_{*i} = 0$, $i = 0, 1$. Since $K_0(M(\mathcal{W})) = \mathbb{R}$ is divisible, $\mathrm{Ext}_{\mathbb{Z}}(K_1(C), K_0(M(\mathcal{W}))) = \{0\}$. By the UCT, $KK(\phi) = 0$. Then $KK(\pi \circ \phi) = 0$ follows. ■

Denote by $\mathcal{D}$ the class of simple C^*- algebras defined in Definition of 8.1 of [18]. Suppose that $A \in \mathcal{D}$. Then for any integer $k \geq 1$, $M_k(A) \in \mathcal{D}$ (see 8.5 of [18]). Moreover, A is stably projectionless (see 9.3 of [18]). We note that $\mathcal{W} \in \mathcal{D}$ (see 9.6 of [18]).

Let us quote the following lemma for the convenience of the reader.

Lemma 7.11 (Theorem 4.4 of [22]). Let A be a separable simple C^*-algebra in $\mathcal{D}$ and let $u \in CU(M_m(\widetilde{A}))$. Then $u \in U_0(M_m(\widetilde{A}))$ and $cel(u) \le 7\pi$.

Proof. Note that, as mentioned above, $M_m(A) \in \mathcal{D}$. Let $\pi : M_m(\widetilde{A}) \to M_m(\mathbb{C})$ be the quotient map. Then $w = \pi(u)$ is a scalar unitary. Denote by $W \in M_m(\mathbb{C} \cdot 1_{\widetilde{A}})$ the same scalar matrix. Then $W^*u \in \widetilde{M_m(A)}$. By Theorem 4.4 of [22], $W^*u \in U_0(M_m(\widetilde{A}))$ and $cel(W^*u) \le 6\pi$. Since $W \in M_m(\mathbb{C} \cdot 1_{\widetilde{A}})$, we conclude that $u \in U_0(M_m(\widetilde{A}))$ and $\mathrm{cel}(u) \le 7\pi$. ■

Lemma 7.12. Let C be a separable amenable C^*-algebra, which is $\mathcal{W}$ embeddable and satisfies the UCT. Let $\sigma : C \to M(\mathcal{W})$ be a $\mathcal{T}_d$ extension, and let $\psi : C \to M(\mathcal{W})$ be a diagonal c.p.c. map of the form

$$\psi = \bigoplus_{n=1}^{\infty} \psi_n$$

as in Proposition 4.5 such that $\pi \circ \psi$ is a non-unital essential extension.

Then there is a diagonal extension $h : C \to \mathcal{C}(\mathcal{W})$ such that

$$\pi \circ \sigma \oplus \pi \circ \psi \sim^u \pi \circ \sigma \oplus h.$$

Proof. Fix a strictly positive element $e_C \in C$ with $\|e_C\| = 1$. By working in $M_2(M(\mathcal{W}))$ if necessary, without loss of generality, we may assume that $ran(\psi) \perp ran(\sigma)$ (see Proposition 3.5).

Since σ is a $\mathcal{T}_d$ extension, using a variation on the notation of Definition 7.6, we write

$$\sigma = \bigoplus_{n=1}^{\infty} \bigoplus^{n+1} \phi_n \circ \sigma_0.$$

We also write $\bigoplus^{n+1} \phi_n \circ \sigma_0 = \sigma_{n,0} \oplus \sigma_{n,1} \oplus \cdots \oplus \sigma_{n,n}$ and $\sigma = \bigoplus_{n=1}^{\infty} \bigoplus_{j=0}^{n} \sigma_{n,j}$. Continuing to follow Definition 7.6, let

$$b_{n,j} := \sigma_{n,j}(e_C)$$

and let b_n be as in Definition 7.6, for all n, j.

Also, let $\{a_n\}$ be the system of quasidiagonal units for $\mathcal{W}$ from Proposition 4.5 that corresponds to $\{\psi_n\}$. Recall (see 4.5) that

$$\lim_{n\to\infty} \|\psi_n(a)\psi_n(b) - \psi_n(ab)\| = 0 \text{ for all } a, b \in C. \tag{7.10}$$

Since σ is a $\mathcal{T}_d$ extension, as in Remark 7.7, there exists a map $F : C_+ \setminus \{0\} \to \mathbb{N} \times (0, \infty)$ such that for all n, j, $\sigma_{n,j} : C \to \overline{b_{n,j}\mathcal{W}b_{n,j}}$ is F-full.

Let $\{\epsilon_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty} \epsilon_n < \infty$.

Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \cdots$ be a sequence of finite subsets of the unit ball of C, whose union is dense in the unit ball of C.

We will apply Theorem 5.8. Note that by Proposition 5.2, $\mathcal{W} \in \mathbf{C}_{0,0,1,T,7}$, with T as in Proposition 5.2. Let $L := 7\pi + 1$. As C is given, we fix maps J, Π_{cu}^{-} and $J^{\sim}$ as in 5.7.

For each n, let $\delta_n > 0$, $\mathcal{G}_n \subset C$ be a finite subset, $\mathcal{P}_n \subset \underline{K}(C)$ be a finite subset, $\mathcal{U}_n \subset \mathcal{U}_{n+1} \subset J^{\sim}(K_1(C))$ be finite subsets, $\mathcal{E}_n \subset C_+ \setminus \{0\}$ be a finite subset, and K_n be an integer associated with $\mathcal{F}_n$ and $\epsilon_n/4$ (as well as F and L above) as provided by Theorem 5.8 (for C^*- algebras in $\mathbf{C}_{0,0,1,T,7}$).

We may assume that $\delta_{n+1} < \delta_n$, $\mathcal{G}_n \subseteq \mathcal{G}_{n+1}$, $K_n < K_{n+1}$, and $\mathcal{U}_n \subset U(M_{m(n)}(\widetilde{C}))$ for all n. Without loss of generality, we may assume that each ψ_n is $\mathcal{G}_n$-δ_n-multiplicative and $\lceil \psi_n(u) \rceil$ is well defined for all $u \in \mathcal{U}_n$.

Moreover, without loss of generality, we may also assume (see Theorem 14.5 of [46]) that for any n, there is a group homomorphism

$$\lambda_n : G(\Pi_1(\mathcal{U}_n)) \to U(M_{m(n)}(\widetilde{\mathrm{Her}(a_n)}))/CU(M_{m(n)}(\widetilde{\mathrm{Her}(a_n)})) \cong \mathrm{Aff}(T(\widetilde{\mathcal{W}}))/\mathbb{Z}$$

such that

$$\mathrm{dist}(\lambda_n(x), \Pi_{cu}(\lceil \psi_n(J^{\sim}(x)) \rceil)) < \frac{1}{16\pi(n+1)} \text{ for all } x \in \Pi_1(\mathcal{U}_n), \tag{7.11}$$

where $G(\Pi_1(\mathcal{U}_n))$ is the subgroup generated by the finite subset $\Pi_1(\mathcal{U}_n)$. Since $\mathrm{Aff}(T(\widetilde{\mathrm{Her}(a_n)}))/\mathbb{Z} = \mathrm{Aff}(T(\widetilde{\mathcal{W}}))/\mathbb{Z}$ is divisible, there is a homomorphism $\bar{\lambda}_n : K_1(C) \to \mathrm{Aff}(T(\widetilde{\mathcal{W}}))/\mathbb{Z}$ such that $\bar{\lambda}_n$ extends λ_n.

It follows from Theorem 6.11 that for each n, there is a monomorphism $h_n : C \to \mathrm{Her}(a_n)$ such that $KL(h_n) = KL(\psi_n) = 0$ and

$$h_n^{\ddagger} = \bar{\lambda}_n. \tag{7.12}$$

Define $H : A \to M(\mathcal{W})$ by $H = \bigoplus_{n=1}^{\infty} h_n$. Note that by 4.1, the sum converges pointwise-strictly and H gives a diagonal extension.

Since $\mathcal{W}$ is KK contractible, we may assume that

$$[\Sigma_{k=n}^{m} h_k]|_{\mathcal{P}_n} = [\Sigma_{k=n}^{m} \psi_k]|_{\mathcal{P}_n} = 0 \text{ for all } m \geq n, \ \ n = 1, 2, \ldots. \tag{7.13}$$

Throwing away finitely many terms and relabelling if necessary, we may assume that

$$\sum_{n=1}^{\infty} d_{\tau_W}(a_n) < d_{\tau_W}(b_{K_1,0}).$$

Let $\{n_k\}_{k=1}^{\infty}$ be a subsequence of $\mathbb{Z}^+$ with $n_1 = 1$ and $n_k + 2 < n_{k+1}$ for all k such that

$$\sum_{l=n_k}^{\infty} d_{\tau_W}(a_l) < d_{\tau_W}(b_{K_k,0}).$$

By (7.12) and (7.11), for any $u \in \mathcal{U}_{n_k}$, for any $n_k \leq l \leq n_{k+1} - 1$, there is a $v_l \in CU(M_{m(l)}(\widetilde{\mathrm{Her}(a_l)}))$ such that

$$h_l(u)\lceil\psi_l(u)\rceil^* \approx_{1/16\pi(l+1)} v_l. \tag{7.14}$$

It follows from Lemma 7.11 that for all $u \in \mathcal{U}_{n_k}$,

$$\mathrm{cel}((\Sigma_{l=n_k}^{n_{k+1}-1} h_l)(u)\lceil(\Sigma_{l=n_k}^{n_{k+1}-1} \psi_l)(u)\rceil^*) \leq 7\pi + 1, \tag{7.15}$$

where the length is computed in $M_{m(n_k)}(\,\mathrm{Her}(\sum_{l=n_k}^{n_{k+1}-1} a_l))$.

Since $\mathcal{W}$ has stable rank one, there is a unitary $U_k' \in \widetilde{\mathcal{W}}$ such that

$$(U_k')^*((\sum_{l=n_k}^{n_{k+1}-1} a_l)\mathcal{W}(\sum_{l=n_k}^{n_{k+1}-1} a_l))U_k' \subseteq \overline{b_{K_k,0}\mathcal{W}b_{K_k,0}}. \tag{7.16}$$

For each k, consider the two maps

$$\mathrm{Ad}\, U_k' \circ (\Sigma_{l=n_k}^{n_{k+1}-1} \psi_l), \ \ \mathrm{Ad}\, U_k' \circ (\Sigma_{l=n_k}^{n_{k+1}-1} h_l) : C \to \mathrm{Her}(b_{K_k,0}) \cong \mathcal{W}.$$

Recall that ψ_n is $\mathcal{G}_n$-δ_n-multiplicative and $\phi_n \circ \sigma_0$ is F-full, for all n. Also, keeping in mind of (7.15) and (7.13) and applying Theorem 5.8, for all k, there is a unitary $u_k' \in M_{K_k+1}(\widetilde{\mathrm{H}er(b_{K_k,0})})$ such that

$$u_k'(U_k'^* \sum_{l=n_k}^{n_{k+1}-1} h_l(c)U_k' + \sum_{l=1}^{K_k} \sigma_{K_k,l}(c))(u_k')^* \approx_{\epsilon_k/4} (U_k')^* \sum_{l=n_k}^{n_{k+1}-1} \psi_l(c)U_k' + \sum_{l=1}^{K_k} \sigma_{K_k,l}(c) \tag{7.17}$$

for all $c \in \mathcal{F}_k$.

For each k, there are $e_k \in \mathrm{Her}(b_{K_k,0})_+$ and $e_k' \in U_k'^*\mathrm{Her}(\sum_{l=n_k}^{n_{k+1}-1} a_l)_+U_k'$ with $\|e_k\| \leq 1$ and $\|e_k'\| \leq 1$ such that for all $c \in \mathcal{F}_k$,

$$e_k'U_k'^* \sum_{l=n_k}^{n_{k+1}-1} \psi_l(c)U_k'e_k' \approx_{\epsilon_k/16} U_k'^* \sum_{l=n_k}^{n_{k+1}-1} \psi_l(c)U_k', \tag{7.18}$$

$$e_k'U_k'^* \sum_{l=n_k}^{n_{k+1}-1} h_l(c)U_k'e_k' \approx_{\epsilon_k/16} U_k'^* \sum_{l=n_k}^{n_{k+1}-1} h_l(c)U_k', \text{ and} \tag{7.19}$$

$$\sum_{l=1}^{K_k} e_k\sigma_{K_k,l}(c)e_k \approx_{\epsilon_k/16} \sum_{l=1}^{K_k} \sigma_{K_k,l}(c), \tag{7.20}$$

where in (7.20), we identify $M_{K_k}(\mathrm{H}er(b_{K_k,0}))$ with $\mathrm{Her}(\sum_{l=1}^{K_k} b_{K_k,l})$.

Set $X_k = U_k'e_k' + diag(\overbrace{e_k, e_k, \cdots, e_k}^{K_k})$, $k = 1, 2, \ldots$. Note that $e_k'U_k'^*d = 0$ for all $d \in ran(\sigma)$. Then for all $c \in \mathcal{F}_k$, by (7.18),

$$X_k \left(U_k'^* \sum_{l=n_k}^{n_{k+1}-1} \psi_l(c)U_k' + \sum_{l=1}^{K_k} \sigma_{K_k,l}(c) \right) X_k^* \approx_{\epsilon_k/16} \sum_{l=n_k}^{n_{k+1}-1} \psi_l(c) + \sum_{l=1}^{K_k} \sigma_{K_k,l}(c) \tag{7.21}$$

and by (7.19),

$$X_k^*(\sum_{l=n_k}^{n_{k+1}-1} h_l(c) + \sum_{l=1}^{K_k} \sigma_{K_k,l}(c))X_k \approx_{\epsilon_k/16} U_k'^* \sum_{l=n_k}^{n_{k+1}-1} h_l(c)U_k' + \sum_{l=1}^{K_k} \sigma_{K_k,l}(c). \tag{7.22}$$

For all k, let $u_k = X_k u_k' X_k^*$. Note that $u_k \in \mathrm{Her}\left(\sum_{j=n_k}^{n_{k+1}-1} a_j + \sum_{l=1}^{K_k} b_{K_k,l}\right)$. Then we have, by (7.22), (7.17), and (7.21), that for all $c \in \mathcal{F}_k$,

$$u_k(\sum_{l=n_k}^{n_{k+1}-1} h_l(c) + \sum_{l=1}^{K_k} \sigma_{K_k,l}(c))u_k^* \approx_{\epsilon_k/2} \sum_{l=n_k}^{n_{k+1}-1} \psi_l(c) + \sum_{l=1}^{K_k} \sigma_{K_k,l}(c). \tag{7.23}$$

Let

$$Y := \sum_{j=1}^{\infty} u_j \in M(\mathcal{W}),$$

where the sum converges strictly. Note that $\|Y\| \le 1$.

For all $c \in C$, let

$$\xi(c)_m = \sum_{k=1}^{m} \left(u_k \left(\sum_{l=n_k}^{n_{k+1}-1} h_l(c) + \sum_{l=1}^{K_k} \sigma_{K_k,l}(c) \right) u_k^* - \sum_{l=n_k}^{n_{k+1}-1} \psi_l(c) + \sum_{l=1}^{K_k} \sigma_{K_k,l}(c) \right),$$

$m = 1, 2, \ldots.,$ and

$$\xi(c) = Y(H(c) \oplus \sigma(c))Y^* - \psi(c) \oplus \sigma(c).$$

Since $\mathcal{F}_j \subset \mathcal{F}_{j+1}$ for all j and since $\sum_{k=m}^{\infty} \epsilon_k \to 0$ as $m \to \infty$, it follows from (7.23) that for all j, for all $c \in \mathcal{F}_j$,

$$\lim_{n\to\infty} \|\xi(c)_n - \xi(c)\| = 0. \tag{7.24}$$

Since $\xi(c)_n \in \mathcal{W}$, one concludes that $\xi(c) \in \mathcal{W}$, for all $c \in \mathcal{F}_j$, $j = 1, 2, \ldots$. Thus, for any $c \in \mathcal{F}_j$,

$$\pi(Y)(\pi \circ H(c) + \pi \circ \sigma(c))\pi(Y)^* = \pi \circ \psi(c) + \pi \circ \sigma(c). \tag{7.25}$$

By a similar argument, for any $c \in \mathcal{F}_j$,

$$\pi \circ H(c) + \pi \circ \sigma(c) = \pi(Y)^*(\pi \circ \psi(c) + \pi \circ \sigma(c))\pi(Y). \tag{7.26}$$

Since $\mathcal{F}_j \subset \mathcal{F}_{j+1}$ for all j and $\cup_{j=1}^{\infty} \mathcal{F}_j$ is dense in the unit ball of C, (7.25) and (7.26) implies that

$$\pi(Y)(\pi \circ H(c) + \pi \circ \sigma(c))\pi(Y)^* = \pi \circ \psi(c) + \pi \circ \sigma(c) \text{ for all } c \in C \tag{7.27}$$

and

$$\pi \circ H(c) + \pi \circ \sigma(c) = \pi(Y)^*(\pi \circ \psi(c) + \pi \circ \sigma(c))\pi(Y) \text{ for all } c \in C. \tag{7.28}$$

Set $d := \sum_{n=1}^{\infty}(a_n + b_n)$. Then $Y \in \mathrm{Her}(d)$. Since $\mathrm{Her}(d)^{\perp} \neq \{0\}$, $\pi(Y)^*\pi(Y)$ and $\pi(Y)\pi(Y)^*$ are each not invertible. Since $C(\mathcal{W})$ is simple purely infinite, it has weak cancellation. Hence, by Corollary 1.10 of [37], $\pi(Y) = u(\pi(Y)^*\pi(Y))^{1/2}$, where $u \in U(C(\mathcal{W}))$. By (7.27) and (7.28),

$$\pi(Y)^*\pi(Y)(\pi \circ H(c) + \pi \circ \sigma(c))\pi(Y)^*\pi(Y) = \pi \circ H(c) + \pi \circ \sigma(c) \text{ for all } c \in C. \tag{7.29}$$

Hence,

$$(\pi(Y)^*\pi(Y))^{1/2}(\pi \circ H(c) + \pi \circ \sigma(c))(\pi(Y)^*\pi(Y))^{1/2} = \pi \circ H(c) + \pi \circ \sigma(c) \text{ for all } c \in C. \tag{7.30}$$

Therefore, by (7.30) and (7.27),

$$\begin{aligned} u(\pi \circ H(c) + \pi \circ \sigma(c))u^* &= \pi(Y)(\pi \circ H(c) + \pi \circ \sigma(c))\pi(Y)^* \\ &= \pi \circ \psi(c) + \pi \circ \sigma(c) \text{ for all } c \in C. \end{aligned}$$

Since $K_1(\mathcal{C}(\mathcal{W})) = 0$ and since $\mathcal{C}(\mathcal{W})$ is simple purely infinite, u can be lifted to a unitary in $M(\mathcal{W})$. ■

Corollary 7.13. In Lemma 7.12, if $\pi \circ \psi$ is in fact a diagonal extension, that is, $\psi = \bigoplus_{n=1}^{\infty} \psi_n$ where each ψ_n is a homomorphism, and if $\psi_k^{\ddagger} = 0$ for all k, then

$$\pi \circ \psi \oplus \pi \circ \sigma \sim^u \pi \circ \sigma.$$

Proof. Note that $KL(\psi_n) = 0$ for all n. Since $\psi_n^{\ddagger} = 0$, $\psi_n(u) \in CU(M_{m(n)}(\mathcal{W}))$ (instead of (7.14)) for all $u \in J^{\sim}(K_1(A)) \cap U(M_{m(n)}(\widetilde{\mathrm{Her}(a_n)}))$. Therefore, in the proof of Lemma 7.12, (7.15) becomes

$$cel\left(\sum_{l=n_k}^{n_{k+1}-1} \psi_l(u)\right) \leq 7\pi + 1 \text{ for all } u \in \mathcal{U}_{n_k}. \tag{7.31}$$

Therefore, the proof of Lemma 7.12 works when we take $h_n = 0$ for all n. ■

Lemma 7.14. Let C be as in Lemma 7.12. Fix a sequence of homomorphism s

$$\lambda_n \in \mathrm{Hom}(K_1(C), U(\mathcal{W})/CU(\mathcal{W})).$$

Then for every system $\{b_n\}$ of quasidiagonal units for $\mathcal{W}$, there is a diagonal homomorphism $H := \bigoplus_{n=1}^{\infty} h_n : C \to M(\mathcal{W})$, where $h_n : A \to \mathrm{H}er(b_n)$ is a homomorphism for every n. Moreover, for each n, $h_n^{\ddagger} = \lambda_m$ for some m; and for each k, there are infinitely many n such that $h_n^{\ddagger} = \lambda_k$.

Proof. Let $\{b_n\}$ be a system of quasidiagonal units for $\mathcal{W}$. Write $\mathbb{N} = \cup_{n=1}^{\infty} S_n$, where each S_n is a countably infinite set, and $S_i \cap S_j = \emptyset$ if $i \neq j$. For each $j \in S_n$, choose a homomorphism $h_j : C \to \mathrm{H}er(b_j)$ such that $h_j^{\ddagger} = \lambda_n$ (see Theorems 6.10 and 6.11). Then it is easy to check that $H := \bigoplus_{k\in\mathbb{N}} h_k$ satisfies the requirements of the lemma. ∎

Lemma 7.15. Let C be a separable amenable C^*-algebra, which is $\mathcal{W}$ embeddable and satisfies the UCT. Let $\sigma : C \to M(\mathcal{W})$ be a $\mathcal{T}_d$ extension, and let $\psi : C \to M(\mathcal{W})$ be a diagonal c.p.c. map with the form

$$\psi = \bigoplus_{n=1}^{\infty} \psi_n$$

as in Proposition 4.5, for which $\pi \circ \psi$ is non-unital essential extension. Then

$$\pi \circ \sigma \oplus \pi \circ \psi \sim^{u} \pi \circ \sigma.$$

As a consequence,

$$KK(\pi \circ \psi) = 0.$$

Proof. By Lemma 7.12, we may assume that ψ is a (non-unital) diagonal extension. So suppose that $\psi = \Sigma_{n=1}^{\infty} \psi_n : C \to M(\mathcal{W})$ is a diagonal homomorphism, where each $\psi_n : C \to \mathrm{H}er(a_n)$ is a homomorphism, and where $\{a_n\}$ is a system of quasidiagonal units for $\mathcal{W}$.

Denote $\lambda_{2n} = \psi_n^{\ddagger}$ and $\lambda_{2n-1} = -\lambda_{2n}$, for $n = 1, 2, \ldots$.

Let $H : C \to M(B)$ be as in Lemma 7.14, associated with the present $\{\lambda_n\}$ and $\{a_n\}$. It is easy to see that there is a permutation $\gamma : \mathbb{N} \to \mathbb{N}$ such that

$$h^{\ddagger}_{\gamma(2n-1)} = -h^{\ddagger}_{\gamma(2n)}, \quad \text{for all } n = 1, 2, \ldots..$$

Define $b_k = a_{\gamma(2k-1)} + a_{\gamma(2k)}$, for all $k = 1, 2, \ldots$. Then $\{b_k\}$ is also a system of quasidiagonal units. Let $h_{n,0} : C \to \mathrm{Her}(a_{\gamma(2n-1)} + a_{\gamma(2n)})$ be defined by $h_{n,0}(c) = h_{\gamma(2n-1)}(c) + h_{\gamma(2n)}(c)$ for all $c \in C$ and for all n. Now define $H_0 : C \to M(B)$ by $H_0(c) = \bigoplus_{n=1}^{\infty} h_{n,0}(c)$ for all $c \in C$. Then H_0 is unitarily equivalent to H (see Lemma 4.2). However, $h^{\ddagger}_{n,0} = 0$ for all n. It follows from Corollary 7.13 that

$$\pi \circ H_0 \oplus \pi \circ \sigma \sim^u \pi \circ \sigma.$$

Therefore,

$$\pi \circ H \oplus \pi \circ \sigma \sim^u \pi \circ \sigma. \tag{7.32}$$

Then $\psi \oplus H$ is another diagonal extension and by the same argument as that for H,

$$\pi \circ \psi \oplus \pi \circ H \oplus \pi \circ \sigma \sim^u \pi \circ \sigma. \tag{7.33}$$

Hence, by (7.32),

$$\pi \circ \psi \oplus \pi \circ \sigma \sim^u \pi \circ \sigma. \tag{7.34}$$

Hence, $KK(\pi \circ \psi) = 0$. ■

Lemma 7.16. Let A be a separable amenable C^*-algebra, which is $\mathcal{W}$ embeddable and satisfies the UCT. Suppose that $\tau : A \to \mathcal{C}(\mathcal{W})$ is an essential extension with $KK(\tau) = 0$. Then τ is a quasidiagonal extension.

Proof. Let $\sigma : A \to M(\mathcal{W})$ be a $\mathcal{T}_d$ extension. Note that $KK(\pi \circ \sigma) = 0$. Consider the unitizations $\widetilde{\pi \circ \sigma}, \widetilde{\tau} : \widetilde{A} \to \mathcal{C}(\mathcal{W})$. Then $KK(\widetilde{\pi \circ \sigma}) = KK(\widetilde{\tau})$. By Theorem 2.5 of [41], there exists a sequence $\{u_n\}$ of unitaries in $\mathcal{C}(\mathcal{W})$ such that $\lim_{n\to\infty} u_n^*(\pi \circ \sigma)(a)u_n = \tau(a)$ for all $a \in A$. By Theorem 4.6, since $\mathcal{W}$ has continuous scale and $\pi \circ \sigma$ is non-unital, τ is quasidiagonal. ■

Lemma 7.17. Let A be a separable amenable C^*-algebra, which is $\mathcal{W}$ embeddable and satisfies the UCT. Suppose that $\tau : A \to \mathcal{C}(\mathcal{W})$ is a quasidiagonal essential extension. Then for any $\mathcal{T}_d$ extension $\sigma : A \to M(\mathcal{W})$,

$$\tau \sim^u \pi \circ \sigma. \tag{7.35}$$

Also, $KK(\tau) = 0$.

Proof. By Theorem 3.8,

$$\tau \sim^u \pi \circ \sigma \oplus \tau_0 \tag{7.36}$$

for some essential extension $\tau_0 : A \to \mathcal{C}(\mathcal{W})$. By [13], τ can be lifted to a c.p.c. map $A \to M(\mathcal{W})$. So by Proposition 4.5 and Lemma 7.15, $KK(\tau) = 0$. Since $KK(\tau) = 0$ and $KK(\pi \circ \sigma) = 0$, $KK(\tau_0) = 0$. By Lemma 7.16, τ_0 is quasidiagonal. By [13], τ_0 also can be lifted to a c.p.c. map $A \to M(\mathcal{W})$. So by Proposition 4.5 and Lemma 7.15,

$$\tau \sim^u \pi \circ \sigma \oplus \tau_0 \sim^u \pi \circ \sigma. \tag{7.37}$$

■

We have the following K theory characterization of quasidiagonality (see the paragraph before Theorem 4.7 for some brief history and references):

Proposition 7.18. Let A be a separable amenable C^*-algebra, which is $\mathcal{W}$ embeddable and satisfies the UCT, and let $\tau : A \to \mathcal{C}(\mathcal{W})$ be an essential extension. Then the following statements are equivalent:

1. $KK(\tau) = 0$.
2. τ is quasidiagonal.
3. τ is unitarily equivalent to an essential trivial diagonal extension.
4. τ is unitarily equivalent to every essential trivial diagonal extension.
5. τ is in the class of zero of $\mathbf{Ext}^u(A, \mathcal{W})$.

Proof. Let us recall that A is non-unital as it is a C^*-subalgebra of the stably projectionless C^*-algebra $\mathcal{W}$. It follows from Theorem 3.7 that $\mathbf{Ext}^u(A, \mathcal{W})$ is a group.

That (1) $\Leftrightarrow$ (2) follows from Lemmas 7.16 and 7.17.

That (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4) follows immediately from Lemma 7.17, which says that every essential quasidiagonal extension (including every essential trivial diagonal extension) is unitarily equivalent to every $\mathcal{T}_d$ extension.

(4) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3) are immediate.

That (4) $\Rightarrow$ (5) follows from the facts that $\mathbf{Ext}^u(A, \mathcal{W})$ is a group and if ρ is a trivial diagonal extension then so is $\rho \oplus \rho$.

(5) $\Rightarrow$ (2): From (4) $\Rightarrow$ (5), we know that the neutral element of $\mathbf{Ext}^u(A, \mathcal{W})$ is the class of an essential trivial diagonal extension. But then, any essential extension, which is unitarily equivalent to a trivial diagonal extension is a trivial diagonal extension. ■

8 Classification of Extensions by $\mathcal{W}$

Lemma 8.1. Let B be a non-unital separable simple C^*-algebra with a unique tracial state t_B such that B stably has almost stable rank one. Suppose that $\mathrm{Cu}(B) = V(B) \sqcup (\mathrm{L}Aff_+(T(B)) \setminus \{0\}) \cong V(B) \sqcup (0, \infty)$.

Let A be a separable exact C^*-algebra with a faithful tracial state, which satisfies the UCT.

Then for any $t \in T_f(A)$ and $r \in (0, 1]$, there is an embedding $\phi_A : A \to M(B)$ such that $t_B \circ \phi_A = rt$ and $\pi \circ \phi_A$ is injective.

Proof. Fix $t \in T_f(A)$ and $r \in (0, 1]$. By Theorem A of [66], let D be a unital simple AF-algebra with a unique tracial state τ_D and let $\psi : A \to D$ be a *-embedding such that $t = \tau_D \circ \psi$.

By Lemma 7.2,

$$K_0(M(B)) \cong \mathbb{R} \ \text{ and } \ V(M(B)) \cong V(B) \sqcup (0, \infty).$$

Let $\lambda : K_0(D) \to K_0(M(B))$ be the homomorphism defined by

$$\lambda([p]) = rt_D([p]) \ \text{ for all } \ p \in Proj(D \otimes \mathcal{K}).$$

Note that this gives an ordered semigroup homomorphism $\lambda_V : V(D) \to (0, \infty) \cup \{0\} \subset V(M(B))$. Note also that $(0, \infty) \cap V(B) = \emptyset$. By Lemma 4.2 of [58], there is a homomorphism $\phi_0 : D \to M(B) \otimes \mathcal{K}$ such that $V(\phi_0) = \lambda_V$. Without loss of generality, one may assume that $\phi_0(D) \subset M_m(M(B))$ for some integer $m \geq 1$. One also has that $[\phi_0(1_D)] = r \leq 1$. It follows, from Lemma 7.2, that there is a unitary $U \in M_m(M(B))$ such that

$$U^*\phi_0(1_D)U \leq 1_{M(B)}.$$

Define $\phi(d) = U^*\phi_0(d)U$ for all $d \in D$. Since D is simple, ϕ is an embedding. Then set

$$\phi_A := \phi \circ \psi.$$

One checks that the embedding ϕ_A meets the requirements. ∎

Remark 8.2. Recall that $\mathrm{Hom}(K_0(A), \mathbb{R})_{T_f}$ is defined in 2.6. Several comments about it are in order. Firstly, under current assumptions, $K_0(A)_+$ might be zero; and also, one may not have traditional order preserving homomorphism s in $\mathrm{Hom}(K_0(A), \mathbb{R})$. Secondly, there could still be a pairing $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$ even in the case that $K_0(A)_+ = \{0\}$. Therefore, an element in $\mathrm{Hom}(K_0(A), \mathbb{R})$ need not be induced by a homomorphism from A (to $M(B)$). Thirdly, there is a possibility that, given two tracial states $t_1, t_2 \in T_f(A)$, one might have $r_A(t_1) = r_A(t_2)$. In other words, $r_A(t_1)$ and $r_A(t_2)$ may give the same element in $\mathrm{Hom}(K_0(A), \mathbb{R})_{T_f}$ and, of course, they will not be distinguished.

We are ready to present the following classification of essential extensions by $\mathcal{W}$:

Theorem 8.3. Let A be a separable amenable C^*-algebra, which is $\mathcal{W}$ embeddable and satisfies the UCT.

(1) If $\tau_1, \tau_2 : A \to \mathcal{C}(\mathcal{W})$ are two essential extensions, then $\tau_1 \sim^u \tau_2$ if and only if $KK(\tau_1) = KK(\tau_2)$.

(2) The map

$$\Lambda : \mathbf{Ext}^u(A, \mathcal{W}) \to KK(A, \mathcal{C}(\mathcal{W})) \cong \mathrm{Hom}(K_0(A), \mathbb{R}) \tag{8.1}$$

defined by $\Lambda([\tau]) = KK(\tau)$ is a group isomorphism.

(3) An essential extension $\tau : A \to \mathcal{C}(\mathcal{W})$ is trivial and diagonal if and only if $KK(\tau) = 0$, and all essential trivial and diagonal extensions of A by $\mathcal{W}$ are unitarily equivalent. In fact, the essential trivial diagonal extensions of A by $\mathcal{W}$ induce the neutral element of $\mathbf{Ext}^u(A, \mathcal{W})$.

(4) An essential extension $\tau : A \to \mathcal{C}(\mathcal{W})$ is trivial if and only if there exist $t \in T_f(A)$ and $r \in (0,1]$ such that

$$\tau_{*0}(x) = r \cdot r_A(t)(x) \text{ for all } x \in K_0(A).$$

(5) Let $\mathcal{T}$ be the set of unitary equivalence classes of essential trivial extensions of A by $\mathcal{W}$. Then

$$\Lambda(\mathcal{T}) = \{r \cdot h : r \in (0,1] \text{ and } h \in \mathrm{Hom}(K_0(A), \mathbb{R})_{T_f(A)}\} \text{ (see Definition 2.6).}$$

(6) All quasidiagonal essential extensions of A by $\mathcal{W}$ are trivial and are unitarily equivalent.

(7) In the case where $\ker\rho_{f,A} = K_0(A)$, all essential trivial extensions of A by $\mathcal{W}$ are unitarily equivalent. Moreover, an essential extension $\tau : A \to \mathcal{C}(\mathcal{W})$ is trivial if and only if $KK(\tau) = \{0\}$.

(8) In the case where $\ker\rho_{f,A} \neq K_0(A)$, there are essential trivial extensions of A by $\mathcal{W}$ which are not quasidiagonal, and not all essential trivial extensions of A by $\mathcal{W}$ are unitarily equivalent (see (5) above).

Proof. Statements (3) and (6) follow from Proposition 7.18.

(2): That $KK(A, \mathcal{C}(\mathcal{W})) = \mathrm{Hom}(K_0(A), \mathbb{R})$ follows from the UCT, since $K_*(\mathcal{C}(\mathcal{W}))$ is divisible and $K_1(\mathcal{W}) = 0$. Note that $\mathrm{Hom}(K_0(A), \mathbb{R})$ is an abelian group. Recall that by Theorem 3.7, $\mathbf{Ext}^u(A, \mathcal{W})$ is also an abelian group. It is obvious that Λ is a semigroup homomorphism. By 7.18, Λ sends zero to zero, and therefore, Λ is a group homomorphism.

We next show that Λ is surjective. Let $x \in KK(A, \mathcal{C}(\mathcal{W}))$ be given. Note that $K_0(\widetilde{A}) = K_0(A) \oplus \mathbb{Z}$. Define $\eta \in \mathrm{Hom}(K_0(\widetilde{A}), K_0(\mathcal{C}(\mathcal{W})))$ by $\eta|_{K_0(A)} = x$ and $\eta([1_{\tilde{A}}]) = [1_{\mathcal{C}(\mathcal{W})}]$. Then η gives an element of $KK(\widetilde{A}, \mathcal{C}(\mathcal{W}))$. It follows from Corollary 8.5 of [41] that there is a homomorphism $\tau_1 : \widetilde{A} \to \mathcal{C}(\mathcal{W})$ such that $KK(\tau_1) = \eta$. Define $\tau = \tau_1|_A$. Then one computes that $KK(\tau) = x$. Since x is arbitrary in $KK(A, \mathcal{C}(\mathcal{W}))$, the map Λ is surjective.

It remains to prove that Λ is injective. But by Proposition 7.18, if $[\psi] \in \mathbf{Ext}^u(A, \mathcal{W})$ is such that $\Lambda([\psi]) = 0$, that is, $KK(\psi) = 0$, then $[\psi] = 0$ in $\mathbf{Ext}^u(A, \mathcal{W})$. Hence, Λ is injective. This completes the proof of (2).

(1) follows from (2).

(4): Say that $\tau : A \to \mathcal{C}(\mathcal{W})$ is an essential trivial extension. Then there is a monomorphism $H : A \to M(\mathcal{W})$ such that $\pi \circ H = \tau$. Let $t_1(a) = t_W \circ H(a)$ for all $a \in A$.

Then t_1 is a faithful trace on A with $\|t_1\| \le 1$. Let $r = \|t_1\|$. Then $t(a) = t_1(a)/r$ is a faithful tracial state on A. Hence, $H_{*0}(x) = r\rho_A(t)(x)$ for all $x \in K_0(A)$. It follows that $\tau_{*0}(x) = r \cdot \rho_A(t)(x)$ for all $x \in K_0(A)$.

Conversely, suppose now that $t \in T_f(A)$, $r \in (0,1]$ and $\tau : A \to \mathcal{C}(\mathcal{W})$ are such that $\tau_{*0}(x) = r \cdot \rho_A(t)(x)$ for all $x \in K_0(A)$. By Lemma 8.1, there is a monomorphism $\psi_A : A \to M(\mathcal{W})$ such that $t_W \circ \psi_A(a) = r \cdot t(a)$ for all $a \in A$. Then $\pi \circ \psi_A : A \to \mathcal{C}(\mathcal{W})$ is an essential trivial extension such that

$$(\pi \circ \psi_A)_{*0} = \tau_{*0}. \tag{8.2}$$

Hence, $KK(\pi \circ \psi_A) = KK(\tau)$. So, by (1), $\tau \sim^u \pi \circ \psi_A$. It follows that τ is trivial. This completes the proof of (4).

(5) follows from (4).

(7): If $K_0(A) = \ker\rho_{f,A}$ and $H : A \to M(\mathcal{W})$ is a monomorphism, then $H_{*0} = 0$. As mentioned before, since $K_0(\mathcal{C}(\mathcal{W})) = \mathbb{R}$ is divisible and $K_1(\mathcal{C}(\mathcal{W})) = \{0\}$, $KK(\pi \circ H) = 0$. Thus, (7) follows from (3).

(8): Suppose that $\ker\rho_{f,A} \neq K_0(A)$. Then there is a $t \in T_f(A)$ such that $\rho_A(t) \neq 0$. Then $\Lambda(\mathcal{T}) \neq \{0\}$. So by (5), there is a trivial essential extension τ such that $KK(\tau) \neq 0$. By (3), τ is not unitarily equivalent to a diagonal trivial extension, and by (6), τ is not even quasidiagonal. ∎

For the second question of the introduction, we offer the following statement:

Corollary 8.4. There is, up to unitary equivalence, only one essential extension of the form

$$0 \to \mathcal{W} \to E \to \mathcal{W} \to 0. \tag{8.3}$$

Moreover, this extension splits.

Proof. By Theorem 9.9,

$$\mathbf{Ext}^u(\mathcal{W}, \mathcal{W}) = \mathrm{Hom}(K_0(\mathcal{W}), \mathbb{R}) = \{0\}. \tag{8.4}$$

∎

As one expected, the C^*- algebras that we were originally interested in do satisfy the hypotheses of Theorem 9.9 and even (7) of Theorem 9.9.

Proposition 8.5. Let X be a connected and locally connected compact metric space, and let $x_0 \in X$ be a point. Then $C := C_0(X \setminus \{x_0\})$ is $\mathcal{W}$ embeddable. Moreover, $K_0(C) = \ker\rho_C$.

Proof. We firstly show that there is an embedding $\iota_C : C_0(X \setminus \{x_0\}) \to C_0((0,1])$. By the Hahn–Mazurkiewicz theorem, there exists a continuous surjection $s_0 : [1/2, 1] \twoheadrightarrow X$. Let $y_0 = s_0(1/2) \in X$. By the assumptions on X, there is a continuous path $s_1 : [0, 1/2] \to X$ such that $s_1(0) = x_0$ and $s_1(1/2) = y_0$. Define $s : [0,1] \to X$ by $s|_{[0,1/2]} = s_1$ and $s|_{[1/2,1]} = s_0$. Then $s : [0,1] \to X$ is a continuous surjection, which induces an embedding $\iota_C : C \to C_0((0,1])$. Since $\mathcal{W}$ is projectionless, one easily embeds $C_0((0,1])$ into $\mathcal{W}$. This shows that C is $\mathcal{W}$-embeddable.

Let $C_1 = C(X) = \widetilde{C}$. Since X is connected, ρ_{C_1} is the rank function and $\rho_{C_1}(K_0(C_1)) = \mathbb{Z}$. The short exact sequence

$$0 \to K_0(C) \to K_0(C_1) \to \mathbb{Z} \to 0$$

also shows that $K_0(C) = \ker\rho_{C_1} = \ker\rho_C$. ∎

The following is a corollary of Theorem 9.9 (and Corollary 8.5).

Theorem 8.6. Let X be a connected and locally connected compact metric space, let $x_0 \in X$ and let $C := C_0(X \setminus \{x_0\})$.

(1) All trivial essential extensions of C by $\mathcal{W}$ are unitarily equivalent and hence, are unitarily equivalent to a diagonal extension $\pi \circ \sigma$ in $\mathcal{T}_d$.

(2) $[\pi \circ \sigma]$ is the class of zero in $\mathbf{Ext}^u(C, \mathcal{W})$.

(3) There is a group isomorphism

$$\mathbf{Ext}^u(C, \mathcal{W}) \cong KK(C, \mathcal{C}(\mathcal{W})) = \mathrm{Hom}(K_0(C), \mathbb{R}).$$

There are many non-commutative C^*-algebras, which are $\mathcal{W}$ embeddable including $\mathcal{W}$ itself. In fact, we have the following:

Proposition 8.7. Let A be a stably projectionless algebraically simple separable C^*-algebra with finite nuclear dimension, which satisfies the UCT. Suppose that $\ker\rho_A = K_0(A)$. Then A is $\mathcal{W}$ embeddable.

Proof. Let $C = A \otimes Q$, where Q is the UHF-algebra with $(K_0(Q), K_0(Q)_+, [1_Q]) = (\mathbb{Q}, \mathbb{Q}_+, 1)$. It follows that there exists an element $c \in C_+^1$ such that $\overline{cCc}$ has continuous scale (see Remark 5.2 of [18]). Let $e_A \in A_+^1$ be a strictly positive element of A. Since A is algebraically simple, $\mathrm{Ped}(A) = A$. It follows that $e_C := e_A \otimes 1_Q$ is in $\mathrm{Ped}(C)$. Note also that $c \in \mathrm{Ped}(C)$. It follows that there is an integer $n \geq 1$ such that $\langle e_C \rangle \leq n\langle c \rangle$. Without loss of generality, we may assume that $e_C \in M_n(\overline{cCc})$. Let $D = M_n(\overline{cCc})$. Then A is embeddable into D. It suffices to show that D is $\mathcal{W}$ embeddable.

Note that D is a stably projectionless simple C^*-algebra with continuous scale and which satisfies the UCT. Moreover, $\ker\rho_D = K_0(D)$. Furthermore, D has finite nuclear dimension. Since D is stably projectionless, $T(D) \neq \emptyset$. By Theorem 15.5 of [22] (see the last line of the proof too), $D \in \mathcal{D}_0$. We then apply Theorem 12.8 of [22] as $\mathcal{W}$ has the form B_T (with $K_i(\mathcal{W}) = 0$, $i = 0, 1$, and $\mathcal{W}$ having unique tracial state). We choose $\kappa = 0$, and $\kappa_T : T(D) \to T(\mathcal{W})$ by mapping all points to one point, and κ_{cu} compatible with (κ, κ_T). Thus, Theorem 12.8 of [22] provides an embedding from D into $\mathcal{W}$. ■

Remark 8.8. Note that A, as in 8.7 but with finite nuclear dimension replaced by nuclearity, can also be embedded into $A \otimes \mathcal{Z}$ (by the map $a \to a \otimes 1_{\mathcal{Z}}$). Also, $A \otimes \mathcal{Z}$ is a separable simple $\mathcal{Z}$-stable nuclear C^*-algebra. Therefore, by a recent result [12], $A \otimes \mathcal{Z}$ has finite nuclear dimension. Also, our assumptions on A imply that $A \otimes \mathcal{Z}$ is stably projectionless and $ker\rho_{A\otimes\mathcal{Z}} = K_0(A \otimes \mathcal{Z})$. Therefore, by Proposition 8.7, $A \otimes \mathcal{Z}$ can be embedded into $\mathcal{W}$. Consequently, A is $\mathcal{W}$ embeddable. So the assumption of finite nuclear dimension in Proposition 8.7 can be replaced by nuclearity.

Recall that there is a separable simple stably projectionless C^*-algebra $\mathcal{Z}_0$ with a unique tracial state τ_z, with finite nuclear dimension and that satisfies the UCT such that $K_0(\mathcal{Z}_0) = \mathbb{Z} = \ker\rho_{\mathcal{Z}_0}$ and $K_1(\mathcal{Z}_0) = 0$. By [22], there is only one such simple C^*-algebra up to isomorphism. Note that for any separable C*-algebra A, $T(A) = T(A \otimes \mathcal{Z}_0)$. Moreover, as abelian groups,

$$K_i(A \otimes \mathcal{Z}_0) \cong K_i(A), \;\; i = 0, 1. \tag{8.5}$$

Proposition 8.9. Let A be a separable exact C^*-algebra, which satisfies the UCT and has a faithful amenable tracial state. Then $C := A \otimes \mathcal{Z}_0$ is $\mathcal{W}$ embeddable and $\ker\rho_{f,C} = K_0(C)$.

Proof. It follows from Theorem A of [66] that there is a unital simple AF-algebra B with a unique tracial state τ_B and a monomorphism $\phi : A \to B$. There is also an embedding $\psi_{z,w} : \mathcal{Z}_0 \to \mathcal{W}$. Thus, we obtain an embedding $\phi_A := \phi \otimes \psi_{z,w} : A \otimes \mathcal{Z}_0 \to B \otimes \mathcal{W}$. Note that $K_*(B\otimes\mathcal{W})=\{0\}$. Since $B\otimes\mathcal{W}$ has only one tracial state (namely $\tau_B\otimes\tau_W$), by Theorem 7.5 of [19], $B\otimes\mathcal{W}\cong\mathcal{W}$. Thus, $A\otimes\mathcal{Z}_0$ is $\mathcal{W}$ embeddable.

To see the last part of the statement, note that every $y\in K_0(A\otimes\mathcal{Z}_0)$ may be written as $x\otimes x_0$, where $x_0\in K_0(\mathcal{Z}_0)=\mathbb{Z}$ is a generator. Note that every faithful tracial state of $A\otimes\mathcal{Z}_0$ has the form $\tau\otimes\tau_z$, where $\tau\in T_f(A)$. But $\tau(x\otimes x_0)=\tau(x)\tau_z(x_0)=0$. ■

Theorem 8.10. Let B be a separable amenable C^*-algebra, which has a faithful tracial state and satisfies the UCT, and let $A=B\otimes\mathcal{Z}_0$.

(1) If $\tau_1,\tau_2:A\to\mathcal{C}(\mathcal{W})$ are two essential extensions, then $\tau_1\sim^u\tau_2$ if and only if $KK(\tau_1)=KK(\tau_2)$.

(2) The map

$$\Lambda:\mathbf{Ext}^u(A,\mathcal{W})\to KK(A,\mathcal{C}(\mathcal{W}))\cong \mathrm{Hom}(K_0(A),\mathbb{R}) \tag{8.6}$$

defined by $\Lambda([\tau])=KK(\tau)$ is a group isomorphism.

(3) An essential extension τ, of A by $\mathcal{W}$, is trivial if and only if $KK(\tau)=0$, and all essential trivial extensions of A by $\mathcal{W}$ are unitarily equivalent.

For the rest of this section, we consider essential extensions of the form

$$0\to\mathcal{W}\to E\to C(X)\to 0,$$

where X is a connected and locally connected compact metric space.

Lemma 8.11. Let $p\in\mathcal{C}(\mathcal{W})$ be a nonzero projection such that $[p]_{K_0(\mathcal{C}(\mathcal{W}))}\in(0,1)$. Then p can be lifted to a projection in $M(\mathcal{W})$.

Moreover, if $p\neq 1_{\mathcal{C}(\mathcal{W})}$ and $[p]_{K_0(\mathcal{C}(\mathcal{W}))}\notin(0,1)$, then p cannot be lifted to a nonzero projection in $M(\mathcal{W})$.

Proof. Say that $[p]_{K_0(\mathcal{C}(\mathcal{W}))}=r\in(0,1)$. By Corollary 4.6 of [47] (see also Section 5 of [30]), let $Q\in M(\mathcal{W})\setminus\mathcal{W}$ be a projection such that $\tau_W(Q)=r$. Therefore, by our computation of $K_0(\mathcal{C}(\mathcal{W}))$, and since $\mathcal{C}(\mathcal{W})$ is simple purely infinite,

$$\pi(Q)\approx p \ \text{ in } \ \mathcal{C}(\mathcal{W}).$$

Since $\mathcal{C}(\mathcal{W})$ is simple purely infinite and since $\pi(Q) \neq 1 \neq p$, there is a unitary $u \in \mathcal{C}(\mathcal{W})$ such that

$$u\pi(Q)u^* = p.$$

Since $\mathcal{C}(\mathcal{W})$ is simple purely infinite and since $K_1(\mathcal{C}(\mathcal{W})) = 0$, u lifts to a unitary $U \in M(\mathcal{W})$. It follows that $\pi(UQU^*) = p$.

The last part follows from the fact that if $P \in M(\mathcal{W})$ is a non-zero projection, then $\tau_W(P) \in (0, 1]$. ■

Lemma 8.12. Let X be a connected and locally connected compact metric space and let $x_0 \in X$. Suppose that $\phi : C(X) \to \mathcal{C}(\mathcal{W})$ is an essential extension. Then there exists a proper subprojection $p \leq \phi(1)$ such that

$$p\phi(f) = \phi(f)p = \phi(f) \text{ for all } f \in C_0(X \setminus \{x_0\}).$$

Moreover, for all $s \in (0, 1)$, we may choose p such that there is a projection $P \in M(\mathcal{W})$ for which $\pi(P) = p$ and $\tau_W(P) = s$.

Proof. Let e_C be a strictly positive element of $C_0(X \setminus \{x_0\})$ for which $\|e_C\| = 1$. Let $B = Her(\phi(e_C)) \subset \mathcal{C}(\mathcal{W})$. Note that $sp(e_C) = [0, 1]$. Write $e = \phi(1)$. If $ey = 0$ for all $y \in B^\perp$, then $(1 - e)y = y = y(1 - e)$ for all $y \in B^\perp$. This implies that $B^\perp = (1 - e)\mathcal{C}(\mathcal{W})(1 - e)$. Then by Theorem 15 of [55], $B = (B^\perp)^\perp = e\mathcal{C}(\mathcal{W})e$. So B is unital. This contradicts that $sp(e_C) = [0, 1]$. Therefore, there is a $y \in (B^\perp)_+$ such that $eye \neq 0$. Since $e\mathcal{C}(\mathcal{W})e$ has real rank zero, there is a projection $p_1 \in e\mathcal{C}(\mathcal{W})e$ such that $p_1 \neq e$ and $p_1 b = bp_1 = b$ for all $b \in B$.

Since $\mathcal{C}(\mathcal{W})$ is simple purely infinite, we can find a projection $q_1 \in (e - p_1)\mathcal{C}(\mathcal{W})(e - p_1)$ with $q_1 \neq e - p_1$ such that $[p_1 + q_1] = s \in (0, 1)$. If we define $p := p_1 + q_1$ then $pb = b$ for all $b \in B$.

Also, by Lemma 8.11, p lifts to a projection $P \in M(W)$. Necessarily, $\tau_W(P) = s$. ■

Theorem 8.13. Let X be a connected and locally connected compact metric space, and let $\phi, \psi : C(X) \to \mathcal{C}(\mathcal{W})$ be essential extensions.

(1) Then $KK(\phi) = KK(\psi)$ if and only if

$$\phi \sim \psi.$$

(2) If both ϕ and ψ are unital or both are non-unital, then $KK(\phi) = KK(\psi)$ if and only if

$$\phi \sim^u \psi.$$

(3) The map

$$\Lambda : \mathbf{Ext}(C(X), \mathcal{W}) \to KK(C(X), \mathcal{C}(\mathcal{W})) = \mathrm{Hom}(K_0(C(X)), \mathbb{R})$$

defined by $\Lambda([\tau]) = KK(\tau)$ is a group isomorphism.

(4) The zero element of $\mathbf{Ext}(C(X), \mathcal{W})$ (or $KK(C(X), \mathcal{C}(\mathcal{W}))$) is not the class of a trivial extension.

(5) Let $\tau : C(X) \to \mathcal{C}(\mathcal{W})$ be an essential extension. Then τ is trivial if and only if

$$KK(\tau|_{C_0(X\setminus\{x_0\})}) = 0 \text{ and either } [\tau(1_{C(X)})] \in (0, 1) \text{ or } \tau(1_{C(X)}) = 1_{\mathcal{C}(\mathcal{W})}.$$

Remark 8.14. Suppose that $p \in \mathcal{C}(\mathcal{W})$ is a projection such that $[p] = [1_{\mathcal{C}(\mathcal{W})}]$ but $p \neq 1_{\mathcal{C}(\mathcal{W})}$. Let $\phi : C(X) \to \mathcal{C}(\mathcal{W})$ be an essential extension with $KK(\phi|_{C_0(X\setminus\{x_0\})}) = 0$ and $\phi(1) = p$. Then $\phi \sim \phi_0$ for some trivial essential extension $\phi_0 : C(X) \to \mathcal{C}(\mathcal{W})$ with $\phi_0(1) = 1_{\mathcal{C}(\mathcal{W})}$. But ϕ is not itself a trivial extension, as p cannot be lifted to a projection in $M(\mathcal{W})$ (see 8.11).

Proof. (1): Suppose that $\phi \sim \psi$. Then there is a $w \in \mathcal{C}(\mathcal{W})$ such that $w^*\phi(c)w = \psi(c)$ for all $c \in C(X)$ with $w^*w = \psi(1)$ and $ww^* = \phi(1)$. Since $M_2(\mathcal{C}(\mathcal{W}))$ is simple and purely infinite, there exists a unitary $W \in M_2(\mathcal{C}(\mathcal{W}))$ such that $W\psi(1) = w$. Then $W^*\phi(c)W = \psi(c)$ for all $c \in C(X)$. It follows that $KK(\phi) = KK(\psi)$.

Conversely, suppose that $KK(\phi) = KK(\psi)$. Fix $x_0 \in X$ and $s \in (0, 1)$. Let $p := \phi(1)$ and $q := \psi(1)$. By Lemma 8.12, choose a proper subprojection $p_0 \leq p$ such that $p_0\phi(f) = \phi(f)p_0 = \phi(f)$ for all $f \in C_0(X \setminus \{x_0\})$, and choose a projection $P_0 \in M(\mathcal{W})$ such that $\pi(P_0) = p_0$ and $\tau_W(P_0) = s$. The same argument shows that there is a proper subprojection $q_0 \leq q$ such that $q_0\psi(f) = \psi(f)q_0 = \psi(f)$ for all $f \in C_0(X \setminus \{x_0\})$, and there exists a projection $Q_0 \in M(\mathcal{W})$ such that $\pi(Q_0) = q_0$ and $\tau_W(Q_0) = s$.

Since $\mathcal{C}(\mathcal{W})$ is purely infinite simple, $p_0 \approx q_0$ and $p - p_0 \approx q - q_0$. Without loss of generality, we may assume that $p_0 = q_0$ and $\phi(1) = p = q = \psi(1)$.

Since $D := P_0\mathcal{W}P_0 \cong \mathcal{W}$, $M(D) = P_0M(\mathcal{W})P_0$ and $\mathcal{C}(D) = p_0\mathcal{C}(\mathcal{W})p_0$, one may view $\phi|_{C_0(X\setminus\{x_0\})}, \psi|_{C_0(X\setminus\{x_0\})}$ as maps from $C_0(X \setminus \{x_0\})$ to $\mathcal{C}(D)$. By applying Proposition 8.5 and

Theorem 9.9, one obtains a unitary $u \in \mathcal{C}(D) = p_0\mathcal{C}(\mathcal{W})p_0$ such that

$$u^*\phi(c)u = \psi(c) \text{ for all } c \in C_0(X \setminus \{x_0\}).$$

If we define $v := u + (p - p_0)$, then v is a partial isometry in $\mathcal{C}(B)$ such that $vv^* = \phi(1)$, $v^*v = \psi(1)$ and

$$v^*\phi(c)v = \psi(c) \text{ for all } c \in C(X).$$

This completes the proof of (1).

(2): If $\phi \sim^u \psi$ then it is immediate that $KK(\phi) = KK(\psi)$. Hence, we only need to prove the converse direction.

So suppose that $KK(\phi) = KK(\psi)$. Suppose that both $\phi(1_{C(X)})$ and $\psi(1_{C(X)})$ are equal to $1_{\mathcal{C}(\mathcal{W})}$. Then by (1), since $KK(\phi) = KK(\psi)$, $\phi \sim \psi$. In other words, there is a $w \in \mathcal{C}(\mathcal{W})$ such that $w^*\phi(x)w = \psi(x)$ for all $x \in C(X)$. Moreover, by Definition 2.1,

$$1_{\mathcal{C}(\mathcal{W})} = \phi(1_{C(X)}) = ww^* \text{ and } 1_{\mathcal{C}(\mathcal{W})} = \psi(1_{C(X)}) = w^*w.$$

Hence, w is a unitary. Since $K_1(\mathcal{C}(\mathcal{W})) = 0$ and $\mathcal{C}(\mathcal{W})$ is simple and purely infinite, every unitary in $\mathcal{C}(\mathcal{W})$ can be lifted to a unitary in $M(\mathcal{W})$. Hence, $\phi \sim^u \psi$.

Now suppose that both ϕ and ψ are non-unital. Then by (1), since $KK(\phi) = KK(\psi)$, $\phi \sim \psi$. So we have a partial isometry $v \in \mathcal{C}(\mathcal{W})$ such that $vv^* = \phi(1)$, $v^*v = \psi(1)$ and $v^*\phi(c)v = \psi(c)$ for all $c \in C(X)$. Since $\phi(1), \psi(1)$ are proper subprojections of $1_{\mathcal{C}(\mathcal{W})}$ and since $\mathcal{C}(\mathcal{W})$ is simple and purely infinite, we can find a unitary $u \in \mathcal{C}(\mathcal{W})$ such that $\phi(1)u = v$. Hence, $u^*\phi(c)u = \psi(c)$ for all $c \in C(X)$. Since $\mathcal{C}(\mathcal{W})$ is simple and purely infinite and $K_1(\mathcal{C}(\mathcal{W})) = 0$, u can be lifted to a unitary in $M(\mathcal{W})$. So $\phi \sim^u \psi$.
This completes the proof of (2).

(3) The injectivity of the group homomorphism Λ follows from (1). Hence, it remains to prove that Λ is surjective.

Let $\alpha \in KK(C(X), \mathcal{C}(\mathcal{W}))$ be given. Fix a point $x_0 \in X$. Let $\iota : C_0(X \setminus \{x_0\}) \to C(X)$ be the inclusion map and $q : C(X) \to \mathbb{C}$ be the corresponding quotient map (point evaluation at x_0). So we have the following split exact sequence:

$$0 \to C_0(X \setminus \{x_0\}) \xrightarrow{\iota} C(X) \xrightarrow{q} \mathbb{C} \to 0, \tag{8.7}$$

which induces the split exact sequence

$$0 \to KK(\mathbb{C}, \mathcal{C}(\mathcal{W})) \xrightarrow{[q]} KK(C(X), \mathcal{C}(\mathcal{W})) \xrightarrow{[\iota]} KK(C_0(X \setminus \{x_0\}), \mathcal{C}(\mathcal{W})) \to 0. \tag{8.8}$$

Consider the Kasparov product

$$\beta := [\iota] \times \alpha \in KK(C_0(X \setminus \{x_0\}), \mathcal{C}(\mathcal{W})).$$

By Proposition 8.5 and Theorem 9.9, let $\phi : C_0(X \setminus \{x_0\}) \to \mathcal{C}(\mathcal{W})$ be an essential extension such that $KK(\phi) = \beta$.

Note that ϕ can be extended to a monomorphism $C(X) \to \mathcal{C}(\mathcal{W})$, which brings $1_{C(X)}$ to $1_{\mathcal{C}(\mathcal{W})}$. So by Lemma 8.12, let $p \leq 1_{\mathcal{C}(\mathcal{W})}$ be a proper subprojection such that $p\phi(f) = \phi(f)$ for all $f \in C_0(X \setminus \{x_0\})$. Since $\mathcal{C}(\mathcal{W})$ is simple and purely infinite, one may choose a proper subprojection $q \leq 1_{\mathcal{C}(\mathcal{W})} - p$ such that

$$[p + q] = \alpha([1_{C(X)}]).$$

Let $\phi_1 : C(X) \to \mathcal{C}(\mathcal{W})$ be the non-unital essential extension given by

$$\phi_1|_{C(X \setminus \{x_0\})} = \phi \text{ and } \phi_1(1_{C(X)}) = p + q.$$

Hence, viewing $[\iota] \in KK(C_0(X \setminus \{x_0\}), C(X))$, $[\iota] \times [\phi_1] = [\phi_1 \circ \iota] = [\phi] = [\iota] \times \alpha$.

Towards seeing that $KK(\phi_1) = \alpha$, consider the inclusion map $j : \mathbb{C} \hookrightarrow C(X)$. The map j splits the exact sequence (8.7), that is, $q \circ j = \mathrm{id}_{\mathbb{C}}$. We have an induced morphism $[j] : KK(C(X), \mathcal{C}(\mathcal{W})) \to KK(\mathbb{C}, \mathcal{C}(\mathcal{W}))$ for which $[j] \circ [q] = \mathrm{id}_{KK(\mathbb{C}, \mathcal{C}(\mathcal{W}))}$. Alternatively, viewing $[j] \in KK(\mathbb{C}, C(X))$ and $[q] \in KK(C(X), \mathbb{C})$, $[j] \times [q] = id_{KK(\mathbb{C},\mathbb{C})}$. Moreover, by this and (8.8), we have a group isomorphism

$$K(C(X), \mathcal{C}(\mathcal{W})) \cong KK(\mathbb{C}, \mathcal{C}(\mathcal{W})) \oplus KK(C_0(X \setminus \{x_0\}), \mathcal{C}(\mathcal{W})) : y \mapsto ([j] \times y, [\iota] \times y). \tag{8.9}$$

Let us note that we have already shown that $[\iota] \times \alpha = [\iota] \times [\phi_1]$. Also, in $KK(\mathbb{C}, \mathcal{C}(\mathcal{W})) = K_0(\mathcal{C}(\mathcal{W}))$, $[j] \times \alpha = \alpha([1_{C(X)}]) = [\phi_1(1_{C(X)})] = [j] \times [\phi_1]$. Hence, by (8.9),

$$\alpha = KK(\phi_1).$$

Therefore, $\Lambda([\phi_1]) = \alpha$ as required. This completes the proof of the surjectivity of Λ and hence, the proof of (3).

(4): Say that $KK(\phi) = 0$. Then $[\phi(1)] = 0 \in \mathbb{R} = K_0(\mathcal{C}(\mathcal{W}))$. By Lemma 8.11, there is no non-zero projection $P \in M(\mathcal{W})$ such that $\pi(P) = \phi(1)$. It follows that ϕ is not liftable.

(5): Say that $\tau : C(X) \to \mathcal{C}(\mathcal{W})$ is an essential trivial extension. Then by Theorem 8.6, $KK(\tau|_{C_0(X\setminus\{x_0\})}) = 0$. Also, $\tau(1_{C(X)}) \in \mathcal{C}(\mathcal{W})$ must be liftable to a nonzero projection in $M(\mathcal{W})$. Hence, by Lemma 8.11, either $[\tau(1_{C(X)})]_{K_0(\mathcal{C}(\mathcal{W}))} \in (0, 1)$ or $\tau(1_{C(X)}) = 1_{\mathcal{C}(\mathcal{W})}$.

Conversely, suppose that $\tau : C(X) \to \mathcal{C}(\mathcal{W})$ is an essential extension such that $KK(\tau|_{C_0(X\setminus\{x_0\})}) = 0$ and either $[\tau(1_{C(X)})]_{K_0(\mathcal{C}(\mathcal{W}))} \in (0, 1)$ or $\tau(1_{C(X)}) = 1_{\mathcal{C}(\mathcal{W})}$.

By the hypotheses on $\tau(1_{C(X)})$ and by Lemma 8.11, $\tau(1_{C(X)})$ can be lifted to a (nonzero) projection $P \in M(\mathcal{W})$.

Consider the extension $\tau|_{C_0(X\setminus\{x_0\})} : C_0(X \setminus \{x_0\}) \to \mathcal{C}(P\mathcal{W}P)$. Since $P\mathcal{W}P \cong \mathcal{W}$ and since $KK(\tau|_{C_0(X\setminus\{x_0\})}) = 0$, it follows, by Theorem 8.6, that there is monomorphism $H_0 : C_0(X \setminus \{x_0\}) \to PM(\mathcal{W})P$ such that

$$\pi \circ H_0 = \tau|_{C_0(X\setminus\{x_0\})}.$$

Let $H : C(X) \to M(\mathcal{W})$ be the monomorphism given by

$$H|_{C_0(X\setminus\{x_0\})} = H_0 \text{ and } H(1_{C(X)}) = P.$$

Then $\pi \circ H = \tau$, that is, τ is trivial. This completes the proof. ■

Corollary 8.15. Let $\mathbb{T}^n$ be the n torus.

1. $\mathbf{Ext}(C_0(\mathbb{T}^n \setminus \{1\}), \mathcal{W}) = \mathbb{R}^{2^{n-1}-1}$.
2. $\mathbf{Ext}(C(\mathbb{T}^n), \mathcal{W}) = \mathbb{R}^{2^{n-1}}$.

9 Classification of Some Non-simple C^*-Algebras

This section is for the second goal of our original research plan. We study non-simple C^*- algebras E, which are essential extensions of the form

$$0 \to \mathcal{W} \to E \xrightarrow{\pi_E} A \to 0, \tag{9.1}$$

where A is some separable stably finite simple C^*-algebra with finite nuclear dimension such that $\ker\rho_A = K_0(A)$. In other words, E has a unique ideal $I \cong \mathcal{W}$ and $E/\mathcal{W}$ is a separable stably finite simple C^*-algebra with finite nuclear dimension such that $\ker\rho_A = K_0(A)$. Denote by $\mathcal{E}$ the class of such C^*-algebras, which satisfy the UCT. (Warning: Here we do not assume that A is fixed, but it is any separable stably finite simple C^*-algebra with finite nuclear dimension such that $\ker\rho_A = K_0(A)$, and which satisfies the UCT.)

Let $\mathcal{E}_c$ be the subclass of those C^*- algebras E in $\mathcal{E}$ such that $A := \pi_E(E)$ has continuous scale, where $\pi_E : E \to E/\mathcal{W}$ is the quotient map.

In general, if E is an essential extension by $\mathcal{W}$ then E is a subalgebra of $M(\mathcal{W})$. Recall that we identify the unique tracial state τ_W on $\mathcal{W}$ with its unique extension to a tracial state on $M(\mathcal{W})$, which we also denote by τ_W. Therefore, τ_W also induces a tracial state on E which, again, we denote by τ_W. There is a group homomorphism $\lambda_E : K_0(E) \to \mathbb{R}$ induced by τ_W, that is, $\lambda_E(x) = \tau_W(x)$ for all $x \in K_0(E)$. Since $A = E/\mathcal{W}$ and since $K_i(\mathcal{W}) = \{0\}$ $(i = 0, 1)$, by the six-term exact sequence in K-theory, one computes that $\pi_{E*i} : K_i(E) \to K_i(A)$ is a group isomorphism $(i = 0, 1)$.

Lemma 9.1. Let E be an essential extension of the form

$$0 \to \mathcal{W} \to E \xrightarrow{\pi_E} A \to 0,$$

where A is a separable amenable C^*-algebra with $K_0(A) = \ker\rho_{f,A}$, which is $\mathcal{W}$ embeddable and satisfies the UCT. Let $\psi : A \to \mathcal{C}(\mathcal{W})$ be the Busby invariant for the above extension.

Then

$$\psi_{*,0} = \lambda_E \circ \pi_{E*0}^{-1} \text{ in } \mathrm{Hom}(K_0(A), \mathbb{R}).$$

Proof. Denote by $\pi : M(\mathcal{W}) \to \mathcal{C}(\mathcal{W})$ the quotient map. One has the following commutative diagram:

$$\begin{array}{ccc} K_0(E) & \xrightarrow{\lambda_E} & K_0(M(\mathcal{W})) = \mathbb{R} \\ \downarrow_{(\pi_E)_{*0}} & & \downarrow_{\pi_{*0}} \\ K_0(A) & \xrightarrow{\psi_{*0}} & K_0(\mathcal{C}(\mathcal{W})) = \mathbb{R}. \end{array}$$

By Theorem 7.4, π_{*0} is a group isomorphism. Since π_{E*0} is also a group isomorphism, the lemma follows. ■

Before defining the classification invariant, we recall some definitions and other items.

Again, let $E \in \mathcal{E}_c$ and let $A := \pi_E(E)$. Then, as per our definitions, A is a separable stably finite simple continuous scale C^*-algebra satisfying the UCT, with finite nuclear dimension such that $\ker\rho_A = K_0(A)$. Since A is stably finite, A is stably projectionless. (Here is a short proof: Suppose, for contradiction, that $p \in M_m(A)$ is a nonzero projection for some integer $m \geq 1$. Since $T(A) \neq \emptyset$, $\tau(p) \neq 0$ for some $\tau \in T(A)$. This contradicts that $\ker\rho_A = K_0(A)$.)

If $\tau \in T(A)$, then $\tau \circ \pi_E \in T(E)$. The map $(\pi_E)_T : T(A) \to T(E)$, defined by $(\pi_E)_T(\tau)(b) = \tau(\pi_E(b))$ for all $b \in E$ and $\tau \in T(A)$, is an affine homeomorphism onto a closed convex subset of $T(E)$. Denote by T_A the closed convex subset $(\pi_E)_T(T(A))$. Then $T(E)$ is the convex hull of T_A and τ_W. Since A has continuous scale, $T(A)$ is compact. It follows that $T(E)$ is compact. Note that T_A is a face of $T(E)$.

Let $S(K_0(\widetilde{E}))$ be the state space of $K_0(\widetilde{E})$, that is, the set of all group homomorphism s $s : K_0(\widetilde{E}) \to \mathbb{R}$ such that $s(x) \geq 0$, for all $x \in K_0(\widetilde{E})_+$, and $s([1_{\widetilde{E}}]) = 1$. Denote $S(K_0(E)) := \{s|_{K_0(E)} : s \in S(K_0(\widetilde{E}))\}$.

The map $r_E : T(E) \to S(K_0(E))$ is defined by $r_E(\tau)(x) = \tau(x)$ for all $x \in K_0(E)$ and $\tau \in T(E)$.

Now we can define our classification invariant.

Definition 9.2. Let $E \in \mathcal{E}_c$. The Elliott invariant $\mathrm{I}nv(E)$ is defined as follows:

$$\mathrm{Inv}(E) = (K_0(E), K_1(E), T(E), r_E). \tag{9.2}$$

Let $E_1, E_2 \in \mathcal{E}_c$. We say that $\mathrm{I}nv(E_1)$ and $\mathrm{Inv}(E_2)$ are isomorphic, and write $\mathrm{Inv}(E_1) \cong \mathrm{Inv}(E_2)$, if there is an isomorphism

$$\Gamma : \mathrm{Inv}(E_1) = (K_0(E_1), K_1(E_1), T(E_1), r_{E_1}) \to \mathrm{Inv}(E_2) = (K_0(E_2), K_1(E_2), T(E_2), r_{E_2}),$$

that is, if there are a group isomorphism $\Gamma_i : K_i(E_1) \to K_i(E_2)$, $i = 0, 1$, and an affine homeomorphism $\Gamma_T : T(E_1) \to T(E_2)$, which maps $T_f(E_1)$ onto $T_f(E_2)$, such that

$$r_{E_2}(\tau)(\Gamma_0(x)) = r_{E_1}(\Gamma_T^{-1}(\tau))(x) \text{ for all } x \in K_0(E_1) \text{ and } \tau \in T(E_2). \tag{9.3}$$

Theorem 9.3. Let $E_1, E_2 \in \mathcal{E}_c$. Then $E_1 \cong E_2$ if and only if

$$\mathrm{Inv}(E_1) \cong \mathrm{Inv}(E_2). \tag{9.4}$$

Moreover, if $\Gamma : \mathrm{I}nv(E_1) \to \mathrm{I}nv(E_2)$ is an isomorphism, then there exists an isomorphism $\Psi : E_1 \to E_2$ such that Ψ induces Γ.

Proof. We have two short exact sequences

$$0 \to \mathcal{W} \to E_1 \xrightarrow{\pi_{E_1}} A \to 0 \text{ and}$$
$$0 \to \mathcal{W} \to E_2 \xrightarrow{\pi_{E_2}} B \to 0.$$

Both A and B are separable simple stably projectionless C^*- algebras with finite nuclear dimension and continuous scale, which satisfy the UCT. Moreover, $K_0(A) = \ker\rho_A$ and $K_0(B) = \ker\rho_B$.

Suppose that $\Gamma : Inv(E_1) \to Inv(E_2)$ is an isomorphism. Then one has a group isomorphism $\gamma_i := \pi_{E_2*i} \circ \Gamma_i \circ \pi_{E_1*i}^{-1} : K_i(A) \to K_i(B)$, $i = 0, 1$. Note that Γ induces an affine homeomorphism $\Gamma_T : T(E_1) \to T(E_2)$, which maps $T_f(E_1)$ to $T_f(E_2)$. Since Γ_T is an affine homeomorphism, it maps extreme points to extreme points. It follows that $\Gamma_T(\tau_W) = \tau_W$. Since the extreme points of the face T_A are also extreme points of $T(E)$, Γ_T maps T_A onto T_B, and $(\pi_{E_2})_T^{-1}\circ\Gamma_T\circ(\pi_{E_1})_T : T(A) \to T(B)$ is also an affine homeomorphism. Since both E_1 and E_2 satisfy the UCT, A and B also satisfy the UCT. It follows that, by the classification results in [22] (see Theorem 13.1 and Theorem 15.5 of [22]), there is an isomorphism $\phi : A \to B$ such that $\phi_{*i} = \gamma_i$, $i = 0, 1$, and $\phi_T = (\pi_{E_1})_T^{-1} \circ \Gamma_T^{-1} \circ (\pi_{E_2})_T$. Recall that $\lambda_{E_i} : K_0(E_i) \to \mathbb{R}$ is defined by $\lambda_{E_i}(x) = \tau_W(x)$ for all $x \in K_0(E_i)$.

In (9.3), with $\tau = \tau_W$, we have that (as Γ_T maps τ_W to τ_W)

$$\tau_W(\Gamma_0(x)) = \tau_W(x) \text{ for all } x \in K_0(E_1). \tag{9.5}$$

In other words,

$$\lambda_{E_2} \circ \Gamma_0 = \lambda_{E_1}. \tag{9.6}$$

Let $\sigma_A : A \to \mathcal{C}(\mathcal{W})$ and $\sigma_B : B \to \mathcal{C}(\mathcal{W})$ be the Busby invariants associated with E_1 and E_2 respectively. Define an essential extension $\psi : A \to \mathcal{C}(\mathcal{W})$ by $\psi = \sigma_B \circ \phi$. Hence,

$$\begin{aligned}
\psi_{*,0} &= (\sigma_B)_{*,0} \circ \phi_{*,0} \\
&= (\lambda_{E_2} \circ \pi_{E_2*,0}^{-1}) \circ (\pi_{E_2*,0} \circ \Gamma_0 \circ \pi_{E_1*,0}^{-1}) && \text{(by Lemma 9.1 and since } \phi_{*,0} = \gamma_0) \\
&= \lambda_{E_2} \circ \Gamma_0 \circ \pi_{E_1*,0}^{-1} = \lambda_{E_1} \circ \pi_{E_1*,0}^{-1} && \text{(by 9.6)} \\
&= (\sigma_A)_{*,0} && \text{(by 9.1).}
\end{aligned}$$

Hence,

$$KK(\psi) = KK(\sigma_A). \tag{9.7}$$

It follows from Theorem 9.9 that there is a unitary $U \in M(\mathcal{W})$ such that

$$\pi(U^*)\psi(a)\pi(U) = \sigma_A(a) \text{ for all } a \in A. \tag{9.8}$$

Note that, by (9.8),

$$U^*eU \in E_1 \text{ for all } e \in E_2. \tag{9.9}$$

Define $\Psi : E_2 \to E_1$ by

$$\Psi(e) = U^*eU \text{ for all } e \in E_2. \tag{9.10}$$

Ψ is a monomorphism. Note that $\Psi(\mathcal{W}) = \mathcal{W}$. By (9.8), Ψ is surjective. So Ψ is an isomorphism. Moreover, from the construction, one checks that Ψ induces Γ.

Conversely, if there is an isomorphism $\Psi : E_1 \to E_2$, then Ψ induces an isomorphism $\Gamma : \mathrm{Inv}(E_1) \to \mathrm{Inv}(E_2)$. ■

Towards classifying C^*- algebras in $\mathcal{E}$, we again recall some terminology and other items.

Let A be a C^*-algebra with $\tilde{T}(A) \neq \{0\}$ and with a strictly positive element e_A. Denote by $\Sigma_A \in \mathrm{L}Aff(\tilde{T}(A))$ the lower semicontinuous affine function defined by

$$\Sigma_A(\tau) = \lim_{n\to\infty} \tau(f_{1/n}(e_A)) \text{ for all } \tau \in \tilde{T}(A).$$

One notes that Σ_A, as a lower semicontinuous affine function on $\tilde{T}(A)$, is independent of the choice of e_A.

Recall also that there is a unique embedding $E \hookrightarrow M(\mathcal{W})$, which is the identity map on $\mathcal{W}$, and also that the group homomorphism $\lambda_E : K_0(E) \to \mathbb{R}$ is defined by $\lambda_E(x) = \tau_W(x)$ for all $x \in K_0(E)$.

Definition 9.4. For any $E \in \mathcal{E}$, define

$$\mathrm{I}nv(E) = (K_0(E), K_1(E), \tilde{T}(E), \Sigma_E, \lambda_E). \tag{9.11}$$

Denote by $\tilde{T}_f(E)$ the set of all faithful traces in $\widetilde{T}(E)$, that is, the set of all $\tau \in \widetilde{T}(E)$ for which $\tau(a) \neq 0$ for every $a \in \mathrm{Ped}(E)_+ \setminus \{0\}$. Write $A = \pi_E(E)$. Let $(\pi_E)_T : \tilde{T}(A) \to \tilde{T}(E)$ be the affine homomorphism defined by $(\pi_E)_T(\tau)(e) = \tau \circ \pi_E(e)$ for all $e \in \mathrm{Ped}(E)$ and $\tau \in \tilde{T}(A)$. The cone $\tilde{T}(E)$ is generated by $(\pi_E)_T(\tilde{T}(A))$ and τ_W.

Let $E_1, E_2 \in \mathcal{E}$. We say that $\mathrm{Inv}(E_1)$ and $\mathrm{I}nv(E_2)$ are isomorphic, and write $\mathrm{I}nv(E_1) \cong \mathrm{I}nv(E_2)$, if there is an isomorphism

$$\Gamma : \mathrm{I}nv(E_1) = (K_0(E_1), K_1(E_1), \tilde{T}(E_1), \Sigma_{E_1}, \lambda_{E_1}) \cong \mathrm{I}nv(E_2) = (K_0(E_2), K_1(E_2), \tilde{T}(E_2), \Sigma_{E_2}, \lambda_{E_2}),$$

that is, if there are a group isomorphism $\Gamma_i : K_i(E_1) \to K_i(E_2)$, $i = 0, 1$, and a topological cone isomorphism $\Gamma_T : \tilde{T}(E_1) \to \tilde{T}(E_2)$, which maps $\tilde{T}_f(E_1)$ onto $\tilde{T}_f(E_2)$ such that

$$\lambda_{E_2} \circ \Gamma_0 = \lambda_{E_1} \text{ and } \Sigma_{E_2} \circ \Gamma_T = \Sigma_{E_1}. \tag{9.12}$$

Lemma 9.5. Let $E \in \mathcal{E}$ be an essential extension of the form:

$$0 \to \mathcal{W} \to E \xrightarrow{\pi_E} A \to 0. \tag{9.13}$$

Suppose that $e_1, e_2 \in E_+$ are such that $d_\tau(\pi_E(e_1)) = d_\tau(\pi_E(e_2))$, for all $\tau \in \tilde{T}(A)$, and $\mathcal{W} \subset \mathrm{Her}_E(e_i)$, $i = 1, 2$. Then there is an isomorphism

$$\psi : \mathrm{Her}_E(e_1) \cong \mathrm{Her}_E(e_2) \tag{9.14}$$

such that $KL(\psi) = KL(\mathrm{id}_E)$ and $\tau \circ \psi(e) = \tau(e)$ for all $\tau \in \tilde{T}(E)$ and $e \in \mathrm{Her}_E(e_1)_+$.

Proof. Since A has stable rank one (see Theorem 11.5 of [22]), it follows from [14] (see Proposition 3.3 of [63]; see also the paragraph above Proposition 3.3 of [63], and [44]) that there is an element $u \in A^{**}$ such that $u\mathrm{Her}_A(\pi_E(e_1))u^* = \mathrm{Her}_A(\pi_E(e_2))$. Moreover, $u\pi_E(e_1), u^*\pi_E(e_2) \in A$ and $u^*u = p$ and $uu^* = q$, where p and q are open projections of A corresponding to $\pi_E(e_1)$ and $\pi_E(e_2)$, respectively. Let $x = u\pi_E(e_1) \in A$. Since A has stable rank one, by Theorem 5 of [56], for each n, there is a unitary $u_n \in \widetilde{A}$ such that

$$u_n\pi_E(f_{1/n}(e_1)) = u\pi_E(f_{1/n}(e_1)). \tag{9.15}$$

Since $K_1(\mathcal{C}(\mathcal{W})) = \{0\}$ (see Theorem 7.4) and $\mathcal{C}(\mathcal{W})$ is purely infinite simple, there is a unitary $w_n \in M(\mathcal{W})$ such that $\pi(w_n) = u_n$. Therefore, $w_n \in \widetilde{E}$. Since

$$u_n \pi_E(f_{1/n}(e_1)) u_n^* \in \mathrm{Her}_A(\pi_E(e_2)), \tag{9.16}$$

$$w_n f_{1/n}(e_1) w_n^* \in \mathrm{Her}_E(e_2). \tag{9.17}$$

(Recall that $\mathcal{W} \subseteq \mathrm{H}er_E(e_2)$.) It follows that, for all n,

$$f_{1/n}(e_1) \lesssim e_2. \tag{9.18}$$

Therefore,

$$e_1 \lesssim e_2. \tag{9.19}$$

Symmetrically,

$$e_2 \lesssim e_1. \tag{9.20}$$

Hence, $e_1 \sim e_2$. By (9.13) and the fact that $K_i(\mathcal{W}) = \{0\}$, $i = 0, 1$, and by applying part (ii) of Proposition 4 of [48], E has stable rank one. It follows from [14] that there is an isomorphism

$$\psi : \mathrm{Her}_E(e_1) \cong \mathrm{Her}_E(e_2)$$

such that $\psi(a) = U^* a U$ for all $a \in \mathrm{Her}_E(e_1)$. Here, $U \in E^{**}$ is a partial isometry such that $U^* a, Ub \in E$ for all $a \in \mathrm{Her}_E(e_1)$ and $b \in \mathrm{Her}_E(e_2)$, $UU^* = P$, and $U^* U = Q$, where P is the open projection corresponding to e_1 and Q is the open projection corresponding to e_2.

Let $z = U^* e_1 \in E$. Since E has stable rank one (which we just proved), by Theorem 5 of [56], for each n, there is a unitary $V_n \in \widetilde{E}$ such that $V_n P_n = U^* P_n$, where P_n is the spectral projection of e_1 in A^{**} corresponding to $(1/(3n), \|z\|]$. It follows that $V_n f_{1/n}(e_1) a f_{1/n}(e_1) V_n^* \in \mathrm{Her}_E(e_2)$ for all $a \in \mathrm{H}er_E(e_1)$, and

$$\lim_{n\to\infty} V_n a V_n^* = \lim_{n\to\infty} V_n f_{1/n}(e_1) a f_{1/n}(e_1) V_n^* = \lim_{n\to\infty} U^* f_{1/n}(e_1) a f_{1/n}(e_1) U = \psi(a) \tag{9.21}$$

for all $a \in \mathrm{Her}_E(e_1)$. It follows that $KL(\psi) = KL(\mathrm{id}_E)$ and $\tau \circ \psi(a) = \tau(a)$ for all $\tau \in \tilde{T}(E)$ and $a \in \mathrm{Her}_E(e_1)_+$. ■

Theorem 9.6. Let $E_1, E_2 \in \mathcal{E}$. Then $E_1 \cong E_2$ if and only if there is an isomorphism

$$\Gamma : \mathrm{Inv}(E_1) = (K_0(E_1), K_1(E_1), \tilde{T}(E_1), \Sigma_{E_1}, \lambda_{E_1}) \cong \mathrm{Inv}(E_2) = (K_0(E_2), K_1(E_2), \tilde{T}(E_2), \Sigma_{E_2}, \lambda_{E_2}).$$

Moreover, if such an isomorphism Γ exists, then there is an isomorphism $\psi : E_1 \to E_2$, which induces Γ.

Proof. Suppose that we have an isomorphism $\Gamma : Inv(E_1) \cong Inv(E_2)$.

We have two short exact sequences

$$0 \to \mathcal{W} \to E_1 \xrightarrow{\pi_{E_1}} A \to 0 \text{ and } 0 \to \mathcal{W} \to E_2 \xrightarrow{\pi_{E_2}} B \to 0.$$

Both A and B are separable simple stably projectionless C*-algebras with finite nuclear dimension and which satisfy the UCT. Moreover, $K_0(A) = \ker\rho_A$ and $K_0(B) = \ker\rho_B$.

Let $e_0 \in (E_1)_+ \setminus \{0\}$ with $\|e_0\| = 1$ be a strictly positive element of E_1. Let $e_1 = f_{1/2}(e_0)$. Choose $a_0 \in A_+ \setminus \{0\}$ such that $a_0 \le \pi_{E_1}(e_1)$ and $d_\tau(a_0)$ is continuous on $\tilde{T}(A)$ (see 11.11 of [18] and Theorem 15.5 of [22]). So $\mathrm{Her}(a_0)$ has continuous scale (see, e.g., Proposition 5.4 of [18]). Choose $a_1' \in (E_1)_+$ such that $f_{1/8}(e_0)a_1' = a_1'$ and $\pi_{E_1}(a_1') = a_0$ (see Lemma 7.2 of [18]). Then $a_1' \in \mathrm{Ped}(E_1)$. Let $e_W \in \mathcal{W}$ be a strictly positive element in $\mathcal{W}$. Since $\mathrm{Ped}(\mathcal{W}) = \mathcal{W}$, $a_1'' = a_1' + e_W \in \mathrm{Ped}(E_1)_+$. Let $a_1 = a_1''/\|a_1''\|$. Then $a_1 \in M(\mathcal{W})_+$. It follows that $\tau_W(f_{1/n}(a_1)) \le 1$ for all n. Obviously,

$$e_W \lesssim a_1'' \sim a_1.$$

We conclude that $d_{\tau_W}(a_1) = 1$. Since every $t \in \tilde{T}(E_1)$ has the form $\alpha \cdot t_A \circ \pi_{E_1} + (1-\alpha) \cdot t_W$, where $t_A \in \tilde{T}(A)$ and $0 \le \alpha \le 1$, one also verifies that $d_\tau(a_1)$ is continuous on $\tilde{T}(E_1)$.

Let $A_1 = \mathrm{Her}(a_0)$ and $E_{1,c} = \mathrm{Her}(a_1)$. Note that $\pi_{E_1}(E_{1,c}) = A_1$. So $E_{1,c} \in \mathcal{E}_c$.

Let $g \in \mathrm{Aff}(\tilde{T}(B))$ be such that $g \circ \Gamma_T(\tau) = d_\tau(\pi_{E_1}(a_1))$ for all $\tau \in \tilde{T}(A)$. By Theorem 11.11 of [18] and Theorem 15.5 of [22], there exists a $b_0 \in B_+$ such that $d_t(b_0) = g(t)$ for all $t \in \tilde{T}(B)$. Let $B_1 = Her(b_0)$. Then B_1 also has continuous scale (see Proposition 5.4 of [18]). Choose $b_1' \in (E_2)_+$ such that $\pi_{E_2}(b_1') = b_0$ and $b_1'' = b_1' + e_W$. Set $b_1 = b_1''/\|b_1''\| \in E_2 \subset M(\mathcal{W})$. Then for any n, $\tau_W(f_{1/n}(b_1)) \le 1$. It follows that $d_{\tau_W}(b_1) \le 1$. Note that $e_W \lesssim b_1'' \sim b_1$. Therefore, $d_{\tau_W}(b_1) = 1$. Note that for each

$\tau = \alpha t_B \circ \pi_{E_2} + (1-\alpha) t_W \in \tilde{T}(E_2)$, where $t_B \in \tilde{T}(B)$ and $0 \le \alpha \le 1$,

$$d_\tau(b_1) = \alpha g(t_B) + (1-\alpha).$$

Define $E_{2,c} = \mathrm{H}er(b_1)$. Then $E_{2,c} \in \mathcal{E}_c$. We note that $E_{j,c}$ is a full hereditary C^*-subalgebra of E_j, which contains $\mathcal{W}$ as an ideal ($j = 1,2$). In particular, $K_i(E_{j,c}) = K_i(E_j)$, $i = 0,1$ and $j = 1,2$.

Consider $T_a = \{\tau \in \tilde{T}(E_1) : d_\tau(a_1) = 1\}$. Since $d_\tau(a_1)$ is continuous on $\tilde{T}(E_1)$, T_a is a compact convex subset of $\tilde{T}(E_1)$. Note that $\Gamma_T(T_a) = T_g = \{t \in \tilde{T}(E_2) : d_\tau(b_1) = 1\}$. Moreover, Γ_T maps T_a affinely and homeomorphically onto T_g.

Let $\gamma_1 : T_a \to T(E_{1,c})$ be defined by $\gamma_1(\tau)(e) = \tau(e)$ for all $e \in E_{1,c}$ and $\tau \in T_a$. Then γ_1 is an affine homeomorphism. Let $\gamma_2 : T_g \to T(E_{2,c})$ be defined by $\gamma_2(t)(d) = t(d)$ for all $d \in E_{2,c}$ and $t \in T_g$. Then γ_2 is also an affine homeomorphism.

Now define

$$\Gamma' : (K_0(E_{1,c}), K_1(E_{1,c}), T(E_{1,c}), r_{E_{1,c}}) \to (K_0(E_{2,c}), K_1(E_{2,c}), T(E_{2,c}), r_{E_{2,c}}) \tag{9.22}$$

as follows: $\Gamma'_i := \Gamma_i : K_i(E_{1,c}) = K_i(E_1) \to K_i(E_2) = K_i(E_{2,c})$, $i = 0,1$, and $\Gamma'_T := \gamma_2 \circ \Gamma_T \circ \gamma_1^{-1}$.

We also check that since $\lambda_{E_2} \circ \Gamma_0 = \lambda_{E_1}$, for any $x \in K_0(E_{1,c})$,

$$\tau_W(\Gamma'_0(x)) = \tau_W(\Gamma_0(x)) = \tau_W(x). \tag{9.23}$$

Since $K_0(A) = \ker \rho_A$ and $K_0(B) = \ker \rho_A$, (9.23) implies that

$$r_{E_{2,c}}(\tau)(\Gamma'_0(x)) = r_{E_{1,c}}(\Gamma_T^{-1}(\tau))(x) \text{ for all } x \in K_0(E_{1,c}) \text{ and } \tau \in T(E_{2,c}). \tag{9.24}$$

Hence, $\Gamma' : Inv(E_{1,c}) \to Inv(E_{2,c})$ is an isomorphism. It follows from Theorem 9.3 that there exists an isomorphism $\Psi : E_{1,c} \to E_{2,c}$, which induces Γ'. This provides (also denoted by Ψ) an isomorphism $\Psi : E_{1,c} \otimes \mathcal{K} \to E_{2,c} \otimes \mathcal{K}$.

By Brown's stable isomorphism theorem [4], we may view E_1 as a full hereditary C^*-subalgebra of $E_{1,c} \otimes \mathcal{K}$. Then we obtain an embedding $\Psi|_{E_1} : E_1 \to E_{2,c} \otimes \mathcal{K}$. Let e'_1 be a strictly positive element of E_1 and $e'_2 = \Psi(e'_1)$. Let e''_2 be a strictly positive element of E_2. Since Ψ induces Γ' and since $\Sigma_{E_2} \circ \Gamma_T = \Sigma_{E_1}$, we have that

$$d_\tau(e'_2) = d_\tau(e''_2) \text{ for all } \tau \in \tilde{T}(E_2).$$

Finally, by applying Lemma 9.5, there is an isomorphism $\psi : \mathrm{H}er(e_2') \cong \mathrm{H}er(e_2'')$ with $KL(\psi) = KL(\mathrm{id}_{E_2})$ and which preserves the traces. Therefore,

$$E_1 \cong E_2 \tag{9.25}$$

and the isomorphism induces Γ. ■

Remark 9.7. In Section 7, we do not include a classification statement for the essential extensions of the form in (9.1), for the case that A does not have continuous scale. Theorem 9.6 is a classification with a different flavor. It should be noted, though, that if A does not have continuous scale, then there may not be any trivial extensions of the form in (9.1). To see this, consider the case where $A = A \otimes \mathcal{K}$. Then A does not have any faithful tracial states. If there were a monomorphism $j : A \to M(\mathcal{W})$ such that

$$\pi_E \circ j = \mathrm{id}_A, \tag{9.26}$$

then $\tau_W \circ j$ would induce a faithful tracial state on A. This is not possible. So no essential extensions of the form in (9.1) splits. This explains, partially, why we choose not to include this case in Section 7.

The following is another version of 7.9. We should keep 7.9 and remove this—we can use this (slightly different) presentation.

Lemma 9.8. Let A be a separable amenable C^*-algebra, which is $\mathcal{W}$ embeddable. Then A has a $\mathcal{T}$ extension $\pi \circ \sigma : A \to \mathcal{C}(\mathcal{W})$.

Proof. Let $\{b_n\}$ be a system of quasidiagonal units for $\mathcal{W}$. We assume, of course, $b_n \neq 0$. Let $D_n = \mathrm{H}er(b_n)$. Then $D_n \cong M_n(\mathcal{W})$. Let $\{e_{i,j}^{(n)} : 1 \leq i,j \leq n\}$ be a system of matrix units for M_n, $n = 1, 2, \ldots$. We may assume (by choosing an diagonal element in $D_n \cong M_n(\mathcal{W})$), that $b_n = \bigoplus_{i=1}^n e_{j,1}^{(n)} b_n e_{1,j}$. Put $b_{n,j} = e_{j,1} b_n e_{1,j}$, $j = 1, 2, \ldots, n$.

Fix an embedding $\iota_A : A \to \mathcal{W}$ which maps strictly positive elements to strictly positive elements. For each n, there is an isomorphism $\psi_n : \mathcal{W} \to \mathrm{H}er(b_{n,1})$. So we have an embedding $\bigoplus^n \psi_n \circ \iota_A : A \to D_n \cong M_n(\mathcal{W})$. Define

$$\sigma : A \to M(\mathcal{W}) \text{ by } \sigma(a) = \bigoplus_{n=1}^{\infty} (\bigoplus^{n} \psi_n \circ \iota_A). \tag{9.27}$$

By Definition 7.6, $\pi \circ \sigma$ is a $\mathcal{T}$ extension. ■

Theorem 9.9. Let A be a separable amenable C^*-algebra, which is $\mathcal{W}$ embeddable and satisfies the UCT. Suppose also that $\ker\rho_A = K_0(A)$.

(1) If $\tau_1, \tau_2 : A \to \mathcal{C}(\mathcal{W})$ are two essential extensions, then $\tau_1 \sim^u \tau_2$ if and only if $KK(\tau_1) = KK(\tau_2)$.

(2) The map $\Lambda : \mathbf{Ext}(A, \mathcal{W}) \to KK(A, \mathcal{C}(\mathcal{W}))$ defined by $\Lambda([\tau]) = KK(\tau)$ is a group isomorphism.

(3) An essential extension τ is trivial if and only if $KK(\tau) = 0$, and all trivial extensions are unitarily equivalent.

Proof. We first show (3). If τ is trivial, then there exists monomorphism $\phi : A \to M(\mathcal{W})$ such that $\pi \circ \phi = \tau$. It follows from 7.10 that $\tau_{*0}(K_0(A)) = \tau_{*0}(\ker\rho_A) = 0$ and $\tau_{*1} = 0$. Since $K_0(\mathcal{C}(\mathcal{W})) = \mathbb{R}$ is divisible and $K_1(\mathcal{C}(\mathcal{W})) = \{0\}$, by the UCT, one computes that $KK(\tau) = 0$. It follows from 7.16 that τ is quasidiagonal. Moreover, by 7.17, $\tau \sim^u \pi \circ \sigma$. Conversely, if $KK(\tau) = 0$, then we just showed that $\tau \sim^u \pi \circ \sigma$.

We now show (2). Define $\Lambda : \mathbf{Ext}^u(A, \mathcal{W}) \to KK(A, \mathcal{C}(\mathcal{W}))$ by $\Lambda([\tau]) = KK(\tau)$. It is a semigroup homomorphism. That the map is injective follows by (1).

Fix $x \in KK(A, \mathcal{C}(\mathcal{W}))$. By the UCT, oen computes that $KK(A, \mathcal{C}(\mathcal{W})) = \mathrm{Hom}(K_0(A), \mathcal{C}(\mathcal{W}))$. Note that $K_0(\widetilde{A}) = K_0(A) \oplus \mathcal{Z}$. Define $\eta \in \mathrm{Hom}(K_0(\widetilde{A}), \mathcal{C}(\mathcal{W}))$ by $\eta|_{K_0(A)} = x$ and $\eta([1_{\tilde{A}}]) = [1_{\mathcal{C}(\mathcal{W})}]$. Then η gives an element in $KL(\widetilde{A}, \mathcal{C}(\mathcal{W}))$. It follows from Corollary 8.5 of [41] that there is a homomorphism $\tau_1 : \widetilde{A} \to \mathcal{C}(\mathcal{W})$ such that $KK(\tau_1) = \eta$. Define $\tau = \tau_1|_A$. Then $KK(\tau) = x$. So the map Λ is surjective. It follows that $\mathbf{Ext}^u(A, \mathcal{W})$ is a group.

We then see (1) follows from (2). ■

10 Extensions by a Simple C^*-Algebra in $\mathcal{I}$

In this section, we consider essential extensions of the form

$$0 \to B \to E \to C \to 0,$$

where B is a separable simple C^*-algebra with a unique tracial state and with finite nuclear dimension and that satisfies the UCT and C is a separable amenable C^*-algebra, which is $\mathcal{W}$ embeddable.

Definition 10.1. Denote by $\mathcal{I}$ the class of all non-unital stably projectionless separable simple amenable $\mathcal{Z}$-stable C^*- algebras with a unique tracial state and that satisfies the UCT.

Note that if $B \in \mathcal{I}$, then $K_0(M(B)) = \mathbb{R}$, $K_1(M(B)) = \{0\}$, $K_1(\mathcal{C}(B)) = K_0(B)$ and $K_0(\mathcal{C}(B)) = \mathbb{R} \oplus K_1(B)$ (see Theorem 7.4). C^*- algebras in $\mathcal{I}$ have been classified by their Elliott invariant in [22]. All C^*- algebras in $\mathcal{I}$ have stable rank one. Moreover, $\ker\rho_B = K_0(B)$ for every C^*-algebra $B \in \mathcal{I}$ (see Lemma 6.1). We will also use the fact that every hereditary C^*-subalgebra , of a C^*-algebra in $\mathcal{I}$, is also in $\mathcal{I}$.

Lemma 10.2. Let $B \in \mathcal{I}$. Let A be a separable amenable C^*-algebra, which is $\mathcal{W}$ embeddable. Then there exists an essential $\mathcal{T}_d$ extension (see 7.6) $\sigma : A \to M(B)$, with model σ_A that factors through $\mathcal{W}$ and, in particular, $KK(\sigma_A) = 0$.

Proof. Fix an embedding $\iota_A : A \to \mathcal{W}$ and an embedding $\iota_{\mathcal{W},B} : \mathcal{W} \to B$ (given, e.g., by 6.4) such that both ι_A and $\iota_{\mathcal{W},B}$ map strictly positive elements to strictly positive elements (see 6.4 and 6.8). Let $\sigma_A = \iota_{\mathcal{W},B} \circ \iota_A : A \to B$. Denote by τ the unique tracial state of B. Fix a system of quasidiagonal units $\{b_k\}$, for B, as in 4.1. Passing to a subsequence if necessary, we may assume that

$$\sum_{k=n+1}^{\infty} d_\tau(b_k) < (1/n)d_\tau(b_n) \text{ for all } n. \tag{10.1}$$

Let $t_n = (1/n)d_\tau(b_n)$, $n \in \mathbb{N}$. There is an element $a_n \in \mathrm{Her}(b_n)$ with $d_\tau(a_n) = t_n$ (since B has strict comparison and since $\mathrm{Cu}(B) \cong V(B) \sqcup (0,\infty]$; see Proposition 11.11 of [18] and Theorem 15.5 of [22]). Moreover, $\mathrm{Her}(b_n) \cong M_n(\mathrm{Her}(a_n))$. By part (2) of 6.6, there is, for each n, an isomorphism $\phi_n : B \to \mathrm{Her}(a_n)$. Define $\sigma : A \to M(B)$ by $\sigma(a) = \sum_{n=1}^{\infty}(\bigoplus^n \phi_n \circ \sigma_A)(a)$ for all $a \in A$. One then checks, from Definition 7.6, that $\pi \circ \sigma$ is a $\mathcal{T}_d$ extension with model σ_A, which factors through $\mathcal{W}$ and $KK(\sigma_A) = 0$. ■

Lemma 10.3. Let C be a separable amenable C^*-algebra, which is $\mathcal{W}$ embeddable and satisfies the UCT. Suppose that $K_i(C)$ is finitely generated ($i = 0, 1$). Let $B \in \mathcal{I}$, and let $\pi \circ \sigma : C \to \mathcal{C}(B)$ be an essential $\mathcal{T}_d$ extension with a model map σ_C, which factors through $\mathcal{W}$. Then for any essential quasidiagonal extension $\tau_q : C \to \mathcal{C}(B)$, there is a trivial diagonal essential extension $\sigma_d : C \to \mathcal{C}(B)$ such that

$$\tau_q \oplus \pi \circ \sigma \sim^u \sigma_d \oplus \pi \circ \sigma. \tag{10.2}$$

Proof. Exactly as at the beginning of the proof of 7.12, without loss of generality, we may assume that $ran(\sigma) \perp ran(\psi)$, where $\psi : C \to M(B)$ is a c.p.c. map for which $\tau_q = \pi \circ \psi$.

We write

$$\sigma = \bigoplus_{n=1}^{\infty} \bigoplus^{n+1} \phi_n \circ \sigma_C$$

as in Definition 7.6 and Lemma 10.2.

Since τ_q is quasidiagonal, we may write $\psi = \bigoplus_{n=1}^{\infty} \psi_n$, and let $\{a_n\}$ be a system of quasidiagonal units from Proposition 4.5 that corresponds to $\{\psi_n\}$. Recall that

$$\lim_{n\to\infty} \|\psi_n(ab) - \psi_n(a)\psi_n(b)\| = 0 \ \ for \ \ all \ \ a, b \in C. \tag{10.3}$$

We now write $\bigoplus^{n+1} \phi_n \circ \sigma_C = \sigma_{n,0} \oplus \sigma_{n,1} \oplus \cdots \oplus \sigma_{n,n}$ and $\sigma = \bigoplus_{n=1}^{\infty} \bigoplus_{j=0}^{n} \sigma_{n,j}$. Following the notation of Definition 7.6, let

$$b_{n,j} := \sigma_{n,j}(e_C) \ \ for \ \ all \ \ n, j.$$

Since σ is a $\mathcal{T}_d$ extension, by 7.7, there exists a map $F : C_+ \setminus \{0\} \to \mathbb{N} \times (0, \infty)$ such that for all n, j, $\sigma_{n,j} : C \to \overline{b_{n,j} B b_{n,j}}$ is F- full.

Let $\{\epsilon_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty} \epsilon_n < \infty$. Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \cdots$ be a sequence of finite subsets of the unit ball of C, whose union is dense in the unit ball of C.

We will apply Theorem 5.8. Note that by Proposition 5.2, every hereditary C^*-subalgebra of B is in $\mathbf{C}_{0,0,1,T,7}$, with T as in Proposition 5.2. Let $L := 7\pi + 1$. Recall that for the given C, we fix maps J, Π_{cu}^{-} and $J^{\sim}$ as in 5.7.

For each n, let $\delta_n > 0$, $\mathcal{G}_n \subset C$ be a finite subset, $\mathcal{P}_n \subset \underline{K}(C)$ be a finite subset, $\mathcal{U}_n \subset J^{\sim}(K_1(C))$ be a finite subset, $\mathcal{E}_n \subset C_+ \setminus \{0\}$ be a finite subset, and K_n be an integer associated with $\mathcal{F}_n$ and ϵ_n (as well as F and L above), as provided by Theorem 5.8 (for C^*- algebras in $\mathbf{C}_{0,0,1,T,7}$).

We may assume that $\delta_{n+1} < \delta_n$, $\mathcal{G}_n \subset \mathcal{G}_{n+1}$, $\mathcal{P}_n \subset \mathcal{P}_{n+1}$, $\mathcal{U}_n \subset \mathcal{U}_{n+1}$, $\mathcal{U}_n \subset U(M_{m(n)}(\widetilde{C}))$, and $K_n < K_{n+1}$, for all n. Without loss of generality, we may assume that each ψ_n is $\mathcal{G}_n$-δ_n-multiplicative and $\lceil \psi_n(u) \rceil$ is well defined for all $u \in \mathcal{U}_n$.

Moreover, without loss of generality, we may also assume that (see Theorem 14.5 of [46]) for any n, there is a group homomorphism

$$\lambda_n : G(\Pi_{cu}(\mathcal{U}_n)) \to U(M_{m(n)}(\widetilde{\mathrm{Her}(a_n)}))/CU(M_{m(n)}(\widetilde{\mathrm{Her}(a_n)}))$$

such that

$$\mathrm{d}ist(\lambda_n(x), \Pi_{cu}(\lceil \psi_n(J^{\sim}(x)) \rceil)) < 1/16\pi(n+1) \text{ for all } x \in \Pi_{cu}(\mathcal{U}_n), \tag{10.4}$$

where $G(\Pi_{cu}(\mathcal{U}_n))$ is the subgroup generated by the finite subset $\Pi_{cu}(\mathcal{U}_n)$. Recall that $\Pi_{cu} \circ J^{\sim}(x) = \Pi_{cu} \circ \Pi_{cu}^{-}(J(x)) = J(x)$ for all $x \in K_1(C)$. Without loss of generality, we may assume that $\mathcal{P}_n \cap K_1(C)$ generates the same group as $\Pi_1 \circ \Pi_{cu}(\mathcal{U}_n)$ does (in the current case, $K_1(C)$ is assumed to be finitely generated).

Moreover, since $K_i(C)$ is finitely generated, we may assume that $KL(\psi_n)$ and $\lambda_n \circ J$ are well defined, and since λ_n is determined by ψ_n, we may also assume that $\lambda_n \circ J$ is compatible with $KK(\psi_n)$.

Again, throwing away finitely many terms and relabelling if necessary, we may assume that

$$\sum_{n=1}^{\infty} d_\tau(a_n) < d_\tau(b_{K_1,0}),$$

where τ is the unique tracial state of B. Let $\{n_k\}_{k=1}^{\infty}$ be a subsequence of $\mathbb{Z}^+$ with $n_1 = 1$ and $n_k + 2 < n_{k+1}$ for all k such that

$$\sum_{l=n_k}^{\infty} d_\tau(a_l) < d_\tau(b_{K_k,0}).$$

Since B has stable rank one, there is a unitary $U_k' \in \widetilde{B}$ such that

$$(U_k')^*((\sum_{l=n_k}^{n_{k+1}-1} a_l)B(\sum_{l=n_k}^{n_{k+1}-1} a_l))U_k' \subset \overline{b_{K_k,0}Bb_{K_k,0}}. \tag{10.5}$$

Let $B_{k,0} = \mathrm{Her}(\sum_{l=n_k}^{n_{k+1}-1} a_l)$. Hence, $(U_k')^* B_{k,0} U_k' \subset \mathrm{Her}(b_{K_k,0})$.

For each n, by Theorem 6.11, there is a homomorphism $h_n : C \to Her(a_n)$ such that $KL(h_n) = KK(\psi_n)$ and $h_n^{\ddagger} = \lambda_n \circ J$.

In other words,

$$[\psi_n]|_{\mathcal{P}_n} = [h_n]|_{\mathcal{P}_n} \text{ and } h_n^{\dagger}|_{\Pi_{cu}(\mathcal{U}_n)} = \lambda_n|_{\Pi_{cu}(\mathcal{U}_n)}. \tag{10.6}$$

By the second part of (10.6) and (10.4), and since $U_n \subset U_{n+1}$ for all n, for any $u \in \mathcal{U}_{n_k}$ and $n_k \le l \le n_{k+1} - 1$, there is a $v_l \in CU(M_{m(l)}(\widetilde{\mathrm{Her}(a_l)}))$ such that

$$h_l(u)\lceil \psi_l(u) \rceil^* \approx_{1/16\pi(l+1)} v_l. \tag{10.7}$$

It follows from Lemma 7.11 that for all $u \in \mathcal{U}_{n_k}$,

$$cel(\mathrm{Ad}\, U_k' \circ (\Sigma_{l=n_k}^{n_{k+1}-1} h_l)(u)\lceil \mathrm{Ad}\, U_k' \circ (\Sigma_{l=n_k}^{n_{k+1}-1} \psi_l)(u)\rceil^*) \le 7\pi + 1, \tag{10.8}$$

where the length is computed inside $M_{m(n_k)}(\mathrm{Her}(b_{K_k,0}))$.

For each k, consider the two maps $\mathrm{Ad}\, U_k' \circ (\Sigma_{l=n_k}^{n_{k+1}-1} \psi_l)$, $\mathrm{Ad} U_k' \circ (\Sigma_{l=n_k}^{n_{k+1}-1} h_l) : C \to \mathrm{Her}(b_{K_k,0})$.

Recall that ψ_n is $\mathcal{G}_n$-δ_n-multiplicative and $\phi_n \circ \sigma_C$ is F-full, for all n. Also, keeping in mind of (10.6) and (10.8), we apply Theorem 5.8 to get that for each k, there is a unitary $u_k' \in M_{K_k+1}(\widetilde{\mathrm{Her}(b_{K_k,0})})$ such that

$$u_k' \left(U_k'^* \sum_{l=n_k}^{n_{k+1}-1} h_l(c) U_k' \bigoplus \sum_{l=1}^{K_k} \sigma_{K_k,l}(c) \right) (u_k')^* \approx_{\epsilon_k} U_k'^* \sum_{l=n_k}^{n_{k+1}-1} \psi_l(c) U_k' \bigoplus \sum_{l=1}^{K_k} \sigma_{K_k,l}(c) \tag{10.9}$$

for all $c \in \mathcal{F}_k$.

Define $H : C \to M(B)$ by $H(c) = \bigoplus_{k=1}^{\infty} h_k(c)$ for all $c \in C$. Note that the sum converges strictly and H is a homomorphism. Set $\sigma_d := \pi \circ H$.

By exactly the same argument as in the later part of the proof of Lemma 7.12, from (10.9), we obtain a unitary $u \in \mathcal{C}(B)$ such that

$$u(\sigma_d(c) \oplus \pi \circ \sigma(c))u^* = \pi \circ \psi(c) + \pi \circ \sigma(c) \text{ for all } c \in C. \tag{10.10}$$

Now since $\mathrm{Her}(\pi \circ \psi(e_C) + \pi \circ \sigma(e_C))^{\perp} \ne \{0\}$, and since $C(B)$ is purely infinite simple, there is a non-zero projection $e_1 \in \mathrm{Her}(\pi \circ \psi(e_C) + \pi \circ \sigma(e_C))^{\perp}$. There is a unitary $v \in e_1 \mathcal{C}(B) e_1$ such that $[v] = [u^*]$. Let $u_1 = (v \oplus (1 - e_1))u$. Replacing u by u_1 if necessary, we may assume that $u \in U_0(\mathcal{C}(B))$. Therefore, we may assume that there is a unitary $U \in M(B)$ such that $\pi(U) = u$. ■

Corollary 10.4. In Lemma 10.3, if τ_q is in fact a diagonal extension, that is, $\tau_q = \pi \circ \Psi$, where $\Psi : C \to M(B)$ is defined by $\Psi(c) = \bigoplus_{n=1}^{\infty} \psi_n(c)$ such that ψ_n is a homomorphism for all n, and if $KK(\psi_n) = 0$ and $\psi_n^{\ddagger} = 0$ for all n, then

$$\pi \circ \Psi \oplus \pi \circ \sigma \sim^u \pi \circ \sigma.$$

Proof. In the proof of Lemma 10.3, let each ψ_n be a homomorphism such that $KK(\psi_n) = 0$ and $\psi_n^{\ddagger} = 0$. Since $\psi_n^{\ddagger} = 0$, $\psi_n(u) \in CU(M_{m(n)}(\mathcal{W}))$ (instead of (10.4)) for all $u \in J^{\sim}(K_1(A)) \cap U(M_{m(n)}(\widetilde{Her(a_n)}))$. Therefore, in the proof of Lemma 10.3, (10.8) becomes

$$cel(\sum_{l=n_k}^{n_{k+1}-1} \psi_l(u)) \leq 7\pi + 1 \text{ for all } u \in \mathcal{U}_{n_k}. \tag{10.11}$$

Therefore, the proof works when we use $h_n = 0$ for all n. In other words,

$$\pi \circ \sigma \sim^u \pi \circ \Psi \oplus \pi \circ \sigma.$$

■

Lemma 10.5. Let C and B be as in 10.3. Fix two sequences

$$\{x_n\} \subset KL(C, B) \text{ and } \{y_n\} \subset \mathrm{Hom}(K_1(C), U(B)/CU(B)) \tag{10.12}$$

such that x_n and y_n are compatible, that is, $x_n(z) = \Pi_{1,cu}(y_n(z))$ for all $z \in K_1(C)$, for all n. Let $\{b_n\}$ be a system of quasidiagonal units for B. Then there is a diagonal monomorphism $h_d := \bigoplus_{n=1}^{\infty} h_n : C \to M(B)$, where $h_n : A \to Her(b_n)$ is a monomorphism for all n, and for each m, $KK(h_m) = x_n$ and $h_m^{\ddagger} = y_n$ at the same time for some n, and for each k, there are infinitely many l such that $KK(h_l) = x_k$ and $h_l^{\ddagger} = y_k$ at the same time.

Proof. Write $\mathbb{N} = \cup_{n=1}^{\infty} S_n$, where each S_n is an infinite countable set, and $S_i \cap S_j = \emptyset$ if $i \neq j$. For each $j \in S_n$, choose a monomorphism $h_j : C \to Her(b_j)$ such that $KK(h_j) = x_n$ and $h_j^{\ddagger} = y_n$ (see Theorems 6.10 and 6.11). Then set $h_d := \bigoplus_{k \in \mathbb{N}} h_k$. One can check that h_d satisfies the requirements of the lemma. ■

Lemma 10.6. Let C, B and σ be as in 10.3. For any essential trivial diagonal extension $\tau : C \to \mathcal{C}(B)$,

$$\tau \oplus \pi \circ \sigma \sim^u \pi \circ \sigma.$$

Proof. Let $\Psi = \bigoplus_{n=1}^{\infty} \psi_n : C \to M(B)$ be any diagonal map, where $\psi_n : C \to \overline{b_n B b_n}$ is a homomorphism for all n, and where $\{b_k\}$ is a system of quasidiagonal units. Let $x_{2n-1} = KK(\psi_n)$, $y_{2n-1} = \psi_n^{\ddagger}$, $x_{2n} = -KK(\psi_n)$ and $y_{2n} = -\psi_n^{\ddagger}$, $n = 1, 2, \ldots$. Note that x_n and y_n are compatible for all n.

Let $h_d : C \to M(B)$ be as in Lemma 10.5 associated with the sequences $\{x_n\}$, $\{y_n\}$ and $\{b_k\}$. We claim that there is a permutation $\lambda : \mathbb{N} \to \mathbb{N}$ such that

$$KK(h_{\lambda(2n-1)}) = -KK(h_{\lambda(2n)}) \text{ and } h^{\ddagger}_{\lambda(2n-1)} = -h^{\ddagger}_{\lambda(2n)}, \;\; n = 1, 2, \ldots..$$

For $n = 1$, there is an integer $\gamma(1) \in \mathbb{N} \setminus \{1\}$ such that $KK(h_{\gamma(1)}) = -KK(h_1)$ and $h^{\ddagger}_{\gamma(1)} = -h^{\ddagger}_1$.

Then define $\lambda(1) = 1$ and $\lambda(2) = \gamma(1)$. Suppose that λ has been defined on $\{1, 2, \ldots, 2n\}$ such that

$$KK(h_{\lambda(2k-1)}) = -KK(h_{\lambda(2k)}) \text{ and } h^{\ddagger}_{\lambda(2k-1)} = -h^{\ddagger}_{\lambda(2k)}, \;\; k = 1, 2, \ldots, n.$$

Choose the smallest integer m such that $m \in \mathbb{N} \setminus \{\lambda(1), \lambda(2), \ldots, \lambda(2n)\}$. Define $\lambda(2n+1) = m$. Note that $(KK(h_m), h_m^{\ddagger}) \in \{(x_n, y_n)\}$. Find an integer $m' \in \mathbb{N} \setminus (\{\lambda(j) : 1 \le j \le 2n\} \cup \{m\})$ such that $KK(h_{m'}) = -KK(h_{\lambda(2n+1)})$ and $h^{\ddagger}_{m'} = -h^{\ddagger}_{\lambda(2n+1)}$. Define $\lambda(2n+2) = m'$. The claim follows by induction.

Define $a_k = b_{\lambda(2k-1)} + b_{\lambda(2k)}$, $k = 1, 2, \ldots$, Then $\{a_k\}$ is also a system of quasidiagonal units. Let $h_{n,0} : C \to Her(b_{\lambda(2n-1)} + b_{\lambda(2n)})$ be defined by $h_{n,0}(c) = h_{\lambda(2n-1)}(c) + h_{\lambda(2n)}(c)$ for all $c \in C$. Now define $H_0 : C \to M(B)$ by $H_0(c) = \bigoplus_{n=1}^{\infty} h_{n,0}(c)$ for all $c \in C$. Then H_0 is unitarily equivalent to h_d (see 4.2). However, $KK(h_{n,0}) = 0$ and $h^{\ddagger}_{n,0} = 0$ for all n. It follows from Corollary 10.4 that with σ as in 10.3,

$$\pi \circ H_0 \oplus \pi \circ \sigma \sim^u \pi \circ \sigma.$$

Therefore,

$$\pi \circ h_d \oplus \pi \circ \sigma \sim^u \pi \circ \sigma. \tag{10.13}$$

Hence, $\Psi \oplus h_d$ is another diagonal map and if we write $\Psi \oplus h_d = \bigoplus_{n=1}^{\infty} h'_n$, then for all n, $(KK(h'_n), (h'_n)^{\ddagger}) \in \{(x_l, y_l)\}$, and for each k, there are infinitely many n with $(KK(h'_n), (h'_n)^{\ddagger}) = (x_k, y_k)$. From what has been proven, we conclude that

$$(\pi \circ \Psi \oplus \pi \circ h_d) \oplus \pi \circ \sigma \sim^u \pi \circ \sigma. \tag{10.14}$$

Then by (10.13),

$$\pi \circ \Psi \oplus \pi \circ \sigma \sim^u \pi \circ \Psi \oplus (\pi \circ h_d \oplus \pi \circ \sigma) \sim^u \pi \circ \sigma.$$

■

Theorem 10.7. Let $B \in \mathcal{I}$. Let A be a separable amenable C^*-algebra, which is $\mathcal{W}$ embeddable and satisfies the UCT. Suppose that $K_i(A)$ is finitely generated $(i = 0, 1)$.

(1) If $\tau_1, \tau_2 : A \to \mathcal{C}(B)$ are two essential extensions, then $\tau_1 \sim^u \tau_2$ if and only if $KK(\tau_1) = KK(\tau_2)$.

(2) The map $\Lambda : \mathbf{Ext}^u(A, B) \to KK(A, \mathcal{C}(B))$, defined by $\Lambda([\tau]) = KK(\tau)$, is a group isomorphism.

(3) An essential extension τ, of A by B, is trivial and diagonal if and only if $KK(\tau) = 0$, and all essential trivial and diagonal extensions of A by B are unitarily equivalent.

(4) All quasidiagonal essential extensions of A by B are trivial and are unitarily equivalent.

(5) If $\ker\rho_{f,A} = K_0(A)$, then all trivial essential extensions of A by B are unitarily equivalent. Moreover, an essential extension τ, of A by B, is trivial if and only if $KK(\tau) = 0$.

(6) If $\ker\rho_{f,A} \neq K_0(A)$, then there are essential trivial extensions of A by B, which are not quasidiagonal, and not all essential trivial extensions of A by B are unitarily equivalent.

Proof. Let us prove (3) first. Suppose that $\tau : A \to \mathcal{C}(B)$ is an essential extension such that $KK(\tau) = 0$. Consider a $\mathcal{T}_d$ extension $\pi \circ \sigma$ as in Lemma 10.2. Then $KK(\sigma) = 0$. It follows that $KK(\pi \circ \sigma) = 0$. Let $\widetilde{\pi \circ \sigma}, \tilde{\tau} : \widetilde{A} \to \mathcal{C}(B)$ be the unital extensions of $\pi \circ \sigma$ and τ, respectively. So $KK(\widetilde{\pi \circ \sigma}) = KK(\tilde{\tau})$. By Theorem 2.5 of [41], there exists a sequence of unitaries $\{v_n\}$ in $\mathcal{C}(B)$ such that

$$\lim_{n\to\infty} v_n^*(\pi \circ \sigma(a))v_n = \tau(a) \text{ for all } a \in A. \tag{10.15}$$

Since A is not unital and $\mathcal{C}(B)$ is simple purely infinite, by 3.4, we obtain a sequence of unitaries $\{u_n\}$ in $M(B)$ such that

$$\lim_{n\to\infty} \pi(u_n)^*(\pi\circ\sigma(a))\pi(u_n) = \tau(a) \text{ for all } a \in A. \tag{10.16}$$

By Theorem 4.6, τ is a quasidiagonal extension. By Lemma 10.3 and Lemma 10.6, there is an essential trivial diagonal extension σ_d such that

$$\tau \oplus \pi\circ\sigma \sim^u \sigma_d \oplus \pi\circ\sigma \sim^u \pi\circ\sigma.$$

On the other hand, by Theorem 3.8,

$$\tau \sim^u \pi\circ\sigma \oplus \tau_1$$

for some essential extension τ_1. We then compute that $KK(\tau_1) = KK(\tau) = KK(\pi\circ\sigma) = 0$. From what has just been proved,

$$\tau_1 \oplus \pi\circ\sigma \sim^u \pi\circ\sigma.$$

It follows that

$$\tau \sim^u \pi\circ\sigma. \tag{10.17}$$

This shows that if $KK(\tau) = 0$ then $\tau \sim^u \pi\circ\sigma$, and in particular, τ is trivial and diagonal.

Conversely, suppose that $\tau : A \to \mathcal{C}(B)$ is an essential trivial diagonal extension. Then, by 10.6, $\tau \oplus \pi\circ\sigma \sim^u \pi\circ\sigma$. It follows that $KK(\tau) = 0$. This proves the converse direction. The above argument also gives that every essential trivial diagonal extension is unitarily equivalent to $\pi\circ\sigma$. This proves (3).

We next prove (1) and (2). Let $\Lambda : \mathbf{Ext}^u(A,B) \to KK(A,\mathcal{C}(B))$ be the map defined by $\Lambda([\tau]) = KK(\tau)$. It is a semigroup homomorphism.

Towards proving that Λ is surjective, let $x \in KK(A,\mathcal{C}(B))$. Note that $KK(\widetilde{A},\mathcal{C}(B)) = KK(A,\mathcal{C}(B)) \oplus KK(\mathbb{C},\mathcal{C}(B)) = KK(A,\mathcal{C}(B)) \oplus K_0(\mathcal{C}(B))$. Let $y := x \oplus [1_{\mathcal{C}(B)}] \in KK(A,\mathcal{C}(B)) \oplus K_0(\mathcal{C}(B))$. By Corollary 8.5 of [41], there exists a monomorphism $\phi : \widetilde{A} \to \mathcal{C}(B)$ such that $KK(\phi) = y$. Let $\phi_0 := \phi|_A : A \to \mathcal{C}(B)$. Then $KK(\phi_0) = x$. This shows that Λ is surjective.

Fix an essential extension $\tau : A \to \mathcal{C}(B)$. Since Λ is surjective, there exists an essential extension τ_{-1} such that $KK(\tau_{-1}) = -KK(\tau)$. Then $KK(\tau \oplus \tau_{-1}) = 0$. By part (3), with σ as in 10.2,

$$\tau \oplus \tau_{-1} \sim^u \pi \circ \sigma. \tag{10.18}$$

Let $\tau_1 : A \to \mathcal{C}(B)$ be any essential extension with $KK(\tau_1) = KK(\tau)$. Then the same argument gives that $\tau_1 \oplus \tau_{-1} \sim^u \pi \circ \sigma$. Then,

$$\tau \oplus \pi \circ \sigma \sim^u \tau \oplus (\tau_{-1} \oplus \tau_1) \sim^u (\tau \oplus \tau_{-1}) \oplus \tau_1 \sim^u \pi \circ \sigma \oplus \tau_1. \tag{10.19}$$

On the other hand, by Theorem 3.8, for some essential extension τ',

$$\tau \sim^u \pi \circ \sigma \oplus \tau'. \tag{10.20}$$

Since $KK(\pi \circ \sigma) = 0$, $KK(\tau') = KK(\tau)$. Therefore, replacing τ_1 with τ' in (10.19), we get that

$$\tau \sim^u \pi \circ \sigma \oplus \tau' \sim^u \pi \circ \sigma \oplus \tau. \tag{10.21}$$

Hence, replacing τ with τ_1 in (10.21), $\tau_1 \sim^u \pi \circ \sigma \oplus \tau_1$. Hence, by (10.19),

$$\tau \sim^u \tau_1. \tag{10.22}$$

This implies that Λ is one-to-one. Since $KK(A, \mathcal{C}(B))$ is a group, this implies that $\mathbf{Ext}^u(A, B)$ is a group with zero $[\pi \circ \sigma]$. Moreover, Λ is a group isomorphism. This proves (1) and (2).

To see that (4) holds, let τ_q be an essential quasidiagonal extension. Then, by Lemma 10.3, with σ as in 10.2,

$$\tau_q \oplus \pi \circ \sigma \sim^u \tau_d \oplus \pi \circ \sigma$$

for some trivial diagonal essential extension τ_d. From this and (3), $KK(\tau_q) = KK(\tau_d) = 0$. It follows from (3) that $\tau_q \sim^u \pi \circ \sigma$. Thus, (4) holds.

To see (5), consider a trivial essential extension with the form $\tau = \pi \circ H$ for some monomorphism $H : A \to M(B)$. Recall that $K_1(M(B)) = 0$ and $K_0(M(B)) = Aff(T(B)) = \mathbb{R}$. Since we now assume that $K_0(A) = \ker \rho_{f,A}$, $H_{*0} = 0$. Hence, $KK(\tau) = 0$. Then (5) follows from (3).

Finally, for (6), we note that, if $\ker\rho_{f,A} \neq K_0(A)$, then there is a $t \in T_f(A)$ such that $\rho_A(t) \neq 0$.

By Lemma 8.1, for any $r \in (0,1)$, there is an monomorphism $\psi_{A,r} : A \to M(B)$, with $\pi \circ \psi_{A,r}$ being injective, such that $t_B \circ \psi_{A,r}(a) = r \cdot t(a)$ for all $a \in A$. Recall that $K_0(M(B)) = \mathrm{Aff}(T(B)) = \mathbb{R}$ and $K_0(\mathcal{C}(B)) = \mathbb{R} \oplus K_1(B)$. Hence, if we let $\lambda : K_0(A) \to \mathbb{R} : z \mapsto t(z)$, then $K_0(\psi_{A,r}) = r\lambda \neq 0$. Hence, $K_0(\pi \circ \psi_{A,r}) = r\lambda \neq 0$. Then $\pi \circ \psi_{A,r} : A \to \mathcal{C}(B)$ is an essential trivial extension such that

$$KK(\pi \circ \psi_{A,r}) \neq 0. \tag{10.23}$$

Thus, we produce an essential trivial extension that is not unitarily equivalent to the trivial diagonal extension $\pi \circ \sigma$ (since $KK(\pi \circ \sigma) = 0$ by (3)). By (4), it is not quasidiagonal. In fact, if $r_1, r_2 \in (0,1)$ and $r_1 \neq r_2$, then $\pi \circ \psi_{A,r_1}$ is not unitarily equivalent to $\pi \circ \psi_{A,r_2}$. ■

Remark 10.8. Note that Theorem 10.7 does not describe exactly what the set $\mathcal{T}$, that is, the set of unitary equivalence classes of trivial essential extensions, looks like. The next statement will do that.

Recall that $KK(A, M(B)) = \mathrm{Hom}(K_0(A), \mathbb{R})$. So we may view $\mathrm{Hom}(K_0(A), \mathbb{R})_{T_f(A)}$ as a subset of $KK(A, M(B))$. Let $[\pi] : KK(A, M(B)) \to KK(A, \mathcal{C}(B))$ be the homomorphism induced by the quotient map $\pi : M(B) \to \mathcal{C}(B)$. Define

$$N = [\pi](\{r \cdot h : r \in (0,1], h \in \mathrm{Hom}(K_0(A), \mathbb{R})_{T_f(A)}\}).$$

Theorem 10.9. Let $B \in \mathcal{I}$ and let A be a separable amenable C^*-algebra, which is $\mathcal{W}$ embeddable and satisfies the UCT. Suppose that $K_i(A)$ is finitely generated ($i = 0, 1$). Then,

(i) the map $[\pi]$ is one-to-one on $KK(A, M(B)) = \mathrm{Hom}(K_0(A), \mathbb{R})$, and

(ii) an essential extension $\tau : A \to \mathcal{C}(B)$ is trivial if and only if

$$\Lambda([\tau]) = KK(\tau) \in N = [\pi](\{r \cdot h : r \in (0,1], h \in \mathrm{Hom}(K_0(A), \mathbb{R})_{T_f(A)}\}).$$

Moreover, Λ is one-to-one on $\mathcal{T}$, the set of unitary equivalence classes of trivial essential extensions of A by B.

Proof. Recall that $K_1(M(B)) = 0$.

Note also that $K_0(M(B)) = \mathbb{R}$, $K_0(\mathcal{C}(B)) = \mathbb{R} \oplus K_1(B)$, and $[\pi]$ maps injectively from $\mathrm{Hom}(K_0(A), K_0(M(B)))$ into $\mathrm{Hom}(K_0(A), K_0(\mathcal{C}(B)))$ as π_{*0} is injective from $\mathbb{R}$ into $\mathbb{R} \oplus K_1(B)$. This proves part (i).

For (ii), suppose that τ is an essential trivial extension. Then there is a monomorphism $H : A \to M(B)$ such that $\tau = \pi \circ H$. Then $t_B \circ H$ is a faithful bounded trace on A. Define $r := \|t_B \circ H\| \in (0, 1]$. Therefore, $t_B \circ H = r \cdot t$ for some $t \in T_f(A)$. It follows that $KK(H) \in r\mathrm{Hom}(K_0(A), \mathbb{R})_{T_f}$. Therefore, $\Lambda([\tau]) \in N$.

For the converse, let $\tau : A \to \mathcal{C}(B)$ be an essential extension such that $KK(\tau) \in N$. So there is a $\lambda \in \mathrm{Hom}(K_0(A), \mathbb{R})_{T_f(A)}$ such that $KK(\tau) = [\pi] \circ r{\cdot}\lambda$ for some $r \in (0, 1]$. Let $t \in T_f(A)$ be a faithful tracial state that induces λ. By Lemma 8.1, let $\psi_{A,r} : A \to M(B)$ be a monomorphism with $ran(\psi_{A,r}) \cap B = \{0\}$ so that $\tau_B \circ \psi_{A,r}(a) = rt(a)$ for all $a \in A$. Hence, $KK(\psi_{A,r}) = r\lambda$. Then $KK(\pi \circ \psi_{A,r}) = KK(\tau)$. By part (2) of Theorem 10.7, $\tau \sim^u \pi \circ \psi_{A,r}$.

The last statement also follows from part (2) of Theorem 10.7. ∎

Remark 10.10. Theorem 10.9 uses N to describe the trivial essential extensions under the assumptions of this section (see also Theorem 10.7). When $\ker \rho_{f,A} \neq K_0(A)$, $N \neq \{0\}$. In fact, there are uncountably many different elements in N. Moreover, $\mathcal{T}$ is not a semigroup. One first notes that, for any $\lambda \in \mathrm{Hom}(K_0(A), \mathbb{R})_{T_f(A)}$ and $r \in (1, \infty)$, if τ is an essential extension with $KK(\tau) = [\pi] \circ (r \cdot \lambda)$, then τ is not a trivial (or splitting) extension, since there is no homomorphism $H : A \to M(B)$ such that $H_{*0} = r \cdot \lambda$. Suppose that $\lambda \in \mathrm{Hom}(K_0(A), \mathbb{R})_{T_f(A)}$ and $\tau \in \mathcal{T}$ such that $\Lambda(\tau) = \lambda$, and suppose that $\tau_0 \in \mathcal{T}$ is another essential trivial extension. Then $\tau \dot{+} \tau_0 \in \mathcal{T}$ if and only if $\Lambda([\tau_0]) = 0$, that is, τ_0 is a trivial diagonal extension. This shows that $\mathcal{T}$ is not a semigroup.

Note that, by the UCT, there is a short exact sequence

$$0 \to \mathrm{ext}_{\mathbb{Z}}(K_*(A), K_{*-1}(B)) \to KK(A, \mathcal{C}(B)) \to \mathrm{Hom}(K_*(A), K_*(\mathcal{C}(B))) \to 0. \tag{10.24}$$

Suppose that τ is an essential extension with $\tau_{*1} = 0$ and $\tau_{*0} \in \mathrm{Hom}(K_0(A), \mathbb{R})_{T_f(A)}$. One realizes that $KK(\tau)$ may not be in N.

Funding

The first named author is partially supported by a NSF grant (DMS-1665183 and DMS-1954600). Both authors would like to acknowledge the support during their visits to the Research Center of Operator Algebras at East China Normal University, which is partially supported by Shanghai Key Laboratory of PMMP, Science and Technology Commission of Shanghai Municipality (STCSM) grant #13dz2260400 and a NNSF grant (11531003).

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