OPTIMAL CONDUIT SHAPE FOR STOKES FLOW *

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Abstract. We consider the problem of the optimal shape of a two dimensional duct which contains a fluid governed by the Stokes equations with mixed boundary conditions. The conduit domain is assumed to be non-smooth and perturbations are allowed only on the walls, while the objective functional aims at minimizing the head loss and to enforce a uniform velocity profile at the outlet. We show existence of solutions to the shape optimization problem and determine the existence of the shape derivative.

Key words. shape optimization, Stokes flow, mixed boundary conditions

1. Introduction. In this paper, we consider the problem of optimal shape of a duct for a two dimensional Stokes flow with mixed boundary conditions and with an objective shape functional involving trace values of velocity and pressure. Specifically, the domain that contains the fluid is assumed to be Lipschitz, and the possible perturbations are located on the walls. Further, the boundary conditions for the Stokes flow are homogeneous Dirichlet (no-slip) on the walls, non-homogeneous Dirichlet on the inlet, and homogeneous Neumann (do-nothing) at the outlet. Finally, the objective functional involves trace values of the state variables at the inlet and outlet, and it is a performance measure of uniformity of the outflow and loss of energy.

The overall problem possesses several intertwined difficulties: The lack of smoothness of the domain, and the mixed boundary conditions of the problem limit the possible regularity of the velocity and pressure of the fluid, which are required for the objective functional evaluation. This implies that possible perturbations of the domain are required to be handled with care in order to prove existence of solutions to the shape optimization problem and for the determination of a shape derivative.

Shape optimization problems with constraints determined by (Navier-)Stokes systems are of great interest with significant complexities. Concerning the fundamentals of shape optimization, we refer the reader to the monographs [3] and [13], and for Navier-Stokes and Stokes equations, to [14] and [5]. In [7], the authors consider the shape problem with Navier-Stokes and homogeneous Dirichlet boundary conditions, and an existence result together with an algorithm for calculation of directional derivatives is provided. For a reference on applied problems involving several fluid equations we refer the reader to [9], the monograph [10] and references therein.

The rest of the paper is organized as follows. In Section 2 we provide complete descriptions of possible domains and the formulation of the Stokes flow with mixed boundary conditions. Subsequently, we present the regularity result that establishes that the velocity-pressure pair has $H^{\frac{3}{2}+\epsilon} \times H^{\frac{1}{2}+\epsilon}$ domain regularity for some small $\epsilon > 0$. In Section 3 we formulate the shape optimization problem and motivate the

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objective functional by means of obtaining a uniform outflow with the minimal loss of energy possible. In addition, we show that there exists solutions to the shape optimization problem if the family of possible domains is properly determined. Next, in Section 4, we show the existence of a shape derivative and establish the structure of the problem that is solved by the derivatives.

2. The Stokes Problem. Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with Lipschitz boundary $\partial\Omega$ divided into a Dirichlet, Γ_D, and a Neumann Γ_N parts, i.e., $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$. Specific regularity and structural assumptions on the boundary and its parts are given in the next section. We assume that the Dirichlet part is the union two parts, an inlet Γ_i and a wall Γ_w, and the outlet Γ_o is identical to the Neumann boundary part; that is

$$\Gamma_D = \Gamma_i \cup \Gamma_w \qquad \text{and} \qquad \Gamma_N = \Gamma_o.$$

For a given viscosity $\nu > 0$, we assume that the fluid and velocity pair $(\mathbf{u}, p) \in H^1(\Omega; \mathbb{R}^2) \times L^2(\Omega)$ satisfy the following Stokes problem

$$-\operatorname{div} \sigma(\mathbf{u}, p) = \mathbf{f} \qquad \text{in } \Omega, \tag{2.1}$$
$$\operatorname{div} \mathbf{u} = 0 \qquad \text{in } \Omega, \tag{2.2}$$
$$\mathbf{u} = \mathbf{u}_i \qquad \text{on } \Gamma_i, \tag{2.3}$$
$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma_w, \tag{2.4}$$
$$\sigma(\mathbf{u}, p)\mathbf{n} = 0 \qquad \text{on } \Gamma_o, \tag{2.5}$$

where $\mathbf{f} : \Omega \to \mathbb{R}^2$ is a distributed forcing term, $\mathbf{u}_i : \Gamma_i \to \mathbb{R}^2$ is a prescribed velocity profile at the inlet, and $\sigma(\mathbf{u}, p) = \nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) - pI$ is the Cauchy stress tensor. For the sake of brevity, we define $\mathbf{u}_D$ as

$$\mathbf{u}_D(x) = \begin{cases} \mathbf{u}_i(x) & \text{if } x \in \Gamma_i, \\ \mathbf{0} & \text{if } x \in \Gamma_w. \end{cases}$$

We consider the Stokes problem over specific domains Ω called *proper* on which needed regularity results hold true, and are defined as follows.

Definition 2.1 (Proper Domains). *The domain Ω is called proper if it is bounded and connected, and satisfies the following assumptions:*

1. *Γ_i and Γ_o are segments, and Γ_w is the union of two C^2 disjoint pieces each of which joins one endpoint of Γ_i to an endpoint of Γ_o.*
2. *There exists $r > 0$ such that at each junction point x_J the set $\{x \in \partial\Omega : \operatorname{dist}(x_J, x) < r\}$ is the union of two segments joining at a right angle.*
3. *There exists a finite tube $\mathcal{T} = \mathcal{T}(\delta; C_1, C_2; S_D, S_N)$ of $\delta > 0$ thickness with C_1, C_2 walls and extremities S_D, S_N (see [12, Definition 2.1]) such that $S_N \subset \Gamma_N$, $S_D \subset \mathbb{R}^2 \setminus \bar{\Omega}$, and $(\mathcal{T} \setminus S_N) \cap \partial\Omega = \mathcal{T} \cap \Gamma_i =: \Gamma_{i,\mathcal{T}}$. Given $x \in \mathcal{T}$, we denote by C_x the unique curve parallel to C_1 and C_2 such that $x \in C_x$, and assume that C_x intersects $\Gamma_{i,\mathcal{T}}$ at exactly one point, say $\gamma_c(x)$, which satisfies $C_x \cap \bar{\Omega} = \{x \in C_x : \ell(x) \geq \ell(\gamma_c(x))\}$. Here $\ell(z)$ denotes the arc length coordinate of $z \in \mathcal{T}$ on the curve C_z. We further assume that there exists a non-negative function $\eta \in C_c^\infty(\mathbb{R})$, with values in $[0, 1]$, such that $\operatorname{supp} \eta \subset (-\delta/2, \delta/2)$, $\eta(0) = 1$, and*

$$\int_{\Gamma_{i,\mathcal{T}}} \eta(\rho(x))\boldsymbol{\tau}(x) \cdot \boldsymbol{n}(x)\, dx \neq 0,$$

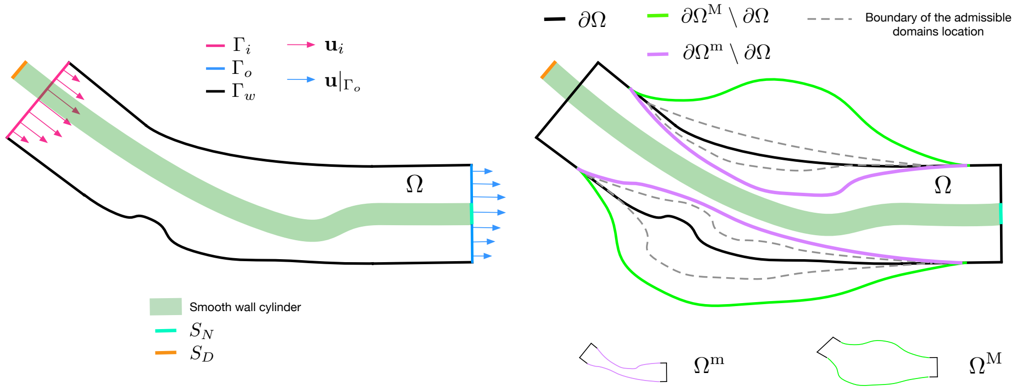

Fig. 2.1: (**Left**) A proper domain Ω, location of Γ_i, Γ_w, and Γ_o, $\mathbf{u}_i$ and velocity profile $\mathbf{u}|_{\Gamma_o}$. (**Right**) Minimal Ω^{m} and maximal Ω^{M} domains, and location for all possible domain perturbations.

where $\rho(x)$ is the transverse coordinate of x and $\boldsymbol{\tau}(x)$ is the unitary vector tangent to C_x, oriented in the sense of increasing arc length coordinates.

It is worth to mentioning that although the definition of *proper domains* is rather technical, it includes natural tubular shapes as straight, and elbow fittings, and variations thereof; it is in place to prevent degenerate-type-domains with low regularity of fluid velocity and pressure.

In order to write our weak formulation of the problem we make use of the following result found in [12, Theorem 2.16] that provides a "lifting" result of the boundary data to the domain.

Lemma 2.2. *Suppose that Ω is a proper domain. Let $\mathbf{u}_D \in H^{j+\frac{1}{2}}(\Gamma_D;\mathbb{R}^2)$ with $j = 0, 1$. Then, there exists $\mathbf{v} \in H^{j+1}(\Omega;\mathbb{R}^2)$ such that*

$$\operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \qquad (\nabla\mathbf{v} + (\nabla\mathbf{v})^T)\mathbf{n} = 0 \text{ on } \Gamma_N, \qquad \text{and} \qquad \mathbf{v} = \mathbf{u}_D \text{ on } \Gamma_D,$$

and further

$$\|\mathbf{v}\|_{H^{j+1}(\Omega;\mathbb{R}^2)} \leq M\|\mathbf{u}_D\|_{H^{j+\frac{1}{2}}(\Gamma_D;\mathbb{R}^2)}, \tag{2.6}$$

where $M > 0$ does not depend on $\mathbf{u}_D$.

We define $V_{\Gamma_D}(\Omega)$ as the closure in $H^1(\Omega;\mathbb{R}^2)$ of smooth divergence free vectorfields that vanish on Γ_D, i.e.,

$$V_{\Gamma_D}(\Omega) := \overline{E_{\Gamma_D}(\Omega)}^{H^1(\Omega;\mathbb{R}^2)},$$

where

$$E_{\Gamma_D}(\Omega) := \{\boldsymbol{\phi} \in C^\infty(\Omega;\mathbb{R}^2) : \operatorname{div}\boldsymbol{\phi} = 0 \text{ in } \Omega \text{ and } \overline{\operatorname{supp}\boldsymbol{\phi}} \cap \Gamma_D = \emptyset\}.$$

The weak formulation associated to (2.1)-(2.5) is given by the following problem: Find $(\mathbf{u}, p) \in H^1(\Omega;\mathbb{R}^2) \times L^2_0(\Omega)$ such that

$$\nu a(\mathbf{u}, \mathbf{z}) := \frac{\nu}{2}\int_\Omega (\nabla\mathbf{u} + (\nabla\mathbf{u})^T) : (\nabla\mathbf{z} + (\nabla\mathbf{z})^T)\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\mathbf{z}\,\mathrm{d}x, \tag{S}$$

for all $\mathbf{z} \in V_{\Gamma_D}(\Omega)$, with $\operatorname{div} \mathbf{u} = 0$ in Ω, $\mathbf{u} = \mathbf{u}_D$ on Γ_D in the trace sense, and where $\nabla p = \mathbf{f} + \frac{\nu}{2} \operatorname{div}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ in $(H^1(\Omega;\mathbb{R}^2))^* \subset H^{-1}(\Omega;\mathbb{R}^2)$. Given $\mathbf{f} \in L^2(\Omega;\mathbb{R}^2)$ and $\mathbf{u}_D \in H^{\frac{1}{2}}(\Gamma_D;\mathbb{R}^2)$, the existence and uniqueness of $\mathbf{u} \in H^1(\Omega;\mathbb{R}^2)$ is provided by the use of Lax-Milgram in combination with the data lifting result from Lemma 2.2. Then the classical result by Nečas (see Proposition 1.2. in Chapter I, §1 of [14]) determines the existence of a unique $p \in L^2_0 := \{q \in L^2(\Omega) : \int_\Omega q \,\mathrm{d}x = 0\}$.

Moreover, the existence of $\mathbf{u}$ above is equivalent to the existence of $\mathbf{w} \in V_{\Gamma_D}(\Omega)$ such that $\mathbf{u} = \mathbf{w} + \mathbf{v}$ and $\mathbf{v}$ is the one associated to Lemma 2.2. It is helpful to note that $\mathbf{w}$ satisfies

$$\nu a(\mathbf{w},\mathbf{z}) = \int_\Omega \mathbf{f} \cdot \mathbf{z} \,\mathrm{d}x - \nu a(\mathbf{v},\mathbf{z}), \qquad \forall \mathbf{z} \in V_{\Gamma_D}(\Omega). \tag{2.7}$$

As it will be clear in the next section, we require better regularity of the pair $(\mathbf{u}, p)$ than the one determined by existence theory. Improved regularity of all variables is established in the following lemma found in [12, Theorem 2.5], see also Lemma 2.2.

Lemma 2.3. *Suppose that Ω is a proper domain. Let $\mathbf{u}_D \in H^{\frac{3}{2}}(\Gamma_D;\mathbb{R}^2)$ and $\mathbf{f} \in L^2(\Omega;\mathbb{R}^2)$, then the weak solution $(\mathbf{u}, p) \in H^1(\Omega;\mathbb{R}^2) \times L^2_0(\Omega)$ to (2.1)-(2.5) belongs to $W^{2,2}_\beta(\Omega;\mathbb{R}^2) \times W^{1,2}_\beta(\Omega)$ for some $\beta \in (0, 1/2)$ and satisfies*

$$\|\mathbf{u}\|_{W^{2,2}_\beta(\Omega;\mathbb{R}^2)} + \|p\|_{W^{1,2}_\beta(\Omega)} \le C(\|\mathbf{f}\|_{L^2(\Omega;\mathbb{R}^2)} + \|\mathbf{u}_D\|_{H^{\frac{3}{2}}(\Gamma_D;\mathbb{R}^2)}). \tag{2.8}$$

Further,

$$\|\mathbf{u}\|_{H^{\frac{3}{2}+\epsilon}(\Omega;\mathbb{R}^2)} + \|p\|_{H^{\frac{1}{2}+\epsilon}(\Omega)} \le C(\|\mathbf{f}\|_{L^2(\Omega;\mathbb{R}^2)} + \|\mathbf{u}_D\|_{H^{\frac{3}{2}}(\Gamma_D;\mathbb{R}^2)}), \tag{2.9}$$

for some $\epsilon \in (0, 1/2)$ and where $C > 0$ does not depend on $\mathbf{u}_D$ nor $\mathbf{f}$.

The spaces $W^{2,2}_\beta(\Omega;\mathbb{R}^2)$ and $W^{2,2}_\beta(\Omega)$ are weighted Sobolev spaces with weights determined by a 2β power of the distances to junction points x_J (cf. [12, page 3013]) endowed with the norms

$$\|\mathbf{v}\|^2_{W^{2,2}_\beta(\Omega;\mathbb{R}^2)} = \sum_{|k|=0}^{2} \sum_{i=1}^{2} \int_\Omega \prod_{J=1}^{4} \operatorname{dist}(x_J, x)^{2\beta} |\partial_k v_i|^2 \,\mathrm{d}x,$$

and

$$\|q\|^2_{W^{1,2}_\beta(\Omega)} = \sum_{|k|=0}^{1} \int_\Omega \prod_{J=1}^{4} \operatorname{dist}(x_J, x)^{2\beta} |\partial_k q|^2 \,\mathrm{d}x.$$

Further, note that the $W^{2,2}_\beta(\Omega;\mathbb{R}^2) \times W^{1,2}_\beta(\Omega)$-regularity of $(\mathbf{u}, p)$ implies that $(\mathbf{u}, p)|_{\Omega'} \in H^2(\Omega';\mathbb{R}^2) \times H^1(\Omega')$ on $\Omega' = \{x \in \Omega : \operatorname{dist}(x, \mathcal{J}) > \delta\}$ for every $\delta > 0$, where $\mathcal{J}$ is the set of all junction points.

It is worth mentioning that the regularity results above are valid in more general domains than the proper ones defined earlier. In fact, we can allow Γ_D and Γ_N to be finite unions of regular connected components of $\partial\Omega$, satisfying certain conditions at the junction points (see assumptions (H1), (H2) and (H3) in [12]). For the sake of simplicity, we consider proper domains as in Definition 2.1.

3. The Shape Optimization Problem. The main objective is to find a regular domain Ω in a set of admissible domains $\mathcal{O}$ such that two criteria are minimized. One target objective is to enforce the outflow to be as uniform as possible, and the second one corresponds to try to minimize the loss of energy of the flow as it travels along the duct. Their mathematical description is given next.

Uniform Outflow Criterion: The normal component of the outflow $\mathbf{u}\cdot\mathbf{n}|_{\Gamma_o}$ should be close to uniform on the entire Γ_o. Since $\mathbf{u}$ on Γ_i is given by $\mathbf{u}_i$, by conservation of mass, if the outflow is constant, its velocity is given by $\overline{q} := -\frac{1}{|\Gamma_o|}\int_{\Gamma_i}\mathbf{u}_i(s)\cdot\mathbf{n}(s)\mathrm{d}S$. This leads us to consider the functional

$$J_1(\mathbf{u},\Omega) := \frac{1}{2}\int_{\Gamma_o}(\mathbf{u}\cdot\mathbf{n}-\overline{q})^2\mathrm{d}S.$$

Total Pressure Loss Criterion: The Bernoulli principle states that, disregarding height differences and dissipation of energy due to nonlinear effects (e.g., friction and turbulence), the total pressure $p+\frac{1}{2}|\mathbf{u}|^2$ is a quantity that remains constant along streamlines. In presence of energy dissipation, a pressure drop (or head loss) occurs. Along a streamline, we would then obtain $(p+\frac{1}{2}|\mathbf{u}|^2)|_{\Gamma_i} > (p+\frac{1}{2}|\mathbf{u}|^2)|_{\Gamma_o}$, and this drop of pressure, related to the loss of energy, is required to be minimized. Based on this, the functional of interest is determined by

$$J_2(\mathbf{u},p,\Omega) := -\int_{\Gamma_i\cup\Gamma_o}(p+\frac{1}{2}|\mathbf{u}|^2)\mathbf{u}\cdot\mathbf{n}\mathrm{d}S.$$

Notice that J_1 and J_2 are well-defined provided that $\mathbf{f}\in L^2(\Omega;\mathbb{R}^2)$ and $\mathbf{u}_D\in H^{\frac{3}{2}}(\Gamma_D;\mathbb{R}^2)$: In this setting, the weak solution $(\mathbf{u},p)\in H^1(\Omega;\mathbb{R}^2)\times L^2_0(\Omega)$ to (2.1)-(2.5) belongs to $H^{\frac{3}{2}+\epsilon}(\Omega;\mathbb{R}^2)\times H^{\frac{1}{2}+\epsilon}(\Omega)$, for some $\epsilon\in(0,1/2)$; see Lemmas 2.2 and 2.3. In particular, $\mathbf{u}_D\in H^{\frac{3}{2}}(\Gamma_D;\mathbb{R}^2)$ yields that $\bar{q}$ is finite, and $\mathbf{u}\in H^{\frac{3}{2}+\epsilon}(\Omega;\mathbb{R}^2)$ implies that trace values of $\mathbf{u}$ are in $L^q(\partial\Omega;\mathbb{R}^2)$ for every $q\geq 1$; see [6]. Thus J_1 is well-defined. Further, since $p\in H^{\frac{1}{2}+\epsilon}(\Omega)$, we have that trace values of p are in $L^q(\partial\Omega)$ for every $1\leq q\leq 2/(1-2\epsilon)$; see [6, Theorems 1.5.1.1 and 1.5.1.2]. This yields that J_2 is well-defined and that

$$\left|\int_{\Gamma_i\cup\Gamma_o}(p+\frac{1}{2}|\mathbf{u}|^2)\mathbf{u}\cdot\mathbf{n}\mathrm{d}S\right| \leq C(\|p\|_{H^{\frac{1}{2}}(\Omega)}+\frac{1}{2}\|\mathbf{u}\|^2_{H^{\frac{3}{2}}(\Omega;\mathbb{R}^2)})\|\mathbf{u}\|_{H^{\frac{3}{2}}(\Omega;\mathbb{R}^2)}. \tag{3.1}$$

The class $\mathcal{O}$ of admissible domains Ω is defined as follows.

Definition 3.1 (Admissible domains $\mathcal{O}$). *The class of admissible domains $\mathcal{O}$ corresponds to the set of all Ω that satisfy:*

1. *There exist $\Omega^{\mathrm{m}},\Omega^{M}$ that are proper in the sense defined in Section 2 such that $\Omega^{\mathrm{m}}\subset\Omega\subset\Omega^{M}$, and we assume that $\Omega^{\mathrm{m}},\Omega^{\mathrm{M}}\in\mathcal{O}$.*
2. *The location of Γ_i and Γ_o is the same for all $\Omega\in\mathcal{O}$. Further, there exists a $r>0$ such that at each junction point x_J the set $\{x\in\partial\Omega:\mathrm{dist}(x_J,x)<r\}$ is identical for each $\Omega\in\mathcal{O}$. In particular, this means that locally at every junction point x_J all domains $\Omega\in\mathcal{O}$ are composed of two identical segments joining at a $\pi/2$ angle.*
3. *All domains in Ω share the same interior smooth wall tube $\mathcal{T}$ as defined in the previous section.*
4. *There exists $k>0$ such that each local parametrization g of Γ_w satisfies $|g'''(t)|\leq k$ for all t. In particular, this means that all $\Omega\in\mathcal{O}$ are piecewise*

C^3. We denote by $\Gamma_{w'} \subset \Gamma_w$ the portion of the wall boundary that may change with respect to elements in $\mathcal{O}$.

Note that that every admissible domain is proper in the sense of Definition 2.1, and the class of domains allows smooth perturbations on the walls (with a uniform bound on the third derivative) of proper domains. The optimization problem of interest is parametrized by $\gamma \in [0,1]$ and given by

$$\text{(P)} \qquad \begin{aligned} &\min_{\Omega\in\mathcal{O}} J^\gamma(\mathbf{u},p,\Omega) := (1-\gamma)J_1(\mathbf{u},\Omega) + \gamma J_2(\mathbf{u},p,\Omega),\\ &\text{s.t. } (\mathbf{u},p) \text{ is a weak solution to (2.1)} - \text{(2.5) in } \Omega. \end{aligned}$$

In order to prove existence of solutions of the above problem we need to address the behavior of weak solutions to (2.1)-(2.5) with respect to variations on the domain $\Omega \in \mathcal{O}$. We do this in the next lemma which shows that constants in Lemma 2.3 can be selected independently of $\Omega \in \mathcal{O}$.

LEMMA 3.2. *Suppose that $\mathbf{u}_D \in H^{\frac{3}{2}}(\Gamma_D;\mathbb{R}^2)$ and $\mathbf{f} \in L^2(\Omega^{\mathrm{M}};\mathbb{R}^2)$. Then, there exist $\beta \in (0,1/2)$, $\epsilon \in (0,1/2)$ and $C>0$ such that* (2.8) *and* (2.9) *hold true independently of $\Omega \in \mathcal{O}$.*

Proof. Let $\mathbf{v}$ be the vector field given in Lemma 2.2 for the domain Ω^{m}. Its extension by zero to $\Omega \in \mathcal{O}$ satisfies the same properties than $\mathbf{v}$, now over Ω. We denote the extension also by $\mathbf{v}$ and consider this vector field to define a weak solution to (2.1)-(2.5) in Ω as in (2.7). In particular, note that the M constant in (2.6) can be selected independently of $\Omega \in \mathcal{O}$.

We have from Lemma 2.3 that the weak solution $(\mathbf{u},p)\in H^1(\Omega;\mathbb{R}^2)\times L^2_0(\Omega)$ to (2.1)-(2.5) in $\Omega\in\mathcal{O}$ belongs to $W^{2,2}_\beta(\Omega;\mathbb{R}^2)\times W^{2,2}_\beta(\Omega)$. Following [12, p. 3013] (see also [8]), we know that the parameter β in the weighted Sobolev spaces $W^{2,2}_\beta(\Omega;\mathbb{R}^2)$ and $W^{2,2}_\beta(\Omega)$ can be selected such that the equation $\lambda^2\sin^2(\pi/2)-\cos^2(\lambda\pi/2)=0$ has no solution λ in the strip $0\le\Re(\lambda)\le 1-\beta$. This equation does not change with domain perturbations since junction points form a right angle for every perturbed domain. We also have from Lemma 2.3 that $(\mathbf{u},p)$ belongs to $H^{\frac{3}{2}+\epsilon}(\Omega;\mathbb{R}^2)\times H^{\frac{1}{2}+\epsilon}(\Omega)$. Exactly as in the proof of Theorem 2.5 in [12] (see also [1, Proposition A.1]), via a Hardy-type inequality, we deduce that the parameter ϵ can be selected as $1/2-\beta$. Thus ϵ remains unchanged with respect to domain perturbations. Therefore, the weak solution to (2.1)-(2.5) enjoys the same regularity in each domain $\Omega\in\mathcal{O}$.

The constant C in (2.8) can be selected as in [12, Theorem 2.5] (see also [8, Theorem 9.4.5]). Since the inlet and outlet parts of the boundary are segments for of every $\Omega\in\mathcal{O}$, and local parametrizations of the walls have uniformly bounded derivatives with respect to $\Omega\in\mathcal{O}$ (see Definition 3.1), we observe that the Lipschitz constant characterizing $\partial\Omega$ can be selected independently of $\Omega\in\mathcal{O}$. In addition, junction points form a right angle for every $\Omega\in\mathcal{O}$ and that $\Omega^{\mathrm{m}}\subset\Omega\subset\Omega^{\mathrm{M}}$ for every $\Omega\in\mathcal{O}$, where Ω^{m} and Ω^{M} are fixed, thus the constant C can be selected independently of the domain perturbations. In a similar way, and via a Hardy-type inequality as before, we obtain that the constant C in (2.9) can be considered to be independent of $\Omega\in\mathcal{O}$ as well. □

THEOREM 3.3. *Problem* (P) *admits solutions.*

Proof. Step 1: Preliminary bounds. Let $(\mathbf{u}_\Omega,p_\Omega)\in H^1(\Omega;\mathbb{R}^2)\times L^2_0(\Omega)$ be the solution of (S) on an arbitrary $\Omega\in\mathcal{O}$. Then, by Lemma 3.2 there exist $\epsilon>0$ and $C>0$ (both independent of Ω) such that

$$\|\mathbf{u}_\Omega\|_{H^{\frac{3}{2}+\epsilon}(\Omega;\mathbb{R}^2)} + \|p_\Omega\|_{H^{\frac{1}{2}+\epsilon}(\Omega)} \le C(\|\mathbf{f}\|_{L^2(\Omega^{\mathrm{M}};\mathbb{R}^2)} + \|\mathbf{u}_D\|_{H^{\frac{3}{2}}(\Gamma_D;\mathbb{R}^2)}). \tag{3.2}$$

In particular, the pair $(\mathbf{u}_\Omega, p_\Omega)$ is uniformly bounded in $H^{\frac{3}{2}+\epsilon}(\Omega;\mathbb{R}^2)\times H^{\frac{1}{2}+\epsilon}(\Omega)$ with respect to Ω.

Step 2: Existence and properties of infimizing sequences. The existence of an infimizing sequence of Problem (P) follows as $\mathcal{O}$ is non-empty, and for each $\Omega\in\mathcal{O}$ the uniform bound in (3.2) holds. From this we obtain that $J^\gamma(\mathbf{u}_\Omega, p_\Omega, \Omega) > M$ for some $M\in\mathbb{R}$ and for all $\Omega\in\mathcal{O}$, see (3.1), where $(\mathbf{u}_\Omega, p_\Omega)$ is the weak solution to the Stokes problem in Ω. Thus, there exists a sequence $\{(\mathbf{u}_k, p_k, \Omega_k)\}$ such that

$$\lim_{k\to\infty} J^\gamma(\mathbf{u}_k, p_k, \Omega_k) = \inf_{\Omega\in\mathcal{O}} J^\gamma(\mathbf{u}_\Omega, p_\Omega, \Omega).$$

Here $(\mathbf{u}_k, p_k)$ denotes the weak solution to the Stokes problem in Ω_k. Note that there exists an extension $(\hat{\mathbf{u}}_k, \hat{p}_k)$ of $(\mathbf{u}_k, p_k)$ on $\Omega^{\mathrm{M}}\setminus\Omega_k$ that belongs to $H^{\frac{3}{2}+\epsilon}(\Omega;\mathbb{R}^2)\times H^{\frac{1}{2}+\epsilon}(\Omega)$ and additionally satisfies

$$\|\hat{\mathbf{u}}_k\|_{H^{\frac{3}{2}+\epsilon}(\Omega^{\mathrm{M}};\mathbb{R}^2)} \le K_1\|\mathbf{u}_k\|_{H^{\frac{3}{2}+\epsilon}(\Omega_k;\mathbb{R}^2)}, \quad\text{and}\quad \|\hat{p}_k\|_{H^{\frac{1}{2}+\epsilon}(\Omega^{\mathrm{M}})} \le K_2\|p_k\|_{H^{\frac{1}{2}+\epsilon}(\Omega_k)},$$

for some $K_1, K_2 > 0$ (both independent of Ω_k); see [4] and recall that the extension is done through Γ_w which is a C^3-smooth boundary part. Thus, $\{(\hat{\mathbf{u}}_k, \hat{p}_k)\}$ is bounded in $H^{\frac{3}{2}+\epsilon}(\Omega^{\mathrm{M}};\mathbb{R}^2)\times H^{\frac{1}{2}+\epsilon}(\Omega^{\mathrm{M}})$.

Step 3: Cluster points of $\{(\hat{\mathbf{u}}_k, \hat{p}_k)\}$ *solve* (S). The boundedness of $\{(\hat{\mathbf{u}}_k, \hat{p}_k)\}$ imply that along a subsequence

$$(\hat{\mathbf{u}}_k, \hat{p}_k) \rightharpoonup (\hat{\mathbf{u}}^*, \hat{p}^*) \quad \text{in } H^{\frac{3}{2}+\epsilon}(\Omega^{\mathrm{M}};\mathbb{R}^2)\times H^{\frac{1}{2}+\epsilon}(\Omega^{\mathrm{M}}),$$

for some $(\hat{\mathbf{u}}^*, \hat{p}^*)$. Further, since $\Omega_k\in\mathcal{O}$, there is $\Omega^*\in\mathcal{O}$ such that

$$\Omega_k \to \Omega^*,$$

along a subsequence and where convergence is in the Hausdorff complementary metric; see [11, Proposition A3.2 and Theorem A3.9] (see also [11, p. 54]). In particular, we have that $\chi_{\Omega_k}\to\chi_{\Omega^*}$ pointwise a.e. in Ω^{M}.

Since $\operatorname{div}\hat{\mathbf{u}}_k = 0$ in Ω_k, we now show that $\operatorname{div}\hat{\mathbf{u}}^* = 0$ in Ω^*. Let O be an open set such that $\overline{O}\subset\Omega^*$, then there exists $k_O > 0$ for which $\overline{O}\subset\Omega_k$ for all $k\ge k_O$, see [11, Proposition A3.8]. It follows then that $\operatorname{div}\hat{\mathbf{u}}^* = 0$ in O and hence also in Ω^*.

Let $\mathbf{v}\in C^\infty(\Omega^*;\mathbb{R}^2)$ be such that $\operatorname{div}\mathbf{v} = 0$ in Ω^* and $\overline{\operatorname{supp}\mathbf{v}}\cap\Gamma_D^* = \emptyset$ where Γ_D^* denotes the Dirichlet boundary of Ω^*, i.e., $\mathbf{v}\in E_{\Gamma_D^*}(\Omega^*)$. Since $\Omega_k\to\Omega^*$ in the Hausdorff complementary metric, there is k^* such that $\overline{\operatorname{supp}\mathbf{v}}\cap\Gamma_D^k = \emptyset$ for $k\ge k^*$ (see [11, Proposition A3.8]) where Γ_D^k is the Dirichlet boundary of Ω_k. Therefore, $\mathbf{v}_k := \mathbf{v}|_{\Omega_k}\in C^\infty(\Omega_k)$, $\operatorname{div}\mathbf{v}_k = 0$ in Ω_k and $\overline{\operatorname{supp}\mathbf{v}}_k\cap\Gamma_D^k = \emptyset$ for $k\ge k^*$ so that

$$\nu a_{\Omega_k}(\mathbf{u}_k, \mathbf{v}_k) = \int_{\Omega_k} \mathbf{f}\cdot\mathbf{v}_k \,\mathrm{d}x \quad \forall k\ge k^*. \tag{3.3}$$

Since $\hat{\mathbf{u}}_k\rightharpoonup\hat{\mathbf{u}}^*$ in $H^{\frac{3}{2}+\epsilon}(\Omega^{\mathrm{M}};\mathbb{R}^2)$, we observe that $\nabla\hat{\mathbf{u}}_k\to\nabla\hat{\mathbf{u}}^*$ in $L^2(\Omega^{\mathrm{M}};\mathbb{R}^{2\times2})$ (see [6, Theorem 1.4.3.2]) and so $\nabla\hat{\mathbf{u}}_k\to\nabla\hat{\mathbf{u}}^*$ pointwise almost everywhere in Ω^{M} along a subsequence. Then,

$$\begin{aligned}\lim_{k\to\infty} a_{\Omega_k}(\mathbf{u}_k, \mathbf{v}_k) &= \lim_{k\to\infty}\int_{\Omega^{\mathrm{M}}} (\nabla\mathbf{u}_k + (\nabla\mathbf{u}_k)^T) : (\nabla\mathbf{v} + (\nabla\mathbf{v})^T)\,\chi_{\Omega_k}\,\mathrm{d}x = \\ &= \int_{\Omega^{\mathrm{M}}} (\nabla\hat{\mathbf{u}}^* + (\nabla\hat{\mathbf{u}}^*)^T) : (\nabla\mathbf{v} + (\nabla\mathbf{v})^T)\,\chi_{\Omega^*}\,\mathrm{d}x = a_{\Omega^*}(\hat{\mathbf{u}}^*, \mathbf{v}),\end{aligned}$$

and

$$\lim_{k\to\infty}\int_{\Omega_k}\mathbf{f}\cdot\mathbf{v}_k\,\mathrm{d}x=\lim_{k\to\infty}\int_{\Omega^{\mathrm{M}}}\mathbf{f}\cdot\mathbf{v}\,\chi_{\Omega_k}\,\mathrm{d}x=\int_{\Omega^{\mathrm{M}}}\mathbf{f}\cdot\mathbf{v}\,\chi_{\Omega^*}\,\mathrm{d}x=\int_{\Omega^*}\mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x.$$

Define $\mathbf{u}^*$ as the restriction of $\hat{\mathbf{u}}^*$ to Ω^*, then by taking the limit in (3.3), we observe

$$\nu a_{\Omega^*}(\mathbf{u}^*,\mathbf{v})=\int_{\Omega^*}\mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x.$$

Since $\mathbf{v}\in C^\infty(\Omega^*;\mathbb{R}^2)$, $\operatorname{div}\mathbf{v}=0$ on Ω^* and that $\overline{\operatorname{supp}\mathbf{v}}\cap\Gamma_D^*=\emptyset$ is arbitrary, then by a density argument we have that the above equality holds true for $\mathbf{v}\in V_{\Gamma_D}(\Omega)$.

The fact that $\mathbf{u}^*=\mathbf{u}_i$ on Γ_i in the trace sense follows from the compact embedding of $H^{\frac{3}{2}+\epsilon}(\Omega^{\mathrm{M}};\mathbb{R}^2)$ into $L^2(\Gamma_i;\mathbb{R}^2)$ and the fact that each $\hat{\mathbf{u}}_k$ is given by $\mathbf{u}_i$ on Γ_i. Now we show that $\mathbf{u}^*=0$ on $\Gamma_w^*:=\partial\Omega^*\setminus(\Gamma_i\cup\Gamma_o)$ in the trace sense. Let O be an open set such that $\overline{O}\subset\Omega^{\mathrm{M}}\setminus\Omega^*$, note that $(\Gamma_i\cup\Gamma_o)\cap\overline{O}=\emptyset$. Then there exists $k_O>0$ for which $\overline{O}\subset\Omega^{\mathrm{M}}\setminus\Omega_k$ for all $k\geq k_O$, see [11, Proposition A3.8]. Let E be the extension by zero operator from Ω_k to Ω^{M}. It follows then that $E\mathbf{u}_k|_O=\mathbf{0}$ a.e. for $k\geq k_O$; hence it is straightforward to observe that $E\mathbf{u}^*|_O=\mathbf{0}$ a.e. and since O is arbitrary, $E\mathbf{u}^*=\mathbf{0}$ on $\Omega^{\mathrm{M}}\setminus\Omega^*$ and so $\hat{\mathbf{u}}^*=\mathbf{0}$ on Γ_w^*. Finally, $\nabla p^*=\mathbf{f}+\frac{\nu}{2}\operatorname{div}(\nabla\mathbf{u}^*+(\nabla\mathbf{u}^*)^T)$ in $\mathscr{D}'(\Omega)$ and hence $(\mathbf{u}^*,p^*)$ solves (S) in Ω^*.

Step 4: Existence of a minimizer. From the lower-semicontinuity of J^γ, and the weak convergences $\mathbf{u}_k\rightharpoonup\mathbf{u}^*$ in $L^q(\Gamma_0\cup\Gamma_i;\mathbb{R}^2)$ for every $q\geq 1$ and $p_k\rightharpoonup p^*$ in $L^q(\Gamma_0\cup\Gamma_i)$ for $q=2/(1-2\epsilon)$, we get

$$J^\gamma(\mathbf{u}^*,p^*,\Omega^*)\leq\liminf_{k\to\infty}J^\gamma(\mathbf{u}_k,p_k,\Omega_k)=\inf_{\Omega\in\mathcal{O}}J^\gamma(\mathbf{u}_\Omega,p_\Omega,\Omega).$$

□

4. Shape derivatives. In this section we consider the differentiability properties of the map $\Omega\mapsto(\mathbf{u}_\Omega,p_\Omega)$ by means of proper perturbations to Ω. We consider a family of perturbed domains $t\mapsto\Omega_t$ given by $\Omega_t=T_t(\Omega)$ where $t\mapsto T_t$ is a family of diffeomorphisms defined as

$$\begin{aligned}\frac{dT_s}{ds}&=\mathrm{V}(s)\circ T_s,\qquad s\in I\\ T_0&=\mathrm{Id}\end{aligned}$$

where I is an real interval such that $0\in I$. Further, we assume that $s\mapsto\mathrm{V}(s)$ is in $C^\infty(I;C^\infty(\overline{\Omega^{\mathrm{M}}};\mathbb{R}^2))$, and that there is a compact set in the region enclosed by $\partial\Omega^{\mathrm{M}}$ and $\partial\Omega^{\mathrm{m}}$ in which the support of $\mathrm{V}(s)$ lies for all $s\in I$. In particular, note that that Γ_i,Γ_0 are left unchanged via T_s as well as a portion of wall boundary part. We assume that $\Omega_0=\Omega$ (and hence each Ω_t) is proper in the sense defined in this paper.

It is convenient to introduce the *Piola transform* $\mathbb{P}_t$: Given $\mathbf{v}:\Omega_0\to\mathbb{R}^2$, then $\mathbb{P}_t(\mathbf{v}):\Omega_t\to\mathbb{R}^2$ is given by

$$\mathbf{v}\mapsto\mathbb{P}_t(\mathbf{v}):=(C_t\cdot\mathbf{v})\circ T_t^{-1},\tag{4.1}$$

for $C_t:=J_t^{-1}DT_t$, $J_t=\det(DT_t)$ and where DT_t is the Jacobian of $x\mapsto T_t(x)$ and for sufficiently small $t>0$, $J_t>0$. It follows that the Piola transform $\mathbb{P}_t$ is an isomorphism between $V_{\Gamma_D}(\Omega_0)$ and $V_{\Gamma_D}(\Omega_t)$. The weak solution in $H^1(\Omega_s;\mathbb{R}^2)\times$

$L^2_0(\Omega_s)$ to (2.1)-(2.5) in Ω_s is denoted as $(\mathbf{u}_s, p_s)$ and we consider a transported pair $(\mathbf{u}^s, p^s)$ defined on Ω as

$$\mathbf{u}^s := \mathbb{P}_t^{-1}(\mathbf{u}_s) \quad \text{and} \quad p^s := p_s \circ T_s.$$

It follows analogously to the approach in [2] that $I \supset J \ni s \mapsto \mathbf{u}^s \in H^1(\Omega_0; \mathbb{R}^2)$ is differentiable for some J; in contrast to the aforementioned reference we do not have continuity of $s \mapsto \mathbf{u}^s$ when considered with values in H^2 as in general $\mathbf{u}^s \notin H^2(\Omega; \mathbb{R}^2)$. However, as we only consider perturbations locally on Γ_w and uniformly away from points of junction with Γ_i and Γ_o, the following can be considered.

We denote by $D^{\mathrm{M}}_{\mathrm{m}}$ the open set in between $\partial\Omega^{\mathrm{M}}$ and $\partial\Omega^{\mathrm{m}}$, and define $\Omega^{\mathrm{M}}_{\mathrm{m}} = \Omega \cap D^{\mathrm{M}}_{\mathrm{m}}$. We say that $\mathbf{u}$ is *shape differentiable* in $H^1(\Omega; \mathbb{R}^2)$ if the following two hold true:

i) $s \mapsto \mathbf{u}_s \circ T_s \in H^1(\Omega; \mathbb{R}^2)$ is differentiable at $s = 0$. The derivative is denoted as $\dot{\mathbf{u}}$ and it is called the *the material derivative*.

ii) The restriction of velocity profile $\mathbf{u}_0$ associated to the initial domain Ω to $\Omega^{\mathrm{M}}_{\mathrm{m}}$ has H^2 regularity, i.e., $\mathbf{u}_0|_{\Omega^{\mathrm{M}}_{\mathrm{m}}} \in H^2(\Omega^{\mathrm{M}}_{\mathrm{m}}; \mathbb{R}^2)$.

Consequently, if $\mathbf{u}$ is shape differentiable in $H^1(\Omega; \mathbb{R}^2)$ then it follows that p is shape differentiable in $L^2(\Omega)$: here i) and ii) hold mutatis mutandis by reducing one order of differentiability on the mentioned Sobolev spaces. Further, the *shape derivative* is denoted as $\mathbf{u}'$ and given by

$$\mathbf{u}' = \dot{\mathbf{u}} - \nabla\mathbf{u} \cdot \mathrm{V}(0).$$

Particularly in this setting, we observe that $\mathbf{u}' \in H^1(\Omega; \mathbb{R}^2)$, as $\mathrm{V}(0) = \mathbf{0}$ outside $D^{\mathrm{M}}_{\mathrm{m}}$. We aim at expressing the shape derivative $\mathbf{u}'$ as the derivative of the extension of $\mathbf{u}_s$, i.e., as the derivative of the map $s \mapsto E\mathbf{u}_s$ where E is a continuous extension from Ω_s to Ω^{M} such that $E : H^1(\Omega; \mathbb{R}^2) \to H^1(\Omega^{\mathrm{M}}; \mathbb{R}^2)$ with a norm uniformly bounded with respect to s, and let $E_0 : L^2(\Omega) \to L^2(\Omega^{\mathrm{M}})$ be the extension by zero operator. In particular, we define

$$\mathbf{U}_s \circ T_s = E(\mathbf{u}_s \circ T_s), \quad \text{and} \quad P_s \circ T_s = E_0(p_s \circ T_s).$$

It can be proven that $J \ni s \mapsto \mathbf{U}_s \in H^1(\Omega^{\mathrm{M}}; \mathbb{R}^2)$ is continuously differentiable: This follows from the fact that $J \ni s \mapsto \mathbf{U}_s \circ T_s \in H^1(\Omega^{\mathrm{M}}; \mathbb{R}^2)$ is continuously differentiable, $\mathrm{V} : I \to \Omega^{\mathrm{M}}$ is zero outside $\Omega^{\mathrm{M}}_{\mathrm{m}}$, and $\mathbf{u}_s \in H^2(\Omega^{\mathrm{M}}_{\mathrm{m}}; \mathbb{R}^2)$ with a uniform norm with respect to s; see [2] and [13, Prop 2.38, p.71]. Hence, it follows that

$$\mathbf{u}' = \frac{\partial \mathbf{U}_s}{\partial s}(0)\Big|_{\Omega}, \quad \text{where} \quad \frac{\partial \mathbf{U}_s}{\partial s}(0) = \frac{\partial \mathbf{U}_s \circ T_s}{\partial s}(0) - \nabla\mathbf{U}_0 \cdot \mathrm{V}(0).$$

It follows that $\mathbf{u}$ is shape differentiable in $H^1(\Omega; \mathbb{R}^2)$ as the regularity of $s \mapsto \mathbf{u}^s$ imply the regularity of $s \mapsto \mathbf{u}_s \circ T_s$ and due to the regularity result in Lemma 2.3. Furthermore, this implies that p is shape differentiable in $L^2(\Omega)$, and in fact we observe the following.

Theorem 4.1. *The shape derivative* $(\mathbf{u}', p')$ *of* $(\mathbf{u}, p) \in H^1(\Omega; \mathbb{R}^2) \times L^2(\Omega)$ *satisfies (weakly) the following Stokes system*

$$-\operatorname{div} \sigma(\mathbf{u}', p') = \mathbf{0} \qquad \text{in } \Omega, \tag{4.2}$$

$$\operatorname{div} \mathbf{u}' = 0 \qquad \text{in } \Omega, \tag{4.3}$$

$$\mathbf{u}' = \mathbf{0} \qquad \text{on } \Gamma_i, \tag{4.4}$$

$$\mathbf{u}' = -(\nabla\mathbf{u}\,\mathbf{n})(\mathrm{V}(0) \cdot \mathbf{n}) \qquad \text{on } \Gamma_w, \tag{4.5}$$

$$\sigma(\mathbf{u}', p')\mathbf{n} = 0 \qquad \text{on } \Gamma_o. \tag{4.6}$$

Proof. Let $\boldsymbol{\phi} \in C^\infty(\Omega;\mathbb{R}^2)$ with $\overline{\text{supp }\boldsymbol{\phi}} \cap \Gamma_D = \emptyset$, then for s sufficiently small we have that $\boldsymbol{\phi} \in C^\infty(\Omega_s;\mathbb{R}^2)$ and $\overline{\text{supp }\boldsymbol{\phi}} \cap \Gamma_D^s = \emptyset$ where Γ_D^s is the Dirichlet part of the boundary of Ω_s. Hence,

$$\int_{\Omega_s} \frac{\nu}{2}(D\mathbf{u}_s : D\boldsymbol{\phi}) - p_s \operatorname{div}\boldsymbol{\phi} - \mathbf{f}\cdot\boldsymbol{\phi}\,\mathrm{d}x = \int_{\Omega^{\mathrm{M}}} \frac{\nu}{2}(D\mathbf{U}_s : D\boldsymbol{\phi}) - P_s \operatorname{div}\boldsymbol{\phi} - \mathbf{f}\cdot\boldsymbol{\phi}\,\mathrm{d}x = 0,$$

where $D\mathbf{z} := \nabla\mathbf{z} + (\nabla\mathbf{z})^T$ and for all $q \in L^2(\Omega^{\mathrm{M}})$

$$\int_{\Omega_s} q \operatorname{div}\mathbf{u}_s\,\mathrm{d}x = \int_{\Omega^{\mathrm{M}}} q \operatorname{div}\mathbf{U}_s\,\mathrm{d}x = 0.$$

Differentiation with respect to s and evaluation at $s = 0$ of the two above equations determine that

$$\int_{\Omega^{\mathrm{M}}} \frac{\nu}{2}(D\mathbf{U}' : D\boldsymbol{\phi}) - P' \operatorname{div}\boldsymbol{\phi}\,\mathrm{d}x = 0 \quad \Longrightarrow \quad \int_{\Omega} \frac{\nu}{2}(D\mathbf{u}' : D\boldsymbol{\phi}) - p' \operatorname{div}\boldsymbol{\phi}\,\mathrm{d}x = 0,$$

and

$$\int_{\Omega^{\mathrm{M}}} q \operatorname{div}\mathbf{U}'\,\mathrm{d}x = 0 \quad \Longrightarrow \quad \int_{\Omega} q \operatorname{div}\mathbf{u}'\,\mathrm{d}x = 0,$$

which show (4.2), (4.3), and (4.6).

The boundary conditions on Γ_i and Γ_w are obtained similarly as in [2]: Let $\boldsymbol{\phi} \in C^\infty(\overline{\Omega^{\mathrm{M}}};\mathbb{R}^2)$ with $\overline{\text{supp }\boldsymbol{\phi}} \cap \Gamma_o = \emptyset$, then

$$\int_{\partial\Omega_s} \mathbf{u}_s \cdot \boldsymbol{\phi}\,\mathrm{d}S = \int_{\partial\Omega} \mathbf{u}_i \cdot \boldsymbol{\phi}\,\mathrm{d}S,$$

where we have used that $\mathbf{u}_s = 0$ on Γ_w^s and $\mathbf{u}_s = \mathbf{u}_i$ on Γ_i, and that Γ_i remains unchanged for each Ω_t. Differentiation with respect to s and evaluation at $s = 0$ of the above expression leads to

$$\int_{\partial\Omega} (\mathbf{u}' + (\nabla\mathbf{u}\,\mathbf{n})(\mathrm{V}(0)\cdot\mathbf{n}))\cdot\boldsymbol{\phi}\,\mathrm{d}S = 0.$$

Given that $\mathrm{V}(0) = 0$ on Γ_i, then $\mathbf{u}' = 0$ on Γ_i, and further $\mathbf{u}' + (\nabla\mathbf{u}\,\mathbf{n})(\mathrm{V}(0)\cdot\mathbf{n}) = \mathbf{0}$ on Γ_w. □

Note that in the previous result, the boundary regularity in Γ_w is the minimal one possible and it is dominated by $\nabla\mathbf{un}$: Since we have that $\mathbf{u}|_{\Omega^{\mathrm{M}}_{\mathrm{m}}} \in H^2(\Omega^{\mathrm{M}}_{\mathrm{m}};\mathbb{R}^2)$ then $\nabla\mathbf{un}|_{\Gamma_w} \in H^{1/2}(\Gamma_w;\mathbb{R}^2)$. Further, the computation of the state variables together with the shape derivatives are suitable to be implemented with a finite element scheme and subsequently a descent algorithm may be used for shape improvements.

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