EXISTENCE, UNIQUENESS, AND STABILIZATION RESULTS FOR PARABOLIC VARIATIONAL INEQUALITIES *

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Abstract. In this paper we consider feedback stabilization for parabolic variational inequalities of obstacle type with time and space depending reaction and convection coefficients and show exponential stabilization to nonstationary trajectories. Based on a Moreau–Yosida approximation, a feedback operator is established using a finite (and uniform in the approximation index) number of actuators leading to exponential decay of given rate of the state variable. Several numerical examples are presented addressing smooth and nonsmooth obstacle functions.

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1. INTRODUCTION

The goal of the paper is the stabilization to trajectories for parabolic variational inequalities (PVIs), in particular towards the solution y to the obstacle problem

$$\langle \tfrac{\partial}{\partial t}y + (-\Delta + \mathbf{1})y + ay + b\cdot\nabla y - f, v - y\rangle \geq 0, \quad \forall v \leq \psi, \;\; t > 0, \tag{1.1a}$$

$$y \leq \psi, \quad \mathcal{G}y|_{\Gamma} = \chi, \quad t > 0, \qquad y(\cdot, 0) = y_{\circ}, \tag{1.1b}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$ with a regular enough boundary $\Gamma := \partial\Omega$, where d is a positive integer. The obstacle $\psi = \psi(x,t)$ and the functions $a = a(x,t) \in \mathbb{R}$, $b = b(x,t) \in \mathbb{R}^d$, $f = f(x,t) \in \mathbb{R}$, $\chi = \chi(\overline{x},t) \in \mathbb{R}$, $v = v(x,t) \in \mathbb{R}$, and $y_{\circ} = y_{\circ}(x)$, are assumed to be sufficiently regular, for $(x,\overline{x},t) \in \Omega\times\Gamma\times(0,+\infty)$; regularity details are specified later. The linear operator $\mathcal{G}$ is determined by either Dirichlet or Neumann boundary conditions. In (1.1) and throughout the manuscript, $\mathbf{1}$ stands for the identity operator $z \mapsto \mathbf{1}z := z$ (for z on a suitable space).

Keywords and phrases: Exponential stabilization, parabolic variational inequalities, oblique projection feedback, Moreau–Yosida approximation

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For some pairs (a,b), the solution w issued from a different initial condition $w_\circ \neq y_\circ$

$$\langle \tfrac{\partial}{\partial t} w + (-\Delta + \mathbf{1})w + aw + b\cdot\nabla w - f, v - w\rangle \geq 0, \quad \forall v \leq \psi, \quad t > 0, \tag{1.2a}$$

$$w \leq \psi, \quad \mathcal{G}w|_\Gamma = \chi, \quad t>0, \qquad w(\cdot,0) = w_\circ, \tag{1.2b}$$

may not converge to y as time increases. Our goal is to show that, by means of a feedback control input $\mathbf{u} = \mathcal{K}(w-y)$, we can track y exponentially fast with an arbitrary exponential rate $-\mu < 0$. That is, we want to construct an input feedback operator $\mathcal{K}$ such that the solution of

$$\langle \tfrac{\partial}{\partial t} w + (-\Delta + \mathbf{1})w + aw + b\cdot\nabla w - f - \mathcal{K}(w-y), v - w\rangle \geq 0, \quad \forall v \leq \psi, \quad t > 0, \tag{1.3a}$$

$$w \leq \psi, \quad \mathcal{G}w|_\Gamma = \chi, \quad t>0, \qquad w(\cdot,0) = w_\circ, \tag{1.3b}$$

satisfies, for a suitable constant $C \geq 1$,

$$|w(t) - y(t)|_{L^2(\Omega)} \leq C\mathrm{e}^{-\mu t}\,|w_\circ - y_\circ|_{L^2(\Omega)}\,, \quad \text{for all} \quad (w_\circ, y_\circ) \in L^2(\Omega)\times L^2(\Omega), \quad t \geq 0. \tag{1.4}$$

Motivated by applications, we are interested in the case $\mathcal{K}\colon L^2(\Omega) \to \mathcal{U}_M$, where $\mathcal{U}_M \subset L^2(\Omega)$ is a finite-dimensional subspace, namely, given by the linear span of a finite set $U_M = \{\Psi_i \mid 1 \leq i \leq M_\mathfrak{m}\} \subset L^2(\Omega)$ of actuators, where $M_\mathfrak{m} \coloneqq \mathfrak{m}(M)$ is a positive integer, defined through a suitable nondecreasing function $\mathfrak{m} : \mathbb{N} \to \mathbb{N}$ which will be appropriately chosen later on. It follows that the control input will be of the form

$$\mathbf{u}(t) = \mathcal{K}(w(t) - y(t)) = \sum_{i=1}^{M_\mathfrak{m}} u_i(t)\Psi_i \in \mathcal{U}_M.$$

We consider the case in which the actuators are determined by indicator functions 1_{ω_i} of small subdomains $\omega_i \subset \Omega$ (cf. [17, Eq. (1.3)], [14, Eq. (2.2)], [15, Ex. III.5]),

$$\Psi_i(x) = 1_{\omega_i}(x) = \begin{cases} 1, & \text{if } x \in \omega_i, \\ 0, & \text{if } x \in \Omega\setminus\omega_i, \end{cases} \qquad 1 \leq i \leq M_\mathfrak{m}.$$

Remark 1.1. Note that for simplicity we have taken the diffusion operator as $-\Delta + \mathbf{1}$. One reason is to facilitate the inclusion of Neumann boundary conditions in our investigation where, in particular, we ask the operator to be injective. This is not a significant restriction, since we can always transform a given dynamics $\frac{\partial}{\partial t}y - \nu\Delta y + \widetilde{a}y + h = 0$ into $\frac{\partial}{\partial \tau}z + (-\Delta + \mathbf{1})z + (\nu^{-1}\widetilde{a} - 1)z + \nu^{-1}h = 0$ simply by rescaling time, $\tau = \nu t$, $z(\tau) = y(\nu^{-1}\tau)$.

1.1. **Main stabilizability result**

Recall that for Dirichlet and Neumann boundary conditions, the operator $\mathcal{G}$ reads, respectively,

$$\mathcal{G} = \mathbf{1} \qquad \text{and} \qquad \mathcal{G} = \tfrac{\partial}{\partial \mathbf{n}} = \mathbf{n}\cdot\nabla,$$

where $\mathbf{n} = \mathbf{n}(\overline{x})$ is the unit outward normal vector to Γ at $\overline{x} \in \Gamma$. In either case we set $L^2(\Omega)$ as a pivot space, that is, we identify $L^2(\Omega)$ with its own dual, $L^2(\Omega)' = L^2(\Omega)$.

Depending on the choice of $\mathcal{G}$, we define the spaces

$$V := \begin{cases} H_0^1(\Omega), & \text{if} \quad \mathcal{G} = \mathbf{1}, \\ H^1(\Omega), & \text{if} \quad \mathcal{G} = \frac{\partial}{\partial \mathbf{n}}, \end{cases}$$

and the symmetric isomorphism

$$A\colon V \to V', \qquad \langle Ay, z\rangle_{V',V} := (\nabla y, \nabla z)_{L^2(\Omega)^d} + (y, z)_{L^2(\Omega)}. \tag{1.5}$$

Under suitable assumptions on the spatial domain Ω, which we shall give in Section 2.2, we have that the domain of A, defined as $\mathrm{D}(A) := \{z \in L^2(\Omega) \mid Az \in L^2(\Omega)\}$, is given by

$$\mathrm{D}(A) = \{z \in H^2(\Omega) \mid \mathcal{G}z|_\Gamma = 0\}. \tag{1.6}$$

It will also follow that A has a compact inverse, and that the restriction $A|_{\mathrm{D}(A)}$ of A to $\mathrm{D}(A) \subset V$ satisfies $A|_{\mathrm{D}(A)} = -\Delta + \mathbf{1}\colon \mathrm{D}(A) \to L^2(\Omega)$.

We shall assume that V and $\mathrm{D}(A)$ are endowed, respectively, with the scalar products

$$(y, z)_V := \langle Ay, z\rangle_{V',V} \quad \text{and} \quad (y, z)_{\mathrm{D}(A)} := (Ay, Az)_{L^2(\Omega)}$$

and associated norms. Note that $(y, z)_V = (y, z)_{H^1(\Omega)}$ coincides with the usual scalar product of $H^1(\Omega)$. Finally, we denote the increasing sequence of eigenvalues of A by $(\alpha_i)_{i\in\mathbb{N}}$, and a complete basis of eigenfunctions by $(e_i)_{i\in\mathbb{N}}$,

$$Ae_i = \alpha_i e_i, \qquad e_i \in \mathrm{D}(A), \qquad 0 < \alpha_i \le \alpha_{i+1} \to +\infty.$$

Throughout this manuscript, for simplicity, we shall denote the Hilbert Sobolev spaces

$$H^s := H^s(\Omega) = W^{s,2}(\Omega) \quad \text{for} \quad s > 0, \quad \text{and} \quad L^2 := L^2(\Omega).$$

We consider sequences of sets of actuators and eigenfunctions E_M of the diffusion operator under homogeneous boundary conditions as follows, for some nondecreasing function $\mathfrak{m} : \mathbb{N} \to \mathbb{N}$

$$(U_M)_{M\in\mathbb{N}}, \quad U_M = \{\Psi_i \mid 1 \le i \le \mathfrak{m}(M)\} \subset L^2(\Omega), \tag{1.7a}$$

$$(E_M)_{M\in\mathbb{N}}, \quad E_M = \{e_i \mid i \in \mathbb{E}_M\} \subset \mathrm{D}(A) \subset L^2(\Omega), \quad \mathbb{E}_M = \{j_k^M \mid 1 \le k \le \mathfrak{m}(M)\} \subset \mathbb{N}, \tag{1.7b}$$

where $\mathbb{N}$ denotes the set of positive integers and the indices j_k^M are specified later. We denote

$$\mathcal{U}_M = \operatorname{span} U_M, \qquad \mathcal{E}_M = \operatorname{span} E_M, \tag{1.7c}$$

and shall take pairs $(\mathcal{U}_M, \mathcal{E}_M)$ satisfying, in particular,

$$\dim \mathcal{U}_M = M_{\mathfrak{m}} = \dim \mathcal{E}_M, \quad L^2(\Omega) = \mathcal{U}_M + \mathcal{E}_M^\perp, \quad \text{and} \quad \mathcal{U}_M \bigcap \mathcal{E}_M^\perp = \{0\}. \tag{1.7d}$$

Due to (1.7d), the oblique projection $P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp}$, in $L^2(\Omega)$ onto $\mathcal{U}_M$ along $\mathcal{E}_M^\perp$, is well defined as follows: we can write an arbitrary $h \in L^2$ in a unique way as $h = h_{\mathcal{U}_M} + h_{\mathcal{E}_M^\perp}$ with $(h_{\mathcal{U}_M}, h_{\mathcal{E}_M^\perp}) \in \mathcal{U}_M \times \mathcal{E}_M^\perp$, then we set $P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp} h := h_{\mathcal{U}_M}$.

Our results will follow under general conditions on the dynamics tuple (a, b, f, χ, ψ), for an appropriately constructed sequence $(\mathcal{U}_M, \mathcal{E}_M)_{M\in\mathbb{N}}$ (explicitly given in Sect. 3.1). Such conditions will be presented and specified later on. Without entering into more details at this point our main result is the following, whose precise statement shall be given in Theorem 4.1.

Main Result. *Let $r = r(t) \coloneqq \min(t, 1)$ for $t \geq 0$. Under sufficient regularity of the data and some assumptions which will be specified in Section 2.2 we have the following:*

(i) *For every $T > 0$, there exists a unique solution y of (1.1), with $y \in L^2((0,T); H^1)$, $\frac{\partial}{\partial t} y \in L^2((0,T); V')$, $ry \in L^2((0,T); H^2)$, and $\frac{\partial}{\partial t}(ry) \in L^2((0,T); L^2)$.*

(ii) *For every $\mu > 0$, there are M and λ large enough such that, with $\mathcal{K}_M^\lambda \coloneqq \lambda P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp} A P_{\mathcal{E}_M}^{\mathcal{U}_M^\perp}$, the solution of the system*

$$\left(\tfrac{\partial}{\partial t} w + (-\nu\Delta + \mathbf{1})w + aw + b\cdot\nabla w - f + \mathcal{K}_M^\lambda(w-y), v-w\right)_{L^2} \geq 0, \quad \forall v \leq \psi, \quad t>0, \tag{1.8a}$$

$$w \leq \psi, \quad w(0) = w_\circ, \quad \mathcal{G}w\big|_\Gamma = \chi. \tag{1.8b}$$

satisfies the inequality (1.4) with $C = 1$. Furthermore,

$$\left|\mathcal{K}_M^\lambda\right|_{\mathcal{L}(L^2)} \leq \lambda\widehat{\alpha}_M \left|P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp}\right|_{\mathcal{L}(L^2)}^2 \quad \text{and} \tag{1.9a}$$

$$\left|\mathcal{K}_M^\lambda(w-y)\right|_{L^2(\mathbb{R}_+, L^2)} \leq \lambda\widehat{\alpha}_M\mu^{-1} \left|P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp}\right|_{\mathcal{L}(L^2)}^2 |w_\circ - y_\circ|_{L^2}, \tag{1.9b}$$

where $\widehat{\alpha}_M = \sup\{\alpha_i \mid e_i \in E_M \text{ and } Ae_i = \alpha_i e_i\}$.

1.2. **Previous literature**

The use of oblique projections has been introduced in Kunisch and Rodrigues [18], in the construction of explicit feedback operators for stabilization of linear parabolic-like systems under homogeneous conditions $(f, \chi) = 0$. Precisely, the feedback in [18] is given by

$$\mathcal{K}_M(t)(y) = P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp}(A + A_{\mathrm{rc}}(t) - \lambda\mathbf{1})y, \tag{1.10}$$

where $\mathcal{U}_M$ is the finite-dimensional actuators space and the auxiliary space $\mathcal{E}_M$ is spanned by a suitable set of eigenfunctions of the diffusion-like operator A. Further A_{rc} is a reaction-convection-like operator. Appropriate variations of such feedback are used in Kunisch and Rodrigues [19] to stabilize coupled parabolic-ODE systems, and in Azmi and Rodrigues [1] to stabilize damped wave equations. In Rodrigues [26], the analogous feedback

$$\mathcal{K}_M(t)(y) = P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp}(Ay + A_{\mathrm{rc}}(t)y + \mathcal{N}(t,y) - \lambda y), \tag{1.11}$$

is used to semiglobally stabilize parabolic equations, where the dynamics includes a given nonlinear term $\mathcal{N}(t,\cdot)$ and the number of actuators is large enough, depending on the norm $|y_0|_V$ of the initial state in a suitable Hilbert space $V \subseteq L^2$.

In this paper we investigate the stabilizability of nonautonomous PVIs through a limiting argument based on Moreau–Yosida approximations. The latter are semilinear parabolic equations and by this reason we could try to use the feedback (1.11). However, due to the nonlinearity and structure of the approximation, the number of actuators required by that feedback may increase with the Moreau–Yosida parameter. As a result, even if a limit feedback operator can be found, it could have an infinite-dimensional range in which case it would be infeasible for real world applications. Therefore, we will use a different feedback operator in (1.8), namely,

$$\mathcal{K}_M^\lambda = -\lambda P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp} A P_{\mathcal{E}_M}^{\mathcal{U}_M^\perp}. \tag{1.12}$$

We shall make use of the monotonicity of the nonlinear term associated with the Moreau–Yosida approximation. Without such monotonicity we do not know whether the feedback in (1.12) is able to stabilize parabolic systems for a general class of nonlinearities as in [26]. Moreover, it is also such monotonicity which will allow us to take the pair (λ, M) in (1.12) independently of the Moreau–Yosida parameter, and this is why we will be able to take such feedback in the limiting PVI.

This manuscript introduces the use of oblique projections in the construction of explicit feedback operators which are able to stabilize PVIs. To the best knowledge of the authors, there are no stabilization results available in the literature for such inequalities. In spite of this fact we would like to refer the reader to previous works dealing with the control problems defined on a *bounded* time interval. Feedback laws for optimal control of PVIs have been addressed in Popa [24] and robust feedback laws in Maksimov [22]. In the first reference the author shows that the optimal control is given by a feedback law constructed from the optimal value function. In the latter reference the author considers a robust control problem in the case of distributed control actions and disturbances, and establishes a feedback law using piecewise (in time) constant control functions being irrespective of the unknown effective perturbation.

Concerning open-loop optimal control (still, in a bounded time interval), Wang [34] considers problems for systems governed by a PVI coupled with a semilinear parabolic differential equation, Ito and Kunisch [13] consider strong and weak solution concepts, study existence, and derive the first order optimality system in a Lagrangian framework. Sensitivity analysis is considered in Christof [8]. Elliptic-parabolic variational inequalities with time-dependent constraints are studied in Hofmann, Kubo, and Yamakaki [12]. Wachsmuth [33] studies quasistatic plasticity with linear kinematic hardening and derives optimality conditions. Chen, Chu, and Tan [7] analyze bilateral obstacle control problems. Barbu [2], derives a variant of the maximum principle for time-optimal trajectories. Finally, we mention Boukrouche and Tarzia [5], where PVIs of second kind have been addressed.

The rest of the paper is organized as follows. In Section 2 we analyze the Moreau–Yosida approximations. The stabilization of such approximations is addressed in Section 3. Section 4 is dedicated to the proof of the main stabilization result for the PVI. Finally, in Section 5 several numerical examples are presented for the case of a regular obstacle fulfilling the theoretical assumptions, and in Section 6 a less regular obstacle ψ is considered for the sake of comparison.

Notation: The subset of positive real numbers shall be denoted $\mathbb{R}_+ := (0, +\infty)$. For an open interval $I \subseteq \mathbb{R}$ and two Banach spaces X, Y, we write $W(I; X, Y) := \{y \in L^2(I; X) \mid \dot{y} \in L^2(I; Y)\}$, where $\dot{y} := \frac{\mathrm{d}}{\mathrm{d}t} y$ is taken in the sense of distributions. This space is a Banach space when endowed with the natural norm $|y|_{W(I;X,Y)} := (|y|^2_{L^2(I;X)} + |\dot{y}|^2_{L^2(I;Y)})^{1/2}$. We also denote the spaces $L^2_{\mathrm{loc}}(\mathbb{R}_+; X) := \{y\colon \mathbb{R}_+ \to X \mid y \in L^2((0,T); X) \text{ for all } T \in \mathbb{R}_+\}$ and $W_{\mathrm{loc}}(\mathbb{R}_+; X, Y) := \{y\colon \mathbb{R}_+ \to X \mid y \in W((0,T); X, Y) \text{ for all } T \in \mathbb{R}_+\}$.

In case we know that $X \cap Y = \{0\}$, we say that $X + Y$ is a direct sum and we write $X \oplus Y$ instead. If the inclusion $X \subseteq Y$ is continuous, we write $X \hookrightarrow Y$.

The space of continuous linear mappings from X into Y is denoted by $\mathcal{L}(X,Y)$. In case $X=Y$ we write $\mathcal{L}(X) \coloneqq \mathcal{L}(X,X)$. The continuous dual of X is denoted $X' \coloneqq \mathcal{L}(X,\mathbb{R})$. The space of continuous functions from X into Y is denoted by $\mathcal{C}(X,Y)$. Given a subset $S \subset H$ of a Hilbert space H, with scalar product $(\cdot,\cdot)_H$, the orthogonal complement of S is denoted $S^\perp \coloneqq \{h \in H \mid (h,s)_H = 0 \text{ for all } s \in S\}$. Given two closed subspaces $F \subseteq H$ and $G \subseteq H$ of the Hilbert space $H = F \oplus G$, we denote by $P_F^G \in \mathcal{L}(H,F)$ the oblique projection in H onto F along G. That is, writing $h \in H$ as $h = h_F + h_G$ with $(h_F, h_G) \in F \times G$, we have $P_F^G h \coloneqq h_F$. The orthogonal projection in H onto F is denoted by $P_F \in \mathcal{L}(H,F)$. Notice that $P_F = P_F^{F^\perp}$. By $\overline{C}_{[a_1,\dots,a_n]}$ we denote a nonnegative function that increases in each of its nonnegative arguments. Finally, C, C_i, $i = 0, 1, \dots$, stand for unessential positive constants.

2. Existence, uniqueness, and approximation of the solution

The reaction term in (1.1) can be written as $y \mapsto ay = (a\mathbf{1})(t)y$, with $(a\mathbf{1})(t)y(x) \coloneqq a(x,t)y(x)$, $x \in \Omega$, where, for (almost) every $t > 0$, $(a\mathbf{1})(t) \in \mathcal{L}(L^2)$. We consider here a more general version of system (1.1), which will allow us to work with the controlled system (1.8) as well. Namely,

$$\left(\tfrac{\partial}{\partial t} y + (-\Delta + \mathbf{1})y + Qy - f, v - y\right)_{L^2} \ge 0, \quad \forall v \le \psi, \;\; t > 0, \tag{2.1a}$$

$$y \le \psi, \quad \mathcal{G}y|_\Gamma = \chi, \quad t > 0, \qquad y(\cdot,0) = y_\circ, \tag{2.1b}$$

with $Q = Q(t)$ in the form $y = y(x) \mapsto Qy(x) \coloneqq (\mathcal{B}(t)y)(x) + b(x,t)\cdot\nabla y(x)$, where $\mathcal{B}(t) \in \mathcal{L}(L^2)$, $y \mapsto \mathcal{B}(t)h$. The precise regularity assumption on the family $\{\mathcal{B}(t) \mid t > 0\}$ and on the vector function b shall be given in Section 2.2.

We show that there exists a solution of (2.1), which can be approximated by the sequence $(y_k)_{k\in\mathbb{N}}$, where y_k is the solution of the system

$$\tfrac{\partial}{\partial t} y_k + (-\Delta + \mathbf{1})y_k + Qy_k + k(y_k - \psi)^+ = f, \qquad y_k(0) = y_\circ, \qquad \mathcal{G}y_k|_\Gamma = \chi, \tag{2.2}$$

with

$$v^+(x) \coloneqq \begin{cases} v(x), & \text{if } v(x) > 0,\\ 0, & \text{if } v(x) \le 0,\end{cases} \quad \text{for} \quad v \in L^2.$$

Remark 2.1. Let $y_{\rm tar}$ be a target trajectory solving (1.1) with a given external forcing $f_{\rm tar}$. Note that we obtain the free-dynamics system (1.1) by taking $Q = a\mathbf{1} + b\cdot\nabla$and $f = f_{\rm tar}$ in (2.1), and we obtain the controlled system (1.8) (with the targetet trajectory $y_{\rm tar}$) by taking $Q = a\mathbf{1} + b\cdot\nabla + \mathcal{K}_M^\lambda$ and $f = f_{\rm tar} - \mathcal{K}_M^\lambda y_{\rm tar}$.

2.1. Trace and lifting operators

For simplicity, we denote

$$\mathcal{W} \coloneqq W_{\rm loc}(\mathbb{R}_+; H^2, L^2) \quad \text{and} \quad \mathcal{W}_0 \coloneqq W_{\rm loc}(\mathbb{R}_+; \mathrm{D}(A), L^2) \subset \mathcal{W}.$$

Let us define the trace spaces on the boundary

$$\mathcal{T} \coloneqq \{\mathcal{G}h|_\Gamma \mid h \in \mathcal{W}\}, \qquad\qquad \mathcal{T}_0 \coloneqq \{\mathcal{G}h|_\Gamma \mid h \in \mathcal{W}_0\}. \tag{2.3}$$

Recall that we have (cf. [21, Ch. 1, Thms. 3.2 and 9.6]) for the trace spaces at initial time,

$$\mathcal{W}^{[t=0]} := \{y(0) \mid y \in \mathcal{W}\} = H^1, \qquad \mathcal{W}_0^{[t=0]} := \{y(0) \mid y \in \mathcal{W}_0\} = V.$$

Now for any finite time interval (t_1, t_2), with $t_2 > t_1$, we define the Hilbert spaces

$$\mathcal{W}_{(t_1,t_2)} := W((t_1,t_2); H^2, L^2) \tag{2.4}$$

and the corresponding traces are denoted by $\mathcal{T}_{(t_1,t_2)} = \mathcal{W}_{(t_1,t_2)}|_\Gamma$.

Next for each positive integer $j \in \mathbb{N}$ we define the time interval $I_j := (j-1, j)$. Observe that for any $\chi \in \mathcal{T}$ we have $\chi|_{I_j} \in \mathcal{T}_{I_j}$. We consider the extension (lifting) function defined, for $\widetilde{\chi} \in \mathcal{T}_{I_j}$ by

$$\mathfrak{E}^j\widetilde{\chi} \in \mathcal{W}_{I_j}, \quad (\mathcal{G}\mathfrak{E}^j\widetilde{\chi})|_\Gamma = \widetilde{\chi}, \quad \text{and} \quad \mathfrak{E}^j\widetilde{\chi} \in \mathcal{W}_{I_j,0}^\perp, \quad \text{with} \quad \mathcal{W}_{I_j,0} := \mathcal{W}_{I_j} \bigcap \mathcal{W}_0|_{I_j},$$

where the orthogonal space $\mathcal{W}_{I_j,0}^\perp$ to $\mathcal{W}_{I_j,0}$ is taken with respect to the scalar product of $\mathcal{W}_{I_j}$. This defines the extension operator, $\mathfrak{E}^j \in \mathcal{L}(\mathcal{T}_{I_j}, \mathcal{W}_{I_j})$, which is a right inverse for the trace operator $(\mathcal{G}(\cdot))|_\Gamma \in \mathcal{L}(\mathcal{W}_{I_j}, \mathcal{T}_{I_j})$. We endow $\mathcal{T}_{I_j}$ with the scalar product induced by the trace mapping

$$(\chi_1, \chi_2)_{\mathcal{T}_{I_j}} := (\mathfrak{E}^j\chi_1, \mathfrak{E}^j\chi_2)_{\mathcal{W}_{I_j}}.$$

This allows to introduce the extension $\mathfrak{E}\colon \mathcal{T} \to \mathcal{W}$ defined by concatenation

$$\mathfrak{E}\chi(t) := (\mathfrak{E}^{\lceil t\rceil}\chi|_{I_{\lceil t\rceil}})(t),$$

where $\lceil t\rceil$ is the positive integer satisfying $\lceil t\rceil - 1 < t \le \lceil t\rceil$.

Remark 2.2. Note that for any $h \in \mathcal{W}$ satisfying $\mathcal{G}h|_\Gamma = \chi$ we have that $\mathfrak{E}\chi - h \in \mathcal{W}_0$. In particular we have that $\mathfrak{E}\chi(t) - h(t) \in V$, for all $t \ge 0$.

2.2. **Assumptions on the data**

We make the following regularity assumptions for the data.

Assumption 2.3. *The subset Ω is bounded, open, and connected, located on one side of its boundary $\Gamma = \partial\Omega$. Furthermore, either Γ is a compact C^2-manifold or Ω is a convex polygonal domain.*

Under Assumption 2.3 we have the characterizations (1.6), this follows from [11, Thms. 2.2.2.3, 2.2.2.5, 3.2.1.3 and 3.2.1.3].

Assumption 2.4. *The operator Q in* (2.1) *is a sum $Q = \mathcal{B} + b\cdot\nabla$ with*

$$\mathcal{B} \in L^\infty(\mathbb{R}_+; \mathcal{L}(L^2)) \quad \textit{and} \quad b \in L^\infty(\Omega\times\mathbb{R}_+)^d.$$

Assumption 2.4 is satisfied if, for example, $\mathcal{B} = a\mathbf{1}$ with $a \in L^\infty(\Omega\times\mathbb{R}_+)$.

Assumption 2.5. *The external forces f and χ, and initial condition $y_\circ$ in* (2.1)*, satisfy*

$$f \in L^2_{\mathrm{loc}}(\mathbb{R}_+; L^2), \quad \chi \in \mathcal{T}, \quad y_\circ \in L^2, \quad \textit{and} \quad y_\circ \le \psi(\cdot, 0),$$

where $\mathcal{T}$ is the trace space as in (2.3)*.*

Assumption 2.6. *The obstacle satisfies $\psi \in W_{\mathrm{loc}}(\mathbb{R}_+; H^2, L^2)$ and $\mathcal{G}\psi|_\Gamma \geq \chi - \eta$ where:*
(i) *for Dirichlet boundary conditions, $\eta = 0$;*
(ii) *for Neumann boundary conditions, $\eta \geq 0$ and $\eta \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}_+)$.*

Remark 2.7. In Assumption 2.6, the function η in the inequality $\mathcal{G}\psi|_\Gamma \geq \chi - \eta$ must be understood as a function $\eta(\overline{x}, t) \coloneqq \eta(t)$ independent of $\overline{x}$. For Dirichlet boundary conditions, since we will be looking for a solution satisfying $y|_\Gamma = \chi$ and $y \leq \psi$, then the requirement $\psi|_\Gamma \geq \chi$ is necessary. Instead, for Neumann boundary conditions, we do not claim the necessity of the requirements in Assumption 2.6. However, the relaxation of those requirements will, probably, involve extra technical difficulties.

Remark 2.8. Several existence results for PVIs can be found in the literature. However, though we borrow some ideas and arguments from classic references (e.g, [3, 4, 6, 10]), we could not find in the literature the existence results for obstacles as general as in Assumption 2.6. For example in [4, Ch. 3, Sect. 2.2, Thm. 2.2], for Dirichlet boundary conditions it is assumed that the boundary trace of the obstacle is static (independent of time). In [6, Sect. II] the triple (a, b, ψ) is time-independent.

2.3. On the Moreau–Yosida approximation

We present the main result concerning Moreau–Yosida approximations for PVIs. We start by denoting, for a given function $\varphi \in L^2_{\mathrm{loc}}(\mathbb{R}_+, L^2)$, the convex sets

$$\mathbf{C}^\varphi_T \coloneqq \{v \in L^2((0,T); H^1) \mid v \leq \varphi\}, \quad \text{for} \quad T > 0, \tag{2.5a}$$

and

$$\mathbf{C}^\varphi_\infty \coloneqq \{v \in L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1) \mid v \leq \varphi\}. \tag{2.5b}$$

We set

$$\mathcal{Z}_r \coloneqq \{z \in W((0,T); H^1, V') \mid rz \in W((0,T); H^2, L^2)\},$$

where $r(t) \coloneqq \min\{t, 1\}$, for $t \geq 0$.

Theorem 2.9. *Let Assumptions 2.3–2.6 hold true, $T > 0$, and suppose $(f_k) \subset L^2((0,T); L^2)$ converges weakly to some f in $L^2((0,T); L^2)$. Then, for a given $k \in \mathbb{N}$, there exists one, and only one, weak solution $y_k \in \mathcal{Z}_r$ for*

$$\tfrac{\partial}{\partial t} y_k + (-\Delta + \mathbf{1})y_k + Qy_k + k(y_k - \psi)^+ = f_k, \qquad \mathcal{G}y_k|_\Gamma = \chi, \qquad y_k(0) = y_\circ. \tag{2.6}$$

Moreover, the sequence (y_k) of solutions satisfy

$$y_k - \mathfrak{E}\chi \xrightharpoonup[L^2((0,T);V)]{} y - \mathfrak{E}\chi, \qquad \tfrac{\partial}{\partial t}(y_k - \mathfrak{E}\chi) \xrightharpoonup[L^2((0,T);V')]{} \tfrac{\partial}{\partial t}(y - \mathfrak{E}\chi), \tag{2.7}$$

for some $y \in \mathcal{Z}_r$ with

$$y \in \mathbf{C}^\psi_T, \qquad y(0) = y_\circ, \qquad \mathcal{G}y|_\Gamma = \chi, \tag{2.8}$$

and, for an arbitrary $v \in \mathcal{Z}_r \bigcap \mathbf{C}^\psi_T$, with $v - y \in \mathcal{C}((0,T]; V)$, we have

$$\langle \tfrac{\partial}{\partial t} y + (-\Delta + \mathbf{1})y + Qy - f, v - y\rangle_{V',V} \geq 0, \quad \textit{almost everywhere in } (0,T). \tag{2.9}$$

Furthermore, we have

$$r(y_k-\mathfrak{E}\chi)\xrightarrow[L^2((0,T);\mathrm{D}(A))]{} r(y-\mathfrak{E}\chi),\quad \tfrac{\partial}{\partial t}(r(y_k-\mathfrak{E}\chi))\xrightarrow[L^2((0,T);L^2)]{}\tfrac{\partial}{\partial t}(r(y-\mathfrak{E}\chi)),\tag{2.10}$$

and, for arbitrary $v\in L^2((0,T);L^2)$,

$$\left(\tfrac{\partial}{\partial t}y+(-\Delta+\mathbf{1})y+Qy-f,v-y\right)_{L^2}\ge 0,\quad \textit{almost everywhere in } (0,T).\tag{2.11}$$

Finally, y is the only element in $\mathcal{Z}_r$ satisfying (2.8) *and* (2.9)*, and we have*

$$y_k\xrightarrow[L^2((0,T);L^2)]{} y\qquad \textit{and}\qquad r(y_k-\mathfrak{E}\chi)\xrightarrow[\mathcal{C}([0,T];L^2)]{} r(y-\mathfrak{E}\chi).\tag{2.12}$$

The proof of Theorem 2.9 is given in several steps, which we include in several lemmas.

Note that by direct computations we can see that

$$(h,h^+)_{L^2}=\left|h^+\right|_{L^2}^2\quad\text{and}\quad(h^+-g^+,h-g)_{L_2}\ge 0,\quad\text{for all}\quad(h,g)\in L^2\times L^2.\tag{2.13}$$

Let us denote

$$C_Q:=|Q|_{L^\infty(\mathbb{R}^+,\mathcal{L}(H^1,L^2))}\,.\tag{2.14}$$

Lemma 2.10. *Let Assumptions 2.3–2.6 hold true. Let us fix $k\in\mathbb{N}$. There exists one, and only one, solution $y_k\in W((0,T);H^1,V')$ for* (2.6)*, furthermore $ry_k\in W((0,T);H^2,L^2)$.*

Proof. We sketch the proof which follows from standard arguments. By a lifting argument (cf. [25, Def. 3.1]), that is, by setting $z_k:=y-\mathfrak{E}\chi$ we reduce the problem to the case of homogeneous boundary conditions. Then, as we briefly sketch next, we establish existence of weak solutions in $W((0,T),V,V')$ as a weak limit of suitable Galerkin approximations for the system

$$\tfrac{\partial}{\partial t}z_k+(-\Delta+\mathbf{1})z_k+Qz_k+k(z_k+\mathfrak{E}\chi-\psi)^+=f_k^\chi,\quad \mathcal{G}z_k|_\Gamma=0,\quad z_k(0)=z_\circ:=y_\circ-\mathfrak{E}\chi(0),$$

with $f_k^\chi:=f_k-\frac{\partial}{\partial t}\mathfrak{E}\chi\in L^2((0,T);L^2)\subset L^2((0,T);V')$. Specifically, such weak (variational) solutions are understood in the classical sense [20,32], and can be found as a weak limit of Galerkin approximations z_k^N taking values $z_k^N(t)\in\mathcal{E}_N^{\mathrm{f}}$, where $\mathcal{E}_N^{\mathrm{f}}$ is the linear span of the first eigenfunctions of A. For the nonlinear term we note that $\mathfrak{E}\chi-\psi\in L^2((0,T);L^2)$ and that

$$2(k(z_k^N+\mathfrak{E}\chi-\psi)^+,z_k^N)_{L^2}\le 2k\left|z_k^N+\mathfrak{E}\chi-\psi\right|_{L^2}\left|z_k^N\right|_{L^2}\le 3k\left|z_k^N\right|_{L^2}^2+k\left|\mathfrak{E}\chi-\psi\right|_{L^2}^2$$

which, together with standard estimates for the linear terms and the forcing f_k^χ, lead us to

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left|z_k^N\right|_{L^2}^2\le -2\left|z_k^N\right|_V^2+2C_Q\left|z_k^N\right|_V\left|z_k^N\right|_{L^2}+2\left|f_k^\chi\right|_{V'}\left|z_k^N\right|_V+3k\left|z_k^N\right|_{L^2}^2+k\left|\mathfrak{E}\chi-\psi\right|_{L^2}^2.$$

Thus, by Young's inequality we obtain

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left|z_k^N\right|_{L^2}^2\le -\left|z_k^N\right|_V^2+(C+3k)\left|z_k^N\right|_{L^2}^2+k\left|\mathfrak{E}\chi-\psi\right|_{L^2}^2+C\left|f_k^\chi\right|_{V'}^2.$$

Following standard arguments (using the Gronwall's inequality and time integration), such estimate allows us to obtain estimates as $\left|z_k^N\right|_{W((0,T),V,V')}\le C_1$ with $C_1\ge 0$ independent of N. Thus, the solution z_k is obtained as a weak limit of z_k^N. The uniqueness of the solution follows by standard arguments concerning the linear terms, together with the monotonicity of the nonlinearity (cf. (2.13)). Indeed, if $\overline{z}_k$ is another solution for the difference $d_k \coloneqq z_k - \overline{z}_k$ we will find

$$\tfrac{\partial}{\partial t} d_k + (-\Delta + \mathbf{1}) d_k + Q d_k = -k(z_k + \mathfrak{E}\chi - \psi)^+ + k(\overline{z}_k + \mathfrak{E}\chi - \psi)^+,$$

and, with $\xi_k \coloneqq z_k + \mathfrak{E}\chi - \psi$ and $\overline{\xi}_k \coloneqq \overline{z}_k + \mathfrak{E}\chi - \psi$,

$$\tfrac{\mathrm{d}}{\mathrm{d}t} |d_k|_{L^2}^2 \le C\, |d_k|_{L^2}^2 - 2k(\xi_k^+ - \overline{\xi}_k^+, d_k)_{L_2} = C\, |d_k|_{L^2}^2 - 2k(\xi_k^+ - \overline{\xi}_k^+, \xi_k - \overline{\xi}_k)_{L_2} \le C\, |d_k|_{L^2}^2,$$

which implies $d_k = 0$, because $d_k(0) = 0$.

Subsequently, the existence of strong solutions $z \in W((0,T); \mathrm{D}(A), L^2)$ can be proven for more regular initial conditions $z_\circ \in V$, see [26, Sect.4.3]. For our initial conditions in $z_\circ \in L^2 \setminus V$, we can use the smoothing property of parabolic-like equations to conclude that $rz_k \in W((0,T); \mathrm{D}(A), L^2)$, see [32, Ch. 3, Thm. 3.10] and [23, Lem. 2.6]. Note that that $w_k \coloneqq rz_k$ solves

$$\tfrac{\partial}{\partial t} w_k + (-\Delta + \mathbf{1}) w_k + Q w_k = f_k^{\chi,r}, \quad \mathcal{G} w_k|_\Gamma = 0, \quad w_k(0) = 0 \in V,$$

with $f_k^{\chi,r} \coloneqq r f_k^\chi + (\tfrac{\mathrm{d}}{\mathrm{d}t} r) z_k - rk(z_k + \mathfrak{E}\chi - \psi)^+ \in L^2((0,T); L^2)$. Recall that $r(0) = 0$, so that by usual estimates (for Galerkin approximations) we obtain

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \left|w_k^N\right|_V^2 \le -\tfrac{1}{2} \left|w_k^N\right|_{\mathrm{D}(A)}^2 + C_2 \left|w_k^N\right|_V^2 + C_2 \left|f_k^{\chi,r}\right|_{L^2}^2,$$

which leads to $\left|z_k^N\right|_{W((0,T);\mathrm{D}(A),L^2)} \le C_3$ with $C_3 \ge 0$ independent of N. Taking a weak limit, we can then conclude that the weak solution is indeed strong, satisfying a similar estimate. □

Lemma 2.11. *Let Assumptions 2.3–2.6 hold true. Then, the solution y_k for* (2.6) *satisfies*

$$\begin{aligned} &2k \left|(y_k - \psi)^+\right|_{L^2((0,T);L^2)}^2 + |y_k|_{L^\infty((0,T);L^2)}^2 + |y_k|_{L^2((0,T);H^1)}^2 \\ &\quad \le \overline{C}_{[C_Q,T]} \left(|y_\circ|_{L^2}^2 + |\mathfrak{E}\chi|_{\mathcal{W}(0,T)}^2 + |f_k|_{L^2((0,T);L^2)}^2 + |\psi|_{W((0,T);H^1,V')}^2 \right), \end{aligned}$$

with $\overline{C}_{[C_Q,T]}$ independent of k.

Proof. Recall that $\psi \in W((0,T); H^2, L^2)$ by Assumption 2.6. Now we set

$$v \coloneqq \mathfrak{E}\chi - (\mathfrak{E}\chi - \psi)^+, \tag{2.15}$$

which implies $v \in W((0,T); H^1, L^2)$. Also, $\psi - v \ge 0$, because

$$\begin{aligned} \psi - v &= 0, && \text{if} \quad \mathfrak{E}\chi \ge \psi, \\ \psi - v &= \psi - \mathfrak{E}\chi, && \text{if} \quad \mathfrak{E}\chi \le \psi. \end{aligned}$$

Furthermore under Dirichlet boundary conditions we also have that $v|_\Gamma = \chi$, because $(\mathfrak{E}\chi - \psi)^+|_\Gamma = 0$, due to $\chi \le \psi|_\Gamma$ in Assumption 2.6. Hence, we have

$$p_k \coloneqq y_k - v \in W((0,T); V, L^2), \qquad v \le \psi, \tag{2.16}$$

and

$$\dot{p}_k + Ap_k + Qp_k + k(y_k - \psi)^+ = h_k, \tag{2.17}$$

with

$$h_k \coloneqq f_k - \tfrac{\mathrm{d}}{\mathrm{d}t}v - (-\Delta + \mathbf{1})v - Qv. \tag{2.18}$$

By multiplying (2.17) with $2p_k$, we obtain

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left|p_k\right|_{L^2}^2 + 2\left|p_k\right|_V^2 + 2k((y_k - \psi)^+, p_k)_{L^2} = 2\langle -Qp_k + h_k, p_k\rangle_{V',V}.$$

Observe that, due to (2.16) we have $p_k \ge y_k - \psi$ and

$$((y_k - \psi)^+, p_k)_{L^2} \ge \left|(y_k - \psi)^+\right|_{L^2}^2,$$

and by using Assumption 2.4, Young's inequality and recalling (2.14), it follows that

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\left|p_k\right|_{L^2}^2 + \left|p_k\right|_V^2 + 2k\left|(y_k - \psi)^+\right|_{L^2}^2 &\le 2C_Q^2\left|p_k\right|_{L^2}^2 + 2\left|h_k\right|_{V'}^2\\ &\le \overline{C}_{[C_Q]}\left(\left|p_k\right|_{L^2}^2 + \left|h_k\right|_{V'}^2\right).\end{aligned} \tag{2.19}$$

By the Gronwall's lemma it follows that

$$\left|p_k\right|_{L^\infty((0,T);L^2)}^2 \le \overline{C}_{[C_Q,T]}\left(\left|p_k(0)\right|_{L^2}^2 + \left|h_k\right|_{L^2((0,T);V')}^2\right), \tag{2.20a}$$

and by integration of (2.19), and using (2.20a), we find

$$\left|p_k\right|_{L^2((0,T);V)}^2 + 2k\left|(y_k - \psi)^+\right|_{L^2((0,T);L^2)}^2 \le \overline{C}_{[C_Q,T]}\left(\left|p_k(0)\right|_{L^2}^2 + \left|h_k\right|_{L^2((0,T);V')}^2\right). \tag{2.20b}$$

Now, note that from (2.16), (2.15), (2.18), and $L^2 \hookrightarrow V'$, we have

$$\left|h_k\right|_{L^2((0,T);V')}^2 \le \overline{C}_{[C_Q]}\left(\left|f_k\right|_{L^2((0,T);V')}^2 + \left|v\right|_{W((0,T);H^1,V')}^2\right) \tag{2.21a}$$

$$\le \overline{C}_{[C_Q]}\left(\left|f_k\right|_{L^2((0,T);V')}^2 + \left|\mathfrak{E}\chi\right|_{\mathcal{W}_{(0,T)}}^2 + \left|\psi\right|_{W((0,T);H^1,V')}^2\right), \tag{2.21b}$$

(cf. (2.4)), and

$$\begin{aligned}&\left|y_k\right|_{L^\infty((0,T);L^2)}^2 + \left|y_k\right|_{L^2((0,T);H^1)}^2\\ &\quad\le 2\left|p_k\right|_{L^\infty((0,T);L^2)}^2 + 2\left|v\right|_{L^\infty((0,T);L^2)}^2 + 2\left|p_k\right|_{L^2((0,T);V)}^2 + 2\left|v\right|_{L^2((0,T);H^1)}^2\\ &\quad\le \overline{C}_{[C_Q,T]}\left(\left|p_k(0)\right|_H^2 + \left|\mathfrak{E}\chi\right|_{\mathcal{W}_{(0,T)}}^2 + \left|f_k\right|_{L^2((0,T);V')}^2 + \left|\psi\right|_{W((0,T);H^1,V')}^2\right).\end{aligned} \tag{2.21c}$$

Notice also that

$$\left|p_k(0)\right|_H^2 = \left|y_k(0) - v(0)\right|_{L^2}^2 \le 2\left|y_\circ\right|_{L^2}^2 + 2\left|\mathfrak{E}\chi(0) - (\mathfrak{E}\chi(0) - \psi(0))^+\right|_{L^2}^2. \tag{2.21d}$$

Hence, the result follows from (2.20) and (2.21). □

The following lemma establishes that we are able to identify a pseudo-distance function with a strictly negative normal derivative.

Lemma 2.12. *Let Assumption 2.3 hold true. Then, there exists $\xi \in H^2(\Omega)\bigcap C^2(\Omega)\bigcap C^1(\overline{\Omega})$ and constant $c_\xi < 0$ satisfying*

$$\xi(x) \ge 0 \quad \textit{for all} \quad x \in \overline{\Omega}, \tag{2.22a}$$

$$\tfrac{\partial}{\partial \mathbf{n}}\xi|_\Gamma(\overline{x}) \le c_\xi \quad \textit{for almost all} \quad \overline{x} \in \overline{\Gamma}. \tag{2.22b}$$

Proof. In the case Ω is of class C^2, we can choose $\xi = \rho d_\Gamma$ as the product of the distance to the boundary function, $d_\Gamma(x) = \min_z\{|x-z|_{\mathbb{R}^d} \min z \in \Gamma\}$, and of a suitable cut-off function ρ. From [9, Appendix, Lem. 1 and Eq. (A7)], see also [16, Sect. 13.3.4], we know that $d_\Gamma \in C^2(\Gamma_\delta)$ for a suitable small enough $\delta > 0$ and $\Gamma_\delta \coloneqq \{x \in \overline{\Omega} \mid d_\Gamma(x) \le \delta\}$, and also that $\frac{\partial d_\Gamma}{\partial \mathbf{n}} = 1$. For ρ we choose a smooth function satisfying $0 \le \rho \le 1$, such that $\rho(x) = 0$ for all $x \in \Omega \setminus \Gamma_{\frac{2\delta}{3}}$, and $\rho(x) = 1$ for all $x \in \Gamma_{\frac{\delta}{3}}$.

In the case Ω is a convex polygonal domain we can choose $x_0 \in \Omega$ and

$$\xi(x) = -\left|x - x_0\right|^2_{\mathbb{R}^d} + \max_{z\in\overline{\Omega}} \left|z - x_0\right|^2_{\mathbb{R}^d}, \quad x \in \overline{\Omega},$$

It is clear that $\xi \in C^2(\overline{\Omega})$ and that $\xi \ge 0$. It remains to prove that ξ strictly decreases on Γ in the direction of the outward normal $\mathbf{n}$. To this purpose let $\overline{x} \in \Gamma$ and let F be a face of Γ contained in the affine hyperplane $\mathbb{H}$ and such that $\overline{x} \in F$. Up to an affine change of variables (a translation and a rotation) we can suppose that $0 \in \Omega$ and

$$x_0 = 0 \quad \text{and} \quad \mathbb{H} = \{(s, x_2, x_3, \dots, x_d) \mid (x_2, x_3, \dots, x_d) \in \mathbb{R}^{d-1}\} \quad \text{with} \quad s > 0.$$

In this case, we find that

$$\xi(x) = -\left|x\right|^2_{\mathbb{R}^d} + \max_{z\in\overline{\Omega}} \left|z\right|^2_{\mathbb{R}^d}, \quad \mathbf{n} = (1, 0, 0, \dots, 0) \quad \text{and} \quad \tfrac{\partial}{\partial \mathbf{n}}\xi|_\Gamma = \tfrac{\partial}{\partial x_1}\xi|_\Gamma = -2x_1.$$

Therefore at an arbitrary point $\overline{x} \in \mathbb{H}$ we find that $\frac{\partial}{\partial \mathbf{n}}\xi|_\Gamma(\overline{x}) = -2\overline{x}_1 = -2s$. Note that s is the distance from 0 to $\mathbb{H}$.

Therefore we can conclude that for every point $\overline{x}$ in the (boundary) interior of a face F we have that $\frac{\partial}{\partial \mathbf{n}}\xi|_\Gamma(\overline{x}) = -2s_F$ where $s_F > 0$ is the distance from x_0 to the hyperplane $\mathbb{H}_F$ containing F. Since the number of faces is finite, $\frac{\partial}{\partial \mathbf{n}}\xi|_\Gamma \le \max\{-2s_F \mid F \text{ is a face of } \Gamma\} \eqqcolon c_\xi < 0$, for all boundary points living in one face only. Note that if $\overline{x}$ lives in the intersection of two faces then the normal derivative is not well defined (not continuously, at least), however the set of such points has vanishing (boundary) measure. That is, $\frac{\partial}{\partial \mathbf{n}}\xi|_\Gamma(\overline{x}) \le c_\xi < 0$ for almost every boundary point $\overline{x}$. □

Lemma 2.13. *Let $c_\xi < 0$ and $\xi \in H^2$ be as in Lemma 2.12, and $\eta \ge \chi - \mathcal{G}\psi|_\Gamma$ be as in Assumption 2.6. Then, for*

$$\zeta_k \coloneqq y_k - \psi + \eta\widehat{\xi}, \quad \textit{with} \quad \widehat{\xi} \coloneqq \begin{cases} 0, & \textit{if } \mathcal{G} = \mathbf{1}, \\ -c_\xi^{-1}\xi, & \textit{if } \mathcal{G} = \frac{\partial}{\partial \mathbf{n}}, \end{cases} \tag{2.23}$$

where y_k is the solution for (2.6)*, we have that*

$$(\tfrac{\partial}{\partial \mathbf{n}}\mathfrak{E}\chi, \zeta_k^+)_{L^2(\Gamma)} - (\psi - \eta\widehat{\xi}, \zeta_k^+)_{H^1} \le 2\left|\psi - \eta\widehat{\xi}\right|_{H^2}\left|\zeta_k^+\right|_{L^2}, \qquad \mathcal{G} \in \{\mathbf{1}, \tfrac{\partial}{\partial \mathbf{n}}\}.$$

Proof. Observe that

$$
\begin{aligned}
&(\tfrac{\partial}{\partial\mathbf{n}}\mathfrak{E}\chi,\zeta_k^+)_{L^2(\Gamma)}-(\psi-\eta\widehat{\xi},\zeta_k^+)_{H^1}\\
&\qquad=(\tfrac{\partial}{\partial\mathbf{n}}\mathfrak{E}\chi,\zeta_k^+)_{L^2(\Gamma)}+((\Delta-\mathbf{1})(\psi-\eta\widehat{\xi}),\zeta_k^+)_{L^2}-(\tfrac{\partial}{\partial\mathbf{n}}(\psi-\eta\widehat{\xi}),\zeta_k^+)_{L^2(\Gamma)}\\
&\qquad=(\tfrac{\partial}{\partial\mathbf{n}}\mathfrak{E}\chi-\tfrac{\partial}{\partial\mathbf{n}}\psi+\eta\tfrac{\partial}{\partial\mathbf{n}}\widehat{\xi},\zeta_k^+)_{L^2(\Gamma)}+((\Delta-\mathbf{1})(\psi-\eta\widehat{\xi}),\zeta_k^+)_{L^2}.
\end{aligned}\tag{2.24}
$$

Note that

$$
\zeta_k^+|_\Gamma=0,\quad\text{if}\quad\mathcal{G}=\mathbf{1},\qquad\text{and}\qquad\tfrac{\partial}{\partial\mathbf{n}}\mathfrak{E}\chi=\chi,\quad\text{if}\quad\mathcal{G}=\tfrac{\partial}{\partial\mathbf{n}}.\tag{2.25a}
$$

Now, by using (2.22b) and (2.23),

$$
\tfrac{\partial}{\partial\mathbf{n}}\mathfrak{E}\chi-\tfrac{\partial}{\partial\mathbf{n}}\psi+\eta\tfrac{\partial}{\partial\mathbf{n}}\widehat{\xi}=\chi-\tfrac{\partial}{\partial\mathbf{n}}\psi|_\Gamma+\eta\tfrac{\partial}{\partial\mathbf{n}}\widehat{\xi}|_\Gamma\le\chi-\tfrac{\partial}{\partial\mathbf{n}}\psi|_\Gamma-\eta\le0,\quad\text{if}\quad\mathcal{G}=\tfrac{\partial}{\partial\mathbf{n}}\tag{2.25b}
$$

and, by (2.25), we have that

$$
(\tfrac{\partial}{\partial\mathbf{n}}\mathfrak{E}\chi-\tfrac{\partial}{\partial\mathbf{n}}\psi+\eta\tfrac{\partial}{\partial\mathbf{n}}\widehat{\xi},\zeta_k^+)_{L^2(\Gamma)}\le0,\quad\text{if}\quad\mathcal{G}\in\{\mathbf{1},\tfrac{\partial}{\partial\mathbf{n}}\},\tag{2.26}
$$

with an equality in the case $\mathcal{G}=\mathbf{1}$. Thus, by (2.24) and (2.26) we obtain

$$
(\tfrac{\partial}{\partial\mathbf{n}}\mathfrak{E}\chi,\zeta_k^+)_{L^2(\Gamma)}-(\psi-\eta\widehat{\xi},\zeta_k^+)_{H^1}\le\left|(\Delta-\mathbf{1})(\psi-\eta\widehat{\xi})\right|_{L^2}\left|\zeta_k^+\right|_{L^2}\le2\left|\psi-\eta\widehat{\xi}\right|_{H^2}\left|\zeta_k^+\right|_{L^2},\tag{2.27}
$$

which ends the proof. □

Lemma 2.14. *Let Assumptions 2.3–2.6 hold true. Then, the solution y_k for* (2.6) *satisfies*

$$
\begin{aligned}
&k^2\left|(y_k-\psi)^+\right|^2_{L^2((0,T);L^2(\Omega))}+\left|\tfrac{\mathrm{d}}{\mathrm{d}t}(y_k-\mathfrak{E}\chi)\right|^2_{L^2((0,T);V')}\\
&\qquad\le\overline{C}_{[C_Q,T]}\left(|y_\circ|^2_{L^2}+|\mathfrak{E}\chi|^2_{\mathcal{W}(0,T)}+|f_k|^2_{L^2((0,T);L^2)}+|\psi|^2_{\mathcal{W}(0,T)}+|\eta|^2_{W^{1,2}(0,T)}\right),
\end{aligned}
$$

with $\overline{C}_{[C_Q,T]}$ independent of k.

Proof. Let us choose $c_\xi<0$ and ξ as in Lemma 2.12 implying in particular that $\xi\in H^2$. We also have $\eta\ge\chi-\mathcal{G}\psi|_\Gamma$, due to Assumption 2.6. Then, we set ζ_k as in (2.23).

Observe that both ζ_k and ζ_k^+ are in H^1. Furthermore, in the case of Dirichlet boundary conditions we also have $\zeta_k^+\in H^1_0$ as a corollary of Assumption 2.6. Therefore,

$$
\zeta_k^+\in V,\quad\text{for}\quad\mathcal{G}\in\{\tfrac{\partial}{\partial\mathbf{n}},\mathbf{1}\}.\tag{2.28}
$$

Let us denote now $\varkappa_k=y_k-\mathfrak{E}\chi$. We find

$$
\dot{\varkappa}_k+A\varkappa_k+Q\varkappa_k+k(y_k-\psi)^+=g_k,\qquad\varkappa_k(0)=\varkappa_\circ,\qquad\mathcal{G}\varkappa_k|_\Gamma=0,\tag{2.29a}
$$

with

$$
\varkappa_\circ=y_\circ-\mathfrak{E}\chi(0),\qquad g_k\coloneqq f_k-\tfrac{\mathrm{d}}{\mathrm{d}t}\mathfrak{E}\chi-(-\Delta+\mathbf{1})\mathfrak{E}\chi-Q\mathfrak{E}\chi.\tag{2.29b}
$$

Testing the dynamics with ζ_k^+, gives us

$$\begin{aligned}
0 &= (\dot{\varkappa}_k, \zeta_k^+)_{L^2} + (\varkappa_k, \zeta_k^+)_V + k((y_k-\psi)^+, \zeta_k^+)_{L^2} + (Q\varkappa_k - g_k, \zeta_k^+)_{L^2}\\
&= (\dot{\varkappa}_k + \tfrac{\mathrm{d}}{\mathrm{d}t}\mathfrak{E}\chi - \dot{\psi} + \dot{\eta}\widehat{\xi}, \zeta_k^+)_{L^2} + (\varkappa_k + \mathfrak{E}\chi - \psi + \eta\widehat{\xi}, \zeta_k^+)_{H^1} + k((y_k-\psi)^+, \zeta_k^+)_{L^2}\\
&\quad + (Q\varkappa_k - g_k - \tfrac{\mathrm{d}}{\mathrm{d}t}\mathfrak{E}\chi + \dot{\psi} - \dot{\eta}\widehat{\xi}, \zeta_k^+)_{L^2} + (-\mathfrak{E}\chi + \psi - \eta\widehat{\xi}, \zeta_k^+)_{H^1}
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
0 &= (\dot{\zeta}_k, \zeta_k^+)_{L^2} + (\zeta_k, \zeta_k^+)_{H^1} + k((y_k-\psi)^+, \zeta_k^+)_{L^2}\\
&\quad + (Q\varkappa_k - g_k - \tfrac{\mathrm{d}}{\mathrm{d}t}\mathfrak{E}\chi + \dot{\psi} - \dot{\eta}\widehat{\xi}, \zeta_k^+)_{L^2} + (-\mathfrak{E}\chi + \psi - \eta\widehat{\xi}, \zeta_k^+)_{H^1}.
\end{aligned}$$

Then, using Stampacchia's lemma [31, Lem. 1.1]) and Lions-Magenes' lemma [32, Ch. 3, Sect. 1.4, Lem. 1.2], we arrive at

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\left|\zeta_k^+\right|_{L^2}^2 + 2\left|\zeta_k^+\right|_V^2 + 2k((y_k-\psi)^+, \zeta_k^+)_{L^2}\\
&\qquad = 2(-Q\varkappa_k + g_k + \tfrac{\mathrm{d}}{\mathrm{d}t}\mathfrak{E}\chi - \dot{\psi} + \dot{\eta}\widehat{\xi}, \zeta_k^+)_{L^2} - 2(-\mathfrak{E}\chi + \psi - \eta\widehat{\xi}, \zeta_k^+)_{H^1}.
\end{aligned}$$

Next, we use the relations in (2.29) to obtain

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\left|\zeta_k^+\right|_{L^2}^2 + 2\left|\zeta_k^+\right|_V^2 + 2k((y_k-\psi)^+, \zeta_k^+)_{L^2}\\
&\qquad = 2(-Qy_k + f_k - (-\Delta + \mathbf{1})\mathfrak{E}\chi - \dot{\psi} + \dot{\eta}\widehat{\xi}, \zeta_k^+)_{L^2} - 2(-\mathfrak{E}\chi + \psi - \eta\widehat{\xi}, \zeta_k^+)_{H^1}\\
&\qquad = 2(-Qy_k + f_k - \dot{\psi} + \dot{\eta}\widehat{\xi}, \zeta_k^+)_{L^2} - 2(\psi - \eta\widehat{\xi}, \zeta_k^+)_{H^1} + 2(\tfrac{\partial}{\partial \mathbf{n}}\mathfrak{E}\chi, \zeta_k^+)_{L^2(\Gamma)}
\end{aligned} \tag{2.30}$$

and, using Lemma 2.13 we find

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left|\zeta_k^+\right|_{L^2}^2 + 2\left|\zeta_k^+\right|_V^2 + 2k((y_k-\psi)^+, \zeta_k^+)_{L^2} \tag{2.31}$$
$$\leq 2(-Qy_k + f_k - \dot{\psi} + \dot{\eta}\widehat{\xi}, \zeta_k^+)_{L^2} + 4\left|\psi - \eta\widehat{\xi}\right|_{H^2}\left|\zeta_k^+\right|_{L^2} \tag{2.32}$$
$$\leq 2\left(\left|-Qy_k + f_k - \dot{\psi} + \dot{\eta}\widehat{\xi}\right|_{L^2} + 2\left|\psi - \eta\widehat{\xi}\right|_{H^2}\right)\left|\zeta_k^+\right|_{H^2}. \tag{2.33}$$

Time integration of (2.33) gives us

$$\left|\zeta_k^+(T)\right|_{L^2}^2 - \left|\zeta_k^+(0)\right|_{L^2}^2 + 2\left|\zeta_k^+\right|_{L^2((0,T);V)}^2 + 2k((y_k-\psi)^+, \zeta_k^+)_{L^2((0,T);L^2)} \leq 2\Xi\left|\zeta_k^+\right|_{L^2((0,T);L^2)}$$

with

$$\Xi := \left(\left|-Qy_k + f_k - \dot{\psi} + \dot{\eta}\widehat{\xi}\right|_{L^2((0,T);L^2)} + 2\left|\psi - \eta\widehat{\xi}\right|_{L^2((0,T);H^2)}\right),$$

from which, together with the fact that, due to Assumption 2.5, at time $t = 0$ we have $\zeta_k^+(0) = (y_\circ - \psi(0))^+ = 0$, we obtain

$$2k\left|((y_k-\psi)^+, \zeta_k^+)_{L^2((0,T);L^2)}\right|_{\mathbb{R}} = 2k((y_k-\psi)^+, \zeta_k^+)_{L^2((0,T);L^2)} \leq 2\Xi\left|\zeta_k^+\right|_{L^2((0,T);L^2)},$$

which, together with $L^2((0,T);L^2) = (L^2((0,T);L^2))'$, give us $|(y_k-\psi)^+|_{L^2((0,T);L^2)} \le k^{-1}\Xi$, thus

$$\begin{aligned} &k\left|(y_k-\psi)^+\right|_{L^2((0,T);L^2)} \\ &\qquad\le \Xi \le \overline{C}_{[C_Q]}\left(|y_k|_{L^2((0,T);L^2)} + |f_k|_{L^2((0,T);L^2)} + |\psi|_{\mathcal{W}(0,T)} + \left|\eta\widehat{\xi}\right|_{\mathcal{W}(0,T)}\right). \end{aligned} \tag{2.34}$$

Next, from (2.29) we also find that

$$|\dot{\varkappa}_k|^2_{V'} = \left|A\varkappa_k + Q\varkappa_k + k(y_k-\psi)^+ - g_k\right|^2_{V'}$$

which together with (2.34), $\varkappa_k = y_k - \mathfrak{E}\chi$, and $L^2 \hookrightarrow V'$, give us

$$|\dot{\varkappa}_k|^2_{L^2((0,T);V')} \le \overline{C}\left(|y_k|^2_{L^2((0,T);H^1)} + |\mathfrak{E}\chi|^2_{\mathcal{W}(0,T)} + |f_k|^2_{L^2((0,T);L^2)} + |\psi|^2_{\mathcal{W}(0,T)} + |\eta|^2_{W^{1,2}(0,T)}\right).$$

with $\overline{C} = \overline{C}_{\left[C_Q,\left|\widehat{\xi}\right|_{H^2}\right]}$. Finally, we can finish the proof by using Lemma 2.11. □

Remark 2.15. We can see that the constant $\overline{C}_{[C_Q,T]}$ in the statement of the Lemma 2.14 will also depend on $\left|\widehat{\xi}\right|_{H^2}$ as $\overline{C}_{\left[C_Q,T,\left|\widehat{\xi}\right|_{H^2}\right]}$, but since essentially $\widehat{\xi}$ depends only on the spatial domain Ω, we omit the dependence on $\left|\widehat{\xi}\right|_{H^2}$ in the statement of Lemma 2.14 and throughout the manuscript.

Lemma 2.16. *Let Assumptions 2.3–2.6 hold true, with in addition $y_\circ - \mathfrak{E}\chi(0) \in V$. Then the solution y_k for* (2.6) *satisfies*

$$\begin{aligned} &|y_k|^2_{L^2((0,T);H^2)} + |y_k|^2_{L^\infty((0,T);H^1)} \\ &\qquad\le \overline{C}\left(|y_\circ|^2_{H^1} + |\mathfrak{E}\chi|^2_{\mathcal{W}_{(0,T)}} + |f_k|^2_{L^2((0,T);L^2)} + |\psi|^2_{\mathcal{W}_{(0,T)}}\right), \end{aligned} \tag{2.35}$$

with a constant $\overline{C}_{[T,C_Q]}$ independent of k.

Proof. Testing the dynamics in (2.29) with $2A\varkappa_k$, where $\varkappa_k = y_k - \mathfrak{E}\chi$, it follows that

$$2\,|\varkappa_k|^2_{\mathrm{D}(A)} + \tfrac{\mathrm{d}}{\mathrm{d}t}\,|\varkappa_k|^2_V = 2(g_k - Q\varkappa_k - k(y_k-\psi)^+, A\varkappa_k)_{L^2}.$$

Then, the Young inequality gives us

$$|\varkappa_k|^2_{\mathrm{D}(A)} + \tfrac{\mathrm{d}}{\mathrm{d}t}\,|\varkappa_k|^2_V \le \left|g_k - Q\varkappa_k - k(y_k-\psi)^+)\right|^2_{L^2},$$

and from the Gronwall's lemma and integration over $(0,T)$ we obtain

$$|\varkappa_k|^2_{L^2((0,T);\mathrm{D}(A))} + |\varkappa_k|^2_{L^\infty((0,T);V)} \le |\varkappa_\circ|^2_V + \left|g_k - Q\varkappa_k - k(y_k-\psi)^+)\right|^2_{L^2((0,T);L^2)}.$$

Finally, we can conclude the proof by using Lemmas 2.11 and 2.14, and recalling the identities in (2.29b). □

In Lemma 2.17 we require the extra regularity for the initial condition in order to have strong solutions for the parabolic equation. This extra requirement is needed due to the compatibility conditions mentioned in Remark 2.2. However, due to the smoothing property of parabolic equations, it turns out that for strictly positive time $t > 0$ we will have that $y_k(t) \in V$ when $y_\circ \in H$. This fact is exploblue in the following result.

Lemma 2.17. *Let Assumptions 2.3–2.6 hold true and let y_k solve* (2.6)*. Then, it follows that*

$$\begin{aligned}&|ry_k|^2_{L^2((0,T);H^2)} + |ry_k|^2_{L^\infty((0,T);H^1)} + \left|\tfrac{\mathrm{d}}{\mathrm{d}t}(ry_k)\right|^2_{L^2((0,T);L^2)}\\ &\qquad\le \overline{C}\left(|y_\circ|^2_{L^2} + |r\mathfrak{E}\chi|^2_{\mathcal{W}_{(0,T)}} + |rf_k|^2_{L^2((0,T);L^2)} + |r\psi|^2_{\mathcal{W}_{(0,T)}}\right),\end{aligned}$$

with a constant $\overline{C}_{[T,C_Q]}$ independent of k.

Proof. Multiplying the dynamics in (2.29) by $2r^2A\varkappa_k$, it follows that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left|r\varkappa_k\right|^2_V - (\tfrac{\mathrm{d}}{\mathrm{d}t}r^2)\left|\varkappa_k\right|^2_V + 2\left|r\varkappa_k\right|^2_{\mathrm{D}(A)} = 2(rg_k - rQ\varkappa_k - rk(y_k-\psi)^+, rA\varkappa_k)_{L^2}.$$

Then, the Young inequality together with $\max\{|r|_{L^\infty(\mathbb{R}_+)}, |\dot r|_{L^\infty(\mathbb{R}_+)}\} = 1$ give us

$$\left|r\varkappa_k\right|^2_{\mathrm{D}(A)} + \tfrac{\mathrm{d}}{\mathrm{d}t}\left|r\varkappa_k\right|^2_V \le \left|g_k - Q\varkappa_k - k(y_k-\psi)^+\right|^2_{L^2} + \left|r\varkappa_k\right|^2_V,$$

and from the Gronwall's lemma and integration over $(0,T)$ we obtain

$$\left|r\varkappa_k\right|^2_{L^2((0,T);\mathrm{D}(A))} + \left|r\varkappa_k\right|^2_{L^\infty((0,T);V)} \le \left|g_k - Q\varkappa_k - k(y_k-\psi)^+\right|^2_{L^2((0,T);L^2)}.$$

Further we have that

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}t}(r\varkappa_k)\right|^2_{L^2} = \left|Ar\varkappa_k + Qr\varkappa_k + rk(\varkappa_k-\phi)^+ - rg_k - (\dot r)\varkappa_k\right|^2_{L^2}.$$

We can conclude the proof by using $ry_k = r\varkappa_k + r\mathfrak{E}\chi$, (2.29b), and Lemmas 2.11 and 2.14. $\square$

We are now ready to conclude the proof of Theorem 2.9.

Proof of Theorem 2.9. **Existence:** From Lemmas 2.11 and 2.17, there exists a subsequence $y_{\mathfrak{n}(k)}$ of y_k, such that the following weak limits hold

$$y_{\mathfrak{n}(k)} - \mathfrak{E}\chi \xrightarrow[L^2((0,T);V)]{} y - \mathfrak{E}\chi, \qquad \dot y_{\mathfrak{n}(k)} - \tfrac{\mathrm{d}}{\mathrm{d}t}\mathfrak{E}\chi \xrightarrow[L^2((0,T);V')]{} \dot y - \tfrac{\mathrm{d}}{\mathrm{d}t}\mathfrak{E}\chi, \tag{2.36a}$$

$$r(y_{\mathfrak{n}(k)} - \mathfrak{E}\chi) \xrightarrow[L^2((0,T);\mathrm{D}(A))]{} z, \qquad \tfrac{\mathrm{d}}{\mathrm{d}t}(r(y_{\mathfrak{n}(k)} - \mathfrak{E}\chi)) \xrightarrow[L^2((0,T);L^2)]{} \dot z, \tag{2.36b}$$

for suitable $y \in W((0,T),H^1,V')$ and $z \in W((0,T),\mathrm{D}(A),L^2)$. Necessarily we have $z = r(y - \mathfrak{E}\chi)$ and the strong limits

$$y_{\mathfrak{n}(k)} \xrightarrow[L^2((0,T);L^2)]{} y, \qquad r(y_{\mathfrak{n}(k)} - \mathfrak{E}\chi) \xrightarrow[L^2((0,T);V)]{} r(y - \mathfrak{E}\chi), \tag{2.37a}$$

$$r(y_{\mathfrak{n}(k)} - \mathfrak{E}\chi) \xrightarrow[\mathcal{C}([0,T];L^2)]{} r(y - \mathfrak{E}\chi), \tag{2.37b}$$

where we have used, in particular the Aubin-Lions-Simon's lemma [30, Sect. 8, Cor. 4].

For the sake of simplicity, let us still denote the subsequence $y_{\mathfrak{n}(k)}$ by y_k. By Lemma 2.11, it follows that $(k^2 \left|(y_k-\psi)^+\right|^2_{L^2((0,T);L^2)})_{k\in\mathbb{N}}$ is bounded, thus

$$\left|(y-\psi)^+\right|^2_{L^2((0,T);L^2)} = \lim_{k\to+\infty} \left|(y_k-\psi)^+\right|^2_{L^2((0,T);L^2)} = 0$$

and, since $y \in L^2((0,T);H^1)$, we obtain that $y \in \mathbf{C}^{\psi}_T$, see (2.5). Now, for an arbitrary $v \in \mathbf{C}^{\psi}_T$, we find, for almost every $t \in (0,T)$,

$$\begin{aligned}
&\left(r\left(\tfrac{\partial}{\partial t} y_k + (-\Delta+\mathbf{1})y_k + Qy_k - f_k\right), r(v-y_k)\right)_{L^2}\\
&\qquad= -k\left(r(y_k-\psi)^+, r(v-y_k)\right)_{L^2}\\
&\qquad= k\left((y_k-\psi)^+, r^2(y_k-\psi)\right)_{L^2} + k\left((y_k-\psi)^+, r^2(\psi-v)\right)_{L^2},
\end{aligned}$$

which gives us

$$\left(r\left(\tfrac{\partial}{\partial t} y_k + (-\Delta+\mathbf{1})y_k + Qy_k - f_k\right), r(v-y_k)\right)_{L^2} \ge 0, \tag{2.38}$$

because $r^2k(y_k-\psi)^+(y_k-\psi) \ge 0$ and $r^2k(y_k-\psi)^+(\psi-v) \ge 0$, due to $v \in \mathbf{C}^{\psi}_T$.

Observe that, with $q_k := r(y_k - \mathfrak{E}\chi)$ and $q := r(y-\mathfrak{E}\chi)$, for the left-factor in (2.38), we find

$$\begin{aligned}
&r\left(\tfrac{\partial}{\partial t} y_k + (-\Delta+\mathbf{1})y_k + Qy_k - f_k\right)\\
&\qquad= \dot q_k + Aq_k + Qq_k - rf_k + r(\tfrac{\mathrm{d}}{\mathrm{d}t} + A + Q)\mathfrak{E}\chi - (\dot r)(y_k - \mathfrak{E}\chi),
\end{aligned}$$

and we have the weak limit in $L^2((0,T);L^2)$ given by

$$\dot q + Aq + Qq - rf + r(\tfrac{\mathrm{d}}{\mathrm{d}t} + A + Q)\mathfrak{E}\chi - (\dot r)(y-\mathfrak{E}\chi) = r\left(\tfrac{\partial}{\partial t} y + (-\Delta+\mathbf{1})y + Qy - f\right)$$

and also the strong limit for the right-factor in (2.38) as follows

$$q_k \xrightarrow[L^2((0,T);L^2)]{} q.$$

These limits allow us to take the limit for the integrated product in (2.38), and obtain

$$\begin{aligned}
&\int_0^T \left(r\left(\tfrac{\partial}{\partial t} y + (-\Delta+\mathbf{1})y + Qy - f\right), r(v-y)\right)_{L^2} \mathrm{d}t\\
&\qquad= \lim_{k\to+\infty} \int_0^T \left(r(\tfrac{\partial}{\partial t} y_k + (-\Delta+\mathbf{1})y_k + Qy_k - f_k), r(v-y_k)\right)_H \,\mathrm{d}t\\
&\qquad\ge 0, \qquad \text{for all} \quad v \in \mathbf{C}^{\psi}_T.
\end{aligned} \tag{2.39}$$

Let us fix arbitrary $v \in \mathbf{C}^{\psi}_T$, $\bar t \in (0,T)$, $\delta \in (0, \min\{\bar t, T-\bar t\})$. Note that the integrand $\xi_v := \left(r\left(\tfrac{\partial}{\partial t} y + (-\Delta+\mathbf{1})y + Qy - f\right), r(v-y)\right)_{L^2}$ is an integrable function, $\xi_v \in L^1(0,T)$. By the Lebesgue differentiation theorem [29, Ch. 7, Thm. 7.7], the set of Lebesgue points

$$\mathfrak{L}_v := \left\{ t^* \in (0,T) \mid \xi_h(t^*) = \lim_{\delta\searrow 0} \frac{1}{2\delta}\int_{t^*-\delta}^{t^*+\delta} \xi_v(t)\,\mathrm{d}t \right\},$$

has full measure. We define the functions

$$v_{\bar{t},\delta} \coloneqq \begin{cases} v, & \text{if } t \in (\bar{t}-\delta, \bar{t}+\delta) \\ y, & \text{if } t \in (0, \bar{t}-\delta) \bigcup (\bar{t}+\delta, T). \end{cases}$$

We have $v_{\bar{t},\delta}(t,x) \in \mathbf{C}_T^{\psi}$. From (2.39), it follows that

$$\int_{\bar{t}-\delta}^{\bar{t}+\delta} \xi_v(t)\,\mathrm{d}t = \int_0^T \left(r\left(\tfrac{\partial}{\partial t}y + (-\Delta+\mathbf{1})y + Qy - f\right), r(v_{\bar{t},\delta} - y)\right)_{L^2}(t)\,\mathrm{d}t \geq 0$$

and as a consequence we have

$$\left(r\left(\tfrac{\partial}{\partial t}y + (-\Delta+\mathbf{1})y + Qy - f\right), r(v-y)\right)_{L^2}(t^*) \geq 0, \quad \text{for all} \quad t^* \in \mathfrak{L}_v,$$

which implies the inequality in (2.8), because $r^2 = \min\{t^2, 1\} > 0$ for time $t > 0$.

Uniqueness: Let us assume that $w \in \mathbf{C}_T^{\psi} \bigcap W((0,T); H^1, V')$, with $rw \in W((0,T); H^2, L^2)$ also satisfies (2.8). In this case we find the relations

$$(\dot{y} + (-\Delta+\mathbf{1})y + Qy - f, w - y)_{L^2} \geq 0, \qquad (\dot{w} + (-\Delta+\mathbf{1})w + Qw - f, y - w)_{L^2} \geq 0,$$

which lead us to, with $z \coloneqq y - w$,

$$(\dot{z} + Az + Qz, z)_{L^2} \leq 0, \quad \text{for almost all} \quad t \in (0,T), \quad z(0) = 0,$$

with $z(t) \in V$ for all $t \in [0,T]$. Thus

$$\tfrac{\mathrm{d}}{\mathrm{d}t} |z|_{L^2}^2 + 2\,|z|_V^2 \leq 2C_Q\,|z|_{H^1}\,|z|_{L^2} \leq |z|_V^2 + C_Q^2\,|z|_{L^2}^2\,, \tag{2.40}$$

and the uniqueness follows from Gronwall's lemma.

Convergence: Finally we show that the strong limits in (2.37) hold for the (entire) sequence y_k. We argue by contradiction. Let us denote $\mathbb{S} \coloneqq \{L^2((0,T),V), \mathcal{C}([0,T],L^2)\}$.

$$\text{Suppose that } r(y_k - \mathfrak{E}\chi) \xrightarrow[\mathcal{S}]{} r(y - \mathfrak{E}\chi) \text{ does not hold, for some } \mathcal{S} \in \mathbb{S}. \tag{2.41}$$

Under assumption (2.41), there would exist $\varepsilon > 0$ and a subsequence $y_{\mathfrak{s}_1(k)}$ of y_k such that

$$\left| r(y_{\mathfrak{s}_1(k)} - \mathfrak{E}\chi) - r(y - \mathfrak{E}\chi) \right|_{\mathcal{S}} \geq \varepsilon. \tag{2.42}$$

However since $\{\overline{y}_k\} \coloneqq \{y_{\mathfrak{s}_1(k)}\}$ is a subsequence of $\{y_k\}$ we would be able to follow the arguments above and arrive to analogous limits as in (2.36) and (2.37), for a suitable subsequence $\{\overline{y}_{\mathfrak{s}_2(k)}\}$ of $\{\overline{y}_k\}$ and a limit $\overline{y}$ in the place of y. In particular, we would arrive to

$$y_{\mathfrak{s}_2(\mathfrak{s}_1(k))} \xrightarrow[\mathcal{S}]{} \overline{y},$$

where moreover $\overline{y}$ solves (2.8). By (2.42) we would have that $\overline{y} \neq y$, which contradicts the uniqueness of the solution proven above. That is, the assumption in (2.41) leads us to a contradiction. Therefore, we can conclude that (2.12) holds true. The proof is finished. $\square$

3. Stabilization of a sequence of parabolic equations

The solution of (1.1) can be approximated by the sequence $(y_k)_{k\in\mathbb{N}}$ as stated in Theorem 2.9, where y_k solves

$$\tfrac{\partial}{\partial t}y_k - \nu\Delta y_k + ay_k + b\cdot\nabla y_k + k(y_k-\psi)^+ = f, \tag{3.1a}$$
$$y_k(0)=y_\circ,\quad \mathcal{G}y|_\Gamma=\chi. \tag{3.1b}$$

This follows from Theorem 2.9 with $Q=a\mathbf{1}+b\cdot\nabla$, and $f_k=f$.

We investigate the stabilizability to trajectories for system (3.1). We consider the sequence $(w_k)_{k\in\mathbb{N}}$, where w_k solves

$$\tfrac{\partial}{\partial t}w_k - \nu\Delta w_k + aw_k + b\cdot\nabla w_k + k(w_k-\psi)^+ = f - \lambda P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp} A P_{\mathcal{E}_M}^{\mathcal{U}_M^\perp}(w_k-y_k), \tag{3.2a}$$
$$w_k(0)=w_\circ,\quad \mathcal{G}w|_\Gamma=\chi, \tag{3.2b}$$

where $P_{\mathcal{E}_M}^{\mathcal{U}_M^\perp}\in\mathcal{L}(L^2)$ and $P_{\mathcal{E}_M}^{\mathcal{U}_M^\perp}\in\mathcal{L}(L^2)$ are suitable oblique projections in L^2, which we shall construct so that $P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp} A P_{\mathcal{E}_M}^{\mathcal{U}_M^\perp}\in\mathcal{L}(L^2)$. Then again from Theorem 2.9, with $Q=a\mathbf{1}+b\cdot\nabla+\lambda P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp} A P_{\mathcal{E}_M}^{\mathcal{U}_M^\perp}$, and $f_k=f+\lambda P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp} A P_{\mathcal{E}_M}^{\mathcal{U}_M^\perp}y_k$, it follows that the solution of (1.3) can be approximated by the sequence $(w_k)_{k\in\mathbb{N}}$. At this point, it is important to underline that the triple $(\lambda,\mathcal{U}_M,\mathcal{E}_M)$ can be chosen independently of k, as we shall show later on.

In this section we will see y_k as our target solution and consider the difference $z_k\coloneqq w_k-y_k$ from the controlled solution w_k to the target. With initial condition $z_\circ\coloneqq w_\circ-y_\circ$, we find that z_k satisfies

$$\tfrac{\partial}{\partial t}z_k - \nu\Delta z_k + az_k + b\cdot\nabla z_k + k\left((z_k+y_k-\psi)^+ - (y_k-\psi)^+\right) = -\lambda P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp} A P_{\mathcal{E}_M}^{\mathcal{U}_M^\perp}z_k, \tag{3.3a}$$
$$z_k(0)=z_\circ,\qquad \mathcal{G}z_k|_\Gamma=0. \tag{3.3b}$$

For a given $\mu>0$, our goal here, see (1.4), is to find a scalar $\lambda>0$, a space of actuators $\mathcal{U}_M$, and an auxiliary space $\mathcal{E}_M$, such that

$$|w_k(t)-y_k(t)|_{L^2}\le C\mathrm{e}^{-\mu t}\,|w_\circ-y_\circ|_{L^2}\,,\quad\text{for all}\quad (w_\circ,y_\circ)\in L^2\times L^2,\quad t\ge 0 \tag{3.4}$$

for a suitable $C\ge 1$.

3.1. The oblique projections

We specify here how we can appropriately choose the spaces of actuators $\mathcal{U}_M$ and auxiliary eigenfunctions $\mathcal{E}_M$, so that the feedback operator $-\lambda P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp} A P_{\mathcal{E}_M}^{\mathcal{U}_M^\perp}$ is stabilizing for large enough $\lambda>0$. Since the stabilization results will hold for large enough M, we will rather consider a sequence of pairs of subspaces $(\mathcal{U}_M,\mathcal{E}_M)_{M\in\mathbb{N}}$ as in (1.7).

In the one-dimensional case, $\Omega^1=(0,L_1)\subset\mathbb{R}$, $L_1>0$, as actuators we take the indicator functions $1_{\omega_j^1}(x_1)$, $j\in\{1,2,\dots,M\}$, defined as follows,

$$1_{\omega_j^1}(x_1)\coloneqq\begin{cases}1, & \text{if } x_1\in\Omega^1\bigcap\overline{\omega_j^1},\\ 0, & \text{if } x_1\in\Omega^1\setminus\overline{\omega_j^1},\end{cases}\quad \omega_j^1\coloneqq(c_j-\tfrac{rL_1}{2M},c_j+\tfrac{rL_1}{2M}),\quad c_j\coloneqq\tfrac{(2j-1)L_1}{2M}. \tag{3.5}$$

As eigenfunctions we take the first M eigenfunctions e^1_j of $-\nu\Delta + \mathbf{1}\colon \mathrm{D}(A) \to L^2(\Omega^1)$ (i.e., the first eigenfunctions of Δ),

$$(-\nu\Delta + \mathbf{1})e^1_j = \alpha^1_j e^1_j, \quad \mathcal{G}e^1_i|_\Gamma = 0, \qquad j \in \{1, 2, \dots, M\}, \tag{3.6}$$

where the α^1_js are the ordered eigenvalues, repeated accordingly to their multiplicity,

$$0 < 1 \le \alpha^1_1 < \alpha^1_2 < \dots < \alpha^1_j < \alpha^1_{j+1} < \dots, \qquad j \in \mathbb{N}.$$

In the higher-dimensional case, for nonempty rectangular domains $\Omega^\times = \prod\limits_{n=1}^{d} (0, L_n) \subset \mathbb{R}^d$, $L_n > 0$ we take Cartesian product actuators of the above actuators $1_{\omega^n_j}$ and eigenfunctions e^n_j as follows. We define $\mathbb{M} := \{1, 2, \dots, M\}$ and take

$$\mathcal{U}_M = \operatorname{span}\{1_{\omega^\times_{\mathbf{j}}} \mid \mathbf{j} \in \mathbb{M}^d\} \quad \text{and} \quad \mathcal{E}_M = \operatorname{span}\{e^\times_{\mathbf{j}} \mid \mathbf{j} \in \mathbb{M}^d\}, \tag{3.7}$$

and $\omega^\times_{\mathbf{j}} := \{(x_1, x_2, \dots, x_d) \in \Omega^\times \mid x_n \in \omega^n_{\mathbf{j}_n}\}$ and $e^\times_{\mathbf{j}}(x_1, x_2, \dots, x_{\mathbf{d}}) := \prod\limits_{n=1}^{d} e^n_{\mathbf{j}_n}(x_n)$. Notice that we can also write $1_{\omega^\times_{\mathbf{j}}} = \prod\limits_{n=1}^{d} 1_{\omega^n_{\mathbf{j}_n}}(x_n)$.

It turns out that the Poincaré-like constant

$$\beta_{M_+} := \min\left\{ \tfrac{|h|_V}{|h|_{L^2}} \;\middle|\; h \in \mathcal{U}_M^\perp \textstyle\bigcap V, \quad h \ne 0 \right\} \tag{3.8a}$$

satisfies

$$\lim_{M\to+\infty} \beta_{M_+} = +\infty. \tag{3.8b}$$

Furthermore, we have

$$L^2 = \mathcal{U}_M \oplus \mathcal{E}_M^\perp \quad \text{and} \quad \sup_{M\ge1} \left| P^{\mathcal{E}_M^\perp}_{\mathcal{U}_M} \right|_{\mathcal{L}(L^2)} =: C_P < +\infty. \tag{3.8c}$$

Finally, we define the following eigenvalue

$$\widehat{\alpha}_M := \max\{\alpha_i \mid \text{there is } \phi \in \mathcal{E}_M \text{ such that } A\phi = \alpha_i\phi\} \tag{3.8d}$$

which we shall need later on.

For details concerning (3.8b) we refer to [27, Sect. 5]. Concerning (3.8c), for the one-dimensional case we refer to [28, Thms. 4.4 and 5.2], for higher-dimensional rectangular domains see [18, Sect. 4.8.1].

Remark 3.1. For nonrectangular domains $\Omega \subset \mathbb{R}^d$, with $d \ge 2$, we still do not know whether we can choose the actuators (as indicator functions) so that the properties in (3.8) are satisfied. So we cannot guarantee that an oblique projection based feedback will stabilize our system. In spite of this fact, we refer the reader to [18, 19], where numerical simulations show the stabilizing performance of such a feedback for equations evolving in a spatial nonrectangular domain.

3.2. **On the nonlinearity**

We gather key properties of the nonlinear operator in (3.3).

$$\mathcal{N}_k(z) \in \mathcal{C}(L^2, L^2), \qquad \mathcal{N}_k(z) \coloneqq k\left((z + y_k - \psi)^+ - (y_k - \psi)^+\right). \tag{3.9}$$

Lemma 3.2. *The nonlinear operator* (3.9) *is bounded, as*

$$\left|\mathcal{N}_k(z_1) - \mathcal{N}_k(z_2)\right|_{L^2} \le k\left|z_1 - z_2\right|_{L^2}, \quad \text{for all} \quad (z_1, z_2) \in L^2 \times L^2.$$

Proof. With $(z_1, z_2) \in L^2 \times L^2$, we find that

$$\mathcal{N}_k(z_1) - \mathcal{N}_k(z_2) = k\left((z_1 + y_k - \psi)^+ - (z_2 + y_k - \psi)^+\right). \tag{3.10}$$

Note that $h \mapsto h^+ = \max(h, 0)$ is a globally Lipschitz continuous functions with unitary Lipschitz constant, and thus $|h_1^+ - h_2^+|_{L^2} \le |h_1 - h_2|_{L^2}$ for all $h_1, h_2 \in L^2$. Therefore,

$$|\mathcal{N}_k(z_1) - \mathcal{N}_k(z_2)|_{L^2} \le k|(z_1 + y_k - \psi) - (z_2 + y_k - \psi)|_{L^2} = k|z_1 - z_2|_{L^2},$$

which finishes the proof. □

Lemma 3.3. *The nonlinear operator* (3.9) *is monotone,*

$$(\mathcal{N}_k(z_1) - \mathcal{N}_k(z_2), z_1 - z_2)_{L^2} \ge 0, \qquad \text{for all} \quad (z_1, z_2) \in L^2 \times L^2.$$

Proof. By (2.13), the mapping $z \mapsto G(z) := z^+$ is monotone in L^2. Hence, $z \mapsto G(z - \zeta_1) - \zeta_2$ is also monotone for arbitrary ζ_1 and ζ_2 in L^2, which finishes the proof. □

3.3. **Stabilizability result**

For simplicity, let us denote

$$\begin{aligned}
&A_{\mathrm{rc}} \coloneqq a\mathbf{1} + b \cdot \nabla, \qquad C_{\mathrm{rc}} \coloneqq |A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_+, \mathcal{L}(V, L^2))}, \\
&\mathcal{K}_M^\lambda \coloneqq -\lambda P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp} A P_{\mathcal{E}_M}^{\mathcal{U}_M^\perp}.
\end{aligned} \tag{3.11}$$

Theorem 3.4. *Let Assumptions 2.3–2.6 hold true, with $\mathcal{B} = a\mathbf{1}$. Let the sequence $(\mathcal{U}_M, \mathcal{E}_M)_{M\in\mathbb{N}}$ be constructed as in Section 3.1. Then, for every given $\mu > 0$, there are large enough constants $\lambda > 0$ and $M \in \mathbb{N}$ such that, for every $k \in \mathbb{N}$, the system*

$$\dot{z}_k + Az_k + A_{\mathrm{rc}} z_k + \mathcal{N}_k(z_k) = \mathcal{K}_M^\lambda z_k, \qquad z_k(0) = z_\circ, \tag{3.12[k]}$$

is exponentially stable with rate $-\mu$. For all $z_\circ \in L^2$, the solution satisfies

$$|z_k(t)|_{L^2} \le \mathrm{e}^{-\mu(t-s)}\, |z_k(s)|_{L^2}, \qquad t \ge s \ge 0. \tag{3.13}$$

Moreover, the feedback operator $\mathcal{K}_M^\lambda$ and control input $\mathcal{K}_M^\lambda z_k$ satisfy the estimate

$$\left|\mathcal{K}_M^\lambda\right|_{\mathcal{L}(L^2)} \le \lambda\widehat{\alpha}_M C_P^2 \quad \text{and} \quad \left|\mathcal{K}_M^\lambda z_k\right|_{L^2(\mathbb{R}_+, L^2)} \le \lambda\widehat{\alpha}_M \mu^{-1} C_P^2\, |z_\circ|_{L^2}, \tag{3.14}$$

where $\widehat{\alpha}_M$ and C_P are as in (3.8)*. Furthermore, we can choose*

$$\lambda \sim \overline{C}_{[\mu,C_{\mathrm{rc}}]} \quad \textit{and} \quad M \sim \overline{C}_{[\mu,C_{\mathrm{rc}}]}. \tag{3.15}$$

Remark 3.5. Note that the feedback operator $\mathcal{K}_M^\lambda$ in (3.11) is independent of (k,ψ), because (λ, M) in (3.15) can be chosen independently of (k,ψ). The upper bound in (3.14) for the norm of the control input $\mathcal{K}_M^\lambda z_k$ is also independent of (k,ψ). The monotonicity stated in Lemma 3.3 plays a key role on such independences on k.

Remark 3.6. Inequality (3.13) implies that $t \mapsto |z_k(t)|_{L^2}^2$ is strictly decreasing at time $t = s$, if $|z_k(s)|_{L^2}^2 > 0$. Of course, if $|z_k(s)|_{L^2}^2 = 0$ then $|z_k(t)|_{L^2}^2 = 0$ for all $t \geq 0$, see [27, Sect. 4].

Proof of Theorem 3.4. Following the arguments in [27, Sect. 4], we decompose the solution of system (3.12[k]) into oblique components as

$$z_k = \theta_k + \Theta_k, \qquad \text{with} \qquad \theta_k \coloneqq P_{\mathcal{E}_M}^{\mathcal{U}_M^\perp} z_k \quad \text{and} \quad \Theta_k \coloneqq P_{\mathcal{U}_M^\perp}^{\mathcal{E}_M} z_k.$$

Observe that form (3.12[k]), Lemma 3.3, and the Young inequality, we obtain that

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} |z_k|_{L^2}^2 &= -2\,|z_k|_V^2 - 2\langle A_{\mathrm{rc}} z_k, z_k\rangle_{V',V} - 2\left(\mathcal{N}_k(z_k), z_k\right)_{L^2} + 2\left(\mathcal{K}_M^\lambda z_k, z_k\right)_{L^2} && (3.16)\\
&\leq -2\,|z_k|_V^2 - 2\langle A_{\mathrm{rc}} z_k, z_k\rangle_{V',V} - 2\lambda\left(A\theta_k, \theta_k\right)_{L^2} && (3.17)\\
&\leq -2\,|z_k|_V^2 + \gamma_1\,|z_k|_V^2 + \gamma_1^{-1} C_{\mathrm{rc}}^2\,|z_k|_{L^2}^2 - 2\lambda\,|\theta_k|_V^2\,,\\
&\leq -(2-\gamma_1)\,|z_k|_V^2 + \gamma_1^{-1} C_{\mathrm{rc}}^2\,|z_k|_{L^2}^2 - 2\lambda\,|\theta_k|_V^2\,, \quad \text{for all} \quad \gamma_1 > 0. && (3.18)
\end{aligned}$$

Now we observe that, by the young inequality, we obtain for all $\gamma_2 > 0$

$$\begin{aligned}
-\,|z_k|_V^2 &= -\,|\Theta_k + \theta_k|_V^2 = -\,|\Theta_k|_V^2 - |\theta_k|_V^2 - 2(\Theta_k, \theta_k)_V\\
&\leq -\,|\Theta_k|_V^2 - |\theta_k|_V^2 + \gamma_2\,|\Theta_k|_V^2 + \gamma_2^{-1}\,|\theta_k|_V^2 = -(1-\gamma_2)\,|\Theta_k|_V^2 - (1-\gamma_2^{-1})\,|\theta_k|_V^2\,. && (3.19)
\end{aligned}$$

Combining (3.18) and (3.19) we obtain, for all $(\gamma_1, \gamma_2) \in (0,2) \times \mathbb{R}_+$,

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} |z_k|_{L^2}^2 &\leq -(2-\gamma_1)(1-\gamma_2)\,|\Theta_k|_V^2 - \left(2\lambda + (2-\gamma_1)(1-\gamma_2^{-1})\right)|\theta_k|_V^2 + \gamma_1^{-1} C_{\mathrm{rc}}^2\,|z_k|_{L^2}^2\\
&\leq -(2-\gamma_1)(1-\gamma_2)\,|\Theta_k|_V^2 - \left(2\lambda - (2-\gamma_1)(\gamma_2^{-1}-1)\right)|\theta_k|_V^2 + 2\gamma_1^{-1} C_{\mathrm{rc}}^2 (|\Theta_k|_{L^2}^2 + |\theta_k|_{L^2}^2)
\end{aligned}$$

Now, we can choose $\gamma_1 \in (0,2)$ and $\gamma_2 \in (0,1)$, and λ satisfying $2\lambda - (2-\gamma_1)(\gamma_2^{-1}-1) > 0$. For such choices, using (3.8), we find

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} |z_k|_{L^2}^2 &\leq -(2-\gamma_1)(1-\gamma_2)\beta_{M_+}\,|\Theta_k|_{L^2}^2 - \left(2\lambda - (2-\gamma_1)(\gamma_2^{-1}-1)\right)\alpha_1\,|\theta_k|_{L^2}^2\\
&\qquad + 2\gamma_1^{-1} C_{\mathrm{rc}}^2 (|\Theta_k|_{L^2}^2 + |\theta_k|_{L^2}^2)\\
&\leq -\Xi_1(M)\,|\Theta_k|_V^2 - \Xi_2(M)\,|\theta_k|_V^2\,, && (3.20)
\end{aligned}$$

where $\alpha_1 \coloneqq \min\left\{ \frac{|h|_V}{|h|_{L^2}} \;\middle|\; h \in V \setminus \{0\} \right\}$, and

$$\Xi_1(M) \coloneqq (2-\gamma_1)(1-\gamma_2)\beta_{M_+} - 2\gamma_1^{-1} C_{\mathrm{rc}}^2, \tag{3.21a}$$

$$\Xi_2(\lambda) \coloneqq \left(2\lambda - (2-\gamma_1)(\gamma_2^{-1}-1)\right)\alpha_1 - 2\gamma_1^{-1} C_{\mathrm{rc}}^2. \tag{3.21b}$$

Recall that, due to (3.8) we have that $\lim\limits_{M\to+\infty}\beta_{M_+}=+\infty$. Let us be given an arbitrary given $\mu>0$ and let us choose γ_1 and γ_2 as above, satisfying

$$\gamma_1\in(0,2)\quad\text{and}\quad\gamma_2\in(0,1). \tag{3.22a}$$

Then, subsequently we can choose $\lambda>0$ and $M\in\mathbb{N}$ large enough satisfying

$$2\lambda-(2-\gamma_1)(\gamma_2^{-1}-1)>0,\quad \Xi_2(\lambda)\ge 4\mu,\quad\text{and}\quad \Xi_1(M)\ge 4\mu. \tag{3.22b}$$

From (3.20), with the choices in (3.22), we arrive at

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left|z_k\right|_{L^2}^2\le -4\mu\left(\left|\Theta_k\right|_{L^2}^2+\left|\theta_k\right|_{L^2}^2\right)\le -2\mu\left|z_k\right|_{L^2}^2, \tag{3.23}$$

which implies (3.13).

It remains to show the boundedness of the feedback control, with $(\gamma_1,\gamma_2,\lambda,M)$ as in (3.22).

We see that $P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp}=P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp}P_{\mathcal{E}_M}$, because $P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp}h=P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp}(P_{\mathcal{E}_M}h+P_{\mathcal{E}_M^\perp}h)=P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp}P_{\mathcal{E}_M}h$, for all $h\in L^2$. Here $P_{\mathcal{E}_M}:=P_{\mathcal{E}_M}^{\mathcal{E}_M\perp}$ stands for the orthogonal projection in L^2 onto $\mathcal{E}_M$. Using (3.13) we obtain that the feedback operator $\mathcal{K}_M^\lambda$ satisfies

$$\begin{aligned}\left|\mathcal{K}_M^\lambda\right|_{\mathcal{L}(L^2)}&=\lambda\left|P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp}AP_{\mathcal{E}_M}^{\mathcal{U}_M^\perp}\right|_{\mathcal{L}(L^2)}=\lambda\left|P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp}P_{\mathcal{E}_M}AP_{\mathcal{E}_M}P_{\mathcal{E}_M}^{\mathcal{U}_M^\perp}\right|_{\mathcal{L}(L^2)}\\&\le\lambda\left|P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp}\right|_{\mathcal{L}(L^2)}\left|P_{\mathcal{E}_M}AP_{\mathcal{E}_M}\right|_{\mathcal{L}(L^2)}\left|P_{\mathcal{E}_M}^{\mathcal{U}_M^\perp}\right|_{\mathcal{L}(L^2)}\le\lambda\widehat{\alpha}_M\left|P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp}\right|_{\mathcal{L}(L^2)}^2\end{aligned} \tag{3.24a}$$

and corresponding control $\mathcal{K}_M^\lambda z_k$

$$\begin{aligned}\left|\mathcal{K}_M^\lambda z_k\right|_{L^2(\mathbb{R}_+,L^2)}&\le\lambda\widehat{\alpha}_M\left|P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp}\right|_{\mathcal{L}(L^2)}^2\left|z_k\right|_{L^2(\mathbb{R}_+,L^2)}\le\lambda\widehat{\alpha}_M\left|P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp}\right|_{\mathcal{L}(L^2)}^2\left|z_\circ\right|_{L^2}\int_0^{+\infty}\mathrm{e}^{-\mu t}\,\mathrm{d}t\\&=\lambda\widehat{\alpha}_M\mu^{-1}\left|P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp}\right|_{\mathcal{L}(L^2)}^2\left|z_\circ\right|_{L^2},\end{aligned} \tag{3.24b}$$

where $\widehat{\alpha}_M$ is as in (3.8). Finally, with C_P is as in (3.8), we also obtain the bounds

$$\left|\mathcal{K}_M^\lambda\right|_{\mathcal{L}(L^2)}\le\lambda\widehat{\alpha}_MC_P^2,\quad\text{and}\quad\left|\mathcal{K}_M^\lambda z_k\right|_{L^2(\mathbb{R}_+,L^2)}\le\lambda\widehat{\alpha}_M\mu^{-1}C_P^2\left|z_\circ\right|_{L^2}. \tag{3.25}$$

The proof is finished. □

4. Stabilization of the variational inequality

Here we prove the main result, which we can write now in a more precise form as follows.

Theorem 4.1. *Let Assumptions 2.3–2.6 hold true, let $\mu>0$, and let the pairs $(\mathcal{U}_M,\mathcal{E}_M)$ be constructed as in Section 3.1. Further let $y\in W_{\mathrm{loc}}(\mathbb{R}_+;H^1,V')$ with $ry\in W_{\mathrm{loc}}(\mathbb{R}_+;H^2,L^2)$ solve* (1.1). *Then for M and λ large enough the solution w of system* (1.8) *satisfies*

$$\left|w(t)-y(t)\right|_{L^2}\le\mathrm{e}^{-\mu t}\left|w_\circ-y_\circ\right|_{L^2},\quad t\ge 0. \tag{4.1}$$

Furthermore, with $\widehat{\alpha}_M$ *and* C_P *as in* (3.8) *the control satisfies*

$$\left|\mathcal{K}_M^\lambda\right|_{\mathcal{L}(L^2)} \le \lambda\widehat{\alpha}_M C_P^2 \quad \textit{and} \quad \left|\mathcal{K}_M^\lambda(w-y)\right|_{L^2(\mathbb{R}^+,L^2)} \le \lambda\widehat{\alpha}_M\mu^{-1}C_P^2\left|w_\circ - y_\circ\right|_{L^2}, \tag{4.2}$$

Proof. Let us fix $\lambda > 0$ and $M \in \mathbb{N}$ so that Theorem 3.4 holds true. Note that $\lambda > 0$ and $M \in \mathbb{N}$ are independent of k.

Let y_k and w_k be the solutions of the Moreau–Yosida approximations (3.1) and (3.2), respectively. For the difference between the solution w of (1.8) and the solution y of (1.1) we find

$$|w(t)-y(t)|_{L^2} \le |w(t)-w_k(t)|_{L^2} + |w_k(t)-y_k(t)|_{L^2} + |y_k(t)-y(t)|_{L^2} \tag{4.3a}$$

Let us now be given arbitrary $\epsilon > 0$, $\varrho > 1$, $T > 0$, and $t \in [0,T]$.

Now for the pair (y_k, y) we apply Theorem 2.9 with $(f_k, Q) = (f, a\mathbf{1} + b\cdot\nabla)$, and for the pair (w_k, w) we apply Theorem 2.9 with $(f_k, Q) = (f + \mathcal{K}_M^\lambda y_k, a\mathbf{1} + b\cdot\nabla + \mathcal{K}_M^\lambda)$. In this way we obtain that, for large enough $k = k(\epsilon, T)$, we have

$$|r(y_k - y)|_{\mathcal{C}([0,T],L^2)} \le \epsilon \quad \text{and} \quad |r(w_k - w)|_{\mathcal{C}([0,T],L^2)} \le \epsilon, \quad \text{with} \quad r(t) = \min\{t,1\}. \tag{4.3b}$$

and, since $z_k \coloneqq w_k - y_k$ satisfies (3.3), that is (3.12[k]), by using Theorem 3.4, we obtain

$$|w_k(t) - y_k(t)|_{L^2} \le \mathrm{e}^{-\mu t}\left|w_\circ - y_\circ\right|_{L^2}, \quad \text{for every } k \in \mathbb{N}. \tag{4.3c}$$

Hence, by selecting k large enough, from (4.3) we obtain that, at time $t = T > 0$,

$$|w(T) - y(T)|_{L^2} \le 2\max\{\tfrac{1}{T}, 1\}\epsilon + \mathrm{e}^{-\mu T}\left|w_\circ - y_\circ\right|_{L^2}.$$

Choosing now $\epsilon \coloneqq \frac{1}{2}\min\{T,1\}(\varrho - 1)\mathrm{e}^{-\mu T}\left|w_\circ - y_\circ\right|_{L^2}$, we arrive at

$$|w(T) - y(T)|_{L^2} \le (\varrho - 1)\mathrm{e}^{-\mu T}\left|w_\circ - y_\circ\right|_{L^2} + \mathrm{e}^{-\mu T}\left|w_\circ - y_\circ\right|_{L^2} = \varrho\mathrm{e}^{-\mu T}\left|w_\circ - y_\circ\right|_{L^2}.$$

Furthermore, since $T > 0$ and $\varrho > 1$ are arbitrary we arrive at

$$|w(t) - y(t)|_{L^2} \le \mathrm{e}^{-\mu t}\left|w_\circ - y_\circ\right|_{L^2}, \quad t \ge 0.$$

Finally proceeding as in (3.24), we find

$$\left|\mathcal{K}_M^\lambda(w-p)\right|_{L^2(\mathbb{R}^+,L^2)} \le \lambda\widehat{\alpha}_M\left|P_{\mathcal{U}_M}^{\mathcal{E}_M^\perp}\right|_{\mathcal{L}(L^2)}^2 |w-p|_{L^2(\mathbb{R}^+,L^2)} \le \lambda\widehat{\alpha}_M\mu^{-1}C_P^2\left|w_\circ - y_\circ\right|_{L^2},$$

with $\widehat{\alpha}_M$ and C_P as in (3.8), which finishes the proof. □

5. NUMERICAL SIMULATIONS

For simplicity, we restrict ourselves to one-dimensional PVIs in the spatial open interval $\Omega = (0,1) \subset \mathbb{R}$, and to the case of homogeneous Neumann boundary conditions. Namely, we consider the Moreau–Yosida approximations

$$\tfrac{\partial}{\partial t}y_k + (-\nu\Delta + \mathbf{1})y_k + ay_k + b\cdot\nabla y_k - f + k(y_k - \psi)^+ = 0, \quad t > 0, \tag{5.1a}$$

$$\tfrac{\partial}{\partial \mathbf{n}}y_k|_\Gamma = 0, \qquad y_k(\cdot, 0) = y_\circ. \tag{5.1b}$$

For the parameters, we have chosen

$$\nu = 0.1, \qquad f(x,t) = -\sin(t)x, \tag{5.2a}$$
$$a(x,t) = -6 + x + 2\left|\sin(t+x)\right|_{\mathbb{R}}, \qquad b(x,t) = \cos(t)x^2 \tag{5.2b}$$

and
$$\psi(x,t) = 2 + \cos(t) + \cos\left(10\pi x(x-1)(x - \tfrac{1}{4}\cos(5t))\right). \tag{5.2c}$$
Recall that by Theorem 2.9, we have that y_k gives us an approximation of the solution y of the PVI with the same data parameters. See also Remark 1.1.

The targeted trajectory y is the one issued, at initial time $t = 0$, from the state

$$y(x,0) = y_\circ(x) = 3\cos(\pi x), \tag{5.3}$$

and we want to target such trajectory starting, again at time $t = 0$, from the state

$$w(x,0) = w_\circ(x) = -1. \tag{5.4}$$

Again by Theorem 2.9, we have that w_k solving

$$\tfrac{\partial}{\partial t} w_k + (-\nu\Delta + \mathbf{1})w_k + a w_k + b \cdot \nabla w_k - f - \mathcal{K}_M^\lambda(w_k - y_k) + k(w_k - \psi)^+ = 0, \quad t > 0, \tag{5.5a}$$
$$\tfrac{\partial}{\partial \mathbf{n}} w_k|_\Gamma = 0, \qquad w_k(\cdot, 0) = w_\circ, \tag{5.5b}$$

gives us an approximation of the solution w of the controlled PVI with the same data parameters.

Initial states are plotted in Figure 1.

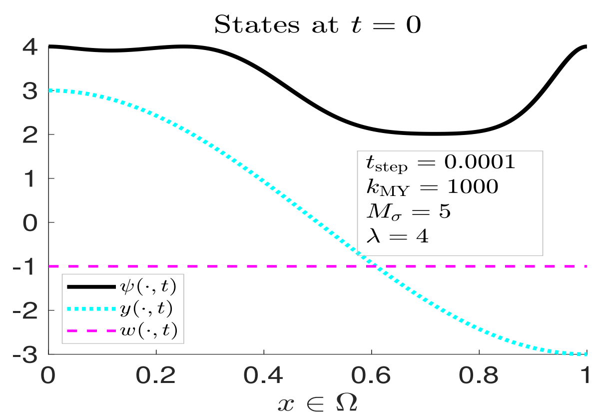

FIGURE 1. Initial states.

For a fixed $M \in \mathbb{N}$ we take $M_\sigma = M$ actuators as in [18] which are indicator functions $1_{\omega_j^M}$ of the subdomains
$$\omega_j^M = \left(\tfrac{2j-1}{2M} - \tfrac{1}{20M}, \tfrac{2j-1}{2M} + \tfrac{1}{20M}\right), \qquad j \in \{1, 2, \dots, M\}.$$
In particular, note that the total volume covered by the actuators is independent of M. It is given by $\frac{1}{10}$, which is 10% of the total volume of the spatial domain.

As auxiliary space of eigenfunctions we take the first eigenfunctions of the Laplace operator, under the imposed Neumann boundary conditions, namely

$$e_j^M = \cos((j-1)\pi x), \qquad j \in \{1, 2, \dots, M\}.$$

The obstacle $\psi(\cdot, t)$ satisfies $\frac{\partial}{\partial \mathbf{n}}\psi = 0$ at every $t \geq 0$. Recall that our Assumption 2.6 requires that $\frac{\partial}{\partial \mathbf{n}}\psi \geq -\eta$ for a suitable positive function $-\eta \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}_+) \geq 0$ hence it is satisfied.

Furthermore, we can see that Assumptions 2.3–2.6 are satisfied. Therefore all the hypotheses of Theorems 3.4 are satisfied. Hereafter we present the results of simulations illustrating the stability result stated in the thesis of Theorem 3.4.

As we have mentioned above, by solving systems (5.1) and (5.5), by Theorem 4.1, with a relatively large Moreau–Yosida parameter $k = k_{MY}$ we expect to obtain a relatively good approximation of the behavior of the limit solutions for the corresponding PVIs. Depending on the simulation example, we have taken k_{MY} in the interval $[500, 20000]$.

For the discretization, we considered a finite element spatial approximation based on the classical piecewise linear hat functions, where the closure $[0, 1]$ of the spatial interval has been discretized with a regular mesh with 2001 equidistant points. Subsequently the closure $[0, +\infty)$ of the temporal interval has been discretized with a uniform time-step $t_{\mathrm{step}} > 0$ and a Crank–Nicolson/Adams–Bashforth scheme was used. Depending on the simulation we have taken $t_{\mathrm{step}} \in \{10^{-4}, 10^{-5}\}$.

In the figures below we denote $H := L^2(\Omega)$.

5.1. Stabilizing performance of the feedback control

In Figure 2 we can see that with 5 actuators and $\lambda = 4$ the explicit oblique projection feedback

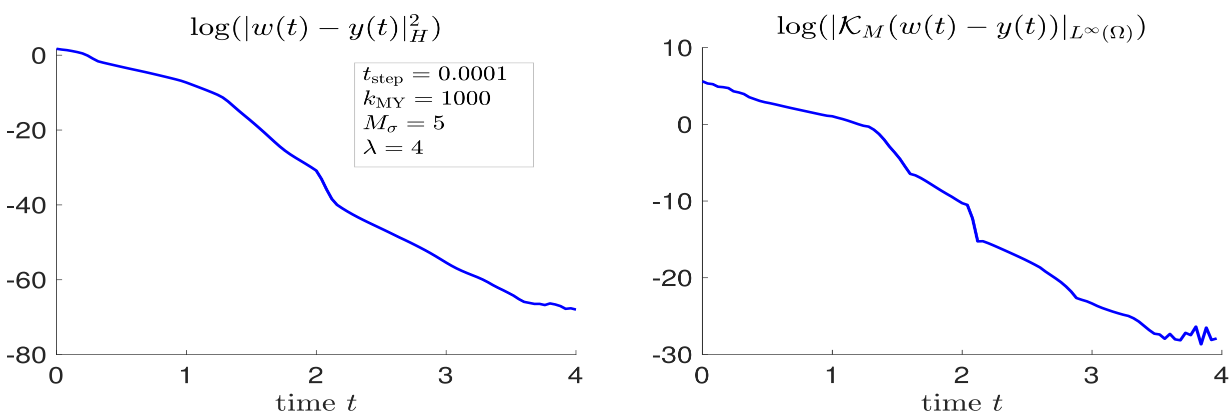

FIGURE 2. Norms of difference to target and control.

control we propose in this manuscript is able to stabilize the solution $w = w_k$ of the Moreau–Yosida approximation, with $k = k_{\mathrm{MY}} = 1000$, to the corresponding targeted uncontrolled solution approximation $y = y_k$. The logarithm in the figures stand for the natural logarithm, $\log(\mathrm{e}^s) := s$, $s \in \mathbb{R}$ (where e is the Napier's constant, Euler's number).

Time snapshots of the corresponding trajectories and control are shown in Figures 3. It is interesting to observe, at time $t = 0.05$, the 5 bumps on the shape of the controlled solution, which are pointing towards the targeted one. The spatial location of these bumps coincide with spatial location of the

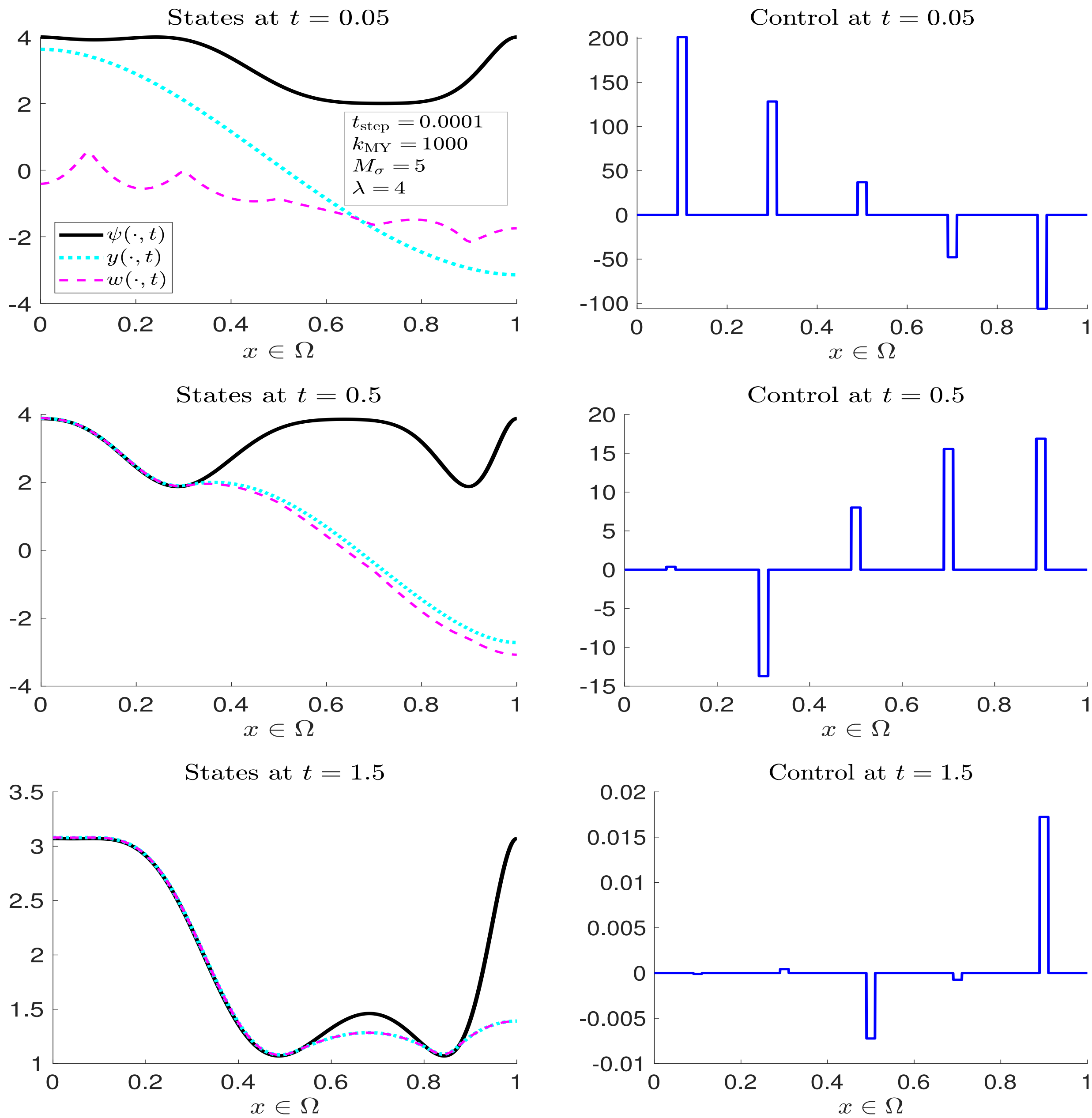

Figure 3. Time snapshots of trajectories and control. Larger time

actuators, and they show the action of the feedback control pushing the controlled solution towards the targeted one.

5.2. **On the Moreau–Yosida parameter** $k_{\rm MY}$

The goal of this section is to show that it is very likely that the Moreau–Yosida approximation with parameter $k_{\rm MY} = 500$ in the above simulation give us already a good approximation of the behavior of the limit solution of the PVI. Indeed, in Figure 4, we can see that the norm of the difference to the target presents an analogous evolution for the considered parameters $k_{\mathrm{MY}}s$.

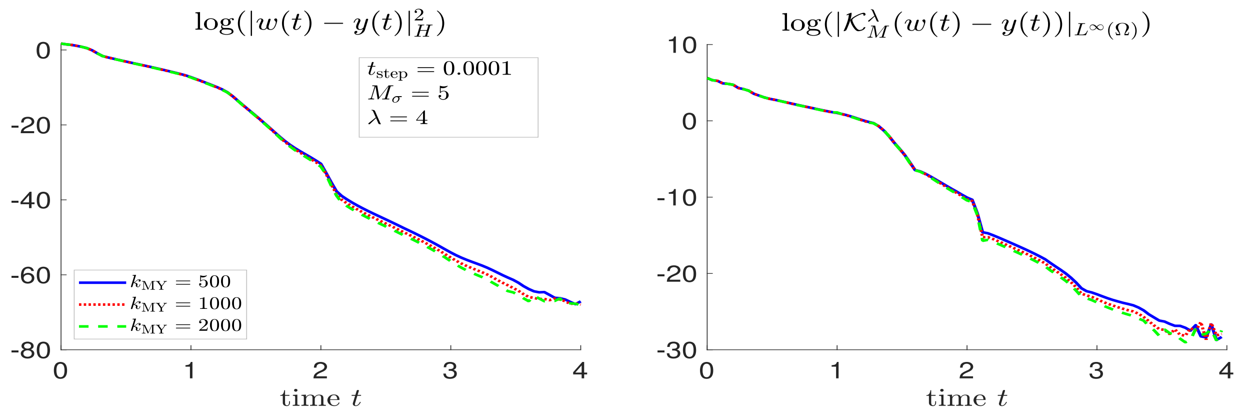

FIGURE 4. Norms of difference to target and control

In Figure 5 we see that the obstacle constraint violation decreases as k_{MY} increases, as we expect,

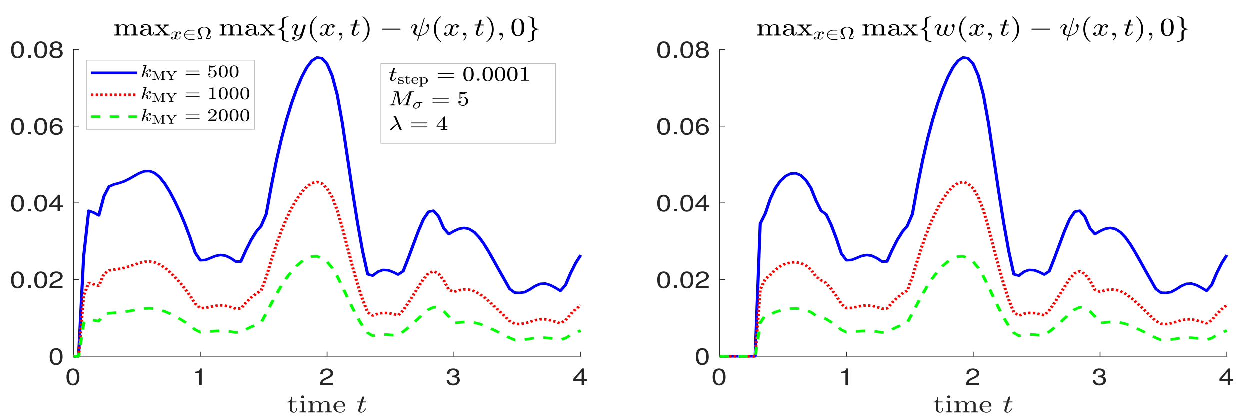

FIGURE 5. Largest magnitude of obstacle constraint violation

since at the limit we must have a vanishing constraint violation. Furthermore, from Lemma 2.14 we have that $k\,|(y_k - \psi)^+|_{L^2(\Omega\times(0,T))} \le C$ for a suitable constant C independent of k. Figure 5 shows that the violation decreases (at each instant of time) as k increases.

In Figure 6 we see a time snapshot of the controlled trajectories and control, where we see a small difference between the controlled trajectories for the several k_{MY}s. A similar behavior was observed for the corresponding targeted trajectories, for simplicity we plotted only the targeted trajectory y corresponding to $k_{MY} = 500$ (which, at that instant of time, is already almost indistinguishable form the controlled states with the naked eye).

5.3. Necessity of both large M and large λ

From our result, for stability it is *sufficient* to take large M and large λ. Here, we present simulations showing that such condition is also *necessary*.

5.3.1. *Necessity of large enough M*

In Figure 7 we see that with a single actuator we cannot stabilize the system, even for the relatively large $\lambda = 50$. Furthermore, for small time we cannot see a considerable change in the norm of the difference to the target for the several λs. This allow us to extrapolate that one actuator is not enough to stabilize the system.

In Figure 8 we present time snapshots of trajectories and control. We see that by taking a larger λ we cannot see a strong enough influence on the evolution of the trajectory to expect (or, hope for) a stabilization effect for large values of λ.

5.3.2. *Necessity of large enough λ*

In Figure 9 we see that with $\lambda = 1$ we cannot stabilize the system, even if we take 20 actuators. Furthermore, for small time we cannot see a considerable change in the norm of the difference to the target for the several M_σs. This allow us to extrapolate that it is necessary to take $\lambda > 1$ if we want to stabilize the system.

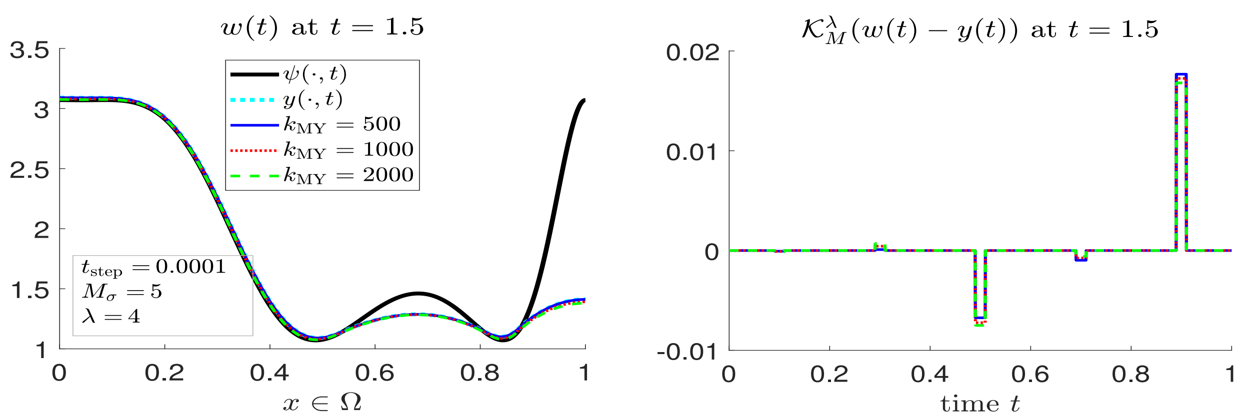

FIGURE 6. Time snapshots of trajectories and controls

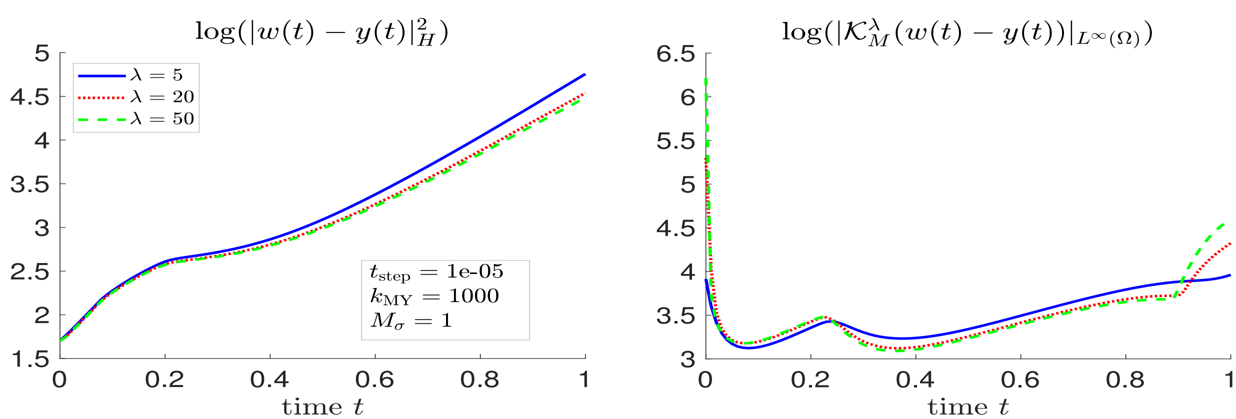

FIGURE 7. Norms of difference to targeted state and of control

In Figure 10 we present time snapshots of trajectories and control. We see that with 10 and 20 actuators we cannot see a strong enough change on the evolution of the trajectory to expect (or, hope for) a stabilization effect for large values of M_σ.

5.3.3. *On the achievement of an arbitrarily small exponential decreasing rate* $-\mu < 0$

From our result we can reach an arbitrarily small exponential decreasing rate $-\mu$, provided we take both M_σ and λ large enough. This is shown in Figure 11, where we see that with $(M_\sigma, \lambda) = (10, 6)$ we obtain a smaller exponential rate than with $(M_\sigma, \lambda) = (4, 3)$. We also observe that with $(M_\sigma, \lambda) = (2, 2)$ we are also able to stabilize the system, however this case does not fully confirm our result, where we can also guarantee that the norm of the difference to the targeted trajectory is strictly decreasing. In the zoomed subplot, in Figure 11, we can see that for small time the norm of the difference is not strictly decreasing, for $(M_\sigma, \lambda) = (2, 2)$.

The time snapshots in Figure 12 also confirm that with a pair (M_σ, λ) with larger coordinates, we obtain a faster convergence of the controlled trajectory w to the targeted one y.

5.4. **The uncontrolled dynamics**

Here we show that the uncontrolled dynamics is unstable. That is, a control is necessary to stabilize the system to the targeted trajectory. In Figure 13 the symbol $FeedOn$ denotes the time interval where the feedback control is switched on. Thus, outside this time interval the free (uncontrolled) dynamics is followed. We see that the free dynamics is exponentially unstable, as the norm of the difference to the target *increases* exponentially when the control is switched off. On the other hand, when the control is switched on we see that such norm *decreases* exponentially, confirming again our theoretical stabilizability results.

Time snapshots in Figure 14 show again that the trajectory w corresponding to the free dynamics $FeedOn = (0, 0)$ is not approaching the targeted one y as time increases (cf. Figure 1).

5.5. **Evolution of the contact set and the Moreau–Yosida parameter**

Here, we investigate the evolution of the contact (or, active) set. In Figure 15 we see that the behavior of the norm of the difference to target and of the control is similar for the several Moreau–Yosida parameters, with some differences for time $t \geq 1.5$. So, the considered parameters give us

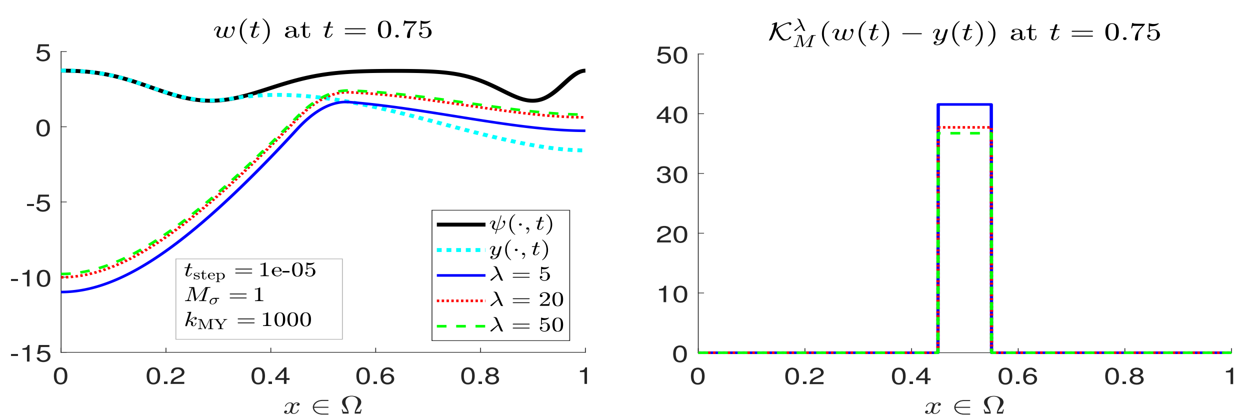

FIGURE 8. Time snapshots of trajectories and controls

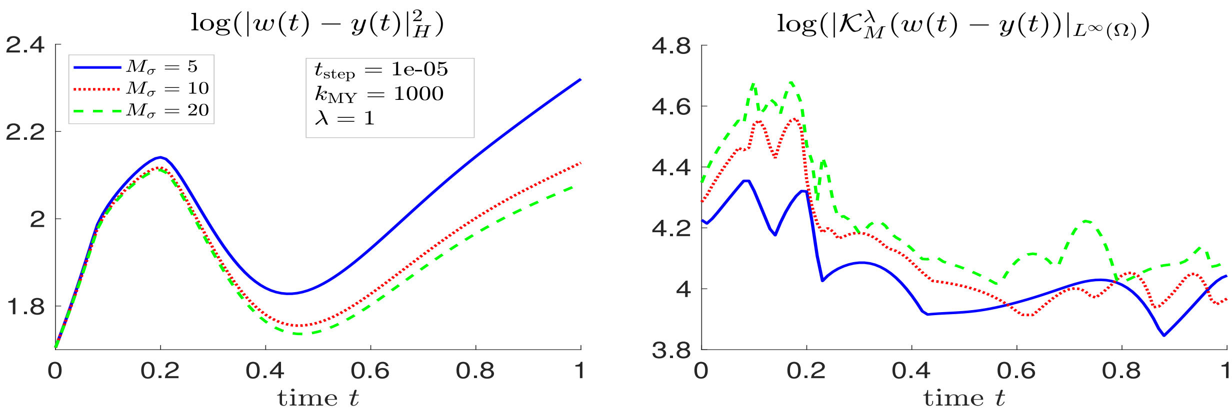

Figure 9. Norms of difference to targeted state and of control

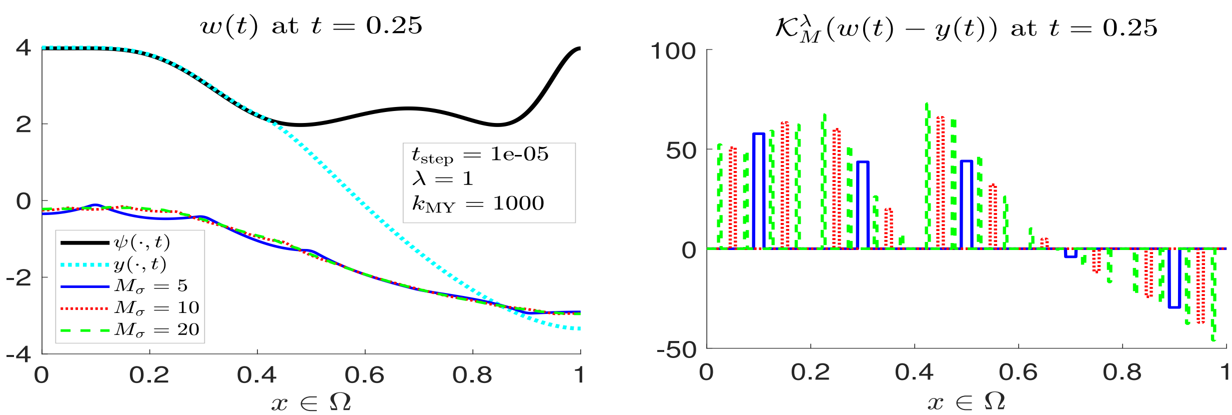

Figure 10. Time snapshots of controlled state

already a good picture of the qualitative behavior of the limit difference and control as $k_{\rm MY}$ diverges to $+\infty$.

The time snapshots in Figure 16 show that the smallest value of $k_{\rm MY}$ already captures a good picture of the likely limit behavior for the PVI.

From Figure 17 we can conjecture also that the magnitude of the violation of the obstacle constraint converges to zero as $k_{\rm MY} \to \infty$. That is, at the limit such magnitude will vanish, as we expect due to the theoretical results.

Finally, in Figures 23 and 24 we can see the evolution of the obstacle constraint violation set. It is interesting to observe that with the smallest value of $k_{\rm MY} = 5000$ considered, we can already capture a good picture of the likely limit contact set evolution for the parabolic variational inequality. The evolution is not simple, for example the number of contact connected components change with time, this can simply be explained from the fact that the moving obstacle and its shape (cf. Figure 3 and other time snapshots) are not simple themselves.

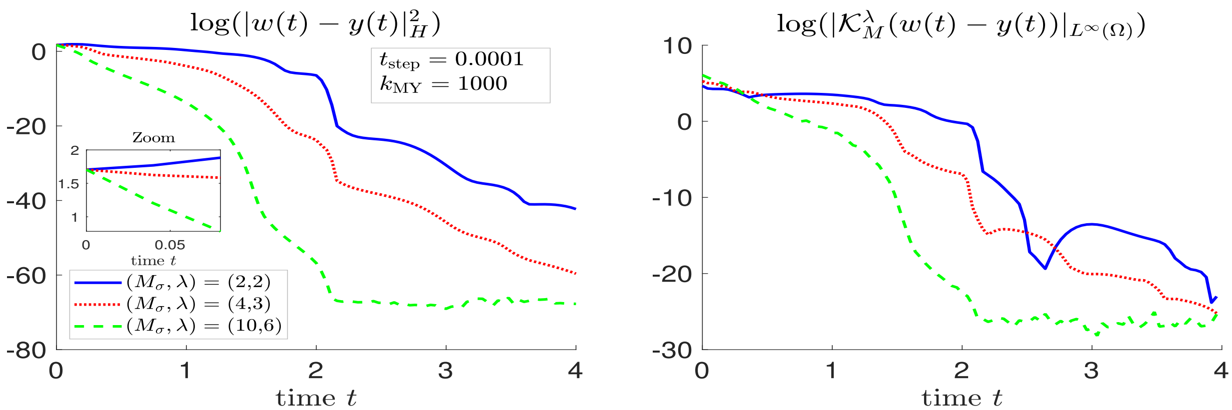

FIGURE 11. Norms of difference to targeted state and of control

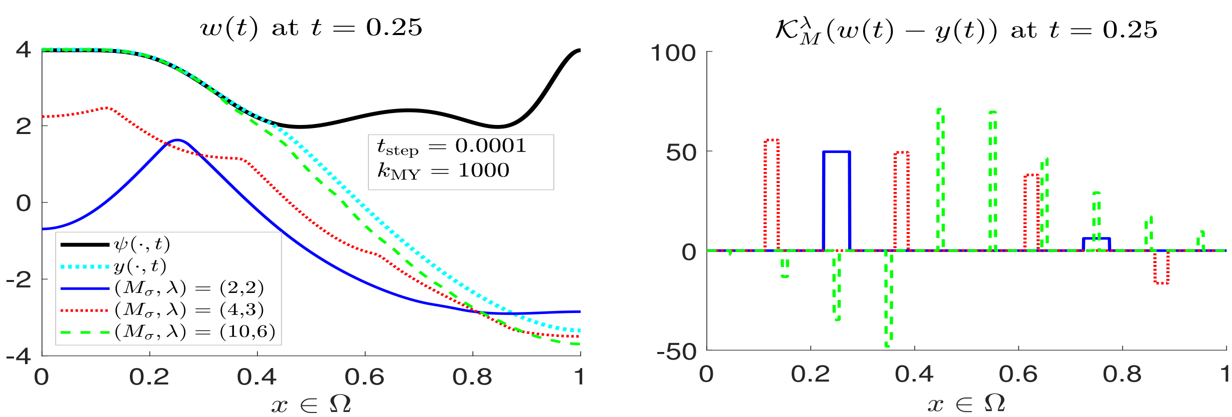

FIGURE 12. Time snapshots of trajectories and controls

6. NUMERICAL SIMULATIONS FOR A NONSMOOTH OBSTACLE. A CONJECTURE.

Note that the stability result for the sequence of k_{MY}-Moreau–Yosida approximations holds true for obstacles which live in $L^2_{\mathrm{loc}}(\Omega \times \mathbb{R}_+)$. In particular, we have a weak limit for the difference $z_k = y_k - w_k$. Thus, we may ask ourselves if y_k and w_k also converge separately and if each of these limits satisfy (a weaker formulation of) the PVI. Next, we present results of simulations which suggest that this may be indeed the case for obstacles in $\mathcal{C}^1([0, +\infty), L^2(\Omega))$. This means that our result can probably be extended to less regular obstacles. Such extension is an interesting problem for future investigation. If possible, such extension is nontrivial and thus will likely require a considerably different proof.

The following simulations correspond to the setting as in (5.2) with the exception that we take a nonsmooth obstacle. Namely, we modify the smooth obstacle in (5.2c), by changing it to constant

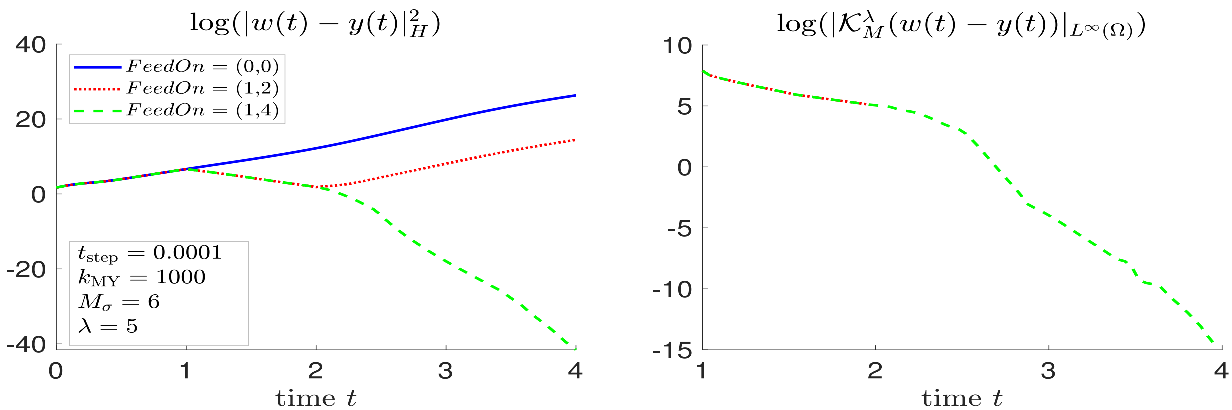

Figure 13. Norms of difference to targeted state and of control

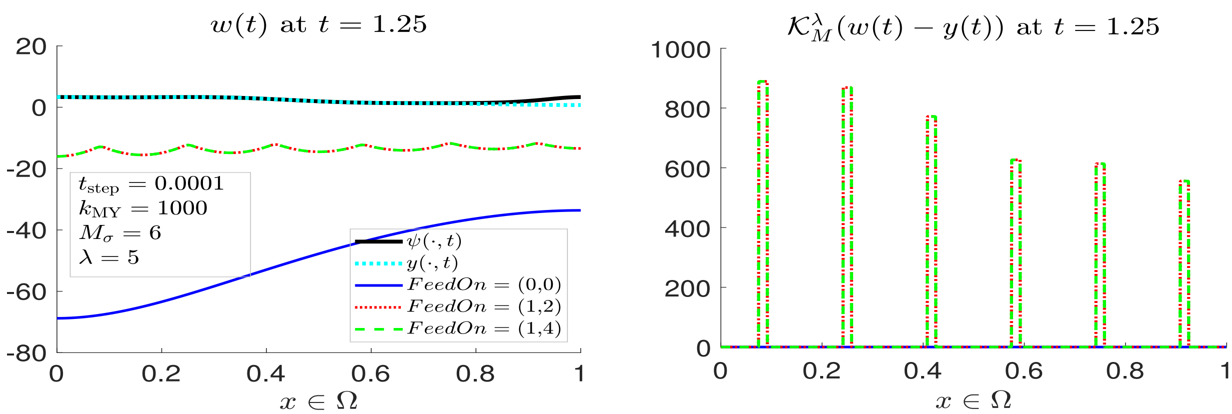

Figure 14. Time snapshots of trajectories and controls

functions on the spatial set $[0,\frac{1}{10}]\bigcup[\frac{8}{10},1]$. More precisely, we take the obstacle

$$\psi(x,t)=\begin{cases}\frac{31}{10}, & \text{if } x\in[0,\frac{1}{10}];\\ 2+\cos(t)+\cos\left(10\pi x(x-1)(x-\frac{1}{4}\cos(5t))\right), & \text{if } x\in(\frac{1}{10},\frac{8}{10});\\ -\frac{5}{10}, & \text{if } x\in[\frac{8}{10},1].\end{cases}$$

In Figure 20 we cannot see a considerable difference in the behavior of the norm of the difference to target and of the control for the several Moreau–Yosida parameters. The same holds for the time snapshots in Figure 21. So we can conjecture that the considered parameters give us already a good picture of the behavior of the limit difference and control as k_{MY} diverges to $+\infty$.

From Figure 22 we can conjecture also that the magnitude of the violation of the obstacle constraint converges to zero as $k_{\text{MY}}\to\infty$.

All the above suggest that a PVI will be satisfied at a limit. But, this remains to be proven for nonsmooth obstacles.

Finally, in Figures 23 and 24 we can see the evolution of the obstacle constraint violation sets.

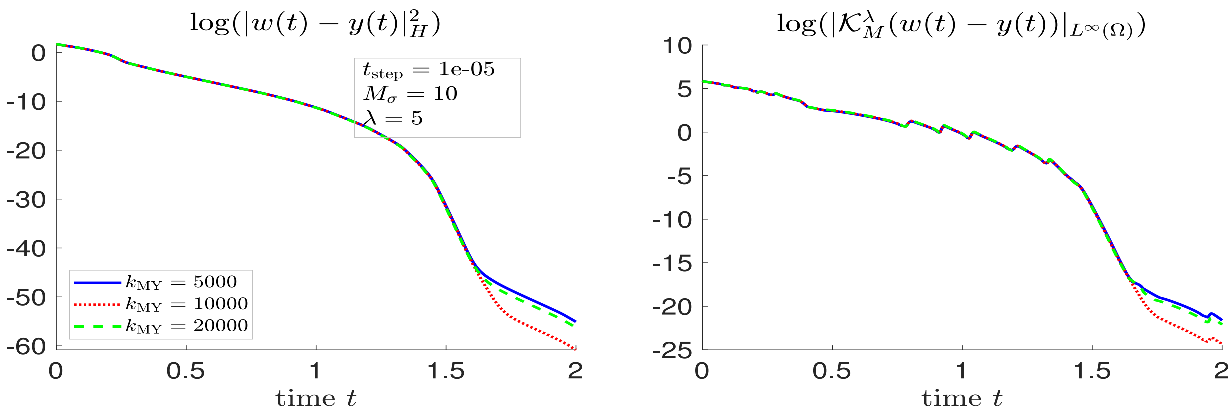

FIGURE 15. Norms of difference to target and control

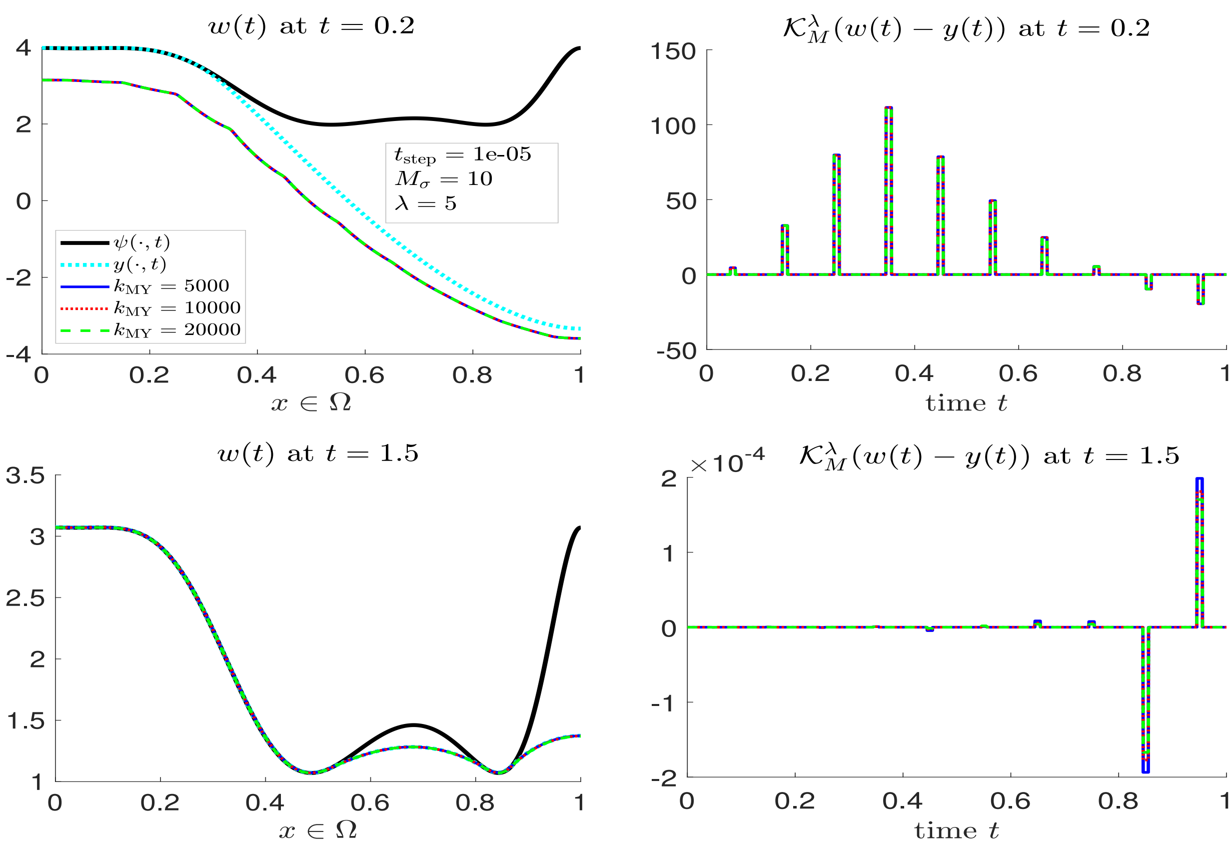

FIGURE 16. Time snapshots of trajectories and control

Again, the smallest value of k_{MY} provides us already with good picture of such evolutions. However, note that by taking the largest value we are able to "sharpen" the picture, in particular it confirms that locally the contact is made at the single (discontinuity) point $x = 0.8$ during a suitable interval

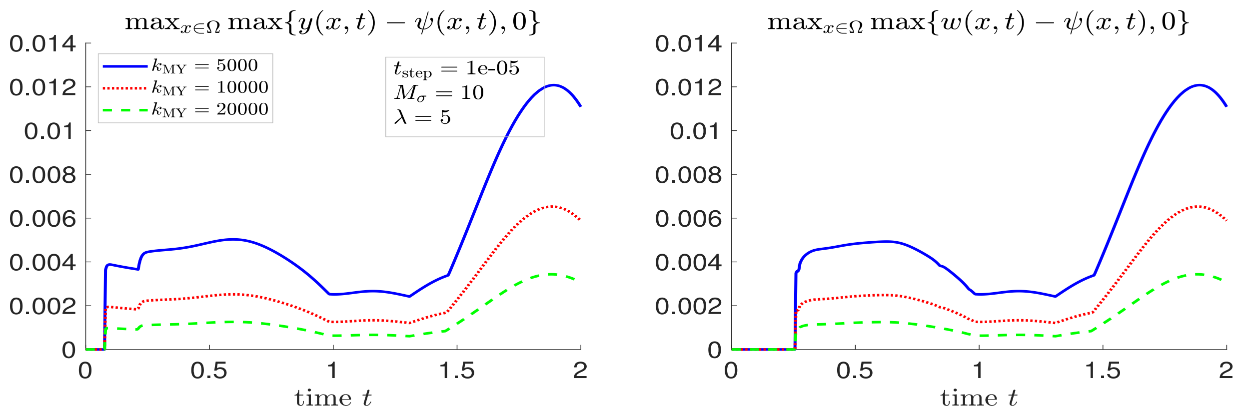

FIGURE 17. Largest magnitude of obstacle constraint violation

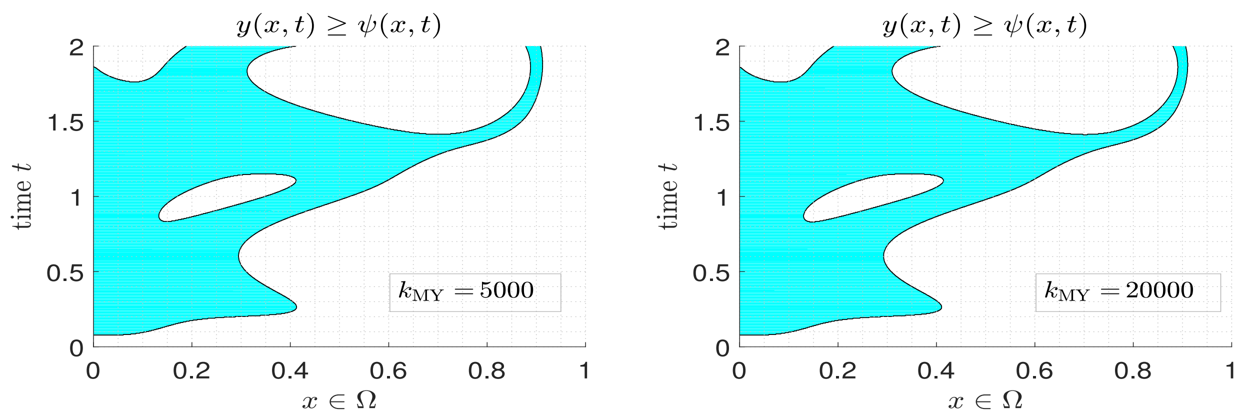

FIGURE 18. Evolution of obstacle constraint violation set for targeted trajectory

of time, where $t = 1.5$ is included, as we see in the snapshot in Figure 21. We also observe that the discontinuity of the obstacle at the spatial points $x \in \{0.1, 0.8\}$ is somehow reflected in Figures 23 and 24.

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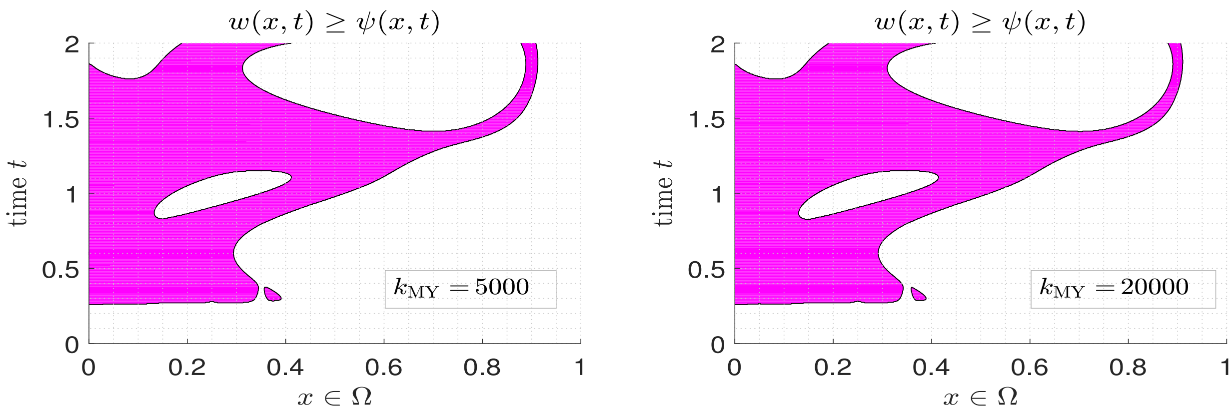

Figure 19. Evolution of obstacle constraint violation set for controlled trajectory

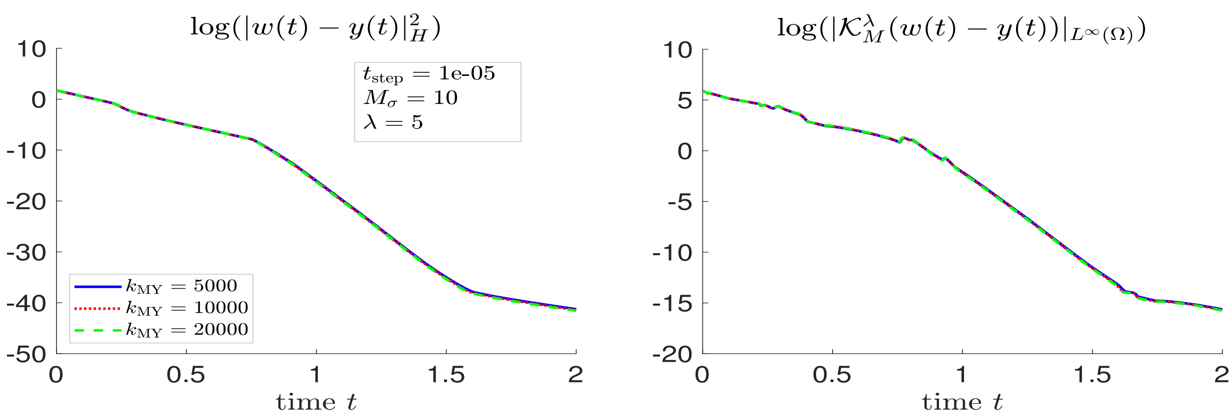

Figure 20. Norms of difference to target and control

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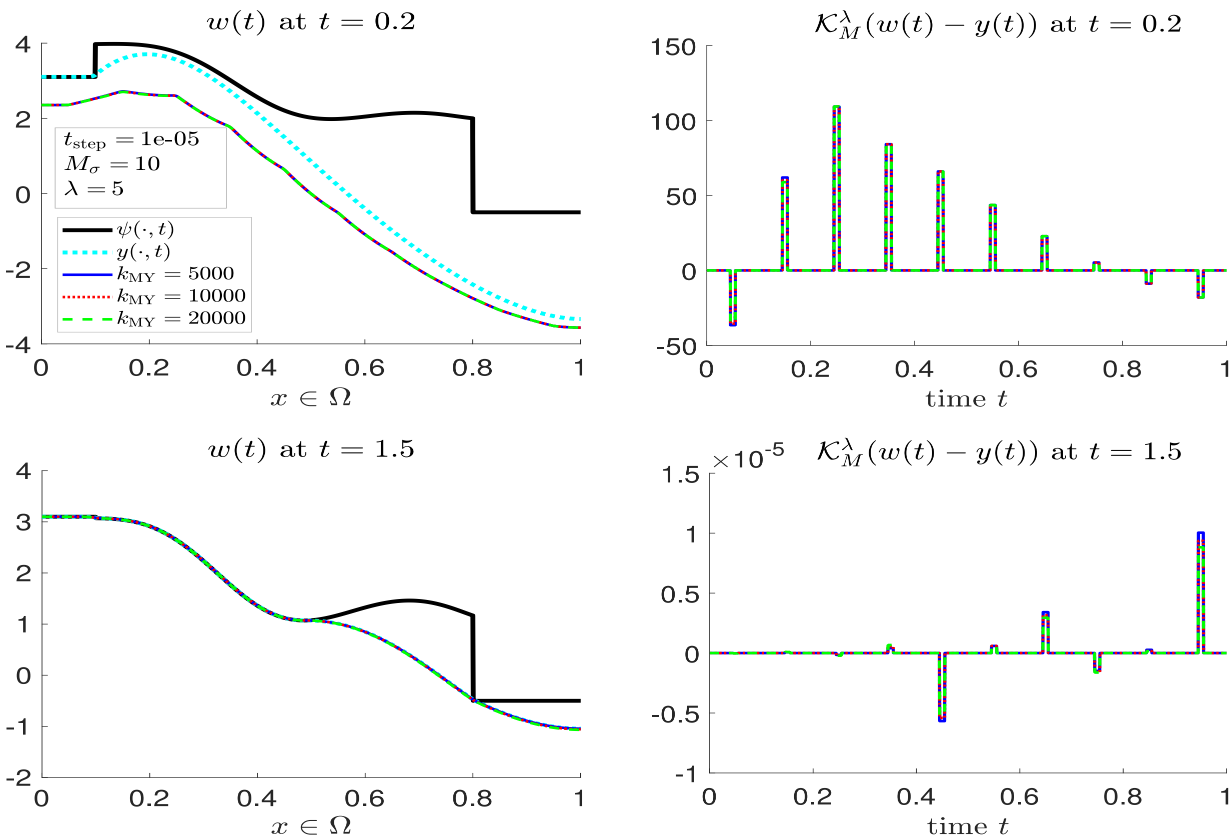

FIGURE 21. Time snapshots of trajectories and control

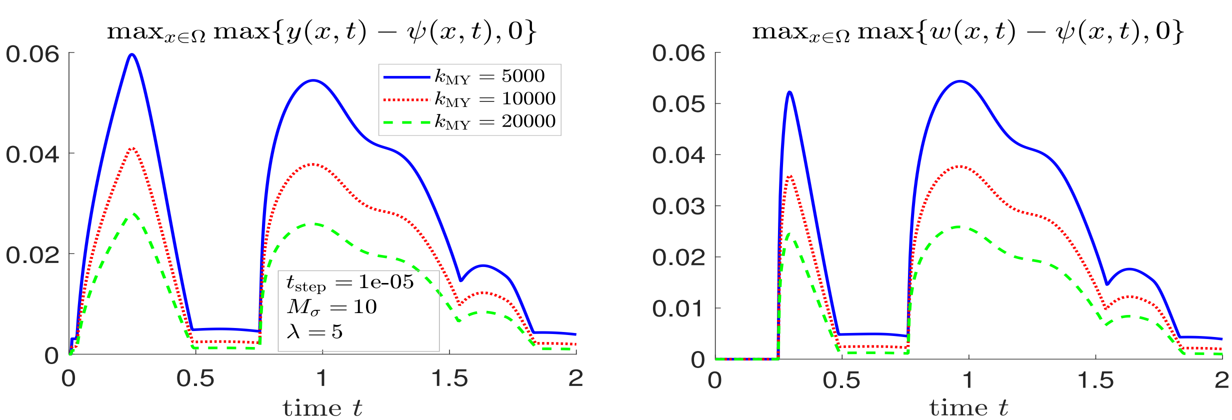

FIGURE 22. Largest magnitude of obstacle constraint violation

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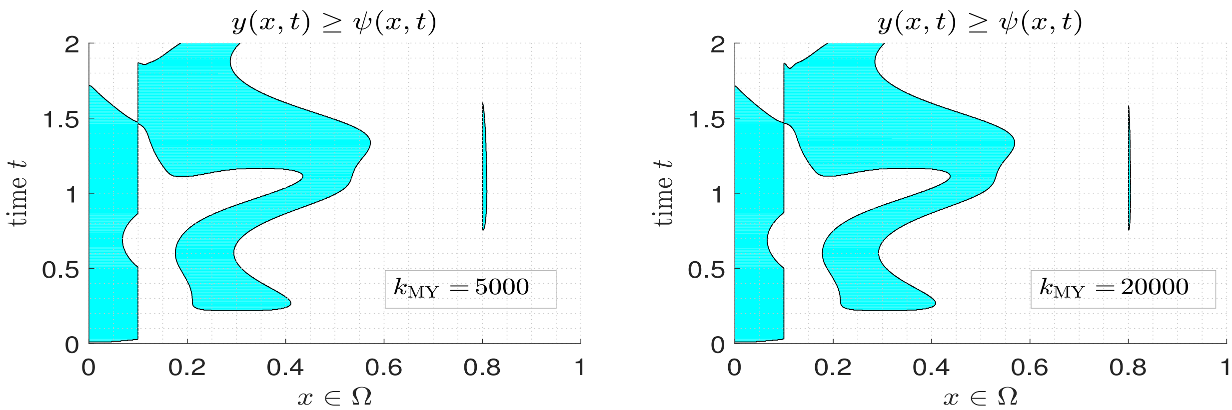

Figure 23. Evolution of obstacle constraint violation set for targeted trajectory

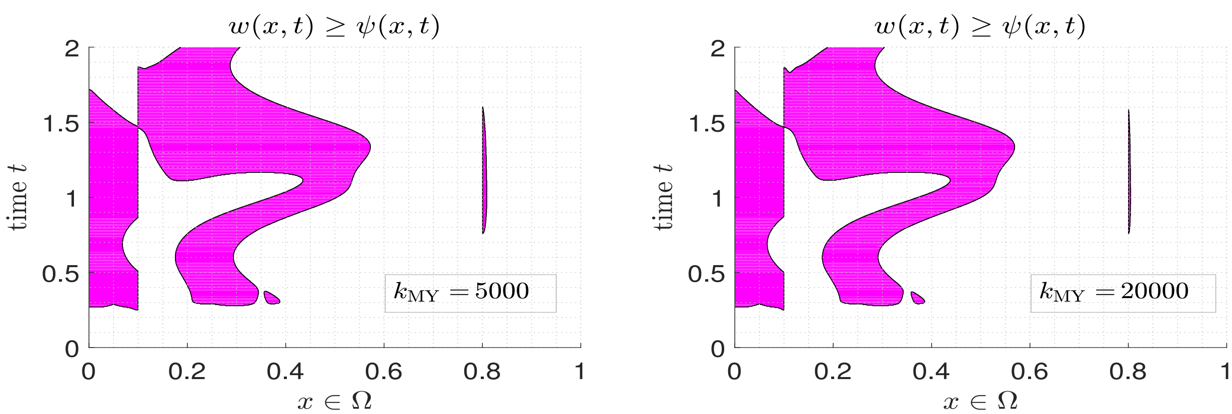

Figure 24. Evolution of obstacle constraint violation set for controlled trajectory

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