La Matematica (2022) 1:317–341
https://doi.org/10.1007/s44007-022-00020-3

ORIGINAL RESEARCH ARTICLE

Optimal L^2 A Priori Error Estimates for the Biot System

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Received: 3 March 2021 / Revised: 14 September 2021 / Accepted: 2 February 2022 /
Published online: 2 March 2022

Abstract

Coupled subsurface fluid flow and geomechanics is receiving growing research interests for applications in geothermal energy, unconventional oil and gas recovery and geological CO_2 sequestration. A key model characterizing these processes is the Biot system. In this paper, we present optimal L^2 error estimates for the Biot system. The flow equation for the pressure is discretized in time by a backward Euler scheme and in space by a continuous Galerkin scheme, while the elastic displacement equation is discretized at all time steps by a continuous Galerkin scheme. We prove optimal L^2 a priori error estimates in space for the resulting Galerkin scheme, provided the domain is a convex polygon or polyhedron according to the dimension and the data and solution spaces have sufficient regularity. The key idea is to introduce suitable auxiliary elliptic projections in the error equations and to use one such projection to approximate the given initial pressure. These theoretical results are confirmed by numerical experiments performed with a fixed-stress split algorithm.

Keywords Biot system · Error estimates · Elliptic projection

1 Introduction

Poromechanics or Biot systems have numerous important applications such as simulating fluid flow in natural (static) and hydraulic (dynamic) fractures, fracture analysis of aging bones, multiple-network poroelastic theory (MPET) arising in dementia and Alzheimer's disease, and evaluation of accelerated degradation of ceramic matrix

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composites (CMC) in aerospace shuttles. Here mathematical modeling is challenging because it involves not only coupled chemical reactions, diffusion, and deformation but also initiation, propagation, and branching of cracks in the bulk matrix as well as fluid flowing through cracks. To address these challenges, high fidelity numerical schemes and multiphysics models must be coupled in order to simulate these processes and their interactions accurately and efficiently. A priori and a posteriori analyses are essential in formulating these schemes. Barbeiro and Wheeler [3] considered mixed finite elements for Darcy flow and Galerkin finite elements for elasticity and established convergence with respect to the L^2-norm for the pressure and for the average fluid velocity and with respect to the H^1-norm for the deformation. Girault et al. [11] considered a poroelastic region embedded into an elastic non-porous region, where a fixed-stress split algorithm is employed, with the elastic displacement equations discretized by a continuous Galerkin scheme, and the flow equations discretized by either a continuous Galerkin scheme or a mixed scheme. The authors established a priori error estimates for the resulting Galerkin scheme as well as the mixed scheme. The work is further extended to Enriched Galerkin scheme for flow [8], where residual-based a posteriori error estimates are established with both lower and upper bounds. To date, most of the error analysis has produced estimates in the energy norm for the pressure and displacement system, even though numerical experiments indicate that higher-order optimal L^2 estimates, such as would be obtained by directly interpolating the exact solution, are also valid. But proving that the error in L^2 of the discrete solution satisfies an improved bound is by no means trivial and requires delicate arguments.

As an illustration, consider the very simple case of a Laplace equation in a bounded polygonal domain Ω with a homogeneous Dirichlet condition on its boundary $\partial\Omega$

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

In variational form, this problem reads: find $u \in H_0^1(\Omega)$, such that

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f\, v. \tag{1.1}$$

To discretize this problem, let $X_h \subset H_0^1(\Omega)$ be a finite element space and replace (1.1) by: find $u_h \in X_h$ such that

$$\forall v_h \in X_h, \quad \int_\Omega \nabla u_h \cdot \nabla v_h = \int_\Omega f\, v_h. \tag{1.2}$$

The similarity between (1.2) and (1.1) explains why u_h is called the elliptic projection of u on X_h. This is confirmed by the following equality

$$\forall v_h \in X_h, \quad \int_\Omega \nabla(u - u_h) \cdot \nabla v_h = 0, \tag{1.3}$$

called Galerkin orthogonality that leads to the following energy estimate (see for instance [5]):

$$\forall v_h \in X_h, \quad \|\nabla(u - u_h)\|_{L^2(\Omega)} \le \inf_{v_h \in X_h} \|\nabla(u - v_h)\|_{L^2(\Omega)}. \tag{1.4}$$

It reduces the interpolation error of u in the H^1 norm. Thinking in terms of interpolation errors, one would then expect the L^2 bound

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch\|\nabla(u - u_h)\|_{L^2(\Omega)}. \tag{1.5}$$

But the proof of this estimate is far from obvious. It was obtained, under suitable assumptions on the domain, via a clever duality argument by Aubin [2] and Nitsche [17]. The idea is to write the L^2 norm as a dual norm,

$$\|u - u_h\|_{L^2(\Omega)} = \sup_{g \in L^2(\Omega)} \frac{\int_\Omega (u - u_h)g}{\|g\|_{L^2(\Omega)}},$$

and introduce the function φ, unique solution of the auxiliary problem

$$-\Delta\varphi = g \text{ in } \Omega, \quad \varphi = 0 \quad \text{on } \partial\Omega.$$

Then, owing to (1.3),

$$\int_\Omega (u - u_h)g = \int_\Omega \nabla(u - u_h) \cdot \nabla\varphi = \int_\Omega \nabla(u - u_h) \cdot \nabla(\varphi - \varphi_h),$$

for any $\varphi_h \in X_h$ and hence

$$\int_\Omega (u - u_h)g \leq \Big(\inf_{\varphi_h \in X_h} \|\nabla(\varphi - \varphi_h)\|_{L^2(\Omega)}\Big)\|\nabla(u - u_h)\|_{L^2(\Omega)}.$$

Thus, a possible gain in accuracy of the error $u - u_h$ in L^2 results from the quality of approximation of φ in X_h for the H^1 norm. This depends solely on the regularity of φ, which in turn depends on the angles of the polygonal boundary $\partial\Omega$. Indeed, as g belongs only to L^2, the function φ belongs at most to H^2, and according to the regularity results of Grisvard [12], this holds if Ω is convex. The Aubin–Nitsche argument that looks deceptively simple heavily relies on the nature of the problem under scrutiny; it must be carefully adapted to the problem and does not always succeed.

In this paper, we present optimal L^2 error estimates for the Biot system when discretized as follows. The flow equation for the pressure is discretized in time by a backward Euler scheme and in space by a continuous Galerkin scheme, while the elastic displacement equation is discretized at all time steps by a continuous Galerkin scheme. We prove optimal L^2 a priori error estimates in space for the resulting Galerkin scheme, which provided the domain is a convex polygon or polyhedron according to the dimension and the data and solution spaces have sufficient regularity. The key idea is to introduce elliptic projections in the spirit of (1.2) [22] in the error equations and to use one such projection to approximate the given initial pressure. These theoretical results, which to the best of our knowledge are new for this problem, are illustrated by numerical experiments performed with a fixed-stress split algorithm.

This article is organized as follows: the notation and statement of the problem are introduced in Sect. 2, its finite element discretization is described in Sect. 3, and error estimates are derived. Section 5 is devoted to numerical experiments and the last section presents a summary of results.

2 Statement of the Problem

Let us recall the notation used in this work.

2.1 Notation

To be specific, the notation is expressed in three dimensions in a bounded connected open set $\Omega \subset \mathbb{R}^3$. The scalar product of $L^2(\Omega)$ is denoted by $(\cdot, \cdot)$

$$\forall f, g \in L^2(\Omega), \quad (f, g) = \int_\Omega f(\boldsymbol{x}) g(\boldsymbol{x}) d\boldsymbol{x}.$$

For any non-negative integer m, the classical Sobolev space $H^m(\Omega)$ is defined by (cf. [1] or [16]),

$$H^m(\Omega) = \{v \in L^2(\Omega) \,:\, \partial^k v \in L^2(\Omega) \,\forall\, |k| \le m\},$$

where

$$\partial^k v = \frac{\partial^{|k|} v}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}},$$

equipped with the following seminorm and norm for which it is a Hilbert space:

$$|v|_{H^m(\Omega)} = \left[\sum_{|k|=m} \int_\Omega |\partial^k v|^2 \,\mathrm{d}\boldsymbol{x} \right]^{\frac{1}{2}}, \quad \|v\|_{H^m(\Omega)} = \left[\sum_{0 \le |k| \le m} |v|^2_{H^k(\Omega)} \right]^{\frac{1}{2}}.$$

The subspace of functions of $H^1(\Omega)$ that vanish on $\partial\Omega$ is $H^1_0(\Omega)$,

$$H^1_0(\Omega) = \{v \in H^1(\Omega) \,:\, v|_{\partial\Omega} = 0\}.$$

We also recall Korn's inequality valid for all functions $\boldsymbol{v}$ in $H^1_0(\Omega)^3$,

$$|\boldsymbol{v}|_{H^1(\Omega)} \le \mathcal{K} \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2(\Omega)}, \tag{2.1}$$

and the generalized Poincaré inequality valid for all functions in $H^1(\Omega)$,

$$\|v\|_{L^2(\Omega)} \le \mathcal{P}\Big(|(v, 1)| + |v|_{H^1(\Omega)}\Big), \tag{2.2}$$

where $\boldsymbol{\varepsilon}(\boldsymbol{v})$ is the strain tensor, and $\mathcal{K}$ and $\mathcal{P}$ are constants depending only on Ω.

As usual, for handling time-dependent problems, it is convenient to consider measurable functions defined on a time interval $]a, b[$ with values in a functional space, say X (cf. [14]). More precisely, let $\|\cdot\|_X$ denote the norm of X; then for any number r, $1 \leq r \leq \infty$, we define

$$L^r(a, b; X) = \left\{ f \text{ measurable in }]a, b[: \int_a^b \|f(t)\|_X^r \mathrm{d}t < \infty \right\},$$

equipped with the norm

$$\|f\|_{L^r(a,b;X)} = \left(\int_a^b \|f(t)\|_X^r \mathrm{d}t \right)^{\frac{1}{r}},$$

with the usual modification if $r = \infty$. It is a Banach space if X is a Banach space, and for $r = 2$, it is a Hilbert space if X is a Hilbert space. Derivatives with respect to time are denoted by ∂_t and we define, for instance,

$$H^1(a, b; X) = \{ f \in L^2(a, b; X) \, : \, \partial_t f \in L^2(a, b; X) \}.$$

2.2 Biot's Model

In a bounded, connected, Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with boundary $\partial\Omega$ and exterior unit normal $\boldsymbol{n}$, and in an interval of time $]0, T[$, we consider Biot's consolidation model for a linear elastic, homogeneous, isotropic, and porous solid saturated with a slightly compressible fluid, see [4]. The unknowns are the solid's displacement $\boldsymbol{u}$ and the fluid's pressure p. This model is based on a *quasi-static* assumption, namely it assumes that the material deformation is much slower than the flow rate, and hence, the second time derivative of the displacement (i.e., the acceleration) is zero. After linearization and simplifications, it leads to the following system of equations a.e. in $\Omega \times]0, T[$

$$\partial_t \Big(\frac{1}{M} p + \alpha \, \nabla \cdot \boldsymbol{u} \Big) - \nabla \cdot \boldsymbol{\kappa} \, \nabla \, p = q, \tag{2.3a}$$

$$- \nabla \cdot \big(\lambda (\nabla \cdot \boldsymbol{u}) \boldsymbol{I} + 2G \boldsymbol{\varepsilon}(\boldsymbol{u}) - \alpha \, p \, \boldsymbol{I} \big) = \boldsymbol{f}. \tag{2.3b}$$

This system is complemented by an initial condition

$$p(0) = p_0 \text{ in } \Omega, \tag{2.4}$$

and boundary conditions on the pressure p and the displacement $\boldsymbol{u}$. In practical situations, these are mixed, in general non-homogeneous Dirichlet and natural boundary conditions. But, except in particular cases, mixed boundary conditions do not lead to optimal error estimates in L^2. For this reason, we assume homogeneous Dirichlet boundary conditions for $\boldsymbol{u}$ and natural boundary conditions for p,

$$\boldsymbol{u} = \boldsymbol{0}, \quad \boldsymbol{\kappa} \nabla \, p \cdot \boldsymbol{n} = 0, \text{ on } \partial\Omega. \tag{2.5}$$

Note that gravity is neglected, but it can easily be incorporated in the problem. Here, $\lambda > 0$ and $G > 0$ are the Lamé coefficients, $\alpha > 0$ is the Biot–Willis constant, which is usually around one, $M > 0$ is the second Biot constant, q is a volumetric fluid source term, and $\boldsymbol{\kappa}$ is the permeability tensor, assumed to be symmetric, uniformly bounded, and uniformly positive definite, i.e., each eigenvalue λ_i of $\boldsymbol{\kappa}$ is real and there exist two constants $\lambda_{\min} > 0$ and $\lambda_{\max} > 0$ such that

$$\text{a.e. } \boldsymbol{x} \in \Omega, \quad \lambda_{\min} \leq \lambda_i(\boldsymbol{x}) \leq \lambda_{\max}. \tag{2.6}$$

Strictly speaking, the initial condition should be given as

$$\Big(\frac{1}{M} p + \alpha \nabla \cdot \boldsymbol{u}\Big)(t = 0) = \frac{1}{M} p_0 + \alpha \nabla \cdot \boldsymbol{u}_0.$$

However, in practice, the pressure is either measured or computed through a hydrostatic assumption and the initial displacement is computed satisfying (2.3b). When the data are sufficiently smooth, as stated below, initializing the pressure is sufficient to determine the solution.

It is well known, see, for instance, [9, 18, 21], that for $\boldsymbol{f} \in L^2(0, T; H^{-1}(\Omega)^d)$ and $q \in L^2(\Omega \times]0, T[)$, problem (2.3)–(2.5) has the equivalent variational formulation: Find $\boldsymbol{u}$ in $L^\infty\big(0, T; H_0^1(\Omega)^d\big)$ and p in $L^\infty\big(0, T; L^2(\Omega)\big) \cap L^2\big(0, T; H^1(\Omega)\big)$ solving

$$\forall \boldsymbol{v} \in H_0^1(\Omega)^d,\ 2G\big(\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})\big) + \lambda\big(\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v}\big) - \alpha\big(p, \nabla \cdot \boldsymbol{v}\big) = \big(\boldsymbol{f}, \boldsymbol{v}\big), \tag{2.7a}$$

$$\begin{aligned} &\forall \theta \in H^1(\Omega),\ \big(\partial_t\big(\frac{1}{M} p + \alpha \nabla \cdot \boldsymbol{u}\big), \theta\big) + \big(\boldsymbol{\kappa} \nabla p, \nabla \theta\big) = \big(q, \theta\big), \\ &p(0) = p_0 \text{ in } \Omega. \end{aligned} \tag{2.7b}$$

Moreover, if in addition the data satisfy $\boldsymbol{f} \in H^1(0, T; L^2(\Omega)^d)$, and $p_0 \in H^1(\Omega)$, this problem has one and only one solution that depends continuously on the data.

3 Finite Element Discretization

From now on, we assume that the boundary of the domain Ω is a polygonal curve in two dimensions or a polyhedral surface in three dimensions, so that Ω can be entirely meshed by triangles or tetrahedra according to the dimension. For $h > 0$, let $\mathcal{T}_h$ be a regular family of conforming simplicial meshes of the domain $\overline{\Omega}$, with h the maximum element diameter. The family of meshes is regular in the sense of Ciarlet [5]: there exists a constant $\sigma > 0$, independent of h, such that

$$\frac{h_E}{\varrho_E} \leq \sigma, \quad \forall E \in \mathcal{T}_h, \tag{3.1}$$

where h_E is the diameter of E and ϱ_E the diameter of the ball inscribed in E.

Let $k \geq 1$ and $m \geq 1$ be two integers. On this mesh, the displacement and pressure are approximated, respectively, by the following finite element spaces:

$$\boldsymbol{X}_h := \{\boldsymbol{v} \in H_0^1(\Omega)^d \ : \ \boldsymbol{v}|_E \in \mathbb{P}_m(E)^d, \forall E \in \mathcal{T}_h\}, \tag{3.2}$$

$$Q_h = \{q \in H^1(\Omega) \ : \ q|_E \in \mathbb{P}_k(E), \forall E \in \mathcal{T}_h\}. \tag{3.3}$$

Regarding approximation in time, the interval $[0, T]$ is divided into N equal subintervals with length Δt and endpoints $t_n = n\Delta t$. The choice of equal time steps is a simplification; the material below extends readily to variable time steps. Observing that $\boldsymbol{f}$ is continuous in time, it is approximated at each time step by its pointwise value,

$$\boldsymbol{f}^n = \boldsymbol{f}(t_n). \tag{3.4}$$

It is convenient to do the same with the source term q; for this, we assume that q belongs to $\mathcal{C}^0([0, T]; L^2(\Omega))$ and approximate it by its pointwise value

$$q^n = q(t_n). \tag{3.5}$$

Starting from

$$p_h^0 = S_h(p(0)), \tag{3.6}$$

where S_h is a suitable approximation operator that will be chosen below, the initial displacement is computed by solving

$$\forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \quad 2G\big(\boldsymbol{\varepsilon}(\boldsymbol{u}_h^0), \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big) + \lambda\big(\nabla \cdot \boldsymbol{u}_h^0, \nabla \cdot \boldsymbol{v}_h\big) = \alpha\big(p_h^0, \nabla \cdot \boldsymbol{v}_h\big) + \big(\boldsymbol{f}^0, \boldsymbol{v}_h\big). \tag{3.7}$$

Then, for $1 \leq n \leq N$, the scheme constructs a sequence $(p_h^n) \in Q_h$ and a sequence $(\boldsymbol{u}_h^n) \in \boldsymbol{X}_h$, solution of

$$\forall \theta_h \in Q_h, \quad \frac{1}{M}\frac{1}{\Delta t}\big(p_h^n - p_h^{n-1}, \theta_h\big) + \big(\boldsymbol{\kappa}\nabla p_h^n, \nabla \theta_h\big) + \frac{\alpha}{\Delta t}\big(\nabla \cdot (\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}), \theta_h\big) = (q^n, \theta_h); \tag{3.8}$$

$$\forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \quad 2G\big(\boldsymbol{\varepsilon}(\boldsymbol{u}_h^n), \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big) + \lambda\big(\nabla \cdot \boldsymbol{u}_h^n, \nabla \cdot \boldsymbol{v}_h\big) = \alpha\big(p_h^n, \nabla \cdot \boldsymbol{v}_h\big) + \big(\boldsymbol{f}^n, \boldsymbol{v}_h\big). \tag{3.9}$$

It has been proved (see, for example, [19, 20]) that, owing to Korn's inequality, the discrete scheme (3.6)–(3.9) generates two unique sequences (p_h^n) and $(\boldsymbol{u}_h^n)$, $0 \leq n \leq N$. These references also establish a priori error estimates of the displacement in $L^\infty\big(0, T; H_0^1(\Omega)^d\big)$ and the pressure in $L^\infty\big(0, T; L^2(\Omega)\big) \cap L^2\big(0, T; H^1(\Omega)\big)$. The following theorem gives typical a priori error estimates:

Theorem 1 *If the data $\boldsymbol{f}$ belong to $H^1(0,T;L^2(\Omega)^d)$, q to $\mathcal{C}^0(0,T;L^2(\Omega))$ and p_0 to $H^1(\Omega)$, then there exists a constant C, independent of h and Δt, such that*

$$\begin{aligned}
&\|r_h(p)(t_n) - p_h^n\|^2_{L^2(\Omega)} + \|\boldsymbol{\varepsilon}(R_h(\boldsymbol{u})(t_n) - \boldsymbol{u}_h^n)\|^2_{L^2(\Omega)} \\
&\quad + \|\nabla\cdot(R_h(\boldsymbol{u})(t_n) - \boldsymbol{u}_h^n)\|^2_{L^2(\Omega)} \\
&\quad + \sum_{m=1}^{n} \|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla(r_h(p)(t_m) - p_h^m)\|^2_{L^2(\Omega)} \\
&\le C\Big(\|r_h(p)(0) - p_h^0\|^2_{L^2(\Omega)} + \|\boldsymbol{\varepsilon}(R_h(\boldsymbol{u})(0) - \boldsymbol{u}_h^0)\|^2_{L^2(\Omega)} \\
&\quad + \|\nabla\cdot(R_h(\boldsymbol{u})(0) - \boldsymbol{u}_h^0)\|^2_{L^2(\Omega)} \\
&\quad + \sum_{m=1}^{n} \Big\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla(r_h(p)(t_m) - p(t_m))\Big\|^2_{L^2(\Omega)} \\
&\quad + \|r_h(\partial_t p) - \partial_t p\|^2_{L^2(\Omega\times(0,t_n))} \\
&\quad + \|R_h(\partial_t \boldsymbol{u}) - \partial_t \boldsymbol{u}\|^2_{L^2(0,t_n;H^1(\Omega)^d)} \\
&\quad + (\Delta t)^2\big(\|p''\|^2_{L^2(\Omega\times(0,t_n))} + \|\boldsymbol{u}''\|^2_{L^2(0,t_n;H^1(\Omega)^d)}\big)\Big),
\end{aligned} \tag{3.10}$$

where r_h and R_h are suitable approximation operators in space with values in Q_h and $\boldsymbol{X}_h$, respectively.

It can be shown that, owing to the above assumptions on the data, all terms in the right-hand side are meaningful. Moreover, when the solution is sufficiently smooth, considering the degree of the polynomial functions of Q_h and $\boldsymbol{X}_h$, (3.10) yields the error bounds, again with a constant C independent of h and Δt,

$$\begin{aligned}
&\|r_h(p)(t_n) - p_h^n\|^2_{L^2(\Omega)} + \|\boldsymbol{\varepsilon}(R_h(\boldsymbol{u})(t_n) - \boldsymbol{u}_h^n)\|^2_{L^2(\Omega)} + \|\nabla\cdot(R_h(\boldsymbol{u})(t_n) - \boldsymbol{u}_h^n)\|^2_{L^2(\Omega)} \\
&\quad + \sum_{m=1}^{n} \|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla(r_h(p)(t_m) - p_h^m)\|^2_{L^2(\Omega)} \le C\Big((\Delta t)^2 + h^{2(m-1)} + h^{2(k-1)}\Big).
\end{aligned} \tag{3.11}$$

4 Error Estimates

We propose to derive sharper a priori estimates *in space* for the displacement and pressure in the L^2 norm. The argument of Aubin [2] and Nitsche [17] discussed in the introduction does not readily adapt to problem (2.7)–(2.7b), which is a complex time-dependent system; but we can try to use the underlying idea of elliptic projection. Thus, following the strategy of [7, 22] for simpler time parabolic equations, the idea is to express all error estimates in terms of the L^2 norm of suitable elliptic projections. They will give more accurate results than the energy norm. More precisely, we shall prove further on, in Theorem 3, the following estimate:

$$\frac{1}{4M}\sup_{1\le n\le N}\|p^n-p_h^n\|^2_{L^2(\Omega)}+\frac{1}{2M}\sum_{n=1}^N\|p^n-p_h^n-(p^{n-1}-p_h^{n-1})\|^2_{L^2(\Omega)}$$
$$\le Ch^2\Big[\Big\|\boldsymbol{\kappa}^{\frac12}\nabla(p-\theta_h)\Big\|^2_{C^0(0,T;L^2(\Omega))}+\Big\|\boldsymbol{\kappa}^{\frac12}\nabla(\partial_t p-\bar\theta_h)\Big\|^2_{L^2(\Omega\times]0,T[)}$$
$$+\Big\|\boldsymbol{\kappa}^{\frac12}\Big\|^2_{L^\infty(\Omega)}\|\boldsymbol{\varepsilon}(\partial_t\boldsymbol{u}-\boldsymbol{v}_h)\|^2_{L^2(\Omega\times]0,T[)}\Big]+C(\Delta t)^2\Big\|\frac{1}{M}p''+\alpha\nabla\cdot(\boldsymbol{u}'')\Big\|^2_{L^2(\Omega\times]0,T[)},$$

that holds for all θ_h and $\bar\theta_h$ in Q_h and all $\boldsymbol{v}_h$ in $\boldsymbol{X}_h$, under suitable assumptions on the solution and the domain.

4.1 Elliptic Projections

Consider the following elliptic projections $S_h(p)\in Q_h$ and $P_h(\boldsymbol{u})\in\boldsymbol{X}_h$ of the exact solution almost everywhere in $]0,T[$:

$$\forall\theta_h\in Q_h,\quad \big(\boldsymbol{\kappa}\nabla(S_h(p(t))-p(t)),\nabla\theta_h\big)=0,\quad \big(S_h(p(t))-p(t),1\big)=0,\ \text{a.e. } t\in]0,T[,\tag{4.1}$$

$$\forall\boldsymbol{v}_h\in\boldsymbol{X}_h,\quad 2G\big(\boldsymbol{\varepsilon}(P_h(\boldsymbol{u}(t))-\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big)+\lambda\big(\nabla\cdot(P_h(\boldsymbol{u}(t))-\boldsymbol{u}(t)),\nabla\cdot\boldsymbol{v}_h\big)=0,\ \text{a.e. } t\in]0,T[.\tag{4.2}$$

Here S_h is the operator used for the initial value $S_h(p(0))$. Note that the second equation is added to the definition of S_h to guarantee uniqueness of the projection. Again, owing to Korn's inequality, (4.2) defines uniquely $P_h(\boldsymbol{u}(t))$. To simplify, we can freeze time since it only acts as a parameter. Being projections, these operators satisfy for all $p\in H^1(\Omega)\cap L^2_0(\Omega)$ and $\boldsymbol{u}\in H^1_0(\Omega)^d$, optimal approximation properties in the energy norm,

$$\forall\theta_h\in Q_h,\quad \Big\|\boldsymbol{\kappa}^{\frac12}\nabla(S_h(p)-p)\Big\|_{L^2(\Omega)}\le\Big\|\boldsymbol{\kappa}^{\frac12}\nabla(\theta_h-p)\Big\|_{L^2(\Omega)},\tag{4.3}$$

$$\forall\boldsymbol{v}_h\in\boldsymbol{X}_h,\quad \Big(2G\|\boldsymbol{\varepsilon}(P_h(\boldsymbol{u})-\boldsymbol{u})\|^2_{L^2(\Omega)}+\lambda\|\nabla\cdot(P_h(\boldsymbol{u})-\boldsymbol{u})\|^2_{L^2(\Omega)}\Big)^{\frac12}$$
$$\le\Big(2G\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h-\boldsymbol{u})\|^2_{L^2(\Omega)}+\lambda\|\nabla\cdot(\boldsymbol{v}_h-\boldsymbol{u})\|^2_{L^2(\Omega)}\Big)^{\frac12}.\tag{4.4}$$

By an Aubin–Nitsche duality argument [2, 17], these projections have the following approximation properties in L^2.

Theorem 2 *Let the domain Ω be a convex polygon or polyhedron according to the dimension. Then, there exists a constant $C_{\boldsymbol{u}}$, independent of h, such that for all $\boldsymbol{u}\in$*

$H_0^1(\Omega)^d$,

$$\forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \quad \|P_h(\boldsymbol{u}) - \boldsymbol{u}\|_{L^2(\Omega)} \leq C_{\boldsymbol{u}} h\Big(2G\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h - \boldsymbol{u})\|^2_{L^2(\Omega)} + \lambda\|\nabla\cdot(\boldsymbol{v}_h - \boldsymbol{u})\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}. \tag{4.5}$$

If in addition, the coefficients of $\boldsymbol{\kappa}$ *belong to* $W^{1,\infty}(\Omega)$, *then there exists a constant* C_p, *independent of* h, *such that, for all* $p \in H^1(\Omega)$,

$$\forall \theta_h \in Q_h, \quad \|S_h(p) - p\|_{L^2(\Omega)} \leq C_p h \left\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla(\theta_h - p)\right\|_{L^2(\Omega)}. \tag{4.6}$$

Proof The proof is sketched because it is an easy variant of the elements of proof given in the introduction. For (4.5), we use the same duality argument with auxiliary function $\boldsymbol{\varphi} \in H_0^1(\Omega)^d$ solution of

$$\forall \boldsymbol{v} \in H_0^1(\Omega)^d, \quad 2G\big(\boldsymbol{\varepsilon}(\boldsymbol{\varphi}), \boldsymbol{\varepsilon}(\boldsymbol{v})\big) + \lambda\big(\nabla\cdot\boldsymbol{\varphi}, \nabla\cdot\boldsymbol{v}\big) = \big(\boldsymbol{g}, \boldsymbol{v}\big),$$

where $\boldsymbol{g}$ is an arbitrary function of $L^2(\Omega)^d$. This linear elasticity system with constant coefficients has a unique solution $\boldsymbol{\varphi}$ that, according to [12], belongs to $H^2(\Omega)^d$ when Ω is convex and there exists a constant C depending only on Ω such that

$$\|\boldsymbol{\varphi}\|_{H^2(\Omega)} \leq C\|\boldsymbol{g}\|_{L^2(\Omega)}.$$

By proceeding as in the Introduction, we readily derive

$$\begin{aligned}\big|\big(P_h(\boldsymbol{u}) - \boldsymbol{u}, \boldsymbol{g}\big)\big| \leq \inf_{\boldsymbol{\varphi}_h \in \boldsymbol{X}_h} &\Big(2G\|\boldsymbol{\varepsilon}(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)\|^2_{L^2(\Omega)} + \lambda\|\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} \\ &\times \Big(2G\|\boldsymbol{\varepsilon}(P_h(\boldsymbol{u}) - \boldsymbol{u})\|^2_{L^2(\Omega)} + \lambda\|\nabla\cdot(P_h(\boldsymbol{u}) - \boldsymbol{u})\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}.\end{aligned}$$

Then (4.5) follows from the above regularity of $\boldsymbol{\varphi}$ and standard approximation properties of $\boldsymbol{X}_h$.

For (4.6), in view of the second part of (4.1), it is convenient to work in the space of L^2 functions with zero mean value,

$$L_0^2(\Omega) = \{\theta \in L^2(\Omega) \,:\, (\theta, 1) = 0\}.$$

Then for any g in $L_0^2(\Omega)$, the relevant auxiliary function φ in $H^1(\Omega) \cap L_0^2(\Omega)$ is the unique solution of

$$-\mathrm{div}\big(\boldsymbol{\kappa}\nabla\varphi\big) = g \quad \text{in } \Omega, \quad \boldsymbol{\kappa}\nabla\varphi\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega,$$

and we easily obtain

$$\big|\big(S_h(p) - p, g\big)\big| \leq \inf_{\theta_h \in Q_h}\left\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla(\varphi - \theta_h)\right\|_{L^2(\Omega)}\left\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla(S_h(p) - p)\right\|^2_{L^2(\Omega)}.$$

The regularity of φ can be assessed by setting $\boldsymbol{w} = \boldsymbol{\kappa}\nabla\varphi$ and observing that if Ω is convex and the coefficients of $\boldsymbol{\kappa}$ are in $W^{1,\infty}(\Omega)$ then $\boldsymbol{w}$ belongs to $H^1(\Omega)^d$ (see [10] for instance). In turn, this implies that φ belongs to $H^2(\Omega)$ and (4.6) follows from standard approximation properties of Q_h.

□

Remark 1 The above proof shows that the extra factor h in (4.5) and (4.6) is obtained when the solution of similar elliptic problems with any data in L^2, and homogeneous boundary conditions, belongs to H^2. It cannot be improved, whatever the degree of the polynomials, because the data for these problems must be measured in L^2. When the domain has corners (and the other data are smooth enough) such regularity holds if the domain is convex, see for example [12, 13]. If the domain is Lipschitz but not convex (and again the other data are smooth enough), then the extra factor has the form h^s where $\frac{1}{2} < s < 1$ depending on the inner angles of the domain, see [12]. The most unfavorable case is that of mixed boundary conditions when the change in conditions does not occur at a corner, see [6].

4.2 Error equalities

In order to derive the error equations, we write the displacement equation at time t_n, $0 \le n \le N$,

$$\forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \quad 2G\big(\boldsymbol{\varepsilon}(\boldsymbol{u}^n), \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big) + \lambda\big(\nabla\cdot\boldsymbol{u}^n, \nabla\cdot\boldsymbol{v}_h\big) = \alpha\big(p^n, \nabla\cdot\boldsymbol{v}_h\big) + \big(\boldsymbol{f}^n, \boldsymbol{v}_h\big), \tag{4.7}$$

and the flow equation at time t_n, $1 \le n \le N$, but it is convenient to express the time derivative as a difference quotient; this gives

$$\forall \theta_h \in Q_h, \quad \frac{1}{M\Delta t}\big(p^n - p^{n-1}, \theta_h\big) + \big(\boldsymbol{\kappa}\nabla p^n, \nabla\theta_h\big) + \frac{\alpha}{\Delta t}\big(\nabla\cdot(\boldsymbol{u}^n - \boldsymbol{u}^{n-1}), \theta_h\big) \\ = (q^n, \theta_h) + E_n(\theta_h), \tag{4.8}$$

where

$$E_n(\theta_h) = \frac{1}{M}\Big[\big(\frac{1}{\Delta t}(p^n - p^{n-1}) - (\partial_t p)^n, \theta_h\big)\Big] \\ +\alpha\Big[\big(\frac{1}{\Delta t}\nabla\cdot(\boldsymbol{u}^n - \boldsymbol{u}^{n-1}) - \nabla\cdot(\partial_t\boldsymbol{u})^n, \theta_h\big)\Big]. \tag{4.9}$$

Here q^n and $\boldsymbol{f}^n$ are defined, respectively, by (3.5) and (3.4); the other superscript n indicates the value at time t_n.

To simplify, it is convenient to denote the time difference of a function v by the symbol δ,

$$\delta v^n = v^n - v^{n-1}. \tag{4.10}$$

With this notation, we subtract (4.8) from (3.8), for all $\theta_h \in Q_h$,

$$\frac{1}{M\Delta t}\big(\delta p_h^n - \delta p^n, \theta_h\big) + \big(\boldsymbol{\kappa}\nabla(p_h^n - p^n), \nabla\theta_h\big) \\ + \frac{\alpha}{\Delta t}\big(\nabla\cdot(\delta \boldsymbol{u}_h^n - \delta\boldsymbol{u}^n), \theta_h\big) = -E_n(\theta_h),$$

and we replace p^n by its elliptic projection in the second term, see (4.1), for all $\theta_h \in Q_h$,

$$\frac{1}{M\Delta t}\big(\delta p_h^n - \delta p^n, \theta_h\big) + \big(\boldsymbol{\kappa}\nabla(p_h^n - S_h(p^n)), \nabla\theta_h\big) \\ + \frac{\alpha}{\Delta t}\big(\nabla\cdot(\delta p_h^n - \delta p^n), \theta_h\big) = -E_n(\theta_h).$$

Next, we introduce $S_h(p^n)$ in the first term and the elliptic projection $P_h(\boldsymbol{u}^n)$ in the third term, see (4.2). This gives a flow error equation, valid for all $1 \le n \le N$, for all $\theta_h \in Q_h$,

$$\frac{1}{M\Delta t}\big(\delta(p_h^n - S_h(p^n)), \theta_h\big) + \big(\boldsymbol{\kappa}\nabla(p_h^n - S_h(p^n)), \nabla\theta_h\big) \\ + \frac{\alpha}{\Delta t}\big(\nabla\cdot(\delta(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n)), \theta_h\big) \\ = -E_n(\theta_h) + \frac{1}{M\Delta t}\big(\delta(p^n - S_h(p^n)), \theta_h\big) + \frac{\alpha}{\Delta t}\big(\nabla\cdot(\delta(\boldsymbol{u}^n - P_h(\boldsymbol{u}^n)), \theta_h\big). \tag{4.11}$$

A similar, but simpler computation gives the displacement error equation

$$\forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \quad 2G\big(\boldsymbol{\varepsilon}(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n)), \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big) + \lambda\big(\nabla\cdot(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n)), \nabla\cdot\boldsymbol{v}_h\big) \\ = \alpha\big(p_h^n - S_h(p^n), \nabla\cdot\boldsymbol{v}_h\big) + \alpha\big(S_h(p^n) - p^n, \nabla\cdot\boldsymbol{v}_h\big). \tag{4.12}$$

Following a standard procedure, see for example [19], (4.11) is tested with $\theta_h = p_h^n - S_h(p^n)$ and the interaction term

$$\frac{\alpha}{\Delta t}\big(\nabla\cdot(\delta(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n)), p_h^n - S_h(p^n)\big)$$

is eliminated by testing (4.12) with $\delta(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))$. This yields a first total error equation,

$$\begin{aligned}
&\frac{1}{2M\Delta t}\Big[\delta\big(\|p_h^n - S_h(p^n)\|_{L^2(\Omega)}^2\big) + \|\delta(p_h^n - S_h(p^n))\|_{L^2(\Omega)}^2\Big] \\
&\quad + \left\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla(p_h^n - S_h(p^n))\right\|_{L^2(\Omega)}^2 \\
&\quad + \frac{1}{2\Delta t}\Big[2G\delta\big(\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\|_{L^2(\Omega)}^2\big) + 2G\|\delta(\boldsymbol{\varepsilon}(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\|_{L^2(\Omega)}^2 \\
&\quad + \lambda\delta\big(\|\nabla\cdot(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\|_{L^2(\Omega)}^2\big) + \|\delta(\nabla\cdot(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n)))\|_{L^2(\Omega)}^2\Big] \\
&= \frac{\alpha}{\Delta t}\big(S_h(p^n) - p^n, \nabla\cdot\delta(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\big) \\
&\quad - \frac{1}{M\Delta t}\big(\delta(S_h(p^n) - p^n), p_h^n - S_h(p^n)\big) \\
&\quad - \frac{\alpha}{\Delta t}\big(\nabla\cdot\delta(P_h(\boldsymbol{u}^n) - \boldsymbol{u}^n), p_h^n - S_h(p^n)\big) - E_n(p_h^n - S_h(p^n)).
\end{aligned}$$

The third term in the above right-hand side is not amenable because it involves the divergence of the displacement's projection error, from which no accuracy can be gained. The divergence can be eliminated by Green's formula in space

$$\begin{aligned}
&-\frac{\alpha}{\Delta t}\big(\nabla\cdot\delta(P_h(\boldsymbol{u}^n) - \boldsymbol{u}^n), p_h^n - S_h(p^n)\big) \\
&\quad = \frac{\alpha}{\Delta t}\big(\delta(P_h(\boldsymbol{u}^n) - \boldsymbol{u}^n), \nabla(p_h^n - S_h(p^n))\big),
\end{aligned}$$

with no contribution from the boundary owing that $P_h(\boldsymbol{u}^n) - \boldsymbol{u}^n = \mathbf{0}$ on $\partial\Omega$. Thus, we have a second total error equality

$$\begin{aligned}
&\frac{1}{2M\Delta t}\Big[\delta\big(\|p_h^n - S_h(p^n)\|_{L^2(\Omega)}^2\big) + \|\delta(p_h^n - S_h(p^n))\|_{L^2(\Omega)}^2\Big] \\
&\quad + \left\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla(p_h^n - S_h(p^n))\right\|_{L^2(\Omega)}^2 \\
&\quad + \frac{1}{2\Delta t}\Big[2G\delta\big(\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\|_{L^2(\Omega)}^2\big) + 2G\|\delta(\boldsymbol{\varepsilon}(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\|_{L^2(\Omega)}^2 \\
&\quad + \lambda\delta\big(\|\nabla\cdot(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\|_{L^2(\Omega)}^2\big) + \|\delta(\nabla\cdot(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n)))\|_{L^2(\Omega)}^2\Big] \\
&= \frac{\alpha}{\Delta t}\big(S_h(p^n) - p^n, \nabla\cdot\delta(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\big) \\
&\quad - \frac{1}{M\Delta t}\big(\delta(S_h(p^n) - p^n), p_h^n - S_h(p^n)\big) \\
&\quad + \frac{\alpha}{\Delta t}\big(\delta(P_h(\boldsymbol{u}^n) - \boldsymbol{u}^n), \nabla(p_h^n - S_h(p^n))\big) - E_n(p_h^n - S_h(p^n)).
\end{aligned} \tag{4.13}$$

Let us multiply both sides of (4.13) by Δt and sum over time from 1 to n. The left-hand side of the resulting equation reads for all $1 \le n \le N$,

$$
\begin{aligned}
\text{LHS} = &\frac{1}{2M}\Big[\|p_h^n - S_h(p^n)\|^2_{L^2(\Omega)} + \sum_{m=1}^{n} \|\delta(p_h^m - S_h(p^m))\|^2_{L^2(\Omega)}\Big]\\
&+ \sum_{m=1}^{n} \Delta t \left\|\boldsymbol{\kappa}^{\frac{1}{2}} \nabla (p_h^m - S_h(p^m))\right\|^2_{L^2(\Omega)}\\
&+ G\Big[\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\|^2_{L^2(\Omega)} - \|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^0 - P_h(\boldsymbol{u}^0))\|^2_{L^2(\Omega)}\\
&+ \sum_{m=1}^{n} \|\delta(\boldsymbol{\varepsilon}(\boldsymbol{u}_h^m - P_h(\boldsymbol{u}^m))\|^2_{L^2(\Omega)}\Big]\\
&+ \frac{\lambda}{2}\Big[\|\nabla \cdot (\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\|^2_{L^2(\Omega)} - \|\nabla \cdot (\boldsymbol{u}_h^0 - P_h(\boldsymbol{u}^0))\|^2_{L^2(\Omega)}\\
&+ \sum_{m=1}^{n} \|\delta(\nabla \cdot (\boldsymbol{u}_h^m - P_h(\boldsymbol{u}^m)))\|^2_{L^2(\Omega)}\Big].
\end{aligned}
\tag{4.14}
$$

Note that the pressure error at initial time vanishes owing to (3.6). Regarding the right-hand side of (4.13), observe that the second argument of the first term cannot be controlled by the LHS, but it could be handled if the δ operator were switched to the first argument. This can be done by a discrete summation by parts,

$$
\begin{aligned}
&\alpha \sum_{m=1}^{n} \big(S_h(p^m) - p^m, \nabla \cdot \delta(\boldsymbol{u}_h^m - P_h(\boldsymbol{u}^m))\big)\\
&= \alpha\Big[-\sum_{m=1}^{n-1} \big(\delta(S_h(p^{m+1}) - p^{m+1}), \nabla \cdot (\boldsymbol{u}_h^m - P_h(\boldsymbol{u}^m))\big)\\
&\quad + \big(S_h(p^n) - p^n, \nabla \cdot (\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\big) - \big(S_h(p^1) - p^1, \nabla \cdot (\boldsymbol{u}_h^0 - P_h(\boldsymbol{u}^0))\big)\Big].
\end{aligned}
$$

Thus, the right-hand side is expressed, for all $2 \le n \le N$, as

$$
\begin{aligned}
\text{RHS} \; = &\alpha\Big[-\sum_{m=1}^{n-1} \big(\delta(S_h(p^{m+1}) - p^{m+1}), \nabla \cdot (\boldsymbol{u}_h^m - P_h(\boldsymbol{u}^m))\big)\\
&+\big(S_h(p^n) - p^n, \nabla \cdot (\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\big)\\
&-\big(S_h(p^1) - p^1, \nabla \cdot (\boldsymbol{u}_h^0 - P_h(\boldsymbol{u}^0))\big)\Big]\\
&-\frac{1}{M}\sum_{m=1}^{n} \big(\delta(S_h(p^m) - p^m), p_h^m - S_h(p^m)\big)\\
&+\alpha \sum_{m=1}^{n} \big(\delta(P_h(\boldsymbol{u}^m) - \boldsymbol{u}^m), \nabla(p_h^m - S_h(p^m))\big) - \sum_{m=1}^{n} \Delta t\, E_m(p_h^m - S_h(p^m)).
\end{aligned}
\tag{4.15}
$$

When $n = 1$, the first sum in (4.15) is empty while the other terms are unchanged. This is summarized in the following proposition.

Proposition 1 *The scheme (3.6)–(3.9) satisfies the following error equality for all* $1 \leq n \leq N$:

$$
\begin{aligned}
&\frac{1}{2M}\Big[\|p_h^n - S_h(p^n)\|^2_{L^2(\Omega)} + \sum_{m=1}^{n} \|\delta(p_h^m - S_h(p^m))\|^2_{L^2(\Omega)}\Big] \\
&\quad + \sum_{m=1}^{n} \Delta t \left\|\boldsymbol{\kappa}^{\frac{1}{2}} \nabla(p_h^m - S_h(p^m))\right\|^2_{L^2(\Omega)} \\
&\quad + G\Big[\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\|^2_{L^2(\Omega)} \\
&\quad - \|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^0 - P_h(\boldsymbol{u}^0))\|^2_{L^2(\Omega)} \\
&\quad + \sum_{m=1}^{n} \|\delta(\boldsymbol{\varepsilon}(\boldsymbol{u}_h^m - P_h(\boldsymbol{u}^m))\|^2_{L^2(\Omega)}\Big] \\
&\quad + \frac{\lambda}{2}\Big[\|\nabla\cdot(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\|^2_{L^2(\Omega)} - \|\nabla\cdot(\boldsymbol{u}_h^0 - P_h(\boldsymbol{u}^0))\|^2_{L^2(\Omega)} \\
&\quad + \sum_{m=1}^{n} \|\delta(\nabla\cdot(\boldsymbol{u}_h^m - P_h(\boldsymbol{u}^m)))\|^2_{L^2(\Omega)}\Big] \\
&= \alpha\Big[-\sum_{m=1}^{n-1}\big(\delta(S_h(p^{m+1}) - p^{m+1}), \\
&\nabla\cdot(\boldsymbol{u}_h^m - P_h(\boldsymbol{u}^m))\big) + \big(S_h(p^n) - p^n, \nabla\cdot(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\big) \\
&\quad - \big(S_h(p^1) - p^1, \nabla\cdot(\boldsymbol{u}_h^0 - P_h(\boldsymbol{u}^0))\big)\Big] \\
&\quad - \frac{1}{M}\sum_{m=1}^{n}\big(\delta(S_h(p^m) - p^m), p_h^m - S_h(p^m)\big) \\
&\quad + \alpha\sum_{m=1}^{n}\big(\delta(P_h(\boldsymbol{u}^m) - \boldsymbol{u}^m), \\
&\nabla(p_h^m - S_h(p^m))\big) - \sum_{m=1}^{n} \Delta t\, E_m(p_h^m - S_h(p^m)).
\end{aligned}
\tag{4.16}
$$

4.3 Error Inequalities

Let us start with the error in time $E_m(p_h^m - S_h(p^m))$. According to (4.9),

$$
\begin{aligned}
\Delta t\, E_m(p_h^m - S_h(p^m)) = \frac{1}{M}&\Big[\big(p^m - p^{m-1} - \Delta t(\partial_t p)^m, p_h^m - S_h(p^m)\big)\Big] \\
&+ \alpha\Big[\big(\nabla\cdot(\boldsymbol{u}^m - \boldsymbol{u}^{m-1}) - \Delta t\nabla\cdot(\partial_t \boldsymbol{u})^m, p_h^m - S_h(p^m)\big)\Big]
\end{aligned}
$$

$$= \frac{1}{M}\Big[- \int_{t_{m-1}}^{t_m} (s - t_{m-1}) p''(s) \mathrm{d}s,\, p_h^m - S_h(p^m)\Big)\Big]$$
$$\alpha\Big[- \int_{t_{m-1}}^{t_m} (s - t_{m-1}) \nabla \cdot (\boldsymbol{u}''(s)) \mathrm{d}s,\, p_h^m - S_h(p^m)\Big)\Big],$$

if p and $\boldsymbol{u}$ are smooth enough. More precisely, if

$$p'' \text{ and } \nabla \cdot (\boldsymbol{u}'') \text{ both belong to } L^2(\Omega \times]0, T[),$$

then

$$\Delta t |E_m(p_h^m - S_h(p^m))| \leq \tfrac{1}{\sqrt{3}} (\Delta t)^{\frac{3}{2}} \| \tfrac{1}{M} p''$$
$$+ \alpha \nabla \cdot (\boldsymbol{u}'')\|_{L^2(\Omega \times]t_{m-1}, t_m[)} \| p_h^m - S_h(p^m)\|_{L^2(\Omega)}.$$

Hence, after summing over m, isolating the term with superscript n and applying Young's inequality, we derive for any $\eta > 0$,

$$\begin{aligned}
&\sum_{m=1}^{n} \Delta t |E_m(p_h^m - S_h(p^m))| \leq \frac{1}{2} \\
&\Big[\frac{1}{4M} \Big\| p_h^n - S_h(p^n)\|_{L^2(\Omega)}^2 \\
&+ \frac{4M}{3} (\Delta t)^3 \Big\| \frac{1}{M} p'' \\
&+ \alpha \nabla \cdot (\boldsymbol{u}'')\|_{L^2(\Omega \times]t_{n-1}, t_n[)}^2\Big] \\
&+ \frac{1}{2}\Big[\frac{1}{4M} \sum_{m=1}^{n-1} \Delta t \Big\| p_h^m - S_h(p^m)\|_{L^2(\Omega)}^2 \\
&+ \frac{4M}{3} (\Delta t)^2 \Big\| \frac{1}{M} p'' \\
&+ \alpha \nabla \cdot (\boldsymbol{u}'')\|_{L^2(\Omega \times]0, t_{n-1}[)}^2\Big].
\end{aligned} \tag{4.17}$$

A similar treatment is applied to the sum in the second line of (4.15),

$$\begin{aligned}
&\frac{1}{M} \Big| \sum_{m=1}^{n} \big(\delta(S_h(p^m) - p^m), \\
&p_h^m - S_h(p^m)\big)\Big| \leq \frac{1}{2M}\Big[\frac{1}{4} \| p_h^n - S_h(p^n)\|_{L^2(\Omega)}^2 \\
&+ 4 \| \delta(p^n - S_h(p^n)\|_{L^2(\Omega)}^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\sum_{m=1}^{n-1}\Delta t\|p_h^m - S_h(p^m)\|_{L^2(\Omega)}^2 \\
&+4\sum_{m=1}^{n-1}\frac{1}{\Delta t}\|\delta(p^m - S_h(p^m)\|_{L^2(\Omega)}^2\Big].
\end{aligned}
\tag{4.18}
$$

The first term in the third line of (4.15) is easily estimated; first we write

$$
\begin{aligned}
|\big(\delta(P_h(\boldsymbol{u}^m) - \boldsymbol{u}^m), \nabla(p_h^m - S_h(p^m))\big)| \le \|\delta(P_h(\boldsymbol{u}^m) - \boldsymbol{u}^m)\|_{L^2(\Omega)} \\
\left\|\boldsymbol{\kappa}^{-\frac{1}{2}}\right\|_{L^\infty(\Omega)} \left\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla(p_h^m - S_h(p^m))\right\|_{L^2(\Omega)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\alpha\Big|\sum_{m=1}^{n}\big(\delta(P_h(\boldsymbol{u}^m) - \boldsymbol{u}^m), \\
&\nabla(p_h^m - S_h(p^m))\big)\Big| \le \frac{1}{2}\Big[\sum_{m=1}^{n}\Delta t\left\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla(p_h^m - S_h(p^m))\right\|_{L^2(\Omega)}^2 \\
&+\alpha^2\left\|\boldsymbol{\kappa}^{-\frac{1}{2}}\right\|_{L^\infty(\Omega)}^2\sum_{m=1}^{n}\frac{1}{\Delta t}\|\delta(P_h(\boldsymbol{u}^m) - \boldsymbol{u}^m)\|_{L^2(\Omega)}^2\Big].
\end{aligned}
\tag{4.19}
$$

Next, we consider the first sum in (4.15),

$$
\begin{aligned}
&\alpha\Big| - \sum_{m=1}^{n-1}\big(\delta(S_h(p^{m+1}) - p^{m+1}), \\
&\nabla\cdot(\boldsymbol{u}_h^m - P_h(\boldsymbol{u}^m))\big)\Big| \le \frac{1}{2}\Big[\frac{\lambda}{2}\sum_{m=1}^{n-1}\Delta t\|\nabla\cdot(\boldsymbol{u}_h^m - P_h(\boldsymbol{u}^m))\|_{L^2(\Omega)}^2 \\
&+\frac{2}{\lambda}\alpha^2\sum_{m=1}^{n-1}\frac{1}{\Delta t}\|\delta(S_h(p^{m+1}) - p^{m+1})\|_{L^2(\Omega)}^2\Big].
\end{aligned}
\tag{4.20}
$$

The second and third terms are straightforward,

$$
\begin{aligned}
&\alpha\big|\big(S_h(p^n) - p^n, \nabla\cdot(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\big)\big| \le \frac{1}{2} \\
&\Big[\frac{\lambda}{2}\|\nabla\cdot(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\|_{L^2(\Omega)}^2 \\
&+\frac{2}{\lambda}\alpha^2\|S_h(p^n) - p^n\|_{L^2(\Omega)}^2\Big],
\end{aligned}
$$

$$\alpha\left|\left(S_h(p^1) - p^1, \nabla\cdot(\boldsymbol{u}_h^0 - P_h(\boldsymbol{u}^0))\right)\right| \le \frac{1}{2}$$
$$\left[\frac{\lambda}{2}\|\nabla\cdot(\boldsymbol{u}_h^0 - P_h(\boldsymbol{u}^0))\|^2_{L^2(\Omega)} + \frac{2}{\lambda}\alpha^2\|S_h(p^1) - p^1\|^2_{L^2(\Omega)}\right]. \tag{4.21}$$

Then, by substituting (4.17)–(4.21) into (4.16) and collecting terms, we derive for all $n \ge 2$,

$$\begin{aligned}
&\frac{1}{4M}\|S_h(p^n) - p_h^n\|^2_{L^2(\Omega)} + G\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\|^2_{L^2(\Omega)} + \frac{\lambda}{4}\|\nabla\cdot(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\|^2_{L^2(\Omega)}\\
&\quad+\frac{1}{2}\sum_{m=1}^{n}\Delta t\left\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla(p_h^m - S_h(p^m))\right\|^2_{L^2(\Omega)}\\
&\quad+\frac{1}{2M}\sum_{m=1}^{n}\|\delta(S_h(p^m) - p_h^m)\|^2_{L^2(\Omega)} + G\sum_{m=1}^{n}\|\delta(\boldsymbol{\varepsilon}(\boldsymbol{u}_h^m - P_h(\boldsymbol{u}^m))\|^2_{L^2(\Omega)}\\
&\quad+\frac{\lambda}{2}\sum_{m=1}^{n}\|\nabla\cdot\delta(\boldsymbol{u}_h^m - P_h(\boldsymbol{u}^m))\|^2_{L^2(\Omega)}\\
&\le\frac{1}{4M}\sum_{m=1}^{n-1}\Delta t\|S_h(p^m) - p_h^m\|^2_{L^2(\Omega)} + \frac{\lambda}{4}\sum_{m=1}^{n-1}\Delta t\|\nabla\cdot(\boldsymbol{u}_h^m - P_h(\boldsymbol{u}^m))\|^2_{L^2(\Omega)}\\
&\quad+\frac{\alpha^2}{\lambda}\left[\|S_h(p^n) - p^n\|^2_{L^2(\Omega)} + \sum_{m=2}^{n}\frac{1}{\Delta t}\|\delta(S_h(p^m) - p^m)\|^2_{L^2(\Omega)}\right]\\
&\quad+\frac{2}{M}\left[\|\delta(S_h(p^n) - p^n)\|^2_{L^2(\Omega)} + \sum_{m=1}^{n-1}\frac{1}{\Delta t}\|\delta(S_h(p^m) - p^m)\|^2_{L^2(\Omega)}\right]\\
&\quad+\frac{\alpha^2}{2}\left\|\boldsymbol{\kappa}^{-\frac{1}{2}}\right\|^2\sum_{m=1}^{n}\frac{1}{\Delta t}\|\delta(\boldsymbol{u}^m - P_h(\boldsymbol{u}^m))\|^2_{L^2(\Omega)}\\
&\quad+\frac{2M}{3}(\Delta t)^2\Big(\Delta t\left\|\frac{1}{M}p'' + \alpha\nabla\cdot(\boldsymbol{u}'')\right\|^2_{L^2(\Omega\times]t_{n-1},t_n[)} + \Big\|\frac{1}{M}p''\\
&\quad+\alpha\nabla\cdot(\boldsymbol{u}'')\Big\|^2_{L^2(\Omega\times]0,t_{n-1}[)}\Big)\\
&\quad+G\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^0 - P_h(\boldsymbol{u}^0))\|^2_{L^2(\Omega)} + \frac{3\lambda}{4}\|\nabla\cdot(\boldsymbol{u}_h^0 - P_h(\boldsymbol{u}^0))\|^2_{L^2(\Omega)}\\
&\quad+\frac{\alpha^2}{\lambda}\|S_h(p^1) - p^1\|^2_{L^2(\Omega)}.
\end{aligned}\tag{4.22}$$

When $n = 1$, all sums except the last one are empty. Let

$$\begin{aligned}
I_h &= G\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^0 - P_h(\boldsymbol{u}^0))\|^2_{L^2(\Omega)} + \frac{3\lambda}{4}\|\nabla\cdot(\boldsymbol{u}_h^0 - P_h(\boldsymbol{u}^0))\|^2_{L^2(\Omega)}\\
&\quad+\frac{\alpha^2}{\lambda}\|S_h(p^1) - p^1\|^2_{L^2(\Omega)},
\end{aligned}\tag{4.23}$$

$$C_h = \frac{\alpha^2}{\lambda}\Big[\|S_h(p^n) - p^n\|^2_{L^2(\Omega)} + \sum_{m=2}^{n} \frac{1}{\Delta t}\|\delta(S_h(p^m) - p^m)\|^2_{L^2(\Omega)}\Big]$$
$$+\frac{2}{M}\Big[\|\delta(S_h(p^n) - p^n)\|^2_{L^2(\Omega)} + \sum_{m=1}^{n-1} \frac{1}{\Delta t}\|\delta(S_h(p^m) - p^m)\|^2_{L^2(\Omega)}\Big]$$
$$+\frac{\alpha^2}{2}\left\|\boldsymbol{\kappa}^{-\frac{1}{2}}\right\|^2 \sum_{m=1}^{n} \frac{1}{\Delta t}\|\delta(\boldsymbol{u}^m - P_h(\boldsymbol{u}^m))\|^2_{L^2(\Omega)}, \tag{4.24}$$
$$C_\Delta = \frac{2M}{3}(\Delta t)^2\Big(\Delta t\left\|\frac{1}{M}p'' + \alpha\nabla\cdot(\boldsymbol{u}'')\right\|^2_{L^2(\Omega\times]t_{n-1},t_n[)} + \left\|\frac{1}{M}p''\right.$$
$$\left.+\alpha\nabla\cdot(\boldsymbol{u}'')\right\|^2_{L^2(\Omega\times]0,t_{n-1}[)}\Big). \tag{4.25}$$

Then by Gronwall's Lemma, we deduce the next proposition.

Proposition 2 *Let both p'' and $\nabla\cdot(\boldsymbol{u}'')$ belong to $L^2(\Omega\times]0,T[)$. The scheme (3.6)–(3.9) satisfies the following error inequality for all $1 \le n \le N$:*

$$\frac{1}{4M}\|S_h(p^n) - p_h^n\|^2_{L^2(\Omega)} + G\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\|^2_{L^2(\Omega)}$$
$$+\frac{\lambda}{4}\|\nabla\cdot(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\|^2_{L^2(\Omega)}$$
$$+\frac{1}{2}\sum_{m=1}^{n}\Delta t\left\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla(p_h^m - S_h(p^m))\right\|^2_{L^2(\Omega)}$$
$$+\frac{1}{2M}\sum_{m=1}^{n}\|\delta(S_h(p^m) - p_h^m)\|^2_{L^2(\Omega)} + G\sum_{m=1}^{n}\|\delta(\boldsymbol{\varepsilon}(\boldsymbol{u}_h^m - P_h(\boldsymbol{u}^m))\|^2_{L^2(\Omega)}$$
$$+\frac{\lambda}{2}\sum_{m=1}^{n}\|\nabla\cdot\delta(\boldsymbol{u}_h^m - P_h(\boldsymbol{u}^m))\|^2_{L^2(\Omega)}$$
$$\le \big(I_h + C_h + C_\Delta\big)\exp(t_n). \tag{4.26}$$

If $n = 1$, the first two sums in C_h are empty.

Note that

$$C_h \le \frac{\alpha^2}{\lambda}\|S_h(p^n) - p^n\|^2_{L^2(\Omega)} + 2\big(\frac{\alpha^2}{\lambda} + \frac{2}{M}\big)\sum_{m=1}^{n}\frac{1}{\Delta t}\|\delta(S_h(p^m) - p^m)\|^2_{L^2(\Omega)}$$
$$+\frac{2}{M}\|\delta(S_h(p^n) - p^n)\|^2_{L^2(\Omega)} + \frac{\alpha^2}{2}\left\|\boldsymbol{\kappa}^{-\frac{1}{2}}\right\|^2\sum_{m=1}^{n}\frac{1}{\Delta t}\|\delta(\boldsymbol{u}^m - P_h(\boldsymbol{u}^m))\|^2_{L^2(\Omega)}.$$

Let us bound the first sum. We have

$$\begin{aligned}\|\delta(S_h(p^m)-p^m)\|^2_{L^2(\Omega)} &= \left\|\int_{t_{m-1}}^{t_m} \partial_t(S_h(p)-p)(s)\mathrm{d}s\right\|^2_{L^2(\Omega)}\\ &\le \Delta t\|\partial_t(S_h(p)-p)\|^2_{L^2(\Omega\times]t_{m-1},t_m[)}.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{m=1}^{n}\frac{1}{\Delta t}\|\delta(S_h(p^m)-p^m)\|^2_{L^2(\Omega)} &\le \|\partial_t(S_h(p)-p)\|^2_{L^2(\Omega\times]0,T[)}\\ &= \|S_h(\partial_t p)-\partial_t p\|^2_{L^2(\Omega\times]0,T[)},\end{aligned}$$

owing that S_h and ∂_t commute. Likewise,

$$\begin{aligned}\sum_{m=1}^{n}\frac{1}{\Delta t}\|\delta(\boldsymbol{u}^m-P_h(\boldsymbol{u}^m))\|^2_{L^2(\Omega)} &\le \|\partial_t(P_h(\boldsymbol{u})-\boldsymbol{u})\|^2_{L^2(\Omega\times]0,T[)}\\ &= \|P_h(\partial_t\boldsymbol{u})-\partial_t\boldsymbol{u}\|^2_{L^2(\Omega\times]0,T[)}.\end{aligned}$$

Finally,

$$\|\delta(S_h(p^n)-p^n)\|^2_{L^2(\Omega)} \le \Delta t\|S_h(\partial_t p)-\partial_t p\|^2_{L^2(\Omega\times]t_{n-1},t_n[)}.$$

Therefore, by substituting these bounds into (4.24), we infer

$$\begin{aligned}C_h \le &\frac{\alpha^2}{\lambda}\|S_h(p^n)-p^n\|^2_{L^2(\Omega)}\\ &+2\Big(\frac{\alpha^2}{\lambda^2}+\frac{2}{M}\Big(1+\frac{\Delta t}{2}\Big)\Big)\|S_h(\partial_t p)-\partial_t p\|^2_{L^2(\Omega\times]0,T[)}\\ &+\frac{\alpha^2}{2}\|\boldsymbol{\kappa}^{-\frac{1}{2}}\|^2\|P_h(\partial_t\boldsymbol{u})-\partial_t\boldsymbol{u}\|^2_{L^2(\Omega\times]0,T[)}.\end{aligned} \tag{4.27}$$

It remains to bound the initial terms. By definition of the projection, $P_h(\boldsymbol{u}^0)$ satisfies

$$2G\big(\boldsymbol{\varepsilon}(P_h(\boldsymbol{u}^0)),\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big)+\lambda\big(\nabla\cdot(P_h(\boldsymbol{u}^0)),\nabla\cdot\boldsymbol{v}_h\big)=\alpha\big(p^0,\nabla\cdot\boldsymbol{v}_h\big)+\big(\boldsymbol{f}^0,\boldsymbol{v}_h\big).$$

Then subtracting this from (3.7) leads to

$$2G\big(\boldsymbol{\varepsilon}(\boldsymbol{u}_h^0-P_h(\boldsymbol{u}^0)),\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big)+\lambda\big(\nabla\cdot(\boldsymbol{u}_h^0-P_h(\boldsymbol{u}^0)),\nabla\cdot\boldsymbol{v}_h\big)=\alpha\big(p_h^0-p^0,\nabla\cdot\boldsymbol{v}_h\big).$$

From this, and the fact that $|\nabla \cdot \boldsymbol{v}|^2 \le d(\boldsymbol{\varepsilon}(\boldsymbol{v}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}))$, we easily derive that

$$\begin{aligned} G\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^0 - P_h(\boldsymbol{u}^0))\|_{L^2(\Omega)}^2 &\le \frac{\alpha^2}{8\lambda}\|p_h^0 - p^0\|_{L^2(\Omega)}^2, \\ \frac{3\lambda}{4}\|\nabla \cdot (\boldsymbol{u}_h^0 - P_h(\boldsymbol{u}^0))\|_{L^2(\Omega)}^2 &\le \frac{3d\alpha^2}{32G}\|p_h^0 - p^0\|_{L^2(\Omega)}^2. \end{aligned} \tag{4.28}$$

Hence

$$I_h \le \frac{3\alpha^2}{8}\Big(\frac{1}{3\lambda} + \frac{d}{4G}\Big)\|S_h(p^0) - p^0\|_{L^2(\Omega)}^2 + \frac{\alpha^2}{\lambda}\|S_h(p^1) - p^1\|_{L^2(\Omega)}^2. \tag{4.29}$$

Finally, by substituting (4.27), (4.29), and (4.25) into (4.26) and applying (4.5) and (4.6), we derive the next lemma.

Lemma 1 *Let the domain Ω be a convex polygon or polyhedron according to the dimension, let the coefficients of $\boldsymbol{\kappa}$ belong to $W^{1,\infty}(\Omega)$. If*

$$\boldsymbol{u} \in H^1(0,T; H_0^1(\Omega)^d), \quad p \in H^1(0,T; H^1(\Omega)), \quad p'' \text{ and } \nabla \cdot \boldsymbol{u}'' \in L^2(\Omega \times]0,T[), \tag{4.30}$$

then, for all n, $1 \le n \le N$,

$$\begin{aligned} &\frac{1}{4M}\|S_h(p^n) - p_h^n\|_{L^2(\Omega)}^2 + G\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\|_{L^2(\Omega)}^2 + \frac{\lambda}{4}\|\nabla \cdot (\boldsymbol{u}_h^n - P_h(\boldsymbol{u}^n))\|_{L^2(\Omega)}^2 \\ &\quad + \frac{1}{2}\sum_{m=1}^{n} \Delta t \left\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla(p_h^m - S_h(p^m))\right\|_{L^2(\Omega)}^2 \\ &\quad + \frac{1}{2M}\sum_{m=1}^{n}\|\delta(S_h(p^m) - p_h^m)\|_{L^2(\Omega)}^2 \\ &\quad + G\sum_{m=1}^{n}\|\delta(\boldsymbol{\varepsilon}(\boldsymbol{u}_h^m - P_h(\boldsymbol{u}^m))\|_{L^2(\Omega)}^2 + \frac{\lambda}{2}\sum_{m=1}^{n}\|\nabla \cdot \delta(\boldsymbol{u}_h^m - P_h(\boldsymbol{u}^m))\|_{L^2(\Omega)}^2 \\ &\le Ch^2\Big[\left\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla(p - \theta_h)\right\|_{\mathcal{C}^0(0,T;L^2(\Omega))}^2 + \left\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla(\partial_t p - \bar{\theta}_h)\right\|_{L^2(\Omega \times]0,T[)}^2 \\ &\quad + \left\|\boldsymbol{\kappa}^{\frac{1}{2}}\right\|_{L^\infty(\Omega)}^2 \|\boldsymbol{\varepsilon}(\partial_t \boldsymbol{u} - \boldsymbol{v}_h)\|_{L^2(\Omega \times]0,T[)}^2\Big] \\ &\quad + C(\Delta t)^2\left\|\frac{1}{M}p'' + \alpha\nabla \cdot (\boldsymbol{u}'')\right\|_{L^2(\Omega \times]0,T[)}^2, \end{aligned} \tag{4.31}$$

for any θ_h, $\bar{\theta}_h$ in Q_h, and $\boldsymbol{v}_h \in \boldsymbol{X}_h$, where C is a constant that depends on α, M, G, λ, and d, but is independent of h, Δt, p and $\boldsymbol{u}$.

Theorem 3 *Under the assumptions of Lemma 1, and with the same notation, we have*

$$
\begin{aligned}
&\frac{1}{4M} \sup_{1\le n\le N} \|p^n - p_h^n\|^2_{L^2(\Omega)} + \frac{1}{2M} \sum_{n=1}^{N} \|\delta(p^n - p_h^n)\|^2_{L^2(\Omega)} \\
&\quad \le Ch^2 \Big[\big\| \boldsymbol{\kappa}^{\frac{1}{2}} \nabla (p - \theta_h) \big\|^2_{C^0(0,T;L^2(\Omega))} + \big\| \boldsymbol{\kappa}^{\frac{1}{2}} \nabla (\partial_t p - \bar{\theta}_h) \big\|^2_{L^2(\Omega\times]0,T[)} \\
&\qquad + \|\boldsymbol{\kappa}^{\frac{1}{2}}\|^2_{L^\infty(\Omega)} \|\boldsymbol{\varepsilon}(\partial_t \boldsymbol{u} - \boldsymbol{v}_h)\|^2_{L^2(\Omega\times]0,T[)} \Big] \\
&\qquad + C(\Delta t)^2 \left\| \frac{1}{M} p'' + \alpha \nabla \cdot (\boldsymbol{u}'') \right\|^2_{L^2(\Omega\times]0,T[)},
\end{aligned}
\tag{4.32}
$$

where C is another constant independent of h, Δt, p and $\boldsymbol{u}$.

5 Numerical Experiments

Consider the following benchmark problem on $\Omega = [0, 1] \times [0, 1]$:

$$
\begin{aligned}
&- \nabla \cdot (\lambda(\nabla \cdot \boldsymbol{u})\boldsymbol{I} + 2G\boldsymbol{\varepsilon}(\boldsymbol{u}) - \alpha p \boldsymbol{I}) = \boldsymbol{f} \quad \text{in } \Omega\times]0, T[, \\
&\partial_t \left(\frac{1}{M} p + \alpha \nabla \cdot \boldsymbol{u} \right) - \frac{1}{\mu_f} \nabla \cdot (\boldsymbol{\kappa} \nabla p) = q \quad \text{in } \Omega\times]0, T[.
\end{aligned}
\tag{5.1}
$$

The system has the following analytical solutions

$$
\begin{aligned}
u(t, x, y) &= -\frac{\exp(-At)}{2\pi} \begin{bmatrix} \cos(\pi x)\sin(\pi y) \\ \sin(\pi x)\cos(\pi y) \end{bmatrix}, \\
p(t, x, y) &= \exp(-At)\sin(\pi x)\sin(\pi y),
\end{aligned}
\tag{5.2}
$$

with $A = \frac{2\pi^2\kappa}{\alpha + \frac{1}{M}}$, $\boldsymbol{\kappa} = \kappa \boldsymbol{I}$, $\kappa = 0.05$, $\alpha = 0.75$, $\frac{1}{M} = \frac{3}{28}$, $\lambda = 0.5$, $G = 0.125$, $\boldsymbol{f} = \boldsymbol{0}$, $q = 0$, $\mu_f = 1$. Dirichlet boundary conditions are imposed on both pressure and displacement of the true solutions; an initial pressure of the true solution is prescribed. Convergence-in-space tests are performed with a small time step Δt= 2.5e−4 and a very tight fixed-stress iteration stopping criterion ε_{fs} = 1e−7 to mitigate the errors caused by the time discretization and the fixed-stress split, we refer to [15] as a reference for the fixed-stress split algorithm. The numerical errors are measured at final time $T = 0.01$ s and summarized in Table 1. These spatial refinement tests show that both the pressure and displacement solutions achieve second-order convergence in the L^2 norm. The results confirm that an extra factor h is obtained when the solution of similar elliptic problems with any data in L^2, and homogeneous boundary conditions, belongs to H^2. The convergence-in-time tests are performed with $h = 1/128$, ε_{fs} = 1e−7, and final time $T = 1$. The results are summarized in Table 2. These results confirm that no accuracy is gained with respect to Δt.

Table 1 Convergence of pressure and displacement solutions under spatial refinement

$1/h$	$\|p^N - p_h^N\|_{L^2(\Omega)}$	Rate	$\|u^N - u_h^N\|_{L^2(\Omega)}$	Rate	$\|u^N - u_h^N\|_e$	Rate
8	1.3441e−02	–	3.2188e−03	–	1.8980e−02	–
16	3.3727e−03	1.9947	8.1489e−04	1.9818	9.3237e−03	1.0255
32	8.2670e−04	2.0116	2.0420e−04	1.9892	4.6401e−03	1.0161
64	1.8876e−04	2.0490	5.0911e−05	1.9944	2.3173e−03	1.0109
128	3.3605e−05	2.1447	1.2557e−05	2.0004	1.1583e−03	1.0077

Table 2 Convergence of pressure and displacement solutions under temporal refinement

Δt	$\|p^N - p_h^N\|_{L^2(\Omega)}$	Rate	$\|u^N - u_h^N\|_{L^2(\Omega)}$	Rate	$\|u^N - u_h^N\|_e$	Rate
0.25	2.9107e−02	–	2.4810e−03	–	7.2982e−03	–
0.2	2.3479e−02	0.9629	2.0112e−03	0.9408	5.9177e−03	0.9397
0.1	1.1924e−02	0.9748	1.0311e−03	0.9596	3.0496e−03	0.9533
0.05	5.9977e−03	0.9817	5.2069e−04	0.9704	1.5741e−03	0.9539

An experiment with a larger time step $\Delta t = 0.025, T = 1$, was conducted. It suggests that when the time step is too large, the time error dominates the spatial discretization error.

A second group of tests was performed with mixed boundary conditions imposed on the flow: namely, a natural boundary condition on $x = 0$ and a Dirichlet boundary condition on the remaining three sides of the domain, so that the change in boundary conditions was located at a boundary corner with a right angle. The other data are the same as in the previous experiments. We observe that the pressure and displacement solutions achieve second-order convergence in the L^2 norm under spatial refinement, and first-order convergence under temporal refinement. This is in agreement with the theory, see the end of Remark 1, indeed the change in boundary conditions occurs at an angle $\pi/2$.

6 Conclusions

Here we have presented optimal L^2 error estimates for Biot system where the flow equation for the pressure is discretized in time by a backward Euler scheme and in space by a continuous Galerkin scheme, while the elastic displacement equation is discretized at all time steps by a continuous Galerkin scheme. To validate these results, we have employed a manufactured solution where Dirichlet or mixed boundary conditions are imposed on both pressure and displacement of the true solutions and an initial pressure of the true solution is prescribed. Convergence-in-space tests are performed with small time steps and a fixed-stress iteration scheme has been utilized. The numerical errors are measured at a final time T and summarized in Tables 1, 2, 3, 4 and 5. These spatial refinement tests show that both the pressure and displacement

Table 3 Convergence of pressure and displacement solutions under spatial refinement

$1/h$	$\|p^N - p_h^N\|_{L^2(\Omega)}$	Rate	$\|u^N - u_h^N\|_{L^2(\Omega)}$	Rate	$\|u^N - u_h^N\|_e$	Rate
8	1.5150e−03	–	1.1021e−03	–	1.2220e−02	
16	4.9272e−04		2.4073e−04		6.0144e−03	
32	8.5265e−04		2.1689e−04		3.2618e−03	
64	9.5567e−04		2.5077e−04		2.1077e−03	
128	9.8192e−04		2.6083e−04		1.7045e−03	

Table 4 Convergence of pressure and displacement solutions under spatial refinement with mixed flow boundary conditions

$1/h$	$\|p^N - p_h^N\|_{L^2(\Omega)}$	Rate	$\|u^N - u_h^N\|_{L^2(\Omega)}$	Rate	$\|u^N - u_h^N\|_e$	Rate
8	4.4032e−03	–	3.2190e−03	–	3.7960e−02	–
16	1.1080e−03	1.9905	8.1504e−04	1.9816	1.8648e−02	1.0254
32	2.7424e−04	2.0025	2.0432e−04	1.9888	9.2801e−03	1.0161
64	6.4691e−05	2.0280	5.1020e−05	1.9934	4.6346e−03	1.0108
128	1.1493e−05	2.1261	1.2582e−05	1.9995	2.3166e−03	1.0077

Table 5 Convergence of pressure and displacement solutions under temporal refinement with mixed flow boundary conditions

Δt	$\|p^N - p_h^N\|_{L^2(\Omega)}$	Rate	$\|u^N - u_h^N\|_{L^2(\Omega)}$	Rate	$\|u^N - u_h^N\|_e$	Rate
0.25	9.5271e−03	–	2.4809e−03	–	1.4596e−02	–
0.2	7.6851e−03	0.9628	2.0111e−03	0.9408	1.1835e−02	0.9396
0.1	3.9027e−03	0.9748	1.0311e−03	0.9595	6.0991e−03	0.9532
0.05	1.9630e−03	0.9817	5.2067e−04	0.9704	3.1480e−03	0.9538

solutions achieve second-order convergence in the L^2 norm. The results confirm that an extra factor h is obtained when the solution of similar elliptic problems with any data in L^2, and homogeneous boundary conditions, belongs to H^2.

Funding Not applicable.

Data Availability Not applicable.

Code Availability Not applicable.

Declarations

Conflict of interest The authors have no conflicts of interest to declare that are relevant to the content of this article.

References

1. Adams, R.A., Fournier, J.J.: Sobolev Spaces, vol. 140. Elsevier, Amsterdam (2003)
2. Aubin, J.-P.: Behavior of the error of the approximate solutions of boundary value problems for linear elliptic operators by Galerkin's and finite difference methods. Anal. Sc. Norm. Super. Pisa **21**, 599–637 (1967)
3. Barbeiro, S., Wheeler, M.F.: A priori error estimates for the numerical solution of a coupled geomechanics and reservoir flow model with stress-dependent permeability. Comput. Geosci. **14**, 755–768 (2010)
4. Biot, M.A.: General theory of three-dimensional consolidation. J. Appl. Phys. **12**, 155–164 (1941)
5. Ciarlet, P.G.: Basic error estimates for elliptic problems. In: Handbook of Numerical Analysis, vol. II, pp. 17–351. North-Holland, Amsterdam (1991)
6. Costabel, M., Dauge, M.: Crack singularities for general elliptic systems. Math. Nachr. **235**, 29–49 (2002)
7. Ern, A., Guermond, J.-L.: Theory and Practice of Finite Elements. Applied Mathematical Sciences, vol. 159. Springer, New York (2004)
8. Girault, V., Lu, X., Wheeler, M.F.: A posteriori error estimates for Biot system using enriched Galerkin for flow. Comput. Methods Appl. Mech. Eng. **369**, 113185 (2020)
9. Girault, V., Pencheva, G.V., Wheeler, M.F., Wildey, T.M.: Domain decomposition for poroelasticity and elasticity with DG jumps and mortars. Math. Models Methods Appl. Sci. **21**, 169–213 (2011)
10. Girault, V., Raviart, P.A.: Finite Element Methods for Navier–Stokes Equations: Theory and Algorithms. Springer Series in Computational Mathematics, vol. 5. Springer, Berlin (1986)
11. Girault, V., Wheeler, M.F., Almani, T., Dana, S.: A Priori Error Estimates for a Discretized Poroelastic–Elastic System Solved by a Fixed-Stress Algorithm. Oil and Gas Science and Technology - Rev, IFP Energies nouvelles (2019)
12. Grisvard, P.: Elliptic Problems in Nonsmooth Domains. Monographs and Studies in Mathematics, vol. 24. Pitman, Boston (1985)
13. Grisvard, P.: Singularities in Boundary Value Problems. Masson, Paris (1992)
14. Magenes, J.-L.L.E.: Non-homogeneous Boundary Value Problems and Applications, vol. I. Springer, New York (1972)
15. Mikelić, A., Wang, B., Wheeler, M.F.: Numerical convergence study of iterative coupling for coupled flow and geomechanics. Comput. Geosci. **18**, 325–341 (2014)
16. Nečas, J.: Les méthodes directes en théorie des équations elliptiques. Masson, Paris (1967)
17. Nitsche, J.: Ein kriterium für die quasi-optimalität des ritzchen verfahrens. Numer. Math. **11**, 346–348 (1968)
18. Phillips, P.J.: Finite element methods in linear poroelasticity: theoretical and computational results. PhD Thesis (2005)
19. Phillips, P.J., Wheeler, M.F.: A coupling of mixed and continuous Galerkin finite element methods for poroelasticity II: the discrete-in-time case. Comput. Geosci. **11**, 145–158 (2007)
20. Phillips, P.J., Wheeler, M.F.: A coupling of mixed and discontinuous Galerkin finite-element methods for poroelasticity. Comput. Geosci. **12**, 417–435 (2008)
21. Showalter, R.E.: Diffusion in poro-elastic media. J. Math. Anal. Appl. **251**, 310–340 (2000)
22. Wheeler, M.F.: A priori L^2 error estimates for Galerkin approximations to parabolic partial differential equations. SIAM J. Numer. Anal. **10**, 723–759 (1973)