Journal of Combinatorics
Volume 13, Number 3, 357–395, 2022

Poset topology of s-weak order via SB-labelings

Stephen Lacina*

Ceballos and Pons generalized weak order on permutations to a partial order on certain labeled trees, thereby introducing a new class of lattices called s-weak order. They also generalized the Tamari lattice by defining a particular sublattice of s-weak order called the s-Tamari lattice. We prove that the homotopy type of each open interval in s-weak order and in the s-Tamari lattice is either a ball or sphere. We do this by giving s-weak order and the s-Tamari lattice a type of edge labeling known as an SB-labeling. We characterize which intervals are homotopy equivalent to spheres and which are homotopy equivalent to balls; we also determine the dimension of the spheres for the intervals yielding spheres.

AMS 2000 subject classifications: 06A07.
Keywords and phrases: s-weak order, s-Tamari lattice, SB-labeling, poset topology.

1. Introduction

In [3], Ceballos and Pons introduced a partial order called **s-weak order** on certain labeled trees known as **s-decreasing trees**. They observed that this partial order generalizes weak order on permutations. They proved s-weak order is a lattice. They also found a particular class of s-decreasing trees which play the role of 231-avoiding permutations. This led them to introduce a sublattice of s-weak order called the s-Tamari lattice, generalizing the Tamari lattice.

Our main result is the following theorem:

Theorem 1.1. *The lattices s-weak order and the s-Tamari lattice each admit an SB-labeling. Thus, the order complex of each open interval in s-weak order and the s-Tamari lattice is homotopy equivalent to a ball or sphere of some dimension.*

We prove this as Theorem 3.19 for s-weak order and Theorem 4.13 for the s-Tamari lattice. In both cases, we prove topological results using the

*The author was supported by NSF grants DMS-1953931 and DMS-1500987.

tool of SB-labelings developed by Hersh and Mészáros in [6]. Our result generalizes another result of Hersh and Mészáros that weak order on permutations and the classical Tamari lattice admit SB-labelings, with our labelings specializing in those cases to SB-labelings distinct from theirs.

In s-weak order and the s-Tamari lattice, the spheres in Theorem 1.1 are not always top dimensional, demonstrating that these posets are not always shellable. See [1] for example for the definition of a shellable poset. We intrinsically characterize which intervals in s-weak order and the s-Tamari lattice are homotopy equivalent to spheres and which are homotopy equivalent to balls. We also determine the dimension of the spheres for the intervals yielding homotopy spheres. As a corollary, we deduce that the Möbius functions of s-weak order and the s-Tamari lattice only take values in $\{-1, 0, 1\}$. It is also known that the existence of an SB-labeling implies that distinct sets of atoms in an interval have distinct joins, giving another consequence of our results.

Part of Ceballos and Pons' interest in s-weak order comes from geometry. They conjecture that the Hasse diagrams of s-weak order are the 1-skeleta of polytopal subdivisions of polytopes. They call these potential polytopal complexes **s-permutahedra**. They also conjecture that in particular cases the polytopes they are subdividing are classical permutahedra. Our result of an SB-labeling for s-weak order, though it considers these lattices from a topological perspective, seems to provide two pieces of evidence for Ceballos and Pons' conjecture. The first piece of evidence is that the Hasse diagrams of many lattices which admit SB-labelings can be realized as the 1-skeleta of polytopes. The second comes from the fact that Ceballos and Pons' geometric perspective is somewhat similar in flavor to one point of view in Hersh's work in [5]. Hersh studied posets which arise as the 1-skeleta of simple polytopes via directing edges by some cost vector. In particular, Hersh's Theorem 4.9 in [5] proves that all open intervals in lattices which are realizable as such 1-skeleta of simple polytopes are either homotopy balls or spheres.

Similarly, Ceballos and Pons' also took a geometric viewpoint on the s-Tamari lattice. They showed that the s-Tamari lattice is isomorphic to another generalization of the classical Tamari lattice, namely the ν-Tamari lattice introduced by Préville-Ratelle and Viennot in [7]. The geometry of the ν-Tamari lattice was recently studied by Ceballos, Padrol, and Sarmiento in [2]. Similarly to how the Hasse diagram of the Tamari lattice is the 1-skeleton of the associahedron, the Hasse diagram of the ν-Tamari lattice is the 1-skeleta of a polytopal subdivision of a polytope. Thus, the s-Tamari lattice also has such a realization. In the context of the s-Tamari lattice,

Ceballos and Pons call these polytopal complexes **s-associahedra**. Further, they conjecture that in particular cases s-associahedra can be obtained from the s-permutahedra by deleting certain facets. The fact that the s-Tamari lattice admits an SB-labeling and has a realization as the 1-skeleton of a polytopal complex seems to strengthen the evidence given by our result for Ceballos and Pons' conjecture of such realizations for s-permutahedra. Additionally, our result contributes two new classes of lattices which admit SB-labelings.

This paper proceeds as follows: Section 2 provides the necessary background on posets, s-decreasing trees, s-weak order, and the s-Tamari lattice. We largely follow the notation and definitions of [3]. We also observe that s-weak order is not always a Cambrian lattice. Section 2 reviews the notion of SB-labeling as well. Section 3 and Section 4 are where we prove our main results, most notably giving SB-labelings for s-weak order and the s-Tamari lattice.

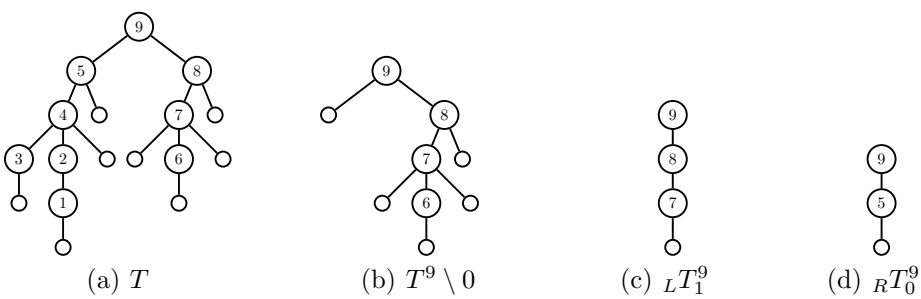
2. Background

2.1. Background on posets

Let $(P, \leq)$ be a poset. For $x \leq y \in P$, the **closed interval** from x to y is the set $[x, y] = \left\{ z \in P \;\middle|\; x \leq z \leq y \right\}$. **The open interval from x to y** is defined analogously with strict inequalities and denoted (x, y). We say that y **covers** x, denoted $x \lessdot y$, if $x \leq z \leq y$ implies $z = x$ or $z = y$. P is a **lattice** if each pair $x, y \in P$ has a unique least upper bound, denoted $x \vee y$, and a unique greatest lower bound, denoted $x \wedge y$. We denote by $\hat{0}$ (respectively $\hat{1}$) the unique minimal (respectively unique maximal) element of a finite lattice. The elements which cover $\hat{0}$ are called **atoms**. For $x, y \in P$ with $x < y$, a **k-chain from x to y** in P is a subset $C = \{x_0, x_1, \ldots, x_k\} \subset P$ such that $x = x_0 < x_1 < \cdots < x_k = y$. A chain C is said to be **saturated** if $x_i \lessdot x_{i+1}$ for all i. The **order complex** of P, denoted $\Delta(P)$, is the abstract simplicial complex with vertices the elements of P and i-dimensional faces the i-chains of P. For $x, y \in P$ with $x < y$, we denote by $\Delta(x, y)$ the order complex of the open interval (x, y) as an induced subposet of P. Thus, when we refer to topological properties of P, we mean the topological properties of a geometric realization of $\Delta(P)$. In particular, the homotopy type of P refers to the homotopy type of $\Delta(P)$. It is well known that the Möbius function of P μ_P satisfies $\mu_P(x, y) = \tilde{\chi}(\Delta(x, y))$. Here, $\tilde{\chi}$ is the reduced Euler characteristic. This provides one of the important connections between the combinatorial and enumerative structure of a poset and its topology.

2.2. Background on s-weak order

A **weak composition** is a sequence of non-negative integers $s = (s(1), \ldots, s(n))$ with $s(i) \in \mathbb{N}$ for all $i \in [n]$. We say the **length** of a weak composition s is $l(s) = n$. Let s be a weak composition. An **s-decreasing tree** is a planar rooted tree T with n internal vertices which are labeled 1 to n (leaves are not labeled and are the only unlabeled vertices) such that internal vertex i has $s(i) + 1$ children and all labeled descendants of i have labels less than i. The $s(i) + 1$ children of i are indexed by 0 to $s(i)$. We denote the full subtree of T rooted at i by T^i, and denote the full subtrees rooted at the $s(i) + 1$ children of i by $T^i_0, \ldots, T^i_{s(i)}$, respectively. For i and $0 \leq j \leq s(i)$, we denote by $T^i \setminus j$, the subtree of T obtained from T^i by replacing T^i_j with a leaf. Also, $T^i_{j_1,\ldots,j_k}$ will denote the forest of the full subtrees rooted at the $j_1, \ldots, j_k$ children of i. Let k be the jth child of i in T. We define the **jth left subtree of i in T**, denoted ${}_LT^i_j$, to be the subtree of T with root i obtained by walking from i to k and then down the left most subtree possible until reaching a leaf. Similarly, we define the **jth right most subtree of i in T**, denoted ${}_RT^i_j$, to be the subtree of T with root i obtained by walking from i to k and then down the right most subtree possible until reaching a leaf. We note that ${}_LT^i_j$ and ${}_RT^i_j$ are both always chains. Fig. 1 is an example of an s-decreasing tree with $s = (0, 0, 0, 2, 1, 0, 2, 1, 1)$, along with some examples of the subtrees just defined.



(a) T (b) $T^9 \setminus 0$ (c) ${}_LT^9_1$ (d) ${}_RT^9_0$

Figure 1: An s-decreasing tree T with $s = (0, 0, 0, 2, 1, 0, 2, 1, 1)$ and examples of some defined subtrees.

Definition 2.1 ([3, Definition 2.1])**.** *Let T be an s-decreasing tree and $1 \leq x < y \leq n$. The **cardinality** of (y, x) in T, denoted $\boldsymbol{\#_T(y, x)}$, is defined by the following rules:*

1. $\#_T(y,x) = 0$ *if* x *is left of* y *in* T *or* $x \in T_0^y$*;*
2. $\#_T(y,x) = i$ *if* $x \in T_i^y$ *with* $0 < i < s(y)$*; and*
3. $\#_T(y,x) = s(y)$ *if* $x \in T_{s(y)}^y$ *or* x *is right of* y *in* T*.*

If $\#_T(y,x) > 0$*, then* (y,x) *is said to be a* ***tree inversion*** *of* T*. We denote by* $\mathbf{inv(T)}$ *the multi-set of tree inversions of* T *counted with multiplicity their cardinality.*

Now we can also formally describe the jth left and right subtrees of i in T, examples of which are found in (c) and (d) of Fig. 1.

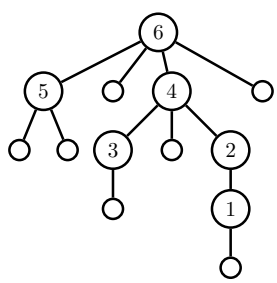
$$ {}_LT_j^i = \left\{ d \in T^i \;\middle|\; d = i, \text{ or } d \in T_j^i \text{ and } \#_T(e,d) = 0 \ \forall e \in T_j^i \text{ such that } d < e \right\}. $$
$$ {}_RT_j^i = \left\{ d \in T^i \;\middle|\; d = i, \text{ or } d \in T_j^i \text{ and } \#_T(e,d) = s(e) \ \forall e \in T_j^i \text{ such that } d < e \right\}. $$

Remark 2.2. For $s = (1, \ldots, 1)$, s-decreasing trees are in by bijection with permutations in $S_{l(s)}$ and tree inversions are precisely inversions of the corresponding permutation.

Remark 2.3. If T is an s-decreasing tree, $1 \le a < b \le n$, and $0 < \#_T(b,a) < s(b)$, then $a \in T^b_{\#_T(b,a)}$.

Remark 2.4. If $e \in T^a$ and $e \in T_i^b$ for some $a < b$, then $a \in T_i^b$. Further, if $e \in T^a$ and $a < b$, then $\#_T(b,e) = \#_T(b,a)$.

Fig. 2 is an s-decreasing tree with the cardinality of each pair of labeled vertices listed.



$\#_T(6,5) = 0$	$\#_T(6,4) = 2$	$\#_T(6,3) = 2$	$\#_T(6,2) = 2$	$\#_T(6,1) = 2$
	$\#_T(5,4) = 1$	$\#_T(5,3) = 1$	$\#_T(5,2) = 1$	$\#_T(5,1) = 1$
		$\#_T(4,3) = 0$	$\#_T(4,2) = 2$	$\#_T(4,1) = 2$
			$\#_T(3,2) = 0$	$\#_T(3,1) = 0$
				$\#_T(2,1) = 0$

Figure 2: An s-decreasing tree and its cardinalities for $s = (0,0,0,2,1,3)$.

Next we establish notation for sets of tree inversions examples of which follow Fig. 3 using s-decreasing trees from those examples of s-weak order.

Definition 2.5 ([3, Definition 2.2]). *A* ***multi-inversion set*** *on* $[n]$ *is a multi-set* I *of pairs* (y, x) *such that* $1 \leq x < y \leq n$. *We write* $\boldsymbol{\#_I(y, x)}$ *for the multiplicity of* (y, x) *in* I *so if* (y, x) *does not appear in* I, $\#_I(y, x) = 0$.

Given multi-inversion sets I *and* J, *we say* I *is* ***included*** *in* J *and write* $I \subseteq J$ *if* $\#_I(y, x) \leq \#_J(y, x)$ *for all* $1 \leq x < y \leq n$. *We also define the* ***multi-inversion set difference***, $\boldsymbol{J - I}$, *to be the multi-inversion set with* $\#_{J-I}(y, x) = \#_J(y, x) - \#_I(y, x)$ *whenever this difference is non-negative and 0 otherwise.*

This leads to a characterization of those multi-inversion sets which are actually sets of tree inversions of s-decreasing trees. Further, it motivates the definition of s-weak order in analogy with the inversion set definition of weak order on permutations.

Proposition 2.6 ([3, Proposition 2.4]). *There is a bijection between* s-*decreasing trees and multi-inversion sets* I *satisfying* $\#_I(y, x) \leq s(y)$ *and the following two properties:*

- ***Transitivity****: if* $a < b < c$ *and* $\#_I(c, b) = i$, *then* $\#_I(b, a) = 0$ *or* $\#_I(c, a) \geq i$.
- ***Planarity****: if* $a < b < c$ *and* $\#_I(c, a) = i$, *then* $\#_I(b, a) = s(b)$ *or* $\#_I(c, b) \geq i$.

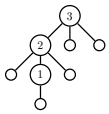
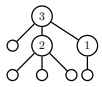
Such multi-inversion sets are called s***-tree inversion sets****.*

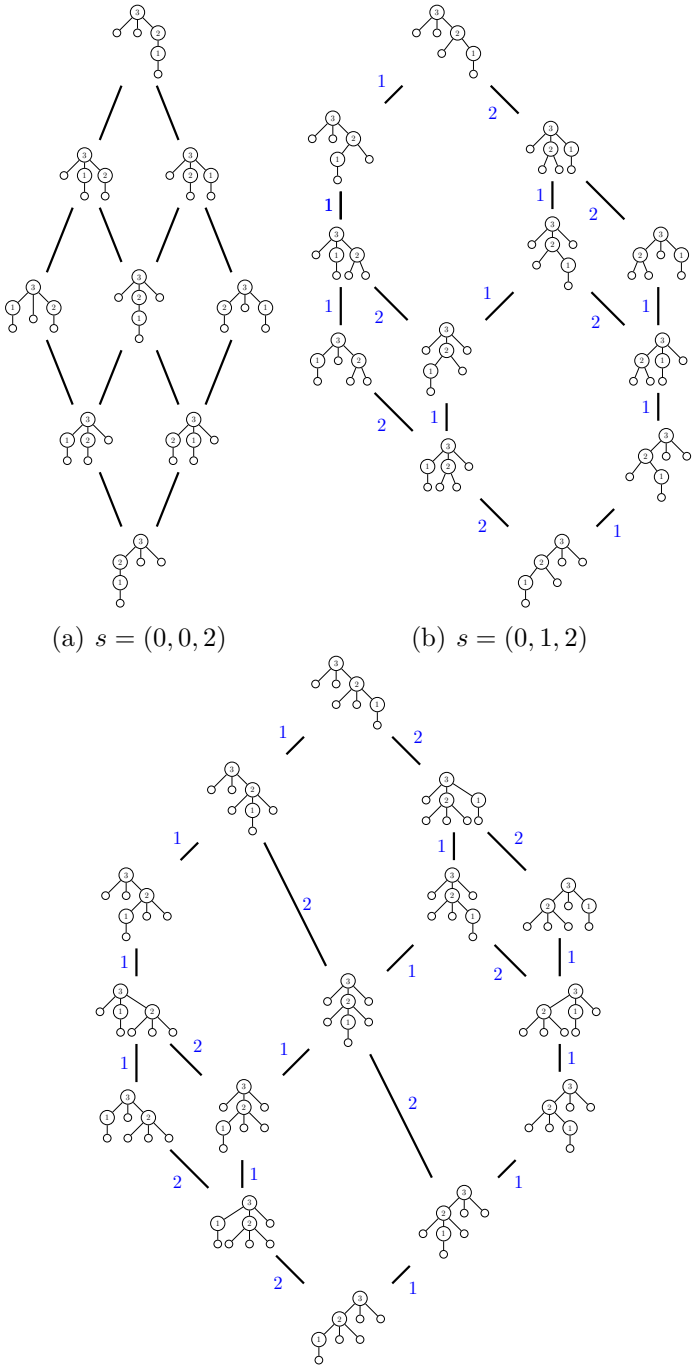
Definition 2.7 ([3, Definition 2.5]). *Let* s *be a weak composition. The* s-***weak order*** *is the partial order on* s*-decreasing trees given by* $T \preceq Z$ *if and only if* $inv(Z) \subseteq inv(T)$ *for* s*-decreasing trees* T *and* Z *using the inclusion of multi-inversion sets from Definition 2.5.*

Fig. 3 shows three examples of s-weak order. The labelings of the last two examples is our SB-labeling which is defined in Section 3.

Below in Example 1, we illustrate Definition 2.5 and Proposition 2.6. We use subscripts on pairs (y, x) to indicate their multiplicity in a multi-inversion set.

Example 1. Illustrating Definition 2.5 and Proposition 2.6, we take

$T_1 =$  and $T_2 =$ 



(a) $s = (0, 0, 2)$ (b) $s = (0, 1, 2)$

(c) $s = (0, 2, 2)$

Figure 3: Examples of s-weak order. The labeling is our SB-labeling in Definition 3.1.

and observe that $\mathrm{inv}\,(T_1) = \{(2,1)_1\}$ and $\mathrm{inv}\,(T_2) = \{(2,1)_2, (3,1)_2, (3,2)_1\}$. Thus, $\mathrm{inv}\,(T_1) \subseteq \mathrm{inv}\,(T_2)$ and $\mathrm{inv}\,(T_2) - \mathrm{inv}\,(T_1) = \{(2,1)_1, (3,1)_2, (3,2)_1\}$. Now we note that while $\mathrm{inv}\,(T_1) = \{(2,1)_1\}$ is transitive, $I = \{(2,1)_1, (3,2)_1\}$ is not transitive because $\#_I(3,2) = 1$ while $\#_I(2,1) = 1 \neq 0$ and $\#_I(3,1) = 0 < \#_I(3,2)$. Similarly, $\mathrm{inv}\,(T_2) = \{(2,1)_2, (3,1)_2, (3,2)_1\}$ is planar while $J = \{(2,1)_1, (3,1)_1\}$ is not planar because $\#_J(3,1) = 1$, but $\#_J(2,1) = 1 \neq 2 = s(2)$ and $\#_J(3,2) = 0 < \#_J(3,1)$.

Remark 2.8. Taking $s = (1, \ldots, 1)$, s-weak order is isomorphic to weak order on the symmetric group $S_{l(s)}$.

The following operations on multi-inversion sets are necessary to formulate the join in s-weak order which we will use in the course of our proofs. We give examples of these operations in Example 2 below.

- For weak composition s and multi-inversion sets I and J satisfying $\#_I(y,x), \#_J(y,x) \leq s(y)$ for all $1 \leq x < y \leq n$, the **union of I and J** is the smallest multi-inversion set by inclusion $\boldsymbol{I \cup J}$ such that $I, J \subseteq I \cup J$, that is $\#_{I \cup J}(y,x) = \max\{\#_I(y,x), \#_J(y,x)\}$ for all $1 \leq x < y \leq n$. Also, the **sum of I and J** is the multi-inversion set $\boldsymbol{I + J}$ with $\#_{I+J}(y,x) = \min\{\#_I(y,x) + \#_J(y,x), s(y)\}$ for all $1 \leq x < y \leq n$. If $J = \{(b,a)\}$, we write $\boldsymbol{I + (b,a)}$ for $I + J$.
- The **transitive closure**, denoted $\boldsymbol{I^{tc}}$, of a multi-inversion set I is the smallest transitive multi-inversion set, in terms of inclusion, containing I.

Theorem 2.9 ([3, Theorem 2.6])**.** *For any weak composition s, the s-weak order on s-decreasing trees is a lattice. The join of two s-decreasing trees T and Z is determined by*

$$inv\,(T \vee Z) = (inv\,(T) \cup inv\,(Z))^{tc}.$$

Example 2. This example illustrates the union and sum of multi-inversion sets as well as the transitive closure. Letting T_1 be the same s-decreasing tree as in Example 1, $\mathrm{inv}\,(T_1) = \{(2,1)_1\}$. Now $\mathrm{inv}\,(T_1) \cup \mathrm{inv}\,(T_1) = \{(2,1)_1\}$ while $\mathrm{inv}\,(T_1) + \mathrm{inv}\,(T_1) = \{(2,1)_2\}$. In Example 1, we saw that the multi-inversion set $\{(2,1)_1, (3,2)_1\}$, which is also $\mathrm{inv}\,(T_1) + (3,2)$, is not transitive. From our observations in Example 1, to satisfy the definition of transitivity in Proposition 2.6, $\{(2,1)_1, (3,2)_1\}^{tc}$ must contain $(3,1)$ with multiplicity at least 1. Thus, $\{(2,1)_1, (3,2)_1\}^{tc} = \{(2,1)_1, (3,1)_1,$
$(3,2)_1\}$. We can check that this is the multi-inversion set of one of the two s-decreasing trees covering T_1 in (c) of Fig. 3.

The cover relations in s-weak order are characterized as a certain type of operations known as tree rotations. We use this characterization heavily in our proofs. We first need a notion of an ascent in an s-decreasing tree. In the case $s = (1, \ldots, 1)$, this notion corresponds to the definition of ascents for permutations. Examples of tree ascents of the s-decreasing tree in Fig. 1 are given in Example 3.

Definition 2.10 ([3, Section 2.2])**.** *Let T be an s-decreasing tree and $1 \leq a < b \leq n$. The pair (a, b) is a* ***tree ascent*** *of T if the following hold:*

(i) $a \in T_i^b$ *for some* $0 \leq i < s(b)$,
(ii) if $a \in T_j^e$ *for any* $a < e < b$, *then* $j = s(e)$,
(iii) if $s(a) > 0$, *then* $T_{s(a)}^a$ *is a leaf, that is,* $T_{s(a)}^a$ *contains no internal vertices.*

Example 3. The tree ascents of the s-decreasing tree in (a) of Fig. 1 are as follows: $\{(1, 4), (2, 4), (3, 4), (4, 5), (5, 9), (6, 7), (7, 8)\}$.

Remark 2.11. If $s(b) = 0$, then (a, b) with $a < b$ is not a tree ascent of any s-decreasing tree. This would contradict (i) of Definition 2.10.

Remark 2.12. An s-decreasing tree, T, cannot have tree ascents (a, b) and (a, c) with $b \neq c$. This would contradict condition (ii) of Definition 2.10 as either $a < b < c$ or $a < c < b$ while $a \notin T_{s(b)}^b, T_{s(c)}^c$ by condition (i) of Definition 2.10. We note that this implies that given an element $c \in [n]$ there is at most one $d \in [n]$ such that (c, d) is a tree ascent of T. Further, whenever (a, b) and (c, d) are distinct tree ascents of T, we may assume $a < c$. We make this assumption throughout our proofs.

Remark 2.13. We observe that by Remark 2.4, conditions (i) and (ii) of Definition 2.10 together are equivalent to $a \in {}_RT_i^b$ for some $0 \leq i < s(b)$. The ith rightmost subtree of b in T ${}_RT_i^b$ is defined at the beginning of Section 2.2.

Definition 2.14 ([3, Section 2.2])**.** *Let T be an s-decreasing tree with tree ascent (a, b). Then $(inv(T) + (b, a))^{tc}$ is an s-tree inversion set. We call the s-decreasing tree Z defined by $inv(Z) = (inv(T) + (b, a))^{tc}$ the* ***s-tree rotation of*** *T* ***along*** *(a, b). We denote this by* $\boldsymbol{T \xrightarrow{(a,b)} Z}$.

Ceballos and Pons characterized cover relations in s-weak order with the following theorem.

Theorem 2.15 ([3, Theorem 2.7])**.** *Let T and Z be s-decreasing trees. Then $T \prec Z$ if and only if there is a unique pair (a, b) which is a tree ascent of T such that $T \xrightarrow{(a,b)} Z$.*

Remark 2.16. $s(1)$ does not change the isomorphism type of s-weak order because no tree ascent of an s-decreasing tree may have larger element 1.

Remark 2.17. We describe an s-tree rotation in terms of an operation on the trees themselves. This is illustrated in Fig. 4. Suppose (a, b) is a tree ascent of T and $T \xrightarrow{(a,b)} Z$. Then $a \in {}_RT^b_j$ for some $j < s(b)$. Let g be the parent of a so $a \in T^g_{s(g)}$ and $g \in T^b_j$ or $g = b$ and a is the jth child of b. Let m be the smallest element of ${}_LT^b_{j+1}$ which is still larger than a. It is possible $m = b$. Then Z is the same as T except for the following changes: $Z^g_{s(g)} = T^a_0$ if $g \neq b$ and $Z^b_j = T^a_0$ if $g = b$ instead of T^a, $Z^a_i = T^a_i$ for $0 < i < s(a)$ if $s(a) > 0$, $Z^a_{s(a)} = T^m_0$ if $m \neq b$ and $Z^a_{s(a)} = T^b_{j+1}$ if $m = b$, Z^a_0 is a leaf is a leaf if $s(a) > 0$, and $Z^m_0 = Z^a$ if $m \neq b$ and $Z^b_{j+1} = Z^a$ if $m = b$.

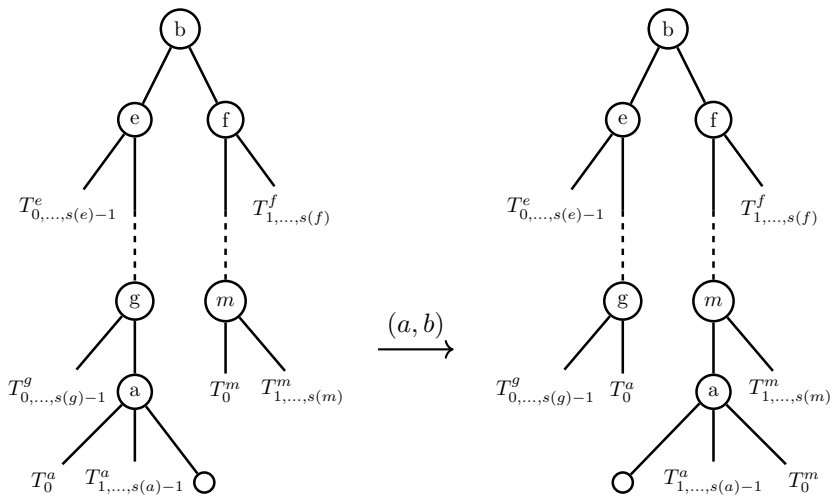


Figure 4: Illustration of the s-tree rotation along the tree ascent (a, b).

Remark 2.18. One might wonder if s-weak order is a Cambrian lattice of some finite Coxeter group. Cambrian lattices were defined by Reading in [8] as certain lattice quotients of weak order. However, from s-weak order with $s = (0, 0, 2)$ (see Fig. 3) we observe that s-weak order is not generally a Cambrian lattice of a finite Coxeter group. The Cambrian lattices of a finite Coxeter group W all have order the Coxeter Catalan number $Cat(W)$. The only W with $Cat(W) = 9$ is the dihedral group $I_2(7)$ see [4]. However, s-weak order with $s = (0, 0, 2)$ has largest anti-chain of cardinality 3 while the largest anti-chain in a Cambrian lattice of $I_2(7)$ has cardinality at most 2.

2.3. Background on the s-Tamari lattice

The Tamari lattice is the sublattice of weak order on permutations generated by the 231-avoiding permutations. Similarly, the s-Tamari lattice is the sublattice of s-weak order generated by certain s-decreasing trees.

Definition 2.19 ([3, Definition 3.1]). *An s-decreasing tree T is called an **s-Tamari tree** if for any $a < b < c$, $\#_T(c,a) \leq \#_T(c,b)$ where $\#_T(c,a)$ is as defined in Definition 2.1. That is, all of the vertex labels in T_i^c are smaller than all of the vertex labels in T_j^c for $i < j$. The multi-inversion set of an s-Tamari tree is called an s-Tamari inversion set.*

We denote the partial order on s-Tamari trees induced by s-weak order by $\preceq_{\boldsymbol{Tam}}$. Similarly, a subscript Tam will be used to denote objects in the s-Tamari lattice. For instance, $[T,Z]_{Tam}$ is the closed interval from T to Z in the s-Tamari lattice.

Theorem 2.20 ([3, Theorem 3.2]). *The collection of s-Tamari trees forms a sublattice of s-weak order, called the s-Tamari lattice.*

Remark 2.21. Taking $s = (1,\ldots,1)$, the s-Tamari lattice is isomorphic to the classical Tamari lattice on $l(s)$.

Similarly to s-weak order, there is a notion of ascent for s-Tamari trees and cover relations in the s-Tamari lattice are characterized as certain tree rotations along these ascents. For $a < b$, we say that (a,b) is a **Tamari tree ascent** of T if a is a non-right most child of b, that is, a is a direct descendant of b and $\#_T(b,a) < s(b)$. Note that in the s-Tamari lattice, $T^a_{s(a)}$ need not be a leaf for some (a,b) to be a Tamari tree ascent. We denote cover relations in the s-Tamari lattice by $\prec_{Tam}$.

Theorem 2.22 ([3, Section 3.1]). *Let T be an s-Tamari tree and let (a,b) be a Tamari tree ascent of T. Then $(inv(T)+(b,a))^{tc}$ is an s-Tamari inversion set. Let Z be the s-Tamari tree such that $inv(Z) = (inv(T)+(b,a))^{tc}$. We say Z is the **s-Tamari rotation of** T **along** (a,b) and write $\boldsymbol{T \xrightarrow{Tam(a,b)} Z}$. Moreover, $T \prec_{Tam} Z$ if and only if there is a unique Tamari tree ascent (a,b) of T such that $T \xrightarrow{Tam(a,b)} Z$.*

An s-Tamari rotation is essentially the same as an s-tree rotation except that the smaller element of the Tamari tree ascent may have right descendants and those right descendants are moved with along with a if $s(a) > 0$. An s-Tamari rotation is illustrated in Fig. 5.

Remark 2.23. Similarly to s-tree rotations, we describe s-Tamari rotations in terms of an operation on the trees themselves. Suppose that (a, b) is a Tamari tree ascent of T and $T \xrightarrow{Tam(a,b)} Z$. Then $a \in T_j^b$ for some $j < s(b)$ and a is a child of b. Recall that every labeled vertex of T_{j+1}^b is greater than a since T is an s-Tamari tree. Let m be the smallest labeled vertex of ${}_LT_{j+1}^b$. Then Z is the same as T except for the following: $Z_j^b = T_0^a$ instead of T^a, $Z_i^a = T_i^a$ for $0 < i \leq s(a)$ if $s(a) > 0$, Z_0^a is a leaf, $Z_0^m = Z^a$.

Remark 2.24. An s-Tamari tree T cannot have Tamari tree ascents (a, b) and (a, c) with $b \neq c$. This follows from the fact that in a rooted tree, every non-root node has exactly one parent. Thus, whenever (a, b) and (c, d) are distinct Tamari tree ascents of T, we may assume $a < c$. We make this assumption throughout our proofs.

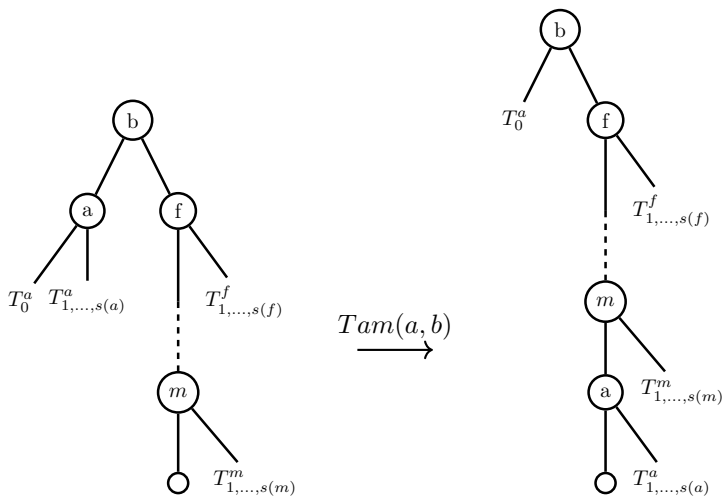


Figure 5: s-Tamari rotation along the Tamari tree ascent (a, b).

2.4. Background on SB-labelings

Hersh and Mészáros developed the notion of an SB-labeling in [6] to show when certain lattices have open intervals which are homotopy balls or spheres.

Definition 2.25 ([6, Definition 3.4])**.** *An **SB-labeling** is an edge labeling λ on a finite lattice L satisfying the following conditions for each $u, v, w \in L$ such that v and w are distinct elements which each cover u:*

(i) $\lambda(u,v) \neq \lambda(u,w)$
(ii) *Each saturated chain from* u *to* $v \vee w$ *uses both of these labels* $\lambda(u,v)$ *and* $\lambda(u,w)$ *a positive number of times.*
(iii) *None of the saturated chains from* u *to* $v \vee w$ *use any other labels besides* $\lambda(u,v)$ *and* $\lambda(u,w)$.

One of Hersh and Mészáros' main theorems in [6] is the following characterization of the homotopy types of intervals in a lattice which admits an SB-labeling.

Theorem 2.26 ([6, Theorem 3.7])**.** *If* L *is a finite lattice which admits an SB-labeling, then each open interval* (u,v) *in* L *is homotopy equivalent to a ball or a sphere of some dimension. Moreover,* $\Delta(u,v)$ *is homotopy equivalent to a sphere if and only if* v *is a join of atoms of* $[u,v]$*, in which case it is homotopy equivalent to a sphere* S^{d-2} *where* d *is the number of atoms in* $[u,v]$.

We will use this theorem to draw our topological conclusions.

3. An SB-labeling of s-weak order

In this section, we prove a series of lemmas on s-decreasing trees and multi-inversion sets which we then use to prove that the following edge labeling of s-weak order is an SB-labeling as Theorem 3.19. We introduce many of the lemmas with a short more intuitive description of the lemma and its proof as well as give reference to an example. In that spirit, we label a cover relation in s-weak order by taking the unique tree ascent (pair of distinct labeled vertices) corresponding to the cover relation by Theorem 2.15 and use the smaller of the two elements of the tree ascent as the label, that is we label cover relations by the label of the root vertex of the subtree moved to achieve the cover relation. Fig. 3 includes two examples of our labeling of s-weak order.

Definition 3.1. *Let* $T \prec Z$ *be a cover relation in* s*-weak order. Let* $T \xrightarrow{(a,b)} Z$ *be the* s*-tree rotation of* T *along the unique tree ascent* (a,b) *associated to* $T \prec Z$ *by Theorem 2.15. Define* λ *to be the edge labeling of* s*-weak order given by* $\lambda(T,Z) = a$.

The notion of tree ascent is defined in Definition 2.10. The notation $T \xrightarrow{(a,b)} Z$ *and corresponding notion of* s*-tree rotation are given in Definition 2.14 and Remark 2.17.*

Remark 3.2. In the case $s = (1, \ldots, 1)$, the SB-labeling of Definition 3.1 gives an SB-labeling of weak order on $S_{l(s)}$. Our labeling is distinct from the one given for symmetric groups by Hersh and Mészáros in [6].

The main point of our proof that Definition 3.1 is an SB-labeling of s-weak order is showing that for any $T \prec Z, Q$, the Hasse diagram of the interval $[T, Z \vee Q]$ is a diamond, a pentagon, or a hexagon. Examples of all three types of such intervals, as well as the underlying reasons they occur which have to do with relationships between tree ascents that are explained in later lemmas, can be seen in Fig. 3. In particular, $[T, Z \vee Q]$ has precisely two maximal chains. Then we verify that, in any case, the labeling on the two maximal chains satisfies Definition 2.25. Many of our proofs are easier with Fig. 4 and Remark 2.17 in mind so it is worth a few moments to internalize those.

The following proposition restricts the possible tree ascents of an s-decreasing tree. In particular, if (a, b) is a tree ascent of some s-decreasing tree T with $s(a) > 0$, then no labeled vertices of T^a besides a can form a tree ascent with b. For instance, $(5, 9)$ is a tree ascent of the s-decreasing tree in Fig. 1, but no vertex below 5 forms a tree ascent with 9 because the rightmost child of 5 must be a leaf. We use this to characterize the multi-inversion set of $Z \vee Q$ for any $T \prec Z, Q$ and to restrict the chains that can appear in $[T, Z \vee Q]$.

Proposition 3.3. *Let T be an s-decreasing tree and let $1 \leq a < b \leq n$ be such that (a, b) is a tree ascent of T with $s(a) > 0$. Then no pair of the form (e, b) such that $e \in T^a$ and $e < a$ is a tree ascent of T.*

Proof. Let (a, b) be a tree ascent of T. Assume (e, b) is also a tree ascent of T with $e \in T^a$ and $e < a$. Then $e \in T^a_{s(a)}$ by (ii) of Definition 2.10 of (e, b) being a tree ascent of T because $e < a < b$. Thus, $T^a_{s(a)}$ is not a leaf. However, since $s(a) > 0$, this contradicts (iii) of Definition 2.10 of (a, b) being a tree ascent of T. Thus, such a pair (e, b) is not a tree ascent of T. □

Remark 3.4. If $s(a) = 0$, it is possible that (a, b) and (e, b) for some $e \in T^a$ with $e < a$ are both tree ascents of T.

The situation precluded by Proposition 3.3 may occur if $s(a) = 0$.

We use the following two definitions to describe $Z \vee Q$ for any $T \prec Z, Q$ in terms of tree inversion sets.

Definition 3.5. *Let T be a s-decreasing tree and let $1 \leq a < b \leq n$ be such that (a, b) is a tree ascent of T. Let Z be the s-decreasing tree obtained by $T \xrightarrow{(a,b)} Z$. Define* ***the set of inversions added by the s-tree rotation***

along $\boldsymbol{(a,b)}$*, denoted* $\boldsymbol{A_T(a,b)}$*, by*

$$A_T(a,b) = \left\{ (f,e) \;\middle|\; \#_Z(f,e) > \#_T(f,e) \right\}.$$

Definition 3.6. *Let T be an s-decreasing tree. Let (a,b) and (c,d) be tree ascents of T with $a < c$. We note that b and d are determined by Remark 2.12 once we know a and c are each the smaller element of a tree ascent. Define the following set valued function:*

$$F_T(a,c) = \begin{cases} \left\{ (d,e) \;\middle|\; e \in T^a \setminus 0 \right\} & \text{if } b = c \text{ and } a \in T_0^c \\ \emptyset & \text{otherwise} \end{cases}$$

Example 4. Let T be the s-decreasing tree in Fig. 1. As we saw in Example 3, $(5,9)$ and $(4,5)$ are both tree ascents of T. Also, $4 \in T_0^5$. If we perform the s-tree rotation of T along $(5,9)$ using Remark 2.17, we observe that $A_T(5,9) = \{(9,5)\}$ and $A_T(4,5) = \{(5,1),(5,2),(5,4)\}$. Also, by definition $F_T(4,5) = \{(9,1),(9,2),(9,4)\}$.

In the next proposition, we explicitly compute the tree inversions added by an s-tree rotation, that is, $A_T(a,b)$ from Definition 3.5. The proposition can be verified on Example 4 above.

Proposition 3.7. *Let T be an s-decreasing tree and let $1 \le a < b \le n$ be such that (a,b) is a tree ascent of T. Suppose $T \xrightarrow{(a,b)} Z$. Then for $1 \le e < f \le n$, $(f,e) \in A_T(a,b)$ if and only if $f = b$ and $e \in T^a \setminus 0$ in which case*

$$\#_Z(f,e) = \#_T(f,e) + 1.$$

The notation $\#_Z(f,e)$ and the corresponding notion of cardinality are given in Definition 2.1.

Proof. First, we note that if $e \in T^a \setminus 0$ and $e < a$, then $s(a) > 0$. Thus, $T^a_{s(a)}$ is a leaf by condition (iii) of Definition 2.10 of (a,b) being a tree ascent of T. Hence, for any $e \in T^a \setminus 0$, $e \notin T^a_{s(a)}$. Then both parts of the statement follow from Remark 2.17 by considering the only subtrees that change in an s-tree rotation (see Fig. 4). □

A particularly simple case of Proposition 3.7 is when the smaller element of a tree ascent has only a single child.

Corollary 3.8. *If (a,b) is a tree ascent of T with $s(a) = 0$ and $T \xrightarrow{(a,b)} Z$, then $A_T(a,b) = \{(b,a)\}$.*

The subsequent lemma essentially shows that the sets of inversions added by s-tree rotations along distinct tree ascents are disjoint. This is illustrated by Example 4 where the particular sets of inversions added are pairwise disjoint. We use this lemma in the proof of one of two different upcoming characterizations of $Z \vee Q$ for any $T \prec Z, Q$. The proof relies on the restrictions on tree ascents from Corollary 3.8 and on our characterization of tree inversions added by an s-tree rotation from Proposition 3.7.

Lemma 3.9. *Let T be an s-decreasing tree. Let $1 \leq a < b \leq n$ and $1 \leq c < d \leq n$ be such that (a, b) and (c, d) are tree ascents of T with $a < c$. Then $A_T(a, b)$, $A_T(c, d)$, and $F_T(a, c)$ are pairwise disjoint.*

The notation $A_T(a, b)$ and $F_T(a, c)$ are given in Definition 3.5 and Definition 3.6, respectively.

Proof. We assume seeking contradiction that there is some $(f, e) \in A_T(a, b) \cap A_T(c, d)$. Then by Proposition 3.7, $f = b = d$ and $e \in T^a, T^c$. Now by Definition 2.10 of (a, b) and (c, d) being tree ascents of T, $a, c \in T^b$. Then, by the fact that e is only below one child of b in T and by Remark 2.4, $a, c \in T_i^b$. Then since (a, b) and (c, b) are both tree ascents of T, $a, c \in {}_RT_i^b$ by Remark 2.13. Now by definition of ${}_RT_i^b$, $a \in T^c$. If $s(c) > 0$, then (a, b) and (c, b) both being tree ascents of T contradicts Proposition 3.3. Thus, $s(c) = 0$. Then by Corollary 3.8, $A_T(c, d) = \{(d, c)\}$ so $(f, e) = (d, c)$. But that contradicts Proposition 3.7 because $a < c$ so $c \notin T^a \setminus 0$.

If $F_T(a, c) \neq \emptyset$, then $b = c \neq d$ and $a \in T_0^c$ by Definition 3.6. Thus, $F_T(a, c)$ is disjoint from $A_T(a, b)$ by Proposition 3.7 since $b \neq d$. Also in this case, $F_T(a, c)$ is disjoint from $A_T(c, d)$ by Proposition 3.7 because each $e \in T^a \setminus 0$ is also in T_0^c. □

Now we have the first of two different descriptions of $Z \vee Q$ for any $T \prec Z, Q$. The second description is $Z \vee Q$ below. Intuitively, this lemma says we can reach $Z \vee Q$ by first performing the s-tree rotation of T along the tree ascent associated with Z and then the s-tree rotation of Z along the tree ascent associated with Q or vice versa. In reality, we run into situations where the tree ascent of T associated with Q is not actually a tree ascent of Z or vice versa. So this intuitive picture is not always defined. We address those situations with later lemmas. We use this description to establish the desired saturated chains in $[T, Z \vee Q]$, while we use the second description in the proofs that there are no other saturated chains to such a join. We prove this lemma by showing double containment of multi-inversion sets using the definition of transitive closure and our characterization of the tree inversions added by an s-tree rotation from Proposition 3.7.

Lemma 3.10. *Let T be an s-decreasing tree and let $1 \leq a < b \leq n$ and $1 \leq c < d \leq n$ be such that (a, b) and (c, d) are distinct tree ascents of T. Suppose $T \xrightarrow{(a,b)} Z$ and $T \xrightarrow{(c,d)} Q$. Then $inv(Z \vee Q) = ((inv(T) + (b, a))^{tc} + (d, c))^{tc}$. Moreover, the order of the pairs in this equality of multi-inversion sets can be reversed.*

The notation $(\cdot)^{tc}$ and the corresponding notion of transitive closure are given just prior to Theorem 2.9. The notion of containment of multi-inversion sets is given in Definition 2.5. The notation $I + J$ and associated idea of the sum of multi-inversion sets are given just after Example 1.

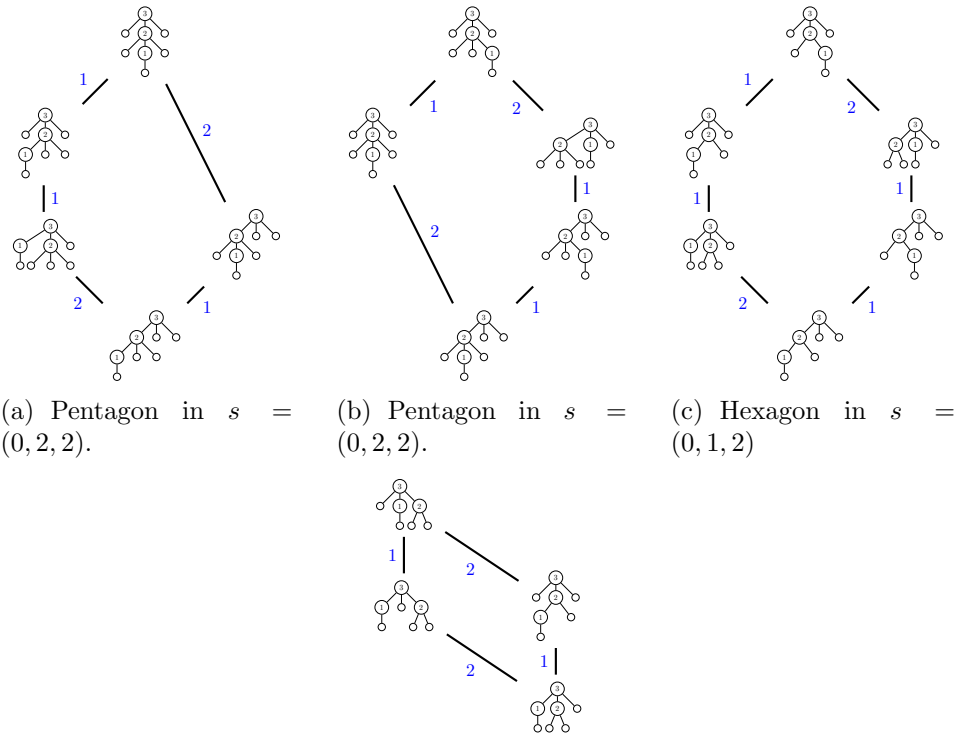
Proof. First, by Theorem 2.9, $\mathrm{inv}\,(Z \vee Q) = (\mathrm{inv}\,(Z) \cup \mathrm{inv}\,(Q))^{tc}$. Let $I = \mathrm{inv}\,(Z) \cup \mathrm{inv}\,(Q)$. By definition of transitive closure, to show

$$((\mathrm{inv}\,(T) + (b, a))^{tc} + (d, c))^{tc} = I^{tc}$$

it suffices to show that $(\mathrm{inv}\,(T) + (b, a))^{tc} + (d, c) \subseteq I$ and $\mathrm{inv}\,(Z)\,, \mathrm{inv}\,(T) + (d, c) \subseteq (\mathrm{inv}\,(T) + (b, a))^{tc} + (d, c)$. We will show the inclusions in that order.

We recall by Definition 2.14 that $\mathrm{inv}\,(Z) = (\mathrm{inv}\,(T) + (b, a))^{tc}$ and $\mathrm{inv}\,(Q) = (\mathrm{inv}\,(T) + (c, d))^{tc}$. By Proposition 3.7 and Lemma 3.9, $\#_Z(d, c) = \#_T(d, c)$ and $\#_Q(d, c) = \#_T(d, c) + 1$ so $\#_I(d, c) = \#_T(d, c) + 1$. Thus, $(\mathrm{inv}\,(T) + (b, a))^{tc} + (d, c) \subset I$. On the other hand, $\mathrm{inv}\,(T) + (d, c) \subset (\mathrm{inv}\,(T) + (b, a))^{tc} + (d, c)$ since $\mathrm{inv}\,(T) \subset (\mathrm{inv}\,(T) + (b, a))^{tc}$. Thus, $\mathrm{inv}\,(T) + (d, c), \mathrm{inv}\,(Z) \subset (\mathrm{inv}\,(T) + (b, a))^{tc} + (d, c)$. Therefore, $\mathrm{inv}\,(Z \vee Q) = ((\mathrm{inv}\,(T) + (b, a))^{tc} + (d, c))^{tc}$. Similarly, the tree ascents may appear in the other order, that is $\mathrm{inv}\,(Z \vee Q) = ((\mathrm{inv}\,(T) + (d, c))^{tc} + (b, a))^{tc}$. □

In the next lemma, we begin with distinct tree ascents (a, b) and (c, d) of an s-decreasing tree T and let Z and Q be the s-tree rotations of T along those tree ascents, respectively. We characterize when one pair ceases to be a tree ascent of the s-tree rotation along the other pair. Intuitively, (c, d) only stops being a tree ascent of Z if the subtree of T rooted at a is moved to the right most child of c by the rotation. Similarly, (a, b) only stops being a tree ascent of Q if the subtree of T rooted at a is left behind while b is moved by the s-tree rotation. The four possibilities turn out to correspond to different relationships between (a, b) and (c, d) in T. These four possibilities end up characterizing the intervals $[T, Z \vee Q]$ which have Hasse diagrams that are diamonds, pentagons, and hexagons. The nature of this interval characterization is essentially contained in the following examples in Fig. 6 of intervals from the s-weak orders in Fig. 3. These intervals act as guides for the remainder of this section.



(a) Pentagon in $s = (0, 2, 2)$.

(b) Pentagon in $s = (0, 2, 2)$.

(c) Hexagon in $s = (0, 1, 2)$

(d) Diamond in $s = (0, 1, 2)$

Figure 6: Examples of pentagonal, hexagonal, and diamond intervals which arise in s-weak order.

In later lemmas, we will show that in particular, when (a, b) is a tree ascent of Q and (c, d) is a tree ascent of Z, $[T, Z \vee Q]$ has Hasse diagram that is a diamond. When exactly one of (a, b) is not a tree ascent of Q or (c, d) is not a tree ascent of Z, $[T, Z \vee Q]$ has Hasse diagram which is a pentagon. When both (a, b) is not a tree ascent of Q and (c, d) is not a tree ascent of Z, $[T, Z \vee Q]$ has Hasse diagram that is a hexagon. Lemma 3.11 below can be illustrated with the s-decreasing tree in Fig. 1. Using Remark 2.17, we can perform the s-tree rotations of the s-decreasing tree in Fig. 1 along the following pairs of tree ascents which exemplify the ways a pair of tree ascents can be related and all of the ways one tree ascent can cease to be a tree ascent after the s-tree rotation along another tree ascent: $(5, 9)$ and $(7, 8)$, $(2, 4)$ and $(3, 4)$, $(3, 4)$ and $(4, 5)$, $(2, 4)$ and $(4, 5)$, $(4, 5)$ and $(5, 9)$.

Lemma 3.11. *Let T be a s-decreasing tree. Let $1 \leq a < b \leq n$ and $1 \leq c < d \leq n$ be such that (a, b) and (c, d) are tree ascents of T with $a < c$. Let $T \xrightarrow{(a,b)} Z$ and $T \xrightarrow{(c,d)} Q$. If either of (a, b) is not a tree ascent of Q or (c, d) is not a tree ascent of Z, then $b = c$ and $s(c) > 0$. Moreover, if (a, c) is not a tree ascent of Q, then $a \in T_0^c$. If (c, d) is not a tree ascent of Z, then $a \in T_{s(c)-1}^c$.*

Proof. We will argue that there are four cases that we must check in more detail for the way in which one of the tree ascents (a, b) or (c, d) can cease to be a tree ascent after the s-tree rotation along the other. We will check these four cases and show that two of them cannot actually occur and that the other two are precisely the conclusions of the lemma. Suppose that either (a, b) is not a tree ascent of Q or (c, d) is not a tree ascent of Z. Then after the s-tree rotation along one of (a, b) or (c, d), at least one of the three conditions of Definition 2.10 must be violated by the other pair.

We begin with three observations with which we show four simpler cases cannot occur leaving us with the four cases mentioned above. First, Remark 2.17 and the fact that $a < c$ imply $Q^a = T^a$ since the only vertices which have changes to the subtrees rooted at them in the s-tree rotation from T to Q are c or have label greater than c. Second, since $a < c$, Remark 2.17 implies the s-tree rotation along (a, b) does not move vertex c or any vertices above c in T. Third, s-tree rotations never decrease the cardinalities of tree inversions by Proposition 3.7.

The first observation shows condition (iii) of Definition 2.10 cannot be violated by (a, b) in Q. This is because $Q^a = T^a$, so if $s(a) > 0$, then $Q^a_{s(a)}$ is a leaf since (a, b) is a tree ascent of T. The second observation shows that condition (i) of Definition 2.10 cannot be violated by (c, d) in Z because the relative positions of c and d in T are not changed by the s-tree rotation along (a, b). The second and third observations together show that condition (ii) of Definition 2.10 cannot be violated by (c, d) in Z. This is because the second observation implies that for any e with $c < e < d$, $c \in T^e$ if and only if $c \in Z^e$. By condition (ii) of Definition 2.10 of (c, d) being a tree ascent of T, $\#_T(e, c) = s(e)$. Then by the third observation, $\#_T(e, c) \leq \#_Z(e, c)$ so $\#_Z(e, c) = s(e)$, which is exactly condition (ii) of Definition 2.10 of (c, d) being a tree ascent of Z. Lastly, the third observation shows that condition (ii) of Definition 2.10 cannot be violated by (a, b) in Q in certain cases, namely by any e such that $a < e < b$, $a \in Q^e$, and $a \in T^e$. This is again because condition (ii) of Definition 2.10 of (a, b) being a tree ascent of T implies $\#_T(e, a) = s(e)$ and the third observation implies

$\#_T(e,a) \leq \#_Q(e,a)$ so $\#_Q(e,a) = s(e)$. The case of $a < e < b$ with $a \in Q^e$, but $a \notin T^e$ is covered as case (1) below.

Thus, there are four possible cases for how conditions (i), (ii), or (iii) of Definition 2.10 might be violated.

(1) (a,b) is not a tree ascent of Q because (ii) is violated by some $a < e < b$ such that $a \notin T^e$, but $a \in Q^e_i$ and $i < s(e)$.
(2) (a,b) is not a tree ascent of Q because (i) is violated by $\#_Q(b,a) = s(b)$,
(3) (a,b) is not a tree ascent of Q because (i) is violated by $a \notin Q^b$,
(4) (c,d) is not a tree ascent of Z because (iii) is violated by $s(c) > 0$ and $Z^c_{s(c)}$ is not a leaf.

We show cases (1) and (2) cannot occur and that cases (3) and (4) give the conclusions of Lemma 3.11.

(1) Assume there is some e such that $a < e < b$, $a \notin T^e$, and $a \in Q^e_i$ with $i < s(e)$. By Remark 2.17, there are two ways that a is below vertex in Q which it was not below in T. Either $a \in Q^c_{s(c)}$ or $a \in T^c \setminus 0$. If $a \in Q^c_{s(c)}$, then the only vertex that a is below in Q which it was not below in T is c. Thus, $e = c$, but $\#_Q(c,a) = s(c)$ so (ii) would not be violated. If $a \in T^c \setminus 0$, then $a \neq c$ implies $s(c) > 0$. However, if $c < b$, then $a \in T^c_{s(c)}$ because (a,b) is a tree ascent of T. This contradicts (c,d) being a tree ascent of T because $s(c) > 0$ and $T^c_{s(c)}$ is not a leaf. If $b \leq c$, then $b \in T^c \setminus 0$ by Remark 2.4. Then by Remark 2.17, if e has $a \in Q^e$ and $a \notin T^e$, then $c \in Q^e$ also. Thus, $e \geq c \geq b$ contradicting $e < b$. Thus, this case cannot occur.
(2) Assume $\#_Q(b,a) = s(b)$. Since (a,b) is a tree ascent of T, $\#_T(b,a) < s(b)$. Thus, $\#_Q(b,a) = s(b)$ implies $(b,a) \in A_T(c,d)$ by Proposition 3.7. However, this contradicts Lemma 3.9. Hence, this case cannot occur.
(3) Suppose $a \notin Q^b$. We note that Remark 2.17 (Fig. 4) implies that $a \in T^b$ and $a \notin Q^b$ if and only if $b = c$ and $a \in T^c_0$ by considering the subtrees which change with the s-tree rotation. Thus, $b = c$ and $a \in T^c_0$. Then since $b = c$ and (a,b) is a tree ascent of T, $s(c) > 0$ by Remark 2.11. This is precisely the first of the two possible conclusions of Lemma 3.11.
(4) Suppose $s(c) > 0$ and $Z^c_{s(c)}$ is not a leaf. We note that $T^c_{s(c)}$ is a leaf by (iii) of Definition 2.10 of (c,d) being a tree ascent of T since $s(c) > 0$. Now Remark 2.17 implies that $T^c_{s(c)}$ is a leaf and $Z^c_{s(c)}$ is not a leaf if and only if $c = b$ and $a \in T^c_{s(c)-1}$ again by considering the subtrees which change with the s-tree rotation. Hence, $b = c$ and $a \in T^c_{s(c)-1}$. This is exactly the second possible conclusion of Lemma 3.11. $\square$

Remark 3.12. Assuming the hypotheses of Lemma 3.11, if $s(c) = 0$, condition (iii) of Definition 2.10 cannot be violated by (c, d) in Z. In this case, (c, d) will be a tree ascent of Z.

In the following lemma, we give a second description of $Z \vee Q$ for any $T \prec Z, Q$. We explicitly find the multi-inversion set difference between $\mathrm{inv}\,(T)$ and $\mathrm{inv}\,(Z \vee Q)$, in contrast with Lemma 3.10 which was the first description of $Z \vee Q$. Similarly to Lemma 3.10 though, $\mathrm{inv}\,(T)$ and $\mathrm{inv}\,(Z \vee Q)$ is obtained from $\mathrm{inv}\,(T)$ by adding the tree inversions necessary to reach Z from T and then the tree inversions needed to reach Q from T but with a correction of some additional tree inversions if (a, b) is not a tree ascent of Z. In practice, this lemma shows the possible pairs that may occur as tree ascents corresponding to cover relations in the interval $[T, Z \vee Q]$. We use this lemma to restrict the chains that can occur in $[T, Z \vee Q]$. To show this lemma, we consider two cases based on relationships between tree ascents from Lemma 3.11. In the proof of the trickier of the two, we construct one of the saturated chains that can occur in $[T, Z \vee Q]$ corresponding to the case from Lemma 3.11 where T has tree ascents (a, b) and (c, d) with $T \xrightarrow{(a,b)} Z$, $T \xrightarrow{(c,d)} Q$, and (a, b) is not a tree ascent of Q. The construction of the chain is illustrated in Fig. 7 below. We can also verify the lemma on the s-decreasing tree in Fig. 1 in the case of the cover relations given by the tree ascents and tree inversions added in Example 4.

Lemma 3.13. *Let T be an s-decreasing tree. Let $1 \leq a < b \leq n$ and $1 \leq c < d \leq n$ be such that (a, b) and (c, d) are tree ascents of T with $a < c$. Suppose $T \xrightarrow{(a,b)} Z$ and $T \xrightarrow{(c,d)} Q$, then $\mathit{inv}(Z \vee Q) - \mathit{inv}(T) = A_T(a, b) \cup A_T(c, d) \cup F_T(a, c)$.*

The notation $\mathit{inv}(\cdot) - \mathit{inv}(\cdot)$ and the corresponding notion of multi-inversion set difference are defined in Definition 2.5. The notations $A_T(\cdot, \cdot)$ and $F_T(\cdot, \cdot)$ are defined in Definition 3.5 and Definition 3.6.

Proof. There are two cases. Either (a, b) is a tree ascent of Q or not.

Suppose (a, b) is a tree ascent of Q. Then either $b \neq c$ or $a \notin T_0^c$ by Lemma 3.11. Either way, $F_T(a, c) = \emptyset$ by definition. Then by Lemma 3.10, $Q \xrightarrow{(a,b)} Z \vee Q$. Thus, by Proposition 3.7, $\mathrm{inv}\,(Z \vee Q) - \mathrm{inv}\,(T) = A_T(c, d) \cup A_Q(a, b)$. Again using Proposition 3.7, $A_Q(a, b) = \left\{ (b, e) \;\middle|\; e \in Q^a \setminus 0 \right\}$. Now since $a < c$, $c \notin T^a$ and $c \notin Q^a$. Thus, Remark 2.17 implies $Q^a \setminus 0 = T^a \setminus 0$. Hence, $A_Q(a, b) = A_T(a, b)$ so $\mathrm{inv}\,(Z \vee Q) - \mathrm{inv}\,(T) = A_T(c, d) \cup A_T(a, b)$.

Next suppose (a, b) is not a tree ascent of Q. Then $b = c$, $a \in T_0^c$, and $s(c) > 0$ by Lemma 3.11. We first argue that the multi-inversion set

difference between $\mathrm{inv}\,(Z \vee Q)$ and $\mathrm{inv}\,(T)$ contains the stated tree inversions. We then produce an s-decreasing tree P' whose multi-inversion set difference with $\mathrm{inv}\,(T)$ actually equals the stated tree inversions. Then the lemma holds because the join is the least upper bound, in this context has the smallest multi-inversion set difference with $\mathrm{inv}\,(T)$ by inclusion. We produce P', which is $Z \vee Q$, in the argument by finding a particular saturated chain starting at T.

We first observe that by Proposition 3.7 and Lemma 3.9, $A_T(a,b) \cup A_T(c,d) \subseteq \mathrm{inv}\,(Z \vee Q) - \mathrm{inv}\,(T)$. Next we show that by transitivity $F_T(a,c) \subset \mathrm{inv}\,(Z \vee Q) - \mathrm{inv}\,(T)$. It suffices to show that $\#_{Z\vee Q}(d,e) \geq \#_T(d,e) + 1$ for all $e \in T^a \setminus 0$. To show this inequality we first note that since $b = c$, $e \in Z_1^c$ for all $e \in T^a \setminus 0$ by Remark 2.17. Thus, $\#_{Z\vee Q}(c,e) \geq 1$ for all $e \in T^a \setminus 0$. Now for any such $e \in T^a \setminus 0$, $e < c < d$ so by transitivity $\#_{Z\vee Q}(d,e) \geq \#_{Z\vee Q}(d,c)$. Next we observe that by Proposition 3.7 and the fact that $Q \preceq Z \vee Q$, $\#_{Z\vee Q}(d,c) \geq \#_T(d,c) + 1$. Lastly, we note that since $a \in T^c$, $\#_T(d,e) = \#_T(d,c)$ for all $e \in T^a \setminus 0$. Thus, $\#_{Z\vee Q}(d,e) \geq \#_T(d,e) + 1$.

It remains to show that there is an s-decreasing tree P' with $\mathrm{inv}\,(P') - \mathrm{inv}\,(T) = A_T(a,b) \cup A_T(c,d) \cup F_T(a,c)$. We claim there is a saturated chain

$$T \xrightarrow{(c,d)} Q \xrightarrow{(a,d)} P \xrightarrow{(a,c)} P'$$

and that $\mathrm{inv}\,(P') - \mathrm{inv}\,(T) = A_T(a,b) \cup A_T(c,d) \cup F_T(a,c)$. Fig. 7 illustrates this chain and guides the proof.

We first show (a,d) is a tree ascent of Q and then that (a,c) is a tree ascent of the s-decreasing tree P resulting from the s-tree rotation of Q along (a,d). We recall that to show that (a,d) is a tree ascent of Q, it suffices to show that $a \in {}_RT_j^d$ for some $j < s(d)$ and that if $s(a) > 0$, then $T_{s(a)}^a$ is a leaf and similarly for (a,c) in P.

We observe that $c \in {}_RT_j^d$ for some $j < s(d)$ since (c,d) is a tree ascent of T. Also, $a \in {}_RT_0^c$ since (a,c) is a tree ascent of T with $a \in T_0^c$. Then by Remark 2.17, $a \in {}_RQ_j^d$ since $f = c$ was the only $a < f < d$ with $a \in T_k^f$ and $k < s(f)$. Further, Remark 2.17 implies $Q^a = T^a$. If $s(a) > 0$, then $T_{s(a)}^a$ is a leaf because (a,c) is a tree ascent of T. So $Q_{s(a)}^a$ would be a leaf also. Hence, (a,d) is a tree ascent of Q.

Next we observe that Q_0^c is a leaf by Remark 2.17 and the fact that $0 < s(c)$ by supposition. Thus, also by Remark 2.17, $P_0^c = P^a$. Hence, $a \in {}_RP_0^c$. Again, since Q_0^c is a leaf, $P_{s(a)}^a$ is a leaf by Remark 2.17. Hence, (a,c) is a tree ascent of P. Therefore, we have the claimed saturated chain.

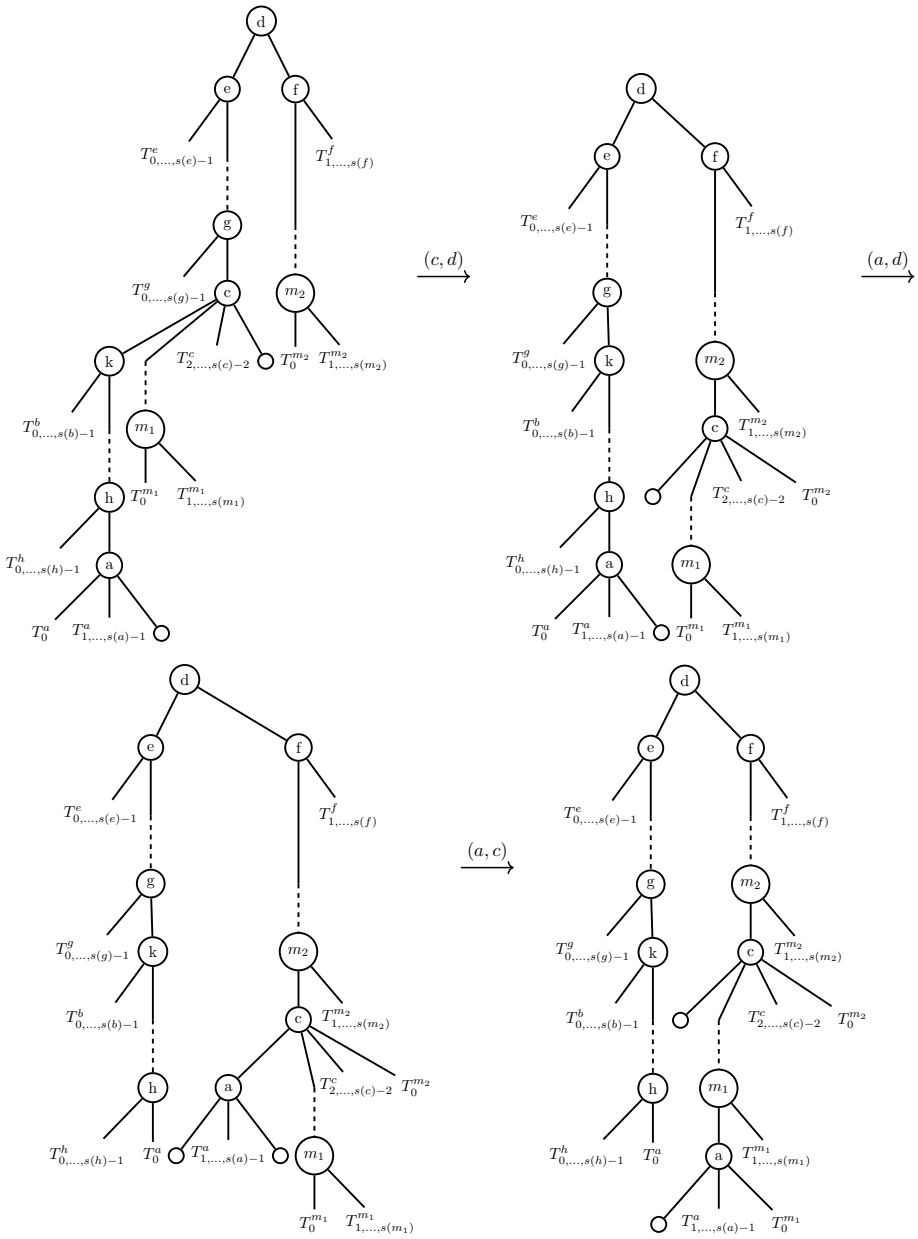


Figure 7: The length three side of an $a \in T^c_0$ pentagon from Lemma 3.13. m_1 is the smallest element of ${}_LT^c_1$ that is larger than a and m_2 is the smallest element of ${}_LT^d_{j+1}$ that is larger than c.

Now by Proposition 3.7, $\mathrm{inv}\,(P') - \mathrm{inv}\,(T) = A_T(c,d) \cup A_Q(a,d) \cup A_P(a,c)$. But by Remark 2.17, we have $Q^a = T^a$ and $P^a = T^a \setminus 0$. Hence, $A_Q(a,d) = F_T(a,c)$. Further, since $b = c$, $A_P(a,c) = A_T(a,b)$. Therefore, $\mathrm{inv}\,(Z \vee Q) - \mathrm{inv}\,(T) = A_T(a,b) \cup A_T(c,d) \cup F_T(a,c)$. □

In the following lemma, we establish that in the interval $[T, Z \vee Q]$ for any $T \prec Z, Q$, the only atoms are Z and Q. We use this in part of the proof that there are only two maximal chains in such an interval. The proof of this lemma relies on Lemma 3.13 and our restrictions on tree ascents from Proposition 3.3. We can visually verify this lemma in the three examples of s-weak order given in Fig. 3.

Lemma 3.14. *Let T be an s-decreasing tree. Let $1 \leq a < b \leq n$ and $1 \leq c < d \leq n$ be such that (a,b) and (c,d) are tree ascents of T with $a < c$. Suppose $T \xrightarrow{(a,b)} Z$ and $T \xrightarrow{(c,d)} Q$, then Z and Q are the only atoms in $[T, Z \vee Q]$.*

Proof. First, Theorem 2.15 implies that atoms of the $[T, Z \vee Q]$ correspond to the tree ascents (e,f) of T such that $(f,e) \in \mathrm{inv}\,(Z \vee Q) - \mathrm{inv}\,(T)$. Thus, by Lemma 3.13 the atoms of $[T, Z \vee Q]$ correspond to pairs $(f,e) \in A_T(a,b) \cup A_T(c,d) \cup F_T(a,c)$ such that (e,f) is a tree ascent of T. By Proposition 3.7 and Proposition 3.3, the only pairs $(f,e) \in A_T(a,b) \cup A_T(c,d)$ such that (e,f) are tree ascents of T are $(f,e) = (b,a), (d,c)$. Further, if $F_T(a,c) \neq \emptyset$ and $(f,e) \in F_T(a,c)$, then $b = c$, $f = d$, and $e \in T^a \setminus 0$ by Definition 3.6. For all $e \in T^a$, $e \in T^b_k$ with $k < s(b)$ since (a,b) is a tree ascent of T. Then since $b < d$, (e,d) such that $e \in T^a$ does not satisfy condition (ii) of Definition 2.10, and so is not a tree ascent of T. Therefore, the only atoms of $[T, Z \vee Q]$ are Z and Q. □

In the next lemma, we consider the case of $T \xrightarrow{(a,b)} Z$ and $T \xrightarrow{(c,d)} Q$ for tree ascents (a,b) and (c,d) of T, but when (c,d) is not a tree ascent of Z. This is one of the cases from Lemma 3.11. We construct a saturated chain from T to $Z \vee Q$. This is similar to the construction of the saturated chain in the proof of Lemma 3.13. This new chain is illustrated in Fig. 8 below. As an example, we can construct this chain using the s-decreasing tree in Fig. 1 and its tree ascents $(2,4)$ and $(4,5)$.

Lemma 3.15. *Let $T \prec Z, Q$ be cover relations in s-weak order corresponding to $T \xrightarrow{(a,b)} Z$ and $T \xrightarrow{(c,d)} Q$ for tree ascents (a,b) and (c,d) of T with $a < c$. Suppose (c,d) is not a tree ascent of Z, then there is a saturated chain of the form $T \xrightarrow{(a,b)} Z \xrightarrow{(a,d)} P \xrightarrow{(c,d)} Z \vee Q$.*

Proof. First, by Lemma 3.11, $b = c$, $a \in T^c_{s(c)-1}$, and $s(c) > 0$. So the two tree ascents of interest in T are (a, c) and (c, d). We claim that there is a saturated chain

$$T \xrightarrow{(a,c)} Z \xrightarrow{(a,d)} P \xrightarrow{(c,d)} P'.$$

We first show that (a, d) is a tree ascent of Z, and then that (c, d) is a tree ascent of the s-decreasing tree P resulting from the s-tree rotation of Z along (a, d). Then we show that $P' = Z \vee Q$. This is illustrated in Fig. 8 which also guides the proof.

First, we note that $c \in {}_RT^d_j$ for some $0 \leq j < s(d)$ since (c, d) is a tree ascent of T. Thus, $c \in {}_RZ^d_j$ because the tree rotation of T along (a, b) does not move any vertices above a in T. Also, $T^c_{s(c)}$ is a leaf because (c, d) is a tree ascent of T and $s(c) > 0$. Then by Remark 2.17, $Z^c_{s(c)} = T^a \setminus 0$ so a is the $s(c)$th child c in Z. Thus, $a \in {}_RZ^d_j$ since a is the $s(c)$th child of c in Z. Further, $Z^a_{s(a)}$ is a leaf again by Remark 2.17 and the fact that $T^c_{s(c)}$ is a leaf. Hence, (a, d) is a tree ascent of Z.

Now, again by Remark 2.17, $c \in {}_RP^d_j$ where j is the same j as above so $0 \leq j < s(d)$. Lastly, $P^c_{s(c)} = Z^a_0$ which is a leaf by Remark 2.17 and the fact that $T^c_{s(c)}$ is a leaf. Thus, (c, d) is a tree ascent of P.

Now we claim $P' = Z \vee Q$. By Proposition 3.7, $\operatorname{inv}(P') - \operatorname{inv}(T) = A_T(a, c) \cup A_Z(a, d) \cup A_P(c, d)$. Thus, by Lemma 3.13, it remains to show that $A_T(a, c) \cup A_Z(a, d) \cup A_P(c, d) = A_T(a, b) \cup A_T(c, d) \cup F_T(a, c)$. Since $b = c$, $A_T(a, b) = A_T(a, c)$. To show $A_Z(a, d) \cup A_P(c, d) = A_T(c, d) \cup F_T(a, c)$, there are two cases because $b = c$. Either $a \in T^c_0$ or $a \notin T^c_0$, that is, $F_T(a, c)$ is possibly non-empty or $F_T(a, c) = \emptyset$, respectively, by Definition 3.6.

Suppose $a \in T^c_0$. Then, as above, by Remark 2.17 and the fact that $T^c_{s(c)}$ is a leaf, $Z^a = T^a \setminus 0$. Thus, by Proposition 3.7 and Definition 3.6, $A_z(a, d) = F_T(a, c)$. Further, by Remark 2.17 along with the fact that $a \in T^c_0$ and our previous observations that $P^c_{s(c)}$ and $T^c_{s(c)}$ are leaves, $P^c \setminus 0 = T^c \setminus 0$. Thus, by Proposition 3.7, $A_P(c, d) = A_T(c, d)$.

Now suppose $a \notin T^c_0$ so $F_T(a, c) = \emptyset$. To show that $A_Z(a, d) \cup A_P(c, d) = A_T(c, d)$, we need to show that $T^c \setminus 0 = Z^a \setminus 0 \cup P^c \setminus 0$ as sets of labeled vertices. We previously argued that $Z^c_s(c) = Z^a \setminus 0 = T^a \setminus 0$. Also, as sets of labeled vertices $P^c \setminus 0 = (T^c \setminus 0) \setminus (T^a \setminus 0)$ by Remark 2.17. This completes the proof. $\square$

In the next three lemmas, we begin with $[T, Z \vee Q]$ for $T \prec Z, Q$ having (a, b) and (c, d) the tree ascents of T associated with Z and Q, respectively. We prove that three of the four relationships given by Lemma 3.11 result

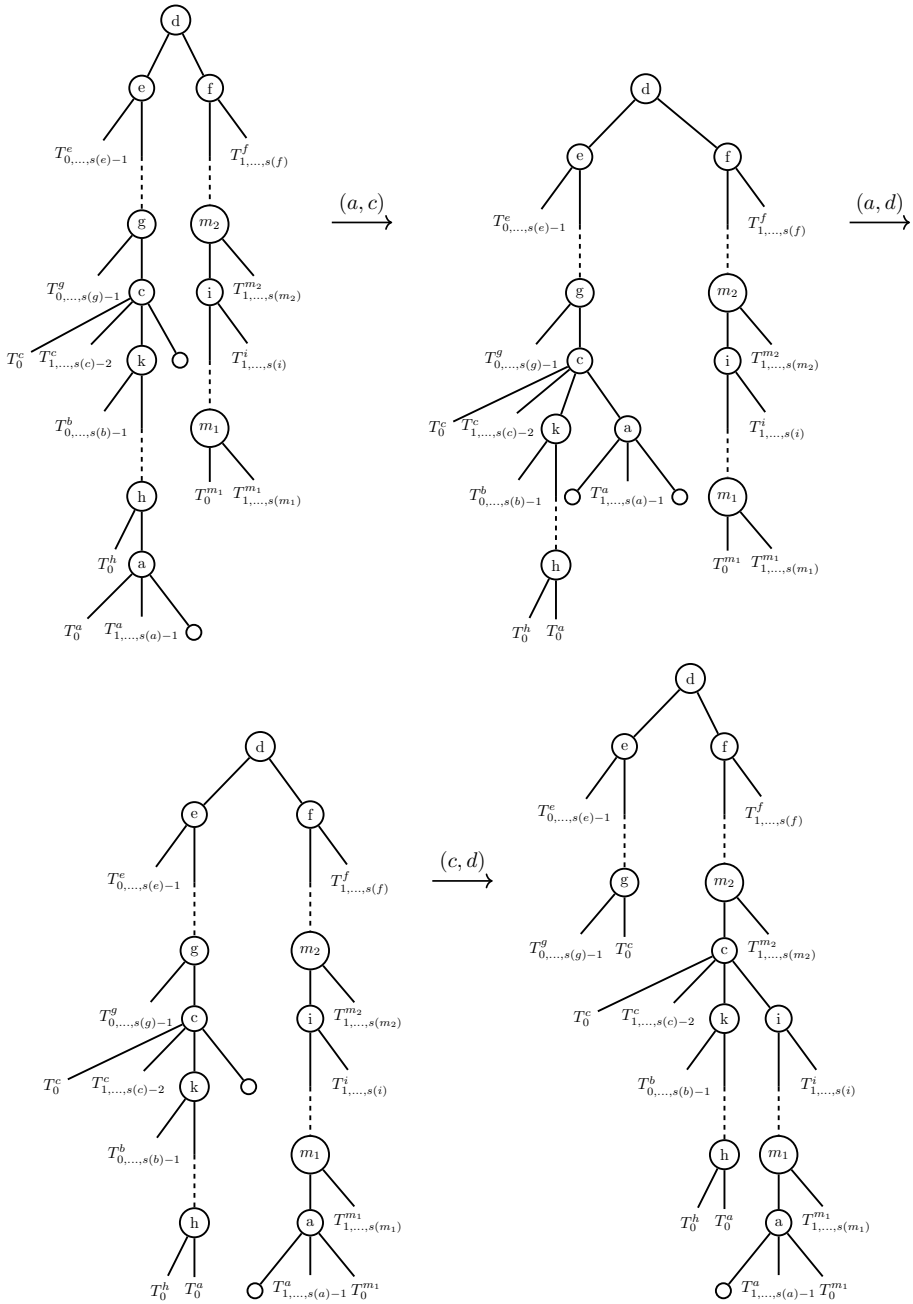


Figure 8: The length three side of an $a \in T^c_{s(c)-1}$ pentagon from Lemma 3.15. m_1 is the smallest element of ${}_LT^d_{j+1}$ that is larger than a and m_2 is the smallest element of ${}_LT^d_{j+1}$ that is larger than c.

in $[T, Z \vee Q]$ having Hasse diagram that is a diamond or a pentagon and that, in any of these three cases, our labeling in Definition 3.1 satisfies the conditions of an SB-labeling. A pentagon arises in two different ways, once each from two of the relationships. The fourth relationship is when both relationships giving pentagons occur at the same time in T. In that case, $[T, Z \vee Q]$ has Hasse diagram a hexagon. We show the hexagonal case in the proof of Theorem 3.19. Theorem 2.15 characterizing cover relations in s-weak order and Lemma 3.10 along with the chains constructed in Lemma 3.13 and Lemma 3.15, establish the two maximal chains of $[T, Z \vee Q]$ in these cases. Thus, the bulk of the proofs the next three lemmas is showing that there are no other maximal chains in $[T, Z \vee Q]$ in these cases using Lemma 3.14 and Proposition 3.3. Moreover, the proofs for the two distinct ways a pentagonal interval can arise combine to prove this about the hexagonal case in our proof of our main result Theorem 3.19. We note our labeling always satisfies the first condition of an SB-labeling by Remark 2.12. We also note that all three lemmas can be verified on the appropriate intervals of the examples of s-weak order in (b) and (c) of Fig. 3.

Lemma 3.16. *Let $T \prec Z, Q$ be cover relations in s-weak order corresponding to $T \xrightarrow{(a,b)} Z$ and $T \xrightarrow{(c,d)} Q$ for tree ascents (a, b) and (c, d) of T with $a < c$. Suppose (a, b) is a tree ascent of Q and (c, d) is a tree ascent of Z. Then $[T, Z \vee Q]$ has Hasse diagram which is a diamond and the edge labeling of Definition 3.1 on its two maximal chains satisfies Definition 2.25.*

Proof. By Lemma 3.10, $\operatorname{inv}(Z \vee Q) = (\operatorname{inv}(Z) + (d, c))^{tc} = (\operatorname{inv}(Q) + (b, a))^{tc}$. Then since (c, d) is a tree ascent of Z and (a, b) is a tree ascent of Q, $R \xrightarrow{(c,d)} Z \vee Q$ and $Q \xrightarrow{(a,b)} Z \vee Q$. Hence, $R, Q \prec Z \vee Q$ Thus, $T \prec Z \prec Z \vee Q$ and $T \prec Q \prec Z \vee Q$ are two distinct saturated chains from T to $Z \vee Q$. Then to show there is not a third such saturated chain, it suffices to show there is not a third atom in the interval $[T, Z \vee Q]$. We showed this fact as Lemma 3.14.

Now we observe that the label sequences of the saturated chains $T \prec Z \prec Z \vee Q$ and $T \prec Q \prec Z \vee Q$ are a, c and c, a, respectively. Therefore, Definition 2.25 is satisfied. □

Lemma 3.17. *Let $T \prec Z, Q$ be cover relations in s-weak order corresponding to $T \xrightarrow{(a,b)} Z$ and $T \xrightarrow{(c,d)} Q$ for tree ascents (a, b) and (c, d) of T with $a < c$. Suppose (a, b) is a tree ascent of Q and (c, d) is not a tree ascent of Z. Then $[T, Z \vee Q]$ has Hasse diagram which is a pentagon and the edge labeling of Definition 3.1 on its two maximal chains satisfies Definition 2.25.*

Proof. Fig. 8 illustrates this case and provides a guide for this proof. First, we observe that $Q \prec Z \vee Q$ by Lemma 3.10 because (a, b) is a tree ascent of Q. This cover relation is given by the s-tree rotation $Q \xrightarrow{(a.b)} Z \vee Q$. Thus, the label sequence for the saturated chain $T \prec Q \prec Z \vee Q$ is c, a.

Next, by Lemma 3.11, $b = c$ and $a \in T^c_{s(c)-1}$ with $s(c) - 1 > 0$. Then by Lemma 3.15 there is a saturated chain of the form $T \xrightarrow{(a,c)} Z \xrightarrow{(a,d)} P \xrightarrow{(c,d)} Z \vee Q$.

Thus, it remains to show that there are no other maximal chains in $[T, Z \vee Q]$ in this case. Proposition 3.7 shows $Q \not\preceq P$. Thus, it suffices to show there are no other elements in $[T, Z \vee Q]$ besides $T, Z, Q, P, Z \vee Q$.

We note the only atoms in $[T, Z \vee Q]$ are Z and Q by Lemma 3.14. Then since $Q \prec Z \vee Q$, the only other possibility of an element in $[T, Z \vee Q]$ besides the five listed above is that there is an atom of $[Z, Z \vee Q]$ distinct from P. Assume there is such an atom, Z'. Then by Theorem 2.15 and Proposition 3.7, there exists $(f, e) \in A_Z(a, d) \cup A_P(c, d)$ such that (e, f) is a tree ascent of Z with $Z \xrightarrow{(e,f)} Z'$. Now by Proposition 3.7 and Proposition 3.3, the only pair $(f, e) \in A_Z(a, d)$ such that (e, f) is a tree ascent of Z is $(f, e) = (d, a)$. However, $(f, e) \neq (d, a)$ since $Z' \neq P$. Next we note that any $(f, e) \in A_P(c, d)$ has the form (d, e) for some $e \in P^c \backslash 0$ by Proposition 3.7. We observe that by Remark 2.17, $P^c = Z^c \setminus s(c)$. Thus, any such any $e \in P^c \setminus 0$ with $e \neq c$ has $e \in Z^c_i$ with $i \neq s(c)$. Thus, (e, d) does not satisfy (ii) of Definition 2.10 of (e, d) being a tree ascent of Z because $e < c < d$. Thus, (c, d) must be the tree ascent of Z corresponding to Z'. However, this contradicts the hypothesis of the lemma that (c, d) is not a tree ascent of Z. Hence, P is the only atom of $[Z, Z \vee Q]$, and there are no other elements of $[T, Z \vee Q]$ besides the five listed earlier.

The two saturated chains have label sequences c, a and a, a, c which satisfy Definition 2.25. □

Lemma 3.18. *Let $T \prec Z, Q$ be cover relations in s-weak order corresponding to $T \xrightarrow{(a,b)} Z$ and $T \xrightarrow{(c,d)} Q$ for tree ascents (a, b) and (c, d) of T with $a < c$. Suppose (a, b) is not a tree ascent of Q, but (c, d) is a tree ascent of Z. Then $[T, Z \vee Q]$ has Hasse diagram which is a pentagon and the edge labeling of Definition 3.1 on its two maximal chains satisfies Definition 2.25.*

Proof. In this case, $Z \prec Z \vee Q$ by Lemma 3.10. This cover relation is given by the s-tree rotation $Z \xrightarrow{(c,d)} Z \vee Q$. Thus, there is a saturated chain $T \prec R \prec Z \vee Q$ with label sequence a, c.

Since (a, b) is not a tree ascent of Q, $b = c$ and $a \in T^c_0$ with $s(c) > 1$ by Lemma 3.11. Then by the proof of Lemma 3.13, there is a saturated chain

of the form

$$T \xrightarrow{(c,d)} Q \xrightarrow{(a,d)} P \xrightarrow{(a,c)} Z \vee Q.$$

Thus, it remains to show these are the only saturated chains in the interval $[T, Z \vee Q]$. Again Proposition 3.7 implies $Z \not\preceq P$. Hence, it suffices to show there are no other elements in $[T, Z \vee Q]$ besides $T, Z, Q, P, Z \vee Q$.

Again the only atoms in $[T, Z \vee Q]$ are Z and Q by Lemma 3.14. Since $Z \prec Z \vee Q$, the only other possibility is that there is an atom Q' in $[Q, Z \vee Q]$ distinct from P. Assume Q' is such an atom. Then by Theorem 2.15 and Proposition 3.7, there exists $(f, e) \in A_Q(a, d) \cup A_P(a, c)$ such that (e, f) is a tree ascent of Q and $Q \xrightarrow{(e,f)} Q'$. By Proposition 3.3 and Proposition 3.7, the only pair $(f, e) \in A_Q(a, d)$ such that (e, f) is a tree ascent of Q is $(f, e) = (d, a)$. But $(f, e) \neq (d, a)$ since $Q' \neq P$. Next we note that any $(f, e) \in A_P(a, c)$, has the form (c, e) for some $e \in P^a \setminus 0$. By Remark 2.17, $P^a \setminus 0 = Q^a \setminus 0 = T^a \setminus 0$ since $a \in T_0^c$ and $s(c) > 1$. Also by Remark 2.17, no element of T^a is in Q^c since $a \in T_0^c$. Thus, for $e \in P^a \setminus 0$, $e \notin Q^c$. Thus, no $(f, e) \in A_P(a, c)$ has (e, f) a tree ascent of Q. Hence, P is the only atom of $[Q, Z \vee Q]$.

Lastly, the label sequences for these two chains are a, c and c, a, a which satisfy Definition 2.25. □

This brings us to the proof of our main theorem, namely that Definition 3.1 is an SB-labeling of s-weak order. In the proof, we must consider the four cases for relationships between two tree ascents of an s-decreasing tree given in Lemma 3.11. The result in the first three cases was proven in Lemma 3.16, Lemma 3.17, and Lemma 3.18. The proof for the fourth case comes from combining the proofs of Lemma 3.17 and Lemma 3.18.

Theorem 3.19. *Let $T \prec Z$ be a cover relation in s-weak order. Let $T \xrightarrow{(a,b)} Z$ be the s-tree rotation of T along the unique tree ascent (a, b) associated to $T \prec Z$ by Theorem 2.15. Let λ to be the edge labeling $\lambda(T, Z) = a$. Then λ is an SB-labeling of s-weak order.*

Proof. Suppose $T \prec Z, Q$ correspond to $T \xrightarrow{(a,b)} Z$ and $T \xrightarrow{(c,d)} Q$ for tree ascents of (a, b) and (c, d) of T with $a < c$. By Remark 2.12, λ satisfies property (i) of Definition 2.25. To verify properties (ii) and (iii) of Definition 2.25, there are four cases we must check:

(1) (a, b) is a tree ascent of Q and (c, d) is a tree ascent of Z, or
(2) (a, b) is a tree ascent of Q while (c, d) is not a tree ascent of Z, or
(3) (c, d) is a tree ascent of Z while (a, b) is not a tree ascent of Q, or
(4) (a, b) is not a tree ascent of Q and (c, d) is not a tree ascent of Z.

Case (1) is Lemma 3.16. Case (2) is Lemma 3.17. Case (3) is Lemma 3.18. Case (4) results in $[T, Z \vee Q]$ having Hasse diagram which is a hexagon and follows from Lemma 3.17 and Lemma 3.18 and their proofs as we show now.

In case (4), Lemma 3.11 implies $b = c$, but this time $a \in T_0^c$ and $s(c) = 1$ so $a \in T_{s(c)-1}^c$. Then the proofs of Lemma 3.17 and Lemma 3.18 imply that there are two distinct maximal chains in $[T, Z \vee Q]$. Both maximal chains are of length three and their label sequences are a, a, c and c, a, a. Additionally, the proofs that there are no other maximal chains in the intervals in Lemma 3.17 and Lemma 3.18 combine to show there are no other maximal chains in $[T, Z \vee Q]$. Thus, (ii) and (iii) of Definition 2.25 are satisfied. Therefore, λ is an SB-labeling of s-weak order. □

Thus, we can characterize the homotopy types of open intervals in s-weak order and the Möbius function of s-weak order as follows.

Corollary 3.20. *Let $T \preceq Z$ in s-weak order. Then $\Delta(T, Z)$, the order complex of the open interval (T, Z), is homotopy equivalent to a ball or a sphere of some dimension. Moreover, the Möbius function of s-weak order satisfies $\mu(T, Z) \in \{-1, 0, 1\}$.*

Proof. The characterization of homotopy type follows from Theorem 2.26 and Theorem 3.19. The result on the Möbius function follows from the fact that $\mu(T, Z) = \tilde{\chi}(\Delta(T, Z))$ along with the fact that the reduced Euler characteristic of a ball is 0 and a d-sphere is $(-1)^d$. □

Lastly, we give an intrinsic characterization of the intervals which are homotopy spheres and the dimension of those spheres.

Lemma 3.21. *If $T \prec Z$ in s-weak order, then Z is the join of the atoms in $[T, Z]$ if and only if*

$$inv(Z) = (inv(T) + A_T(a_1, b_1) + \cdots + A_T(a_l, b_l))^{tc}$$

where $(a_1, b_1), \ldots, (a_l, b_l)$ are the tree ascents of T such that $(b_i, a_i) \in inv(Z) - inv(T)$. Moreover, the number of atoms in the interval $[T, Z]$ is l regardless of whether or not Z is the join of atoms in the interval.

Proof. Let $T \preceq Z$ in s-weak order. The number of atoms in $[T, Z]$ follows from the characterization of cover relations in s-weak order.

Let $(a_1, b_1), \ldots, (a_l, b_l)$ be all the tree ascents of T contained in $\mathrm{inv}\,(Z) - \mathrm{inv}\,(T)$. Let $T_1, \ldots, T_l$ be the corresponding atoms of $[T, Z]$, respectively. Then to prove the characterization of the join of atoms, it suffices to show $\mathrm{inv}\left(\bigvee_{i=1}^{l} T_i\right) = (\mathrm{inv}\,(T) + A_T(a_1, b_1) + \cdots + A_T(a_l, b_l))^{tc}$. We note that by

induction, $\operatorname{inv}\left(\bigvee_{i=1}^{l} T_i\right) = (\operatorname{inv}(T_1) \cup \cdots \cup \operatorname{inv}(T_l))^{tc}$. Now by Proposition 3.7, $\operatorname{inv}(T_i) = \operatorname{inv}(T) + A_T(a_i, b_i)$. By Lemma 3.9, the sets $A_T(a_i, b_i)$ are pairwise disjoint. Thus,

$$\operatorname{inv}(T) + A_T(a_1, b_1) + \cdots + A_T(a_l, b_l) \subset \operatorname{inv}(T_1) \cup \cdots \cup \operatorname{inv}(T_l)$$

so

$$(\operatorname{inv}(T) + A_T(a_1, b_1) + \cdots + A_T(a_l, b_l))^{tc} \subset \operatorname{inv}\left(\bigvee_{i=1}^{l} T_i\right).$$

On the other hand, $\operatorname{inv}(T) + A_T(a_i, b_i) \subset \operatorname{inv}(T) + A_T(a_1, b_1) + \cdots + A_T(a_l, b_l)$ for each $i \in [l]$ so $\operatorname{inv}(T_i) \subset (\operatorname{inv}(T) + A_T(a_1, b_1) + \cdots + A_T(a_l, b_l))^{tc}$ for each $i \in [l]$. Thus, $\operatorname{inv}\left(\bigvee_{i=1}^{l} T_i\right) \subset (\operatorname{inv}(T) + A_T(a_1, b_1) + \cdots + A_T(a_l, b_l))^{tc}$ which gives the result. □

Lemma 3.21 combined with Theorem 2.26 implies the following intrinsic description of intervals which are homotopy spheres and the dimensions of those spheres.

Theorem 3.22. *If $T \prec Z$, then $\Delta(T, Z)$ is homotopy equivalent to a sphere if and only if*

$$inv(Z) = (inv(T) + A_T(a_1, b_1) + \cdots + A_T(a_l, b_l))^{tc}$$

where $(a_1, b_1), \ldots, (a_l, b_l)$ are the tree ascents of T such that $(b_i, a_i) \in inv(Z) - inv(T)$. Moreover, in this case the dimension of the sphere is $l - 2$.

4. An SB-labeling of the s-Tamari lattice

In this section, we prove that a quite similar edge labeling of the s-Tamari Lattice is an SB-labeling. The notation and notions we need to work with the s-Tamari lattice are defined in Section 2.3 and are quite similar to those for s-weak order. We use a subscript of Tam to differentiate between s-weak order and the s-Tamari lattice, for instance $\prec_{Tam}$ instead of $\prec$ for cover relations. For the join however, we still use $\vee$ as in s-weak order because the s-Tamari lattice is a sublattice of s-weak order. We follow a quite similar structure of lemmas as in the proof for s-weak order. The proofs are quite similar to the case of s-weak order with the only major difference being that $[T, Z \vee Q]_{Tam}$ for any $T \prec_{Tam} Z, Q$ have Hasse diagrams which are only diamonds or pentagons. Further, there is only one way that pentagonal intervals arise. There are also some minor differences in the details we must

check, but these details are usually simpler than in the case of s-weak order because Tamari tree ascents are always a pair of a parent and child as defined just after Theorem 2.20. Because of the similarities, the proofs presented here are more cursory.

Intuitively, we label cover relations in the s-Tamari lattice by the label of the root vertex of the subtree that is moved to obtain the cover relation, that is we label by the smaller element of the Tamari tree ascent associated to the cover relation by Theorem 2.22, just as in s-weak order.

Definition 4.1. *Let $T \prec_{Tam} Z$ be a cover relation in the s-Tamari lattice. Let $T \overset{Tam(a,b)}{\longrightarrow} Z$ be the s-Tamari rotation of T along the Tamari tree ascent (a,b) of T associated to $T \prec_{Tam} Z$ by Theorem 2.22. Define λ be the edge labeling $\lambda(T,Z) = a$.*

For $T \prec_{Tam} Z, Q$, we prove that $[T, Z \vee Q]_{Tam}$ has Hasse diagram which is either a diamond or a pentagon, and that the labeling on the two maximal chains satisfies Definition 2.25 in either case. In the s-Tamari lattice, there is only one type of pentagonal interval instead of two. Similarly to s-weak order, our first proposition restricts the Tamari tree ascents which can occur in an s-Tamari tree. We use it to characterize when $[T, z \vee Q]_{Tam}$ has Hasse diagram which is a diamond or which is a pentagon, as well as to describe the atoms in such intervals.

Proposition 4.2. *Let T be an s-Tamari tree and let $1 \leq a < b \leq n$ be such that (a,b) is a Tamari tree ascent of T. Then no pair of the form (c,b) such that $c \in T^a$ and $c < a$ is a Tamari tree ascent of T.*

Proof. Since (a,b) is a Tamari tree ascent of T, a is a child of b in T. No other $c \in T^a$ is a child of b in T. □

Just as in the s-weak order case, the next two definitions let us describe $\mathrm{inv}\,(Z \vee Q)$ when $T \prec TamZ, Q$. The subsequent proposition explicitly computes the tree inversions added by an s-Tamari rotation along a Tamari tree ascent.

Definition 4.3. *Let T be a s-Tamari tree and let $1 \leq a < b \leq n$ be such that (a,b) is a Tamari tree ascent of T. Let Z be the s-Tamari tree obtained by $T \overset{Tam(a,b)}{\longrightarrow} Z$. Define* ***the set of inversions added by the s-Tamari rotation along*** *(a,b), denoted* $\boldsymbol{A_T^{Tam}(a,b)}$*, by*

$$A_T^{Tam}(a,b) = \left\{ (f,e) \;\middle|\; \#_Z(f,e) > \#_T(f,e) \right\}.$$

Definition 4.4. *Let T be an s-Tamari tree and let (a,b) and (c,d) be Tamari tree ascents of T with $a<c$. We note that b and d are determined by a and c since they are the parents of a and c, respectively. Define the following set valued function:*

$$F_T^{Tam}(a,c) = \begin{cases} \left\{(d,e) \;\middle|\; e \in T^a \setminus 0\right\} & \text{if } b=c \text{ and } a \in T_0^c \\ \\ \emptyset & \text{otherwise} \end{cases}$$

Proposition 4.5. *Let T be an s-Tamari tree and let $1 \le a < b \le n$ be such that (a,b) is a Tamari tree ascent of T. Suppose $T \overset{Tam(a,b)}{\longrightarrow} Z$. Then $(f,e) \in A_T^{Tam}(a,b)$ if and only if $f=b$ and $e \in T^a \setminus 0$ in which case*

$$\#_Z(f,e) = \#_T(f,e) + 1.$$

Proof. This follows from Remark 2.23 by keeping track of the only subtrees that change in an s-Tamari rotation. □

Again as in the s-weak order case, we use the following lemma in one of two different characterizations of $Z \vee Q$ for $T \prec_{Tam} Z, Q$.

Lemma 4.6. *Let T be an s-Tamari tree. Let $1 \le a < b \le n$ and $1 \le c < d \le n$ be such that (a,b) and (c,d) are Tamari tree ascents of T with $a<c$. Then $A_T^{Tam}(a,b)$, $A_T^{Tam}(c,d)$, and $F_T^{Tam}(a,c)$ are pairwise disjoint.*

Proof. Assume seeking contradiction that $A_T^{Tam}(a,b) \cap A_T^{Tam}(c,d) \neq \emptyset$. Then by Proposition 4.5, $b=d$. Then $a \in T_i^b$ and $c \in T_j^b$ with $i \neq j$ since a and c are distinct children of d. Thus, T^a and T^c are disjoint. However, the intersection being non-empty then contradicts Proposition 4.5.

If $F_T^{Tam}(a,c) \neq \emptyset$, then $b=c$ and $a \in T_0^c$ by Definition 4.4. Thus, $F_T^{Tam}(a,c)$ is disjoint from $A_T^{Tam}(a,b)$ since $b \neq d$. $F_T^{Tam}(a,c)$ is also disjoint from $A_T^{Tam}(c,d)$ by Proposition 4.5 because every $e \in T^a \setminus 0$ is in T_0^c since $a \in T_0^c$. □

In the following lemma, we show the first of two descriptions of $Z \vee Q$ for $T \prec_{Tam} Z, Q$. The second description of $Z \vee Q$ is Lemma 4.9 below. Our proof of Lemma 4.7 is nearly identical to the proof of Lemma 3.10 since the s-Tamari lattice is a sublattice of s-weak order.

Lemma 4.7. *Let T be an s-Tamari tree and let $1 \le a < b \le n$ and $1 \le c < d \le n$ be such that (a,b) and (c,d) are distinct Tamari tree ascents of T. Suppose $T \overset{Tam(a,b)}{\longrightarrow} Z$ and $T \overset{Tam(c,d)}{\longrightarrow} Q$, then $inv(Z \vee Q) = ((inv(T) + (b,a))^{tc} + (d,c))^{tc}$.*

Proof. Since the s-Tamari lattice is a sublattice of s-weak order, $Z \vee Q$ is the same s-decreasing tree in the s-Tamari lattice as in s-weak order. Thus, this proof is the same as the proof of Lemma 3.10, but with Proposition 3.7 and Lemma 3.9 replaced by Proposition 4.5 and Lemma 4.6, respectively. □

In the next lemma, we begin with s-Tamari Tree T with distinct Tamari tree ascents (a, b) and (c, d) with $a < c$. We show (c, d) is always a Tamari tree ascent of the s-Tamari rotation of T along (a, b). We also show that the only way that (a, b) ceases to be a Tamari tree ascent of the s-Tamari rotation of T along (c, d) is if $b = c$ and a is the 0th child of c in T. In contrast with the four possibilities we say in Lemma 3.11 for s-weak order, there are only two possibilities in the s-Tamari lattice. These turn out to characterize which s-Tamari lattice intervals have Hasse diagrams that are diamonds and that are pentagons. The proof is simpler than that of Lemma 3.11 because Tamari tree ascents are pairs of a parent and child.

Lemma 4.8. *Let T be a s-Tamari tree. Let $1 \leq a < b \leq n$ and $1 \leq c < d \leq n$ be such that (a, b) and (c, d) are Tamari tree ascents of T with $a < c$. Let $T \overset{Tam(a,b)}{\longrightarrow} Z$ and $T \overset{Tam(c,d)}{\longrightarrow} Q$. If (a, b) is not a Tamari tree ascent of Q, then $b = c$ and a is the 0th child of c. Moreover, (c, d) is a Tamari tree ascent of Z.*

Proof. By Remark 2.23, the s-Tamari rotation along (a, b) changes nothing above c in T. Thus, c is still a non-right most child of d in Z so (c, d) is a Tamari tree ascent of Z. Because $a < c$, there are only two ways that (a, b) might not be a Tamari tree ascent of Q. Either (1) $a \in Q^b_{s(b)}$ or (2) a is not a child of b in Q.

For (1), we note that $a \in T^b_j$ for some $j < s(b)$ since (a, b) is a Tamari tree ascent of T. Then by Proposition 4.5, $a \in Q^b_{s(b)}$ implies $b = d$ and $a \in T^c$. Then, however, since $a < c$, (a, d) being a Tamari tree ascent of T contradicts Proposition 4.2. Thus, (1) cannot occur. For (2), Remark 2.23 implies a is a child of b in T, but not a child of b in Q if and only if $b = c$ and a is the 0th child of b in T. This is precisely the conclusion of this lemma. □

Next we give a second description of $Z \vee Q$ for $T \prec_{Tam} Z.Q$, this time in terms of explicit multi-inversion sets instead of the transitive closure. The first description of $Z \vee Q$ we Lemma 4.7 above. We use the same main idea as in the proof of Lemma 3.13 for s-weak order and construct the chain of length three that occurs in the intervals $[T, Z \vee Q]_{Tam}$ which have Hasse diagrams that are pentagons.

Lemma 4.9. *Let T be an s-Tamari tree. Let $1 \leq a < b \leq n$ and $1 \leq c < d \leq n$ be such that (a,b) and (c,d) are Tamari tree ascents of T with $a < c$. Suppose $T \overset{Tam(a,b)}{\longrightarrow} Z$ and $T \overset{Tam(c,d)}{\longrightarrow} Q$. Then $inv(Z \vee Q) - inv(T) = A_T^{Tam}(a,b) \cup A_T^{Tam}(c,d) \cup F_T^{Tam}(a,c)$.*

Proof. If (a,b) is a Tamari tree ascent of Q, then a similar argument to that in the proof of Lemma 3.13, but with the corresponding lemmas for s-Tamari trees shows that the result holds.

If (a,b) is not a Tamari tree ascent of Q, then $b = c$ and a is the $0th$ child of c by Lemma 4.8. A similar argument to that in the proof of Lemma 3.13 using transitivity shows that $A_T^{Tam}(a,b) \cup A_T^{Tam}(c,d) \cup F_T^{Tam}(a,c) \subseteq \mathrm{inv}\,(Z \vee Q) - \mathrm{inv}\,(T)$. Thus, it suffices two show there is an s-Tamari tree P' with $\mathrm{inv}\,(P') - \mathrm{inv}\,(T) = A_T^{Tam}(a,b) \cup A_T^{Tam}(c,d) \cup F_T^{Tam}(a,c)$. We claim there is a saturated chain

$$T \overset{Tam(c,d)}{\longrightarrow} Q \overset{Tam(a,d)}{\longrightarrow} P \overset{Tam(a,c)}{\longrightarrow} P'.$$

Since a is the $0th$ child of c, a is the $0th$ child of d in Q by Remark 2.23. Thus, (a,d) is a tree ascent of Q. Then, again by Remark 2.23, a is the $0th$ child of c in P. Hence, (a,c) is a Tamari tree ascent of P. Thus, we have the claimed saturated chain. Now we apply Proposition 4.5 at each step of the chain which gives $\mathrm{inv}\,(P') - \mathrm{inv}\,(T) = A_T^{Tam}(c,d) \cup A_Q^{Tam}(a,d) \cup A_P^{Tam}(a,c)$. Now by Remark 2.23 we have $A_Q^{Tam}(a,d) = F_T(a,c)$ and $A_P^{Tam}(a,c) = A_T^{Tam}(a,c)$. Thus, $\mathrm{inv}\,(P') - \mathrm{inv}\,(T) = A_T^{Tam}(c,d) \cup A_T^{Tam}(a,c) \cup F_T^{Tam}(a,c)$ and these sets are pairwise disjoint by Lemma 4.6. $\square$

In the next lemma, we show that the only atoms in $[T, Z \vee Q]_{Tam}$ with $T \prec_{Tam} Z, Q$ are Z and Q using Lemma 4.9.

Lemma 4.10. *Let T be an s-decreasing tree. Let $1 \leq a < b \leq n$ and $1 \leq c < d \leq n$ be such that (a,b) and (c,d) are Tamari tree ascents of T with $a < c$. Suppose $T \overset{Tam(a,b)}{\longrightarrow} Z$ and $T \overset{Tam(c,d)}{\longrightarrow} Q$, then Z and Q are the only atoms in $[T, Z \vee Q]_{Tam}$.*

Proof. Assume $T' \in [T, Z \vee Q]_{Tam}$ and $T \prec_{Tam} T'$ with $T' \neq Z, Q$. Let (e,f) be the Tamari tree ascent of T corresponding to T'. By Lemma 4.9, $(f,e) \in A_T^{Tam}(a,b) \cup A_T^{Tam}(c,d) \cup F_T^{Tam}(a,c)$. $(e,f) \neq (a,b), (c,d)$ since $T' \neq Z, Q$. Any other pair $(f,e) \in A_T^{Tam}(a,b) \cup A_T^{Tam}(c,d) \cup F_T^{Tam}(a,c)$ being a Tamari tree ascent of T contradicts Proposition 4.2 because either $f = b$ or $f = d$ and e is below a or c in T and so cannot be a child of f. $\square$

In the subsequent two lemmas, we show the s-Tamari lattice intervals of the form $[T, Z \vee Q]_{Tam}$ where $T \prec_{Tam} Z, Q$ have Hasse diagrams that are either diamonds or pentagons and that the labeling of Definition 4.1 satisfies the definition of SB-labeling. These two lemmas combine to prove our labeling is an SB-labeling of the s-Tamari lattice.

Lemma 4.11. *Let $T \prec_{Tam} Z, Q$ be cover relations in the s-Tamari lattice corresponding to $T \overset{Tam(a,b)}{\longrightarrow} Z$ and $T \overset{Tam(c,d)}{\longrightarrow} Q$ for distinct Tamari tree ascents of (a, b) and (c, d) of T. Suppose (a, b) is a Tamari tree ascent of Q. Then $[T, Z \vee Q]_{Tam}$ has Hasse diagram which is a diamond and the edge labeling of Definition 4.1 on its two maximal chains satisfies Definition 2.25.*

Proof. Similarly to the corresponding proof in s-weak order, we use Lemma 4.7 to show $Z \overset{Tam(c,d)}{\longrightarrow} Z \vee Q$ and $Q \overset{Tam(a,b)}{\longrightarrow} Z \vee Q$. Hence, $Z, Q \prec_{Tam} Z \vee Q$ Thus, $T \prec_{Tam} Z \prec_{Tam} Z \vee Q$ and $T \prec_{Tam} Q \prec_{Tam} R \vee Q$ are two distinct saturated chains from T to $Z \vee Q$. To show there is not a third such saturated chain it suffices to show there is not a third atom in the interval $[T, Z \vee Q]_{Tam}$, but this is Lemma 4.10. Hence, the above chains are the only two saturated chains from T to $Z \vee Q$.

Now we only need observe that the label sequences of the saturated chains $T \prec_{Tam} Z \prec_{Tam} Z \vee Q$ and $T \prec_{Tam} Q \prec_{Tam} Z \vee Q$ are a, c and c, a, respectively. Therefore, Definition 2.25 is satisfied. □

Lemma 4.12. *Let $T \prec_{Tam} Z, Q$ be cover relations in the s-Tamari lattice corresponding to $T \overset{Tam(a,b)}{\longrightarrow} Z$ and $T \overset{Tam(c,d)}{\longrightarrow} Q$ for Tamari tree ascents (a, b) and (c, d) of T with $a < c$. Suppose (a, b) is not a Tamari tree ascent of Q. Then $[T, Z \vee Q]_{Tam}$ has Hasse diagram which is a pentagon and the edge labeling of Definition 4.1 on its two maximal chains satisfies Definition 2.25.*

Proof. By Lemma 4.8, $b = c$ and a is the 0th child of c. Again by Lemma 4.7, we have the saturated chain $T \prec_{Tam} Z \prec_{Tam} Z \vee Q$ given by the s-Tamari rotations $T \overset{Tam(a,b)}{\longrightarrow} Z$ and $Z \overset{Tam(c,d)}{\longrightarrow} Z \vee Q$. By the proof of Lemma 4.9, we have a saturated chain

$$T \overset{Tam(c,d)}{\longrightarrow} Q \overset{Tam(a,d)}{\longrightarrow} P \overset{Tam(a,c)}{\longrightarrow} Z \vee Q.$$

We note that by Proposition 4.5 $Z \preceq_{Tam} P$. Thus, to show the Hasse diagram of $[T, Z \vee Q]_{Tam}$ is a pentagon, it suffices to show there are no other elements in the interval besides $T, Z, Q, P, Z \vee Q$. To show there are no other elements in the interval, it suffices to show there are no other atoms in $[T, Z \vee Q]_{Tam}$ besides Z and Q and that there are no other atoms in

$[Q, Z \vee Q]_{Tam}$ besides P. The fact that there are no atoms of $[T, Z \vee Q]_{Tam}$ besides Z and Q is Lemma 4.10. Similarly to the proof of Lemma 3.18 for s-weak order, Lemma 4.9 implies the existence of an atom in $[Q, Z \vee Q]_{Tam}$ besides P would contradict Proposition 4.2. Hence, the Hasse diagram of the interval is a pentagon whose only maximal chains are the two already shown.

The label sequences for the maximal chains $T \prec_{Tam} Z \prec_{Tam} R \vee Q$ and $T \prec_{Tam} Q \prec_{Tam} P \prec_{Tam} Z \vee Q$ are a, c and c, a, a, respectively. These label sequences satisfy Definition 2.25. □

The previous two lemmas together prove the labeling of Definition 4.1 is an SB-labeling.

Theorem 4.13. *Let $T \prec_{Tam} Z$ be a cover relation in the s-Tamari lattice. Let $T \xrightarrow{Tam(a,b)} Z$ be the s-Tamari rotation of T along the Tamari tree ascent (a, b) of T associated to $T \prec_{Tam} Z$ by Theorem 2.22. Let λ be the edge labeling $\lambda(T, Z) = a$. Then λ is an SB-labeling of the s-Tamari lattice.*

Proof. Condition (i) of Definition 2.25 is satisfied by Remark 2.12. Lemma 4.8, Lemma 4.11, and Lemma 4.12 together imply conditions (ii) and (iii) of Definition 2.25 are satisfied proving the theorem. □

Theorem 4.13 and Theorem 2.26 prove a characterization of the homotopy type of open intervals in the s-Tamari lattice and so also characterize its Möbius function.

Corollary 4.14. *Let $T \preceq_{Tam} Z$ in the s-Tamari lattice. Then $\Delta(T, Z)_{Tam}$, the order complex of the open interval $(T, Z)_{Tam}$, is homotopy equivalent to a ball or a sphere of some dimension. Moreover, the Möbius function of the s-Tamari lattice satisfies $\mu_{Tam}(T, Z) \in \{-1, 0, 1\}$.*

Furthermore, we give the analogous intrinsic description of open s-Tamari intervals whose order complexes are homotopy spheres as for s-weak order. We begin with a lemma characterizing the join of atoms in a closed interval $[T, Z]_{Tam}$.

Lemma 4.15. *If $T \prec_{Tam} Z$, then Z is the join of the atoms in $[T, Z]_{Tam}$ if and only if*

$$inv(Z) = \left(inv(T) + A_T^{Tam}(a_1, b_1) + \cdots + A_T^{Tam}(a_l, b_l)\right)^{tc}$$

where $(a_1, b_1), \ldots, (a_l, b_l)$ are the Tamari tree ascents of T such that $(b_i, a_i) \in inv(Z) - inv(T)$. Moreover, the number of atoms in the interval $[T, Z]_{Tam}$ is l regardless of whether or not Z is the join of atoms in the interval.

Proof. The number of atoms follows from Theorem 2.22, the characterization of cover relations in the s-Tamari lattice. The rest of the statement follows from the same argument as in the proof of Lemma 3.21 with the lemmas about s-weak order replaced by the corresponding lemmas for the s-Tamari lattice because the s-Tamari lattice is a sublattice of s-weak order. □

We conclude with the theorem characterizing the open s-Tamari intervals which are homotopy equivalent to spheres.

Theorem 4.16. *If $T \prec Z$, then $\Delta(T, Z)_{Tam}$ is homotopy equivalent to a sphere if and only if*

$$inv(Z) = \left(inv(T) + A_T^{Tam}(a_1, b_1) + \cdots + A_T^{Tam}(a_l, b_l)\right)^{tc}$$

where $(a_1, b_1), \ldots, (a_l, b_l)$ are the Tamari tree ascents of T such that $(b_i, a_i) \in inv(Z) - inv(T)$. Moreover, in this case the dimension of the sphere is $l - 2$.

Proof. This follows from combining Lemma 4.15 and Theorem 2.26. □

Acknowledgements

The author is grateful to Patricia Hersh for guidance and many helpful discussions throughout the course of this work. The author also thanks Joseph Doolittle for helpful discussions about the classical Tamari lattice at FPSAC 2019 where the author first encountered the work of Ceballos and Pons. The author additionally thanks the reviewer whose comments improved the quality of this paper.

References

[1] Björner, A. (1980). Shellable and Cohen-Macaulay Partially Ordered Sets. *Transactions of the American Mathematical Society* **260** 159–183. MR0570784

[2] Ceballos, C., Padrol, A. and Sarmiento, C. (2019). Geometry of ν-Tamari Lattices in Types A and B. *Transactions of the American Mathematical Society* **371** 2575–2622. MR3896090

[3] Ceballos, C. and Pons, V. (2019). The s-weak order and s-permutahedra. *Séminaire Lotharingien de Combinatoire* **82B**. MR4098297

[4] Grätzer, G. and Wehrung, F. e. (2016). *Lattice Theory: Special Topics and Applications*. Birkhäuser, Basel. MR1670580

[5] Hersh, P. (2018). Posets arising as 1-skeleta of simple polytopes, the nonrevisiting path conjecture, and poset topology. *arXiv:1802.04342*.

[6] Hersh, P. and Mészáros, K. (2017). SB-labelings and posets with each interval homotopy equivalent to a sphere or a ball. *Journal of Combinatorial Theory, Series A* **152** 104–120. MR3682728

[7] Préville-Ratelle, L.-F. and Viennot, X. (2017). The Enumeration of Generalized Tamari Intervals. *Transactions of the American Mathematical Society* **369** 5219–5239. MR3632566

[8] Reading, N. (2006). Cambrian lattices. *Advances in Mathematics* **205** 313–353. MR2258260

Stephen Lacina
Department of Mathematics
University of Oregon
Eugene, OR 97403
USA
E-mail address: slacina@uoregon.edu

Received September 5, 2020