

METRIC GRAPHS, CROSS RATIOS, AND RAYLEIGH'S LAWS

ROBIN DE JONG AND FARBOD SHOKRIEH

We systematically study the notion of cross ratios and energy pairings on metric graphs and electrical networks. We show that several foundational results on electrical networks and metric graphs immediately follow from the basic properties of cross ratios. For example, the projection matrices of Kirchhoff have natural (and efficiently computable) expressions in terms of cross ratios. We prove a generalized version of Rayleigh's law, relating energy pairings and cross ratios on metric graphs before and after contracting an edge segment. Quantitative versions of Rayleigh's law for effective resistances, potential kernels, and cross ratios will follow as immediate corollaries.

1. Introduction

Let Γ be a metric graph (i.e., a length metric space homeomorphic to a topological graph) which is compact and connected. We may think of Γ as a (resistive) electrical network. We are interested in the *j*-function on Γ defined informally as follows: for $x, y, z \in \Gamma$, let $j_z(x, y; \Gamma)$ denote the electric potential at x when one unit of current enters the network Γ at y and exits at z , with z “grounded” (i.e., having zero potential). The *effective resistance* between two points $x, y \in \Gamma$ is then defined as $r(x, y; \Gamma) := j_y(x, x; \Gamma)$.

More formally, the *j*-function is the kernel of integration that inverts the (distributional) Laplacian operator Δ . As such, it appears naturally in the harmonic analysis of Γ . We observe that there is an alternative interpretation of *j*-functions in the language of Gromov hyperbolic spaces: let $(x|y)_z$ denote the *Gromov product* on Γ with respect to the effective resistance distance function $r: \Gamma \times \Gamma \rightarrow \mathbb{R}$ (see Section 4B). Then $(x|y)_z = j_z(x, y; \Gamma)$ (see Lemma 4.4 and Example 6.5).

Motivated by this latter point of view, and following [5, Remark B.12], we define the *cross ratio function* ξ on Γ as follows: fix a point $q \in \Gamma$. For $x, y, z, w \in \Gamma$, let

$$\xi(x, y, z, w; \Gamma) := j_q(x, z; \Gamma) + j_q(y, w; \Gamma) - j_q(x, w; \Gamma) - j_q(y, z; \Gamma).$$

The resulting function is easily seen to be independent of the base point q (see Lemma 6.2). The *j*-function and the *r*-function are themselves evaluations of the cross ratio function:

$$j_z(x, y; \Gamma) = \xi(x, z, y, z; \Gamma) \quad \text{and} \quad r(x, y; \Gamma) = \xi(x, y, x, y; \Gamma).$$

The second-named author was partially supported by NSF CAREER DMS-2044564 grant.

2020 AMS *Mathematics subject classification*: 94C05, 14T15, 35J05, 05C50.

Keywords and phrases: metric graph, electrical network, Laplacian, energy pairing, potential kernel, cross ratio, resistance, spanning tree, Rayleigh's law.

Received by the editors on December 3, 2020.

The cross ratio function is, in turn, an evaluation of the so-called *energy pairing*. Let $D\text{Meas}_0(\Gamma)$ denote the real vector space of discrete measures on Γ with zero total mass. For $\nu_1, \nu_2 \in D\text{Meas}_0(\Gamma)$, the energy pairing is defined by

$$\langle \nu_1, \nu_2 \rangle_{\text{en}}^\Gamma := \int_{\Gamma \times \Gamma} j_q(x, y; \Gamma) \, d\nu_1(x) \, d\nu_2(y),$$

which is, again, independent of the base point $q \in \Gamma$. One can easily check that the following identity holds:

$$\xi(x, y, z, w; \Gamma) = \langle \delta_x - \delta_y, \delta_z - \delta_w \rangle_{\text{en}}^\Gamma.$$

In this paper, we argue that cross ratios and the energy pairing play a foundational role in the study of electrical networks and metric graphs (abstract tropical curves). We show that many important and well-known results in the theory of electrical networks are immediate consequences of their basic properties. For example, the “reciprocity theorem” in electrical networks can be seen as a consequence of the fact that cross ratios are independent of the choice of base points (see [Example 6.7](#)). We will actually show that, more generally, the energy pairing can be computed using *any* generalized inverse of the Laplacian operator. This fact gives more flexibility in proofs and computations, and can be seen as a vast generalization of the reciprocity theorem (see [Proposition 5.2](#), [Example 6.7](#), and [Remark 6.8](#)).

We remark that our notion of cross ratios is closely related to the notion of *transpedance* introduced and studied by Brooks, Smith, Stone, and Tutte in the seminal paper [\[11\]](#) (see [Remark 6.4](#)).

Kirchhoff’s projection matrices. As another example of our point of view, we show in an elementary way that Kirchhoff’s famous projection matrices in electrical network theory have a simple and convenient description in terms of cross ratios. Let G be a model of Γ , which is a finite graph together with a length function ℓ on its edge set. Fix an orientation \mathbb{O} on G . The space of 1-chains $C_1(G, \mathbb{R}) \simeq \bigoplus_{e \in \mathbb{O}} \mathbb{R}e$ is endowed with a canonical bilinear form: for distinct $e, f \in \mathbb{O}$, we let $[e, e] = \ell(e)$ and $[e, f] = 0$. The first homology group $H_1(G, \mathbb{R})$ is naturally a subspace of the inner product space $C_1(G, \mathbb{R})$. As is well known, the theory of electrical networks is essentially the study of the orthogonal projections (see [Section 7A](#)):

$$\begin{aligned} \pi_G : C_1(G, \mathbb{R}) &\rightarrow H_1(G, \mathbb{R}), \\ \pi'_G : C_1(G, \mathbb{R}) &\rightarrow H_1(G, \mathbb{R})^\perp. \end{aligned}$$

Let m denote the number of edges of G , and let Ξ be the $m \times m$ matrix of cross ratios:

$$\Xi := (\xi(e^-, e^+, f^-, f^+))_{e, f \in \mathbb{O}}.$$

Let \mathbf{D} be the $m \times m$ diagonal matrix whose diagonal entries are $\ell(e)$ for $e \in \mathbb{O}$, and let \mathbf{I} be the identity matrix. We have the following elementary result:

Theorem A (see [Theorem 7.8](#)). *In the basis given by \mathbb{O} , the matrix of π_G is $\mathbf{I} - \mathbf{D}^{-1} \Xi$, and the matrix of π'_G is $\mathbf{D}^{-1} \Xi$.*

[Theorem A](#) is proved by a very straightforward linear algebra argument. We would like to mention though that the simple expression of Kirchhoff’s projection matrices in terms of cross ratios is quite useful for applications. For example, unlike Kirchhoff’s classical description in terms of a certain average over spanning trees (reviewed below in [Section 7](#)), the description provided by [Theorem A](#) is very efficient for computations. Indeed, both π_G and π'_G can be computed in time at most $O(n^\omega)$, where n is the number of vertices of G and ω is the exponent for the matrix multiplication algorithm (see [Remark 7.9](#)). By contrast, note that a given graph G can have a super-exponential number of spanning trees.

Rayleigh's laws. As a further illustration of our point of view, we revisit *Rayleigh's laws*. The classical Rayleigh monotonicity law is the statement that if the resistances in an electrical network are decreased, the effective resistance between any two points can only decrease.

We will show the following quantitative *Rayleigh's law for energy pairings*. Let Γ again be a metric graph. For an edge segment e of Γ , let Γ/e denote the network obtained by “short-circuiting” e .

Theorem B (see Theorem 8.2). *Let e be an edge segment of Γ with endpoints $\partial e = \{e^-, e^+\}$, and let $v_1, v_2 \in \text{DMeas}_0(\Gamma)$. Then*

$$\langle v_1, v_2 \rangle_{\text{en}}^{\Gamma/e} = \langle v_1, v_2 \rangle_{\text{en}}^{\Gamma} - \frac{\langle v_1, \delta_{e^+} - \delta_{e^-} \rangle_{\text{en}}^{\Gamma} \langle \delta_{e^+} - \delta_{e^-}, v_2 \rangle_{\text{en}}^{\Gamma}}{r(e^-, e^+; \Gamma)}.$$

From this, one immediately obtains Rayleigh's laws for cross ratios, j -functions, and r -functions:

$$(1-1) \quad \xi(x, y, z, w; \Gamma/e) = \xi(x, y, z, w; \Gamma) - \frac{\xi(x, y, e^-, e^+; \Gamma) \xi(z, w, e^-, e^+; \Gamma)}{r(e^-, e^+; \Gamma)},$$

$$(1-2) \quad j_z(x, y; \Gamma/e) = j_z(x, y; \Gamma) - \frac{\xi(x, z, e^-, e^+; \Gamma) \xi(y, z, e^-, e^+; \Gamma)}{r(e^-, e^+; \Gamma)},$$

$$(1-3) \quad r(x, y; \Gamma/e) = r(x, y; \Gamma) - \frac{\xi(x, y, e^-, e^+; \Gamma)^2}{r(e^-, e^+; \Gamma)}.$$

We remark that one can find expressions equivalent to (1-1), (1-2), and (1-3) in the paper [11] mentioned before. Note that (1-3) may be thought of as a quantitative version of the classical Rayleigh's monotonicity law, and it is equivalent to a quantitative version of the classical statement that edges in a uniform spanning tree are negatively correlated (see, e.g., [20, §4.2]). As such, (1-3) is also a special case of the *transfer-current theorem* (see Section 7D(iv)).

Structure of the paper. In Section 2, we set the notation and terminology for graphs and electrical networks and discuss their correspondences. In Section 3, various notions of Laplacian operators and their compatibilities are reviewed. The notion of j -functions and their relation to Gromov products are discussed in Section 4. In Section 5, we study the notions of energy and Dirichlet pairings and discuss their relationship. We show, in particular, that energy pairings can be computed using *any* generalized inverse of the Laplacian matrix. In Section 6, cross ratios are defined and some of their basic properties are established. In Section 7, we will review Kirchhoff's classical work on projection matrices arising in electrical networks. We then present our description of Kirchhoff's projection matrices in terms of cross ratios (Theorem A). In Section 8, we state and prove our generalized and quantitative versions of Rayleigh's law (Theorem B).

2. Graphs and networks

2A. Weighted graphs. By a *weighted graph* we mean a finite weighted connected multigraph G with no loop edges. We denote the set of vertices of G by $V(G)$ and the set of edges of G by $E(G)$, and let $n = |V(G)|$ and $m = |E(G)|$. We assume both $V(G)$ and $E(G)$ to be nonempty, and this gives $n \geq 2$. The weights of edges are determined by a *length function*

$$\ell : E(G) \rightarrow \mathbb{R}_{>0}.$$

We let $\mathbb{E}(G) = \{e, \bar{e} : e \in E(G)\}$ denote the set of *oriented edges*. We have $\bar{\bar{e}} = e$. An *orientation* \mathbb{O} on G is a partition $\mathbb{E}(G) = \mathbb{O} \cup \bar{\mathbb{O}}$, where $\bar{\mathbb{O}} = \{\bar{e} : e \in \mathbb{O}\}$. We have an obvious extension of the length function

$$\ell : \mathbb{E}(G) \rightarrow \mathbb{R}_{>0}$$

by requiring $\ell(e) = \ell(\bar{e})$. There is a map $\mathbb{E}(G) \rightarrow V(G) \times V(G)$, sending an oriented edge e to (e^+, e^-) .

2B. Metric graphs and models. A *metric graph* is a pair (Γ, ℓ) consisting of a compact connected topological graph Γ , together with an *inner metric* ℓ . We will always assume Γ is not a single point. In this case, one can alternatively define a metric graph as a compact connected metric space Γ such that every point has a neighborhood isometric to a star-shaped set, endowed with the path metric. We often assume ℓ is implicitly defined, and refer to Γ as the metric graph.

The points of Γ that have valency different from 2 are called the *branch points* of Γ . A *vertex set* for Γ is a finite set $V \subset \Gamma$ containing all branch points such that each connected component c of $\Gamma \setminus V$ has the property that the closure of c in Γ is isometric with a closed interval. Each vertex set V of Γ naturally determines a weighted graph G with nonempty set of vertices $V(G) = V$ and with nonempty set of edges $E(G)$ given by the connected components of $\Gamma \setminus V$. We call such a weighted graph G determined by a vertex set a *model* of Γ .

Given a vertex set V of Γ , the closure e in Γ of a connected component of $\Gamma \setminus V$ is called an *edge segment* of Γ . Note that there is a natural bijective correspondence between the set of edge segments determined by V and the edge set $E(G)$ of the associated weighted graph. Given an edge segment e of Γ , we denote by $\partial e = \{e^-, e^+\} \subset V$ the set of boundary points of e . We use the notation $\partial e = \{e^-, e^+\}$ for the set of boundary points even when there is no orientation present. We hope that this does not lead to confusion.

2C. Electrical networks. Let Γ be a metric graph and G be a model of Γ . We will think of Γ (or G) as an electrical network in which each edge $e \in E(G)$ is a resistor having resistance $\ell(e)$. The vertex set corresponding to G may be thought of as the set of *access points* of the network, i.e., the points at which the external current or voltage sources can be attached or measurements can be done. See Figure 1.

When studying the “potential theory” on a metric graph Γ , it is convenient to always fix an (arbitrary) model G , and think of it as an electrical network as above. This will often allow us to give concrete linear algebraic formulas for quantities and functions of interest. We refer to [9, Chapter II] and [8] for an introduction to the theory of electrical networks from this point of view.

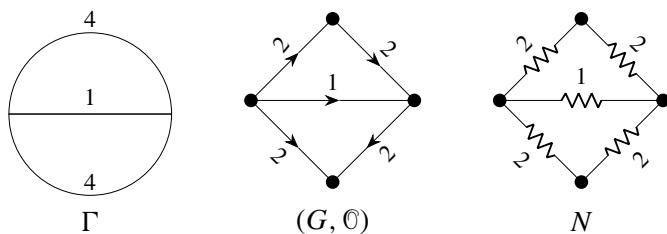


Figure 1. A metric graph Γ , a model G with an orientation \mathbb{O} , and the corresponding electrical network N .

3. Laplacian operators

3A. Distributional Laplacians on metric graphs. Let Γ be a metric graph. We let $\text{PL}(\Gamma)$ be the real vector space consisting of all continuous piecewise affine real valued functions on Γ that can change slope finitely many times on each closed edge segment. Let Δ be the Laplacian operator in the sense of distributions; for $\phi \in \text{PL}(\Gamma)$, its Laplacian $\Delta(\phi)$ is the discrete measure

$$\Delta(\phi) = \sum_{p \in \Gamma} \sigma_p(\phi) \delta_p,$$

where δ_p is the usual delta (Dirac) measure centered at p , and $\sigma_p(\phi)$ is the sum of incoming slopes of ϕ in all tangent directions at p .

Let $\text{DMeas}_0(\Gamma)$ denote the real vector space of discrete measures ν on Γ with $\nu(\Gamma) = 0$. Any $\nu \in \text{DMeas}_0(\Gamma)$ is of the form $\nu = \sum_{p \in \Gamma} a_p \delta_p$, where $a_p \in \mathbb{R}$, all but finitely many a_p 's are zero, and $\sum_{p \in \Gamma} a_p = 0$. One can easily check $\Delta(\phi) \in \text{DMeas}_0(\Gamma)$. Let $\mathbb{R} \subset \text{PL}(\Gamma)$ denote the space of constant functions on Γ . Then Δ induces an isomorphism of vector spaces $\text{PL}(\Gamma)/\mathbb{R} \xrightarrow{\sim} \text{DMeas}_0(\Gamma)$.

3B. Combinatorial Laplacians on weighted graphs. Let G be a weighted graph. We denote by $\mathcal{M}(G) = \text{Hom}(V(G), \mathbb{R}) = C^0(G, \mathbb{R})$ the real vector space of \mathbb{R} -valued functions on $V(G)$. Let Δ be the (combinatorial) Laplacian operator; for $\psi \in \mathcal{M}(G)$, its Laplacian $\Delta(\psi)$ is the measure

$$\Delta(\psi) = \sum_{p \in \Gamma} \Delta_p(\psi) \delta_p,$$

where

$$\Delta_p(\psi) := \sum_{\substack{e^+ = p \\ e \in \mathbb{E}(G)}} (\psi(p) - \psi(e^-)) / \ell(e).$$

Let $\text{DMeas}_0(G)$ denote the real vector space of discrete measures ν on $V(G)$ with $\nu(V(G)) = 0$. Any $\nu \in \text{DMeas}_0(G)$ is of the form $\nu = \sum_{p \in V(G)} a_p \delta_p$, with $a_p \in \mathbb{R}$ and $\sum_{p \in V(G)} a_p = 0$. One can easily check that $\Delta(\psi) \in \text{DMeas}_0(G)$. Let $\mathbb{R} \subset \mathcal{M}(G)$ denote the space of constant functions on $V(G)$. Then Δ induces an isomorphism of vector spaces $\mathcal{M}(G)/\mathbb{R} \xrightarrow{\sim} \text{DMeas}_0(G)$.

3C. Compatibilities and Laplacian matrices. The (distributional) Laplacian Δ and the (combinatorial) Laplacian Δ are compatible in the following sense. Let $\phi \in \text{PL}(\Gamma)$, and let G be a model of Γ such that $V(G)$ contains all those points of Γ at which ϕ changes slopes. Let $\psi \in \mathcal{M}(G)$ denote the function obtained from ϕ by restriction. Then $\sigma_p(\phi) = \Delta_p(\psi)$.

The (combinatorial) Laplacian operator on a weighted graph G can be conveniently presented by its *Laplacian matrix*. Let $\{v_1, \dots, v_n\}$ be a labeling of $V(G)$. The Laplacian matrix \mathbf{Q} associated to G is the $n \times n$ matrix $\mathbf{Q} = (q_{ij})$, where for $i \neq j$,

$$-q_{ij} = \sum_{e=\{v_i, v_j\} \in E(G)} 1/\ell(e).$$

The diagonal entries are determined by forcing the matrix to have zero-sum rows:

$$q_{ii} = - \sum_{j \neq i} q_{ij} = \sum_{e=\{v_i, v\} \in E(G)} 1/\ell(e).$$

It is well known that the symmetric matrix \mathbf{Q} has rank $n - 1$ and that its kernel consists of constant functions (see, e.g., [8]).

Let $H \subset \mathbb{R}^n$ be the subspace of vectors (x_1, \dots, x_n) such that $\sum_{i=1}^n x_i = 0$. The labeling $\{v_1, \dots, v_n\}$ allows one to fix isomorphisms

$$[\cdot] : \mathcal{M}(G) \xrightarrow{\sim} \mathbb{R}^n \quad \text{and} \quad [\cdot] : \text{DMeas}_0(G) \xrightarrow{\sim} H.$$

Then for all $\psi \in \mathcal{M}(G)$, we have

$$[\Delta(\psi)] = Q[\psi].$$

The Laplacian matrix of G can also be expressed in terms of the *incidence matrix* of G . Let $\{v_1, \dots, v_n\}$ be a labeling of $V(G)$ as before. Fix an orientation $\mathbb{O} = \{e_1, \dots, e_m\}$ on G . The incidence matrix B associated to G and \mathbb{O} is the $n \times m$ matrix $B = (b_{ij})$, where

$$b_{ij} = \begin{cases} +1, & e_j^+ = v_i, \\ -1, & e_j^- = v_i, \\ 0, & \text{otherwise.} \end{cases}$$

Let D denote the $m \times m$ diagonal matrix with diagonal entries $\ell(e_i)$ for $e_i \in \mathbb{O}$. We have

$$(3-1) \quad Q = BD^{-1}B^T,$$

where $(\cdot)^T$ denotes the usual matrix transpose operation.

4. Potential kernels

4A. The j -function. A fundamental solution of the Laplacian is given by j -functions. We follow the notation of [13]. See also [2; 4; 6; 22; 25] for more details and formulas.

Let Γ be a metric graph and fix two points $y, z \in \Gamma$. We denote by $j_z(\cdot, y; \Gamma)$ the unique function in $\text{PL}(\Gamma)$ satisfying:

- (i) $\Delta(j_z(\cdot, y; \Gamma)) = \delta_y - \delta_z$,
- (ii) $j_z(z, y; \Gamma) = 0$.

If the metric graph Γ is clear from the context, we will write $j_z(x, y)$ instead of $j_z(x, y; \Gamma)$. Observe that $j_z(\cdot, y)$ is a harmonic function on $\Gamma \setminus \{y, z\}$. It follows from (i) that the j -function is the kernel of integration that inverts the Laplacian operator.

Remark 4.1. If we think of Γ as an electrical network, the function $j_z(x, y)$ has a nice interpretation (Figure 2): it denotes the electric potential at x if one unit of current enters the network at y and exits at z , with z “grounded” (i.e., has zero potential). See Section 7A.3.

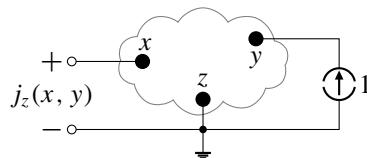


Figure 2. Electrical network interpretation of the j -function.

It follows directly from electrical network theory that $j_z(\cdot, y; \Gamma) \in \text{PL}(\Gamma)$ exists and is unique. For a modern exposition, as well as an explicit integral formula for the j -function, see [22, §3.3] and Remark 7.16.

Remark 4.2. The following properties of the j -function are expected from the electrical network interpretation and are easy to prove (see, e.g., [13, Lemma 2.10]):

- (i) $j_z(x, y)$ is jointly continuous in all three variables $x, y, z \in \Gamma$.
- (ii) $j_z(x, y) = j_z(y, x)$.
- (iii) $0 \leq j_z(x, y) \leq j_z(x, x)$.
- (iv) $j_z(x, x) = j_x(z, z)$.

4B. Effective resistance and Gromov products. Following the electrical network interpretation, it makes sense to make the following definition:

Definition 4.3. The *effective resistance* between two points $x, y \in \Gamma$ is

$$r(x, y) := j_y(x, x) = j_x(y, y).$$

If we want to clarify that the effective resistance is measured on Γ , we will use the notation $r(x, y; \Gamma)$ instead.

Let X be a set and let $d : X \times X \rightarrow \mathbb{R}$ be a symmetric map. For $x, y, z \in X$, one defines (see, e.g., [10, Definition 1.19]) the *Gromov product* $(x|y)_z$ of x and y relative to z by the formula

$$(x|y)_z := \frac{1}{2}(d(x, z) + d(y, z) - d(x, y)).$$

Note that d satisfies the triangle inequality if and only if for all $x, y, z \in X$ the Gromov product $(x|y)_z$ is nonnegative.

Lemma 4.4. *Let Γ be a metric graph, and let $r : \Gamma \times \Gamma \rightarrow \mathbb{R}$ be the effective resistance function. Then $j_z(x, y)$ is precisely the Gromov product $(x|y)_z$ applied to the pair (Γ, r) .*

Proof. For “tripods”, the equality $j_z(x, y) = (x|y)_z$ is immediate (as observed in [5, Remark B.7]) — see Figure 3. The general case follows from the tripod version by applying standard “circuit reduction” techniques (see, e.g., [16, §5.5]). \square

Remark 4.5. We will give a direct proof (avoiding circuit reductions) of Lemma 4.4 using cross ratios in Example 6.5.

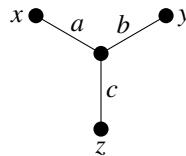


Figure 3. A “tripod” with edge lengths a, b, c . Clearly $r(x, z) = a + c$, $r(y, z) = b + c$, $r(x, y) = a + b$, and $j_z(x, y) = c$.

Remark 4.6. By Remark 4.2(iii) we have $j_z(x, y) \geq 0$. Therefore, Lemma 4.4 has the immediate corollary that the effective resistance function satisfies the triangle inequality. The maximum principle for harmonic functions together with Remark 4.2(iii) furthermore implies that $r(x, y) \geq 0$ with equality if and only if $x = y$. We obtain the well-known fact (see, e.g., [19]) that the effective resistance is a distance function on Γ .

4C. Computing j -functions. Let G be an arbitrary model of Γ . One can explicitly compute the quantities $j_q(p, v) \in \mathbb{R}$ for $q, p, v \in V(G)$ using linear algebra (see [6, §3]) as follows: fix a labeling of $V(G)$ as before, and let \mathbf{Q} be the corresponding Laplacian matrix. Let \mathbf{Q}_q be the $(n-1) \times (n-1)$ matrix obtained from \mathbf{Q} by deleting the row and column corresponding to $q \in V(G)$ from \mathbf{Q} . It is well known that the matrix \mathbf{Q}_q is invertible, see, e.g., Remark 7.6(ii). Let \mathbf{L}_q be the $n \times n$ matrix obtained from \mathbf{Q}_q^{-1} by inserting zeros in the row and column corresponding to q . One can easily check that

$$(4-1) \quad \mathbf{Q}\mathbf{L}_q = \mathbf{I} + \mathbf{R}_q,$$

where \mathbf{I} is the $n \times n$ identity matrix and \mathbf{R}_q has all -1 entries in the row corresponding to q and has zeros elsewhere. It follows from (4-1) and the compatibility of various Laplacians in Section 3 that

$$\mathbf{L}_q = (j_q(p, v))_{p, v \in V(G)}.$$

Remark 4.7. (i) In Corollary 7.13 we give another explicit formula for computing the j -function on weighted graphs.

(ii) Clearly $\mathbf{R}_q \mathbf{Q} = \mathbf{0}$, where $\mathbf{0}$ is the $n \times n$ zero matrix. Therefore, \mathbf{L}_q is a *generalized inverse* of \mathbf{Q} , in the sense that $\mathbf{Q}\mathbf{L}_q \mathbf{Q} = \mathbf{Q}$.

(iii) Computing \mathbf{L}_q takes time at most $O(n^\omega)$, where ω is the exponent for the matrix multiplication algorithm (currently, $\omega < 2.38$).

5. Energy pairings

In this section, we briefly study two useful pairings. We remark that both pairings can be defined and studied on larger vector spaces (see, e.g., [4] for more general statements). Here we restrict our attention to those spaces that are relevant to our work, and give more explicit descriptions, statements, and proofs.

Definition 5.1. Let Γ be a metric graph. The *energy pairing*

$$\langle \cdot, \cdot \rangle_{\text{en}} : \text{DMeas}_0(\Gamma) \times \text{DMeas}_0(\Gamma) \rightarrow \mathbb{R}$$

is defined by

$$\langle v_1, v_2 \rangle_{\text{en}} := \int_{\Gamma \times \Gamma} j_q(x, y) \, dv_1(x) \, dv_2(y).$$

for a fixed $q \in \Gamma$.

If we want to clarify Γ , we will use the notation $\langle v_1, v_2 \rangle_{\text{en}}^\Gamma$ instead.

It follows from Proposition 5.2 below that the energy pairing is indeed independent of the choice of q . A closely related concept is the *Dirichlet pairing*

$$\langle \cdot, \cdot \rangle_{\text{Dir}} : \text{PL}(\Gamma) \times \text{PL}(\Gamma) \rightarrow \mathbb{R}$$

defined by

$$\langle \phi_1, \phi_2 \rangle_{\text{Dir}} := \int_{\Gamma} \phi_1 \Delta(\phi_2) = \int_{\Gamma} \phi_2 \Delta(\phi_1).$$

If $v_1 = \Delta(\phi_1)$ and $v_2 = \Delta(\phi_2)$, then

$$\langle v_1, v_2 \rangle_{\text{en}} = \langle \phi_1, \phi_2 \rangle_{\text{Dir}}.$$

Note that this equality does not depend on the choice of ϕ_1 and ϕ_2 , which are well defined only up to constant functions.

The energy pairing (and the Dirichlet pairing) can be computed using linear algebra: let G be a model of Γ such that $v_1, v_2 \in \text{DMeas}_0(G)$. Then $\langle v_1, v_2 \rangle_{\text{en}}$ can be computed using the (combinatorial) energy pairing

$$\langle \cdot, \cdot \rangle_{\text{en}} : \text{DMeas}_0(G) \times \text{DMeas}_0(G) \rightarrow \mathbb{R}$$

defined by

$$(5-1) \quad \langle v_1, v_2 \rangle_{\text{en}} := \sum_{p, v \in V(G)} v_1(p) j_q(p, v) v_2(v) = [v_1]^T \mathbf{L}_q [v_2].$$

Likewise, let G be a model of Γ such that $V(G)$ contains all those points of Γ at which ϕ_1 or ϕ_2 changes slopes. Let $\psi_i \in \mathcal{M}(G)$ denote the function obtained from ϕ_i by restriction. Then $\langle \phi_1, \phi_2 \rangle_{\text{Dir}}$ can be computed using the (combinatorial) Dirichlet pairing

$$\langle \cdot, \cdot \rangle_{\text{Dir}} : \mathcal{M}(G) \times \mathcal{M}(G) \rightarrow \mathbb{R}$$

defined by

$$\langle \psi_1, \psi_2 \rangle_{\text{Dir}} := [\psi_1]^T \mathbf{Q} [\psi_2].$$

Proposition 5.2. *Let G be a weighted graph. Fix a labeling of $V(G)$ and let \mathbf{L} be any generalized inverse of the Laplacian matrix \mathbf{Q} (i.e., $\mathbf{Q} \mathbf{L} \mathbf{Q} = \mathbf{Q}$). Then the symmetric bilinear form on $\text{DMeas}_0(G)$ defined by*

$$\langle v_1, v_2 \rangle_{\text{en}} = [v_1]^T \mathbf{L} [v_2]$$

is independent of the choice of the generalized inverse \mathbf{L} and is positive definite.

Proof. Independence from \mathbf{L} follows from the fact that if $[v_i] = \mathbf{Q}[\psi_i]$, then

$$\langle v_1, v_2 \rangle_{\text{en}} = [v_1]^T \mathbf{L} [v_2] = [\psi_1]^T \mathbf{Q} \mathbf{L} \mathbf{Q} [\psi_2] = [\psi_1]^T \mathbf{Q} [\psi_2] = \langle \psi_1, \psi_2 \rangle_{\text{Dir}}.$$

Positive definiteness follows from the factorization (3-1) (see also [6, Lemma 3.5]). Let $[\nu] = \mathbf{Q}[\psi]$. Then

$$\langle \nu, \nu \rangle_{\text{en}} = [\psi]^T \mathbf{Q} [\psi] = \|\mathbf{D}^{-1/2} \mathbf{B}^T [\psi]\|_2.$$

The kernel of $\mathbf{D}^{-1/2} \mathbf{B}^T$ is the space of constant functions in $\mathcal{M}(G)$. □

Remark 5.3. A canonical choice for the generalized inverse of \mathbf{Q} in Proposition 5.2 is the “Moore–Penrose pseudoinverse” \mathbf{Q}^+ . In fact, it is straightforward to check $\mathbf{Q}^+ = (1/n) \sum_{q \in V(G)} \mathbf{L}_q$ (see [6, Constructions 3.2–3.3]). In what follows, we find it more natural to work with \mathbf{L}_q ’s directly.

The energy and Dirichlet pairings have the following interpretation in the language of electrical networks: consider $v \in \text{DMeas}_0(G)$ as an external “current source” attached to the network G . Then $\langle v, v \rangle_{\text{en}}$ is precisely the total energy dissipated (per unit time) in G .

6. Cross ratios

In this section, we introduce the notion of cross ratios for metric graphs. We remark that this notion is already mentioned (in passing) in [5, Remark B.12]. We establish some basic properties of these cross ratios and provide some basic examples.

Definition 6.1. Let Γ be a metric graph and fix $q \in \Gamma$. We define the *cross ratio* function (with respect to the base point q) $\xi_q : \Gamma^4 \rightarrow \mathbb{R}$ by

$$\xi_q(x, y, z, w) := j_q(x, z) + j_q(y, w) - j_q(x, w) - j_q(y, z).$$

If we want to clarify Γ , we will use the notation $\xi_q(x, y, z, w; \Gamma)$ instead.

Lemma 6.2. (a) $\xi(x, y, z, w) := \xi_q(x, y, z, w)$ is independent of the choice of q .

(b) $\xi(x, y, z, w) = \xi(z, w, x, y)$.

(c) $\xi(y, x, z, w) = -\xi(x, y, z, w)$.

Proof. Parts (b) and (c) are immediate from [Definition 6.1](#).

Let G be a model for Γ such that $x, y, z, w, q_1, q_2 \in V(G)$. Note that (see [\(5-1\)](#))

$$\xi_{q_i}(x, y, z, w) = \langle \delta_x - \delta_y, \delta_z - \delta_w \rangle_{\text{en}}.$$

Part (a) then follows from the fact that the energy pairing is independent of the choice of the base point q_i , see [Proposition 5.2](#). See also [5, Remark B.12] for an outline of a different proof of part (a). \square

Remark 6.3. (i) Yet another proof of [Lemma 6.2](#) (a) can be obtained from [Lemma 4.4](#) and an explicit computation. Namely, one finds the relation

$$(6-1) \quad -2\xi_q(x, y, z, w) = r(x, z) + r(y, w) - r(x, w) - r(y, z).$$

(ii) As is evident from the proof of [Lemma 6.2](#), we could define the cross ratio with the more canonical expression

$$\xi(x, y, z, w) = \langle \delta_x - \delta_y, \delta_z - \delta_w \rangle_{\text{en}}.$$

See also [Corollary 7.12](#) and [Remark 7.16](#) for other explicit formulas for computing cross ratios on weighted graphs and on metric graphs.

Remark 6.4. Our notion of cross ratio is related to that of *transpedance* (or *generalized transfer impedance*) introduced and studied by Brooks, Smith, Stone, and Tutte in the seminal paper [11] (see also [24, §VI.5]).

Let G be a model of Γ whose vertex set contains $x, y, z, w \in \Gamma$. Fix a labeling $\{v_1, \dots, v_n\}$ of vertices of G and assume $x = v_i$, $y = v_j$, $z = v_g$, $w = v_h$. The transpedance $[xy, zw]$ for $x \neq y$ and $z \neq w$ is defined to be $(-1)^{i+g}$ times the (j, h) -cofactor in the matrix obtained from the Laplacian matrix \mathbf{Q} by removing row i and column g . They separately define $[xy, zw] = 0$ if $x = y$ or $z = w$.

It is easy to show (see, e.g., [11, (2.13)]) and [Example 6.6](#) that

$$\xi(x, y, z, w) = \frac{[xy, zw]}{w'(G)},$$

where $w'(G)$ is the coweight of G (see [Section 7B.2](#)). We remark that both $w'(G)$ and $[xy, zw]$ depend on the choice of the model G , whereas $\xi(x, y, z, w)$ only depends on Γ .

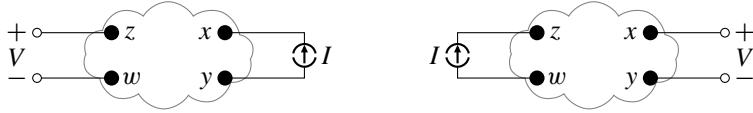


Figure 4. Reciprocity theorem for electrical networks: (left) $V = (j_y(x, z) - j_y(x, w))I$, (right) $V = (j_w(x, z) - j_w(y, z))I$.

Example 6.5 (proof of Lemma 4.4 using cross ratios). Let us compute the cross ratio $\xi(x, y, x, y)$ with respect to two different base points:

$$\xi_x(x, y, x, y) = j_x(x, x) + j_x(y, y) - j_x(x, y) - j_x(y, x) = j_x(y, y) = r(x, y),$$

$$\xi_z(x, y, x, y) = j_z(x, x) + j_z(y, y) - j_z(x, y) - j_z(y, x) = r(x, z) + r(y, z) - 2j_z(x, y).$$

By Lemma 6.2 (a), we must have $\xi_x(x, y, x, y) = \xi_z(x, y, x, y)$, and therefore,

$$j_z(x, y) = \frac{1}{2}(r(x, z) + r(y, z) - r(x, y)).$$

Example 6.6. Let us compute the cross ratio $\xi(x, y, z, w)$ with respect to w as the base point:

$$\xi(x, y, z, w) = \xi_w(x, y, z, w) = j_w(x, z) + j_w(y, w) - j_w(x, w) - j_w(y, z) = j_w(x, z) - j_w(y, z).$$

This means $\xi(x, y, z, w)$ can be interpreted as the *potential difference* between x and y when a unit of current enters the network at z and exits the network at w .

Example 6.7 (reciprocity theorem in electrical networks). By Lemma 6.2 (a), we have $\xi_y(x, y, z, w) = \xi_w(x, y, z, w)$. Therefore,

$$j_y(z, x) - j_y(w, x) = j_w(x, z) - j_w(y, z).$$

This is the celebrated ‘reciprocity theorem’ for electrical networks (see, e.g., [21, §5.3], [16, §17.2], [23, Theorem 4], [2, Theorem 8]): informally, the location of the current source and the resulting voltage may be interchanged without a change in voltage. See Figure 4.

Remark 6.8. One may think of Proposition 5.2 as a vast generalization of the reciprocity theorem.

7. Projections

Throughout this section, we fix a model G for a metric graph Γ and fix an orientation \mathbb{O} on G . It is convenient to define the real 1-chains by

$$C_1(G, \mathbb{R}) := \frac{\bigoplus_{e \in \mathbb{E}(G)} \mathbb{R}e}{\langle e + \bar{e} : e \in \mathbb{O} \rangle} \simeq \bigoplus_{e \in \mathbb{O}} \mathbb{R}e.$$

So, for $e \in \mathbb{E}(G)$, the above presentation implies $\bar{e} = -e$ inside $C_1(G, \mathbb{R})$. Note that \mathbb{O} is a basis for $C_1(G, \mathbb{R})$.

Definition 7.1. For any subset $\mathcal{A} \subseteq \mathbb{E}(G)$, we define its *associated 1-chain* as

$$\gamma_{\mathcal{A}} = \sum_{e \in \mathcal{A}} e = \sum_{e \in \mathbb{O}} \text{sign}(\mathcal{A}, e) e.$$

where

$$\text{sign}(\mathcal{A}, e) = \begin{cases} +1, & \text{if } e \in \mathcal{A}, \\ -1, & \text{if } \bar{e} \in \mathcal{A}, \\ 0, & \text{otherwise.} \end{cases}$$

We will work with the usual definition of real 0-chains:

$$C_0(G, \mathbb{R}) := \bigoplus_{v \in V(G)} \mathbb{R}v.$$

Let $\mathbb{O} = \{e_1, \dots, e_m\}$ be a labeling of the orientation \mathbb{O} of G . The real vector space $C_1(G, \mathbb{R})$ has a canonical inner product

$$[\cdot, \cdot] : C_1(G, \mathbb{R}) \times C_1(G, \mathbb{R}) \rightarrow \mathbb{R}$$

defined by $[e_i, e_j] = \delta_i(j)\ell(e_i)$.

Consider the usual boundary map $\partial : C_1(G, \mathbb{R}) \rightarrow C_0(G, \mathbb{R})$ defined by $\partial(e) = e^+ - e^-$. The first homology group coincides with the space of 1-cycles

$$H_1(G, \mathbb{R}) = \text{Ker } \partial.$$

The inner product $[\cdot, \cdot]$ restricts to an inner product, also denoted by $[\cdot, \cdot]$, on $H_1(G, \mathbb{R})$. It is easy to check that the pair $(H_1(G, \mathbb{R}), [\cdot, \cdot])$ is a canonical inner product space associated to Γ ; it is independent of the choice of the model G .

Remark 7.2. (i) The labeling $\mathbb{O} = \{e_1, \dots, e_m\}$ fixes an isomorphism

$$[\cdot, \cdot] : C_1(G, \mathbb{R}) \xrightarrow{\sim} \mathbb{R}^m.$$

- (ii) The incidence matrix \mathbf{B} in [Section 3C](#) is precisely the matrix of ∂ with respect to bases \mathbb{O} for $C_1(G, \mathbb{R})$, and $V(G)$ for $C_0(G, \mathbb{R})$.
- (iii) The matrix \mathbf{D} in [Section 3C](#) is precisely the Gram matrix associated to the pair $(C_1(G, \mathbb{R}), [\cdot, \cdot])$, with respect to the basis \mathbb{O} .

We are interested in the two orthogonal projection maps

$$\begin{aligned} \pi : C_1(G, \mathbb{R}) &\rightarrow H_1(G, \mathbb{R}), \\ \pi' : C_1(G, \mathbb{R}) &\rightarrow H_1(G, \mathbb{R})^\perp \end{aligned}$$

and their matrix representations. Here, $H_1(G, \mathbb{R})^\perp$ denotes the orthogonal complement of $H_1(G, \mathbb{R}) \subseteq C_1(G, \mathbb{R})$ with respect to $[\cdot, \cdot]$.

7A. Electrical network problems. The original motivation for computing projection matrices comes from electrical network theory.

7A.1. The Kirchhoff problem. Consider the *Kirchhoff problem*: Given $\mathbf{c} \in C_1(G, \mathbb{R})$, find $\mathbf{i} \in H_1(G, \mathbb{R})^\perp$ such that $\mathbf{c} - \mathbf{i} \in H_1(G, \mathbb{R})$. Here, \mathbf{c} should be thought of as an *external current source*, and \mathbf{i} should be thought of as the induced *internal current*.

The condition $\mathbf{c} - \mathbf{i} \in H_1(G, \mathbb{R})$ is precisely the *Kirchhoff's current law*. The condition $\mathbf{i} \in H_1(G, \mathbb{R})^\perp$ is precisely the *Kirchhoff's voltage law*. These laws are equivalent to computing the orthogonal decomposition $\mathbf{c} = \mathbf{i} + (\mathbf{c} - \mathbf{i})$, so the solution is provided by computing

$$\mathbf{i} = \pi'(\mathbf{c}).$$

Remark 7.3. The contribution of the external current source \mathbf{c} only depends on its boundary $\partial\mathbf{c}$. It is customary to only refer to $\partial\mathbf{c}$ as the external current source. See, e.g., [Remark 4.1](#).

7A.2. Coboundaries and Ohm's law. The space of coboundaries is, by definition,

$$\text{Im}(d : C^0(G, \mathbb{R}) \rightarrow C^1(G, \mathbb{R})).$$

We may define an isomorphism from $C_1(G, \mathbb{R})$ to $C^1(G, \mathbb{R})$ using the bilinear form $[\cdot, \cdot]$. More precisely, we may think of $C^1(G, \mathbb{R}) \simeq \bigoplus \mathbb{R}e^*$ with $e^*(e) = 1$. The isomorphism is defined by $e/\ell(e) \mapsto e^*$. Under this isomorphism $H_1(G, \mathbb{R})^\perp$ is identified with the space of coboundaries.

Under this identification \mathbf{i} corresponds to a coboundary element \mathbf{v} , referred to as *internal voltage*. Explicitly, the internal voltage $\mathbf{v} = \sum_{e \in \mathbb{E}} v(e) e^*$ is identified with $\sum_{e \in \mathbb{E}} v(e)/\ell(e) e = \mathbf{i} = \sum_{e \in \mathbb{E}} i(e) e$, so $v(e) = \ell(e)i(e)$. This is *Ohm's law*.

7A.3. The Dirichlet problem. Since \mathbf{v} in [Section 7A.2](#) is a coboundary, $\mathbf{v} = d\psi$ for some $\psi \in C^0(G, \mathbb{R})$ (well-defined up to a constant function). The 0-cochain ψ is called the *potential* associated to \mathbf{v} . One might be interested to directly compute this ψ given the external source $\partial\mathbf{c}$. It is easy to check that this problem boils down to solving the *Dirichlet problem*

$$\Delta(\psi) = \partial\mathbf{c}.$$

Here, Δ is as in [Section 3B](#) and $\partial\mathbf{c}$ is thought of as a discrete measure on $V(G)$, after identifying $v \in C_0(G, \mathbb{R})$ with $\delta_v \in \text{DMeas}(G)$. The j -function defined in [Section 4A](#) provides the fundamental solutions for this Dirichlet problem: if $\partial\mathbf{c} = \sum_{v \in V(G)} a_v \delta_v$, then $\psi = \sum_{v \in V(G)} a_v j_q(\cdot, v)$ for any fixed $q \in V(G)$.

7B. Projections using spanning trees. Before presenting the projection formulas in terms of cross ratios, we will review Kirchhoff's beautiful description of these projections (in the basis \mathbb{O}) as a certain average over spanning trees. This was introduced in the seminal paper [\[18\]](#).

Recall that a spanning tree T of G is a maximal subset of $E(G)$ that contains no circuit (closed simple path). Equivalently, T is a minimal subset of $E(G)$ that connects all vertices.

7B.1. Fundamental circuits and cocircuits. Let T be a spanning tree of G . Each (unoriented) edge $e \notin T$ determines a *fundamental circuit*, i.e., a unique circuit $\mathcal{C}(T, e) \subseteq E(G)$ in $T \cup e$. Let e also denote the oriented edge in the fixed orientation \mathbb{O} corresponding to the (unoriented) edge e . Note that every edge in $\mathcal{C}(T, e)$ comes with a preferred choice of orientation, namely the orientations that agree with the direction of the oriented edge $e \in \mathbb{O}$ as one travels along the circuit.

Definition 7.4. (i) For $e \notin T$, we let $\text{circ}(T, e)$ be the associated 1-chain of $\mathcal{C}(T, e)$ endowed with its preferred orientation ([Definition 7.1](#)). For $e \in T$ we define $\text{circ}(T, e) = 0$.
(ii) We let \mathbf{M}_T be the $m \times m$ matrix whose columns are $[\text{circ}(T, e)]$ for $e \in \mathbb{O}$, where $[\cdot]$ is as in [Remark 7.2\(i\)](#).

It is well known (and easy to check) that $\{\text{circ}(T, e) : e \notin T\}$ forms a basis for $H_1(G, \mathbb{R})$.

Each (unoriented) edge $e \in T$ determines a *fundamental cocircuit*, i.e., the unique minimal subset $\mathcal{B}(T, e) \subseteq (E(G) \setminus T) \cup e$ such that $E(G) \setminus \mathcal{B}(T, e)$ is disconnected. Let e also denote the oriented edge in the fixed orientation \mathbb{O} corresponding to the (unoriented) edge e . Note that every edge in $\mathcal{B}(T, e)$ comes with a preferred choice of orientation, namely the orientation that agrees with the direction of $e \in \mathbb{O}$ in the cut-set $\mathcal{B}(T, e)$.

Definition 7.5. (i) For $e \in T$, we let $\text{cocirc}(T, e)$ be the associated 1-chain of $\mathcal{B}(T, e)$ endowed with its preferred orientation (Definition 7.1). For $e \notin T$ we define $\text{cocirc}(T, e) = 0$.
(ii) We let N_T be the $m \times m$ matrix whose columns are $[\text{cocirc}(T, e)]$ for $e \in \mathbb{O}$, where $[\cdot]$ is as in Remark 7.2(i).

7B.2. Weights and coweights. The *weight* of a spanning tree T of G is the product $w(T) := \prod_{e \notin T} \ell(e)$. The *coweight* of a spanning tree T of G is the product $w'(T) := \prod_{e \in T} \ell(e)^{-1}$. The *weight* and *coweight* of G are

$$w(G) := \sum_T w(T) \quad \text{and} \quad w'(G) := \sum_T w'(T),$$

where the sums are over all spanning trees of G .

Remark 7.6. (i) For any spanning tree T of G we have $w(T)/w(G) = w'(T)/w'(G)$. Moreover, the quantity $w(G)$ depends only on the underlying metric graph Γ . This is not true of $w'(G)$.
(ii) By (Tutte's version of) Kirchhoff's matrix tree theorem ([24, Theorem VI.27]), both $w(G)$ and $w'(G)$ can be expressed in terms of certain determinants. For example, $w'(G) = \det(Q_q)$, where Q_q is as defined in Section 4C. These are simple consequences of the ‘‘Cauchy–Binet formula’’ for determinants. See [1, Section 5] for more details and for a geometric (or tropical) proof.

7B.3. Kirchhoff's projection formulas. Consider the following matrix averages:

$$\mathbf{P} = \sum_T \frac{w(T)}{w(G)} \mathbf{M}_T \quad \text{and} \quad \mathbf{P}' = \sum_T \frac{w'(T)}{w'(G)} \mathbf{N}_T,$$

the sums being over all spanning trees T of G .

Theorem 7.7 (Kirchhoff). (a) *The matrix of $\pi : C_1(G, \mathbb{R}) \rightarrow H_1(G, \mathbb{R})$, with respect to \mathbb{O} , is \mathbf{P} .*
(b) *The matrix of $\pi' : C_1(G, \mathbb{R}) \rightarrow H_1(G, \mathbb{R})^\perp$, with respect to \mathbb{O} , is $(\mathbf{P}')^\top$.*

Proof. By [8, Proposition 15.2], we know the matrix of π' is $\mathbf{D}^{-1} \mathbf{P}' \mathbf{D}$. Since $\mathbf{D}^{-1} \mathbf{P}'$ is symmetric ([8, Proposition 15.1]), we have

$$\mathbf{D}^{-1} \mathbf{P}' \mathbf{D} = (\mathbf{P}')^\top \mathbf{D}^{-1} \mathbf{D} = (\mathbf{P}')^\top.$$

This proves part (b). Part (a) follows from part (b) and the fact that $\mathbf{I} - (\mathbf{P}')^\top = \mathbf{P}$. See the computation in the first paragraph of [8, §16]. \square

We note that a given graph G can have many (super-exponential number of) spanning trees. This makes the computation of projection matrices using Theorem 7.7 quite inefficient. The situation is better if one uses cross ratios, as we propose in the next section.

7C. Projections using cross ratios. We now show that Kirchhoff's projection matrices have expressions in terms of cross ratios. They are efficient for computations, and useful for proving theorems.

Let Ξ be the $m \times m$ matrix of cross ratios:

$$\Xi := (\xi(e^-, e^+, f^-, f^+))_{e, f \in \mathbb{O}}.$$

It follows from [Proposition 5.2](#) and [Remark 6.3\(ii\)](#) that

$$(7-1) \quad \Xi = \mathbf{B}^T \mathbf{L} \mathbf{B}$$

for any generalized inverse \mathbf{L} of the Laplacian matrix \mathbf{Q} .

Theorem 7.8. *Let \mathbf{D} be as in [Section 3C](#).*

- (a) *The matrix of $\pi : C_1(G, \mathbb{R}) \rightarrow H_1(G, \mathbb{R})$, with respect to \mathbb{O} , is $\mathbf{I} - \mathbf{D}^{-1} \Xi$.*
- (b) *The matrix of $\pi' : C_1(G, \mathbb{R}) \rightarrow H_1(G, \mathbb{R})^\perp$, with respect to \mathbb{O} , is $\mathbf{D}^{-1} \Xi$.*

Remark 7.9. One can compute Ξ and both these projection matrices in time at most $O(n^\omega)$, where ω is the exponent for the matrix multiplication algorithm (currently $\omega < 2.38$). See [Remark 4.7\(iii\)](#).

Proof. It suffices to prove (b), which follows from a straightforward linear algebraic argument. We identify $C_1(G, \mathbb{R})$ with \mathbb{R}^m using \mathbb{O} . Recall $H_1(G, \mathbb{R}) = \text{Ker } \partial = \text{Ker } \mathbf{B}$. Therefore, see [Remark 7.2](#), we have $H_1(G, \mathbb{R})^\perp = \text{Im } \mathbf{D}^{-1} \mathbf{B}^T$.

For $\mathbf{b} \in \mathbb{R}^m$, let $\hat{\mathbf{b}} = \mathbf{D}^{-1} \mathbf{B}^T \mathbf{x}$ denote its orthogonal projection onto $\text{Im } \mathbf{D}^{-1} \mathbf{B}^T$. From $\mathbf{b} - \hat{\mathbf{b}} \in \text{Ker } \mathbf{B}$ we obtain the Dirichlet problem $\mathbf{Q}\mathbf{x} = \mathbf{B}\mathbf{b}$ which has $\mathbf{x} = \mathbf{L}_q \mathbf{B}\mathbf{b}$ as a solution (see [Section 7A.3](#)). Therefore, by (7-1), we have $\hat{\mathbf{b}} = \mathbf{D}^{-1} \mathbf{B}^T \mathbf{L}_q \mathbf{B}\mathbf{b} = \mathbf{D}^{-1} \Xi \mathbf{b}$. \square

The following is a restatement of [Theorem 7.8](#) in a more canonical language:

Corollary 7.10. *For any $f \in \mathbb{O}$, we have*

- (a) $\pi(f) = \sum_{e \in \mathbb{O}} F(e, f) e$, where

$$F(e, f) := \begin{cases} 1 - r(e^-, e^+)/\ell(e), & \text{if } e = f, \\ -\xi(e^-, e^+, f^-, f^+)/\ell(e), & \text{if } e \neq f. \end{cases}$$

- (b) $\pi'(f) = \sum_{e \in \mathbb{O}} F'(e, f) e$, where

$$F'(e, f) = \xi(e^-, e^+, f^-, f^+)/\ell(e).$$

7D. Relations and consequences. The two different descriptions of projection matrices ([Theorem 7.7](#) and [Theorem 7.8](#)) have some important consequences.

- (i) We have the equalities

$$(7-2) \quad \mathbf{P} = \mathbf{I} - \mathbf{D}^{-1} \Xi \quad \text{and} \quad \mathbf{P}' = \Xi \mathbf{D}^{-1}.$$

- (ii) Using (7-1) and (7-2) we obtain

$$\mathbf{P}' = \mathbf{B}^T \mathbf{L} \mathbf{B} \mathbf{D}^{-1}.$$

This refines (and generalizes) the “canonical factorization” of Biggs in [8, §8 and §15].

(iii) The *Foster coefficient* of $e \in \mathbb{O}$ is, by definition,

$$F(e) := F(e, e) = 1 - r(e^-, e^+)/\ell(e).$$

Clearly $F(e) = F(\bar{e})$, so the Foster coefficient is also defined for $e \in E(G)$. It measures the probability $\Pr\{e \notin T\}$, where T is a weighted uniform spanning tree. It is easy to see that $\sum_{e \in E(G)} F(e) = \dim_{\mathbb{R}} H_1(G, \mathbb{R})$. In fact, both sides of this equality represent the trace of the orthogonal projection matrix P . This is the theorem of Ronald Foster in [15]. See also [14], [23, Theorem 6], and [3, Corollary 6.5].

(iv) Recall, by [Example 6.6](#), that we have

$$\xi_{f^-}(e^-, e^+, f^-, f^+) = j_{f^-}(e^+, f^+) - j_{f^-}(e^-, f^+).$$

Therefore, $F'(e, f)$ can be interpreted as the current that flows across e when a unit current is imposed between the endpoints of f . In this way, we recover the well-known description as a “transfer-current matrix” for π' in probability theory (see, e.g., [12], [20, §2.4, §4.2], [17, §4.3.2], [7, §4]). The “transfer-current theorem” states that the weighted uniform spanning tree of G is a determinantal point process on $E(G)$ with kernel π' .

7E. Energy pairing and cross ratios using projections. We have already seen the entries of projection matrices are computed from certain cross ratios. We now show that an arbitrary cross ratio can be computed using these projection matrices.

A *path* γ in G is an alternating sequence of distinct vertices v_i and oriented edges e_i ,

$$v_0, e_0, v_1, e_1, v_2, \dots, v_{k-1}, e_{k-1}, v_k$$

such that $e_i^- = v_i$ and $e_i^+ = v_{i+1}$. A *closed path* is one that starts and ends at the same vertex. One can associate a 1-chain γ to the path γ by applying [Definition 7.1](#) to the set of oriented edges $\{e_0, \dots, e_{k-1}\}$. By construction, we have $\partial\gamma = \delta_{v_k} - \delta_{v_0}$.

More generally, for any $v \in \text{DMeas}_0(G)$, it is easy to see there exists $\gamma \in C_1(G, \mathbb{R})$, well-defined up to an element of $H_1(G, \mathbb{R})$, such that $\partial(\gamma) = v$. To see this, let $v = \sum_{v \in V(G)} a_v \delta_v$. Then for any fixed $q \in V(G)$, we have $v = \sum_{v \in V(G)} a_v (\delta_v - \delta_q)$. Let γ_{qv} be an arbitrary path in G from q to v . Then $\gamma = \sum_{v \in V(G)} a_v \gamma_{qv}$ has the property that $\partial(\gamma) = v$.

Proposition 7.11. *Let $v_1, v_2 \in \text{DMeas}_0(G)$. Fix a model G compatible with v_1, v_2 , and let $\gamma_1, \gamma_2 \in C_1(G, \mathbb{R})$ be such that $\partial(\gamma_i) = v_i$. Then*

$$\langle v_1, v_2 \rangle_{\text{en}} = [\gamma_1, \pi'(\gamma_2)].$$

Here, $\pi' : C_1(G, \mathbb{R}) \rightarrow H_1(G, \mathbb{R})^\perp$ denotes the orthogonal projection as before.

Proof. We use the basis \mathbb{O} and do the computations with the help of corresponding matrices:

$$\begin{aligned}
 [\gamma_1, \pi'(\gamma_2)] &= [\gamma_1]^T \mathbf{D}[\pi'(\gamma_2)] \quad (\text{Remark 7.2(iii)}) \\
 &= [\gamma_1]^T \mathbf{D}\mathbf{D}^{-1} \Xi[\gamma_2] \quad (\text{Theorem 7.8}) \\
 &= [\gamma_1]^T \mathbf{B}^T \mathbf{L} \mathbf{B}[\gamma_2] \quad (7-1) \\
 &= [\partial \gamma_1]^T \mathbf{L}[\partial \gamma_2] \quad (\text{Remark 7.2(ii)}) \\
 &= [\nu_1]^T \mathbf{L}[\nu_2] \\
 &= \langle \nu_1, \nu_2 \rangle_{\text{en}} \quad (\text{Proposition 5.2}). \quad \square
 \end{aligned}$$

Corollary 7.12. Fix arbitrary paths in G from y to x , and from w to z . Let γ_{yx} and γ_{wz} denote their associated 1-chains. Then

$$\xi(x, y, z, w) = [\gamma_{yx}, \pi'(\gamma_{wz})].$$

Proof. This follows from [Proposition 7.11](#), applied to $\nu_1 = \delta_x - \delta_y$ and $\nu_2 = \delta_z - \delta_w$. See [Remark 6.3\(ii\)](#). \square

An explicit integral formula for j -functions is given in [\[22, Proposition 3.17\]](#). The following is a discrete version of that result:

Corollary 7.13. Fix arbitrary paths from z to x , and from z to y . Let γ_{zx} and γ_{zy} denote their associated 1-chains. Then

$$j_z(x, y) = [\gamma_{zx}, \pi'(\gamma_{zy})].$$

Proof. This follows from [Corollary 7.12](#), applied to $\xi(x, z, y, z) = j_z(x, y)$. \square

Corollary 7.14. Fix two arbitrary paths from y to x . Let γ_{yx} and γ'_{yx} denote the associated 1-chains. Then

$$r(x, y) = [\gamma'_{yx}, \pi'(\gamma_{yx})].$$

Proof. This follows from [Corollary 7.12](#), applied to $\xi(x, y, x, y) = r(x, y)$. \square

Remark 7.15. Since $\pi'^2 = \pi'$ and π' is self-adjoint with respect to $[\cdot, \cdot]$, one can write the expression in [Corollary 7.14](#) as $r(x, y) = [\pi'(\gamma_{yx}), \pi'(\gamma_{yx})]$. So $r(x, y)$ is the *norm squared* of the projected vector $\pi'(\gamma_{yx})$. This is equivalent to *Thomson's principle* for electrical networks (see [\[8, §18\]](#)). [Proposition 7.11](#) may be thought of as a generalized version of Thomson's principle.

Remark 7.16. [Corollary 7.12](#) can easily be proved assuming [Corollary 7.13](#) (using [Definition 6.1](#)). Similarly, one can use the explicit integral formula for j -functions in [\[22, Proposition 3.17\]](#) to write down an explicit integral formula for cross ratios on metric graphs. Namely, for all $x, y, z, w \in \Gamma$, we have

$$(7-3) \quad \xi(x, y, z, w) = \int_{\gamma_{yx}} (\omega_{\gamma_{wz}} - \pi(\omega_{\gamma_{wz}})).$$

Here $\pi : \Omega^1(\Gamma) \rightarrow \mathcal{H}(\Gamma)$ denotes the orthogonal projection from the space of piecewise constant 1-forms to the subspace of harmonic 1-forms on Γ . For $p, q \in \Gamma$, the 1-form $\omega_{\gamma_{pq}}$ is associated to a piecewise linear path from p to q . We refer to [\[22, §3\]](#) for more details.

8. Rayleigh's law

8A. Contractions. Let Γ be a metric graph. Let $e \subseteq \Gamma$ be an edge segment with boundary points $\partial e = \{e^-, e^+\}$. We denote by Γ/e the quotient metric graph whose equivalence classes are e and all one point subsets $\{x\}$ for $x \notin e$. Geometrically, one is contracting (collapsing) e to a single point p_e . From the point of view of electrical networks, it is best to think of setting $\ell(e) = 0$, which can be interpreted as “short-circuiting” the segment e .

Let G be a model of Γ so that $e \in E(G)$. Possibly upon making a refinement of the vertex set underlying G we can take $V/\{e^-, e^+\}$ as a vertex set of Γ/e , yielding a model G/e of Γ/e with the property that $E(G/e)$ is canonically identified with $E(G) \setminus e$. We consider $C_1(G/e, \mathbb{R})$ as a subspace of $C_1(G, \mathbb{R})$ via the natural map

$$\iota : C_1(G/e, \mathbb{R}) \hookrightarrow C_1(G, \mathbb{R}).$$

Let $\pi'_G : C_1(G, \mathbb{R}) \rightarrow H_1(G, \mathbb{R})^\perp$ and $\pi'_{G/e} : C_1(G/e, \mathbb{R}) \rightarrow H_1(G/e, \mathbb{R})^\perp$ denote the orthogonal projections.

Let W be the orthogonal complement of $\pi'_G(e)$ inside $H_1(G, \mathbb{R})^\perp$:

$$W = \text{span}\{\pi'_G(e)\}^\perp \cap H_1(G, \mathbb{R})^\perp.$$

Let $\text{Proj} : H_1(G, \mathbb{R})^\perp \rightarrow W$ denote the corresponding orthogonal projection map.

Proposition 8.1. *The map ι restricts to an isomorphism $\bar{\iota} : H_1(G/e, \mathbb{R})^\perp \xrightarrow{\sim} W$. Moreover, the following diagram commutes:*

$$(8-1) \quad \begin{array}{ccc} C_1(G/e, \mathbb{R}) & \xhookrightarrow{\iota} & C_1(G, \mathbb{R}) \\ \pi'_{G/e} \downarrow & & \downarrow \pi'_G \\ H_1(G/e, \mathbb{R})^\perp & \xrightarrow{\sim} & W \\ \downarrow \text{Proj} & & \downarrow \\ H_1(G, \mathbb{R})^\perp & & \end{array}$$

Proof. Let $\pi_* : C_1(G, \mathbb{R}) \rightarrow C_1(G/e, \mathbb{R})$ denote the canonical projection, and observe that $\text{Ker } \pi_* = \text{span}\{e\}$. Write $K = \text{Ker}(\text{Proj} \circ \pi'_G) = \text{span}\{e\} + H_1(G, \mathbb{R})$ and $L = \pi_*^{-1} H_1(G/e, \mathbb{R})$. The map π_* induces an isomorphism $H_1(G, \mathbb{R}) \xrightarrow{\sim} H_1(G/e, \mathbb{R})$ upon restriction. Since clearly $K \subseteq L$ and $\dim K = m - n + 2 = \dim L$, we find the equality $K = L$. As ι splits π_* , we have $\iota^{-1} K = \iota^{-1} L = H_1(G/e, \mathbb{R})$. This shows that the inclusion $\iota : C_1(G/e, \mathbb{R}) \hookrightarrow C_1(G, \mathbb{R})$ induces an injective map $\bar{\iota} : H_1(G/e, \mathbb{R})^\perp \hookrightarrow W$. As $\dim H_1(G/e, \mathbb{R})^\perp = n - 2 = \dim W$, we conclude that $\bar{\iota}$ is an isomorphism. This proves the proposition. \square

8B. Generalized Rayleigh's law. We are now ready to prove our version of Rayleigh's law for energy pairings.

Theorem 8.2 (Rayleigh's law for energy pairings). *Let Γ be a metric graph. Let e be an edge segment of Γ , and let $v_1, v_2 \in \text{DMeas}_0(\Gamma)$. Then*

$$\langle v_1, v_2 \rangle_{\text{en}}^{\Gamma/e} = \langle v_1, v_2 \rangle_{\text{en}}^\Gamma - \frac{\langle v_1, \delta_{e^+} - \delta_{e^-} \rangle_{\text{en}}^\Gamma \langle \delta_{e^+} - \delta_{e^-}, v_2 \rangle_{\text{en}}^\Gamma}{r(e^-, e^+; \Gamma)}.$$

In particular, for $v \in \text{DMeas}_0(\Gamma)$, we have $\langle v, v \rangle_{\text{en}}^{\Gamma/e} \leq \langle v, v \rangle_{\text{en}}^\Gamma$.

Proof. Let G be a model of Γ determined by a vertex set V so that $e \in E(G)$ and $V/\{e^-, e^+\}$ is a vertex set of Γ/e (see [Section 8A](#)). Assume, moreover, that V is taken fine enough so that $v_1, v_2 \in \text{DMeas}_0(G)$. We choose an orientation $\mathbb{O} = \{e_1, \dots, e_m\}$. Let e also denote the corresponding oriented edge in \mathbb{O} . Let $\gamma_1, \gamma_2 \in C_1(G, \mathbb{R})$ be such that $\partial(\gamma_i) = v_i$. We have a well-defined model G/e of Γ/e .

By [Proposition 7.11](#), we know that

$$(8-2) \quad \langle v_1, v_2 \rangle_{\text{en}}^{\Gamma/e} = [\gamma_1, \pi'_{G/e}(\gamma_2)].$$

By [Proposition 8.1](#), we know that $\pi'_{G/e}$ corresponds to $\text{Proj} \circ \pi'_G$ via ι .

For $e_j \in \mathbb{O}$, we have:

$$\begin{aligned} \text{Proj} \circ \pi'_G(e_j) &= \pi'_G(e_j) - \frac{[\pi'_G(e), \pi'_G(e_j)]}{[\pi'_G(e), \pi'_G(e)]} \pi'_G(e) = \pi'_G(e_j) - \frac{[e, \pi'_G(e_j)]}{[e, \pi'_G(e)]} \pi'_G(e) \\ &= \pi'_G(e_j) - \frac{F'(e, e_j) \ell(e)}{F'(e, e) \ell(e)} \pi'_G(e) = \pi'_G(e_j) - \frac{F'(e, e_j)}{F'(e, e)} \sum_{i=1}^m F'(e_i, e) e_i \\ &= \pi'_G(e_j) - \frac{1}{r(e^-, e^+)} \sum_{i=1}^m \frac{\xi(e_i^-, e_i^+, e^-, e^+) \xi(e^-, e^+, e_j^-, e_j^+)}{\ell(e_i)} e_i. \end{aligned}$$

We used [Corollary 7.10](#) for the third, fourth, and fifth equalities.

It follows from this computation that the matrix of $\pi'_{G/e} = \text{Proj} \circ \pi'_G$, with respect to the basis \mathbb{O} , is given by

$$(8-3) \quad S = D^{-1} \Xi - \frac{1}{r(e^-, e^+)} D^{-1} (\Xi[e]) (\Xi[e])^T.$$

Recall from [Remark 7.2\(i\)](#) that $[e]$ denotes the column vector with a 1 on the row corresponding to e , and 0's everywhere else. The result now follows from (8-2) and the following straightforward matrix computation:

$$\begin{aligned} [\gamma_1, \pi'_{G/e}(\gamma_2)] &= [\gamma_1]^T D (S[\gamma_2]) \\ &= [\gamma_1]^T \Xi[\gamma_2] - \frac{1}{r(e^-, e^+)} [\gamma_1]^T (\Xi[e]) (\Xi[e])^T [\gamma_2] \\ &= [\gamma_1]^T B^T L B [\gamma_2] - \frac{1}{r(e^-, e^+)} ([\gamma_1]^T B^T L B [e]) ([e]^T B^T L B [\gamma_2]) \\ &= [v_1]^T L [v_2] - \frac{1}{r(e^-, e^+)} ([v_1]^T L [\delta_{e^+} - \delta_{e^-}]) ([\delta_{e^+} - \delta_{e^-}]^T L [v_2]) \\ &= \langle v_1, v_2 \rangle_{\text{en}} - \frac{1}{r(e^-, e^+)} (\langle v_1, \delta_{e^+} - \delta_{e^-} \rangle_{\text{en}} \langle \delta_{e^+} - \delta_{e^-}, v_2 \rangle_{\text{en}}). \end{aligned}$$

We used [Remark 7.2\(iii\)](#), (8-3), (7-1), [Remark 7.2\(ii\)](#), [Proposition 5.2](#), and [Remark 6.3\(ii\)](#) in this computation. \square

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ROBIN DE JONG: rdejong@math.leidenuniv.nl
 Mathematisch Instituut, Universiteit Leiden, Leiden, Netherlands

FARBOD SHOKRIEH: farbod@uw.edu
 Department of Mathematics, University of Washington, Seattle, WA, United States