

Tracial approximate divisibility and stable rank one

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Abstract

In this paper, we show that every separable simple tracially approximately divisible C^* -algebra has strict comparison, and it is either purely infinite or has stable rank one. As a consequence, we show that every (non-unital) finite simple $\mathcal{Z}$ -stable C^* -algebra has stable rank one.

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1 | INTRODUCTION

Approximate divisibility for C^* -algebras was introduced in [5] in the study of noncommutative tori following the earlier work of M. Rørdam (see [37] and [38]). It is shown in Theorem 1.4 of [5] that a unital separable simple C^* -algebra A which is approximately divisible has strict comparison, and is either purely infinite or has stable rank one.

Tracial approximation was introduced in the Elliott program of classification for simple C^* -algebras at the end of last century (see, e.g., [24, 25] and [26]). The term of tracially approximate divisibility appeared at the same time as the study of simple C^* -algebras of tracial rank one (see Definition 5.3 and the proof of Theorem 5.4 of [28]). It was shown, for example, that every unital non-elementary simple C^* -algebra with tracial rank at most one is tracially approximately divisible. A more general version of tracially approximate divisibility was given in [16, Definition 5.2]. Similar variations of tracially approximate divisibility also occurred (see Definition 4.4 below, and also in [12, Definition 10.1]). A concept with the same nature was also given in [18] which was called tracially $\mathcal{Z}$ -absorbing (see also [9] and [1]). As a continuation of [16] (and also of [15]), we first show that these notions of tracially approximate divisibility are all equivalent for (not necessarily unital) non-elementary separable simple C^* -algebras (see Theorem 4.11).

With the same spirit of [5], we also show that a separable simple C^* -algebra which is tracially approximately divisible has strict comparison, and is either purely infinite or has stable rank one (see Theorem 5.2 and Corollary 6.5). Moreover, we show that if A is a non-elementary separable simple C^* -algebra which is tracially approximately divisible, then its Cuntz semigroup can be written as $\text{Cu}(A) = (V(A) \setminus \{0\}) \sqcup \text{LAff}_+(\widetilde{QT}(A))$ (see Theorem 5.7 and Remark 5.8).

We would like to point out that the Jiang-Su algebra $\mathcal{Z}$ is not approximately divisible since it has no non-zero projection other than the unit [20]. However, using [16, Theorem 5.9], every simple C^* -algebra which can be essentially tracially approximated by separable $\mathcal{Z}$ -stable C^* -algebras (see [16, Definition 3.1] and Definition 6.6 below) is tracially approximately divisible. In particular, simple $\mathcal{Z}$ -stable C^* -algebras are tracially approximately divisible. In Example 7.1, we observe that there are a whole set of non-nuclear separable simple C^* -algebras which are tracially approximately divisible. Since every unital simple C^* -algebra which has tracial rank zero is tracially approximately divisible, there exist tracially approximately divisible C^* -algebras which are not $\mathcal{Z}$ -stable by [31].

Gong, Jiang and Su showed in [17] that a unital simple $\mathcal{Z}$ -stable C^* -algebra A , that is, $A \cong A \otimes \mathcal{Z}$, is either purely infinite, or is stably finite, and has weakly unperforated $K_0(A)$. In [39], Rørdam showed that a unital simple $\mathcal{Z}$ -stable C^* -algebra A is either purely infinite, or has stable rank one, and has almost unperforated Cuntz semigroup. If A is a separable simple $\mathcal{Z}$ -stable C^* -algebra and contains a non-zero projection p , then pAp is also $\mathcal{Z}$ -stable [43, Corollary 3.1]. One then quickly concludes that A has stable rank one if it is finite. In [36], L. Robert showed that every stably projectionless simple C^* -algebra A which is $\mathcal{Z}$ -stable has almost stable rank one. It left open whether a stably projectionless simple $\mathcal{Z}$ -stable C^* -algebra has stable rank one (see [36, Question 3.5]). As a by-product, we show that every finite simple $\mathcal{Z}$ -stable C^* -algebra always has stable rank one (Corollary 6.8). In particular, we answer Robert's question affirmatively. Some applications and examples to dynamical systems can be found at the end of this paper. We also refer the reader to the recent papers [41] and [2] for further related results about C^* -algebras of stable rank one.

The paper is organized as follows. Section 2 is a preliminary. Section 3 discusses the so-called Cuntz-null sequences. In Section 4, we discuss several variations of tracial approximate divisibility and we show in Theorem 4.11 that they are actually all equivalent. In Section 5, we show that a separable simple tracially approximately divisible C^* -algebra has strict comparison (see Theorem 5.2). Moreover, we also show that the canonical map from the purely non-compact elements in Cuntz semigroup $\text{Cu}(A)$ to the set of strictly positive lower semi-continuous affine functions in $\text{LAff}_+(\widetilde{QT}(A))$ is an order-isomorphism (see Theorem 5.7). In Section 6, we show that a separable simple tracially approximately divisible C^* -algebra is either purely infinite, or has stable rank one (see Corollary 6.5). We end Section 6 by showing that every (non-unital) simple $\mathcal{Z}$ -stable

C^* -algebra is either purely infinite or has stable rank one (see Corollary 6.8). Finally, we include some examples in Section 7.

2 | PRELIMINARY

In this paper, the set of all positive integers is denoted by $\mathbb{N}$. The set of all compact operators on a separable infinite-dimensional Hilbert space is denoted by $\mathcal{K}$.

Notation 2.1. Let A be a normed space and $F \subset A$ be a subset. Let $\epsilon > 0$. Let $a, b \in A$, we write $a \approx_\epsilon b$ if $\|a - b\| < \epsilon$. We write $a \in_\epsilon F$, if there is $x \in F$ such that $a \approx_\epsilon x$.

Notation 2.2. Let A be a C^* -algebra and let $S \subset A$ be a subset of A . Denote by $\text{Her}_A(S)$ (or just $\text{Her}(S)$, when A is clear) the hereditary C^* -subalgebra of A generated by S . Denote by A^1 the closed unit ball of A , by A_+ the set of all positive elements in A . Put $A_+^1 := A_+ \cap A^1$. Denote by $\tilde{A}$ the minimal unitization of A . When A is unital, denote by $GL(A)$ the group of invertible elements of A , and denote by $U(A)$ the unitary group of A . Let $\text{Ped}(A)$ denote the Pedersen ideal of A and $\text{Ped}(A)_+ := \text{Ped}(A) \cap A_+$. Denote by $T(A)$ the tracial state space of A .

Definition 2.3. Let A and B be C^* -algebras and $\varphi : A \rightarrow B$ be a linear map. The map φ is positive, if $\varphi(A_+) \subset B_+$. The map φ is completely positive contractive, abbreviated as c.p.c., if $\|\varphi\| \leq 1$ and $\varphi \otimes \text{id} : A \otimes M_n \rightarrow B \otimes M_n$ are positive for all $n \in \mathbb{N}$. A c.p.c. map $\varphi : A \rightarrow B$ is said to have order zero, if for any $x, y \in A_+, xy = 0$ implies $\varphi(x)\varphi(y) = 0$.

In what follows, $\{e_{i,j}\}_{i,j=1}^n$ (or just $\{e_{i,j}\}$, if there is no confusion) is a system of matrix unit for M_n , and, $\iota \in C_0((0,1])$ is the identity function on $(0,1]$, i.e., $\iota(t) = t$ for all $t \in (0,1]$.

Definition 2.4. A C^* -algebra A is said to have stable rank one [34], if $\tilde{A} = \overline{GL(\tilde{A})}$, that is, $GL(\tilde{A})$ is dense in $\tilde{A}$. A C^* -algebra A is said to have almost stable rank one [36], if for any hereditary C^* -subalgebra $B \subset A$, $B \subset \overline{GL(B)}$.

Notation 2.5. Let $\epsilon > 0$. Define a continuous function $f_\epsilon : [0, +\infty) \rightarrow [0, 1]$ by

$$f_\epsilon(t) = \begin{cases} 0 & t \in [0, \epsilon/2], \\ 1 & t \in [\epsilon, \infty), \\ \text{linear} & t \in [\epsilon/2, \epsilon]. \end{cases}$$

Definition 2.6. Let A be a C^* -algebra and let $M_\infty(A)_+ := \bigcup_{n \in \mathbb{N}} M_n(A)_+$. For $x \in M_n(A)$, we identify x with $\text{diag}(x, 0) \in M_{n+m}(A)$ for all $m \in \mathbb{N}$. Let $a \in M_n(A)_+$ and $b \in M_m(A)_+$. We may write $a \oplus b := \text{diag}(a, b) \in M_{n+m}(A)_+$. If $a, b \in M_n(A)$, we write $a \lesssim b$ if there are $x_i \in M_n(A)$ such that $\lim_{i \rightarrow \infty} \|a - x_i^* b x_i\| = 0$. We write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$ hold. The Cuntz relation $\sim$ is an equivalence relation. Set $W(A) := M_\infty(A)_+ / \sim$. Let $\langle a \rangle$ denote the equivalence class of a . We write $\langle a \rangle \leq \langle b \rangle$ if $a \lesssim b$. $(W(A), \leq)$ is a partially ordered abelian semigroup. Let $\text{Cu}(A) = W(A \otimes \mathcal{K})$. $W(A)$ (resp. $\text{Cu}(A)$) is called almost unperforated, if for any $\langle a \rangle, \langle b \rangle \in W(A)$ (resp. $\text{Cu}(A)$), and for any $k \in \mathbb{N}$, if $(k+1)\langle a \rangle \leq k\langle b \rangle$, then $\langle a \rangle \leq \langle b \rangle$ (see [38]). Denote by $V(A)$ the subset of those elements in $W(A)$ represented by projections.

Remark 2.7. It is known to experts that $W(A)$ is almost unperforated is equivalent to say that $\text{Cu}(A)$ is almost unperforated. To see this briefly, let $a, b \in (A \otimes \mathcal{K})_+$ such that $(k+1)\langle a \rangle \leq k\langle b \rangle$. Let $\{e_{i,j}\}$ be the system of matrix units for $\mathcal{K}$ and $E_n = \sum_{i=1}^n 1_{\bar{A}} \otimes e_{i,i}$ and let $\varepsilon > 0$. Note that $E_n a E_n \in M_n(A)_+$ for all $n \in \mathbb{N}$. Moreover, $a \approx_{\varepsilon/8} E_n a E_n$ for some large $n \in \mathbb{N}$. It follows from [38, Proposition 2.2] that $(a - \varepsilon)_+ \lesssim (E_n a E_n - \varepsilon/4)_+$ and $(E_n a E_n - \varepsilon/4)_+ \lesssim (a - \varepsilon/8)_+$. By [38, Proposition 2.4], there exists $\delta > 0$ such that $(k+1)\langle (a - \varepsilon/8)_+ \rangle \leq k\langle (b - \delta)_+ \rangle$. Repeating Rørdam's results ([38]), one obtains that $\langle (b - \delta)_+ \rangle \leq \langle E_m b E_m \rangle$ for some even larger m . Now one has $(k+1)\langle (E_n a E_n - \varepsilon/4)_+ \rangle \leq k\langle E_m b E_m \rangle$. Since $W(A)$ is almost unperforated, $(a - \varepsilon)_+ \lesssim (E_n a E_n - \varepsilon/4)_+ \lesssim E_m b E_m$. Then, $(a - \varepsilon)_+ \lesssim E_m b E_m \lesssim b$. It follows that $a \lesssim b$. Therefore, $W(A)$ is almost unperforated implies $\text{Cu}(A)$ is almost unperforated.

To see the converse, just notice that A is a hereditary C^* -subalgebra of $A \otimes \mathcal{K}$ and $\langle a \rangle \leq \langle b \rangle$ in $\text{Cu}(A) = W(A \otimes \mathcal{K})$ implies $\langle a \rangle \leq \langle b \rangle$ in $W(A)$.

Definition 2.8. Let A be a C^* -algebra. A densely defined 2-quasi-trace is a 2-quasi-trace defined on $\text{Ped}(A)$ (see [4, Definition II.1.1]). Denote by $\widetilde{QT}(A)$ the set of densely defined 2-quasi-traces on $A \otimes \mathcal{K}$. In what follows we will identify A with $A \otimes e_{1,1}$, whenever it is convenient. Let $\tau \in \widetilde{QT}(A)$. Note $\tau(a) \neq \infty$ for any $a \in \text{Ped}(A)_+ \setminus \{0\}$.

Note, for each $a \in (A \otimes \mathcal{K})_+$ and $\varepsilon > 0$, $f_\varepsilon(a) \in \text{Ped}(A \otimes \mathcal{K})_+$. Define

$$d_\tau(a) = \lim_{\varepsilon \rightarrow 0} \tau(f_\varepsilon(a)) \text{ for all } \tau \in \widetilde{QT}(A). \quad (\text{e2.1})$$

A simple C^* -algebra A is said to have (Blackadar's) strict comparison, if, for any $a, b \in (A \otimes \mathcal{K})_+$, one has $a \lesssim b$, if

$$d_\tau(a) < d_\tau(b) \text{ for all } \tau \in \widetilde{QT}(A) \setminus \{0\}. \quad (\text{e2.2})$$

Let A be a simple C^* -algebra. By [39, Proposition 3.2] (and [13, Proposition 6.2]), if $\text{Cu}(A)$ is almost unperforated then A has strict comparison (see also Proposition 4.2 of [13]).

We endow $\widetilde{QT}(A)$ with the topology in which a net $\{\tau_i\}$ converges to τ if $\{\tau_i(a)\}$ converges to $\tau(a)$ for all $a \in \text{Ped}(A)$ (see also (4.1) on page 985 of [13]).

Let A be a simple C^* -algebra. Note that, if τ is a lower semicontinuous quasitrace on $A \otimes \mathcal{K}$ defined in [13], and if $\tau(a) < \infty$ for some $a \in \text{Ped}(A)_+ \setminus \{0\}$, then $\tau(c) \in \mathbb{C}$ for all $c \in \text{Ped}(A)$. In other words, $\tau \in \widetilde{QT}(A)$. If $\tau(a) = \infty$ for some $a \in \text{Ped}(A)_+ \setminus \{0\}$, then, in this case, $\tau(c) = \infty$ for all $c \in \text{Ped}(A)_+ \setminus \{0\}$. Note that we exclude the constant ∞ from $\widetilde{QT}(A)$.

Choose any $e \in \text{Ped}(A)_+ \setminus \{0\}$. Put $T_e = \{\tau \in \widetilde{QT}(A) : \tau(e) = 1\}$. Then T_e is compact (see [13, Theorem 4.4], note that $T_e = \{\tau \in QT_2(A) : \tau(e) = 1\}$ is a closed subset in the compact space $QT_2(A)$, which is used in [13]). Suppose that $\widetilde{QT}(A) \neq \{0\}$. Since A is simple, if $\tau \in \widetilde{QT}(A) \setminus \{0\}$, then $\tau(e) > 0$. Note that, for any $\tau \in \widetilde{QT}(A) \setminus \{0\}$, $\tau(\cdot)/\tau(e) \in T_e$. In other words, $\tau(a) < \tau(b)$ for all $\tau \in \widetilde{QT}(A) \setminus \{0\}$ if and only if $\tau(a) < \tau(b)$ for all $\tau \in T_e$.

Definition 2.9. Let $\text{Aff}_+(\widetilde{QT}(A))$ be the set of all continuous affine functions f on $\widetilde{QT}(A)$ such that $f(\tau) > 0$ for all $\tau \in \widetilde{QT}(A) \setminus \{0\}$ and $f(0) = 0$, and, the zero function.

Let $\text{LAff}_+(\widetilde{QT}(A))$ be the set of those lower semi-continuous affine functions $f : \widetilde{QT}(A) \rightarrow [0, \infty]$ such that there exists an increasing sequence of functions $f_n \in \text{Aff}_+(\widetilde{QT}(A))$ such that $f(\tau) = \lim_{n \rightarrow \infty} f_n(\tau)$ for all $\tau \in \widetilde{QT}(A)$.

The canonical map from $\text{Cu}(A)$ to $\text{LAff}_+(\widetilde{QT}(A))$ is defined as follows: for each $a \in (A \otimes \mathcal{K})_+$, the map $\langle a \rangle \rightarrow \langle \widehat{a} \rangle$ is defined by $\langle \widehat{a} \rangle(\tau) = d_\tau(a)$ for all $\tau \in \widetilde{QT}(A)$.

3 | CUNTZ-NULL SEQUENCES

In this section, we will introduce the notion of Cuntz-null sequence and discuss the related properties of the (closed two-sided) ideal consisting of Cuntz-null sequences.

Definition 3.1. Let A be a C^* -algebra. A bounded sequence $\{a_n\}$ in A is said to be Cuntz-null, if for any $a \in A_+ \setminus \{0\}$ and any $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $f_\varepsilon(a_n^* a_n) \lesssim a$ for all $n \geq n_0$.

Let $l^\infty(A)$ be the C^* -algebra of bounded sequences of A . Recall that $c_0(A) := \{\{a_n\} \in l^\infty(A) : \lim_{n \rightarrow \infty} \|a_n\| = 0\}$ is a (closed) two-sided ideal of $l^\infty(A)$. Let $A_\infty := l^\infty(A)/c_0(A)$. Let $\pi_\infty : l^\infty(A) \rightarrow A_\infty$ be the quotient map. We view A as a subalgebra of $l^\infty(A)$ via the canonical map $a \mapsto \{a, a, \dots\}$ for all $a \in A$. In what follows, we will identify a with the constant sequence $\{a, a, \dots\}$ in $l^\infty(A)$ without further warning. Denote by $N_{cu}(A)$ (or just N_{cu}) the set of all Cuntz-null sequences in $l^\infty(A)$. Let us also write that

$$\pi_\infty^{-1}(A') = \{\{x_n\} \in l^\infty(A) : \lim_{n \rightarrow \infty} \|x_n a - a x_n\| = 0 \text{ for all } a \in A\}. \quad (\text{e3.1})$$

Remark 3.2. For a free (ultra)filter ω on $\mathbb{N}$, we may similarly define ω -Cuntz-null sequences as follows: the set of those $\{a_n\} \in l^\infty(A)$ such that, for any $a \in A_+ \setminus \{0\}$ and any $\varepsilon > 0$, there is $W \in \omega$ satisfying $f_\varepsilon(a_n^* a_n) \lesssim a$ for all $n \in W$. Similar results in this section also works for ω -Cuntz-null sequences. But we will not explore this further in this paper.

Proposition 3.3. Let A be a C^* -algebra and $B \subset A$ be a C^* -subalgebra. Let

$$I := \{a \in A : a^* a \lesssim b \text{ for all } b \in B_+ \setminus \{0\}\}.$$

If B has no one-dimensional hereditary C^* -subalgebras, then I is a closed two-sided ideal of A .

Proof. First, let us show that I is a $*$ -invariant linear space. To see this, let $a \in I$. If $\lambda \in \mathbb{C}$ and $b \in B_+ \setminus \{0\}$, then $|\lambda|^2 a^* a \lesssim a^* a \lesssim b$. Thus $\lambda a \in I$. Since $a^* a \sim aa^*$, we also have $a^* \in I$. Now let $a_1, a_2 \in I$ and $b \in B_+ \setminus \{0\}$. By our assumption, $\text{Her}_B(b) \not\subset \mathbb{C}$. Then there exist $b_1, b_2 \in \text{Her}_B(b)_+ \setminus \{0\}$ such that $b_1 b_2 = 0$. By the definition of I , we have $a_i^* a_i \lesssim b_i$, ($i = 1, 2$). Then $(a_1 + a_2)^*(a_1 + a_2) \leq 2(a_1^* a_1 + a_2^* a_2) \lesssim b_1 + b_2 \lesssim b$. It follows $a_1 + a_2 \in I$. Consequently, I is a $*$ -invariant linear space.

Next, let us show that I is a two-sided (algebraic) ideal. Let $a \in I$ and $x \in A$. For any $b \in B_+ \setminus \{0\}$, we have $(ax)^* ax = x^* a^* ax \lesssim a^* a \lesssim b$. Similarly, $(xa)^* xa = a^* x^* xa \lesssim a^* a \lesssim b$. Thus $ax, xa \in I$. This shows that I is a two-sided (algebraic) ideal of A .

It remains to show that I is norm closed. Let $\{a_n\} \subset I$ and $a \in A$ such that $\|a_n - a\| \rightarrow 0$ ($n \rightarrow \infty$). Let $b \in B_+ \setminus \{0\}$. For any $\varepsilon > 0$, there is $m \in \mathbb{N}$ such that $a^* a \approx_{\varepsilon/2} a_m^* a_m$. Since $a_m \in I$, we have $a_m^* a_m \lesssim b$. Thus there exist $r \in A$ such that $a_m^* a_m \approx_{\varepsilon/2} r b r^*$. Then $a^* a \approx_\varepsilon r b r^*$. Since ε is arbitrary, we have $a^* a \lesssim b$. Since $b \in B_+ \setminus \{0\}$ is arbitrary, we have $a \in I$. Thus I is norm closed as desired. $\square$

Proposition 3.4. *An element $\{a_n\}$ in $l^\infty(A)$ is a Cuntz-null sequence if and only if $\pi_\infty(\{a_n^*a_n\}) \lesssim \pi_\infty(a)$ for all $a \in A_+ \setminus \{0\}$.*

Proof. To see the “if” part, let us assume that $\{a_n\} \in l^\infty(A)$ has the property that, for any $a \in A_+ \setminus \{0\}$,

$$\pi_\infty(\{a_n^*a_n\}) = \pi_\infty(\{a_n\})^* \pi_\infty(\{a_n\}) \lesssim \pi_\infty(a). \quad (\text{e3.2})$$

For any $\varepsilon > 0$, there is $r = \pi_\infty(\{r_n\}) \in l^\infty(A)/c_0(A)$ such that $\|\pi_\infty(\{a_n^*a_n\}) - r^* \pi_\infty(a) r\| < \varepsilon/2$. Then there is $n_0 \in \mathbb{N}$ such that $\|a_n^*a_n - r_n^*ar_n\| < \varepsilon/2$ for all $n \geq n_0$. By [38, Proposition 2.2], we have $f_\varepsilon(a_n^*a_n) \lesssim r_n^*ar_n \lesssim a$ for all $n \geq n_0$. So $\{a_n\} \in N_{cu}(A)$.

Conversely, we assume that $\{a_n\} \in N_{cu}(A)$. Let $a \in A_+ \setminus \{0\}$ and $\varepsilon > 0$.

Choose $\delta > 0$ such that $f_\delta(a) \neq 0$. Since $\{a_n\}$ is a Cuntz-null sequence, there is $n_1 \in \mathbb{N}$ satisfying $f_{\varepsilon/2}(a_n^*a_n) \lesssim f_\delta(a)$ for all $n \geq n_1$. By [38, Proposition 2.4 (iv)], for each $n \geq n_0$, there is $r_n \in A$ such that $f_\varepsilon(a_n^*a_n) = r_n^*f_\delta(a)r_n$. Note that $\|f_\delta(a)^{1/2}r_n\| = \|r_n^*f_\delta(a)r_n\|^{1/2} = \|f_\varepsilon(a_n^*a_n)\|$ is bounded for all $n \geq n_1$. For $n \in \mathbb{N}$, let $s_n = f_\delta(a)^{1/2}r_n$ if $n \geq n_1$, and let $s_n = 0$ if $n < n_1$. Then $s = \{s_n\} \in l^\infty(A)$. Moreover, $f_\varepsilon(a_n^*a_n) = s_n^*f_{\delta/2}(a)s_n$ for all $n \geq n_1$. Then

$$\|f_\varepsilon(\pi_\infty(\{a_n^*a_n\})) - s^*f_{\delta/2}(\pi_\infty(a))s\| = \|\pi_\infty(f_\varepsilon(\{a_n^*a_n\}) - s^*f_{\delta/2}(\pi_\infty(a))s)\| \quad (\text{e3.3})$$

$$\leq \sup_{n \geq n_1} \|f_\varepsilon(a_n^*a_n) - s_n^*f_{\delta/2}(a)s_n\| = 0. \quad (\text{e3.4})$$

Therefore $f_\varepsilon(\pi_\infty(\{a_n^*a_n\})) \lesssim f_{\delta/2}(\pi_\infty(a)) \lesssim \pi_\infty(a)$. It follows $\pi_\infty(\{a_n^*a_n\}) \lesssim \pi_\infty(a)$ as desired. $\square$

Proposition 3.5. *If A is a C^* -algebra which has no one-dimensional hereditary C^* -subalgebras, then $N_{cu}(A)$ is a closed two-sided ideal of $l^\infty(A)$ and $c_0(A) \subset N_{cu}(A)$. If A is a non-elementary separable simple C^* -algebra, then $c_0(A) \neq N_{cu}(A)$.*

Proof. Let $J := \{x \in l^\infty(A)/c_0(A) : x^*x \lesssim a \text{ for all } a \in \pi_\infty(A)_+ \setminus \{0\}\}$. Since A has no one-dimensional hereditary C^* -subalgebra, by Proposition 3.3, J is a norm closed two-sided ideal of $l^\infty(A)/c_0(A)$. Then, by Proposition 3.4, $N_{cu}(A) = \pi_\infty^{-1}(J)$ is a norm closed two-sided ideal of $l^\infty(A)$. Moreover, $c_0(A) = \pi_\infty^{-1}(0) \subset \pi_\infty^{-1}(J) = N_{cu}(A)$.

Suppose now that A is non-elementary, separable and simple, by [15, Lemma 4.3], there exists a sequence $\{s_n\} \subset A_+$ with $\|s_n\| = 1$ such that, for any $a \in A_+ \setminus \{0\}$, there exists $n_0 \geq 1$ such that $s_n \lesssim a$ for all $n \geq n_0$. In other words, $\{s_n\} \in N_{cu}(A)$. Note that $\{s_n\} \notin c_0(A)$. $\square$

Definition 3.6. Let A be a C^* -algebra which has no one-dimensional hereditary C^* -subalgebras. Let $A_{cu} := l^\infty(A)/N_{cu}(A)$ and $\pi_{cu} : l^\infty(A) \rightarrow A_{cu}$ be the quotient map. Let $\pi_\infty(A)' := \{x \in A_\infty : xa = ax \text{ for all } a \in \pi_\infty(A)\}$. Let $\pi_\infty(A)^\perp := \{x \in A_\infty : xa = 0 = ax \text{ for all } a \in \pi_\infty(A)\}$. Let $\pi_{cu}(A)' := \{x \in A_{cu} : xa = ax \text{ for all } a \in \pi_{cu}(A)\}$. Let $\pi_{cu}(A)^\perp := \{x \in A_{cu} : xa = 0 = ax \text{ for all } a \in \pi_{cu}(A)\}$. Recall that $c_0(A) \subset N_{cu}(A)$. Denote by $\pi : A_\infty \rightarrow A_{cu}, \pi_\infty(\{a_n\}) \mapsto \pi_{cu}(\{a_n\})$ the well-defined quotient map. Moreover, $\pi : A_\infty \rightarrow A_{cu}$ induces canonical maps $\pi' : \pi_\infty(A)' \rightarrow \pi_{cu}(A)'$ and $\pi^\perp : \pi_\infty(A)^\perp \rightarrow \pi_{cu}(A)^\perp$.

Remark 3.7. It is obvious that $\pi_\infty(A)^\perp$ is a closed two-sided ideal of $\pi_\infty(A)'$, and $\pi_{cu}(A)^\perp$ is a closed two-sided ideal of $\pi_{cu}(A)'$.

Proposition 3.8. Let A be a non-elementary separable simple C^* -algebra with $\widetilde{QT}(A) \neq \{0\}$. Let $e \in \text{Ped}(A)_+ \setminus \{0\}$ and let $T_e = \{\tau \in \widetilde{QT}(A) : \tau(e) = 1\}$. Define

$$I_{T,0} = \{\{x_n\} \in l^\infty(A) : \lim_{n \rightarrow \infty} \sup\{\tau(x_n^* x_n) : \tau \in T_e\} = 0\} \text{ and} \quad (\text{e3.5})$$

$$I_T := \{\{x_n\} \in l^\infty(A) : \lim_{n \rightarrow \infty} \sup\{\tau((x_n^* x_n - \varepsilon)_+) : \tau \in T_e\} = 0 \text{ for all } \varepsilon > 0\}. \quad (\text{e3.6})$$

Suppose that A has strict comparison. Then

$$\overline{I_{T,0}} \subset I_T = N_{cu}(A). \quad (\text{e3.7})$$

Moreover, if $A = \text{Ped}(A)$, then $I_{T,0} = N_{cu}(A) = I_T$.

Proof. To see (e3.7), we first notice that $I_{T,0} \subset I_T$. Let $\{x_n\} \in I_T$. Fix $a \in A_+^1 \setminus \{0\}$. Choose $0 < \eta_1 < \|a\|/2$. Then $(a - \eta_1)_+ \in \text{Ped}(A)_+ \setminus \{0\}$. Since T_e is compact, A is simple and $\tau \mapsto \tau((a - \eta_1)_+)$ is continuous, we have

$$\sigma_0 := \inf\{\tau((a - \eta_1)_+) : \tau \in T_e\} > 0.$$

Fix any $\varepsilon \in (0, \eta_1)$. Then

$$f_{\varepsilon/2}(x_n^* x_n) \leq \left(\frac{16}{\varepsilon}\right)(x_n^* x_n - \varepsilon/8)_+ \text{ for all } n \geq 1. \quad (\text{e3.8})$$

There exists $N \geq 1$ such that, for all $n \geq N$, $\tau((x_n^* x_n - \varepsilon/8)_+) < \frac{\varepsilon \cdot \sigma_0}{16}$ for all $\tau \in T_e$. By (e3.8), we have

$$d_\tau(f_\varepsilon(x_n^* x_n)) \leq \tau(f_{\varepsilon/2}(x_n^* x_n)) < \sigma_0 \leq d_\tau(a) \text{ for all } \tau \in T_e \text{ and for all } n \geq N. \quad (\text{e3.9})$$

Since A has strict comparison (see Definition 2.8), for all $n \geq N$, $f_\varepsilon(x_n^* x_n) \lesssim a$. Thus $\{x_n\} \in N_{cu}(A)$. It follows that $I_T \subset N_{cu}(A)$. Since $N_{cu}(A)$ is closed, we conclude $\overline{I_{T,0}} \subset N_{cu}(A)$.

Now let $\{x_n\} \in N_{cu}(A)$ and $\varepsilon > 0$. We may assume that $\|\{x_n\}\| \leq 1$. Fix $\varepsilon_1 \in (0, \varepsilon)$. For any $\eta > 0$, since A is simple, infinite-dimensional, and non-elementary, $\text{Her}_A(e)$ is also simple, infinite-dimensional, and non-elementary. It follows from [15, Lemma 4.3] (see also [23, Lemma 2.4]) that there is $a_\eta \in \text{Her}_A(e)_+ \setminus \{0\}$ such that $d_\tau(a_\eta) < \eta$ for all $\tau \in T_e$. There exists an integer $N_1 \geq 1$ such that

$$(x_n^* x_n - \varepsilon_1)_+ \lesssim a_\eta \text{ for all } n \geq N_1. \quad (\text{e3.10})$$

It follows that, for all $n \geq N_1$,

$$\sup\{\tau((x_n^* x_n - \varepsilon_1)_+) : \tau \in T_e\} \leq \sup\{d_\tau((x_n^* x_n - \varepsilon_1)_+) : \tau \in T_e\} \leq \sup\{d_\tau(a_\eta) : \tau \in T_e\} \leq \eta.$$

Therefore $\lim_{n \rightarrow \infty} \sup\{\tau((x_n^* x_n - \varepsilon_1)_+) : \tau \in T_e\} = 0$. In other words, $\{x_n\} \in I_T$. This proves the first part of the proposition.

If, in addition, we assume $A = \text{Ped}(A)$, then, by [12, Proposition 5.6], there are $M(e) > 0$ and an integer $N(e) \geq 1$ such that, for any $b \in A_+^1$, there are $y_1, y_2, \dots, y_m \in A$ with $\|y_i\| \leq M(e)$ and $m \leq N(e)$ such that

$$\sum_{i=1}^m y_i^* e y_i = b. \quad (\text{e3.11})$$

Let τ be a 2-quasi-trace on $A = \text{Ped}(A)$, which extends to a 2-quasi-trace on $M_m(A)$. Let $Z := (z_{i,j})_{m \times m}$, where, for each i , $z_{i,1} = y_i$ and $z_{i,j} = 0$ for $1 < j \leq m$ and $\bar{e}_m = \text{diag}(e, e, \dots, e)$. We then estimate

$$\tau(b) = \tau\left(\sum_{i=1}^m y_i^* e y_i\right) = \tau(Z^* \bar{e}_m Z) = \tau((\bar{e}_m)^{1/2} Z Z^* (\bar{e}_m)^{1/2}) \quad (\text{e3.12})$$

$$\leq \|Z Z^*\| \tau(\bar{e}_m) = \|Z\|^2 \cdot m \tau(e) \leq N(e)^3 M(e)^2 \tau(e). \quad (\text{e3.13})$$

It follows that

$$\Delta := \sup\{\|\tau\| : \tau \in T_e\} \leq N(e)^3 M(e)^2. \quad (\text{e3.14})$$

Let $\{x_n\} \in I_T$ and let $\varepsilon, \delta > 0$. Choose $\varepsilon_1 := \varepsilon / N(e)^3 M(e)^2$. Then there is $N \in \mathbb{N}$ such that

$$\sup\{\tau((x_n^* x_n - \varepsilon_1)_+) : \tau \in T_e\} < \delta \quad \text{for all } n \geq N. \quad (\text{e3.15})$$

Let $n \geq N$ and $\tau \in T_e$. Consider the commutative C^* -subalgebra B of A generated by $x_n^* x_n$. Then τ extends to a positive linear functional on $\tilde{B}$. Then,

$$\tau(x_n^* x_n) = \tau((x_n^* x_n - \varepsilon_1)_+) - \tau((x_n^* x_n - \varepsilon_1)_-) + \tau(\varepsilon_1) \quad (\text{e3.16})$$

$$\leq \tau((x_n^* x_n - \varepsilon_1)_+) + \varepsilon_1 \|\tau\| < \delta + \varepsilon. \quad (\text{e3.17})$$

This implies $\lim_{n \rightarrow \infty} \sup\{\tau(x_n^* x_n) : \tau \in T_e\} = 0$. It follows that $\{x_n\} \in I_{T,0}$. Therefore, we have shown $I_T \subset I_{T,0}$.

In conclusion, when $A = \text{Ped}(A)$, we have $I_{T,0} = N_{cu}(A) = I_T$. $\square$

Remark 3.9. In Proposition 3.8, if $A = \text{Ped}(A)$, for any $e \in A_+ \setminus \{0\}$,

$$I_{T,0} = \{\{x_n\} \in l^\infty(A) : \lim_{n \rightarrow \infty} \sup\{\|x_n\|_{2,\tau} : \tau \in T_e\} = 0\} = N_{cu}(A), \quad (\text{e3.18})$$

where $\|x_n\|_{2,\tau} = \tau(x_n^* x_n)^{1/2}$. Note that, $I_{T,0}$ and I_T are independent of the choice of e in $\text{Ped}(A)_+ \setminus \{0\}$.

However, $N_{cu}(A) \neq I_{T,0}$ in general. To see this, let B be a unital separable simple C^* -algebra which has a nontrivial 2-quasi-trace and $A = B \otimes \mathcal{K}$. Let $\{e_{i,j}\}$ be a system of matrix units for $\mathcal{K}$.

Fix $e \in \text{Ped}(A)_+ \setminus \{0\}$. Note that $1/2 < \sum_{i=n}^{2n} (1/i) < 1$. Define $y_n = \sum_{i=n}^{2n} (1/\sqrt{i})(1_B \otimes e_{i,i}) \in A$, $n \in \mathbb{N}$. Note $y_n^* y_n \in \text{Ped}(A)_+$ and $\|y_n\| \leq 1$. For any $\varepsilon > 0$, let $m \in \mathbb{N}$ such that $1/m < \varepsilon$. Then $(y_n^* y_n - \varepsilon)_+ = 0$ for all $n > m$. Therefore $\{y_n\} \in N_{cu}(A)$. Also, $\tau(y_n^* y_n) > (1/2)\tau(1_B \otimes e_{1,1})$ for all $\tau \in T_e$ (but $\tau(y_n^* y_n) < \tau(1_B \otimes e_{1,1})$ for each n). So $\{y_n\} \notin I_{T,0}$.

Recall Definition 3.6. We have the following version of central surjectivity (c.f. [22, Proposition 4.5(iii) and Proposition 4.6], see also [30, Theorem 3.1]). Note that the following proposition is related to the so-called σ -ideal [21, Definition 1.5, Proposition 1.6].

Proposition 3.10. *For a non-elementary separable simple C^* -algebra A , the canonical maps $\pi' : \pi_\infty(A)' \rightarrow \pi_{cu}(A)'$ and $\pi^\perp : \pi_\infty(A)^\perp \rightarrow \pi_{cu}(A)^\perp$ are surjective.*

Proof. Let $\{d_n\} \subset A_+^1$ be a sequence of positive contractive elements with $\|d_n\| = 1$ for all $n \in \mathbb{N}$ such that for any $a \in A_+ \setminus \{0\}$, there exists $N \in \mathbb{N}$ satisfying $d_n \lesssim a$ for all $n \geq N$ (see [15, Lemma 4.3]). Let $x = \{x_n\} \in l^\infty(A)^1$ such that $\pi_{cu}(x) \in \pi_{cu}(A)'$ and $g = \{g_n\} \in l^\infty(A)^1$ such that $\pi_{cu}(g) \in \pi_{cu}(A)^\perp$. Let $F_1 \subset F_2 \subset \dots \subset A$ be a sequence of finite subsets with $\bigcup_m F_m = A$. Let $m \in \mathbb{N}$, then $\pi_{cu}(x)\pi_{cu}(y) - \pi_{cu}(y)\pi_{cu}(x) = 0$ and $\pi_{cu}(g)\pi_{cu}(y) = 0 = \pi_{cu}(y)\pi_{cu}(g)$ for all $y \in F_m$, that is, $xy - yx, gy, yg \in N_{cu}(A)$. By the existence of quasi-central approximate identity, there exists $e^{(m)} = \{e_n^{(m)}\}_{n=1}^\infty \in N_{cu}(A)_+^1$ such that

$$\|(1 - e^{(m)})(xy - yx)(1 - e^{(m)})\| < 1/3m \text{ and } \|e^{(m)}y - ye^{(m)}\| < 1/3m \text{ for all } y \in F_m, \quad (\text{e3.19})$$

which implies

$$\|(1 - e^{(m)})x(1 - e^{(m)})y - y(1 - e^{(m)})x(1 - e^{(m)})\| < 1/m \text{ for all } y \in F_m. \quad (\text{e3.20})$$

Similarly, we also assume that

$$\|(1 - e^{(m)})g(1 - e^{(m)})y\| < 1/m \text{ and } \|y(1 - e^{(m)})g(1 - e^{(m)})\| < 1/m \text{ for all } y \in F_m. \quad (\text{e3.21})$$

Let $z^{(m)} := x - (1 - e^{(m)})x(1 - e^{(m)}) = e^{(m)}x + xe^{(m)} - e^{(m)}xe^{(m)} \in N_{cu}(A)$ and let $\zeta^{(m)} := g - (1 - e^{(m)})g(1 - e^{(m)}) = e^{(m)}g + ge^{(m)} - e^{(m)}ge^{(m)} \in N_{cu}(A)$. Write $z^{(m)} = \{z_k^{(m)}\}_{k=1}^\infty$ and $\zeta^{(m)} = \{\zeta_k^{(m)}\}_{k=1}^\infty$. Then, for any $m \in \mathbb{N}$, there is $K(m) \in \mathbb{N}$ such that

$$f_{1/m}(z_k^{(m)*} z_k^{(m)}) \lesssim d_m \text{ and } f_{1/m}(\zeta_k^{(m)*} \zeta_k^{(m)}) \lesssim d_m \text{ for all } k \geq K(m). \quad (\text{e3.22})$$

We may assume that $K(m+1) > K(m) > 0$ for all $m \in \mathbb{N}$. For each $k \geq K(1)$, define $m_k := \max\{m \in \mathbb{N} : K(m) \leq k\} < \infty$. Note that

$$K(m_k) \leq k. \quad (\text{e3.23})$$

For $k < K(1)$, define $w_k = 0 = v_k$. For $k \geq K(1)$, define

$$w_k := z_k^{(m_k)} \text{ and } v_k := \zeta_k^{(m_k)}. \quad (\text{e3.24})$$

For any $\varepsilon > 0$ and any $a \in A_+ \setminus \{0\}$, let $r_1 \in \mathbb{N}$ such that $d_m \lesssim a$ for all $m \geq r_1$. Then, for any $k \geq K(\max\{[1/\varepsilon] + 1, r_1\})$, we have $m_k \geq \max\{1/\varepsilon, r_1\}$, and by (e3.23) and (e3.22), we have

$$f_\varepsilon(w_k^* w_k) = f_\varepsilon(z_k^{(m_k)*} z_k^{(m_k)}) \lesssim f_{1/m_k}(z_k^{(m_k)*} z_k^{(m_k)}) \lesssim d_{m_k} \lesssim a \quad \text{and} \quad (\text{e3.25})$$

$$f_\varepsilon(v_k^* v_k) = f_\varepsilon(\zeta_k^{(m_k)*} \zeta_k^{(m_k)}) \lesssim f_{1/m_k}(\zeta_k^{(m_k)*} \zeta_k^{(m_k)}) \lesssim d_{m_k} \lesssim a, \quad (\text{e3.26})$$

which shows $\{w_k\}, \{v_k\} \in N_{cu}(A)$.

Now define

$$\bar{x}_k := x_k - w_k = x_k - z_k^{(m_k)} = (1 - e_k^{(m_k)})x_k(1 - e_k^{(m_k)}) \quad \text{and} \quad (\text{e3.27})$$

$$\bar{g}_k := g_k - v_k = g_k - \zeta_k^{(m_k)} = (1 - e_k^{(m_k)})g_k(1 - e_k^{(m_k)}). \quad (\text{e3.28})$$

Since $\{w_k\}, \{v_k\} \in N_{cu}(A)$, we have

$$\begin{aligned} \pi(\pi_\infty(\{\bar{x}_k\})) - \pi_{cu}(x) &= \pi_{cu}(\{\bar{x}_k - x_k\}) = -\pi_{cu}(\{w_k\}) = 0 \quad \text{and} \\ \pi(\pi_\infty(\{\bar{g}_k\})) - \pi_{cu}(g) &= \pi_{cu}(\{\bar{g}_k - g_k\}) = -\pi_{cu}(\{v_k\}) = 0. \end{aligned} \quad (\text{e3.29})$$

Fix $r \in \mathbb{N}$ and $y \in F_r$. Let $\delta > 0$. Then, for any $k \geq K(\max\{r, [1/\delta] + 1\})$, we have $m_k \geq r, y \in F_{m_k}$, and $1/m_k \leq \delta$. In particular,

$$\lim_{k \rightarrow \infty} 1/m_k = 0. \quad (\text{e3.30})$$

By (e3.20) and (e3.21), for $y \in F_r$ and $k \geq K(\max\{r, [1/\delta] + 1\})$,

$$\|\bar{x}_k y - y \bar{x}_k\| = \|(1 - e_k^{(m_k)})x_k(1 - e_k^{(m_k)})y - y(1 - e_k^{(m_k)})x_k(1 - e_k^{(m_k)})\| \leq 1/m_k, \quad (\text{e3.31})$$

$$\|\bar{g}_k y\| = \|(1 - e_k^{(m_k)})g_k(1 - e_k^{(m_k)})y\| \leq 1/m_k, \quad \text{and} \quad (\text{e3.32})$$

$$\|y \bar{g}_k\| = \|y(1 - e_k^{(m_k)})g_k(1 - e_k^{(m_k)})\| \leq 1/m_k. \quad (\text{e3.33})$$

Combining with (e3.30), this implies that, for each $y \in F_r$,

$$\|\pi_\infty(\{\bar{x}_k\})\pi_\infty(y) - \pi_\infty(y)\pi_\infty(\{\bar{x}_k\})\| = 0 \quad \text{and} \quad (\text{e3.34})$$

$$\|\pi_\infty(\{\bar{g}_k\})\pi_\infty(y)\| = 0 = \|\pi_\infty(y)\pi_\infty(\{\bar{g}_k\})\|. \quad (\text{e3.35})$$

Since $\overline{U_r F_r} = A$, we have

$$\pi_\infty(\{\bar{x}_k\}) \in \pi_\infty(A)' \quad \text{and} \quad \pi_\infty(\{\bar{g}_k\}) \in \pi_\infty(A)^\perp. \quad (\text{e3.36})$$

Then (e3.29) and (e3.36) show that π' and $\pi^\perp$ are surjective. $\square$

4 | TRACIAL APPROXIMATE DIVISIBILITY

In this section we will discuss several versions of tracial approximate divisibility. The main result of this section is Theorem 4.11 which states that these versions are all equivalent.

Definition 4.1. Let A be a simple C^* -algebra. We say that A has property (TAD) if the following holds: for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$, any $s \in A_+ \setminus \{0\}$, and any integer $n \geq 1$, there are $\theta \in A_+^1$ and a C^* -subalgebra $D \otimes M_n \subset A$ such that

- (i) $\theta x \approx_\varepsilon x\theta$ for all $x \in \mathcal{F}$,
- (ii) $(1 - \theta)x \in_\varepsilon D \otimes 1_n$ for all $x \in \mathcal{F}$, and
- (iii) $\theta \lesssim s$.

Remark 4.2.

- (1) It is straightforward to show that if A has (TAD), we may further require that
 - (iv) $(1 - \theta)x \approx_\varepsilon (1 - \theta)^{1/2}x(1 - \theta)^{1/2} \in_\varepsilon D \otimes 1_n$ for all $x \in \mathcal{F}$, and
 - (v) $x \approx_\varepsilon \theta^{1/2}x\theta^{1/2} + (1 - \theta)^{1/2}x(1 - \theta)^{1/2}$ for all $x \in \mathcal{F}$.
- (2) It is also easy to see that if $A \neq \mathbb{C}$ has property (TAD), then for any integer $n \geq 1$, $M_n(A)$ has the property (TAD) as well.
- (3) If $A = \bigcup_{n=1}^\infty A_n$, where each A_n has the property (TAD), then A has property (TAD). To see this, let $\varepsilon > 0$, $\mathcal{F} \subset A$ be a finite subset and $s \in A_+ \setminus \{0\}$. Choose $n \geq 1$ such that $x \in \varepsilon/4 A_n$ for all $x \in \mathcal{F}$ and $a \in (A_n^1)_+$ such that $s_1 := (a - \varepsilon\|a\|/4)_+ \lesssim s$. Using the assumption that A_n has property (TAD), one concludes that A has property (TAD). As a consequence, if $A \neq \mathbb{C}$ and has property (TAD), then $A \otimes \mathcal{K}$ has property (TAD).

Next we would like to recall the definition of tracial approximate divisibility which is slightly different from [16]. One should note that if a simple C^* -algebra A is tracially approximately divisible in the sense of Definition 4.3, then A is tracially approximately divisible in the sense of [16, Definition 5.2] below. Of course these two definitions of tracial approximate divisibility are equivalent for non-elementary separable simple C^* -algebras, see Remark 4.12 (1).

Definition 4.3 [16, Definition 5.2]. Let A be a simple C^* -algebra. A is said to be tracially approximately divisible, if, for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$, any element $e_F \in A_+^1$ with $e_F x \approx_{\varepsilon/4} x \approx_{\varepsilon/4} x e_F$ for all $x \in \mathcal{F}$, any $s \in A_+ \setminus \{0\}$, and any integer $n \geq 1$, there are $\theta \in A_+^1$, a C^* -subalgebra $D \otimes M_n \subset A$ and a c.p.c. map $\beta : A \rightarrow A$ such that

- (1) $x \approx_\varepsilon x_1 + \beta(x)$ for all $x \in \mathcal{F}$, where $\|x_1\| \leq \|x\|$, $x_1 \in \text{Her}(\theta)$,
- (2) $\beta(x) \in_\varepsilon D \otimes 1_n$ and $e_F \beta(x) \approx_\varepsilon \beta(x) \approx_\varepsilon \beta(x) e_F$ for all $x \in \mathcal{F}$, and
- (3) $\theta \lesssim s$.

Suppose that A has property (TAD). Let $\varepsilon > 0$, $\mathcal{F}$ be a finite subset and $e_F \in A_+^1$ such that $e_F x \approx_{\varepsilon/4} x \approx_{\varepsilon/4} x e_F$ for all $x \in \mathcal{F}$. Let $s \in A_+ \setminus \{0\}$ and $n \in \mathbb{N}$ be given. Put $\mathcal{F}_1 = \mathcal{F} \cup \{e_F\}$. Then, by Definition 4.1 and Remark 4.2, (i), (iii), (iv) and (v) hold for $\mathcal{F}_1$ with $\varepsilon/4$ (in place of ε).

Put $x_1 = \theta^{1/2}x\theta^{1/2}$ for $x \in \mathcal{F}$ and define a c.p.c. map $\beta : A \rightarrow A$ by $\beta(y) = (1 - \theta)^{1/2}y(1 - \theta)^{1/2}$ for all $y \in A$. Then (1) and (3) in Definition 4.3 hold. Moreover, $\beta(x) \in_\varepsilon D \otimes 1_n$ for all $x \in \mathcal{F}$.

Note that, if $x \in \mathcal{F}$, then

$$e_F \beta(x) = e_F(1 - \theta)^{1/2} x (1 - \theta)^{1/2} \approx_{\varepsilon/4} e_F(1 - \theta)x \approx_{\varepsilon/4} (1 - \theta)e_F x \quad (\text{e4.1})$$

$$\approx_{\varepsilon/4} (1 - \theta)x \approx_{\varepsilon/4} (1 - \theta)^{1/2} x (1 - \theta)^{1/2} = \beta(x). \quad (\text{e4.2})$$

Similarly $\beta(x)e_F \approx_\varepsilon \beta(x)$.

In other words, we have just shown that if A has property (TAD), then A is tracially approximately divisible. This is essentially the same as [16, Proposition 5.3].

In [18] (Definition 2.1), a unital C^* -algebra A which is not $\mathbb{C}$ is called tracially $\mathcal{Z}$ -absorbing, if for any finite subset $\mathcal{F} \subset A$, any $\varepsilon > 0$, any $s \in A_+ \setminus \{0\}$, and any integer $n \geq 1$, there is an order zero c.p.c. map $\varphi : M_n \rightarrow A$ such that the following condition hold:

- (i) $\varphi(g)x \approx_\varepsilon x\varphi(g)$ for all $x \in \mathcal{F}$ and $g \in M_n^1$, and
- (ii) $1 - \varphi(1_n) \lesssim s$.

We state a non-unital variation of this notion (taken from [1], see also [14, Definition 6.6] and [9, Definition 2.1]) as follows.

Definition 4.4. (c.f. [1]) Let A be a simple C^* -algebra. We say that A has property (TAD-2) if the following holds: for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$, any $e_F \in A_+^1$ with $e_F x \approx_\varepsilon x \approx_\varepsilon x e_F$, any $s \in A_+ \setminus \{0\}$, any integer $n \geq 1$, and any finite subset $\mathcal{G} \subset C_0((0, 1]) \otimes M_n$, there is a homomorphism $\varphi : C_0((0, 1]) \otimes M_n \rightarrow A$ such that (recall that ι is the identity function on $(0, 1]$)

- (1) $\varphi(g)x \approx_\varepsilon x\varphi(g)$ for all $x \in \mathcal{F}$ and $g \in \mathcal{G}$, and
- (2) $((e_F - e_F^{1/2} \varphi(\iota \otimes 1_n) e_F^{1/2}) - \varepsilon)_+ \lesssim s$.

Remark 4.5. When A has a unit 1_A , then (2) in the above definition is equivalent to $((1_A - \varphi(\iota \otimes 1_n)) - \varepsilon)_+ \lesssim s$. Indeed, by choosing $e_F = 1_A$, (2) becomes $((1_A - \varphi(\iota \otimes 1_n)) - \varepsilon)_+ \lesssim s$. Conversely, let $1/2 > \varepsilon > 0$, $\mathcal{F}$, e_F , s , n and $\mathcal{G}$ be given. Define $\iota_1 \in C_0((0, 1])_+$ by $\iota_1(t) = 1$ if $t \in [1 - \varepsilon, 1]$, $\iota_1(0) = 0$ and $\iota_1(t)$ is linear on $[0, 1 - \varepsilon]$. Define a homomorphism $\alpha : C_0((0, 1]) \otimes M_n \rightarrow C_0((0, 1]) \otimes M_n$ by $\alpha(f(\iota) \otimes e_{i,j}) = f(\iota_1) \otimes e_{i,j}$ for all $f \in C_0((0, 1])$. Put $\mathcal{G}_1 = \{\alpha(g) : g \in \mathcal{G}\}$. Suppose that φ is as in the definition associated with $\varepsilon, \mathcal{F}$, s , n and $\mathcal{G}_1$. Define $\psi : C_0((0, 1]) \otimes M_n \rightarrow A$ by $\psi \circ \alpha$. Then, we have

- (i) $\psi(g)x = \varphi \circ \alpha(g)x \approx_\varepsilon x\psi(g)$ for all $g \in \mathcal{G}$. Moreover, (ii)

$$1_A - \psi(\iota \otimes 1_n) = 1_A - \varphi \circ \alpha(\iota \otimes 1_n) = \frac{1}{1 - \varepsilon} (1_A - \varphi(\iota \otimes 1_n) - \varepsilon)_+ \lesssim s. \quad (\text{e4.3})$$

In other words, in the case that A is unital, the property (TAD-2) is equivalent to the property of tracially $\mathcal{Z}$ -absorbing in the sense of [18, Definition 2.1].

Let $\psi' : M_n \rightarrow A$ be the c.p.c. order zero map defined by $\psi'(e_{i,j}) = \varphi(\iota \otimes e_{i,j})$ ($1 \leq i, j \leq 1$). Since the unit ball of M_n is compact, with a large $\mathcal{G}$, (1) is equivalent to that $\|\psi'(g), x\| < \varepsilon$ for all $x \in \mathcal{F}$ and $g \in M_n^1$.

The following Lemma 4.6 and Corollary 4.7 are taken from [21]. We include proofs here for the reader's convenience.

Lemma 4.6 (c.f. [21, Proposition 1.9 (3)]). *Let A be a C^* -algebra, $B \subset A$ be a subset, $B' := \{a \in A : ab = ba \text{ for all } b \in B\}$, $B^\perp := \{a \in A : ab = 0 = ba \text{ for all } b \in B\}$, then $B^\perp$ is a closed two-sided ideal of B' . Let $\pi : B' \rightarrow B'/B^\perp$ be the quotient map. Suppose that $e \in B'$ satisfies $eb = be = b$ for all $b \in B$, then $B'/B^\perp$ is unital and $\pi(e)$ is the unit.*

Proof. It is straightforward to see that $B^\perp$ is a closed two-sided ideal of B' . If $x \in B'$, then for any $b \in B$, $(ex - x)b = b(ex - x) = bex - bx = bx - bx = 0$. Also, $(xe - x)b = xeb - xb = xb - xb = 0$. Then $xe - x, ex - x \in B^\perp$. Thus $\pi(e)\pi(x) = \pi(x) = \pi(x)\pi(e)$. This completes the proof. $\square$

Corollary 4.7 (c.f. [21, Proposition 1.9 (3)]). *Let A be a σ -unital C^* -algebra without one-dimensional hereditary C^* -subalgebras. Then both $\pi_\infty(A)'/\pi_\infty(A)^\perp$ and $\pi_{cu}(A)'/\pi_{cu}(A)^\perp$ are unital.*

Proof. Let $e \in A_+^1$ be a strictly positive element. Set $e_0 = \{e^{1/n}\} \in l^\infty(A)$. Then $\pi_\infty(e_0)$ (resp. $\pi_{cu}(e_0)$) is a local unit of $\pi_\infty(A)$ (resp. $\pi_{cu}(A)$). Hence, by Lemma 4.6, the corollary holds. $\square$

Definition 4.8. Let A be a σ -unital non-elementary simple C^* -algebra. We say that A has property (TAD-3), if, for any $n \in \mathbb{N}$, there is a unital homomorphism $\varphi : M_n \rightarrow \pi_{cu}(A)'/\pi_{cu}(A)^\perp$.

Lemma 4.9. *Let A be a non-elementary separable simple C^* -algebra.*

- (1) *If A is tracially approximately divisible, then A has property (TAD-3).*
- (2) *If A has property (TAD-3), then A has the property (TAD-2).*

Proof. *Proof of (1).* Note that if A is tracially approximately divisible, then A is tracially approximately divisible in the sense of [16]. In what follows, we will show that if A is tracially approximately divisible in the sense of [16], then A has property (TAD-3).

Fix $N \in \mathbb{N}$. By [15, Lemma 4.3], there exists a sequence $\{s_n\} \subset A_+^1 \setminus \{0\}$ such that, for any $a \in A_+ \setminus \{0\}$, there exists $n_0 \geq 1$ such that $s_n \lesssim a$ for all $n \geq n_0$. Choose $0 < \varepsilon_n < \varepsilon$ such that $\sum_{n=1}^\infty \varepsilon_n < 1$ and an increasing sequence of finite subsets $\mathcal{F}_n \subset A^1$ such that $\bigcup_{n=1}^\infty \mathcal{F}_n$ is dense in A^1 .

Since A is separable, for each $n \in \mathbb{N}$, there are $e'_n, e_n \in A_+^1$ such that $e'_n x \approx_{\varepsilon_n/3} x \approx_{\varepsilon_n/3} x e'_n$ and $e_n e'_n = e'_n$ for all $x \in \mathcal{F}_n$. Define $\mathcal{F}'_n = \{e'_n x e'_n : x \in \mathcal{F}_n\}$. Note that $x \approx_{2\varepsilon_n/3} e'_n x e'_n$ and $e_n(e'_n x e'_n) = (e'_n x e'_n)e_n$, $n \in \mathbb{N}$.

If A is tracially approximately divisible in the sense of [16], then there exist a sequence of C^* -subalgebras $D_n \otimes M_N \subset A$, a sequence of c.p.c. maps $\beta_n : A \rightarrow A$, and a sequence of positive elements $\theta_n \in A^1$ satisfying the following: for any $y \in \mathcal{F}'_n$, there is $y^{(n)} \in \text{Her}(\theta_n)^1$ such that

- (i) $y \approx_{\varepsilon_n/3} y^{(n)} + \beta_n(x)$,
- (ii) $\beta_n(y) \in_{\varepsilon_n/3} D_n \otimes 1_N$, and
- (iii) $\theta_n \lesssim s_n$.

In particular, we have that $\{\theta_n\} \in N_{cu}(A)$. For each $x \in \mathcal{F}_n$, let $x^{(n)} = (e'_n x e'_n)^{(n)} \in \text{Her}(\theta)^1$ as above. Then we also have (i') $x \approx_{\varepsilon_n} x^{(n)} + \beta_n(x)$ and (ii') $\beta_n(x) \in_{\varepsilon_n} D_n \otimes 1_N$.

For convenience, we put $J = \pi_{cu}(A)^\perp$. By Lemma 4.6, J is an ideal of $\pi_{cu}(A)'$. Denote by $\pi_{cu,J} : \pi_{cu}(A)' \rightarrow \pi_{cu}(A)'/J$ the quotient map.

For each $x \in \mathcal{F}_k$, we have $\{x^{(n)}\} \in N_{cu}(A)$. It follows from (i) that, for each k and each $x \in \mathcal{F}_k$,

$$\pi_{cu}(\{\beta_n(x)\}) = \pi_{cu}(x). \quad (\text{e4.4})$$

Choose $d_n \in D_{n+}^1$ such that, for any $x \in \mathcal{F}_k$,

$$(d_n \otimes e_{i,j})\beta_n(x) \approx_{3\varepsilon_n} \beta_n(x)(d_n \otimes e_{i,j}) \text{ and} \quad (\text{e4.5})$$

$$(d_n \otimes 1_N)\beta_n(x) \approx_{3\varepsilon_n} \beta_n(x) \approx_{3\varepsilon_n} \beta_n(x)(d_n \otimes 1_N). \quad (\text{e4.6})$$

Since $\cup_{n=1}^\infty \mathcal{F}_n$ is dense in A^1 , combining this with (e4.4), we obtain that

$$\pi_{cu}(\{d_n \otimes e_{i,j}\})\pi_{cu}(x) = \pi_{cu}(x)\pi_{cu}(\{d_n \otimes e_{i,j}\}) \text{ and} \quad (\text{e4.7})$$

$$\pi_{cu}(\{d_n \otimes 1_N\})\pi_{cu}(x) = \pi_{cu}(x) = \pi_{cu}(x)\pi_{cu}(\{d_n \otimes 1_N\}) \text{ for all } x \in A. \quad (\text{e4.8})$$

In other words, $\{\pi_{cu}(\{d_n \otimes e_{i,j}\}) : 1 \leq i, j \leq N\} \subset \pi_{cu}(A)'$, and $\pi_{cu}(\{d_n \otimes 1_N\})$ is a local unit of $\pi_{cu}(A)$.

By (e4.8) and Lemma 4.6, $\pi_{cu,J}(\{d_n \otimes 1_N\})$ is the unit of $\pi_{cu}(A)'/J$. Then the map $\varphi : M_N \rightarrow \pi_{cu}(A)'/J$ defined by $\varphi(e_{i,j}) = \pi_{cu,J} \circ \pi_{cu}(\{d_n \otimes e_{i,j}\})$, $1 \leq i, j \leq N$, is a unital homomorphism. Since, for every $N \in \mathbb{N}$, there is a unital embedding $\varphi : M_N \rightarrow \pi_{cu}(A)'/\pi_{cu}(A)^\perp$, A has property (TAD-3).

Proof of (2). Let $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, any $s \in A_+ \setminus \{0\}$, and any integer $N \geq 1$ be given. Choose $e_F \in A_+^1$ such that

$$e_F x \approx_{\varepsilon/32} x e_F \approx_{\varepsilon/32} x \text{ for all } x \in \mathcal{F}. \quad (\text{e4.9})$$

Recall that $J = \pi_{cu}(A)^\perp$. Since A is separable and A has (TAD-3), there exists a unital embedding $\varphi : M_N \rightarrow \pi_{cu}(A)'/J$ such that

$$\varphi(1_N)\pi_{cu,J}(a) = \pi_{cu,J}(a) = \pi_{cu,J}(a)\varphi(1_N) \text{ for all } a \in A. \quad (\text{e4.10})$$

Define a homomorphism $\Phi : C_0((0, 1]) \otimes M_N \rightarrow \pi_{cu}(A)'/J$ by $\Phi(f \otimes e_{i,j}) = \varphi(f(1) \otimes e_{i,j})$ for all $f \in C_0((0, 1])$, $1 \leq i, j \leq N$.

By Proposition 3.10, $\pi(\pi_\infty(A)') = \pi_{cu}(A)'$. Since $C_0((0, 1]) \otimes M_N$ is projective, there is a homomorphism $\Psi : C_0((0, 1]) \otimes M_N \rightarrow \pi_\infty(A)'$ such that $\pi \circ \Psi = \Phi$. We may write $\Psi = \pi_\infty \circ \{\psi_n\}$, where $\psi_n : C_0((0, 1]) \otimes M_N \rightarrow A$ is a homomorphism for all $n \in \mathbb{N}$. Thus, for any finite subset $\mathcal{G} \subset C_0((0, 1]) \otimes M_N$, there exists $n_1 \geq 1$ such that

$$\psi_n(g)x \approx_\varepsilon x\psi_n(g) \text{ for all } x \in \mathcal{F}, \text{ all } g \in \mathcal{G} \text{ and all } n \geq n_1. \quad (\text{e4.11})$$

Put $e_n := \psi_n(\iota \otimes 1_N)$ and $\eta_n := e_F^{1/2} - e_F^{1/4} e_n e_F^{1/4}$. By (e4.10),

$$\pi_{cu}(\{\eta_n\}) = \pi_{cu}\left(\left\{e_F^{1/2}\right\} - \left\{e_F^{1/4} e_n e_F^{1/4}\right\}\right) \in J. \quad (\text{e4.12})$$

Thus, by Proposition 3.10, there are $y = \{y_n\} \in N_{cu}(A)$ and $z = \{z_n\} \in l^\infty(A)$ with $\pi_\infty(z) \in \pi_\infty(A)^\perp$, such that $\{\eta_n\} = y + z$. So $\pi_\infty(aza) = 0$ in $l^\infty(A)/c_0(A)$ for all $a \in A$. It follows

$$\pi_\infty\left(e_F - e_F^{1/2}\{\psi_n(\iota \otimes 1_N)\}e_F^{1/2}\right) = \pi_\infty\left(e_F^{1/4}\{\eta_n\}e_F^{1/4}\right) = \pi_\infty\left(e_F^{1/4}ye_F^{1/4}\right) \in \pi_\infty(N_{cu}(A)) \quad (e4.13)$$

In other words,

$$e_F - e_F^{1/2}\{\psi_n(\iota \otimes 1_N)\}e_F^{1/2} \in N_{cu}(A). \quad (e4.14)$$

It follows that there exists $n_2 \geq n_1$ such that, for all $n \geq n_2$,

$$\left(e_F - e_F^{1/2}\psi_n(\iota \otimes 1_N)e_F^{1/2} - \varepsilon\right)_+ \lesssim s. \quad (e4.15)$$

The lemma then follows (see also (e4.11)). $\square$

Proposition 4.10. *Let A be a simple C^* -algebra which has the property (TAD-2). Then A has the property (TAD).*

Proof. Fix $\varepsilon > 0$, a finite subset $F \subset A$ and an integer $n \geq 1$. We may assume that $F \subset A^1$. Choose $e_F \in A_+^1$ such that, for all $x \in F$,

$$e_F^{1/2}x \approx_{\varepsilon/32n} x \approx_{\varepsilon/32n} xe_F^{1/2} \text{ and } e_F x \approx_{\varepsilon/32n} x \approx_{\varepsilon/32n} xe_F. \quad (e4.16)$$

Put $F_1 = F \cup \{e_F, e_F^{1/2}\}$. Let $0 < \eta < \varepsilon$. Put

$$\mathcal{G} = \{\iota \otimes 1_n, \iota^{1/2} \otimes 1_n, \iota \otimes e_{i,j}, \iota^{1/2} \otimes e_{i,j}, \iota \otimes 1_n : 1 \leq i, j \leq n\} \subset C_0((0, 1]) \otimes M_n$$

Let $e \in A_+ \setminus \{0\}$. Since A has (TAD-2), there is a homomorphism $\varphi : C_0((0, 1]) \otimes M_n \rightarrow A$ such that

$$\| [x, \varphi(f)] \| < \eta/32n \text{ for all } f \in \mathcal{G} \text{ and } ((e_F - e_F^{1/2}\varphi(\iota e_F^{1/2}) - \varepsilon/32n)_+ \lesssim e. \quad (e4.17)$$

Put $d_1 = \varphi(\iota \otimes e_{1,1})$. Define $D := \overline{d_1 A d_1}$. Put

$$C_0 = \left\{ \sum_{i,j=1}^n \varphi(\iota^{1/2} \otimes e_{i,1}) a \varphi(\iota^{1/2} \otimes e_{1,j}) : a \in A \right\}.$$

Note that C_0 is a $*$ -subalgebra of A . Put $C = \overline{C_0}$. Define a homomorphism $\Phi : C \rightarrow D \otimes M_n$ by

$$\Phi(\varphi(\iota^{1/2} \otimes e_{i,1}) a \varphi(\iota^{1/2} \otimes e_{1,j})) = d_1^{1/2} a d_1^{1/2} \otimes e_{i,j}, \quad (e4.18)$$

for all $a \in A$. It is easy to verify that Φ is an isomorphism. So $C \cong D \otimes M_n$. Put $\theta = (e_F - e_F^{1/2}\varphi(\iota \otimes 1_n)e_F^{1/2} - \varepsilon/64n)_+$. Then by (e4.16) and (e4.17), we have the following (i)–(iii).

$$(i) \quad \theta x \approx_{\varepsilon/32n} (e_F - e_F^{1/2}\varphi(\iota \otimes 1_n)e_F^{1/2})x \approx_{4\varepsilon/32n} x(e_F - e_F^{1/2}\varphi(\iota \otimes 1_n)e_F^{1/2}) \quad (e4.19)$$

$$\approx_{\varepsilon/32n} x\theta \quad \text{for all } x \in \mathcal{F}. \quad (\text{e4.20})$$

$$(ii) (1 - \theta)x \approx_{\varepsilon/32n} (1 - (e_F - e_F^{1/2} \varphi(\iota \otimes 1_n) e_F^{1/2}))x \quad (\text{e4.21})$$

$$\approx_{\varepsilon/32n} (e_F^{1/2} \varphi(\iota \otimes 1_n) e_F^{1/2})x \approx_{2\varepsilon/64n} \varphi(\iota \otimes 1_n)x \quad (\text{e4.22})$$

$$= \sum_{i=1}^n \varphi(\iota \otimes e_{i,i})x = \sum_{i=1}^n \varphi(\iota^{1/2} \otimes e_{i,1})\varphi(\iota^{1/2} \otimes e_{1,i})x \quad (\text{e4.23})$$

$$\approx_{\eta/32} \sum_{i=1}^n \varphi(\iota^{1/2} \otimes e_{i,1})x\varphi(\iota^{1/2} \otimes e_{1,i}) \in D \otimes 1_n. \quad (\text{e4.24})$$

(iii) $\theta \lesssim e$. □

Now we can unify different variations of tracial approximate divisibility for separable simple C^* -algebras in the following theorem.

Theorem 4.11. *Let A be a non-elementary separable simple C^* -algebra. Then the following are equivalent.*

- (1) A is tracially approximately divisible,
- (2) A has the property (TAD),
- (3) A has the property (TAD-2), and,
- (4) A has the property (TAD-3).

Proof. The implications (1) $\Rightarrow$ (4) and (4) $\Rightarrow$ (3) follow from Lemma 4.9. That (3) $\Rightarrow$ (2) follows from Proposition 4.10. Finally, we note that (2) $\Rightarrow$ (1) is shown earlier right after Definition 4.3. □

Remark 4.12.

- (1) Let us point out that [16, Definition 5.2] and Definition 4.3 are equivalent for non-elementary separable simple C^* -algebras. Suppose that A is a non-elementary separable simple C^* -algebra. As we mentioned earlier, if A is tracially approximate divisible in the the sense of Definition 4.3, then A is tracially approximate divisible in the the sense of [16, Definition 5.2]. Now assume that A is tracially approximately divisible in the sense of [16, Definition 5.2]. By the first paragraph of the proof of (1) of Lemma 4.9, we see that A has property (TAD-3). Then, by Theorem 4.11, A is tracially approximately divisible in the sense of Definition 4.3.
- (2) Let A be a non-elementary separable simple C^* -algebra which satisfies one of four conditions in Theorem 4.11 and $B \subset A$ be a non-zero hereditary C^* -subalgebra. Then, by [16, Theorem 5.5] and Theorem 4.11 above, B satisfies all conditions in Theorem 4.11. We note that the element e_F plays important role in [16, Theorem 5.5]. One might notice that neither property (TAD) nor property (TAD-3) are straightforward to pass to hereditary C^* -subalgebras. However, one of the purposes of this section is to simplify Definition 4.3. In fact, we have the next remark.

- (3) Suppose that A is a non-elementary separable simple C^* -algebra which is tracially approximate divisible (in the sense of Definition 4.3). Then, for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$, any $s \in A_+ \setminus \{0\}$, and any integer $N \geq 1$, there are $\theta \in A_+^1$, a C^* -subalgebra $D \otimes M_N \subset A$ and a c.p.c. map $\beta : A \rightarrow A$ such that
- (i) $x \approx_\varepsilon x_1 + \beta(x)$ for all $x \in \mathcal{F}$, where $\|x_1\| \leq \|x\|$, $x_1 \in \text{Her}(\theta)$,
 - (ii) $\beta(x) \in_\varepsilon D \otimes 1_N$ for all $x \in \mathcal{F}$, and
 - (iii) $\theta \lesssim s$.

Conversely, suppose that a non-elementary separable simple C^* -algebra A satisfies conditions above. Choose $\mathcal{F}_n$ and ε_n as in the proof of Lemma 4.9 (1). Then (i') (ii') and (iii) in the proof of Lemma 4.9 (1) hold. As in the proof of Lemma 4.9 (1), these imply that A has (TAD-3). In other words, by Theorem 4.11, one may drop the reference to e_F in Definition 4.3.

We would like to include the following statement. Note that, by Theorem 5.2 in the next section, if A has (TAD-2), A automatically has strict comparison.

Proposition 4.13. *Let A be a non-elementary separable simple exact C^* -algebra which has strict comparison and at least one nonzero densely defined trace. Then A is uniformly McDuff in the sense of [9, Definition 1.3] if and only if A has the property (TAD-2) (or (TAD-3)).*

Proof. Suppose that A is uniformly McDuff and has strict comparison. Let $a \in \text{Ped}(A)_+ \setminus \{0\}$ with $0 \leq a \leq 1$. We will show that $B = \text{Her}(a)$ has property (TAD-2). Let $\mathcal{F} \subset B$ be a finite subset and $e_F \in B_+^1$ such that $e_F x \approx_{\varepsilon/4} x \approx_{\varepsilon/4} x e_F$ for all $x \in \mathcal{F}$. Fix $\varepsilon > 0$ and $s \in B_+ \setminus \{0\}$.

We will retain notions used in [9, Section 1] regarding the notion of uniform McDuff. Fix a free ultrafilter $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$. Let $n \in \mathbb{N}$. Since A is uniformly McDuff, choose a sequence of order zero c.p.c. maps $\{\psi_k\} : M_n \rightarrow \pi_\infty^{-1}(A')$ (see (e3.1) for the notation) such that $\Pi_\omega \circ \{\psi_k\} = \psi : M_n \rightarrow F_\omega$ is an order zero map, where $\Pi_\omega : \pi_\infty^{-1}(A') \rightarrow F_\omega = (\pi_\infty^{-1}(A')/c_\omega(A))/(\pi_\infty^{-1}(A')/c_\omega(A))$ is the quotient map. Moreover, $\tau_\omega((1 - \psi(1_n))b) = 0$ for $\tau_\omega \in T_\omega^+(A)$ for any $b \in B_+$.

Let $\{e_k\} \in \text{Her}(a)$ be an approximate identity of B and $\{\delta_k\} \subset (0, 1/2)$ be a given sequence such that $\delta_n \searrow 0$. Since $\{\psi_k\}$ maps M_n to $\pi_\infty^{-1}(A')$, for each $k \in \mathbb{N}$, there is n_k such that $\|e_k^{1/2} \varphi_m(b) - \varphi_m(b) e_k^{1/2}\| < \delta_k$ for all $b \in M_n^1$ (recall that M_n^1 is compact) if $m \geq n_k$. We may assume that $n_k < n_{k+1}$. Define $c_j = e_1$ if $j \leq n_2$, $c_j = e_k$, if $n_k < j \leq n_{k+1}$, $k \geq 2$. Then $\{c_j\}$ is an approximate identity for B and $\|c_k^{1/2} \varphi_m(b) - \varphi_m(b) c_k^{1/2}\| < \delta_k$ if $m > k$ for all $b \in M_n^1$.

Denote by $T_{e_F} = \{\tau \in T^+(A) : \tau(e_F) = 1\}$. We claim that

$$\limsup_{k \rightarrow \omega} \{\tau((1 - \psi_k(1_n))e_F) : \tau \in T_{e_F}\} = 0. \quad (\text{e4.25})$$

Otherwise, there exists $d > 0$ satisfying the following: for each $\mathcal{P} \in \omega$, there exists $m_{\mathcal{P}} \in \mathcal{P}$ and $\tau_{\mathcal{P}} \in T_{e_F}$ such that

$$\tau_{\mathcal{P}}((1 - \psi_{m_{\mathcal{P}}}(1_n))e_F) \geq d. \quad (\text{e4.26})$$

Fix one $\tau_1 \in T_{e_F}$. Let $S = \{s = m_{\mathcal{P}} : \mathcal{P} \in \omega\}$. Then $S \cap \mathcal{P} \neq \emptyset$ for any $\mathcal{P} \in \omega$. Fix $s \in S$. Let $\omega_s = \{\mathcal{P} \in \omega : s = m(\mathcal{P})\}$. For each $s \in S$, choose one $\mathcal{P}_s \in \omega_s$. Find a sequence $\tau_k \in T_{e_F}$ such that, if

$s \in S, \tau_s = \tau_{\mathcal{P}_s}$ (if $s = m(\mathcal{P}_s)$), and $\tau_k = \tau_1$ if $k \notin S$. Then, if $s \in S$,

$$\tau_s((1 - \psi_s(1_n))e_F) \geq d. \quad (\text{e4.27})$$

Since $S \cap \mathcal{P} \neq \emptyset$ for any $\mathcal{P} \in \omega$, this contradicts the fact that $\tau_\omega((1 - \psi(1_n))e_F) = 0$ for all $\tau \in T_\omega^+(A)$. This proves the claim.

Define $\{\varphi'_k\} : M_n \rightarrow \pi_\infty^{-1}(B')$ by $\varphi'_k(b) = c_k^{1/2}\psi_k(b)c_k^{1/2}$ for all $b \in M_n$. Then $\psi'_k : M_n \rightarrow \text{Her}(e_k^{1/2})$ is a c.p.c. map and $\Pi_\omega \circ \{\varphi_k\} : M_n \rightarrow F'_\omega = B_\omega \cap B'/B_\omega \cap B^\perp$ is an order zero c.p.c. map. By choosing δ_k sufficiently small at the beginning, using the projectivity of $C_0((0, 1]) \otimes M_n$, we obtain a sequence of order zero c.p.c. maps $\varphi_k : M_n \rightarrow \text{Her}(c_k^{1/2}) \subset B$ such that

$$\lim_{k \rightarrow \infty} \|\psi'_k - \varphi_k\| = 0, \quad (\text{e4.28})$$

(see [44, Lemma 1.2.5]). Since $\{c_k^{1/2}\}$ is an approximate identity of B , we have that

$\lim_{k \rightarrow \infty} \|(e_F^{1/2}(c_k^{1/2}\psi_k(c)c_k^{1/2} - \psi_k(c))e_F^{1/2})\| = 0$ for each $c \in M_n$. Hence, by the claim (and (e4.28)),

$$\limsup_{k \rightarrow \infty} \left\{ \tau \left(e_F - e_F^{1/2} \varphi_k(1_n) e_F^{1/2} \right) : \tau \in T_{e_F} \right\} = 0. \quad (\text{e4.29})$$

Define $\varphi_{c,k} : C_0((0, 1]) \otimes M_n \rightarrow B$ by $\varphi_{c,k}(\iota \otimes y) = \varphi_k(y)$ for all $y \in M_n$. Thus we obtain a subsequence $\{n(k)\}$ such that

$$\begin{aligned} \lim_{k \rightarrow \infty} \|\varphi_{c,n(k)}(z)y - y\varphi_{c,n(k)}(y)\| &= 0 \text{ for all } y \in A \text{ and } y \in C_0((0, 1]) \otimes M_n \text{ and} \\ \limsup_{k \rightarrow \infty} \left\{ \tau(e_F - e_F^{1/2} \varphi_{c,n(k)}(\iota \otimes 1_n) e_F^{1/2}) : \tau \in T_{e_F} \right\} &= 0. \end{aligned} \quad (\text{e4.30})$$

It follows that $\{e_F - e_F^{1/2} \varphi_{c,n(k)}(\iota \otimes 1_n) e_F^{1/2}\} \in I_{T,0}$. Since A has strict comparison, by Proposition 3.8, $\{e_F - e_F^{1/2} \varphi_{c,n(k)}(\iota \otimes 1_n) e_F^{1/2}\} \in N_{cu}(A)$. Hence, for a fixed finite subset $\mathcal{G} \subset C_0((0, 1]) \otimes M_n$, by choosing $\varphi = \varphi_{c,n(k)}$ for some large k , we obtain

$$\|\varphi(z)y - y\varphi(z)\| < \varepsilon \text{ for all } y \in \mathcal{F} \text{ and } z \in \mathcal{G} \text{ and} \quad (\text{e4.31})$$

$$(e_F - e_F^{1/2} \varphi(\iota \otimes 1_n) e_F^{1/2} - \varepsilon)_+ \lesssim s. \quad (\text{e4.32})$$

Hence B has property (TAD-2). By Theorem 4.11, B has property (TAD). It follows (3) of Remark 4.2 and Brown's stable isomorphism [6] that $A \otimes \mathcal{K}$ has property (TAD) and hence has property (TAD-2), by Theorem 4.11 again. By (2) of Remark 4.12, A has (TAD-2).

Conversely, let A have strict comparison and property (TAD-3). Let $\varphi : M_n \rightarrow A'_{cu}/A_{cu}^\perp$ be a unital homomorphism. By [44, Proposition 1.2.4] and the central surjectivity (see 3.10), there exists a sequence of order zero c.p.c. maps $\{\psi_k\} : M_n \rightarrow \pi_\infty^{-1}(A')$ such that $\Pi \circ \{\psi_k\} = \varphi$, where $\Pi : l^\infty(A) \cap A' \rightarrow A'_{cu}/A_{cu}^\perp$ is the quotient map. Since $c_\omega(A)$ is an ideal of $c_0(A)$, the map $\psi = \Pi_\omega \circ \{\psi_k\} : M_n \rightarrow F_\omega$ is also an order zero c.p.c. map. The fact that φ is unital implies that $\{(1 - \psi_k(1_n))b\}_{k \in \mathbb{N}} \in N_{cu}(A)$ for any $b \in A_+$. We choose $b \in \text{Ped}(A)_+ \setminus \{0\}$. Then

$\{b^{1/2}(1 - \psi_k(1_n))b^{1/2}\}_{k \in \mathbb{N}} \in N_{cu}(\overline{bAb})$. Since $\overline{bAb}$ is algebraically simple, by Proposition 3.8,

$$\lim_{k \rightarrow \infty} \sup\{\tau((1 - \psi_k(1_n))b) : \tau \in T_b\} = \lim_{k \rightarrow \infty} \sup\{\tau(b^{1/2}(1 - \psi_k(1_n))b^{1/2}) : \tau \in T_b\} = 0.$$

It follows that, for any $\{\tau_k\} \in T_b$, $\lim_{k \rightarrow \infty} \tau_k((1 - \psi_k(1_n))b) = 0$. Thus A has uniform McDuff in the sense of [9]. $\square$

Remark 4.14. We note that the implication from (TAD-2) to the uniform McDuff property is also proved in [9, Proposition 4.6].

In view of Proposition 4.13, one may make the following definition: Let A be a separable simple (exact) C^* -algebra with at least one nontrivial densely defined trace. Let I_T be the closed ideal defined in Proposition 3.8 and let $\pi_{I_T} : l^\infty(A) \rightarrow l^\infty(A)/I_T$ be the quotient map. We say that A has the uniform McDuff property, if for each $n \in \mathbb{N}$, there is a unital embedding $\varphi : M_n \rightarrow \pi_{I_T}(A)'/\pi_{I_T}(A)^\perp$.

When A is exact and also has strict comparison, A is uniformly McDuff if and only if A has property (TAD-3) (see Proposition 4.13 and Proposition 3.8).

5 | STRICT COMPARISON AND CUNTZ SEMIGROUP

In this section, we show that the Cuntz semigroups of simple C^* -algebras which are tracially approximately divisible behave nicely as those of simple $\mathcal{Z}$ -stable C^* -algebras. The main result of this section is stated in Theorem 5.7 which will also be used in the proof of Corollary 6.5 in next section.

Proposition 5.1. *Let A be a simple C^* -algebra with the property (TAD). Then, for any integer $n \geq 1$, any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$, and any $s \in A_+ \setminus \{0\}$, there are $\theta \in A_+^1$ and C^* -subalgebra $D \otimes M_n \subset A$ such that*

- (i) $\theta x \approx_\varepsilon x\theta$ for all $x \in \mathcal{F}$,
- (ii) $(1 - \theta)x \in_\varepsilon D \otimes 1_n$ for all $x \in \mathcal{F}$,
- (iii) $\theta \lesssim s$,
- (iv) $(1 - \theta)^{1/2}x(1 - \theta)^{1/2} \in_\varepsilon D \otimes 1_n$ for all $x \in \mathcal{F}$,
- (v) $x \approx_\varepsilon \theta^{1/2}x\theta^{1/2} + (1 - \theta)^{1/2}x(1 - \theta)^{1/2}$ for all $x \in \mathcal{F}$,
- (vi) for any finite subset $\mathcal{G} \subset C_0((0, 1])$, there is $d \in D_+^1 \setminus \{0\}$ such that, for all $x \in \mathcal{F}$,

$$(1 - \theta)^{1/2}x(1 - \theta)^{1/2} \approx_{\varepsilon/64n^2} (d \otimes 1_n)^{1/2}x(d \otimes 1_n)^{1/2} \quad \text{and} \quad (\text{e5.1})$$

$$x(f(d) \otimes e_{i,j}) \approx_\varepsilon (f(d) \otimes e_{i,j})x \quad \text{for all } f \in \mathcal{G}, 1 \leq i, j \leq k, \quad (\text{e5.2})$$

(vii) if $x \in \mathcal{F}$ and $x \geq 0$, we may choose d such that

$$x - x^{1/2}(d \otimes 1_n)x^{1/2} \approx_{\varepsilon/4} x^{1/2}\theta x^{1/2}. \quad (\text{e5.3})$$

Proof. Fix $a \in A_+ \setminus \{0\}$, $\varepsilon \in (0, 1)$, and a finite subset $\mathcal{F}_1 \subset A$. Without loss of generality, we may assume that for all $x \in \mathcal{F}_1$, $\|x\| \leq 1$. By a standard perturbation, we may assume that there is $e_F, e_A \in A_+^1$ satisfying $xe_F = e_Fx = x$ for all $x \in \mathcal{F}$ and $e_Ae_F = e_Fe_A = e_F$. Put $\mathcal{F} = \mathcal{F}_1 \cup \{e_F, e_F^{1/2}, e_A\}$.

Let $m \in \mathbb{N}$ such that for any $c \in A_+^1$, $c \approx_{\varepsilon/4} c \cdot c^{1/m}$. Let $\eta \in (0, \varepsilon)$ be such that, for any element $z \in A^1$, any $c \in A_+^1$, $\|zc - cz\| < 20\eta$ implies $\|zc^{1/m} - c^{1/m}z\| < \varepsilon/4$.

By definition of (TAD), there are $\theta \in A_+^1$ and C^* -subalgebra $D \otimes M_n \subset A$ such that

- (1) $\theta x \approx_\eta x\theta$ for all $x \in \mathcal{F}$,
- (2) $(1 - \theta)x \in_\eta D \otimes 1_n$ for all $x \in \mathcal{F}$, and
- (3) $\theta \lesssim a$.

This implies that (i), (ii), and (iii) in the proposition hold (even for η). By Remark 4.2, we may further assume that (iv) and (v) also hold (even for η).

By (2), for each $x \in \mathcal{F}$, there exists $y \otimes 1_n \in D \otimes 1_n$ such that

$$\|(1 - \theta)x - y \otimes 1_n\| < \eta. \quad (\text{e5.4})$$

Since $e_A \in \mathcal{F}$ and (iv) holds (for η), we can choose $d \in D_+^1 \setminus \{0\}$ such that

$$(1 - \theta)e_A \approx_\eta (1 - \theta)^{1/2}e_A(1 - \theta)^{1/2} \approx_\eta d \otimes 1_n. \quad (\text{e5.5})$$

It follows that

$$(y \otimes 1_n)(d \otimes 1_n) \approx_{3\eta} (1 - \theta)x(1 - \theta)e_A \quad (\text{e5.6})$$

$$\approx_\eta (1 - \theta)xe_A(1 - \theta) = (1 - \theta)e_Ax(1 - \theta) \quad (\text{e5.7})$$

$$\approx_\eta (1 - \theta)e_A(1 - \theta)x \approx_{3\eta} (d \otimes 1_n)(y \otimes 1_n). \quad (\text{e5.8})$$

Thus $\|yd - dy\| < 8\eta$. Note that, by (e5.5), (1), (e5.4) and the choice of e_A , for all $x \in \mathcal{F}_1 \cup \{e_F, e_F^{1/2}\}$,

$$x(d \otimes 1_n) \approx_{2\eta} x(1 - \theta)e_A \approx_\eta (1 - \theta)xe_A = (1 - \theta)x \approx_\eta y \otimes 1_n. \quad (\text{e5.9})$$

Similarly,

$$(d \otimes 1_n)x \approx_{2\eta} (1 - \theta)e_Ax = (1 - \theta)x \approx_{3\eta} x(d \otimes 1_n). \quad (\text{e5.10})$$

We compute that (recall $e_Ax = xe_A = x$ if $x \in \mathcal{F}_1$), for all $x \in \mathcal{F}_1$,

$$\begin{aligned} x(d \otimes e_{i,j}) &\approx_{\varepsilon/4} x(d \otimes 1_n)(d^{1/m} \otimes e_{i,j}) \approx_{3\eta} (1 - \theta)x(d^{1/m} \otimes e_{i,j}) \\ &\approx_\eta (y \otimes 1_n)(d^{1/m} \otimes e_{i,j}) = (yd^{1/m} \otimes e_{i,j}) \\ &\approx_{\varepsilon/4} (d^{1/m} \otimes e_{i,j})(y \otimes 1_n) \\ &\approx_{3\eta} (d^{1/m} \otimes e_{i,j})(d \otimes 1_n)x \approx_{\varepsilon/4} (d \otimes e_{i,j})x. \end{aligned}$$

Thus, by choosing smaller η (and ε) if necessary, we conclude that the second part ((e5.2)) of (vi) holds. The first part of (vi) follows from (e5.5) and a choice of small η .

To see (vii), combining (e5.9) and (e5.10), with sufficiently small η , we have, if $x \in F \cup \{e_F, e_F^{1/2}\}$ and $x \geq 0$,

$$x^{1/2}(d \otimes 1_n)x^{1/2} \approx_{\varepsilon/4} x^{1/2}(1 - \theta)x^{1/2}, \text{ or } x - x^{1/2}(d \otimes 1_n)x^{1/2} \approx_{\varepsilon/4} x^{1/2}\theta x^{1/2}. \quad \square$$

The following statement is already mentioned in [16, Remark 5.8].

Theorem 5.2 (c.f. [18, Theorem 3.3]). *Let A be a simple C^* -algebra which has property (TAD). Then A has the strict comparison for positive elements (or A is purely infinite).*

Proof. Following the original idea of Rørdam, we will modify the argument in the proof of Lemma 3.2 of [18]. Let us assume that A is not elementary. Let $a, b \in M_\infty(A)_+$. Let us first assume that 0 is not an isolated point of $\text{sp}(b) \cup \{0\}$, and $k\langle a \rangle \leq k\langle b \rangle$ for some integer $k \geq 1$, we wish to show that $a \lesssim b$. Without loss of generality, we may assume that $a, b \in M_N(A)_+$ for some large N . Since $M_N(A)$ also has the property (TAD), we may assume $a, b \in A_+$. We may further assume $\|a\| = \|b\| = 1$.

Fix $\delta > 0$. By [38, Proposition 2.4 (iv)], we can choose $c = (c_{i,j})_{k \times k} \in M_k(A)$ and $\varepsilon \in (0, \delta)$, such that

$$c((b - \delta)_+ \otimes 1_k)c^* = (a - \varepsilon)_+ \otimes 1_k. \quad (\text{e5.11})$$

Since 0 is not an isolated point of $\text{sp}(b) \cup \{0\}$, there is a $f_0 \in C_0((0, 1])$ such that $f_0(t) = 0$ for all $t \in (\delta/2, 1]$ and $d := f_0(b) \neq 0$. So $d \perp (b - \delta)_+$. By replacing $c_{i,j}$ with $c_{i,j}q(b)$ for some $q \in C_0((0, 1])$ which vanishes in $(0, \delta/2]$ and $q(t) = 1$ for all $t \in [\delta, 1]$, we may assume that

$$c_{i,j}d = 0, \quad i, j = 1, 2, \dots, k. \quad (\text{e5.12})$$

By (e5.11), we compute

$$\sum_{l=1}^k c_{i,l}((b - \delta)_+ \otimes 1_k)c_{j,l}^* = \begin{cases} (a - \varepsilon)_+ & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases} \quad (\text{e5.13})$$

Now let $\eta \in (0, \varepsilon)$ be arbitrary. Put $F_0 = \{(a - \varepsilon)_+, (b - \delta)_+\} \cup \{c_{i,j}, c_{i,j}^* : 1 \leq i, j \leq k\}$. Let $g \in C_0((0, 1])_+^1$ such that $g(t) = 1$ for $t \in [\eta, 1]$. Let $M := 1 + \max\{\|x\| : x \in F_0\}$.

By Proposition 5.1, there are $\theta \in A_+^1$ and C^* -subalgebra $D \otimes M_k \subset A$ such that

- (i) $\theta x \approx_\eta x\theta$ for all $x \in F_0$,
- (ii) $(1 - \theta)x \in_\eta D \otimes 1_n$ for all $x \in F_0$,
- (iii) $\theta \lesssim d$,
- (iv) $(1 - \theta)^{1/2}x(1 - \theta)^{1/2} \in_\eta D \otimes 1_k$ for all $x \in F_0$,
- (v) $x \approx_\eta \theta^{1/2}x\theta^{1/2} + (1 - \theta)^{1/2}x(1 - \theta)^{1/2}$ for all $x \in F_0$,
- (vi) there is $e \in D_+^1$ such that, for all $x \in F_0$,

$$(1 - \theta)^{1/2}x(1 - \theta)^{1/2} \approx_\eta (e \otimes 1_k)^{1/2}x(e \otimes 1_k)^{1/2}, \text{ and} \quad (\text{e5.14})$$

$$\|x(g(e) \otimes e_{i,j}) - (g(e) \otimes e_{i,j})x\| < \eta/(kM)^4 \quad (1 \leq i, j \leq k). \quad (\text{e5.15})$$

Put $a_1 = (1 - \theta)^{1/2}(a - \varepsilon)_+(1 - \theta)^{1/2}$ and $a_2 = \theta^{1/2}(a - \varepsilon)_+\theta^{1/2}$. Then by (v),

$$\|(a - \varepsilon)_+ - (a_1 + a_2)\| \leq \eta. \quad (\text{e5.16})$$

Denote $\bar{c} := \sum_{i,j=1}^k (e^{1/2} \otimes 1_k)(g(e) \otimes e_{i,j})c_{i,j}$. We compute that (using (e5.15) and (e5.13))

$$\bar{c}((b - \delta)_+)\bar{c}^* \quad (\text{e5.17})$$

$$= (e^{1/2} \otimes 1_k) \left(\sum_{i,j,l,m}^k (g(e) \otimes e_{i,j})c_{i,j}((b - \delta)_+)c_{l,m}^*(g(e) \otimes e_{m,l}) \right) (e^{1/2} \otimes 1_k) \quad (\text{e5.18})$$

$$\approx_{3\eta} (e^{1/2} \otimes 1_k) \left(\sum_{i,j,l,m}^k (g(e) \otimes e_{i,j})(g(e) \otimes e_{m,l})c_{i,j}((b - \delta)_+)c_{l,m}^* \right) (e^{1/2} \otimes 1_k) \quad (\text{e5.19})$$

$$= (e^{1/2} \otimes 1_k) \left(\sum_{i,j,l}^k (g(e)^2 \otimes e_{i,l})c_{i,j}((b - \delta)_+)c_{l,j}^* \right) (e^{1/2} \otimes 1_k) \quad (\text{e5.20})$$

$$= (e^{1/2} \otimes 1_k) \left(\sum_{i,l=1}^k (g(e)^2 \otimes e_{i,l}) \left(\sum_{j=1}^k c_{i,j}((b - \delta)_+)c_{l,j}^* \right) \right) (e^{1/2} \otimes 1_k) \quad (\text{e5.21})$$

$$= (e^{1/2} \otimes 1_k) \left(\sum_{i=1}^k (g(e)^2 \otimes e_{i,i})(a - \varepsilon)_+ \right) (e^{1/2} \otimes 1_k) \quad (\text{e5.22})$$

$$= (e^{1/2} \otimes 1_k)(g(e)^2 \otimes 1_k)(a - \varepsilon)_+(e^{1/2} \otimes 1_k)$$

$$\approx_{\eta} (e^{1/2} \otimes 1_k)(a - \varepsilon)_+(e^{1/2} \otimes 1_k) \approx_{\eta} a_1.$$

In other words, $\|\bar{c}(b - \delta)_+\bar{c}^* - a_1\| \leq 5\eta$.

Since $a_2 \lesssim \theta \lesssim d$, there exists $c_0 \in A$ such that $\|c_0dc_0^* - a_2\| < \eta$. Since $d \perp (b - \delta)_+$, we may assume that $c_0((b - \delta)_+) = 0$. Now put $z = \bar{c} + c_0$. Recall (e5.12), we have

$$\|z((b - \delta)_+ + d)z^* - a\| = \|(\bar{c}(b - \delta)_+\bar{c}^* - a_1) + (c_0dc_0^* - a_2) + (a_1 + a_2 - a)\| < 6\eta + \varepsilon.$$

Since ε and η can be arbitrary small, it follows that $a \lesssim b$.

The case that 0 is an isolated point of $\text{sp}(b) \cup \{0\}$ can be reduced to the case above by applying, for example, [18, Lemma 3.1].

To show that $W(A)$ is almost unperforated, suppose that $k\langle a \rangle \leq (k - 1)\langle b \rangle$. Then $k\langle a \rangle \leq k\langle b \rangle$. From what has been proved, $\langle a \rangle \leq \langle b \rangle$.

By [39, Corollary 5.1] (see also Proposition 4.9 of [16] as well as the end of Definition 2.6 and Definition 2.8 of the current paper), A has strict comparison (or A is purely infinite). $\square$

Remark 5.3. It is worth noting that Theorem 5.2 has also been independently proved in [9] with a different point of view. More precisely, [9, Theorem 3.2] shows that (non-unital) σ -unital simple C^* -algebras with (TAD-2) have strict comparison.

Corollary 5.4. *Let A be a unital stably finite simple C^* -algebra with the property (TAD). Then A has strict comparison and has stable rank one.*

Proof. This is a corollary of Theorem 5.2 and [16, Theorem 5.7]. $\square$

In Section 6, we will show that the condition that A is unital in Corollary 5.4 can be removed, if we additionally assume that A is separable.

Recall that, for $x, y \in \text{Cu}(A)$, we write $x \ll y$, if for any increasing sequence $\{y_n\}$ with $y \leq \sup\{y_n\}$, there exists n_0 such that $x \leq y_{n_0}$.

The following property is introduced by L. Robert [35]. It may be viewed as a tracial version of almost divisibility which is closely related to Winter's tracial 0-almost divisibility [45, Definition 3.5].

Definition 5.5 [35, Proposition 6.2.1]. Let A be a C^* -algebra. We say that $\text{Cu}(A)$ has property (D), if for any $x \in \text{Cu}(A)$, $x' \ll x$, and any integer $n \in \mathbb{N}$, there exists $y \in \text{Cu}(A)$ such that $n\hat{y} \leq \hat{x}$ and $\hat{x}' \leq (n+1)\hat{y}$ (see Definition 2.9 for $\hat{z}, z \in \text{Cu}(A)$).

From [13, Corollary 5.8] as observed by L. Robert, he shows the following (in the proof of Proposition 6.2.1 of [35]).

Lemma 5.6. *Let A be a finite simple C^* -algebra with strict comparison. Suppose that A has property (D). Then the canonical map from $\text{Cu}(A)$ to $\text{LAff}_+(\widetilde{QT}(A))$ is surjective.*

Proof. The proof is contained in the second paragraph of the proof of Proposition 6.2.1 of [35]. $\square$

The following is an analogue of [13, Theorem 6.6]. Recall ([13, Proposition 6.4 (iv)]) that, in a simple C^* -algebra A , every element of $\text{Cu}(A)$ is purely non-compact except for the elements $[p]$, where p is a non-zero finite projection. In particular, if A has no infinite projections, then the set of purely non-compact elements of $\text{Cu}(A)$ is precisely those elements which cannot be represented by a projection.

Theorem 5.7. *Let A be a non-elementary separable simple C^* -algebra which is tracially approximately divisible. Then the map $\langle a \rangle \rightarrow \widehat{\langle a \rangle}$ is an isomorphism between ordered semigroups of purely non-compact elements of $\text{Cu}(A)$ and $\text{LAff}_+(\widetilde{QT}(A))$.*

Proof. If A is purely infinite, then, $\widetilde{QT}(A) = \{0\}$, and every element in A is purely non-compact, and, all non-zero elements are Cuntz-equivalent. So, in this case, the conclusion uninterestingly holds. Now we assume that A is not purely infinite. Recall, from Theorem 5.2, $W(A)$ is almost unperforated. By [39, Corollary 5.1] (see also [16, Proposition 4.9]), A is stably finite. Consequently, A has no infinite projections.

Thus, from now on in this proof, we assume that purely non-compact elements are precisely those which cannot be represented by projections.

Let $a, b \in (A \otimes \mathcal{K})_+$ such that $\langle a \rangle$ and $\langle b \rangle$ are two purely non-compact elements and $d_\tau(a) = d_\tau(b)$ for all $\tau \in \widetilde{QT}(A)$. Then, since A is simple and 0 is not an isolated point of $\text{sp}(a) \cup \{0\}$, for any $\varepsilon > 0$,

$$d_\tau((a - \varepsilon)_+) < d_\tau(a) = d_\tau(b) \text{ for all } \tau \in \widetilde{QT}(A) \setminus \{0\}. \quad (\text{e5.23})$$

Hence, by Theorem 5.2 and Theorem 4.11, we have $(a - \varepsilon)_+ \lesssim b$ for any $\varepsilon > 0$. It follows that

$$a \lesssim b. \quad (\text{e5.24})$$

Symmetrically, $b \lesssim a$. So $a \sim b$. This proves the map $\langle a \rangle \rightarrow \widehat{\langle a \rangle}$ is injective.

To prove the surjectivity, by the first paragraph of the proof of Lemma 6.5 of [13], it suffices to show that the canonical map $\langle a \rangle \mapsto \widehat{a}$ is surjective from $\text{Cu}(A)$ to $\text{LAff}_+(\widetilde{QT}(A))$. Therefore, by Lemma 5.6, it suffices to show that A has property (D).

To see A has property (D), let $x, x' \in \text{Cu}(A)$ such that $x' \ll x$. Let $a, b \in (A \otimes \mathcal{K})_+^1$ such that $\langle a \rangle = x'$ and $\langle b \rangle = x$. By Remark 4.2 and [16, Proposition 5.3], $A \otimes \mathcal{K}$ is tracially approximately divisible.

Then, for some $1/16 > \varepsilon > 0$,

$$a \lesssim f_{2\varepsilon}(b). \quad (\text{e5.25})$$

In particular, we assume that $(b - \varepsilon)_+ \neq 0$. Note that $f_{\varepsilon/128}(b) \ll b$.

Choose $0 < \eta < \varepsilon$ such that $(b - \eta)_+ \neq 0$. Since A is a non-elementary simple C^* -algebra, there are $n + 1$ mutually orthogonal and mutually Cuntz-equivalent elements $s_1, s_2, \dots, s_{n+1} \in \text{Her}((b - \eta)_+) \setminus \{0\}$. Since $A \otimes \mathcal{K}$ has property (TAD), there are $d_0, d_1 \in (A \otimes \mathcal{K})_+^1$ and a C^* -subalgebra $D \otimes M_n \subset A \otimes \mathcal{K}$ such that

- (1) $b \approx_{\eta/64} d_0 + d_1$,
- (2) $d_1 \in_{\eta/64} D \otimes 1_n$, and
- (3) $d_0 \lesssim s_1$.

Choose $d \in D_+$ such that

$$d_1 \approx_{\eta/64} \sum_{i=1}^n d \otimes e_{i,i}. \quad (\text{e5.26})$$

Then

$$(d_1 - \eta/64)_+ \approx_{\eta/32} \sum_{i=1}^n d \otimes e_{i,i} \text{ and } (d_1 - \eta/32)_+ \approx_{\eta/16} \sum_{i=1}^n (d - \eta/64)_+ \otimes e_{i,i}. \quad (\text{e5.27})$$

By applying [38, Proposition 2.2], we have

$$\sum_{i=1}^n (d - \eta/32)_+ \otimes e_{i,i} \lesssim (d_1 - \eta/64)_+ \text{ and } (d_1 - 3\eta/16)_+ \lesssim \sum_{i=1}^n (d - \eta/64)_+ \otimes e_{i,i}. \quad (\text{e5.28})$$

From (1) above and by [33, Lemma 1.7],

$$(d_1 - \eta/64)_+ \lesssim ((d_0 + d_1) - \eta/64)_+ \lesssim b. \quad (\text{e5.29})$$

Put $y := (d - \eta/32)_+ \otimes e_{1,1}$. Then, for all $\tau \in \widetilde{QT}(A)$, by the first inequality in (e5.28),

$$nd_\tau(y) \leq d_\tau((d_1 - \eta/64)_+) \leq d_\tau(((d_0 + d_1) - \eta/64)_+) \leq d_\tau(b). \quad (\text{e5.30})$$

On the other hand, by (1) and (e5.26), we have

$$b \approx_{\eta/32} d_0 + \sum_{i=1}^n (d \otimes e_{i,i}) \approx_{\eta/32} d_0 + \sum_{i=1}^n (d - \eta/32)_+ \otimes e_{i,i}. \quad (\text{e5.31})$$

It follows that

$$(b - \eta/16)_+ \lesssim d_0 + \sum_{i=1}^n (d - \eta/32)_+ \otimes e_{i,i} \lesssim s_1 \oplus \sum_{i=1}^n (d - \eta/32)_+ \otimes e_{i,i}. \quad (\text{e5.32})$$

Recall that $(b - \eta/16)_+ \in \text{Ped}(A)$. So $d_\tau((b - \eta/16)_+) < \infty$ for all $\widetilde{QT}(A)$. It follows from (e5.32), (3), and the choice of s_1 that

$$nd_\tau(y) \geq d_\tau((b - \eta/16)_+) - d_\tau(s_1) \geq d_\tau((b - \eta/16)_+) - \frac{d_\tau((b - \eta/16)_+)}{n+1} \quad (\text{e5.33})$$

$$\geq \left(\frac{n}{n+1} \right) d_\tau((b - \eta/16)_+) \quad \text{for all } \tau \in \widetilde{QT}(A). \quad (\text{e5.34})$$

In other words,

$$\langle (b - \eta/16)_+ \rangle^\wedge \leq (n+1) \hat{y}. \quad (\text{e5.35})$$

By (e5.25), $a \lesssim f_{2\varepsilon}(b) \lesssim (b - \varepsilon/16)_+$. It follows that (recall $\eta < \varepsilon$)

$$\hat{x}' = \langle \hat{a} \rangle \leq \langle (b - \eta/16)_+ \rangle^\wedge \leq (n+1) \hat{y}.$$

Combining this with (e5.30), we conclude that A has property (D) as desired. $\square$

Remark 5.8. In Theorem 5.7, we may write

$$\text{Cu}(A) = (V(A) \setminus \{0\}) \sqcup \text{LAff}_+(\widetilde{QT}(A)).$$

Note that, here, $0 \in \text{LAff}_+(\widetilde{QT}(A))$ is the zero element, and, if $[p] \in V(A)$ and $z \in \text{LAff}_+(\widetilde{QT}(A)) \setminus \{0\}$, then $[p] + z = \hat{p} + z \in \text{LAff}_+(\widetilde{QT}(A))$. Moreover, for $x = \langle a \rangle$ and $y = \langle b \rangle$, then $x < y$ if and only if $\langle \hat{a} \rangle < \langle \hat{b} \rangle$ (see also [41, Corollary 8.12] for the unital case).

6 | STABLE RANK ONE

This section is devoted to prove that a separable simple tracially approximately divisible C^* -algebra is either purely infinite, or has stable rank one (see Corollary 6.5). As a consequence, we show that (non-unital) simple $\mathcal{Z}$ -stable C^* -algebras have stable rank one which answers a question of L. Robert.

Lemma 6.1. *Let A be a C^* -algebra, $a, b, c \in A$. Assume $c^n = 0$ for some $n \geq 1$, and $ab = ac = ca = cb = b^2 = 0$. Then $\text{sp}((a + b + c)^{n+1}) \setminus \{0\} = \text{sp}(a^{n+1}) \setminus \{0\}$.*

Proof. We first claim that $(a + b + c)^{n+1} = a^{n+1} + ba^n$.

To see this, let x be a non-zero term in the expansion of $(a + b + c)^{n+1}$. Note that x is a product of factors a, b , and c .

Case 1: $x = a \cdot y$, where y is a product of n elements in $\{a, b, c\}$. If y has a factor b or c , then x must have a factor ab or ac . Then, by the assumption $ac = ab = 0$, x would be zero. Therefore y has no factor b or c . Consequently, $x = a^{n+1}$.

Case 2: $x = b \cdot y$.

Case 2.1: If $y = a \cdot z$, where z is still a product of elements in $\{a, b, c\}$. Since, again, $ac = ab = 0$, z could not have a factor b or c . Therefore the only possible non-zero x , in this case, must be ba^n .

Case 2.2: $y = b \cdot z$. This actually is impossible, since $b^2 = 0$.

Case 2.3: $y = c \cdot z$. Then, by the assumption, $ca = cb = 0$, z could not have a factor a or b . Thus $x = bc^n$. However, by the assumption $c^n = 0$, Case 2.3 will not occur.

Case 3: $x = c \cdot y$. If y contains factor a or b , then x contains factor ca or cb . Then, by the assumption $ca = cb = 0$, we have $x = 0$. Hence y could not contain factor a or b . Hence the only possible non-zero x , in this case, must be c^{n+1} . However, by the assumption $c^{n+1} = 0$, this case could not occur.

Thus, it leaves two terms: $x = a^{n+1}$ and $x = ba^n$. In other words, $(a + b + c)^{n+1} = a^{n+1} + ba^n$. This proves the claim.

As $a^n b = 0$, we see that

$$\text{sp}((a + b + c)^{n+1}) \setminus \{0\} = \text{sp}((a + b)a^n) \setminus \{0\} = \text{sp}(a^n(a + b)) \setminus \{0\} = \text{sp}(a^{n+1}) \setminus \{0\}. \quad (\text{e6.1})$$

□

Corollary 6.2. *Let a, b, c be as in Lemma 6.1. If, in addition, a is positive, then $a + b + c$ can be approximated by invertible elements in $\tilde{A}$.*

Proof. By Lemma 6.1 and the fact that $a \in A_+$, we have $\text{sp}((a + b + c)^{n+1}) \setminus \{0\} = \text{sp}(a^{n+1}) \setminus \{0\} \subset \mathbb{R}_+$. By the spectral mapping theorem, $\text{sp}(a + b + c) \setminus \{0\}$ lies in the union of $n + 1$ rays, which is

$$\{e^{t+2ik\pi/(n+1)} : t \in \mathbb{R}, k = 1, \dots, n + 1\}. \quad (\text{e6.2})$$

Hence 0 is not an interior point of $\text{sp}(a + b + c) \cup \{0\}$. Therefore $a + b + c$ can be approximated by invertible elements. □

Recall the the definition of continuous scale from [23].

Definition 6.3. [23, Definition 2.5] Let A be a σ -unital, non-unital and non-elementary simple C^* -algebra. A is said to have continuous scale, if there is an approximate identity $\{e_n\}$ such that, for any $a \in A_+ \setminus \{0\}$, there is $N \in \mathbb{N}$, such that for any $n > m \geq N$, $e_n - e_m \lesssim a$.

Note that, by [23, Theorem 2.8] and [27, Theorem 2.4], A has continuous scale if and only if the corona algebra $M(A)/A$ is simple. It also follows from [27, Theorem 2.4] that if A has continuous scale, then any approximate identity $\{e_n\}$ with $e_{n+1}e_n = e_n e_{n+1} = e_n$ for all $n \in \mathbb{N}$ has the property that, for any $a \in A_+ \setminus \{0\}$, there exists $N \geq 1$ such that, for all $m > n \geq N$, $e_m - e_n \lesssim a$.

Moreover, for a σ -unital non-elementary simple exact C^* -algebra A with $T(A) \neq \emptyset$, if A has strict comparison, then A has continuous scale if and only if $T(A)$ is compact (see [12, Proposition 5.4], see also the proof of Theorem 5.3 of [12], and, an early version, [27, Proposition 2.2]). See also the third paragraph of the proof of Corollary 6.5 below.

Theorem 6.4. Let A be a σ -unital projectionless simple C^* -algebra with continuous scale. Suppose that, for any σ -unital hereditary C^* -subalgebra $B \subset A$, any non-invertible element in B can be approximated (in norm) by products of two nilpotent elements in B . Then A has stable rank one.

Proof. Let $\{e^{(k)}\}$ be an approximate identity of A such that $e^{(k+1)}e^{(k)} = e^{(k)}e^{(k+1)} = e^{(k)}$. By passing to a subsequence, without loss of generality, we may assume $e^{(k_2)} - e^{(k_1)} \neq 0$ for all $k_2 > k_1 \in \mathbb{N}$. Let $\tilde{A}$ be the unitization of A and $y + \lambda \in \tilde{A}$, where $y \in A$ and $\lambda \in \mathbb{C}$.

Fix $\delta \in (0, 1)$. To show A has stable rank one, it suffices to show the following:

Goal: there is $z \in GL(\tilde{A})$ such that

$$\|(y + \lambda) - z\| < \delta. \quad (\text{e6.3})$$

To achieve the goal, we first choose $k_0 \in \mathbb{N}$ such that $e^{(k_0)}ye^{(k_0)} \approx_{\delta/4} y$. Put $B := \text{Her}_A(e^{(k_0)})$ and $y_0 = e^{(k_0)}ye^{(k_0)} \in B$. Then

$$\|(y_0 + \lambda) - (y + \lambda)\| < \delta/4. \quad (\text{e6.4})$$

Note, since A has no non-zero projection, B is non-unital. Put $A_1 := \mathbb{C} \cdot 1_{\tilde{A}} + B \cong \tilde{B}$. Let $v \in A_1$ be such that $v^*v = 1$. Then $1 - vv^* \in A$ is a projection. Since A has no non-zero projections, $v^*v - vv^* = 0$. In other words, A_1 is finite and hence every one-sided invertible element in A_1 is invertible. If $y_0 + \lambda \in GL(A_1) \subset GL(\tilde{A})$, then, by (e6.4), our goal is achieved. So we assume that $y_0 + \lambda \notin GL(A_1)$. By [37, Proposition 3.2], there is a two-sided zero divisor $y_1 + \lambda_1 \in A_1$ such that $\|y + \lambda - (y_1 + \lambda_1)\| < \delta/4$, for some $y_1 \in B$ and $\lambda_1 \in \mathbb{C}$. Therefore, to achieve the goal above, it suffices to show that $y_1 + \lambda_1 \in GL(\tilde{A})$.

By [37, Lemma 3.5], working in $A_1 \cong \tilde{B}$, we can find $a_1 \in A_{1+} \setminus \{0\}$ and a unitary $u \in A_1$ such that

$$a_1 u(y_1 + \lambda_1) = u(y_1 + \lambda_1) a_1 = 0. \quad (\text{e6.5})$$

Since B is an essential ideal of A_1 , there is $b \in B_+$ such that $a_1 b a_1 \neq 0$. Put $a = a_1 b a_1 \in B$, and we may assume $\|a\| = 1$. We write $u(y_1 + \lambda_1) = x_1 + \eta$, where $x_1 \in B$ and $|\eta| = |\lambda_1|$. Since u is invertible, to show $y_1 + \lambda_1 \in GL(\tilde{A})$, it suffices to show that $x_1 + \eta \in GL(\tilde{A})$.

If $\eta = 0$, then $\lambda_1 = 0$, and then $y_1 + \lambda_1 = y_1 \in B \subset A$. By our assumption, y_1 can be approximated by products of two nilpotent elements in A , which can be approximated by invertible elements in $\tilde{A}$. Thus we may assume that $\eta \neq 0$.

Put $x = \frac{x_1}{\eta} \in \text{Her}_A(e^{(k_0)})$. If $x + 1 = \frac{x_1}{\eta} + 1 \in \overline{GL(\tilde{A})}$, then $x_1 + \eta \in \overline{GL(\tilde{A})}$.

Hence, it suffices to show that $x + 1 \in \overline{GL(\tilde{A})}$.

To do that, let us fix $\varepsilon > 0$. As A has continuous scale, we may choose $e_0, e_1 \in A_+^1$ of the form $e^{(k)}$ with $k > k_0$, such that $e_0 e_1 = e_1 = e_1 e_0$, $e_0 - e_1 \lesssim a$. Note, since $x \in \text{Her}_A(e^{(k_0)})$, and e_1 is of the form $e^{(k)}$ with $k > k_0$, we also have

$$e_1 x = x = x e_1 \text{ and, hence } e_1(x + 1) = (x + 1)e_1. \quad (\text{e6.6})$$

Note, by (e6.5) and the choice of a and $x + 1$, we have $a(x + 1) = (x + 1)a = 0$. We also have

$$x + 1 = (1 - e_0) + (e_0 - e_1) + e_1(x + 1). \quad (\text{e6.7})$$

Since $e_0 - e_1 \lesssim a \sim a^2$, there is $r \in A$ such that

$$(e_0 - e_1 - \varepsilon)_+ = r^* a^2 r. \quad (\text{e6.8})$$

Note that, since $(e_0 - e_1)e^{(k_0)} = e^{(k_0)}(e_0 - e_1) = 0$, and $a \in \text{Her}_A(e^{(k_0)})$, we have

$$(ar)^2 = 0. \quad (\text{e6.9})$$

Let $C = \{z \in A : za = az = 0\}$. Then C is a hereditary C^* -subalgebra of A . Since e_1 commutes with $x + 1$ and $a(x + 1) = (x + 1)a = 0$, we have $ae_1(x + 1) = a(x + 1)e_1 = 0$, and $e_1(x + 1)a = 0$. Thus $e_1(x + 1) \in C$. Let $D = \text{Her}_A(e_1(x + 1)) \subset C$, which is a σ -unital hereditary C^* -subalgebra. Note that, since D is projectionless, $e_1(x + 1)$ is not invertible in D . By the assumption of the theorem, there are two nilpotent elements $s_1, s_2 \in D$ such that

$$s_1 s_2 \approx_\varepsilon e_1(x + 1). \quad (\text{e6.10})$$

Since $e_0 e_1 = e_1$, and $D \subset \text{Her}_A(e_1)$, we have

$$s_1(1 - e_0) = (1 - e_0)s_1 = 0 = (1 - e_0)s_2 = s_2(1 - e_0). \quad (\text{e6.11})$$

Also note that (recall, $s_1, s_2 \in D \subset C = \{a\}^\perp$)

$$(1 - e_0)a = 0 = a(1 - e_0), \quad as_1 = s_1a = 0, \quad as_2 = s_2a = 0. \quad (\text{e6.12})$$

Then by (e6.7), (e6.8), (e6.10), (e6.11), and (e6.12),

$$x + 1 = (1 - e_0) + (e_0 - e_1) + e_1(x + 1) \quad (\text{e6.13})$$

$$\approx_{2\varepsilon} (1 - e_0) + (e_0 - e_1 - \varepsilon)_+ + s_1 s_2 \quad (\text{e6.14})$$

$$= (1 - e_0) + r^* a^2 r + s_1 s_2 \quad (\text{e6.15})$$

$$= ((1 - e_0)^{1/2} + r^* a + s_1)((1 - e_0)^{1/2} + ar + s_2). \quad (\text{e6.16})$$

Let $\alpha = (1 - e_0)^{1/2}$, $\beta = ar$, $\gamma = s_2$. Then $(1 - e_0)^{1/2} + ar + s_2 = \alpha + \beta + \gamma$. Note that α is a positive element, γ is a nilpotent element, and by (e6.12), $\alpha\beta = \gamma\beta = 0$, by (e6.11), $\alpha\gamma = \gamma\alpha = 0$, by (e6.9), $\beta^2 = 0$. Then by Lemma 6.1 and Corollary 6.2, $\alpha + \beta + \gamma = (1 - e_0)^{1/2} + ar + s_2$ can be approximated by invertible elements in $\tilde{A}$.

The same argument also holds for $(1 - e_0)^{1/2} + ar + s_1^* = ((1 - e_0)^{1/2} + r^*a + s_1)^*$. Thus $(1 - e_0)^{1/2} + r^*a + s_1$ also can be approximated by invertible elements in $\tilde{A}$. Then by (e6.16), we obtain $z' \in GL(\tilde{A})$ such that $\|(x + 1) - z'\| < 2\varepsilon$. Since ε is arbitrary, this implies that

$$x + 1 \in \overline{GL(\tilde{A})} \quad (\text{e6.17})$$

as desired. Therefore A has stable rank one. $\square$

We have the following dichotomy for separable simple tracially approximately divisible C^* -algebras.

Corollary 6.5. *Let A be a separable simple C^* -algebra which is tracially approximately divisible. Then either A is purely infinite or A has stable rank one.*

Proof. We assume that A is not purely infinite. By Theorem 5.2 and Corollary 5.1 of [39] (see also Proposition 4.9 of [16]), A is stably finite. So from now on we will assume that A is stably finite.

We will use the fact that every hereditary C^* -subalgebra of A is tracially approximately divisible (by [16, Theorem 5.5]). Suppose that A contains a non-zero projection p . Then the unital hereditary C^* -subalgebra pAp has stable rank one, by Corollary 5.4. It follows that A also has stable rank one.

We now assume that A is projectionless.

By Theorem 5.7, we can choose $e \in \text{Ped}(A)_+ \setminus \{0\}$ such that $\widehat{\langle e \rangle}$ is continuous on $\widetilde{QT}(A)$. Let $A_0 = \overline{eAe}$ and let $QT(A_0) = \{\tau \in \widetilde{QT}(A) : d_\tau(e) = 1\}$. Recall that $\widehat{\langle e \rangle} (= d_\tau(e))$ is continuous on $\widetilde{QT}(A)$. It follows that $QT(A_0)$ is compact (see also the last paragraph of Definition 2.8). We claim that A_0 has continuous scale (see the proof of Proposition 5.4 of [12]). Indeed, let $\{e_n\}$ be an approximate identity with property that $e_{n+1}e_n = e_ne_{n+1} = e_n$. Then $\langle e_n \rangle$ converges uniformly on $QT(A_0)$. Therefore, by strict comparison, for any $a \in A_+ \setminus \{0\}$, there exists $N \geq 1$ such that, for all $m > n \geq N$,

$$e_m - e_n \lesssim a. \quad (\text{e6.18})$$

This proves the claim.

Next we claim that, by the proof of [16, Theorem 5.7], every element in a projectionless simple C^* -algebra which is tracially approximately divisible can be approximated by the products of two nilpotent elements. To see this, let $x' \in A$. Since A is a non-unital separable C^* -algebra, for any $\varepsilon > 0$, there are $a \in A_+ \setminus \{0\}$ and $x \in A$ such that $x \approx_\varepsilon x'$ and $ax = xa = 0$. It then suffices to show that x can be approximated by products of two nilpotents. Then the proof of [16, Theorem 5.7] from the second paragraph can be applied. Note that in the last estimate (e 5.29) at the end of that proof, v and w are nilpotents. This proves the claim.

As we pointed out at the beginning of the proof, every hereditary C^* -subalgebra of A_0 is tracially approximately divisible. So Theorem 6.4 implies that A_0 has stable rank one. By [34, Theorem 3.6], $A_0 \otimes \mathcal{K}$ has stable rank one, so does $A \otimes \mathcal{K}$ (by [6]). It follows from Corollary 3.6 of [7], A itself has stable rank one. $\square$

Definition 6.6 [16, Definition 3.1]. A simple C^* -algebra A is essentially tracially in the class of $\mathcal{Z}$ -stable C^* -algebras, if for any finite subset $\mathcal{F} \subset A$, any $\varepsilon > 0$, any $s \in A_+ \setminus \{0\}$, there exist an element $e \in A_+^1$ and a non-zero C^* -subalgebra B of A which is $\mathcal{Z}$ -stable, such that

- (1) $\|ex - xe\| < \varepsilon$ for all $x \in \mathcal{F}$,
- (2) $(1 - e)x \in_{\varepsilon} B$ and $\|(1 - e)x\| \geq \|x\| - \varepsilon$ for all $x \in \mathcal{F}$, and
- (3) $e \lesssim s$.

Theorem 6.7. Let A be a separable simple C^* -algebra which is essentially tracially in the class of $\mathcal{Z}$ -stable C^* -algebras. Then A is purely infinite, or A has stable rank one and $\text{Cu}(A) = (V(A) \setminus \{0\}) \sqcup \text{LAff}_+(\widetilde{QT}(A))$.

Proof. It follows from [16, Theorem 5.9] that A is tracially approximately divisible. Then, by Theorem 5.7 and Corollary 6.5, A is purely infinite, or has stable rank one, and $\text{Cu}(A) = (V(A) \setminus \{0\}) \sqcup \text{LAff}_+(\widetilde{QT}(A))$. $\square$

Rørdam showed that every unital simple $\mathcal{Z}$ -stable C^* -algebra is either purely infinite, or has stable rank one (see [39]). In [36], L. Robert showed that every stably projectionless simple $\mathcal{Z}$ -stable C^* -algebra has almost stable rank one and left open whether it actually has stable rank one. The following corollary answers his question affirmatively.

Corollary 6.8. Let A be a simple $\mathcal{Z}$ -stable C^* -algebra. Then A is either purely infinite or has stable rank one.

Proof. If A contains a non-zero projection p , then by [39, Theorem 6.7], pAp is either purely infinite, or has stable rank one. So the corollary follows by [6] and [34]. Therefore we may assume that A is projectionless. Let $x + \lambda \in \widetilde{A}$, where $x \in A$ and $\lambda \in \mathbb{C}$. Let $\varepsilon > 0$. Since $\mathcal{Z} \cong \bigotimes_{n=1}^{\infty} \mathcal{Z}$ (see [20, Corollary 8.8]) and A is $\mathcal{Z}$ -stable, there is an isomorphism $\alpha : A \otimes \mathcal{Z} \rightarrow A$ such that $\alpha(x \otimes 1_{\mathcal{Z}}) \approx_{\varepsilon/2} x$ (see [39, Lemma 4.4]). Note that α extends to an isomorphism $\widetilde{\alpha} : (A \otimes \mathcal{Z})^{\sim} \rightarrow \widetilde{A}$.

By the fact that A is simple and by [3, Proposition 2.2], we obtain a separable simple C^* -algebra $B \subset A$ containing $C^*(x)$. Then $B \otimes \mathcal{Z}$ is separable, simple, $\mathcal{Z}$ -stable and projectionless. By Theorem 6.7, there is an invertible element $z \in GL((B \otimes \mathcal{Z})^{\sim}) \subset GL((A \otimes \mathcal{Z})^{\sim})$ such that $z \approx_{\varepsilon/2} x \otimes 1_{\mathcal{Z}} + \lambda$. Then $\widetilde{\alpha}(z) \in GL(\widetilde{A})$ and $\widetilde{\alpha}(z) \approx_{\varepsilon/2} \widetilde{\alpha}(x \otimes 1_{\mathcal{Z}} + \lambda) \approx_{\varepsilon/2} x + \lambda$. It follows that A has stable rank one. $\square$

Proposition 6.9. Let A be a unital infinite-dimensional separable simple C^* -algebra with tracial rank zero. Then A is essentially tracially in the class of $\mathcal{Z}$ -stable C^* -algebras.

Proof. Let $\varepsilon > 0$, let $\mathcal{F} \subset A$ be a finite subset of A and let $a \in A_+^1 \setminus \{0\}$. Since A has tracial rank zero, there is a non-zero projection $p \in A$ and a finite-dimensional C^* -subalgebra $D \subset A$ with $1_D = p$ such that

- (1) $xp \approx_{\varepsilon/2} px$ for all $x \in \mathcal{F}$,
- (2) $pxp \in_{\varepsilon/2} D$, and
- (3) $1 - p \lesssim a$.

Write $D = M_{r(1)} \oplus M_{r(2)} \oplus \cdots \oplus M_{r(m)}$. Let $\{e_{i,j}^{(k)}\}_{1 \leq i, j \leq r(k)}$ be a system of matrix units for $M_{r(k)}$, $1 \leq k \leq m$. By [32, Corollary 4.4], for each k , there is a unital simple AF-algebra B_k and unital embedding $\varphi_k : B_k \rightarrow \overline{e_{1,1}^{(k)} A e_{1,1}^{(k)}}$ such that $V(\varphi) : V(B_k) \rightarrow V(\overline{e_{1,1}^{(k)} A e_{1,1}^{(k)}}) = V(A)$ is surjective.

We claim that $V(A)$ is not finitely generated. To see this, suppose that $V(A)$ is generated by $[p_1], [p_2], \dots, [p_m]$. We may assume that $p_i \in M_l(A)$ (for some $l \geq 1$) is a non-zero projection, $1 \leq i \leq m$. Note that $M_l(A)$ is also a unital infinite-dimensional simple C^* -algebra of real rank zero. By repeatedly applying [47, Lemma 1.1], for example, we obtain a sequence of non-zero projections $\{q_n\} \subset M_l(A)$ such that $\lim_{n \rightarrow \infty} \sup\{\tau(q_n) : \tau \in T(A)\} = 0$. Then, for any non-negative integers $k_1, k_2, \dots, k_m$ (not all zero), there is an integer $N \geq 1$ such that $\tau(q_N) < \sum_{i=1}^m k_i \tau(p_i)$ for all $\tau \in T(A)$. Hence $[q_N]$ is not in $V(A)$. This proves the claim.

Since $V(A)$ can not be finitely generated, we deduce that each $V(B_k)$ is not finitely generated either, and in particular each B_k is infinite-dimensional. Define $C_k := \{\varphi_k(b) \otimes e_{i,j}^{(k)} : 1 \leq i, j \leq r(k), b \in B_k\} \cong B_k \otimes M_{r(k)}$ ($1 \leq k \leq m$) and $C := \bigoplus_{k=1}^m C_k$. Then, $D \subset C$. By (2),

$$p x p \in_{\varepsilon/2} C. \quad (\text{e6.19})$$

Since each C_k is a unital simple infinite-dimensional AF-algebra, C_k is $\mathcal{Z}$ -stable, $1 \leq k \leq m$ (see [20, Corollary 6.3]). Therefore C is $\mathcal{Z}$ -stable. By (1), (e6.19) and (3), A is essentially tracially in the class of $\mathcal{Z}$ -stable C^* -algebras. $\square$

Example 6.10. In [31], Niu and Wang constructed a class of separable simple exact non-nuclear C^* -algebras which have tracial rank zero but not $\mathcal{Z}$ -stable. Then, by Proposition 6.9, Niu and Wang's examples are unital separable simple exact C^* -algebras which are essentially tracially $\mathcal{Z}$ -stable but not $\mathcal{Z}$ -stable. By [16, Theorem 5.9], these C^* -algebras are particularly tracially approximately divisible.

7 | EXAMPLES

Example 7.1. It is shown in [16, Theorem 5.9] that a simple C^* -algebra A which is essentially tracially in $C_{\mathcal{Z}}$ (see [16, Notation 4.1]), then A is tracially approximately divisible. Any simple C^* -algebras A_z^C constructed in Theorem 8.4 of [16] and any hereditary C^* -subalgebra of A_z^C (by [16, Proposition 3.5]) are tracially approximately divisible. Therefore all (non-unital hereditary C^* -subalgebras) of C^* -algebras in [16, Theorem 8.6] are tracially approximately divisible and non-nuclear C^* -algebras. By Corollary 6.5, all these C^* -algebras have stable rank one.

Recall that a II_1 factor (N, τ) is said to have property Γ , if there is a sequence of unitaries $\{u_n\} \subset N$ satisfying $\lim_{n \rightarrow \infty} \|u_n x - x u_n\|_2 = 0$ for all $x \in N$, and $\tau(u_n) = 0$ for all $n \in \mathbb{N}$.

The following is well-known to experts.

Proposition 7.2. *Let A be a unital infinite-dimensional separable simple C^* -algebra with a unique tracial state τ which is also tracially approximately divisible. Let π_τ be the GNS representation with respect to τ , and $N := \pi_\tau(A)''$ the weak closure of $\pi_\tau(A)$. Then (N, τ) is a II_1 factor with property Γ .*

Proof. Since τ is an extreme point of $T(A) = \{\tau\}$, N is a II_1 factor ([10, Theorem 6.7.3]). From Theorem 4.11 we know that if A is tracially approximately divisible then A has property

(TAD-3). Thus there is a unital embedding $\hat{\psi} : M_2 \rightarrow \pi_{cu}(A)'$. By Proposition 3.10 and projectivity of $C_0((0, 1]) \otimes M_2$, there is a homomorphism $\bar{\psi} : C_0((0, 1]) \otimes M_2 \rightarrow \pi_\infty(A)'$ such that $\pi_\infty \circ \bar{\psi}(\iota \otimes e_{i,j}) = \hat{\psi}(e_{i,j})$, $1 \leq i, j \leq 1$. Again, using projectivity of $C_0((0, 1]) \otimes M_2$, there is a homomorphism $\psi : C_0((0, 1]) \otimes M_2 \rightarrow l^\infty(A) \subset l^\infty(N)$ that lifts $\bar{\psi}$. We may represent ψ by a sequence of homomorphisms $\psi_n : C_0((0, 1]) \otimes M_2 \rightarrow A \subset N$. Then $\{\psi_n\}$ satisfies the following:

- (1) $\lim_{n \rightarrow \infty} \|\psi_n(x)a - a\psi_n(x)\| = 0$ for all $x \in C_0((0, 1]) \otimes M_2$ and all $a \in A$, and
- (2) $\{1_A - \psi_n(\iota \otimes 1_{M_2})\} \in N_{cu}(A)$.

By Proposition 3.8, (2) implies $\lim_{n \rightarrow \infty} \tau(\psi_n(1_{M_2})) = 1$, hence

$$\lim_{n \rightarrow \infty} \tau(\psi_n(\iota \otimes e_{1,1})) = \lim_{n \rightarrow \infty} \tau(\psi_n(\iota \otimes e_{2,2})) = 1/2. \quad (\text{e7.1})$$

Let $y \in N$ and $\varepsilon > 0$. Let $\|x\|_2 = \tau(x^*x)^{1/2}$ for all $x \in N$. Since A is dense in N in the strong operator topology, there is $z \in A$ such that $\|y - z\|_2 < \varepsilon/4$. By (1), there is $K \in \mathbb{N}$ such that $\|\psi_n(\iota \otimes e_{i,i})z - z\psi_n(\iota \otimes e_{i,i})\| < \varepsilon/2$ for all $n \geq K$ and $i \in \{1, 2\}$. Then

$$\|\psi_n(\iota \otimes e_{i,i})y - y\psi_n(\iota \otimes e_{i,i})\|_2 \leq \|\psi_n(\iota \otimes e_{i,i})z - z\psi_n(\iota \otimes e_{i,i})\|_2 + \|y - z\|_2 < \varepsilon. \quad (\text{e7.2})$$

It follows (e7.1) and (e7.2) that $\{\psi_n(\iota \otimes e_{1,1})\}$ and $\{\psi_n(\iota \otimes e_{2,2})\}$ are two mutually orthogonal nontrivial central sequences of N . Therefore N has property Γ (see, for example [40, Lemma A.7.3]). $\square$

We now present an example of unital non-elementary separable simple exact (but non-nuclear) C^* -algebra that has stable rank one, a unique tracial state, strict comparison, and 0-almost divisible Cuntz semigroup, and contains a unital embedded copy of the Jiang-Su algebra $\mathcal{Z}$, but is not tracially approximately divisible.

Example 7.3. Let $C_r^*(\mathbb{F}_\infty)$ be the reduced group C^* -algebra of the free group on countably infinitely many generators. It is well known that $C_r^*(\mathbb{F}_\infty)$ is a unital infinite-dimensional separable simple C^* -algebra with a unique tracial state τ . It is also well known that $C_r^*(\mathbb{F}_\infty)$ is exact. Moreover, $C_r^*(\mathbb{F}_\infty)$ has stable rank one ([11]) and has strict comparison for positive elements (see [35, Proposition 6.3.2]). Hence, the Cuntz semigroup of $C_r^*(\mathbb{F}_\infty)$ is almost divisible by [41, Corollary 8.12]. By [35, Proposition 6.3.1], The Jiang-Su algebra $\mathcal{Z}$ can be unitaly embedded into $C_r^*(\mathbb{F}_\infty)$. On the other hand, the group von Neumann algebra $L(\mathbb{F}_\infty)$ does not have property Γ (see, e.g., [40, Theorem A.7.2]). It follows from Proposition 7.2 that $C_r^*(\mathbb{F}_\infty)$ can not be tracially approximately divisible.

From a recent result of Ma and Wu in [29] on groupoid C^* -algebras, let us restate the following.

Theorem 7.4 (c.f. [29, Theorem 9.7]). *Let G be a locally compact, second countable and Hausdorff étale minimal groupoid on a compact metrizable space without isolated points. Suppose G is almost elementary. Then $C_r^*(G)$ is unital, separable, simple, tracially $\mathcal{Z}$ -absorbing, and, is either purely infinite, or has stable rank one.*

Proof. By [29, Theorem 9.7], $C_r^*(G)$ is unital, separable, simple and is tracially $\mathcal{Z}$ -absorbing in the sense of [18]. It follows from Theorem 4.11 ((3) $\Rightarrow$ (2)) and Corollary 6.5 that $C_r^*(G)$ either has stable rank one or is purely infinite. $\square$

We end this section by the following dichotomy result on flow actions (see [35, Section 7] for examples of both cases).

Theorem 7.5. *Let A be a separable C^* -algebra with finite nuclear dimension, and let $\alpha : \mathbb{R} \rightarrow \text{Aut}(A)$ be a flow with no α -invariant ideals and with finite Rokhlin dimension. Then $A \rtimes_{\alpha} \mathbb{R}$ is either purely infinite or has stable rank one.*

Proof. By [19, Theorem 4.5], $A \rtimes_{\alpha} \mathbb{R}$ has finite nuclear dimension. Since A is separable and has no α -invariant ideals, and α has finite Rokhlin dimension, we deduce that $A \rtimes_{\alpha} \mathbb{R}$ is separable and simple (see [19, Corollary 3.12]). Then by [42, Corollary 8.7], $A \rtimes_{\alpha} \mathbb{R}$ is $\mathcal{Z}$ -stable. Then by Corollary 6.8, $A \rtimes_{\alpha} \mathbb{R}$ is either purely infinite or has stable rank one. $\square$

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