

Differentiability and Control of a Model for Granular Material Accumulation

Rafael Arndt and Carlos N. Rautenberg

Abstract—We consider differentiability properties associated to the problem of minimizing the accumulation of a granular cohesionless material on a given surface. The design variable or control is determined by source locations and intensity thereof. The control problem is described by an optimization one in function space constrained by a variational inequality or a non-smooth equation. We address a regularization approach via a family of nonlinear partial differential equations, and provide a novel result of Newton differentiability of the control-to-state map. Further, we discuss solution algorithms for the state equation as well as for the optimization problem.

I. INTRODUCTION

The growth description of piles made of granular cohesionless and homogeneous materials is a complex, nonconvex and nonsmooth process. In this problem, we assume that material is deposited by a known source on a supporting structure u_0 that may not be flat. The material is characterized by its angle of repose $\alpha > 0$: The steepest angle at which a sloping surface formed from a point source of material is stable. The source (assumed constant in time) is given by $g : \Omega \rightarrow \mathbb{R}$, where $\Omega \subset \mathbb{R}^d$ with $d = 1, 2$, and represents the (density) rate of a granular material being deposited on the smooth supporting structure $u_0 : \Omega \rightarrow \mathbb{R}$ with $u_0|_{\partial\Omega} = 0$. The latter boundary condition translates into the ability of material to escape the surface freely when it reaches the boundary $\partial\Omega$. In the limit time $\rightarrow \infty$, the function $u : \Omega \rightarrow \mathbb{R}$ describing the surface of the outmost layer of material is approximated by the solution to the stationary variational inequality: Find $u \in K$ such that

$$\langle -\epsilon \Delta u - f, v - u \rangle_{H^{-1}, H_0^1} \geq 0, \quad (1)$$

for all $v \in K$ with

$$K := \{v \in H_0^1(\Omega) : |Dv| \leq \varphi \text{ a.e.}\}$$

and where $f = g + \epsilon \Delta u_0$, and $H_0^1(\Omega)$ is the space of $L^2(\Omega)$ functions such that their weak gradients belong to $L^2(\Omega)^d$, together with function values vanishing at the boundary $\partial\Omega$ in the sense of the trace; see [1].

We assume that D is either the weak gradient or an approximation thereof, and $0 < \epsilon \ll 1$. The function $\varphi : \Omega \rightarrow \mathbb{R}$ is strictly and uniformly above zero, i.e.,

$$\varphi(x) \geq \nu > 0, \quad (2)$$

C. N. R. was supported by NSF grant DMS-2012391, and acknowledges the support of Germany's Excellence Strategy - The Berlin Mathematics Research Center MATH+ (EXC-2046/1, project ID: 390685689) within project AA4-3.

R. Arndt (tarndt@gmu.edu) and C. N. Rautenberg (crautenb@gmu.edu) are members of the Department of Mathematical Sciences at George Mason University, and of the Center for Mathematics and Artificial Intelligence (CMAI), Fairfax VA, USA.

for almost all $x \in \Omega$. If the pile is homogeneous and $|\nabla u_0| \leq \tan(\alpha)$, then $\varphi \equiv \tan(\alpha)$. In the case of an inhomogeneous pile (if multiple materials are present), α is not longer a constant and therefore neither is φ ; see [2]. There is a further more complex case (not treated within this paper) when $|\nabla u_0| > \tan(\alpha)$ on a positive measure set within Ω ; in this case φ is actually dependent on u and the problem is a *quasi-variational inequality*. This approach was pioneered by Prigozhin [3], [4], [5].

In the limiting case of $\epsilon \downarrow 0$, the process can be described as follows. Initially, and provided that the slope of the supporting structure is smaller than φ (a material dependent function) material accumulates there, friction forces keep the pile connected. Material is only allowed to flow, as the pile grows, when the slope of the pile reached the φ -bound and then it flows in direction of steepest descent, and only on the outermost layer. The latter can be seen as gravitational forces overcoming frictional ones. A schematic of this situation can be seen in Figure 1.

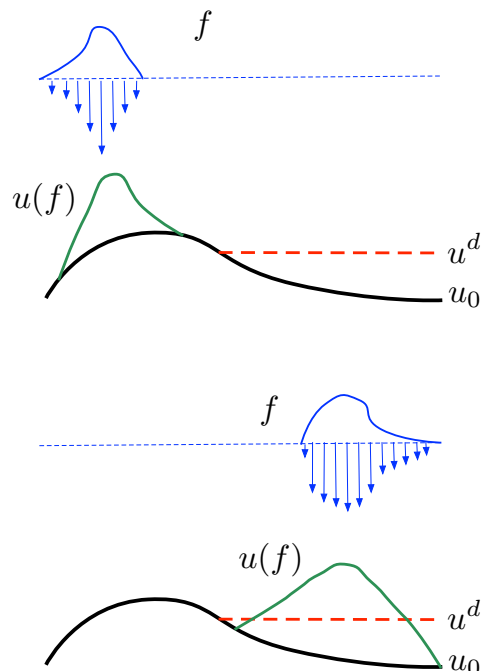


Fig. 1. Two accumulation profiles $u(f)$ associated to two instances of f , the intensity of material being poured. The lower f provides an $u(f)$ closer to u^d as desired.

A control problem of interest associated to problem (1) corresponds to the selection of a source of material f so that the accumulation of material $u = u(f)$ is close to some

desired surface structure u^d while the amount of material deposited is minimized in some sense. This leads to the minimization of the functional

$$J(u(f), f) := \frac{1}{2} \int_{\Omega} |u(f) - u^d|^2 dx + \lambda \|f\|_{Y'}^2, \quad (3)$$

for some space $Y' \subset H^{-1}(\Omega)$ where $H^{-1}(\Omega)$ is the topological dual to $H_0^1(\Omega)$. In the same vein, the design of algorithms for the minimization of (3) and the study of optimality conditions require information on the differentiability of the map $f \mapsto u(f)$. This is an extremely complex task due to the constraint K . In light of this, problem (1) is replaced by the regularized version

$$-\epsilon \Delta u + \gamma \mathcal{P}(u) = f \quad \text{in } H^{-1}(\Omega), \quad (4)$$

for some $\mathcal{P} : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ monotone, nonsmooth, and vanishing at K , whose specific form is given later in the paper. Further, we assume that $\gamma > 0$ and we recover (in some sense explained later) the original problem when $\gamma \rightarrow \infty$.

The abstract version of the problem above can be formulated as the following minimization problem with a nonsmooth equation as constraint:

$$\begin{aligned} & \text{Minimize } J(u, f) \\ & \text{subject to } A(u) + F(u) = f \end{aligned} \quad (\text{P})$$

for some objective function $J : X \times Y' \rightarrow \mathbb{R}$, with $Y' \subset X'$, and where $A : X \rightarrow X'$ is a strongly monotone, and continuously Fréchet differentiable operator. Additionally, $F : X \rightarrow X'$ is monotone, and Newton differentiable (see the next section for a definition). Existence of solutions to (P) is available under mild conditions.

In this paper we study the Newton differentiability properties of the solution map $f \mapsto u(f)$ associated to the constraint in (P) and how this permeates to $f \mapsto J(u(f), f)$. In particular, the obtained results imply that (P) is suitable to be tackled by semismooth Newton approaches.

The rest of the paper is organized as follows. In Section II we study results involving differentiability and monotonicity, and further establish an abstract result, Theorem 1, for the characterization of the sensitivity of the control-to-state map. Subsequently, in Section III, we apply the abstract results to our specific application. There we improve known results for the Newton differentiability of convex regularization of gradient type constraints, and also consider analogous results for operators approximating the gradient. Several lemmata are proven that culminate in Theorem 2 which establishes the control-to-state differentiability result for the application example. The paper ends with a discussion on solution algorithms and future research directions.

II. PRELIMINARIES AND THEORETICAL RESULTS

We assume throughout this section that X is a reflexive and real Banach space, and additionally assume that the control space Y' is such that $Y' \subset X'$ for a reflexive and real Banach space Y . The typical example of application here is $X = H_0^1(\Omega)$ and $Y' \simeq Y = L^2(\Omega)$.

We start with monotonicity definitions used throughout the paper. We say that $A : X \rightarrow X'$ is *strongly monotone* if there exist $c > 0$ and $q > 1$ such that

$$\langle A(u+h) - A(u), h \rangle \geq c \|h\|^q, \quad (5)$$

for all $u, h \in X$. Further, we say that it is *monotone* if (5) holds for $c = 0$; see for example [6].

Two differentiability concepts are used in this work: Fréchet, and Newton (or slant) differentiability. While for the definition of Fréchet differentiability we refer the reader to [7] or a nonlinear functional analysis textbook, we now introduce the Newton differentiability concept; see [8] for a solid introduction on the subject. Let X, Y be real Banach spaces and $D \subset X$ be an open set. Then $F : D \subset X \rightarrow Y$ is called *Newton differentiable* at u if there exists an open neighborhood $\mathcal{N}(u) \subset D$ and mappings

$$G_F : \mathcal{N}(u) \rightarrow \mathcal{L}(X, Y)$$

such that

$$\lim_{\|h\|_X \rightarrow 0} \frac{\|F(u+h) - F(u) - G_F(u+h)h\|_Y}{\|h\|_X} = 0.$$

Note that Newton derivatives are in general not unique, and it is direct to prove that if $F : D \subset X \rightarrow Y$ is continuously Fréchet differentiable, then it is Newton differentiable.

In order to simplify notation, we use the Landau o notation: We denote $\|r(h)\|_Y = o(\|h\|_X)$ for a map $r : X \rightarrow Y$ if

$$\lim_{\|h\|_X \rightarrow 0} \frac{\|r(h)\|_Y}{\|h\|_X} = 0,$$

i.e., $\|r(h)\|_Y = o(\|h\|_X)$ implies that r vanishes faster than h as $h \rightarrow 0$.

We start now with a result that establishes that for a strongly monotone differentiable map, its derivative is also strongly monotone under relatively mild conditions.

Lemma 1: Let $A : D \subset X \rightarrow X'$, with D open, satisfy for some fixed $c > 0$ and $2 \leq q < 3$

$$\langle A(u+h) - A(u), h \rangle \geq c \|h\|^q, \quad (6)$$

for all $u, u+h \in D$.

In addition, suppose that

$$\langle A(u+h) - A(u), h \rangle = \langle A'(u)h, h \rangle + \langle w(u, h), h \rangle \quad (7)$$

with $\|w(u, h)\|_{X^*} \leq M \|h\|_X^2$, for all $u \in D$, and all $h \in B_r(0, X)$ for some sufficiently small $r = r(u) > 0$.

Then for each $u \in D$, $A'(u)$ is strongly monotone with $q = 2$, i.e.,

$$\langle A'(u)h, h \rangle \geq \tilde{c} \|h\|_X^2$$

for some $\tilde{c} > 0$ and all $h \in X$.

Proof: Let $u \in D$ be fixed. From (6) and (7) we obtain for all $h \in B_s(0, X)$ with $s > 0$ sufficiently small that

$$c \|h\|_X^q \leq \langle A'(u)h, h \rangle + M \|h\|_X^3.$$

Let $h = t\tilde{h}$ for some $\tilde{h} \in B_s(0, X)$ with $\|\tilde{h}\|_X = s$, and $t \in [0, 1]$, then

$$t^{q-2} c s^q \leq \langle A'(u)\tilde{h}, \tilde{h} \rangle + M t s^3.$$

If $q = 2$, simply take $t = 0$ and the result follows. If $q \in (2, 3)$, define $g(t) = t^{q-2}cs^q - Mts^3$, and choose s sufficiently small so that the positive maximum of g is achieved in $(0, 1)$; this can be done since $q < 3$. In fact, since $q \in (2, 3)$, the maximum value is given by $g(t^*) = \tilde{c}s^2$ for some $\tilde{c} > 0$ independent of s . Thus, $\tilde{c}s^2 \leq \langle A'(u)\tilde{h}, \tilde{h} \rangle$ and hence

$$\tilde{c}\|\tilde{h}\|_X^2 \leq \langle A'(u)\tilde{h}, \tilde{h} \rangle.$$

Scaling by $\gamma > 0$ and using that $A'(u)$ is linear, completes the proof. ■

A few words are in order on the assumption (7). This condition is satisfied if A is twice Fréchet differentiable on the open set D and its second order derivative A'' is uniformly bounded in D ; the result follows from the Taylor remainder theorem, see [9].

We show next that in some cases the Newton derivative also inherits monotonicity properties of the original map. In particular, this requires some continuity assumption on the Newton derivative with respect to the base point.

Lemma 2: Let $F : D \subset X \rightarrow X'$ be Newton differentiable with Newton derivative G_F for an open set D and suppose that F is monotone, i.e.,

$$\langle F(u+h) - F(u), h \rangle \geq 0, \quad (8)$$

for all $u, u+h \in D$. If $w \mapsto \langle G_F(w)h, h \rangle$ is continuous at $w = u \in D$ for each h , then $G_F(u)$ is monotone, i.e.,

$$\langle G_F(u)h, h \rangle \geq 0$$

for all $h \in X$.

Proof: By (8) and the definition of Newton derivative, we observe that

$$\langle G_F(u+h)h, h \rangle \geq \langle r(h), h \rangle,$$

for some $r(h) = o(\|h\|_X)$. Let $h = t\tilde{h}$ with $\tilde{h} \in X$ fixed, then

$$\langle G_F(u+t\tilde{h})\tilde{h}, \tilde{h} \rangle \geq \frac{\langle r(t\tilde{h}), t\tilde{h} \rangle}{\|t\tilde{h}\|_X^2} \|\tilde{h}\|_X^2,$$

and the result follows by taking the limit of $t \rightarrow 0$. ■

We now establish the main theorem of the section and the tool which later allows us to determine differentiability properties of the control-to-state map in the introduction.

Theorem 1: Let $A : X \rightarrow X'$ satisfy the assumptions of Lemma 1 for $q = 2$ and $D = X$, let $F : X \rightarrow X'$ be monotone, continuous and Newton differentiable, and suppose that either F satisfies the assumptions of Lemma 2 or that its Newton derivative $G_F(u) : X \rightarrow X'$ is monotone.

Then, for $f \in Y'$, $u(f)$ is well-defined as the unique solution $u \in X$ to the equation

$$A(u) + F(u) = f \quad \text{in } X'. \quad (9)$$

In addition,

$$Y' \ni f \mapsto u(f) \in X$$

is Newton differentiable with Newton derivative $Y' \ni f \mapsto G_u(f) \in \mathcal{L}(Y', X)$ defined as follows: For $h \in Y'$, $w(f) = G_u(f)h$ is the unique solution $w \in X$ to

$$A'(u(f))w + G_F(u(f))w = h \quad \text{in } X' \quad (10)$$

where A' is the Fréchet derivative of A .

Proof: Note first that $u(f)$ is well-defined for any $f \in Y'$ given that there exists a unique solution to (9); the result follows by standard methods in monotone operator theory, see [6, Theorem 2.1]. Further, by Lemma 1, we have that for any $u \in X$, $A'(u)$ is strongly monotone, and $G_F(u)$ is monotone by initial assumption or by Lemma 2, so that w , the unique solution to (10), is also well-defined; again by [6, Theorem 2.1].

Since A is continuously Fréchet differentiable, it also is Newton differentiable. Then $E := A + F$ is Newton differentiable with derivative $G_E := A' + G_F$ satisfying

$$\langle G_E(u)v, v \rangle \geq \langle A'(u)v, v \rangle \geq \tilde{c}\|v\|_X^2,$$

for all $u, v \in X$.

For any $f, h \in Y'$, define $d(h) = u(f+h) - u(f)$. Considering (9) with f and $f+h$ and subtracting the results, we obtain

$$E(u(f) + d(h)) - E(u(f)) = h.$$

Since A satisfies (6) with $q = 2$, and F is monotone, it follows that $Y' \ni f \mapsto u(f) \in X$ is Lipschitz continuous. Then, for a map $r : Y' \rightarrow X'$ satisfying $\|r(h)\|_{X'} = o(\|d(h)\|_X)$, we have $\|r(h)\|_{X'} = o(\|h\|_{Y'})$. Subsequently, from the definition of Newton derivative, it holds true that

$$G_E(u(f) + d(h))d(h) = h + r(h), \quad (11)$$

where

$$\|r(h)\|_{X'} = o(\|h\|_{Y'}).$$

In contrast, from (10), for some $w = w(f+h)$ we have that

$$G_E(u(f+h))w(f+h) = h,$$

and subtracting this from (11), we obtain

$$G_E(u(f+h))R(h) = r(h), \quad (12)$$

where

$$R(h) := u(f+h) - u(f) - w(f+h).$$

By testing in (12) with $R(h)$, we observe due to the strong monotonicity of $G_E(u(f+h))$ that

$$\tilde{c}\|R(h)\|_X^2 \leq \langle G_E(u(f+h))R(h), R(h) \rangle = \langle r(h), R(h) \rangle,$$

and since

$$\langle r(h), R(h) \rangle \leq \|r(h)\|_{X'}\|R(h)\|_X,$$

we observe

$$\|R(h)\|_X = o(\|h\|_{Y'}),$$

and the proof is finished. ■

We now aim at applying the results in this section to our sandpile control problem.

III. APPLICATION TO THE SANDPILE CONTROL PROBLEM

In the framework of (P), we consider the problem in the introduction associated to the control of the stationary accumulation of granular material. In this section, we fix $X = H_0^1(\Omega)$, and $X \subset Y \subset L^2(\Omega)$ with Y a real Banach space, e.g., $Y = L^2(\Omega)$. Note that this implies that $L^2(\Omega) \subset Y' \subset X'$ since we identify $L^2(\Omega)$ with its topological dual.

In (4), we define the constraint regularization operator $\mathcal{P} : X \rightarrow X'$ as

$$\begin{aligned} \langle \mathcal{P}(u), w \rangle_{X', X} &= \int_{\Omega^+(u)} P(Du) \cdot Dw \, dx \\ &= \int_{\Omega^+(u)} (|Du| - \varphi)^\pm \frac{(Du \cdot Dw)}{|Du|} \, dx, \end{aligned} \quad (13)$$

where

$$\Omega^+(u) := \{x \in \Omega : |Du(x)| > 0 \text{ a.e.}\}.$$

Additionally, $P(u) := q(u)b(u)$ for

$$q(u) := \frac{u}{|u|}, \quad \text{and} \quad b(u) := (|u| - \varphi)^\pm,$$

and $(\cdot)^\pm := \min(1, \max(0, \cdot))$ in the pointwise sense, i.e., for $g : \Omega \rightarrow \mathbb{R}$,

$$(g(x))^\pm = \begin{cases} 0, & \text{if } g(x) \leq 0, \\ g(x), & \text{if } 0 \leq g(x) \leq 1, \\ 1, & \text{if } 1 \leq g(x). \end{cases}$$

Two possible choices for D are considered: $D = \nabla$, the weak gradient, and $D = D_\mu$, where $D_\mu : L^p(\Omega) \rightarrow L^p(\Omega)^d$ is a bounded linear operator for all $1 \leq p \leq +\infty$, approximating the gradient (e.g., by means of incremental quotients). The parameter $\mu > 0$ can be considered in order to obtain $D_\mu \rightarrow \nabla$ in some sense, as $\mu \downarrow 0$. In addition, note that (formally) we can write $\mathcal{P}(u) = D'P(Du)$.

For both cases, we have that $\mathcal{P}$ corresponds to the derivative of the convex functional

$$J_{\mathcal{P}}(u) = \int_{\Omega} K(|Du(x)| - \varphi(x)) \, dx,$$

where

$$K(t) = \begin{cases} \int_0^t (y)^\pm \, dy, & \text{if } t \geq 0; \\ 0, & \text{if } t < 0. \end{cases}$$

It follows that $\mathcal{P}$ is monotone given that $J_{\mathcal{P}}$ is convex, and further from its definition we observe that $\mathcal{P}$ is continuous. Thus, in both the cases $D \in \{\nabla, D_\mu\}$, we obtain that for every $f \in Y'$, since $-\Delta$ is strongly monotone, equation (4) has a unique solution by the same argument as in the beginning of the proof of Theorem 1. Hence, for each $\gamma > 0$, there exists a $u_\gamma \in X$ solving (4), and standard arguments exploiting the monotonicity of $J_{\mathcal{P}}$ yield that $u_\gamma \rightarrow u^*$ in X as $\gamma \rightarrow \infty$, where u^* is the solution to (1); see for example [10].

Next we show that $P(u) = q(u)b(u)$ is Newton differentiable between appropriate spaces, and later we use the

result to obtain a Newton differentiability result for $\mathcal{P}(u) = D'P(Du)$.

Lemma 3: The operator $P : L^p(\Omega)^d \rightarrow L^q(\Omega)^d$ with $2 \leq 2q \leq p$ is Newton differentiable. A Newton derivative G_P can be defined as

$$G_P(u) = q(u)G_b(u) + b(u)Q(u), \quad (14)$$

where

$$Q(u)(x) := \frac{1}{|u(x)|} \left(\text{id} - \frac{u(x)u^T(x)}{|u(x)|^2} \right),$$

and

$$G_b(u)(x) := G_{\max}^{\min}(|u(x)| - \varphi(x)) \frac{u^T(x)}{|u(x)|}$$

is a Newton derivative of $b : L^p(\Omega)^d \rightarrow L^q(\Omega)$, and

$$G_{\max}^{\min}(w)(x) := \chi_{(0,1)}(w(x))$$

is a Newton derivative of $(\cdot)^\pm : L^p(\Omega) \rightarrow L^q(\Omega)$.

Proof: The fact that $G_b(u)$ is a Newton derivative of $b : L^p(\Omega)^d \rightarrow L^q(\Omega)$ follows from $G_{\max}^{\min}(u)(x) = \chi_{(0,1)}(u(x))$ being a Newton derivative of $(\cdot)^\pm : L^p(\Omega) \rightarrow L^q(\Omega)$, and this is similarly obtained as the Newton derivative of the max function alone, see [8] and compare to [10].

Initially, we observe that

$$\begin{aligned} P(u+h) - P(u) - G_P(u+h)h &= b(u+h)(q(u+h) - q(u) - Q(u+h)h) \\ &\quad + (q(u) - q(u+h))(b(u+h) - b(u)) \\ &\quad + q(u+h)(b(u+h) - b(u) - G_b(u+h)h) \\ &= I + II + III, \end{aligned}$$

and in what follows we show that

$$I + II + III = o(\|h\|_p).$$

Initially consider I. Note that we have

$$\begin{aligned} I &= b(u+h) \left(-\frac{u}{|u+h||u|} \left(|u+h| - |u| - \frac{(u+h)^T h}{|u+h|} \right) \right. \\ &\quad \left. + \frac{(u+h)^T h}{|u+h|^2} \left(\frac{u+h}{|u+h|} - \frac{u}{|u|} \right) \right). \end{aligned}$$

Since

$$\left| b(u+h) \frac{u}{|u+h||u|} \right| \leq \frac{1}{\nu},$$

due to the fact that $\varphi \geq \nu$ a.e. in Ω , and because

$$G_{|\cdot|}(u+h)h = \frac{(u+h)^T h}{|u+h|},$$

when $u+h \geq \nu$, where $G_{|\cdot|}$ is a Newton derivative of $|\cdot| : L^p(\Omega) \rightarrow L^q(\Omega)$, we observe that the first part of the sum in I is $o(\|h\|_p)$.

Regarding the second term of I , if $u+h \leq \nu$ then $b(u+h) = 0$, and if $u+h \geq \nu$, then

$$\left| \frac{(u+h)^T h}{|u+h|^2} \right| \leq \frac{|h|}{\nu} \quad \text{and} \quad \left(\frac{u+h}{|u+h|} - \frac{u}{|u|} \right) \leq 2 \frac{|h|}{\nu},$$

and since $b(u + h) \leq 1$, we get a bound of $2\frac{|h|^2}{\nu^2}$. An application of Hölder's inequality implies that the second term of I is also $o(\|h\|_p)$ as we see next. Suppose that $z : \Omega \rightarrow \mathbb{R}$ satisfies $|z| \leq |h|^2$. Thus by applying Hölder's inequality to $|h|^{2q}$ and 1 for the exponents $\frac{p}{2q} \geq 1$ and $\left(\frac{p}{2q}\right)'$ we get

$$\int_{\Omega} |z|^q dx \leq \int_{\Omega} |h|^{2q} dx \leq C_1(\Omega) \left(\int_{\Omega} |h|^p dx \right)^{2q/p}.$$

Therefore (by taking the q -th root), we get

$$\|z\|_{L^q(\Omega)} \leq C_2(\Omega) \left(\int_{\Omega} |h|^p dx \right)^{2/p} = C_2(\Omega) \|h\|_{L^p(\Omega)}^2,$$

which proves the statement.

We turn our attention now to II. For a.e. $x \in \Omega$ it holds that

$$|b(u(x) + h(x)) - b(u(x))| \leq |h(x)| \chi_{\Omega_{\nu}}(x) \quad (15)$$

where

$$\Omega_{\nu} := \{x \in \Omega : |u(x)| < \nu \wedge |u(x) + h(x)| < \nu \text{ a.e.}\}.$$

Thus for $x \in \Omega \setminus \Omega_{\nu}$ we observe that

$$|q(u) - q(v)| \leq 2 \min \left(\frac{|h|}{|u|}, \frac{|h|}{|u + h|} \right) \leq 2 \frac{|h|}{\nu},$$

thus II is bounded by $|h|^2/\nu$. As above, this implies $II = o(\|h\|_p)$.

Finally, we consider III. Since G_b is the Newton derivative of $b : L^p(\Omega)^d \rightarrow L^q(\Omega)$, and $|q|$ is bounded by 1, we have that $III = o(\|h\|_p)$. ■

Let $\mathcal{P}^{\nabla} : X \rightarrow X'$ and $\mathcal{P}^{D_{\mu}} : X \rightarrow X'$ be defined as $\mathcal{P}$ in (13) for $D = \nabla$ and $D = D_{\mu}$, respectively. Although these operators are well-defined as maps from X to X' , in order to obtain differentiability properties in the case of $\mathcal{P}^{\nabla}$, the operator needs to be defined in slightly different spaces as we see next.

Lemma 4: The maps $\mathcal{P}^{\nabla}$ and $\mathcal{P}^{D_{\mu}}$ are Newton differentiable when defined as $\mathcal{P}^{\nabla} : X \rightarrow (W_0^{1,\infty}(\Omega))'$ and $\mathcal{P}^{D_{\mu}} : X \rightarrow X'$. The general expression of a Newton derivative $G_{\mathcal{P}}$ in these cases is given by

$$\langle G_{\mathcal{P}}(u)v, w \rangle = \int_{\Omega^+(u)} (G_P(Du)Dv) \cdot Dw \, dx, \quad (16)$$

for

- (i) all $u, v \in X$, $w \in W_0^{1,\infty}(\Omega)$, and the duality pairing considered between $(W_0^{1,\infty}(\Omega))'$ and $W_0^{1,\infty}(\Omega)$ in the case $D = \nabla$ and $\mathcal{P} = \mathcal{P}^{\nabla}$.
- (ii) all $u, v, w \in X$, and the duality pairing considered between X' and X in the case of $D = D_{\mu}$ and $\mathcal{P} = \mathcal{P}^{D_{\mu}}$.

Proof: Consider (i) first. The map $\nabla : X \rightarrow L^2(\Omega)^d$ is Fréchet differentiable with derivative ∇ , and by Lemma 3 the map $P : L^2(\Omega)^d \rightarrow L^1(\Omega)^d$ is Newton differentiable. Thus $u \mapsto P(\nabla u)$ is Newton differentiable (since it is the composition of a Newton and a Fréchet differentiable

mapping [8]) as map from $X \rightarrow L^1(\Omega)^d$ with Newton derivative $u \mapsto G_P(\nabla u)\nabla$. From here, an application of Hölder's inequality can be used to show that $\mathcal{P}^{\nabla} : X \rightarrow (W_0^{1,\infty}(\Omega))'$ is Newton differentiable (analogously as in [10, Corollary A.3]), with Newton derivative given by (16).

Next consider (ii). In the case of D_{μ} , we use that for $d = 1, 2$, by the Sobolev Embedding Theorem (e.g. see [11]), X is embedded in $L^p(\Omega)$ for any $2 \leq p < \infty$, and by the same result we have that $L^q(\Omega)$ is continuously embedded in $H^{-1}(\Omega)$ for $q > 1$. In addition, $D_{\mu} : X \rightarrow L^p(\Omega)^d$ is Fréchet differentiable with derivative D_{μ} for $2 \leq p < \infty$, and $P : L^p(\Omega)^d \rightarrow L^q(\Omega)$ with $2 \leq 2q \leq p$ is Newton differentiable. Then, as in the previous item, $u \mapsto P(D_{\mu}u)$ is Newton differentiable as map from $X \rightarrow L^q(\Omega)$ with Newton derivative $u \mapsto G_P(D_{\mu}u)D_{\mu}$. By choosing a $q > 1$, an application of Hölder's inequality shows that $\mathcal{P}^{\nabla} : X \rightarrow X'$ is Newton differentiable with Newton derivative given by (16). ■

Next we prove the existence of a solution to the sensitivity equation: Note that this requires to show the monotonicity of $G_{\mathcal{P}}(u)$ directly. The reason for this is that $\mathcal{P}$ does not satisfy the continuity assumption of Lemma 2, i.e., continuity of $w \mapsto \langle G_{\mathcal{P}}(w)h, h \rangle$.

Lemma 5: There exists a unique $w \in X$ such that

$$\langle -\epsilon \Delta w, z \rangle_{X', X} + \langle G_{\mathcal{P}}(u)w, z \rangle_{X', X} = \langle h, z \rangle_{X', X} \quad (17)$$

for all $z \in X$, for $\mathcal{P} = \mathcal{P}^{\nabla}$ and for $\mathcal{P} = \mathcal{P}^{D_{\mu}}$.

Proof: The map $-\Delta$ is strongly monotone and linear, then we only need to show that $G_{\mathcal{P}}(u) \in \mathcal{L}(X, X')$ is monotone, i.e.,

$$\langle G_{\mathcal{P}}(u)z, z \rangle_{X', X} \geq 0,$$

for all $z \in X$.

Note that $|G_P(Du)| \in L^{\infty}(\Omega)$ for each $u \in X$, hence we are only left to prove that $D'G_P(Du)D \geq 0$. Exploiting the structure of $G_P(Du)$ in (14) we have

$$\begin{aligned} \langle G_{\mathcal{P}}(u)z, z \rangle_{X', X} = \\ (q(Du)G_b(Du)Dz, Dz) + (b(Du)Q(Du)Dz, Dz). \end{aligned}$$

For the first term we have

$$G_{\min}^{\max}(|Du| - \varphi) \frac{(Du)^T Dz}{|Du|^2} \cdot \frac{(Du)^T Dz}{|Du|^2} \geq 0,$$

and since $b(Du) \geq 0$ and

$$Q(Du)Dz \cdot Dz = \frac{1}{|Du|} \left(|Dz|^2 - \frac{((Du)^T Dz)^2}{|Du|^2} \right) \geq 0,$$

the second term is also monotone and the proof is complete. ■

We are now in shape to apply the results from Section II to this specific control problem.

Theorem 2: For $\mathcal{P} = \mathcal{P}^{D_{\mu}}$, the solution map $Y' \ni f \mapsto u(f) \in X$ of (4) is Newton differentiable.

Proof: The result follows as direct application of the abstract result in Theorem 1 where the hypotheses are covered by Lemma 4 and 5. ■

A significant obstacle of the “non- ϵ differentiability gap” in Lemma 3 results in the lack of an analogous result for the case $\mathcal{P} = \mathcal{P}^\nabla$. However, some of these issues can be resolved in the algorithmic area with the introduction of a “lifting operator”, see [12].

A direct corollary of the previous result concerns the Newton differentiability of the reduced functional for the control problem under study.

Corollary 1: Provided that $Y' \ni f \mapsto \|f\|_{Y'}^2$ is Newton differentiable, the functional $Y' \ni f \mapsto J(u(f), f)$ where J is defined in (3) is Newton differentiable.

Proof: Note that $u \mapsto \frac{1}{2} \int_{\Omega} |u - u^d|^2 dx$ is Fréchet differentiable. Then, the proof is an application of the previous theorem and the composition result of Newton differentiable maps in [13]. ■

IV. SOLUTION ALGORITHMS

The idea of this section is to provide a solid idea of the applicability of the results of the previous section for the development of solution algorithms for (4) and (P).

As defined previously, the map $E : X \rightarrow X'$ is given by

$$E(u) = -\epsilon \Delta u + \gamma \mathcal{P}(u) - f,$$

so that the state equation (4) can be written as

$$E(u) = 0.$$

It is known that for every $f \in H^{-1}(\Omega)$, this equation is uniquely solvable in all cases contemplated in this paper, namely for $D = \nabla$ and $D = D_\mu$.

In the case $D = D_\mu$, and given that E is Newton differentiable as a map $X \rightarrow X'$, a Newton derivative of E is given by

$$G_E(u)v = -\epsilon \Delta v + \gamma G_{\mathcal{P}}(u)v,$$

where $G_{\mathcal{P}}$ is made explicit in (16). Further, since the constant in the strong monotonicity of $-\epsilon \Delta$ is identical to ϵ , and because we have proven that $G_{\mathcal{P}}(u)$ is monotone, it follows that $G_E(u)$ is nonsingular and its inverse is uniformly bounded by ϵ^{-1} , i.e.,

$$\|G(u)^{-1}\| \leq \frac{1}{\epsilon}.$$

Hence, solutions to $E(u) = 0$ are suitable to be approximated by a function space version of a *semismooth Newton method*; see [8, Theorem 8.16].

The semismooth Newton iteration is then given by

$$u^{n+1} := u^n + v^*,$$

where the Newton step v^* is defined as the solution of

$$G_E(u^n)v = -E(u^n). \quad (18)$$

This yields (see [8, Theorem 8.16]) that the sequence $\{u^n\}_{n \in \mathbb{N}}$ defined by the iteration (18) converges superlinearly to the unique solution $u^* \in X$ of the state equation $E(u) = 0$, provided that the initial iterate u_0 is sufficiently close to u^* . The fact that $D = D_\mu$ is considered, leads to the

idea to further apply a path-following method simultaneously on γ and μ .

Under mild conditions we have proven that $Y' \ni f \mapsto J(u(f), f)$ is Newton differentiable. In the case where Y' does not have a large number of degrees of freedom, a descent approach for the overall problem is possible. However the computation of the entire derivative is prohibitive if the dimension of Y' is not small. On the other hand, if the approach is such that the entire derivative is not needed (only a small number of components are), a descent method is directly suitable.

V. CONCLUSION AND FUTURE RESEARCH

We have provided several abstract results that link notions of differentiability to monotonicity ones. In fact, in Theorem 1, the hypotheses contemplate the possibility of heavily nonlinear operators, e.g., the p -Laplacian defined as $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2} \nabla u)$ with $p \in [2, 3)$ is within the scope of the theorem. The application of the abstract results to the specific example of control of the pile of granular material is then tackled in Theorem 2 and its corollary. There, a function space approach is suitable provided that the gradient is considered in its approximated version. The result of Newton differentiability in Lemma 4 is of interest in its own right; it provides a better result than the one standing in the literature for the case $D = \nabla$. The short discussion on the algorithmic development is a current area of active research and topic of a future publication.

REFERENCES

- [1] R. A. Adams and J. J. F. Fournier, *Sobolev spaces*, 2nd ed., ser. Pure and Applied Mathematics (Amsterdam). Elsevier/Academic Press, Amsterdam, 2003, vol. 140.
- [2] H. Antil, R. Arndt, C. N. Rautenberg, and D. Verma, “Nondiffusive variational problems with distributional and weak gradient constraints,” *Advances in Nonlinear Analysis*, vol. 11, no. 1, pp. 1466–1495, 2022.
- [3] L. Prigozhin, “Sandpiles and river networks: extended systems with non-local interactions,” *Phys. Rev. E*, vol. 49, pp. 1161–1167, 1994.
- [4] —, “Sandpiles, river networks, and type-II superconductors,” *Free Boundary Problems News*, vol. 10, pp. 2–4, 1996.
- [5] —, “Variational model of sandpile growth,” *Euro. J. Appl. Math.*, vol. 7, pp. 225–236, 1996.
- [6] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*. American Mathematical Society, 1997.
- [7] R. F. Curtain and A. J. Pritchard, *Functional analysis in modern applied mathematics*. Academic press, 1977.
- [8] K. Ito and K. Kunisch, *Lagrange Multiplier Approach to Variational Problems and Applications*. SIAM, 2008.
- [9] E. Zeidler, *Nonlinear functional analysis and its applications: III: variational methods and optimization*. Springer Science & Business Media, 2013.
- [10] M. Hintermüller and C. N. Rautenberg, “A sequential minimization technique for elliptic quasi-variational inequalities with gradient constraints,” *SIAM Journal on Optimization*, vol. 22, no. 4, pp. 1224–1257, 2012.
- [11] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, 2nd ed. Academic Press, 2003.
- [12] M. Hintermüller and J. Rasch, “Several path-following methods for a class of gradient constrained variational inequalities,” *Comput. Math. Appl.*, vol. 69, no. 10, pp. 1045–1067, 2015. [Online]. Available: <http://dx.doi.org/10.1016/j.camwa.2014.12.001>
- [13] M. Hintermüller, “Semismooth newton methods and applications,” *Department of Mathematics, Humboldt-University of Berlin*, 2010.