



Half-Space Solutions with $7/2$ Frequency in the Thin Obstacle Problem

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Abstract

For the thin obstacle problem in $\mathbb{R}^3$, we show that half-space solutions form an isolated family in the space of $\frac{7}{2}$ -homogeneous solutions. For a general solution with one blow-up profile in this family, we establish the rate of convergence to this profile. As a consequence, we obtain the regularity of the free boundary near such contact points.

1. Introduction

Motivated by applications in linear elasticity [21] and reverse osmosis [8], the *thin obstacle problem* studies minimizers of the Dirichlet energy over functions that lie above a lower-dimensional obstacle. In the most basic formulation, a minimizer satisfies the following system:

$$\begin{cases} \Delta u \leq 0 & \text{in } B_1, \\ u \geq 0 & \text{in } B_1 \cap \{x_n = 0\}, \\ \Delta u = 0 & \text{in } B_1 \cap (\{u > 0\} \cup \{x_n \neq 0\}). \end{cases} \quad (1.1)$$

Here B_1 is the unit ball in the Euclidean space $\mathbb{R}^n$. The coordinate of this space is decomposed as $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Note that the odd part of the solution, $(u(x', x_n) - u(x', -x_n))/2$, is harmonic and vanishes along the hyperplane $\{x_n = 0\}$; by removing it, we assume that the solution is *even with respect to* $\{x_n = 0\}$.

Remark 1.1. The thin obstacle problem enjoys several invariances. For instance, if u is a solution, then rotations of u around the x_n -axis also solve the problem. The

same happens for positive multiples of u . For simplicity, we identify two solutions u and v up to a *normalization* if a rotation of u around the x_n -axis equals a positive multiple of v .

After works by RICHARDSON [16] and URALTSEVA [22], Athanasopoulos and Caffarelli obtained the optimal regularity of the solution u [2], namely,

$$u \in C_{loc}^{0,1}(B_1) \cap C_{loc}^{1,\frac{1}{2}}(B_1 \cap \{x_n \geq 0\}).$$

The next step is to address the regularity of the *contact set* $\Lambda(u) := \{u = 0\} \cap \{x_n = 0\}$ and the *free boundary* $\partial_{\mathbb{R}^{n-1}} \Lambda(u)$. To this end, we need precise information about the solution near a contact point.

Applying Almgren's monotonicity formula [1], ATHANASOPOULOS–CAFFARELLI–SALSA [3] showed that for each $q \in \Lambda(u)$, there is a constant λ_q , called the *frequency of the solution at q* , such that

$$\|u\|_{L^2(\partial B_r(q))} \sim r^{\frac{n-1}{2} + \lambda_q}$$

as $r \rightarrow 0$. Moreover, *along a subsequence* of $r \rightarrow 0$, the normalized solution converges to a *blow-up profile at q* , that is,

$$u_{q,r} := r^{\frac{n-1}{2}} \frac{u(r \cdot + q)}{\|u\|_{L^2(\partial B_r(q))}} \rightarrow u_0. \quad (1.2)$$

The limit u_0 is a λ_q -homogeneous solution to (1.1), also known as a λ_q -*cone*.

This opened up two interesting directions of research. The first concerns the space of homogeneous solutions, and the goal is to classify admissible frequencies and cones, namely, to classify

$$\Phi := \{\lambda \in \mathbb{R} : \text{there is a non-trivial } \lambda\text{-homogeneous solution to (1.1)}\},$$

and

$$\mathcal{P}_\lambda := \{u : u \text{ solves (1.1) with } x \cdot \nabla u = \lambda u\}$$

for each $\lambda \in \Phi$. The second direction concerns the regularity of the contact set $\Lambda(u)$ for a general solution. Here the central issue is to quantify the rate of convergence in (1.2), as this leads to uniqueness of the blow-up profile as well as regularity of the contact set. This often requires sorting contact points into

$$\Lambda_\lambda(u) := \{q \in \Lambda(u) : \lambda_q = \lambda\}. \quad (1.3)$$

1.1. Admissible frequencies and homogeneous solutions

The program along the first direction is complete when $n = 2$. See, for instance, PETROSYAN–SHAHGHOLIAN–URALTSEVA [15]. In this case, it is known that

$$\Phi = \mathbb{N} \cup \{2k - \frac{1}{2} : k \in \mathbb{N}\}.$$

Corresponding to integer frequencies, the homogeneous solutions are (even reflections of) polynomials. To be precise, we have that

$$\mathcal{P}_{2k-1} = \{a(-1)^k \operatorname{Re}(|x_2| + ix_1)^{2k-1} : a \geq 0\} \quad (1.4)$$

and

$$\mathcal{P}_{2k} = \{a \operatorname{Re}(x_1 + ix_2)^{2k} : a \geq 0\}, \quad (1.5)$$

where $\operatorname{Re}(\cdot)$ denotes the real part of a complex number. In particular, all $(2k - 1)$ -cones vanish along the line $\{x_2 = 0\}$, and $2k$ -cones are harmonic in the entire space.

On the other hand, homogeneous solutions with $(2k - \frac{1}{2})$ frequencies vanish along half-lines. Up to a normalization, they satisfy

$$\operatorname{spt}(\Delta u) = \Lambda(u) = \{x_1 \leq 0, x_2 = 0\},$$

where we denote by $\operatorname{spt}(\cdot)$ the support of a measure. Up to a normalization, the $(2k - \frac{1}{2})$ -cone is given by

$$u_{2k-\frac{1}{2}}(r, \theta) := r^{2k-\frac{1}{2}} \cos\left(\left(2k - \frac{1}{2}\right)\theta\right), \quad (1.6)$$

where $r \geq 0$ and $\theta \in (-\pi, \pi]$ are the polar coordinates of the plane.

In general dimensions, the classification of admissible frequencies and cones remains incomplete. By extending the solutions from $\mathbb{R}^2$, we see that

$$\Phi \supset \mathbb{N} \cup \left\{2k - \frac{1}{2} : k \in \mathbb{N}\right\}.$$

Thanks to FOCARDI–SPADARO [11, 12], we know that $\bigcup_{\lambda \in \mathbb{N} \cup \{2k - \frac{1}{2} : k \in \mathbb{N}\}} \Lambda_\lambda(u)$ makes up most of the contact points, in the sense that its complement in $\bar{\Lambda}(u)$ has dimension at most $(n - 3)$.

Athanasopoulos–Caffarelli–Salsa classified the lowest three frequencies [3], namely,

$$\Phi \subset \left\{1, \frac{3}{2}\right\} \cup [2, +\infty).$$

COLOMBO–SPOLAOR–VELICHKOV [4] and SAVIN–YU [17] showed the existence of a frequency gap around each integer, that is, for each $m \in \mathbb{N}$, there is $\alpha_m > 0$, depending only on m and n , such that

$$\Phi \cap (m - \alpha_m, m + \alpha_m) = \{m\}.$$

For the classification of cones, most results center around frequencies in $\{\frac{3}{2}\} \cup \mathbb{N}$. By ATHANASOPOULOS–CAFFARELLI–SALSA [3], it is known that

$$\mathcal{P}_{3/2} = \{\text{Normalizations of } u_{3/2} \text{ as in (1.6)}\}^1$$

Note that $u_{\frac{3}{2}}$ is monotone along any direction in $\{x_n = 0\}$, a fact used extensively for the classification of $\frac{3}{2}$ -cones as well as free boundary regularity near points with $\frac{3}{2}$ frequency.

Extensions of (1.4) and (1.5) to general dimensions were obtained by FIGALLI–ROS–OTON–SERRA [10] and GAROFALO–PETROSYAN [13], respectively. Similar to their counterparts in $\mathbb{R}^2$, all $(2k - 1)$ -cones vanish in the hyperplane $\{x_n = 0\}$, and $2k$ -cones are harmonic in the entire space. Consequently, if we let v denote a solution to the linearized equation around an integer-frequency cone, then either $v|_{\{x_n=0\}}$ or $\Delta v|_{\{x_n=0\}}$ has a sign. The vanishing property of $(2k - 1)$ -cones implies that $v|_{\{x_n=0\}} \geq 0$. The harmonicity of $2k$ -cones implies that $\Delta v|_{\{x_n=0\}} \leq 0$. These are the key observations behind the regularity of contact points with integer frequencies [18].

1.2. Regularity of the contact set

By the classification of $\frac{3}{2}$ -cones, if $q \in \Lambda_{\frac{3}{2}}(u)$, then after a normalization, we have $u_{q,r} \rightarrow u_{\frac{3}{2}}$ along a subsequence of $r \rightarrow 0$. Here we are using the notations from (1.2) and (1.6). With the monotone property of $u_{\frac{3}{2}}$, Athanasopoulos–Caffarelli–Salsa proved that the blow-up profile is independent of the subsequence of $r \rightarrow 0$, and that $\Lambda_{\frac{3}{2}}(u)$ is locally a $(n - 2)$ -dimensional $C^{1,\alpha}$ -manifold in $\{x_n = 0\}$ [3]. Recently, this manifold has been shown to be smooth in [6] and analytic in [14].

For points in $\Lambda_{2k}(u)$, uniqueness of the blow-up profile was established by GAROFALO–PETROSYAN [13], who also showed that $\Lambda_{2k}(u)$ is contained in countably many C^1 -manifolds. Regularity of the covering manifolds was improved to $C^{1,\log}$ by COLOMBO–SPOLAOR–VELICHKOV [4]. For points in $\Lambda_{2k-1}(u)$, uniqueness of the blow-up profile was obtained by FIGALLI, ROS–OTON and SERRA [10]. Recently, a unified approach was developed to quantify the rate of convergence in (1.2) at points in $\Lambda_{2k-1}(u)$ and $\Lambda_{2k}(u)$ [18]. In particular, we proved that $\Lambda_{2k-1}(u)$ is locally covered by $C^{1,\alpha}$ -manifolds.

On a different note, Fernández-Real and Ros-Oton showed that for generic boundary data, the free boundary is smooth outside a set of dimension at most $(n - 3)$ [9]. In general, the free boundary is always countably $(n - 2)$ -rectifiable, a result by FOCARDI–SPADARO [11, 12].

1.3. Main results

In this paper, we study contact points with $\frac{7}{2}$ frequency in $\mathbb{R}^3$. The example $u_{\frac{7}{2}}$ from (1.6) illustrates that these points can make up the entire free boundary as well

¹ In $\mathbb{R}^n$, the pair (r, θ) is understood as the polar coordinates of the (x_{n-1}, x_n) -plane.

as the entire line $\{r = 0\}$. Unfortunately, not much is known about them in terms of the classification of $\frac{7}{2}$ -cones and the regularity of $\Lambda_{\frac{7}{2}}(u)$.

Unlike $\frac{3}{2}$ -cones, homogeneous solutions with $\frac{7}{2}$ frequency are not monotone along directions in $\{x_n = 0\}$. On top of that, for a solution, v , to the linearized equation around $u_{\frac{7}{2}}$, neither $v|_{\{x_n=0\}} \geq 0$ nor $\Delta v|_{\{x_n=0\}} \leq 0$ is necessarily true. Thus the observation behind the study of integer-frequency points is no longer applicable. As a result, it requires new ideas to study contact points with $\frac{7}{2}$ frequency.

With the full classification of $\frac{7}{2}$ -cones seemingly out of reach, we focus on the family of *half-space cones*. Up to a rotation in $\{x_n = 0\}$, these are homogeneous solutions satisfying

$$\text{either } \text{spt}(\Delta u) \subset \{x_{n-1} \leq 0, x_n = 0\}, \text{ or } \text{spt}(\Delta u) \supset \{x_{n-1} \leq 0, x_n = 0\}.$$

With notations from (1.6) and footnote 1, half-space $\frac{7}{2}$ -cones in $\mathbb{R}^3$ belong to, up to a normalization, the family

$$\mathcal{F}_1 := \left\{ u_{\frac{7}{2}} + a_1 x_1 u_{\frac{5}{2}} + a_2 \left(x_1^2 - \frac{1}{5} r^2 \right) u_{\frac{3}{2}} : 0 \leq a_2 \leq 5, \text{ and } a_1^2 \leq \Gamma(a_2) \right\}, \quad (1.7)$$

where

$$\Gamma(a_2) := \min \left\{ 4a_2 \left(1 - \frac{1}{5} a_2 \right), \frac{24}{25} a_2 \left(\frac{7}{2} - \frac{3}{10} a_2 \right) \right\}. \quad (1.8)$$

The subscript in $\mathcal{F}_1$ is to indicate that the coefficient of $u_{\frac{7}{2}}$ is 1. The parameters $a = (a_1, a_2)$ lie in the region

$$\mathcal{A} = E_1 \cap E_2,$$

where E_1 and E_2 are two ellipses

$$E_1 = \left\{ \frac{a_1^2}{5} + \frac{(a_2 - 5/2)^2}{25/4} \leq 1 \right\} \text{ and } E_2 = \left\{ \frac{a_1^2}{49/5} + \frac{(a_2 - 35/6)^2}{(35/6)^2} \leq 1 \right\}. \quad (1.9)$$

Their boundaries intersect at $(0, 0)$ and $(\pm \frac{\sqrt{15}}{2}, 5/4)$; see Fig. 1.

Remark 1.2. Up to a normalization, this family $\mathcal{F}_1$ contains all examples of $\frac{7}{2}$ -cones currently known.

For future reference, we divide $\mathcal{A}$ further into three subregions

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3,$$

according to the location of $a = (a_1, a_2)$ relative to $\partial \mathcal{A}$. See Fig. 2.

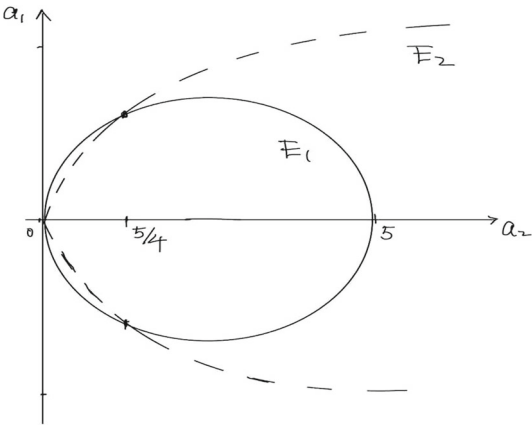


Fig. 1. Range of (a_1, a_2) in $\mathcal{F}_1$ is $\mathcal{A} = E_1 \cap E_2$

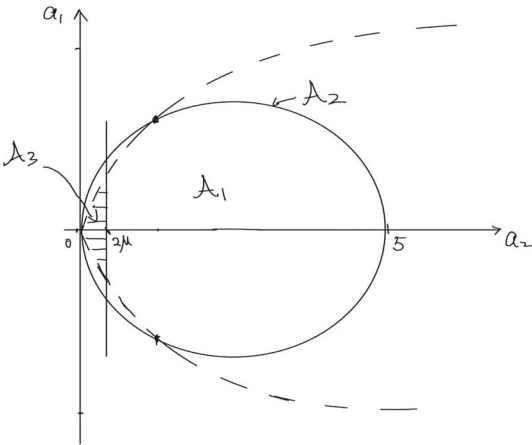


Fig. 2. $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$

Let $\mu > 0$ be a small parameter². These subregions are defined as:

$$\begin{aligned} \mathcal{A}_1 &:= \{a \in \mathcal{A} : a_2 \geq \mu, \quad a_1^2 < \Gamma(a_2)\}; \\ \mathcal{A}_2 &:= \{a \in \mathcal{A} : a_2 \geq \mu, \quad a_1^2 = \Gamma(a_2)\}; \text{ and} \\ \mathcal{A}_3 &:= \{a \in \mathcal{A} : a_2 \leq 2\mu\}. \end{aligned} \tag{1.10}$$

With an abuse of notation, we also write

$$p \in \mathcal{A}_j \text{ for } j = 1, 2, 3 \tag{1.11}$$

when the coefficients of p belong to the corresponding region.

We now describe the main results of this work.

² The parameter μ is chosen in Sections 5. See Remark 5.1.

Although there could be other $\frac{7}{2}$ -cones, they cannot be connected to the half-space cones. This is the content of our first result.

Theorem 1.1. *Suppose that u is a $\frac{7}{2}$ -homogeneous solution to (1.1) in $\mathbb{R}^3$ with*

$$|u - p| \leq d \text{ in } B_1$$

for some $p \in \mathcal{F}_1$.

There is a universal constant $d_0 > 0$ such that if $d < d_0$, then up to a normalization, we have that

$$u \in \mathcal{F}_1.$$

A *universal constant* is a constant whose value is independent of the particular solution under consideration.

The following two results address the behavior of the solution near a contact point where at least one blow-up belongs to $\mathcal{F}_1$. For brevity, let us denote these points by $\Lambda_{\frac{7}{2}}^{HS}(u)$, that is,

$$\Lambda_{\frac{7}{2}}^{HS}(u) := \{q \in \Lambda(u) : \text{Up to a normalization, one blow-up profile at } q \text{ is in } \mathcal{F}_1\}.$$

The next result quantifies the rate of convergence in (1.2) at a point in $\Lambda_{\frac{7}{2}}^{HS}(u)$.

Theorem 1.2. *Let u be a solution to (1.1) in $B_1 \subset \mathbb{R}^3$ with $0 \in \Lambda_{\frac{7}{2}}^{HS}(u)$. Then up to a normalization, we have the following two possibilities:*

(1) *either*

$$|u - u_{\frac{7}{2}}| \leq O(r^{\frac{7}{2}} |\log(r)|^{-c_0}) \text{ in } B_r \text{ for all small } r;$$

(2) *or*

$$|u - p| \leq O(r^{\frac{7}{2}+c_0}) \text{ in } B_r \text{ for all small } r$$

for some $p \in \mathcal{F}_1 \setminus \{u_{\frac{7}{2}}\}$.

The parameter $c_0 > 0$ is universal.

In particular, blow-up profiles at points in $\Lambda_{\frac{7}{2}}^{HS}(u)$ are independent of the subsequence $r \rightarrow 0$.

Theorem 1.2 also leads to a stratification result concerning $\Lambda_{\frac{7}{2}}^{HS}(u)$, we have

Theorem 1.3. *Let u be a solution to (1.1) in $B_1 \subset \mathbb{R}^3$. Then we have the decomposition*

$$\Lambda_{\frac{7}{2}}^{HS}(u) \cap B_1 = \Sigma_0 \cup \Sigma_1,$$

where Σ_0 is locally discrete, and Σ_1 is locally covered by a $C^{1,\log}$ -curve.

Remark 1.3. Suppose $0 \in \Lambda_{\frac{7}{2}}^{HS}(u)$, we actually have regularity of the entire $\Lambda_{\frac{7}{2}}(u)$ near 0 (instead of just $\Lambda_{\frac{7}{2}}^{HS}(u)$). If $u_{\frac{7}{2}}$ is a blow-up profile at 0, then the free boundary $\partial_{\mathbb{R}^{n-1}} \Lambda(u)$ is $C^{\frac{1}{2}, \log}$ at 0. If u blows up to some $p \in \mathcal{F}_1 \setminus \{u_{\frac{7}{2}}\}$, then $\Lambda_{\frac{7}{2}}(u) \cap B_\rho(0) = \{0\}$ for some small $\rho > 0$.

These results are proven through an improvement-of-flatness argument. Roughly, if the solution u is approximated in B_1 by a profile p with error d , then we need to reduce the error at a smaller scale, say in B_ρ , by picking another profile p' . The natural candidate is

$$p' = p + dv, \quad (1.12)$$

where v is the solution to the linearized problem around p .

This strategy has been successful in many free boundary problems, for instance, the Bernoulli problem [5], the obstacle problem [19] and the triple membrane problem [20]. In these problems, the solutions have a fixed homogeneity at free boundary points. This is not the case for the thin obstacle problem. Consequently, we cannot always reduce the error in our problem. When this happens, however, we can ‘improve the homogeneity’ in terms of the Weiss energy functional [23].

This is the content of the main lemma of this work, which is as follows:

Lemma 1.1. *There are constants, $\tilde{d}$, ρ , c small, and C big, such that If $u \in \mathcal{S}(p, d, 1)$ with $p \in \mathcal{F}_1$ and $d < \tilde{d}$, then we have the following dichotomy: a) either*

$$W_{\frac{7}{2}}(u; 1) - W_{\frac{7}{2}}(u; \rho) \geq cd^2,$$

and

$$u \in \mathcal{S}(p, Cd, \rho);$$

b) or

$$u \in \mathcal{S}(p', \frac{1}{2}d, \rho),$$

where $p' \in \mathcal{F}_1$ up to a normalization, and

$$\|p' - p\|_{L^\infty(\mathbb{S}^2)} \leq Cd.$$

The space $\mathcal{S}(p, d, \rho)$ consists of d -approximated solutions at scale ρ , and $W_{\frac{7}{2}}(\cdot)$ is the Weiss energy functional, defined in (2.6). A similar lemma was established for integer-frequency points in [18].

The proof of Lemma 1.1 is divided into three cases, corresponding to the three subregions of $\mathcal{A}$ as in Fig. 2.

when $p \in \mathcal{A}_1$, the modification p' from (1.12) solves the thin obstacle problem for small d . In this case, Lemma 1.1 follows from a standard compactness argument.

Extra care is needed when $p \in \mathcal{A}_2 \cup \mathcal{A}_3$. Here the profile p could become degenerate at certain points. The same p' might violate the constraints $p'|_{\{x_3=0\}} \geq 0$ and $\Delta p' \leq 0$.

For $p \in \mathcal{A}_2$, there are two possibilities. If the coefficients (a_1, a_2) are bounded away from $(\pm\sqrt{15}/2, 5/4)$, the intersection of ∂E_1 and ∂E_2 , then only one of the two constraints might fail. If they are very close to $(\pm\sqrt{15}/2, 5/4)$, both constraints can fail, but the locations of failure are well-separated from $\{r = 0\}$. In both cases, we replace p' by solving a *boundary-layer problem* around the place where the constraints fail. This is the same strategy adapted to study integer-frequency points [18].

New challenges arise when $p \in \mathcal{A}_3$. When p is very close to $u_{\frac{7}{2}}$, both constraints might fail along $\{r = 0\}$. Indeed Lemma 1.1 needs to be modified to be a trichotomy. See Lemma 5.1. For this, we need to study an ‘inner problem’ in small spherical caps near $\{r = 0\}$, which reduces to the thin obstacle problem in $\mathbb{R}^2$ with data at infinity. This is the main reason why we restrict to three dimension in this work.

Although this restriction to three dimension seems crucial, we hope similar ideas would work for half-space solutions with higher frequencies.

This paper is organized as follows: In Section 2, we collect some preliminary results. In Sections 3, we establish Lemma 1.1 when $p \in \mathcal{A}_1$. The same lemma is proved in Section 4 for p near $\mathcal{A}_2$. In Section 5, a modified lemma is proved for profiles in $\mathcal{A}_3$. This is the most involved part of this paper, and requires several technical preparations that are left to the Appendices. Finally in Section 6, all these are combined to show the main results Theorems 1.1, 1.2 and 1.3.

2. Preliminaries

In this section, we gather some useful notations and results.

Unless otherwise specified, in this paper we denote by u a solution to the thin obstacle problem (1.1) in some domain in $\mathbb{R}^3$. For this space, we have the standard coordinate system $\mathbb{R}^3 = \{(x_1, x_2, x_3) : x_j \in \mathbb{R}\}$, decomposed as

$$x = (x', x_3) \text{ where } x' = (x_1, x_2).$$

A subset of $\mathbb{R}^3$ is decomposed as $E = E^+ \cup E' \cup E^-$, where

$$E' = E \cap \{x_3 = 0\}, \text{ and } E^\pm = E \cap \{\pm x_3 > 0\}. \quad (2.1)$$

With this notation, the contact set is $\Lambda(u) = \{u = 0\}'$.

Recall that the solution u is assumed to be even with respect to $\{x_3 = 0\}$. As such it may fail to be differentiable in the x_3 -direction at points in $\Lambda(u)$. Nevertheless, it is still differentiable from either side of the domain. In this paper, for a function $w \in C^1(B_1 \cap \{x_3 \geq 0\})$, we use $\frac{\partial}{\partial x_3} w(x)$ to denote its one-sided derivative at $x \in B'_1$, namely,

$$\frac{\partial}{\partial x_3} w(x) = \lim_{t \rightarrow 0^+} \frac{w(x', t) - w(x', 0)}{t}. \quad (2.2)$$

In general, if Ω is a domain in $\mathbb{R}^3$, we denote by ν the inner unit normal along $\partial\Omega$. For $w \in C^1(\overline{\Omega})$ and $x \in \partial\Omega$, we denote by $w_\nu(x)$ the one-sided normal derivative at x_0 with respect to Ω , that is,

$$w_\nu(x) = \lim_{t \rightarrow 0^+} \frac{w(x + t\nu) - w(x)}{t}. \quad (2.3)$$

To utilize the rotational symmetry of the problem, we introduce the *rotation operator* with respect to the x_n -axis. For $\tau \in (-\pi, \pi)$, this operator U_τ acts on points, sets, and functions in the following manner:

$$\begin{aligned} U_\tau(x) &= (x_1 \cos(\tau) - x_2 \sin(\tau), x_1 \sin(\tau) + x_2 \cos(\tau), x_3), \\ U_\tau(E) &= \{x : U_{-\tau}x \in E\}, \\ U_\tau(f)(x) &= f(U_{-\tau}x). \end{aligned} \quad (2.4)$$

The problem is also scaling invariant. For $\rho > 0$ and $q \in \Lambda_{\frac{7}{2}}(u)$, defined as in (1.3), the *rescaled function*

$$u_{(q,\rho)}(x) := u(q + \rho x) / \rho^{\frac{7}{2}} \quad (2.5)$$

solves the problem in a rescaled domain with $0 \in \Lambda_{\frac{7}{2}}(u_{(q,\rho)})$. When $q = 0$, we simplify the notation by

$$u_{(\rho)} := u_{(0,\rho)}.$$

2.1. Weiss monotonicity formula and consequences

The Weiss monotonicity formula was used by Weiss to treat the obstacle problem [23], and was adapted to the thin obstacle problem by GAROFALO–PETROSYAN [13]. Its decay is used in this paper to quantify an ‘improvement of homogeneity’ between scales.

Since we are concerned with contact points with $\frac{7}{2}$ frequency in $\mathbb{R}^3$, we include here only the $\frac{7}{2}$ -Weiss energy functional in $3d$

$$W_{\frac{7}{2}}(u; \rho) = \frac{1}{\rho^8} \int_{B_\rho} |\nabla u|^2 - \frac{7}{2\rho^9} \int_{\partial B_\rho} u^2. \quad (2.6)$$

We collect some of its properties in the following lemma. For its proof, see Theorem 1.4.1 and Theorem 1.5.4 in [13].

Lemma 2.1. *Suppose that u solves the thin obstacle problem in $B_1 \subset \mathbb{R}^3$. Then for $\rho \in (0, 1)$, we have*

$$\frac{d}{d\rho} W_{\frac{7}{2}}(u; \rho) = \frac{2}{\rho} \int_{\partial B_1} \left(\nabla u_{(\rho)} \cdot \nu - \frac{7}{2} u_{(\rho)} \right)^2. \quad (2.7)$$

In particular, $\rho \mapsto W_{\frac{7}{2}}(u; \rho)$ is non-decreasing.

If $0 \in \Lambda_{\frac{7}{2}}(u)$, then $\lim_{\rho \rightarrow 0} W_{\frac{7}{2}}(u; \rho) = 0$.

The rescaling $u_{(\rho)}$ is defined as in (2.5).

Under the same assumptions as in Lemma 2.1, we can integrate (2.7) and apply Hölder's inequality to get that

$$\int_{\partial B_1} |u_{(\rho_1)} - u_{(\rho_2)}| \leq (\log(\rho_1/\rho_2))^{\frac{1}{2}} [W_{\frac{7}{2}}(u; \rho_1) - W_{\frac{7}{2}}(u; \rho_2)]^{\frac{1}{2}} \quad (2.8)$$

for $0 < \rho_2 < \rho_1 < 1$.

2.2. Harmonic functions in slit domains and half-space cones

Motivated by thin free boundary problems, harmonic functions in slit domains were studied in great detail by DE SILVA-SAVIN [6, 7]. In this work, we only need certain basic elements when the slit is flat.

Let (r, θ) denote the polar coordinate for the (x_2, x_3) -domain with $r \geq 0$ and $\theta \in (-\pi, \pi]$. The *slit* is defined as

$$\mathcal{S} := \{\theta = \pi\} = \{x_2 \leq 0, x_3 = 0\}. \quad (2.9)$$

For a subset of $\mathbb{R}^3$, we decompose it relative to the slit as $E = \widehat{E} \cup \widetilde{E}$, where

$$\widehat{E} = E \setminus \mathcal{S}, \text{ and } \widetilde{E} = E \cap \mathcal{S}. \quad (2.10)$$

Given a domain $\Omega \subset \mathbb{R}^3$, a *harmonic function in the slit domain* $\widehat{\Omega}$ is a continuous function that is even with respect to $\{x_3 = 0\}$ and satisfies

$$\begin{cases} \Delta v = 0 & \text{in } \widehat{\Omega}, \\ v = 0 & \text{in } \widetilde{\Omega}. \end{cases} \quad (2.11)$$

As is the case for regular domains, homogeneous solutions play an important role. Given a non-negative integer m , let's define the following space:

$$\mathcal{H}_{m+\frac{1}{2}} := \left\{ v : v \text{ is a harmonic function in } \widehat{\mathbb{R}^3}, \quad x \cdot \nabla v = (m + \frac{1}{2})v \right\}. \quad (2.12)$$

Functions in $\mathcal{H}_{m+\frac{1}{2}}$ satisfy

$$\begin{cases} (\Delta_{\mathbb{S}^2} + \lambda_{m+\frac{1}{2}})v = 0 & \text{in } \widehat{\mathbb{S}^2}, \\ v = 0 & \text{in } \widetilde{\mathbb{S}^2}, \end{cases} \quad (2.13)$$

where

$$\Delta_{\mathbb{S}^2} \text{ is the spherical Laplacian, and } \lambda_{m+\frac{1}{2}} = \left(m + \frac{1}{2}\right) \left(m + \frac{3}{2}\right). \quad (2.14)$$

These functions are the basic building blocks for general solutions to (2.11). For instance, we have the following theorem from [6]:

Theorem 2.1. (Theorem 4.5 from [6]) *Let v be a solution to (2.11) with $\Omega = B_1$ and $\|v\|_{L^\infty(B_1)} \leq 1$.*

Given $m \geq 0$, we can find $v_{k+\frac{1}{2}} \in \mathcal{H}_{k+\frac{1}{2}}$ for $k = 0, 1, \dots, m$, such that

$$\|v_{k+\frac{1}{2}}\|_{L^\infty(B_1)} \leq C$$

and

$$\left| v - \sum_{k=0}^m v_k \right| (x) \leq C|x|^{m+1} u_{\frac{1}{2}} \text{ for } x \in B_{\frac{1}{2}}.$$

Here $u_{\frac{1}{2}}$ is defined as in (1.6), and C depends only on m .

The functions from (1.6) are homogeneous harmonic functions in $\widehat{\mathbb{R}^3}$. The following proposition states that, in some sense, these functions generate all homogeneous harmonic functions; its elementary proof is left to the reader.

Proposition 2.1. *If $v \in \mathcal{H}_{m+\frac{1}{2}}$, then we have the following expansion:*

$$v = a_0 u_{m+\frac{1}{2}} + p_1(x_1, r) u_{m-\frac{1}{2}} + \dots + p_k(x_1, r) u_{m+\frac{1}{2}-k} + \dots + p_m(x_1, r) u_{\frac{1}{2}},$$

where $a_0 \in \mathbb{R}$, and p_k is a k -homogeneous polynomial in (x_1, r) .

The following orthogonality follows from standard argument:

Proposition 2.2. *Suppose $m \neq n$ are two non-negative integers, then we have the following:*

a) If p and q are polynomials of (x_1, r) , then

$$\int_{\mathbb{S}^2} p u_{m+\frac{1}{2}} \cdot q u_{n+\frac{1}{2}} = 0.$$

2) If $v \in \mathcal{H}_{m+\frac{1}{2}}$ and $w \in \mathcal{H}_{n+\frac{1}{2}}$, then

$$\int_{\mathbb{S}^2} v \cdot w = 0.$$

In this work, we are most interested in the space of harmonic functions with $\frac{7}{2}$ homogeneity, namely, $\mathcal{H}_{\frac{7}{2}}$. Following Proposition 2.1, we see that this space is spanned by the following four functions:

$$\begin{aligned} u_{\frac{7}{2}} &= r^{\frac{7}{2}} \cos\left(\frac{7}{2}\theta\right), \quad v_{\frac{5}{2}} := x_1 u_{\frac{5}{2}}, \\ v_{\frac{3}{2}} &:= (x_1^2 - r^2/5) u_{\frac{3}{2}}, \quad \text{and } v_{\frac{1}{2}} := (x_1^3 - x_1 r^2) u_{\frac{1}{2}}. \end{aligned} \quad (2.15)$$

The same space is also spanned by $u_{\frac{7}{2}}$ and its first three rotational derivatives. Using the notation from (2.4), they are

$$\begin{aligned} u_{\frac{7}{2}}, \quad w_{\frac{5}{2}} &:= \frac{d}{d\tau} \Big|_{\tau=0} U_\tau(u_{\frac{7}{2}}), \quad w_{\frac{3}{2}} := \frac{d}{d\tau} \Big|_{\tau=0} U_\tau(w_{\frac{5}{2}}), \quad \text{and} \\ w_{\frac{1}{2}} &:= \frac{d}{d\tau} \Big|_{\tau=0} U_\tau(w_{\frac{3}{2}}). \end{aligned} \quad (2.16)$$

Remark 2.1. These two bases are related by

$$w_{\frac{5}{2}} = \frac{7}{2}v_{\frac{5}{2}}, \quad w_{\frac{3}{2}} = \frac{35}{4}v_{\frac{3}{2}} - \frac{7}{4}u_{\frac{7}{2}}, \quad \text{and} \quad w_{\frac{1}{2}} = \frac{105}{8}v_{\frac{1}{2}} - \frac{133}{8}v_{\frac{5}{2}}.$$

With these preparations, we classify half-space solutions to the thin obstacle problem in $\mathbb{R}^3$ that are $\frac{7}{2}$ -homogeneous.

Proposition 2.3. *Suppose that u is a nontrivial $\frac{7}{2}$ -homogeneous solution to (1.1) in $\mathbb{R}^3$. The followings are equivalent:*

- (1) $\text{spt}(\Delta u) \subset \mathcal{S}$;
- (2) $\text{spt}(\Delta u) \supset \mathcal{S}$;
- (3) $u \in \mathcal{F}_1$ up to a normalization.

Recall the definition of $\mathcal{F}_1$ from (1.7). See also Remark 1.1 for the notion of normalization.

Proof. By definition of $\mathcal{F}_1$, statement (3) implies the other two. Here we show that statement (1) implies statement (3). A similar argument gives the implication (2) $\implies$ (3).

By Green's formula and homogeneity of the functions involved, we have that

$$\int_{B_1} u_{\frac{7}{2}} \Delta u - u \Delta u_{\frac{7}{2}} = \frac{7}{2} \int_{\mathbb{S}^2} u_{\frac{7}{2}} u - u u_{\frac{7}{2}} = 0.$$

By statement (1), $u_{\frac{7}{2}} = 0$ on $\text{spt}(\Delta u)$, thus $\int_{B_1} u \Delta u_{\frac{7}{2}} = 0$.

Since $u \geq 0$ on $\mathcal{S} = \text{spt}(\Delta u_{\frac{7}{2}})$, this implies $u = 0$ on $\mathcal{S}$. With statement (1), we see that u is a $\frac{7}{2}$ -homogeneous harmonic function in $\widehat{\mathbb{R}^3}$. Consequently, it is a linear combination of functions from (2.15), that is,

$$u = a_0 u_{\frac{7}{2}} + a_1 v_{\frac{5}{2}} + a_2 v_{\frac{3}{2}} + a_3 v_{\frac{1}{2}}$$

for $a_j \in \mathbb{R}$. Such a function satisfies the constraints $u|_{\{x_3=0\}} \geq 0$ and $\Delta u|_{\{x_3=0\}} \leq 0$ if and only if $a_0 > 0$ and $u/a_0 \in \mathcal{F}_1$. $\square$

For our purpose, we also need homogeneous harmonic functions in slit domains with singularities. In $\mathbb{R}^2$, typical examples are given by

$$u_{-n+\frac{1}{2}}(r, \theta) := r^{-n+\frac{1}{2}} \cos\left(\left(n - \frac{1}{2}\right)\theta\right) \text{ for } n \in \mathbb{N}. \quad (2.17)$$

Each $u_{-n+\frac{1}{2}}$ is $(-n + \frac{1}{2})$ -homogeneous and harmonic in $\widehat{\mathbb{R}^2}$.

In $\mathbb{R}^3$, we will need the following two functions:

$$v_{-\frac{1}{2}} := (x_1^4 - 6x_1^2 r^2 - r^4) \cdot u_{-\frac{1}{2}}, \quad \text{and} \quad v_{-\frac{3}{2}} := (x_1^5 + 10x_1^3 r^2 - 15x_1 r^4) \cdot u_{-\frac{3}{2}}. \quad (2.18)$$

Both of these are $\frac{7}{2}$ -homogeneous functions in $\mathbb{R}^3$ and harmonic in $\widehat{\mathbb{R}^3}$. Near the poles $\mathbb{S}^2 \cap \{r = 0\}$, they have a singularity of order $-\frac{1}{2}$ and $-\frac{3}{2}$ respectively.

Correspondingly, we have that

$$w_{-\frac{1}{2}} := \frac{d}{d\tau} \Big|_{\tau=0} U_{\tau}(w_{\frac{1}{2}}), \text{ and } w_{-\frac{3}{2}} := \frac{d}{d\tau} \Big|_{\tau=0} U_{\tau}(w_{-\frac{1}{2}}), \quad (2.19)$$

which are also $\frac{7}{2}$ -homogeneous and harmonic in $\widehat{\mathbb{R}}^3$.

2.3. A Double-sequence lemma

We conclude this section with a lemma dealing with two numerical sequences. It is a slight modification of Lemma 5.1 from [18].

Lemma 2.2. *Let (w_n) and (e_n) be two sequences of real numbers between 0 and 1. Suppose that for some constants, A big, a small and $\gamma \in (0, 1]$, we have*

$$w_{n+1} \leq A e_n^{1+\gamma} \quad \forall n \in \mathbb{N}$$

and the following dichotomy:

- *either $w_{n+1} \leq w_n - a e_n^2$ and $e_{n+1} = A e_n$;*
- *or $w_{n+1} \leq w_n$ and $e_{n+1} = \frac{1}{2} e_n$.*

Then we have that

$$e_n \leq C e_1^{\frac{1+\gamma}{2}} \quad (2.20)$$

for all $n \in \mathbb{N}$, and

$$\sum e_n < +\infty.$$

Moreover, we have that

$$\sum_{n \geq N} e_n \leq C(w_N + e_N^2)^{\frac{1}{2}} \text{ if } \gamma = 1; \quad (2.21)$$

and

$$\sum_{n \geq 2^N} e_n \leq C 2^{\frac{-\gamma}{1-\gamma} N} \text{ if } \gamma \in (0, 1). \quad (2.22)$$

Here $c \in (0, 1)$ and C are constants depending only on A , a and γ .

Proof. The only modification from Lemma 5.1 in [18] is the right-hand side of (2.21). To see this, let $\alpha_n := w_n + \mu e_n^2$. For $\mu > 0$ small, it was shown in [18] that $\alpha_n \leq (1 - c)\alpha_{n-1}$, which gives that

$$\sum_{n \geq N} \alpha_n^{1/2} \leq C \alpha_N^{1/2}.$$

From here, we simply note that $e_n \leq \alpha_n^{1/2}$. $\square$

3. Dichotomy for $p \in \mathcal{A}_1$

In this section, we prove Lemma 1.1 when $p \in \mathcal{A}_1$; see (1.10), (1.11) and Fig. 2.

Starting with such a profile p , the natural modification $p' = p + dv$ from (1.12) solves the thin obstacle problem if d is small. Consequently, the improvement-of-flatness result follows by a classical argument.

Nevertheless, we include the argument here. Readers less familiar with the subject might take this section as a roadmap for the strategy. Contrasting this section with the next two, we hope to illustrate the challenges that arise in each different case.

3.1. Well-approximated solutions

Throughout this section, we consider profiles $p = a_0 u_{\frac{7}{2}} + a_1 v_{\frac{5}{2}} + a_2 v_{\frac{3}{2}}$ with

$$a_0 \in [1/2, 2], \text{ and } (a_1/a_0, a_2/a_0) \in \mathcal{A}_1;$$

that is, for a small parameter $\mu > 0$,

$$1/2 \leq a_0 \leq 2, \quad \mu \leq a_2/a_0 \leq 5, \text{ and } (a_1/a_0)^2 < \Gamma(a_2/a_0). \quad (3.1)$$

Recall the basis $\{u_{\frac{7}{2}}, v_{\frac{5}{2}}, v_{\frac{3}{2}}, v_{\frac{1}{2}}\}$ from (2.15), and the function Γ from (1.8).

To simplify our discussions, let us denote

$$\mu_p := \Gamma(a_2/a_0) - (a_1/a_0)^2. \quad (3.2)$$

The space of well-approximated solutions is defined as follows:

Definition 3.1. Suppose that the coefficients of p satisfy (3.1).

For $d, \rho \in (0, 1]$, we say that u is a *solution d -approximated by p at scale ρ* if u solves the thin obstacle problem (1.1) in B_ρ , and

$$|u - p| \leq d\rho^{\frac{7}{2}} \text{ in } B_\rho.$$

In this case, we write

$$u \in \mathcal{S}(p, d, \rho).$$

Being well-approximated implies the localization of the contact set, as follows:

Lemma 3.1. Suppose that $u \in \mathcal{S}(p, d, 1)$ with d small.

We have

$$\Delta u = 0 \text{ in } \widehat{B}_1 \cap \{r > Cd^{\frac{2}{7}}\}, \text{ and } u = 0 \text{ in } \widetilde{B}_{\frac{7}{8}} \cap \{r > Cd^{\frac{2}{13}}\},$$

where C depends only on μ and μ_p from (3.2).

Recall the notations for slit domains from (2.10), and that (r, θ) denotes the polar coordinate of the (x_2, x_3) -plane.

Proof. Using (3.1) and direct computations, we have that

$$p \geq c_{\mu, \mu_p} r^{\frac{7}{2}} \text{ in } \{\theta = 0\}.$$

With $u \geq p - d$ in B_1 , it follows $u > 0$ in $\{\theta = 0, r > Cd^{\frac{2}{7}}\} \cap B_1$. This gives the first conclusion.

To see the second conclusion, we note that

$$\frac{\partial}{\partial x_3} p \leq -c_{\mu, \mu_p} r^{\frac{5}{2}} \text{ in } \{\theta = \pi\}. \quad (3.3)$$

Recall our convention from (2.2).

Now for some large A to be chosen, let $x_0 \in \widetilde{B_{7/8}} \cap \{r > Ad^{\frac{2}{15}}\}$, and $\Omega := \{|x' - x_0| < d^{\frac{2}{3}}, |x_3| < d^{\frac{2}{3}}\}$.

With (3.3) and the $C^{1, \frac{1}{2}}$ -regularity of p , we have that

$$\frac{\partial}{\partial x_3} p \leq -\frac{1}{2} c_{\mu, \mu_p} A^{\frac{5}{2}} d^{\frac{1}{3}} \text{ in } \Omega^+$$

if A is large, depending only on μ and μ_p .

Define the barrier $\varphi(x', x_3) = (|x' - x_0|^2 - 2x_3^2)/d^{\frac{1}{3}}$, then φ is a solution to the thin obstacle problem. Inside Ω^+ , we have that

$$\begin{aligned} \varphi - p &\geq |x' - x_0|^2/d^{\frac{1}{3}} - 2x_3^2/d^{\frac{1}{3}} + \frac{1}{2} c_{\mu, \mu_p} A^{\frac{5}{2}} d^{\frac{1}{3}} \cdot x_3 \\ &\geq |x' - x_0|^2/d^{\frac{1}{3}} + \frac{1}{4} c_{\mu, \mu_p} A^{\frac{5}{2}} d^{\frac{1}{3}} \cdot x_3 \end{aligned}$$

for A large. It follows from even symmetry that

$$\varphi \geq p + d \text{ along } \partial\Omega$$

for A large.

Together with $u \leq p + d$ in B_1 , this implies $u \leq \varphi$ in Ω . The second conclusion follows. $\square$

Since the profile p solves the thin obstacle problem, by the maximum principle and Caccioppoli's estimate, we have the following:

Lemma 3.2. *Suppose that u solves (1.1) in B_1 . Then*

$$\|u - p\|_{L^\infty(B_{1/2})} + \|u - p\|_{H^1(B_{1/2})} \leq C \|u - p\|_{L^1(B_1)}$$

for a universal constant C .

Recall the Weiss energy functional from (2.6). This energy is controlled for well-approximated solutions.

Lemma 3.3. *Suppose that $u \in \mathcal{S}(p, d, 1)$, then*

$$W_{\frac{7}{2}}(u, 3/4) \leq Cd^2$$

for a universal constant C .

Proof. The homogeneity of p implies that

$$\begin{aligned}
 W_{\frac{7}{2}}(p; 1) &= \int_{B_1} |\nabla p|^2 - \frac{7}{2} \int_{\partial B_1} p^2 \\
 &= \int_{B_1} -p \Delta p - \int_{\partial B_1} \left(p p_\nu + \frac{7}{2} p^2 \right) \\
 &= \int_{B_1} -p \Delta p = 0.
 \end{aligned} \tag{3.4}$$

Recall from (2.3) that p_ν denotes the inner normal derivative along ∂B_1 . For the last equality, we used the fact that p is harmonic in $\widehat{\mathbb{R}^3}$.

The rest of the proof is identical to the case for integer-frequency points; see Lemma 2.7 in [18]. $\square$

3.2. The dichotomy

With these preparations, we state the main lemma of this section.

Lemma 3.4. *Suppose that $u \in \mathcal{S}(p, d, 1)$ with p satisfying (3.1).*

There is small $\tilde{\delta} > 0$, depending only on μ and μ_p , such that if $d < \tilde{\delta}$, then we have the following dichotomy:

(1) *either*

$$W_{\frac{7}{2}}(u; 1) - W_{\frac{7}{2}}(u; \rho_0) \geq c_0^2 d^2$$

and

$$u \in \mathcal{S}(p, Cd, \rho_0);$$

(2) *or*

$$u \in \mathcal{S}(p', \frac{1}{2}d, \rho_0)$$

for some

$$p' = U_\tau [a'_0 u_{\frac{7}{2}} + a'_1 v_{\frac{5}{2}} + a'_2 v_{\frac{3}{2}}]$$

$$\text{with } |\tau| + \sum |a'_j - a_j| \leq Cd.$$

The constants c_0 , ρ_0 and C depend only on μ .

Recall the rotation operator U from (2.4), and the basis $\{u_{\frac{7}{2}}, v_{\frac{5}{2}}, v_{\frac{3}{2}}, v_{\frac{1}{2}}\}$ from (2.15).

Proof. Let c_0 and ρ_0 be small constants to be chosen.

Note that for any $u \in \mathcal{S}(p, d, 1)$, we always have $u \in \mathcal{S}(p, \rho_0^{-\frac{7}{2}}d, \rho_0)$.

Suppose, on the contrary, that the conclusion is false. Then we find a sequence (u_n, p_n, d_n) satisfying

$$\liminf \mu_{p_n} > 0$$

and

$$u_n \in \mathcal{S}(p_n, d_n, 1) \text{ with } d_n \rightarrow 0,$$

but

$$W_{\frac{7}{2}}(u_n; 1) - W_{\frac{7}{2}}(u_n; \rho_0) \leq c_0^2 d_n^2 \text{ for all } n, \quad (3.5)$$

and

$$u_n \notin \mathcal{S}\left(p', \frac{1}{2}d_n, \rho_0\right) \quad (3.6)$$

for any p' satisfying the properties as in alternative (2) from the lemma.

Step 1: Compactness.

Define $\hat{u}_n = \frac{u_n - p_n}{d_n}$. Then $|\hat{u}_n| \leq 1$ in B_1 .

With Lemma 3.1, we have that

$$\Delta \hat{u}_n = 0 \text{ in } \widehat{B}_1 \cap \{r > Cd_n^{\frac{2}{7}}\}, \text{ and } \hat{u}_n = 0 \text{ in } \widetilde{B}_{7/8} \cap \{r > Cd_n^{\frac{2}{15}}\}.$$

As a result, up to a subsequence, the functions $\hat{u}_n$ converge locally uniformly in $\widetilde{B}_{7/8} \setminus \{r = 0\}$ to some $\hat{u}_\infty$. The limit $\hat{u}_\infty$ is a harmonic function in the slit domain $\widetilde{B}_{7/8}$, defined as in (2.11). Since the set $\{r = 0\}$ has zero capacity, we have that

$$\|\hat{u}_n - \hat{u}_\infty\|_{L^2(B_{7/8})} = o(1) \text{ as } n \rightarrow \infty. \quad (3.7)$$

With Theorem 2.1, for $k = 0, 1, 2, 3$, we find $h_{k+\frac{1}{2}}$, a $(k + \frac{1}{2})$ -homogeneous harmonic function in $\widehat{\mathbb{R}^3}$, such that

$$|\hat{u}_\infty - (h_{\frac{1}{2}} + h_{\frac{3}{2}} + h_{\frac{5}{2}} + h_{\frac{7}{2}})|(x) \leq C|x|^{\frac{9}{2}} \text{ for } x \in B_{7/8}. \quad (3.8)$$

Moreover, each $\|h_{k+\frac{1}{2}}\|_{L^\infty(B_1)}$ is universally bounded.

In the remaining of this proof, we omit the subscripts in u_n , p_n , $\hat{u}_n$ and d_n .

Step 2: Almost homogeneity.

With (3.7), we find $\rho \in [\rho_0, 4\rho_0]$ such that

$$\|\hat{u} - \hat{u}_\infty\|_{L^2(\partial B_\rho)} + \|\hat{u} - \hat{u}_\infty\|_{L^2(\partial B_{2\rho})} = o(1).$$

Combined with (3.8), this implies

$$\begin{aligned} & \|\hat{u} - (h_{\frac{1}{2}} + h_{\frac{3}{2}} + h_{\frac{5}{2}} + h_{\frac{7}{2}})\|_{L^2(\partial B_\rho)} + \|\hat{u} - (h_{\frac{1}{2}} + h_{\frac{3}{2}} + h_{\frac{5}{2}} + h_{\frac{7}{2}})\|_{L^2(\partial B_{2\rho})} \\ & \leq C\rho^{\frac{11}{2}} + o(1). \end{aligned}$$

As a result, we have

$$\begin{aligned} & \|[\hat{u} - (h_{\frac{1}{2}} + h_{\frac{3}{2}} + h_{\frac{5}{2}} + h_{\frac{7}{2}})]_{(\frac{1}{2})} - [\hat{u} - (h_{\frac{1}{2}} + h_{\frac{3}{2}} + h_{\frac{5}{2}} + h_{\frac{7}{2}})]\|_{L^2(\partial B_\rho)} \\ & \leq C\rho^{\frac{11}{2}} + o(1), \end{aligned}$$

where $f_{(\frac{1}{2})}$ denotes the rescaling of the function f as in (2.5). With the homogeneity of p and $h_{k+\frac{1}{2}}$, this gives that

$$\left\| \frac{1}{d} [u_{(\frac{1}{2})} - u] - (7h_{\frac{1}{2}} + 3h_{\frac{3}{2}} + h_{\frac{5}{2}}) \right\|_{L^2(\partial B_\rho)} \leq C\rho^{\frac{11}{2}} + o(1). \quad (3.9)$$

Meanwhile, applying (2.8) together with (3.5), we have that

$$\begin{aligned} \int_{\partial B_1} |u_{(\rho)} - u_{(2\rho)}| &\leq \sqrt{\log(2)} \sqrt{W_{\frac{7}{2}}(u; 2\rho) - W_{\frac{7}{2}}(u; \rho)} \\ &\leq C \sqrt{W_{\frac{7}{2}}(u; 1) - W_{\frac{7}{2}}(u; \rho_0)} \\ &\leq Cc_0d \end{aligned}$$

for a universal constant C . Note that we used our choice of $\rho \in [\rho_0, 4\rho_0]$.

This implies, by the maximum principle, that $|u_{(\rho)} - u_{(2\rho)}| \leq Cc_0d$ in $B_{1/2}$. As a result,

$$\|u_{(\frac{1}{2})} - u\|_{L^2(\partial B_\rho)} \leq C\rho^{\frac{9}{2}} \|u_{(\rho)} - u_{(2\rho)}\|_{L^2(\partial B_{\frac{1}{2}})} \leq Cc_0d\rho^{\frac{9}{2}}.$$

Together with (3.9), this gives that

$$\|7h_{\frac{1}{2}} + 3h_{\frac{3}{2}} + h_{\frac{5}{2}}\|_{L^2(\partial B_\rho)} \leq C(\rho^{\frac{11}{2}} + c_0\rho^{\frac{9}{2}} + o(1)).$$

Using Proposition 2.2 and homogeneity of the functions involved, we have that

$$\begin{aligned} \|h_{\frac{1}{2}}\|_{L^\infty(B_1)} &\leq C(\rho^4 + c_0\rho^3 + o(1)), \\ \|h_{\frac{3}{2}}\|_{L^\infty(B_1)} &\leq C(\rho^3 + c_0\rho^2 + o(1)), \\ \|h_{\frac{5}{2}}\|_{L^\infty(B_1)} &\leq C(\rho^2 + c_0\rho + o(1)). \end{aligned}$$

With (3.7) and (3.8), we have that

$$\|\hat{u} - h_{\frac{7}{2}}\|_{L^1(B_{2\rho_0})} \leq C(\rho_0 + c_0)\rho_0^{\frac{13}{2}} + o(1),$$

since $\rho \in [\rho_0, 4\rho_0]$.

Step 3: Improvement of flatness.

The last estimate from the previous step gives that

$$\|u - (p + dh_{\frac{7}{2}})\|_{L^1(B_{2\rho_0})} \leq Cd[(\rho_0 + c_0)\rho_0^{\frac{13}{2}} + o(1)].$$

We temporarily switch to the basis $\{u_{\frac{7}{2}}, w_{\frac{5}{2}}, w_{\frac{3}{2}}, w_{\frac{1}{2}}\}$ from (2.16). Suppose, in this basis, we have that

$$p = b_0u_{\frac{7}{2}} + b_1w_{\frac{5}{2}} + b_2w_{\frac{3}{2}}, \text{ and } h_{\frac{7}{2}} = \alpha_0u_{\frac{7}{2}} + \alpha_1w_{\frac{5}{2}} + \alpha_2w_{\frac{3}{2}} + \alpha_3w_{\frac{1}{2}}.$$

Thus

$$p + dh_{\frac{7}{2}} = (b_0 + d\alpha_0)u_{\frac{7}{2}} + (b_1 + d\alpha_1)w_{\frac{5}{2}} + (b_2 + d\alpha_2)w_{\frac{3}{2}} + d\alpha_3w_{\frac{1}{2}}.$$

Using Remark 2.1 and (3.1), we have the lower bounds

$$b_0 + d\alpha_0 \geq \frac{1}{2} - Cd, \text{ and } b_2 + d\alpha_2 \geq \frac{2}{35}\mu - Cd. \quad (3.10)$$

Now we let (β_1, β_2, τ) be the solution to the following system:

$$\begin{cases} \beta_1 + (b_0 + d\alpha_0)\tau = b_1 + d\alpha_1 \\ \beta_2 + \beta_1\tau + \frac{1}{2}(b_0 + d\alpha_0)\tau^2 = b_2 + d\alpha_2 \\ \beta_2\tau + \frac{1}{2}\beta_1\tau^2 + \frac{1}{6}(b_0 + d\alpha_0)\tau^3 = d\alpha_3 \end{cases}$$

Using (3.10), it is elementary that this system has a solution when d is small. Moreover, we have that

$$|\tau| + |\beta_1 - (b_1 + \alpha_1 d)| + |\beta_2 - (b_2 + \alpha_2 d)| \leq C \left| \frac{\alpha_3 d}{b_2 + d\alpha_2} \right| \leq Cd. \quad (3.11)$$

Using Taylor's Theorem and the integrability of $\frac{d}{d\tau} U_\tau(w_{\frac{1}{2}})$, we have that

$$\|(p + dh_{\frac{7}{2}}) - [(b_0 + \alpha_0 d)u_{\frac{7}{2}} + \beta_1 w_{\frac{5}{2}} + \beta_2 w_{\frac{3}{2}}](U_\tau \cdot)\|_{L^1(\mathbb{S}^2)} \leq Cd^2$$

for C depending on μ . Switching back to the basis $\{u_{\frac{7}{2}}, v_{\frac{5}{2}}, v_{\frac{3}{2}}, v_{\frac{1}{2}}\}$, we have that

$$\|(p + dh_{\frac{7}{2}}) - U_{-\tau}[a'_0 u_{\frac{7}{2}} + a'_1 v_{\frac{5}{2}} + a'_2 v_{\frac{3}{2}}]\|_{L^1(\mathbb{S}^2)} \leq Cd^2,$$

with $|a'_j - a_j| \leq Cd$ by (3.11). By homogeneity, we have that

$$\|(p + dh_{\frac{7}{2}}) - U_{-\tau}[a'_0 u_{\frac{7}{2}} + a'_1 v_{\frac{5}{2}} + a'_2 v_{\frac{3}{2}}]\|_{L^1(B_{2\rho_0})} \leq Cd^2 \rho_0^{\frac{13}{2}}.$$

Combining this with the first estimate in this step, we have that

$$\|u - U_{-\tau}[a'_0 u_{\frac{7}{2}} + a'_1 v_{\frac{5}{2}} + a'_2 v_{\frac{3}{2}}]\|_{L^1(B_{2\rho_0})} \leq Cd\rho_0^{\frac{13}{2}}[\rho_0 + c_0 + o(1)].$$

Since p lies in the interior of $\mathcal{A}$ and $|a'_j - a_j| \leq Cd$, we see that $a'_0 u_{\frac{7}{2}} + a'_1 v_{\frac{5}{2}} + a'_2 v_{\frac{3}{2}}$ solves the thin obstacle problem when d is small, depending on μ_p from (3.2). As a result, we can apply Lemma 3.2 to get that

$$\|u - U_{-\tau}[a'_0 u_{\frac{7}{2}} + a'_1 v_{\frac{5}{2}} + a'_2 v_{\frac{3}{2}}]\|_{L^\infty(B_{\rho_0})} \leq Cd\rho_0^{\frac{7}{2}}[\rho_0 + c_0 + o(1)].$$

Consequently, if we choose ρ_0 and c_0 small, depending only on μ , such that $C(\rho_0 + c_0) < \frac{1}{4}$, then

$$\|u - U_{-\tau}[a'_0 u_{\frac{7}{2}} + a'_1 v_{\frac{5}{2}} + a'_2 v_{\frac{3}{2}}]\|_{L^\infty(B_{\rho_0})} \leq d\rho_0^{\frac{7}{2}}\left(\frac{1}{4} + Co(1)\right) < \frac{1}{2}d\rho_0^{\frac{7}{2}}$$

eventually. This contradicts (3.6). $\square$

4. Dichotomy for p Near $\mathcal{A}_2$

In this section, we focus on profiles near $\mathcal{A}_2$ from (1.10).

To illustrate the ideas, let's take $p = u_{\frac{7}{2}} + a_1 v_{\frac{5}{2}} + a_2 v_{\frac{3}{2}} \in \mathcal{A}_2$ with

$$\mu \leq a_2 \leq 5 \text{ and } a_1^2 = \Gamma(a_2)$$

for a small parameter $\mu > 0$, to be chosen in Section 5; see Remark 5.1. The function Γ was defined in (1.8). Recall also the basis $\{u_{\frac{7}{2}}, v_{\frac{5}{2}}, v_{\frac{3}{2}}, v_{\frac{1}{2}}\}$ from (2.15) for the space $\mathcal{H}_{\frac{7}{2}}$ from (2.12).

We further assume that

$$a_1 \geq 0.$$

The other case is symmetric.

Although p solves the thin obstacle problem, the two constraints $p|_{\{x_3=0\}} \geq 0$ and $\Delta p|_{\{x_3=0\}} \leq 0$ become degenerate as

- (1) when $a_2 \leq \frac{5}{4}$,

$$\Delta p = 0 \text{ along } R_p^+;$$

- (2) when $a_2 \geq \frac{5}{4}$,

$$p = 0 \text{ along } R_p^-,$$

where

$$\begin{aligned} R_p^+ &:= \left\{ t \cdot \left(1, \frac{-5a_1}{14 - \frac{6}{5}a_2}, 0 \right) : t \geq 0 \right\} \text{ and} \\ R_p^- &:= \left\{ t \cdot \left(-1, \frac{a_1}{2 - \frac{2}{5}a_2}, 0 \right) : t \geq 0 \right\}.^3 \end{aligned} \quad (4.1)$$

Let us denote by $A_p^\pm$ the intersections of these two rays with the sphere

$$\{A_p^\pm\} = R_p^\pm \cap \mathbb{S}^2. \quad (4.2)$$

It is crucial that both points are bounded away from $\{r = 0\}$ with

$$\text{dist}(A_p^\pm, \{r = 0\}) \geq c_\mu > 0, \quad (4.3)$$

where c_μ depends only on μ .

Due to the degeneracy of p , the modified $p' = p + dv$ as in (1.12) may fail to solve the thin obstacle problem, and is no longer a suitable profile to approximate our solution (for instance, a result similar to Lemma 3.2 is not necessarily true).

³ This ray R_p^- is understood to be $\{(0, s, 0) : s \geq 0\}$ if $a_2 = 5$.

We tackle this issue by solving the thin obstacle problem in small spherical caps around $A_p^\pm$, and replace p' with this solution. Along the boundary of the caps, this procedure creates an error. With (4.3), we show that this error has a significant projection into $\mathcal{H}_{\frac{7}{2}}$ from (2.12). This allows us to control the error in terms of the decay of the Weiss energy.

In most part of this section, we deal with profiles near the ‘doubly critical’ profile

$$p_{dc} := u_{\frac{7}{2}} + \frac{\sqrt{15}}{2} v_{\frac{5}{2}} + \frac{5}{4} v_{\frac{3}{2}}. \quad (4.4)$$

This is the only profile in $\mathcal{A}_2$ for which both $\Delta p_{dc}(A^+)$ and $p_{dc}(A^-)$ vanish. As a result, for profiles nearby, we need to find replacements in spherical caps near both $A^\pm$.

For other profiles $p \in \mathcal{A}_2$, only one of the two constraints is degenerate. The treatment is more straightforward, and is only sketched near the end of this section.

4.1. The boundary layer problem around p_{dc}

We study homogeneous harmonic functions near p_{dc} . For a small universal constant $\delta > 0$, suppose that

$$p = a_0 u_{\frac{7}{2}} + a_1 v_{\frac{5}{2}} + a_2 v_{\frac{3}{2}} + a_3 v_{\frac{1}{2}}$$

satisfy

$$\frac{1}{2} \leq a_0 \leq 2, \text{ and } \left| \frac{a_1}{a_0} - \frac{\sqrt{15}}{2} \right| + \left| \frac{a_2}{a_0} - \frac{5}{4} \right| + \left| \frac{a_3}{a_0} \right| \leq \delta. \quad (4.5)$$

Recall from (4.2) that $A^+ = (\sqrt{5/8}, -\sqrt{3/8}, 0)$ and $A^- = (-\sqrt{3/8}, \sqrt{5/8}, 0)$ are the points of degeneracy for p_{dc} . For a universal small $\eta > 0$, define two spherical caps

$$\mathcal{C}_\eta^+ := \{x \in \mathbb{S}^2 : |x - A^+| < \eta\}, \text{ and } \mathcal{C}_\eta^- := \{x \in \mathbb{S}^2 : |x - A^-| < \eta\}.$$

Thanks to (4.3), both are bounded away from $\{r = 0\}$ for small η . The same notations are used to denote the cones generated by the two caps. See Fig. 3.

In general, for $\ell > 0$ we define

$$\mathcal{C}_\ell^\pm := \{x \in \mathbb{S}^2 : |x - A^\pm| < \ell\}. \quad (4.6)$$

Inside the caps $\mathcal{C}_\eta^\pm$, we solve the thin obstacle problem for the operator $(\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})$ from (2.14) with p as boundary data:

$$\begin{cases} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}}) v_p^\pm \leq 0 & \text{in } \mathcal{C}_\eta^\pm, \\ v_p^\pm \geq 0 & \text{on } \mathcal{C}_\eta^\pm \cap \{x_3 = 0\}, \\ (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}}) v_p^\pm = 0 & \text{in } \{x_3 \neq 0\} \cup \{v_p^\pm > 0\}, \\ v_p^\pm = p & \text{along } \partial \mathcal{C}_\eta^\pm. \end{cases} \quad (4.7)$$

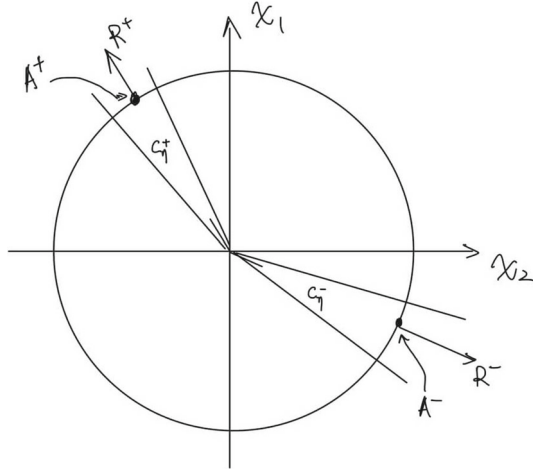


Fig. 3. Boundary layers for p_{dc} in (x_1, x_2) -plane

Note that when η is universally small, the maximum principle holds for $(\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})$ in $C_\eta^\pm$, and problem (4.7) is well-posed.

The maximum principle also implies

$$v_p^\pm \geq p \text{ in } C_\eta^\pm. \quad (4.8)$$

With the symmetry of p , the solutions $v_p^\pm$ are even with respect to $\{x_3 = 0\}$.

Definition 4.1. Given p satisfying (4.5), our *replacement* for p , to be denoted by $\tilde{p}$, is the following function

$$\tilde{p} = \begin{cases} p & \text{outside } C_\eta^\pm, \\ v_p^\pm & \text{in } C_\eta^\pm. \end{cases}$$

Equivalently, the replacement $\tilde{p}$ is the unique minimizer of

$$v \mapsto \int_{\mathbb{S}^2} |\nabla_{\mathbb{S}^2} v|^2 - \lambda_{\frac{7}{2}} v^2$$

over

$$\{v : v = p \text{ outside } C_\eta^\pm, \text{ and } v \geq 0 \text{ on } \{x_3 = 0\}\}.$$

Here $\nabla_{\mathbb{S}^2}$ denotes the tangential gradient on $\mathbb{S}^2$.

We also denote the $\frac{7}{2}$ -homogeneous extension of $\tilde{p}$ by the same notation.

Recall the notations for slit domains from (2.10). We have, by definition,

$$\begin{cases} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})\tilde{p} = f_p^\pm dH^1|_{\partial C_\eta^\pm} + g_p^\pm dH^1|_{C_\eta^\pm \cap \{x_3=0\}} & \text{in } \mathbb{S}^2 \setminus (\tilde{\mathbb{S}^2} \setminus C_\eta^+), \\ \tilde{p} = 0 & \text{on } \tilde{\mathbb{S}^2} \setminus C_\eta^+. \end{cases} \quad (4.9)$$

Here $f_p^\pm$ arise from the gluing of $v_p^\pm$ and p along $\partial\mathcal{C}_\eta^\pm$, and $g_p^\pm$ are a consequence of the thin obstacle problem (4.7). In particular, following our convention for one-sided derivatives (2.3), we have

$$f_p^\pm = (\tilde{p} - p)_\nu|_{\partial\mathcal{C}_\eta^\pm} \quad (4.10)$$

and

$$g_p^\pm \leq 0 \text{ is supported in } \{\tilde{p} = 0\} \cap \{x_3 = 0\}.$$

Remark 4.1. The replacement $\tilde{p}$ does not necessarily satisfy the two constraints $\tilde{p}|_{\{x_3=0\}} \geq 0$ and $\Delta\tilde{p}|_{\{x_3=0\}} \leq 0$ outside $\mathcal{C}_\eta^\pm$. This is due to the possible presence of $v_{\frac{1}{2}}$ in p , which becomes dominant near $\{r = 0\}$.

On the other hand, suppose that $p = a_0 u_{\frac{7}{2}} + a_1 v_{\frac{5}{2}} + a_2 v_{\frac{3}{2}} + a_3 v_{\frac{1}{2}}$ satisfies (4.5) with $a_3 = 0$, then p satisfies $p|_{\{x_3=0\}} \geq 0$ and $\Delta p|_{\{x_3=0\}} \leq 0$ outside $\mathcal{C}_\eta^\pm$, and the same holds for $\tilde{p}$.

One essential ingredient of this section is that $f_p^\pm$ have significant projections into $\mathcal{H}_{\frac{7}{2}}$ from (2.12). To measure this, we introduce some auxiliary functions.

Let $\varphi_p : \mathbb{S}^2 \rightarrow \mathbb{R}$ denote the projection of $f_p^\pm$ into $\mathcal{H}_{\frac{7}{2}}$ from (2.12), namely,

$$\varphi_p := c_{\frac{7}{2}} u_{\frac{7}{2}} + c_{\frac{5}{2}} v_{\frac{5}{2}} + c_{\frac{3}{2}} v_{\frac{3}{2}} + c_{\frac{1}{2}} v_{\frac{1}{2}}, \quad (4.11)$$

where

$$c_{\frac{7}{2}} = \frac{1}{\|u_{\frac{7}{2}}\|_{L^2(\mathbb{S}^2)}} \cdot \int_{\mathbb{S}^2} u_{\frac{7}{2}} (f_p^+ dH^1|_{\partial\mathcal{C}_\eta^+} + f_p^- dH^1|_{\partial\mathcal{C}_\eta^-})$$

and

$$c_{m+\frac{1}{2}} = \frac{1}{\|v_{m+\frac{1}{2}}\|_{L^2(\mathbb{S}^2)}} \cdot \int_{\mathbb{S}^2} v_{m+\frac{1}{2}} (f_p^+ dH^1|_{\partial\mathcal{C}_\eta^+} + f_p^- dH^1|_{\partial\mathcal{C}_\eta^-})$$

for $m = 0, 1, 2$. It follows that

$$(f_p^+ dH^1|_{\partial\mathcal{C}_\eta^+} + f_p^- dH^1|_{\partial\mathcal{C}_\eta^-} - \varphi_p) \perp \mathcal{H}_{\frac{7}{2}}.$$

By Fredholm alternative, there is a unique function $H_p : \mathbb{S}^2 \rightarrow \mathbb{R}$ satisfying

$$\begin{cases} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})H_p = f_p^+ dH^1|_{\partial\mathcal{C}_\eta^+} + f_p^- dH^1|_{\partial\mathcal{C}_\eta^-} - \varphi_p & \text{on } \widehat{\mathbb{S}^2}, \\ H_p = 0 & \text{on } \widetilde{\mathbb{S}^2}. \end{cases}$$

The natural extensions of $f_p^\pm$, $g_p^\pm$, φ_p and H_p into $\mathbb{R}^3$ are denoted by the same symbols.

With this convention, we define

$$\Phi_p := H_p \left(\frac{x}{|x|} \right) |x|^{\frac{7}{2}} + \frac{1}{8} \varphi_p \left(\frac{x}{|x|} \right) |x|^{\frac{7}{2}} \log(|x|), \quad (4.12)$$

which satisfies

$$\begin{cases} \Delta \Phi_p = f_p^\pm dH^2|_{\partial \mathcal{C}_\eta^\pm} & \text{in } \widetilde{\mathbb{R}^3}, \\ \Phi_p = 0 & \text{on } \widetilde{\mathbb{R}^3}, \end{cases} \quad (4.13)$$

where we have used the notations for slit domains from (2.10).

Finally, let us denote that

$$\kappa_p := \|\Phi_p\|_{L^\infty(B_1)}, \quad (4.14)$$

which measures the size of the error coming from the gluing procedure along $\partial \mathcal{C}_\eta^\pm$.

For all the functions and constants defined so far, the subscript p is often omitted when there is no ambiguity.

We collect some properties of the replacement $\tilde{p}$ from Definition 4.1.

We have the following localization of the contact set of $\tilde{p}$:

Lemma 4.1. *For p satisfying (4.5), we have that*

$$\tilde{p} > 0 \text{ in } (\mathcal{C}_\eta^- \setminus \mathcal{C}_{C\delta^{\frac{1}{2}}}^-)' , \text{ and } \tilde{p} = 0 \text{ in } (\mathcal{C}_\eta^+ \setminus \mathcal{C}_{C\delta^{\frac{1}{2}}}^+)' ,$$

where C is a universal constant.

Recall our notations from (2.1) and (4.6).

Proof. With (4.8), we have $\tilde{p} \geq p \geq p_{dc} - C\delta$ in $\mathcal{C}_\eta^-$. The first statement follows from direct computation.

Note that p_{dc} and $\tilde{p}$ both solve (4.7) in $\mathcal{C}_\eta^+$ with $\tilde{p} = p \leq p_{dc} + C\delta$ along $\partial \mathcal{C}_\eta^+$, it follows from the maximum principle that $p \leq p_{dc} + C\delta$ in $\mathcal{C}_\eta^+$. The second conclusion follows from a barrier argument similar to the proof for Lemma 3.1. $\square$

The next lemma controls the change in $\tilde{p}$ when p is modified.

Lemma 4.2. *Suppose that p satisfies (4.5), and take $q = \alpha_0 u_{\frac{7}{2}} + \alpha_1 v_{\frac{5}{2}} + \alpha_2 v_{\frac{3}{2}} + \alpha_3 v_{\frac{1}{2}}$ with $|\alpha_j| \leq 1$ for $j = 0, 1, 2, 3$.*

Then we can find a universal modulus of continuity, $\omega(\cdot)$, such that

$$\begin{aligned} \|p \widetilde{+} dq - (\tilde{p} + dq)\|_{L^\infty(\mathcal{C}_\eta^+)} &\leq \omega(\delta + d) \cdot d, \\ \|p \widetilde{+} dq - (\tilde{p} + dq)\|_{L^\infty(\mathcal{C}_\eta^- \setminus \mathcal{C}_{\eta/2}^-)} &\leq \omega(\delta + d) \cdot d, \end{aligned}$$

and

$$\|p \widetilde{+} dq - (\tilde{p} + dq)\|_{L^1(\mathbb{S}^2)} \leq \omega(d + \delta) \cdot d.$$

Proof. The distinction between the estimates in $\mathcal{C}_\eta^+$ and $\mathcal{C}_\eta^-$ is due to the fact that $q = 0$ in $\mathcal{C}_\eta^+ \cap \{x_3 = 0\}$, while $q \neq 0$ in $\mathcal{C}_\eta^- \cap \{x_3 = 0\}$.

Step 1: The estimate in $\mathcal{C}_\eta^+$.

Let $\Omega := C_\eta^+ \cap \{x_3 > 0\}$. We build a barrier by solving the following system

$$\begin{cases} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{1}{2}})w = 0 & \text{in } \Omega, \\ w = 1 & \text{in } (C_{C\sqrt{\delta+d}}^+)', \\ w = 0 & \text{in } \partial\Omega \setminus (C_{C\sqrt{\delta+d}}^+)', \end{cases}$$

where C is the universal constant from Lemma 4.1. We extend w to $C_\eta^+ \cap \{x_3 < 0\}$ by evenly reflecting it with respect to $\{x_3 = 0\}$.

It follows from the maximum principle and the second statement in Lemma 4.1 that

$$p \widetilde{+} dq - (\tilde{p} + dq) \leq Cd \cdot w \text{ in } C_\eta^+.$$

For $\ell > C(\delta + d)^{\frac{1}{2}}$ to be chosen, it follows that

$$p \widetilde{+} dq - \tilde{p} \leq Cd \cdot (\ell + \sup_{\partial C_\ell^+} w) \text{ along } \partial C_\ell^+,$$

where we used the Lipschitz regularity of q and $q = 0$ along $C_\eta^+ \cap \{x_3 = 0\}$. From here the maximum principle implies $p \widetilde{+} dq - \tilde{p} \leq Cd \cdot (\ell + \sup_{\partial C_\ell^+} w)$ in C_ℓ^+ , and consequently,

$$p \widetilde{+} dq - (\tilde{p} + dq) \leq Cd \cdot (\ell + \sup_{\partial C_\ell^+} w) \text{ in } C_\ell^+.$$

Using Lemma 4.1, for small $d + \delta$, it follows from the maximum principle in $C_\eta^+ \setminus C_\ell^+$ that

$$p \widetilde{+} dq - (\tilde{p} + dq) \leq Cd \cdot (\ell + \sup_{\partial C_\ell^+} w) \text{ in } C_\eta^+.$$

A symmetric argument gives that

$$p \widetilde{+} dq - (\tilde{p} + dq) \geq -Cd \cdot (\ell + \sup_{\partial C_\ell^+} w) \text{ in } C_\eta^+.$$

By choosing ℓ small, and noting that $\sup_{\partial C_\ell^+} w \leq \omega_\ell(d + \delta)$ for a modulus of continuity depending only on ℓ , we get the desired estimate in C_η^+ .

Step 2: The estimate in $C_\eta^- \setminus C_{\eta/2}^-$.

The main difference with the previous case is that q no longer vanishes along $\{x_3 = 0\}$ in the cap C_η^- .

We build a barrier by solving

$$\begin{cases} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{1}{2}})w = 0 & \text{in } C_\eta^- \setminus (C_{C\sqrt{\delta+d}}^-)', \\ w = 1 & \text{in } (C_{C\sqrt{\delta+d}}^-)', \\ w = 0 & \text{in } \partial C_\eta^-. \end{cases}$$

With the first statement in Lemma 4.1, it follows from the maximum principle that

$$p \widetilde{+} dq - (\tilde{p} + dq) \leq Cd \cdot w \text{ in } \mathcal{C}_\eta^-.$$

In particular, we have that

$$p \widetilde{+} dq - (\tilde{p} + dq) \leq Cd \cdot \sup_{\partial \mathcal{C}_{\eta/2}^-} w \text{ in } \mathcal{C}_\eta^- \setminus \mathcal{C}_{\eta/2}^-.$$

A symmetric argument gives $p \widetilde{+} dq - (\tilde{p} + dq) \geq -Cd \cdot \sup_{\partial \mathcal{C}_{\eta/2}^-} w \text{ in } \mathcal{C}_\eta^- \setminus \mathcal{C}_{\eta/2}^-.$

Note that $\sup_{\partial \mathcal{C}_{\eta/2}^-} w \rightarrow 0$ as $d + \delta \rightarrow 0$, the previous two estimates gives the desired results in $\mathcal{C}_\eta^- \setminus \mathcal{C}_{\eta/2}^-.$

Step 3: The $L^1(\mathbb{S}^2)$ estimate.

For $\ell > 0$, with the same barrier from Step 2, we have for small $d + \delta$

$$|p \widetilde{+} dq - (\tilde{p} + dq)| \leq Cd \cdot \sup_{\partial \mathcal{C}_\ell^-} w \text{ in } \mathcal{C}_\eta^- \setminus \mathcal{C}_\ell^-.$$

Thus

$$\|p \widetilde{+} dq - (\tilde{p} + dq)\|_{L^1(\mathcal{C}_\eta^-)} \leq Cd \cdot \sup_{\partial \mathcal{C}_\ell^-} w + Cd\ell^2.$$

By choosing ℓ small, and noting $\sup_{\partial \mathcal{C}_\ell^-} w \rightarrow 0$ as $d + \delta \rightarrow 0$, we have that

$$\|p \widetilde{+} dq - (\tilde{p} + dq)\|_{L^1(\mathcal{C}_\eta^-)} \leq \omega(d + \delta) \cdot d.$$

A similar estimate in $\mathcal{C}_\eta^+$ follows directly from the conclusion in Step 1. Since $p \widetilde{+} dq - (\tilde{p} + dq) = 0$ outside $\mathcal{C}_\eta^\pm$, the $L^1(\mathbb{S}^2)$ estimate follows. $\square$

As a consequence, we can control the change of κ_p , defined in (4.14), when p is modified.

Corollary 4.1. *Under the same assumption as in Lemma 4.2, we have that*

$$\kappa_{p+dq} \leq \kappa_p + \omega(\delta + d) \cdot d.$$

Proof. Define $w_p = \tilde{p} - p$ and $w_{p+dq} = p \widetilde{+} dq - (p + dq)$, then $w_{p+dq} - w_p$ vanishes along $\partial \mathcal{C}_\eta^-$, and satisfies $(\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})(w_{p+dq} - w_p) = 0$ in $\mathcal{C}_\eta^- \setminus \mathcal{C}_{\eta/2}^-.$

Boundary regularity estimate gives that

$$(w_{p+dq} - w_p)_\nu \leq C \|w_{p+dq} - w_p\|_{L^\infty(\mathcal{C}_\eta^- \setminus \mathcal{C}_{\eta/2}^-)} \text{ along } \partial \mathcal{C}_\eta^-.$$

Similarly, with $w_{p+dq} - w_p$ vanishing along $\partial \mathcal{C}_\eta^+ \cup (\mathcal{C}_\eta^+ \setminus \mathcal{C}_{\eta/2}^+)',$ and $(\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})(w_{p+dq} - w_p) = 0$ in $(\mathcal{C}_\eta^+ \setminus \mathcal{C}_{\eta/2}^+) \cap \{x_3 > 0\},$ we have that

$$(w_{p+dq} - w_p)_\nu \leq C \|w_{p+dq} - w_p\|_{L^\infty(\mathcal{C}_\eta^+ \setminus \mathcal{C}_{\eta/2}^+)} \text{ along } \partial \mathcal{C}_\eta^+.$$

The conclusion follows from (4.10) and Lemma 4.2. $\square$

The following lemma is the key estimate of this section:

Lemma 4.3. *Suppose that p satisfies (4.5) with $\delta > 0$ universally small. Then*

$$\|\varphi\|_{L^\infty(\mathbb{S}^2)} \geq c\kappa$$

for a universal $c > 0$.

Recall the definitions of φ and κ from (4.11) and (4.14), respectively.

Proof. Define $\Omega = (\mathcal{C}_\eta^+ \setminus \mathcal{C}_{\eta/4}^+) \cap \{x_3 > 0\}$, and let w denote the solution to

$$\begin{cases} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})w = 0 & \text{in } \Omega, \\ w = 1 & \text{on } \partial\mathcal{C}_{\eta/4}^+ \cap \{x_3 > 0\}, \\ w = 0 & \text{on } \partial\Omega \setminus \partial\mathcal{C}_{\eta/4}^+. \end{cases}$$

For $\delta > 0$ small, Lemma 4.1 implies that $(\tilde{p} - p)$ solves the same equation as w in Ω , and both vanish long $\partial\Omega \setminus \partial\mathcal{C}_{\eta/4}^+$.

With (4.8), we can apply boundary Harnack principle to get that

$$c \cdot \frac{\tilde{p} - p}{w} \left(A^+ + \frac{\eta}{2} e_3 \right) \leq \frac{\tilde{p} - p}{w}(x) \leq C \cdot \frac{\tilde{p} - p}{w} \left(A^+ + \frac{\eta}{2} e_3 \right)$$

for any $x \in (\mathcal{C}_\eta^+ \setminus \mathcal{C}_{\eta/2}^+) \cap \{x_3 > 0\}$. Here we denoted by $(A^+ + \frac{\eta}{2} e_3)$ the point on $\mathbb{S}^2$ we get by moving from A^+ along the x_3 -direction by distance $\eta/2$.

With (4.10), this implies that

$$c \cdot \frac{\tilde{p} - p}{w} \left(A^+ + \frac{\eta}{2} e_3 \right) \leq \frac{f^+}{w_v} \leq C \cdot \frac{\tilde{p} - p}{w} \left(A^+ + \frac{\eta}{2} e_3 \right) \text{ along } \partial\mathcal{C}_\eta^+.$$

With a similar argument, we have that

$$c \cdot \frac{\tilde{p} - p}{v} \left(A^- + \frac{\eta}{2} e_3 \right) \leq \frac{f^-}{v_v} \leq C \cdot \frac{\tilde{p} - p}{v} \left(A^- + \frac{\eta}{2} e_3 \right) \text{ along } \partial\mathcal{C}_\eta^-,$$

where v denotes the solution to

$$\begin{cases} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})v = 0 & \text{in } \mathcal{C}_\eta^- \setminus \mathcal{C}_{\eta/4}^-, \\ v = 1 & \text{on } \partial\mathcal{C}_{\eta/4}^-, \\ v = 0 & \text{on } \partial\mathcal{C}_\eta^-. \end{cases}$$

Now for a constant $\beta > 0$ to be chosen, let us define that

$$q = u_{\frac{7}{2}} + \beta v_{\frac{5}{2}} \in \mathcal{H}_{\frac{7}{2}}. \quad (4.15)$$

A direct computation gives that

$$\frac{\partial}{\partial x_3} q(A^+) = a \left(\beta - \frac{7}{5} \sqrt{\frac{3}{5}} \right), \text{ and } q(A^-) = b \left(\sqrt{\frac{5}{3}} - \beta \right),$$

where a and b are positive constants. With $\frac{7}{5} \sqrt{\frac{3}{5}} < \sqrt{\frac{5}{3}}$, we find β such that

$$\Delta q \geq c > 0 \text{ along } \mathcal{C}_\eta^+ \cap \{x_3 = 0\}, \quad q \geq c|x_3| \text{ in } \mathcal{C}_\eta^+, \text{ and } q \geq c \text{ in } \mathcal{C}_\eta^- \quad (4.16)$$

for a universal $c > 0$ if η is small.

Consequently,

$$\begin{aligned} \int_{\partial C_\eta^+} f^+ q dH^1 &\geq c \int_{\partial C_\eta^+} w_\nu |x_3| dH^1 \cdot \frac{\tilde{p} - p}{w} \left(A^+ + \frac{\eta}{2} e_3 \right) \\ &\geq c \frac{\tilde{p} - p}{w} \left(A^+ + \frac{\eta}{2} e_3 \right) \\ &\geq c \|f^+\|_{L^\infty(\partial C_\eta^+)}. \end{aligned}$$

Similarly,

$$\int_{\partial C_\eta^-} f^- q dH^1 \geq c \frac{\tilde{p} - p}{w} \left(A^- + \frac{\eta}{2} e_3 \right) \geq c \|f^-\|_{L^\infty(\partial C_\eta^-)}. \quad (4.17)$$

Note that $q \in \mathcal{H}_{\frac{7}{2}}$, we have $\|q\|_{L^\infty(\mathbb{S}^2)} \geq c(\|f^+\|_{L^\infty(\partial C_\eta^+)} + \|f^-\|_{L^\infty(\partial C_\eta^-)})$, which gives the desired estimate. $\square$

We also have

Lemma 4.4. *Under the same assumptions as in Lemma 4.3, we have that*

$$|\tilde{p} - p| \leq C\kappa \text{ in } C_\eta^-$$

for a universal constant C .

Proof. In this proof, define $w = \tilde{p} - p$. By (4.8), it suffices to get an upper bound for w in C_η^- .

Suppose $\varepsilon := \max_{C_\eta^-} w > 0$, and that x_0 is a point where this maximum is achieved, then $x_0 \in \{(\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})w < 0\}$. As a result, $x_0 \in \{\tilde{p} = 0\}'$. This implies that $p(x_0) = -\varepsilon$, and $\frac{\partial}{\partial x_1} w(x_0) = \frac{\partial}{\partial x_1} \tilde{p}(x_0) = 0$. Thus $\frac{\partial}{\partial x_1} p(x_0) = 0$ and we have that

$$p \leq -\frac{7}{8}\varepsilon \text{ in } (B_{c\sqrt{\varepsilon}}(x_0))'.$$

Since $\tilde{p} \geq 0$ along $\{x_3 = 0\}$, this implies $w \geq \frac{7}{8}\varepsilon$ in $(B_{c\sqrt{\varepsilon}}(x_0))'$. With the super-harmonicity of w in C_η^- , we have that

$$w \geq \frac{1}{2}\varepsilon \text{ in } B_{c\sqrt{\varepsilon}}(x_0).$$

Comparing with the Green's function for C_η^- with a pole at x_0 , we see that

$$(\tilde{p} - p)(A^- + \frac{\eta}{2} e_3) \geq c\varepsilon/|\log \varepsilon|.$$

With (4.17) from the proof of Lemma 4.3, this implies $\kappa \geq c\varepsilon$, the desired estimate. $\square$

We also have the following control on the Weiss energy of the replacement $\tilde{p}$:

Lemma 4.5. *Suppose that p satisfies (4.5), then*

$$W_{\frac{7}{2}}(\tilde{p}; 1) \leq C\kappa^2$$

for a universal constant C .

Recall the definition of the Weiss energy from (2.6).

Proof. We first control the total mass of g^- from (4.9).

Take the auxiliary function q from (4.15), we have that

$$\begin{aligned} \int_{\mathbb{S}^2} \tilde{p} \cdot (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})q &= \int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})\tilde{p} \cdot q \\ &= \int_{\partial C_{\eta}^{\pm}} f^{\pm} q + \int_{C_{\eta}^{\pm} \cap \{x_3=0\}} g^{\pm} q \\ &= \int_{\partial C_{\eta}^{\pm}} f^{\pm} q + \int_{C_{\eta}^{-} \cap \{x_3=0\}} g^{-} q. \end{aligned}$$

For the last equality, we used that $q = 0$ along $(C_{\eta}^+)'$, which contains $\text{spt}(g^+)$.

Now we note that $(\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})q$ is supported in $\tilde{\mathbb{S}}^2$. On this set, $\tilde{p} \geq 0$ is supported in $(C_{\eta}^+)'$. Recall from (4.16), we have $(\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})q \geq 0$ in $(C_{\eta}^+)'$. Thus $\tilde{p} \cdot (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})q \geq 0$ and we conclude that

$$-\int_{C_{\eta}^{-} \cap \{x_3=0\}} g^{-} q \leq \int_{\partial C_{\eta}^+} f^{+} q + \int_{\partial C_{\eta}^-} f^{-} q \leq C\kappa.$$

With $q \geq c$ in C_{η}^- and $g^{-} \leq 0$, we conclude that

$$\int_{\mathbb{S}^2} |g^{-}| \leq C\kappa. \quad (4.18)$$

Using (3.4) and the homogeneity of $\tilde{p}$, we have that

$$\begin{aligned} W_{\frac{7}{2}}(\tilde{p}; 1) &= C[\int_{\mathbb{S}^2} (|\nabla_{\mathbb{S}^2} \tilde{p}|^2 - \lambda_{\frac{7}{2}} \tilde{p}^2) - \int_{\mathbb{S}^2} (|\nabla_{\mathbb{S}^2} p|^2 - \lambda_{\frac{7}{2}} p^2)] \\ &= C[\int_{C_{\eta}^+} (|\nabla_{\mathbb{S}^2} \tilde{p}|^2 - \lambda_{\frac{7}{2}} \tilde{p}^2) - \int_{C_{\eta}^+} (|\nabla_{\mathbb{S}^2} p|^2 - \lambda_{\frac{7}{2}} p^2)] \\ &\quad + C[\int_{C_{\eta}^-} (|\nabla_{\mathbb{S}^2} \tilde{p}|^2 - \lambda_{\frac{7}{2}} \tilde{p}^2) - \int_{C_{\eta}^-} (|\nabla_{\mathbb{S}^2} p|^2 - \lambda_{\frac{7}{2}} p^2)], \end{aligned}$$

since $\text{spt}(\tilde{p} - p) \subset C_{\eta}^{\pm}$.

With $p = 0$ along $(C_{\eta}^+)'$, the profile p is admissible in the minimization problem defining $\tilde{p}$ in C_{η}^+ . See Definition 4.1. Using the harmonicity of p in C_{η}^- , we continue

with the previous estimate to get that

$$\begin{aligned}
 W_{\frac{7}{2}}(\tilde{p}; 1) &\leq C \left[\int_{C_{\eta}^-} (|\nabla_{\mathbb{S}^2} \tilde{p}|^2 - \lambda_{\frac{7}{2}} \tilde{p}^2) - \int_{C_{\eta}^-} (|\nabla_{\mathbb{S}^2} p|^2 - \lambda_{\frac{7}{2}} p^2) \right] \\
 &= -C \int_{C_{\eta}^-} (\tilde{p} - p)(\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}}) \tilde{p} \\
 &= -C \int_{C_{\eta}^-} (\tilde{p} - p) g^-.
 \end{aligned}$$

Combining this with (4.18) and Lemma 4.4, we have the desired estimate. $\square$

4.2. Well-approximated solutions near p_{dc}

For p satisfying (4.5), we define the space of well-approximated solutions in a similar manner as in Subsection 3.1. The main difference is that now the solutions are approximated by the replacement $\tilde{p}$ as in Definition 4.1.

Definition 4.2. Suppose that the coefficients of p satisfy (4.5).

For $d, \rho \in (0, 1]$, we say that u is a *solution d -approximated by p at scale ρ* if u solves the thin obstacle problem (1.1) in B_{ρ} , and

$$|u - \tilde{p}| \leq d\rho^{\frac{7}{2}} \text{ in } B_{\rho}.$$

In this case, we write that

$$u \in \mathcal{S}(p, d, \rho).$$

Similarly to Lemma 3.1, we can localize the contact set of a well-approximated solution if we assume $a_3 = 0$ in the expansion of p ; see Remark 4.1.

Lemma 4.6. Suppose that $u \in \mathcal{S}(p, d, 1)$ with d small, and that p satisfies (4.5) with $a_3 = 0$.

We have that

$$\begin{aligned}
 \Delta u &= 0 \text{ in } \widehat{B}_1 \cap \{r > C(d + \delta)^{\frac{2}{7}}\} \\
 &\cap \left\{ (x_1, x_2, 0) : \left| x_2 + \sqrt{\frac{5}{3}}x_1 \right| > C(d + \delta)^{\frac{1}{2}}, x_1 \leq 0 \right\},
 \end{aligned}$$

and

$$u = 0 \text{ in } \widetilde{B}_{\frac{7}{8}} \cap \{r > C(d + \delta)^{\frac{2}{15}}\} \cap \{(x_1, x_2, 0) : |x_2 + \sqrt{\frac{3}{5}}x_1| > C(d + \delta)^{\frac{1}{2}}, x_1 \geq 0\}.$$

Recall that $\delta > 0$ is the small parameter from (4.5), and that $\{(x_1, x_2, 0) : |x_2 + \sqrt{\frac{3}{5}}x_1| = 0, x_1 \geq 0\}$ and $\{(x_1, x_2, 0) : |x_2 + \sqrt{\frac{5}{3}}x_1| = 0, x_1 \leq 0\}$ are the two rays of degeneracy $R^{\pm}$ from (4.1) for p_{dc} .

Proof. Under our assumptions, we have, inside B_1 ,

$$u \geq \tilde{p} - d \geq p - d \geq Cp_{dc} - C\delta - d, \text{ and } u \leq \tilde{p} + d \leq Cp_{dc} + C\delta + d.$$

The conclusion follows from the same argument as in the proof for Lemma 3.1. $\square$

The following is similar to Lemma 3.2 and Lemma 3.3; it is the main reason why it is preferable to work with $\tilde{p}$ instead of directly with p :

Lemma 4.7. *Suppose that p satisfies (4.5) with $a_3 = 0$. Let u be a solution to (1.1) in B_1 , then*

$$\|u - \tilde{p}\|_{L^\infty(B_{1/2})} + \|u - \tilde{p}\|_{H^1(B_{1/2})} \leq C(\|u - \tilde{p}\|_{L^1(B_1)} + \kappa)$$

and

$$W_{\frac{7}{2}}(u; \frac{1}{2}) \leq C(\|u - \tilde{p}\|_{L^1(B_1)}^2 + \kappa^2).$$

Proof. With Remark 4.1 and $a_3 = 0$, $\tilde{p}$ satisfies the constraints $\tilde{p}|_{\{x_3=0\}} \geq 0$ and $\Delta \tilde{p}|_{\{x_3=0\}} \leq 0$ outside $\mathcal{C}_\eta^\pm$. The same constraints are satisfied inside $\mathcal{C}_\eta^\pm$ by Definition 4.1.

As a result, inside $\{u > \tilde{p}\}$, we have that

$$\Delta(u - \tilde{p}) \geq -f^\pm d H^2|_{\partial \mathcal{C}_\eta^\pm} \geq -C\kappa|x|^{\frac{5}{2}} d H^2|_{\partial \mathcal{C}_\eta^\pm}.$$

This implies

$$u - \tilde{p} \leq (\|u - \tilde{p}\|_{L^1(B_1)} + \kappa) \text{ in } B_{1/2}.$$

A symmetric argument gives the corresponding lower bound, and we have the control on $\|u - \tilde{p}\|_{L^\infty(B_{1/2})}$.

The other estimates follow from the same argument as for Lemma 3.2 and Lemma 3.3, together with Lemma 4.5. $\square$

4.3. The dichotomy near p_{dc}

With these preparations, we state the dichotomy for profiles near p_{dc} from (4.4).

Lemma 4.8. *Suppose that*

$$u \in \mathcal{S}(p, d, 1)$$

for some $p = a_0 u_{\frac{7}{2}} + a_1 v_{\frac{5}{2}} + a_2 v_{\frac{3}{2}}$ satisfying that

$$a_0 \in [\frac{1}{2}, 2], \text{ and } |a_1/a_0 - \frac{\sqrt{15}}{2}| + |a_2/a_0 - \frac{5}{4}| \leq \delta.$$

There is a universal small constant $\tilde{\delta} > 0$, such that if $d < \tilde{\delta}$ and $\delta < \tilde{\delta}$, then we have the following dichotomy:

(1) either

$$W_{\frac{7}{2}}(u; 1) - W_{\frac{7}{2}}(u; \rho_0) \geq c_0^2 d^2$$

and

$$u \in \mathcal{S}(p, Cd, \rho_0);$$

(2) or

$$u \in \mathcal{S}\left(p', \frac{1}{2}d, \rho_0\right)$$

for some

$$p' = U_\tau[a'_0 u_{\frac{7}{2}} + a'_1 v_{\frac{5}{2}} + a'_2 v_{\frac{3}{2}}]$$

with

$$\kappa_{p'} < d, \text{ and } |\tau| + \sum |a'_j - a_j| \leq Cd.$$

The constants c_0 , ρ_0 and C are universal.

Recall the rotation operator U from (2.4), and the basis $\{u_{\frac{7}{2}}, v_{\frac{5}{2}}, v_{\frac{3}{2}}, v_{\frac{1}{2}}\}$ from (2.15).

Remark 4.2. For $p' = U_\tau[a'_0 u_{\frac{7}{2}} + a'_1 v_{\frac{5}{2}} + a'_2 v_{\frac{3}{2}}]$, all the constructions in Subsection 4.1, leading to $\tilde{p}'$, are performed with respect to the rotated coordinate system.

Proof. We apply a similar strategy as in the proof for Lemma 4.8.

For c_0 and ρ_0 to be chosen, suppose that the lemma is false, then we find a sequence $(u_n, p_n, d_n, \delta_n)$ satisfying the assumptions as in the lemma with $d_n, \delta_n \rightarrow 0$, but both alternatives fail, namely,

$$W_{\frac{7}{2}}(u_n; 1) - W_{\frac{7}{2}}(u_n; \rho_0) \leq c_0^2 d_n^2 \text{ for all } n, \quad (4.19)$$

and

$$u_n \notin \mathcal{S}\left(p', \frac{1}{2}d_n, \rho_0\right) \quad (4.20)$$

for any p' satisfying the properties as in alternative (2) from the lemma.

Step 1: Compactness.

Define

$$\hat{u}_n := \frac{1}{d_n + \kappa_{p_n}}(u_n - \tilde{p}_n + \Phi_{p_n}),$$

where Φ and κ are defined in (4.12) and (4.14) respectively. Then $|\hat{u}_n| \leq 2$ in B_1 .

With Lemma 4.1, Lemma 4.6, (4.9) and (4.13), we have, up to a subsequence,

$$\|\hat{u}_n - \hat{u}_\infty\|_{L^2(B_{7/8})} \rightarrow 0.$$

The limit $\hat{u}_\infty$ is a harmonic function in the slit domain $\widehat{B}_1$ according to (2.11).

For $m = 0, 1, 2, 3$, Theorem 2.1 allows us to find $h_{m+\frac{1}{2}}$, an $(m + \frac{1}{2})$ -homogeneous harmonic function that is universally bounded in B_1 and satisfies

$$\begin{aligned} & \|[\hat{u}_n - (h_{\frac{1}{2}} + h_{\frac{3}{2}} + h_{\frac{5}{2}} + h_{\frac{7}{2}})]_{(\frac{1}{2})} - [\hat{u}_n - (h_{\frac{1}{2}} + h_{\frac{3}{2}} + h_{\frac{5}{2}} + h_{\frac{7}{2}})]\|_{L^2(\partial B_\rho)} \\ & \leq C\rho^{\frac{11}{2}} + o(1) \end{aligned}$$

for some $\rho \in [\rho_0, 4\rho_0]$.

Recall that $f_{(\frac{1}{2})}$ denotes the rescaling of f as in (2.5).

We omit the subscripts in $u_n, \hat{u}_n, p_n, d_n$ and δ_n in the remaining of the proof.

Step 2: Almost homogeneity.

Using the homogeneity of the functions as well as the definition of Φ from (4.12), the last estimate from the previous step gives that

$$\|(7h_{\frac{1}{2}} + 3h_{\frac{3}{2}} + h_{\frac{5}{2}}) + \frac{\log 2}{8(d + \kappa)}\varphi\left(\frac{x}{|x|}\right)|x|^{\frac{7}{2}}\|_{L^2(\partial B_\rho)} \leq C\rho^{\frac{9}{2}}(c_0 + \rho) + o(1). \quad (4.21)$$

For this, we used (4.19) in the same way as in Step 2 from the proof of Lemma 3.4 to control $(u_{(\frac{1}{2})} - u)$.

By definition, we have $\varphi \in \mathcal{H}_{\frac{7}{2}}$ from (2.12). With (4.21), we apply Proposition 2.2 to get that

$$\|h_{\frac{1}{2}}\|_{L^\infty(B_1)} + \rho\|h_{\frac{3}{2}}\|_{L^\infty(B_1)} + \rho^2\|h_{\frac{5}{2}}\|_{L^\infty(B_1)} \leq C(\rho_0^4 + c_0\rho_0^3 + o(1)),$$

and

$$\left\| \frac{1}{d + \kappa} \varphi \right\|_{L^\infty(B_1)} \leq C[(c_0 + \rho_0) + o(1)]. \quad (4.22)$$

These imply

$$\|\hat{u} - h_{\frac{7}{2}}\|_{L^1(B_{2\rho_0})} \leq C(\rho_0 + c_0)\rho_0^{\frac{13}{2}} + o(1). \quad (4.23)$$

Moreover, with Lemma 4.3, we use (4.22) to conclude that

$$\kappa \leq Cd[(c_0 + \rho_0) + o(1)] \quad (4.24)$$

for small $(c_0 + \rho_0)$ and large n . Consequently,

$$\|\Phi\|_{L^1(B_{2\rho_0})} \leq Cd[(c_0 + \rho_0)\rho_0^{\frac{13}{2}} |\log \rho_0| + o(1)].$$

Combining this with the definition of $\hat{u}$ and (4.23), we have that

$$\|u - [\tilde{p} + (d + \kappa)h_{\frac{7}{2}}]\|_{L^1(B_{2\rho_0})} \leq Cd[(\rho_0 + c_0)\rho_0^{\frac{13}{2}} |\log \rho_0| + o(1)].$$

With Lemma 4.2, we get that

$$\|u - \tilde{q}\|_{L^1(B_{2\rho_0})} \leq Cd[(\rho_0 + c_0)\rho_0^{\frac{13}{2}} |\log \rho_0| + o(1)],$$

where $q = p + (d + \kappa)h_{\frac{7}{2}}$.

Step 3: Improvement of flatness.

With the same technique in Step 3 from the proof of Lemma 4.8, we find

$$p' = a'_0 u_{\frac{7}{2}} + a'_1 v_{\frac{5}{2}} + a'_2 v_{\frac{3}{2}}$$

such that

$$\|q - U_\tau(p')\|_{L^\infty(\mathbb{S}^2 \cap \{r > \frac{1}{8}\})} \leq Cd^2, \quad (4.25)$$

where $|\tau| + \sum |a_j - a'_j| \leq Cd$. In this step, it is crucial that a_2 is bounded away from 0.

If η and δ are small, then $\{r > \frac{1}{8}\}$ contains $\mathcal{C}_\eta^\pm$ and $U_\tau(\mathcal{C}_\eta^\pm)$. By definition of replacements and (4.25), we have that

$$|\tilde{q} - U_\tau(\widetilde{p'})| \leq Cd^2 \text{ on } \mathbb{S}^2.$$

Combining this with the last estimate from the previous step, we have that

$$\|u - U_\tau(\widetilde{p'})\|_{L^1(B_{2\rho_0})} \leq Cd[(\rho_0 + c_0)\rho_0^{\frac{13}{2}} |\log \rho_0| + o(1)]. \quad (4.26)$$

On the other hand, with Lemma 4.1, we have

$$\kappa_{p'} \leq \kappa_p + d \cdot o(1) \leq Cd[(c_0 + \rho_0) + o(1)], \quad (4.27)$$

where we used (4.24) for the last comparison.

With Lemma 4.7 and (4.26), this implies

$$\|u - U_\tau(\widetilde{p'})\|_{L^\infty(B_{\rho_0})} \leq Cd\rho_0^{\frac{7}{2}}[(\rho_0 + c_0)|\log \rho_0| + o(1)],$$

which implies

$$u \in \mathcal{S}(p', \frac{1}{2}d, \rho_0)$$

if ρ_0, c_0 are chosen small universally and n is large. The bound on $\kappa_{p'}$ follows from (4.27).

This contradicts (4.20). $\square$

4.4. Dichotomy near general $p^* \in \mathcal{A}_2$

In this subsection, we sketch the ideas for dealing with other profiles in $\mathcal{A}_2$. Let's take $p^* \in \mathcal{A}_2$, namely,

$$p^* = u_{\frac{7}{2}} + a_1^* v_{\frac{5}{2}} + a_2^* v_{\frac{3}{2}}$$

with

$$\mu \leq a_2^* \leq 5 \text{ and } (a_1^*)^2 = \Gamma(a_2^*)$$

for a small parameter $\mu > 0$ to be fixed in the next section (see Remark 5.1), and the function Γ defined in (1.8). Let's further assume $a_1^* \geq 0$ as the other case is symmetric.

Thanks to results from Subsection 4.3, it suffices to consider the case when

$$|a_2^* - \frac{5}{4}| \geq \tilde{\delta}. \quad (4.28)$$

For such p^* , we define

$$C_\eta^\pm(p^*) = \{x \in \mathbb{S}^2 : |x - A_{p^*}^\pm| < \eta\},$$

with the notation from (4.2). With (4.3), these caps are bounded away from $\{r = 0\}$ if η is small, depending on μ .

For small $\delta > 0$ and a profile

$$p = a_0 u_{\frac{7}{2}} + a_1 v_{\frac{5}{2}} + a_2 v_{\frac{3}{2}}$$

with

$$a_0 \in [1/2, 2] \text{ and } |a_1/a_0 - a_1^*| + |a_2/a_0 - a_2^*| < \delta,$$

we define its replacement $\tilde{p}$ by solving (4.7) in $C_\eta^\pm(p^*)$, as in Definition 4.1. With (4.28), we see that when $\delta > 0$ is small, the replacement in one of the caps is identical to p . The auxiliary functions in Subsection 4.1 can be defined in a similar fashion.

With this construction, results from Subsection 4.1 follow from the same argument, with the constants possibly depending on μ . The class of well-approximated solutions can be defined similarly to Definition 4.2 with similar properties. The same argument in the previous section leads to a similar dichotomy for profiles near p^* .

Combining these with Lemma 4.8, we have

Lemma 4.9. *Suppose that*

$$u \in \mathcal{S}(p, d, 1)$$

for some $p = a_0 u_{\frac{7}{2}} + a_1 v_{\frac{5}{2}} + a_2 v_{\frac{3}{2}}$ satisfying

$$a_0 \in \left[\frac{1}{2}, 2\right] \text{ and } |a_1/a_0 - a_1^*| + |a_2/a_0 - a_2^*| \leq \delta$$

with

$$\mu \leq a_2^* \leq 5 \text{ and } (a_1^*)^2 = \Gamma(a_2^*).$$

There is a small constant $\tilde{\delta} > 0$, depending only on μ , such that if $d < \tilde{\delta}$ and $\delta < \tilde{\delta}$, then we have the following dichotomy:

(1) either

$$W_{\frac{7}{2}}(u; 1) - W_{\frac{7}{2}}(u; \rho_0) \geq c_0^2 d^2$$

and

$$u \in \mathcal{S}(p, Cd, \rho_0);$$

(2) or

$$u \in \mathcal{S}\left(p', \frac{1}{2}d, \rho_0\right)$$

for some

$$p' = U_\tau[a'_0 u_{\frac{7}{2}} + a'_1 v_{\frac{5}{2}} + a'_2 v_{\frac{3}{2}}]$$

with

$$\kappa_{p'} < d, \text{ and } |\tau| + \sum |a'_j - a_j| \leq Cd.$$

The constants c_0 , ρ_0 and C depend only on μ .

5. Trichotomy for $p \in \mathcal{A}_3$

In this section, we study profiles near $\mathcal{A}_3$ from (1.10), that is, profiles near $u_{\frac{7}{2}}$. To some extent, this section contains the main contribution of the work.

Similarly to the case studied in Section 4, the constraints $p|_{\{x_3=0\}} \geq 0$ and $\Delta p|_{\{x_3=0\}} \leq 0$ become degenerate for profiles in $\mathcal{A}_3$. Contrary to the previous case, the points of degeneracy can be arbitrarily close to two poles P^+ , $P^- \in \mathbb{S}^2 \cap \{r = 0\}$. As a result, the cornerstone for the previous case, Lemma 4.3, no longer holds.

To tackle this issue, we need to study the thin obstacle problem on spherical caps near $P^\pm$. At the infinitesimal level, this reduces to the problem in $\mathbb{R}^2$ studied in Appendix 6. This is one of the main reasons why we need to restrict to three dimensions in this work. This infinitesimal information influences the solution at unit scale through two higher Fourier coefficients along small spherical caps near $P^\pm$. With these two extra Fourier coefficients, we can define two extended profiles, one for each semi-sphere. These extended profiles approximate the solution with finer accuracy.

Arising from this procedure are two errors, say E_1 and E_2 . The former E_1 happens along the big circle $\mathbb{S}^2 \cap \{x_1 = 0\}$, where the extended profiles from the two semi-spheres are glued. The latter E_2 happens along the boundary of small spherical caps near the poles $P^\pm$. We establish that E_1 has a projection into $\mathcal{H}_{\frac{7}{2}}$ with size proportional to E_1 . Therefore, if E_1 is dominating, a similar strategy as in Section 4 can be carried out.

This leads to the following trichotomy, the main result in this section:

Lemma 5.1. *Suppose that*

$$u \in \mathcal{S}(p, d, 1)$$

for some $p = a_0 u_{\frac{7}{2}} + a_1 v_{\frac{5}{2}} + a_2 v_{\frac{3}{2}} + a_3 v_{\frac{1}{2}}$ with $a_0 \in [\frac{1}{2}, 2]$ and

$$\varepsilon_p := \max\{|a_1/a_0|, |a_2/a_0|^{\frac{1}{2}}, |a_3/a_0|^{\frac{1}{3}}\} \quad (5.1)$$

small.

Given small $\sigma > 0$, we can find $\tilde{\delta}$, c_0, ρ_0 small, and C big, depending only on σ , such that if $d < \tilde{\delta}$ and $\varepsilon_p < \tilde{\delta}$, then we have the following three possibilities:

(1)

$$W_{\frac{7}{2}}(u; 1) - W_{\frac{7}{2}}(u; \rho_0) \geq c_0^2 d^2;$$

and

$$u \in \mathcal{S}(p, Cd, \rho_0);$$

(2) or

$$u \in \mathcal{S}\left(p', \frac{1}{4}d, \rho_0\right)$$

for $p' = a'_0 u_{\frac{7}{2}} + a'_1 v_{\frac{5}{2}} + a'_2 v_{\frac{3}{2}} + a'_3 v_{\frac{1}{2}}$ with

$$\kappa_{p'} < \sigma \kappa_p, \text{ and } \sum |a'_j - a_j| \leq Cd;$$

(3) or

$$d \leq \varepsilon_p^5.$$

Recall the basis $\{u_{\frac{7}{2}}, v_{\frac{5}{2}}, v_{\frac{3}{2}}, v_{\frac{1}{2}}\}$, the rotation operator U , and the Weiss energy $W_{\frac{7}{2}}$ from (2.15), (2.4) and (2.6) respectively. The solution class $\mathcal{S}(\dots)$ and the measurement for error κ , similarly to their counterparts from Section 4, will be defined later in this section.

Remark 5.1. We will fix σ universally, which leads to universally defined $\tilde{\delta}$ as in the lemma. If we choose μ , the parameter from (1.10), small depending on $\tilde{\delta}$, then Lemma 5.1 holds true for p near $\mathcal{A}_3$ from (1.10). Once this choice of μ is made, estimates from previous sections become universal.

Remark 5.2. With the definition of ε_p , we can write $p = a_0 u_{\frac{7}{2}} + \tilde{a}_1 \varepsilon_p v_{\frac{5}{2}} + \tilde{a}_2 \varepsilon_p^2 v_{\frac{3}{2}} + \tilde{a}_3 \varepsilon_p^3 v_{\frac{1}{2}}$ with $|\tilde{a}_j| \leq 1$. Such coordinates are more convenient for profiles near $u_{\frac{7}{2}}$.

In this section, we often write

$$p = a_0 u_{\frac{7}{2}} + a_1 \varepsilon v_{\frac{5}{2}} + a_2 \varepsilon^2 v_{\frac{3}{2}} + a_3 \varepsilon^3 v_{\frac{1}{2}},$$

assuming implicitly that

$$a_0 \in [\frac{1}{2}, 2] \text{ and } |a_j| \leq 1 \text{ for } j = 1, 2, 3 \text{ with } \varepsilon \text{ small.}$$

Equivalently, in the basis $\{u_{\frac{7}{2}}, w_{\frac{5}{2}}, w_{\frac{3}{2}}, w_{\frac{1}{2}}\}$ from (2.16),

$$p = \tilde{a}_0 u_{\frac{7}{2}} + \tilde{a}_1 \varepsilon w_{\frac{5}{2}} + \tilde{a}_2 \varepsilon^2 w_{\frac{3}{2}} + \tilde{a}_3 \varepsilon^3 w_{\frac{1}{2}}.^4$$

Remark 5.3. By comparing with (1.7) and (1.8) we see that $p = a_0 u_{\frac{7}{2}} + a_1 \varepsilon v_{\frac{5}{2}} + a_2 \varepsilon^2 v_{\frac{3}{2}} + a_3 \varepsilon^3 v_{\frac{1}{2}}$ solves the thin obstacle problem if and only if

$$a_0 > 0, \quad a_2 \geq 0, \quad a_3 = 0, \quad \text{and} \quad \left(\frac{a_1}{a_0}\right)^2 \leq \min \left\{ 4 \frac{a_2}{a_0} \left(1 - \frac{1}{5} \frac{a_2 \varepsilon^2}{a_0}\right), \frac{24}{25} \frac{a_2}{a_0} \left(\frac{7}{2} - \frac{3}{10} \frac{a_2 \varepsilon^2}{a_0}\right) \right\}.$$

If we assume only

$$a_0 > 0, \quad a_2 \geq 0, \quad a_3 = 0, \quad \text{and} \quad \left(\frac{a_1}{a_0}\right)^2 \leq 4 \frac{a_2}{a_0} \left(1 - \frac{1}{5} \frac{a_2 \varepsilon^2}{a_0}\right),$$

and further that $a_1 \geq 0$, then p solves the thin obstacle problem outside a cone with $O(\frac{a_1 \varepsilon}{a_0})$ -opening near $\{r = 0, x_1 > 0\}$, where $\Delta p|_{\{x_3=0\}}$ might become positive.

The first half of this section is devoted to the proof of Lemma 5.1.

If the last possibility in Lemma 5.1 happens, then E_1 is not necessarily dominating (see the paragraph before the statement of the lemma). In this case, we need finer information of the solution using information from Appendix 6. This is carried out in the second half of this section.

5.1. The extended profile p_{ext}

Recall that (r, θ) denotes the polar coordinate of the (x_2, x_3) -plane. For a small *universal* constant $\eta > 0$, we define two small spherical caps near $\mathbb{S}^2 \cap \{r = 0\}$

$$\mathcal{C}_\eta^\pm := \{r < \eta, \pm x_1 > 0\} \cap \mathbb{S}^2. \quad (5.2)$$

In general, for small $r_0 > 0$, let

$$\mathcal{C}_{r_0}^\pm := \{r < r_0, \pm x_1 > 0\} \cap \mathbb{S}^2.$$

The cones generated by these caps are denoted by the same notations.

In this subsection, we focus on $\mathcal{C}_\eta^+$. The other case is symmetric. As a result, we often omit the superscript in $\mathcal{C}_\eta^+$

⁴ The coefficients are related by $a_0 = \tilde{a}_0 - \frac{7}{4} \tilde{a}_2 \varepsilon^2$, $a_1 = \frac{7}{2} \tilde{a}_1 - \frac{133}{8} \tilde{a}_3 \varepsilon^2$, $a_2 = \frac{35}{4} \tilde{a}_2$, and $a_3 = \frac{105}{8} \tilde{a}_3$.

Given a profile $p = a_0 u_{\frac{7}{2}} + a_1 \varepsilon v_{\frac{5}{2}} + a_2 \varepsilon^2 v_{\frac{3}{2}} + a_3 \varepsilon^3 v_{\frac{1}{2}}$, if we denote by v the solution to the thin obstacle problem in $\mathcal{C}_\eta$ with p as boundary data, then in general we have $|v - p| \leq C\varepsilon^{\frac{7}{2}}$. This is not precise enough for later development.

The main task of this subsection is to show that we can find an extended profile, p_{ext} , so that if we solve a similar problem with p_{ext} as boundary data, then the error can be improved to $O(\varepsilon^6)$ -order. This p_{ext} will be an essential building block for our replacement of p .

Throughout this subsection, we assume that

$$a_0 \in [\frac{1}{2}, 2]; \quad |a_j| \leq 1 \text{ for } j = 1, 2, 3; \quad |b_j| \leq A + 1 \text{ for } j = 1, 2; \text{ and } \varepsilon < \tilde{\varepsilon} \quad (5.3)$$

where A is the universal constant from Proposition B.1, and $\tilde{\varepsilon}$ is a small universal constant.

Corresponding to these parameters, let us denote that

$$\begin{aligned} p[a_0, a_1, a_2, a_3; \varepsilon] &:= a_0 u_{\frac{7}{2}} + a_1 \varepsilon v_{\frac{5}{2}} + a_2 \varepsilon^2 v_{\frac{3}{2}} + a_3 \varepsilon^3 v_{\frac{1}{2}}, \text{ and} \\ p_{ext}[a_0, a_1, a_2, a_3; b_1, b_2; \varepsilon] &:= p[a_0, a_1, a_2, a_3; \varepsilon] + b_1 \varepsilon^4 v_{-\frac{1}{2}} + b_2 \varepsilon^5 v_{-\frac{3}{2}}. \end{aligned} \quad (5.4)$$

Recall the basis $\{u_{\frac{7}{2}}, v_{\frac{5}{2}}, v_{\frac{3}{2}}, v_{\frac{1}{2}}\}$ from (2.15) and the functions $v_{-\frac{1}{2}}, v_{-\frac{3}{2}}$ from (2.18).

Let $v = v[a_0, a_1, a_2, a_3; b_1, b_2; \varepsilon]$ denote the solution to

$$\begin{cases} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})v \leq 0 & \text{in } \mathcal{C}_\eta, \\ v \geq 0 & \text{in } \mathcal{C}_\eta \cap \{x_3 = 0\}, \\ (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})v = 0 & \text{in } \mathcal{C}_\eta \cap (\{v > 0\} \cup \{x_3 \neq 0\}) \\ v = p_{ext}[a_0, a_1, a_2, a_3; b_1, b_2; \varepsilon] & \text{along } \partial\mathcal{C}_\eta. \end{cases} \quad (5.5)$$

The $\frac{7}{2}$ -homogeneous extension of v is denoted by the same notation.

We often omit some of the parameters in $p[\dots]$, $p_{ext}[\dots]$ and $v[\dots]$ when there is no ambiguity.

We begin with localization of the contact set of v .

Lemma 5.2. *Assuming (5.3), we have that*

$$\begin{aligned} |v - p| &\leq C\varepsilon^{\frac{7}{2}} \text{ in } \mathcal{C}_\eta, \\ (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})v &= 0 \text{ in } \widehat{\mathcal{C}}_\eta \cap \{r > M\varepsilon\}, \text{ and } v = 0 \text{ in } \widetilde{\mathcal{C}}_\eta \cap \{r > M\varepsilon\} \end{aligned}$$

for universal constants C and M .

Here we are using the notations for slit domains from (2.10).

Proof. The key is to construct a barrier similar to the one in Step 2 from the proof of Proposition B.1. We omit some details.

With the basis from (2.16), rewrite p_{ext} as

$$p_{ext} = \tilde{a}_0 u_{\frac{7}{2}} + \tilde{a}_1 \varepsilon w_{\frac{5}{2}} + \tilde{a}_2 \varepsilon^2 w_{\frac{3}{2}} + \tilde{a}_3 \varepsilon^3 w_{\frac{1}{2}} + b_1 \varepsilon^4 v_{-\frac{1}{2}} + b_2 \varepsilon^5 v_{-\frac{3}{2}}.$$

By taking τ large universally and (α_1, α_2) solving

$$\alpha_1 + \tilde{a}_0\tau = \tilde{a}_1, \text{ and } \alpha_2 + \alpha_1\tau + \frac{1}{2}\tilde{a}_0\tau^2 = \tilde{a}_2,$$

the function $q = \tilde{a}_0u_{\frac{7}{2}} + \alpha_1\varepsilon w_{\frac{5}{2}} + \alpha_2\varepsilon^2w_{\frac{3}{2}}$ satisfies

$$U_{-\tau\varepsilon}(q) \geq p_{ext} \text{ along } \partial\mathcal{C}_\eta$$

for ε small. Recall the rotation operator U from (2.4). By taking τ larger, if necessary, it can be verified that q solves the thin obstacle problem in $\mathbb{R}^3$ (see Remark 5.3).

By the maximum principle, we have that

$$v \leq U_{-\tau\varepsilon}(q) \text{ in } \mathcal{C}_\eta. \quad (5.6)$$

With $q = 0$ along $\widetilde{\mathbb{R}^3}$, we have that

$$v = 0 \text{ in } \tilde{\mathcal{C}}_\eta \cap \{r > M\varepsilon\}.$$

Using again (5.6), we have that

$$v - p \leq U_{-\tau\varepsilon}(q) - p \leq C\varepsilon^{\frac{7}{2}} \text{ in } \mathcal{C}_{M\varepsilon}.$$

With a similar argument, we can construct a lower barrier of the form $U_{\tau\varepsilon}(\tilde{a}_0u_{\frac{7}{2}} + \beta_1\varepsilon w_{\frac{5}{2}} + \beta_2\varepsilon^2w_{\frac{3}{2}})$, which implies

$$(\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})v = 0 \text{ in } \widehat{\mathcal{C}}_\eta \cap \{r > M\varepsilon\},$$

and

$$v - p \geq -C\varepsilon^{\frac{7}{2}} \text{ in } \mathcal{C}_{M\varepsilon}.$$

Finally, we apply the maximum principle to $v - p$ in $\mathcal{C}_\eta \setminus \mathcal{C}_{M\varepsilon}$ to get the desired bound on $|v - p|$. $\square$

We now link the behavior of v near $\{r = 0\} \cap \mathbb{S}^2$ to the problem studied in Appendix 6. Before the precise statement, we introduce some notations.

For a function $w : \mathbb{R}^3 \rightarrow \mathbb{R}$, let us denote by $\tilde{w}_\varepsilon$ the following rescaling of its restriction to the plane $\{x_1 = 1\}$

$$\tilde{w}_\varepsilon : \mathbb{R}^2 \rightarrow \mathbb{R}, \text{ and } \tilde{w}_\varepsilon(x_2, x_3) := \frac{1}{\varepsilon^{\frac{7}{2}}} w(1, \varepsilon x_2, \varepsilon x_3). \quad (5.7)$$

For the solution v from (5.5), it follows that $\tilde{v}_\varepsilon$ solves

$$\begin{cases} \mathcal{L}_\varepsilon \tilde{v}_\varepsilon \leq 0 & \text{in } B_{R_\varepsilon}, \\ \tilde{v}_\varepsilon \geq 0 & \text{in } B'_{R_\varepsilon}, \\ \mathcal{L}_\varepsilon \tilde{v}_\varepsilon = 0 & \text{in } B_{R_\varepsilon} \cap (\{\tilde{v}_\varepsilon > 0\} \cup \{x_3 \neq 0\}), \\ \tilde{v}_\varepsilon = (\tilde{p}_{ext})_\varepsilon & \text{on } \partial B_{R_\varepsilon}, \end{cases} \quad (5.8)$$

where $R_\varepsilon = \frac{\eta}{\varepsilon\sqrt{1-\eta^2}}$, and $\mathcal{L}_\varepsilon$ is the operator defined as

$$\mathcal{L}_\varepsilon w = \Delta_{\mathbb{R}^2} w + \varepsilon^2[x \cdot (D_{\mathbb{R}^2}^2 w - x) - 5x \cdot \nabla_{\mathbb{R}^2} w + \frac{35}{4}w].$$

Proposition 5.1. For $\{a_j\}$, $\{b_j\}$, ε satisfying (5.3), let $v = v[a_j; b_j; \varepsilon]$ be the solution to (5.5).

For $a_0^* \in [\frac{1}{2}, 2]$ and $|a_j^*| \leq 1$ for $j = 1, 2, 3$, let v^* be the solution to (B.3) with data $p^* = a_0^* u_{\frac{7}{2}} + a_1^* u_{\frac{5}{2}} + a_2^* u_{\frac{3}{2}} + a_3^* u_{\frac{1}{2}}$ at infinity.

Given $R > 0$, there is a modulus of continuity, ω_R , depending only on R , such that

$$\|\tilde{v}_\varepsilon - v^*\|_{L^\infty(B_R)} \leq \omega_R(\sum |a_j - a_j^*| + \varepsilon),$$

where $\tilde{v}_\varepsilon$ is defined in (5.7).

Proof. With the compactness of the region for (a_j^*) , it suffices to find a modulus of continuity for a fixed (a_j^*) .

Suppose there is no such ω , we find a sequence $(a_j^n, b_j^n, \varepsilon_n)$ with $\varepsilon_n \rightarrow 0$ and $a_j^n \rightarrow a_j^*$ such that

$$\|\tilde{v}_{\varepsilon_n}^n - v^*\|_{L^\infty(B_R)} \geq \delta > 0, \quad (5.9)$$

where $v^n = v^n[a_j^n; b_j^n; \varepsilon_n]$ as in (5.5).

With the bound on $|v - p|$ from Lemma 5.2, we have, for a universal C ,

$$|\tilde{v}_{\varepsilon_n}^n - [p_n + \varepsilon_n^2(-a_2^n \frac{r^2}{5} u_{\frac{3}{2}} + a_3^n r^2 u_{\frac{1}{2}})]| \leq C \text{ in } B_{R_{\varepsilon_n}},$$

where $R_{\varepsilon_n} = \frac{\eta}{\varepsilon_n \sqrt{1-\eta^2}}$ and

$$p_n = a_0^n u_{\frac{7}{2}} + a_1^n u_{\frac{5}{2}} + a_2^n u_{\frac{3}{2}} + a_3^n u_{\frac{1}{2}}.$$

Consequently, for any compact $K \subset \mathbb{R}^2$, there is a constant C_K depending only on K , such that

$$|\tilde{v}_{\varepsilon_n}^n - p_n| \leq C + C_K \varepsilon_n^2 \text{ in } K \text{ if } n \text{ is large.}$$

With $\tilde{v}_{\varepsilon_n}^n$ solving (5.8), we have, up to a subsequence,

$$\tilde{v}_{\varepsilon_n}^n \rightarrow u_\infty \text{ locally uniformly in } \mathbb{R}^2, \quad (5.10)$$

with the limit u_∞ solving the thin obstacle problem (1.1) in $\mathbb{R}^2$.

Now for any $x \in \mathbb{R}^2$, we find a compact set $K \ni x$, then

$$\begin{aligned} |u_\infty - p^*|(x) &\leq \limsup[|u_\infty - \tilde{v}_{\varepsilon_n}^n|(x) + |\tilde{v}_{\varepsilon_n}^n - p_n|(x) + |p_n - p^*|(x)] \\ &\leq \limsup(C + C_K \varepsilon_n^2) \leq C. \end{aligned}$$

With this, we have $\sup_{\mathbb{R}^2} |u_\infty - p^*| < +\infty$, and by the uniqueness result in Proposition B.1, we have $u_\infty = v^*$. This contradicts (5.9) and (5.10). $\square$

If we apply Proposition 5.1 to the special case when $a_j = a_j^*$, we see that the infinitesimal behavior of v near $\{r = 0\} \cap \mathbb{S}^2$ is not affected by the coefficient b_j . This allows the fixed argument in the following lemma:

Lemma 5.3. *Let a_j satisfy $a_0 \in [\frac{1}{2}, 2]$ and $|a_j| \leq 1$ for $j = 1, 2, 3$, and A be the universal constant from Proposition B.1.*

We can find b_1 and b_2 with $|b_j| \leq A + 1$ such that

$$|v - p_{ext}|_{L^\infty(\mathcal{C}_\eta \setminus \mathcal{C}_{\eta/2})} \leq C\varepsilon^6,$$

where $v = v[a_j; b_j; \varepsilon]$ and $p_{ext} = p_{ext}[a_j; b_j; \varepsilon]$ are defined in (5.5) and (5.4), and C is universal.

Moreover, there is a universal modulus of continuity, ω , such that

$$|b_j - a_0 \cdot b_j^{\mathbb{R}^2} [a_1/a_0, a_2/a_0, a_3/a_0]| \leq \omega(\varepsilon)$$

for $j = 1, 2$.

Recall the definition of $b_j^{\mathbb{R}^2}[\dots]$ from Remark B.1.

Proof. With Lemma 5.2, we have that

$$\begin{cases} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})(v - p_{ext}) = 0 & \text{in } \mathcal{C}_\eta \setminus \widehat{\mathcal{C}}_{M\varepsilon}, \\ v - p_{ext} = 0 & \text{on } \partial\mathcal{C}_\eta \cup \mathcal{C}_\eta \setminus \widetilde{\mathcal{C}}_{M\varepsilon}, \end{cases}$$

and $|v - p_{ext}| \leq C\varepsilon^{\frac{7}{2}}$ along $\partial\mathcal{C}_{M\varepsilon}$. By Lemma A.1 with $m = 2$, to get the bound on $|v - p_{ext}|$, it suffices to choose b_j so that

$$\int_{\partial\mathcal{C}_{M\varepsilon}} (v - p_{ext}) \cdot \cos\left(\frac{1}{2}\theta\right) = 0, \text{ and } \int_{\partial\mathcal{C}_{M\varepsilon}} (v - p_{ext}) \cdot \cos\left(\frac{3}{2}\theta\right) = 0. \quad (5.11)$$

By a change of variable, we have that

$$\int_{\partial\mathcal{C}_{M\varepsilon}} (v - p_{ext}) \cdot \cos\left(\frac{1}{2}\theta\right) = \varepsilon^{\frac{9}{2}}(1 - M^2\varepsilon^2)^{\frac{9}{4}} \int_{\partial B_{R_\varepsilon}} (\tilde{v}_\varepsilon - (\tilde{p}_{ext})_\varepsilon) \cdot \cos\left(\frac{1}{2}\theta\right), \quad (5.12)$$

where $R_\varepsilon = \frac{M}{\sqrt{1-M^2\varepsilon^2}}$, and $\tilde{v}_\varepsilon$, and $(\tilde{p}_{ext})_\varepsilon$ are defined in (5.7).

Now let us define

$$p^* = a_0 u_{\frac{7}{2}} + a_1 u_{\frac{5}{2}} + a_2 u_{\frac{3}{2}} + a_3 u_{\frac{1}{2}},$$

v^* is the solution to (B.3) with data p^* at infinity,

$$\beta_j = a_0 \cdot b_j^{\mathbb{R}^2} [a_1/a_0, a_2/a_0, a_3/a_0]$$

and

$$p_{ext}^* = p^* + \beta_1 u_{-\frac{1}{2}} + \beta_2 u_{-\frac{3}{2}}.$$

Then we can compute the last term in (5.12) as

$$\begin{aligned} \int_{\partial B_{R_\varepsilon}} (\tilde{v}_\varepsilon - (\tilde{p}_{ext})_\varepsilon) \cdot \cos\left(\frac{1}{2}\theta\right) &= \int_{\partial B_{R_\varepsilon}} (\tilde{v}_\varepsilon - v^*) \cdot \cos\left(\frac{1}{2}\theta\right) \\ &+ \int_{\partial B_{R_\varepsilon}} (v^* - p_{ext}^*) \cdot \cos\left(\frac{1}{2}\theta\right) + \int_{\partial B_{R_\varepsilon}} (p_{ext}^* - (\tilde{p}_{ext})_\varepsilon) \cdot \cos\left(\frac{1}{2}\theta\right) \end{aligned}$$

With Corollary B.1, the second term vanishes. With Proposition 5.1 and $R_\varepsilon \leq 2M$ for ε small, the first term is of order $\omega_{2M}(\varepsilon) \int_{\partial B_{R_\varepsilon}} \cos(\frac{1}{2}\theta)$. We can use the definition of p_{ext} and p_{ext}^* to continue

$$\begin{aligned} \int_{\partial B_{R_\varepsilon}} (\tilde{v}_\varepsilon - (\tilde{p}_{ext})_\varepsilon) \cdot \cos\left(\frac{1}{2}\theta\right) &= \int_{\partial B_{R_\varepsilon}} \cos\left(\frac{1}{2}\theta\right) \cdot \omega_{2M}(\varepsilon) \\ &+ (\beta_1 - b_1) R_\varepsilon^{-\frac{1}{2}} \int_{\partial B_{R_\varepsilon}} \cos^2\left(\frac{1}{2}\theta\right). \end{aligned}$$

By adjusting b_1 , we can make the right-hand side 0. Moreover, this choice of b_1 satisfies

$$|b_1 - \beta_1| \leq 2M^{\frac{1}{2}} \omega_{2M}(\varepsilon), \quad (5.13)$$

where ω_{2M} is the modulus of continuity from Proposition 5.1, and M is the constant from Lemma 5.2.

In particular, for ε small, we can find b_1 satisfying $|b_1| \leq A + 1$ as in (5.3) such that

$$\int_{\partial C_{M\varepsilon}} (v - p_{ext}) \cdot \cos\left(\frac{1}{2}\theta\right) = 0.$$

A similar argument gives b_2 satisfying $|b_2| \leq A + 1$ such that

$$|b_2 - \beta_2| \leq C M^{\frac{3}{2}} \omega_{2M}(\varepsilon), \quad (5.14)$$

and

$$\int_{\partial C_{M\varepsilon}} (v - p_{ext}) \cdot \cos\left(\frac{3}{2}\theta\right) = 0.$$

Therefore, we have (5.11) and the bound on $|v - p_{ext}|$ follows. The control on $|b_j - \beta_j|$ is a consequence of (5.13) and (5.14). $\square$

With these, we finally define the extended profile

Definition 5.1. Corresponding to $p = a_0 u_{\frac{7}{2}} + a_1 \varepsilon v_{\frac{5}{2}} + a_2 \varepsilon^2 v_{\frac{3}{2}} + a_3 \varepsilon^3 v_{\frac{1}{2}}$ with a_j, ε satisfying (5.3), we define the *extended profile*, p_{ext} , by

$$p_{ext} = \begin{cases} p + b_1^+ \varepsilon^4 v_{-\frac{1}{2}} + b_2^+ \varepsilon^5 v_{-\frac{3}{2}} & \text{in } \{x_1 > 0\}, \\ p + b_1^- \varepsilon^4 v_{-\frac{1}{2}} + b_2^- \varepsilon^5 v_{-\frac{3}{2}} & \text{in } \{x_1 < 0\}, \end{cases}$$

where b_j^+ are the coefficients from Lemma 5.3, and b_j^- are given by a similar procedure in $\mathcal{C}_\eta^-$.

Remark 5.4. Corresponding to a_j, ε , we refer the coefficients $b_j^\pm$ from Definition 5.1 by

$$b_j^{\pm, \mathbb{S}^2}[a_j; \varepsilon] := b_j^\pm.$$

5.2. Boundary layer near $\mathcal{A}_3$ and well-approximated solutions

In this section we construct the replacement of profiles near $\mathcal{A}_3$. It is illustrative to compare this subsection with the construction from Subsection 4.1.

Given $p = a_0 u_{\frac{7}{2}} + a_1 \varepsilon v_{\frac{5}{2}} + a_2 \varepsilon^2 v_{\frac{3}{2}} + a_3 \varepsilon^3 v_{\frac{1}{2}}$, satisfying (5.3), and the corresponding p_{ext} as in Definition 5.1, we define $v_p^\pm$ as the solution to (5.5) in $\mathcal{C}_\eta^\pm$.

If $b_j^+ \neq b_j^-$ in Definition 5.1, the extended profile p_{ext} is discontinuous along $\{x_1 = 0\}$. To fix this issue, we make another replacement in the following layer

$$\mathcal{L}_\eta := \{|x_1| < \eta\} \cap \mathbb{S}^2.$$

The cone generated by $\mathcal{L}_\eta$ is denoted by the same notation.

In this layer, we solve

$$\begin{cases} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})h_p = 0 & \text{in } \widehat{\mathcal{L}}_\eta, \\ h_p = p_{ext} & \text{in } \partial\mathcal{L}_\eta \cup \widetilde{\mathcal{L}}_\eta. \end{cases}$$

Recall the notation for slit domains from (2.10).

With this, we define the replacement of p as:

Definition 5.2. For p satisfying (5.3), its *replacement*, $\bar{p}$, is defined as

$$\bar{p} = \begin{cases} v_p^\pm & \text{in } \mathcal{C}_\eta^\pm, \\ h_p & \text{in } \mathcal{L}_\eta, \\ p_{ext} & \text{in } \mathbb{S}^2 \setminus (\mathcal{C}_\eta^\pm \cup \mathcal{L}_\eta). \end{cases}$$

Equivalently, the replacement $\bar{p}$ is the minimizer of the energy

$$v \mapsto \int_{\mathbb{S}^2} |\nabla_{\mathbb{S}^2} v|^2 - \lambda_{\frac{7}{2}} v^2$$

over

$$\{v : v = p_{ext} \text{ outside } \mathcal{C}_\eta^\pm \cup \mathcal{L}_\eta, \text{ and } v \geq 0 \text{ on } \{x_3 = 0\} \cap \mathcal{C}_\eta^\pm\}.$$

This replacement satisfies

$$\begin{cases} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})\bar{p} = f_{1,p} dH^1|_{\partial\mathcal{L}_\eta} + f_{2,p}^{\pm} dH^1|_{\partial\mathcal{C}_\eta^\pm} + g_p^{\pm} dH^1|_{\mathcal{C}_\eta^\pm \cap \{x_3=0\}} & \text{in } \widehat{\mathbb{S}^2} \cup \{r < \eta\}, \\ \bar{p} = 0 & \text{in } \widetilde{\mathbb{S}^2} \cap \{r \geq \eta\}. \end{cases} \quad (5.15)$$

Similarly to Subsection 4.1, we define some auxiliary functions.

The projection of $f_{1,p}$ into $\mathcal{H}_{\frac{7}{2}}$ (see (2.12)) is denoted by $\varphi_{1,p}$, namely,

$$\varphi_{1,p} := c_{\frac{7}{2}} u_{\frac{7}{2}} + c_{\frac{5}{2}} v_{\frac{5}{2}} + c_{\frac{3}{2}} v_{\frac{3}{2}} + c_{\frac{1}{2}} v_{\frac{1}{2}}, \quad (5.16)$$

where

$$c_{\frac{7}{2}} = \frac{1}{\|u_{\frac{7}{2}}\|_{L^2(\mathbb{S}^2)}} \cdot \int_{\mathbb{S}^2} u_{\frac{7}{2}} \cdot f_{1,p} dH^1|_{\partial\mathcal{L}_\eta}, \text{ and } c_{m+\frac{1}{2}} = \frac{1}{\|v_{m+\frac{1}{2}}\|_{L^2(\mathbb{S}^2)}} \cdot \int_{\mathbb{S}^2} v_{m+\frac{1}{2}} \cdot f_{1,p} dH^1|_{\partial\mathcal{L}_\eta}$$

for $m = 0, 1, 2$.

By Fredholm alternative, there is a unique function $H_{1,p} : \mathbb{S}^2 \rightarrow \mathbb{R}$ satisfying

$$\begin{cases} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})H_{1,p} = f_{1,p}dH^1|_{\partial\mathcal{L}_\eta} - \varphi_{1,p} & \text{on } \widehat{\mathbb{S}^2}, \\ H_{1,p} = 0 & \text{on } \widetilde{\mathbb{S}^2}. \end{cases}$$

Corresponding to these, we define

$$\Phi_{1,p} := H_{1,p} \left(\frac{x}{|x|} \right) |x|^{\frac{7}{2}} + \frac{1}{8} \varphi_{1,p} \left(\frac{x}{|x|} \right) |x|^{\frac{7}{2}} \log(|x|), \quad (5.17)$$

which satisfies

$$\begin{cases} \Delta \Phi_{1,p} = f_{1,p}dH^2|_{\partial\mathcal{L}_\eta} & \text{in } \widehat{\mathbb{R}^3}, \\ \Phi_{1,p} = 0 & \text{on } \widetilde{\mathbb{R}^3}. \end{cases} \quad (5.18)$$

Let us denote that

$$\kappa_{1,p} := \|\Phi_{1,p}\|_{L^\infty(B_1)}, \quad (5.19)$$

which measures the size of the error coming from the gluing procedure along $\partial\mathcal{L}_\eta$.

Similarly, corresponding to $f_{2,p}^+ dH^1|_{\partial\mathcal{C}_\eta^+} + f_{2,p}^- dH^1|_{\partial\mathcal{C}_\eta^-}$ from (5.15), we define

$$\varphi_{2,p} := \text{Proj}_{\mathcal{H}_{\frac{7}{2}}} (f_{2,p}^+ dH^1|_{\partial\mathcal{C}_\eta^+} + f_{2,p}^- dH^1|_{\partial\mathcal{C}_\eta^-}).$$

The function $H_{2,p}$ is the unique solution to

$$\begin{aligned} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})H_{2,p} &= f_{2,p}^+ dH^1|_{\partial\mathcal{C}_\eta^+} + f_{2,p}^- dH^1|_{\partial\mathcal{C}_\eta^-} - \varphi_{2,p} \text{ on } \widehat{\mathbb{S}^2}, \text{ and} \\ H_{2,p} &= 0 \text{ on } \widetilde{\mathbb{S}^2}. \end{aligned}$$

Finally, we let

$$\Phi_{2,p} := H_{2,p} \left(\frac{x}{|x|} \right) |x|^{\frac{7}{2}} + \frac{1}{8} \varphi_{2,p} \left(\frac{x}{|x|} \right) |x|^{\frac{7}{2}} \log(|x|),$$

$$\kappa_{2,p} := \|\Phi_{2,p}\|_{L^\infty(B_1)}, \quad (5.20)$$

and

$$\kappa_p := \kappa_{1,p} + \kappa_{2,p}. \quad (5.21)$$

The subscript p is often omitted when there is no ambiguity.

We estimate the change in $\bar{p}$ when p is modified.

Lemma 5.4. Suppose that $p = a_0 u_{\frac{7}{2}} + a_1 \varepsilon v_{\frac{5}{2}} + a_2 \varepsilon^2 v_{\frac{3}{2}} + a_3 \varepsilon^3 v_{\frac{1}{2}}$ satisfies (5.3), and that $q = \alpha_0 u_{\frac{7}{2}} + \alpha_1 v_{\frac{5}{2}} + \alpha_2 v_{\frac{3}{2}} + \alpha_3 v_{\frac{1}{2}}$ satisfies $|\alpha_j| \leq 1$.

Given small $a > 0$, we have, for $C_a > 0$ big depending on a ,

$$\|\overline{p + dq} - (\overline{p} + dq)\|_{L^\infty(\mathbb{S}^2)} \leq ad + C_a \varepsilon^6$$

for small $d, \varepsilon > 0$.

Remark 5.5. It suffices to consider the case when $d \leq \varepsilon^3$. In case $d > \varepsilon^3$, we define $\bar{\varepsilon} = d^{\frac{1}{3}} > \varepsilon$, and rewrite $p = a_0 u_{\frac{7}{2}} + \tilde{a}_1 \bar{\varepsilon} v_{\frac{5}{2}} + \tilde{a}_2 \bar{\varepsilon}^2 v_{\frac{3}{2}} + \tilde{a}_3 \bar{\varepsilon}^3 v_{\frac{1}{2}}$. Then $|\tilde{a}_j| \leq |a_j|$. Consequently, conditions from (5.3) are satisfied if d is small enough.

Once the case for $d \leq \varepsilon^3$ is established, to deal with the case $d > \varepsilon^3$, we can apply the conclusion to get an upper bound of the form $ad + C_a \bar{\varepsilon}^6 = ad + C_a d^2$, leading to the desired conclusion if we slightly decrease a and choose d small enough.

Proof. We first focus on the estimate in $\{x_1 > 0\}$. In this region, suppose

$$\begin{aligned} p_{ext} &= p + b_1 \varepsilon^4 v_{-\frac{1}{2}} + b_2 \varepsilon^5 v_{-\frac{3}{2}}, \text{ and} \\ (p + dq)_{ext} &= p + dq + \beta_1 \varepsilon^4 v_{-\frac{1}{2}} + \beta_2 \varepsilon^5 v_{-\frac{3}{2}}. \end{aligned}$$

Step 1: Estimate on $|b_j - \beta_j|$.

For $r_0 > 0$ to be chosen, with the same argument as in Lemma 5.3, we have

$$|\overline{p} - p_{ext}| + |\overline{p + dq} - (p + dq)_{ext}| \leq C_{r_0} \varepsilon^6 \text{ in } C_\eta \setminus C_{r_0}, \quad (5.22)$$

for a constant C_{r_0} depending on r_0 .

Since $\overline{p}$ and $\overline{p + dq}$ both solve the thin obstacle problem (5.5) in C_η , the maximum principle implies that the function $r \mapsto \|\overline{p} - p + dq\|_{L^\infty(\partial C_r)}$ is increasing for $r \in (0, \eta)$. With the previous estimate, this gives that

$$\begin{aligned} \|(b_1 - \beta_1) \varepsilon^4 v_{-\frac{1}{2}} + (b_2 - \beta_2) \varepsilon^5 v_{-\frac{3}{2}}\|_{L^\infty(\partial C_{r_0})} &\leq \|(b_1 - \beta_1) \varepsilon^4 v_{-\frac{1}{2}} \\ &+ (b_2 - \beta_2) \varepsilon^5 v_{-\frac{3}{2}}\|_{L^\infty(\partial C_{r_1})} + C_{r_0} \varepsilon^6 + C_{r_1}^{\frac{1}{2}} d \end{aligned}$$

if $r_0 < r_1 < \eta$. The last term is a consequence of the bound $|q| \leq C r_1^{\frac{1}{2}}$ in C_{r_1} .

With the orthogonality between $v_{-\frac{1}{2}}$ and $v_{-\frac{3}{2}}$, this implies

$$\begin{aligned} |b_1 - \beta_1| \varepsilon^4 r_0^{-\frac{1}{2}} + |b_2 - \beta_2| \varepsilon^5 r_0^{-\frac{3}{2}} &\leq C[|b_1 - \beta_1| \varepsilon^4 r_1^{-\frac{1}{2}} + |b_2 - \beta_2| \varepsilon^5 r_1^{-\frac{3}{2}}] \\ &+ C_{r_0} \varepsilon^6 + C_{r_1}^{\frac{1}{2}} d, \end{aligned}$$

where C is universal. By choosing $r_1/r_0 \gg 1$, universally, we have that

$$|b_1 - \beta_1| \varepsilon^4 r_0^{-\frac{1}{2}} + |b_2 - \beta_2| \varepsilon^5 r_0^{-\frac{3}{2}} \leq C_{r_0} \varepsilon^6 + C_{r_1}^{\frac{1}{2}} d.$$

Step 2: Estimate in $\{x_1 > \eta\}$.

With the final estimate from the previous step, we have that

$$|(p_{ext} + dq) - (p + dq)_{ext}| \leq C_{r_0} \varepsilon^6 + Cr_1^{\frac{1}{2}} d \text{ in } \{x_1 > 0\} \setminus C_{r_0}.$$

With the maximum principle and (5.22), this gives that

$$|\overline{p + dq} - (\overline{p} + dq)| \leq C_{r_0} \varepsilon^6 + Cr_1^{\frac{1}{2}} d \text{ in } C_\eta.$$

Meanwhile, $\overline{p + dq} = (p + dq)_{ext}$ and $\overline{p} = p_{ext}$ in $\{x_1 > \eta\} \setminus C_\eta$ by definition, we conclude

$$|\overline{p + dq} - (\overline{p} + dq)| \leq C_{r_0} \varepsilon^6 + Cr_1^{\frac{1}{2}} d \text{ in } \{x_1 > \eta\}.$$

Step 3: Conclusion.

With a symmetric argument, we have that

$$|\overline{p + dq} - (\overline{p} + dq)| \leq C_{r_0} \varepsilon^6 + Cr_1^{\frac{1}{2}} d \text{ in } \{x_1 < -\eta\}.$$

Using the maximum principle, we have that

$$|\overline{p + dq} - (\overline{p} + dq)| \leq C_{r_0} \varepsilon^6 + Cr_1^{\frac{1}{2}} d \text{ in } \mathbb{S}^2.$$

From here, we choose r_1 small, depending on a such that $Cr_1^{\frac{1}{2}} < a$. Then we choose $r_0 \ll r_1$, which fixes the constant C_{r_0} . $\square$

Lemma 5.4 leads to a control over the change in $\kappa_{1,p}$ and $\kappa_{2,p}$ from (5.19) and (5.20) when p is modified. The proof is similar to the proof for Corollary 4.1 and is omitted.

Corollary 5.1. *With the same assumption and the same notation as in Lemma 5.4, we have that*

$$\kappa_{j,p+dq} \leq \kappa_{j,p} + ad + C_a \varepsilon^6 \text{ for } j = 1, 2.$$

Among the terms on the right-hand side of (5.15), the term f_1 has a significant projection into $\mathcal{H}_{\frac{7}{2}}$. While the similar result is not necessarily true for $f_2^\pm$, we can show that $f_2^\pm$ is small. This is the content of the following lemma.

Lemma 5.5. *Suppose that p satisfies (5.3) and p_{ext} is as in Definition 5.1, then we have that*

$$c\kappa_1 \leq |b_1^+ - b_1^-| \varepsilon^4 + |b_2^+ - b_2^-| \varepsilon^5 \leq C \|\varphi_1\|_{L^\infty(\mathbb{S}^2)},$$

and

$$\kappa_2 \leq C \varepsilon^6$$

for universal constants c small and C large.

Proof. With our convention for one-sided derivatives from (2.3), we have $f_1 = (\bar{p} - p_{ext})_\nu|_{\partial\mathcal{L}_\eta}$. By definition of $\bar{p}$ and the maximum principle, we have $|\bar{p} - p_{ext}| \leq C(|b_1^+ - b_1^-|\varepsilon^4 + |b_2^+ - b_2^-|\varepsilon^5)$ in $\mathcal{L}_\eta$. The upper bound on κ_1 follows from boundary regularity of harmonic functions.

With Lemma 5.3, the bound on κ_2 follows from a similar argument.

It remains to prove the lower bound for $\|\varphi_1\|_{L^\infty(\mathbb{S}^2)}$.

For this let us define

$$q := \begin{cases} \bar{p} - p & \text{in } \mathbb{S}^2 \setminus \mathcal{C}_\eta^\pm, \\ p_{ext} - p & \text{in } \mathcal{C}_\eta^\pm. \end{cases}$$

Then for small $r_0 > 0$, we have that

$$\begin{aligned} \int_{\mathbb{S}^2} f_1 v_{\frac{1}{2}} dH^1|_{\partial\mathcal{L}_\eta} &= \int_{\mathbb{S}^2 \setminus \mathcal{C}_{r_0}^\pm} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}}) q \cdot v_{\frac{1}{2}} \\ &= \int_{\mathbb{S}^2 \setminus \mathcal{C}_{r_0}^\pm} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}}) q \cdot v_{\frac{1}{2}} - q \cdot (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}}) v_{\frac{1}{2}} \\ &= \int_{\partial\mathcal{C}_{r_0}^\pm} q \cdot (v_{\frac{1}{2}})_\nu - v_{\frac{1}{2}} \cdot q_\nu. \end{aligned}$$

With the definition of q and orthogonality, we have that

$$\begin{aligned} \int_{\mathbb{S}^2} f_1 v_{\frac{1}{2}} dH^1|_{\partial\mathcal{L}_\eta} &= b_1^+ \varepsilon^4 \int_{\partial\mathcal{C}_{r_0}^+} [v_{-\frac{1}{2}}(v_{\frac{1}{2}})_\nu - v_{\frac{1}{2}}(v_{-\frac{1}{2}})_\nu] \\ &\quad + b_1^- \varepsilon^4 \int_{\partial\mathcal{C}_{r_0}^-} [v_{-\frac{1}{2}}(v_{\frac{1}{2}})_\nu - v_{\frac{1}{2}}(v_{-\frac{1}{2}})_\nu]. \end{aligned} \quad (5.23)$$

By direct computation, we have that

$$\lim_{r_0 \rightarrow 0} \int_{\partial\mathcal{C}_{r_0}^+} v_{-\frac{1}{2}}(v_{\frac{1}{2}})_\nu = -A_1 = - \lim_{r_0 \rightarrow 0} \int_{\partial\mathcal{C}_{r_0}^+} v_{\frac{1}{2}}(v_{-\frac{1}{2}})_\nu,$$

where A_1 is a positive constant. Thus

$$\int_{\partial\mathcal{C}_{r_0}^+} [v_{-\frac{1}{2}}(v_{\frac{1}{2}})_\nu - v_{\frac{1}{2}}(v_{-\frac{1}{2}})_\nu] \rightarrow -2A_1 \text{ as } r_0 \rightarrow 0.$$

Now note that $v_{\frac{1}{2}}$ is odd with respect to the x_1 -variable and $v_{-\frac{1}{2}}$ is even with respect to the x_1 -variable, we have that

$$\int_{\partial\mathcal{C}_{r_0}^-} [v_{-\frac{1}{2}}(v_{\frac{1}{2}})_\nu - v_{\frac{1}{2}}(v_{-\frac{1}{2}})_\nu] \rightarrow 2A_1 \text{ as } r_0 \rightarrow 0.$$

Consequently, sending $r_0 \rightarrow 0$ in (5.23), we have that

$$\int_{\mathbb{S}^2} f_1 v_{\frac{1}{2}} dH^1|_{\partial\mathcal{L}_\eta} = 2A_1(b_1^- - b_1^+) \varepsilon^4.$$

A similar argument gives $\int_{\mathbb{S}^2} f_1 v_{\frac{3}{2}} dH^1|_{\partial\mathcal{L}_\eta} = 2A_2(b_2^- - b_2^+) \varepsilon^5$ for a positive constant A_2 . The lower bound for $\|\varphi_1\|_{L^\infty(\mathbb{S}^2)}$ follows from these two equations together with the definition in (5.16). $\square$

We now control the Weiss energy from (2.6) for the replacement $\bar{p}$ as follows:

Lemma 5.6. *Suppose that p satisfies (5.3) and $\bar{p}$ is as in Definition 5.2. Then*

$$W_{\frac{7}{2}}(\bar{p}; 1) \leq C\varepsilon^6$$

for a universal C .

Proof. Similarly to Lemma 4.5, it suffices to prove the upper bound for the following quantity

$$\begin{aligned} \int_{\mathbb{S}^2} (|\nabla_{\mathbb{S}^2} \bar{p}|^2 - \lambda_{\frac{7}{2}} \bar{p}^2) &= \int_{\mathbb{S}^2} (|\nabla_{\mathbb{S}^2} \bar{p}|^2 - \lambda_{\frac{7}{2}} \bar{p}^2) - (|\nabla_{\mathbb{S}^2} p|^2 - \lambda_{\frac{7}{2}} p^2) \\ &= \int_{\mathbb{S}^2} -(\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}}) \bar{p} \cdot (\bar{p} - p) - \int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}}) p \cdot (\bar{p} - p) \\ &= - \int_{\mathbb{S}^2} (f_1 + f_2^\pm + g^\pm) (\bar{p} - p) - \int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}}) p \cdot (\bar{p} - p). \end{aligned} \quad (5.24)$$

For the second term, we note that $(\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}}) p \cdot (\bar{p} - p)$ is supported in $\{-M\varepsilon \leq x_2 \leq 0\} \cap \{x_3 = 0\}$ by Lemma 5.2. On this segment, we have $(\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}}) p \sim r^{\frac{5}{2}} + \varepsilon r^{\frac{3}{2}} + \varepsilon^2 r^{\frac{1}{2}} + \varepsilon^3 r^{-\frac{1}{2}}$, thus

$$\begin{aligned} & \left| \int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}}) p \cdot (\bar{p} - p) \right| \\ & \leq C \|\bar{p} - p\|_{L^\infty(\mathcal{C}_\eta^\pm)} \cdot \int_0^\varepsilon r^{\frac{5}{2}} + \varepsilon r^{\frac{3}{2}} + \varepsilon^2 r^{\frac{1}{2}} + \varepsilon^3 r^{-\frac{1}{2}} \leq C\varepsilon^7 \end{aligned} \quad (5.25)$$

since $\|\bar{p} - p\|_{L^\infty(\mathcal{C}_\eta^\pm)} \leq C\varepsilon^{\frac{7}{2}}$ by Lemma 5.2.

For $\int_{\mathbb{S}^2} (f_1 + f_2^\pm) (\bar{p} - p)$ from (5.24), we notice that on the support of f_j , we have $\bar{p} = p_{ext}$ and $|p_{ext} - p| = O(\varepsilon^4)$ by definition. With $|f_j| \leq C\varepsilon^4$ from Lemma 5.5, we have that

$$\left| \int_{\mathbb{S}^2} (f_1 + f_2^\pm) (\bar{p} - p) \right| \leq C\varepsilon^8. \quad (5.26)$$

It remains to control $-\int_{\mathbb{S}^2} g^\pm \cdot (\bar{p} - p)$. To this end, we have that

$$\begin{aligned} - \int_{\mathbb{S}^2} g^\pm \cdot (\bar{p} - p) &= - \int_{\mathcal{C}_\eta^\pm} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}}) (\bar{p} - p) \cdot (-p) \\ &= \int_{\mathcal{C}_\eta^\pm} (\bar{p} - p) (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}}) p - \int_{\partial \mathcal{C}_\eta^\pm} (\bar{p} - p)_\nu \cdot p \\ &\quad + \int_{\partial \mathcal{C}_\eta^\pm} (\bar{p} - p) \cdot p_\nu \\ &\leq - \int_{\partial \mathcal{C}_\eta^\pm} (\bar{p} - p)_\nu \cdot p + \int_{\partial \mathcal{C}_\eta^\pm} (\bar{p} - p) \cdot p_\nu + C\varepsilon^7 \end{aligned}$$

$$\begin{aligned}
&= - \int_{\partial C_\eta^\pm} (\bar{p} - p_{ext})_v \cdot p - \int_{\partial C_\eta^\pm} (p_{ext} - p)_v \cdot p \\
&\quad + \int_{\partial C_\eta^\pm} (\bar{p} - p_{ext}) \cdot p_v + \int_{\partial C_\eta^\pm} (p_{ext} - p) \cdot p_v + C\varepsilon^7
\end{aligned}$$

where we used (5.25) for the second to last line.

Note that $(\bar{p} - p_{ext})_v = O(\varepsilon^6)$ along $\partial C_\eta^\pm$ by Lemma 5.5, this implies

$$- \int_{\mathbb{S}^2} g^\pm \cdot (\bar{p} - p) \leq - \int_{\partial C_\eta^\pm} (p_{ext} - p)_v \cdot p + \int_{\partial C_\eta^\pm} (p_{ext} - p) \cdot p_v + C\varepsilon^6. \quad (5.27)$$

On the other hand, using orthogonality, we have that

$$\begin{aligned}
\int_{\partial C_\eta^\pm} (p_{ext} - p)_v \cdot p &= \int_{\partial C_\eta^\pm} (b_1^\pm \varepsilon^4 v_{-\frac{1}{2}})_v \cdot a_3 \varepsilon^3 v_{\frac{1}{2}} \\
&\quad + (b_2^\pm \varepsilon^5 v_{-\frac{3}{2}})_v \cdot a_2 \varepsilon^2 v_{\frac{3}{2}} = O(\varepsilon^7).
\end{aligned} \quad (5.28)$$

Similarly,

$$\int_{\partial C_\eta^\pm} (p_{ext} - p) \cdot p_v = O(\varepsilon^7). \quad (5.29)$$

Putting (5.28) and (5.29) into (5.27), we have that

$$- \int_{\mathbb{S}^2} g^\pm \cdot (\bar{p} - p) \leq C\varepsilon^6.$$

Together with (5.24), (5.25) and (5.26), this implies the desired control. $\square$

The following lemma explains the main reason why it is preferable to work with $\bar{p}$ instead of p or p_{ext} .

Lemma 5.7. *Suppose that p satisfies (5.3). Let u be a solution to (1.1) in B_1 , then*

$$\|u - \bar{p}\|_{L^\infty(B_{1/2})} + \|u - \bar{p}\|_{H^1(B_{1/2})} \leq C(\|u - \bar{p}\|_{L^1(B_1)} + \kappa_p)$$

and

$$W_{\frac{7}{2}}\left(u; \frac{1}{2}\right) \leq C(\|u - \bar{p}\|_{L^1(B_1)}^2 + \kappa_p^2 + \varepsilon^6).$$

Similarly to Definition 4.2, we define the class of well-approximated solutions as follows:

Definition 5.3. Suppose that the coefficients of p satisfy (5.3).

For $d, \rho \in (0, 1]$, we say that u is a *solution d -approximated by p at scale ρ* if u solves the thin obstacle problem (1.1) in B_ρ , and

$$|u - \bar{p}| \leq d\rho^{\frac{7}{2}} \text{ in } B_\rho.$$

In this case, we write

$$u \in \mathcal{S}(p, d, \rho).$$

Similarly to Lemma 3.1 and Lemma 4.6, we can localize the contact set of well-approximated solutions.

Lemma 5.8. *Suppose that $u \in \mathcal{S}(p, d, 1)$ with p satisfying (5.3) and $d \leq \varepsilon^3$. We have that*

$$\Delta u = 0 \text{ in } \widehat{B}_1 \cap \{r > C\varepsilon\}, \text{ and } u = 0 \text{ in } \widetilde{B}_{\frac{7}{8}} \cap \{r > C\varepsilon^{\frac{2}{5}}\}.$$

5.3. The trichotomy near $\mathcal{A}_3$

With all these preparations, we prove the trichotomy as stated in Lemma 5.1. Steps similar to those in the proofs of Lemma 3.4 and Lemma 4.8 are omitted.

Proof of Lemma 5.1. For c_0 and ρ_0 to be chosen, depending on σ , suppose that the lemma fails, we find a sequence (u_n, p_n, d_n) satisfying the assumptions from the lemma with $d_n \rightarrow 0$ and $\varepsilon_{p_n} \rightarrow 0$, but none of the three possibilities happens, namely,

$$W_{\frac{7}{2}}(u_n; 1) - W_{\frac{7}{2}}(u_n; \rho_0) \leq c_0^2 d_n^2 \text{ for all } n, \quad (5.30)$$

$$u_n \notin \mathcal{S}(p', \frac{1}{4}d_n, \rho_0) \quad (5.31)$$

for any p' satisfying the properties as in alternative (2) from the lemma, and

$$d_n \geq \varepsilon_{p_n}^5. \quad (5.32)$$

A direct consequence of Lemma 5.5 is that

$$\kappa_{2,p_n} \leq C d_n \cdot \varepsilon_{p_n} = d_n o(1). \quad (5.33)$$

With the same reasoning as in Remark 5.5, it suffices to consider the case when

$$d_n \leq \varepsilon_{p_n}^3.$$

We omit the subscript in the remaining of the proof.

Define $\hat{u} = \frac{1}{d+\kappa_1+\kappa_2}(u - \bar{p} + \Phi_1 + \Phi_2)$, with the parameters and auxiliary functions from Subsection 5.2. Similar to the proofs for Lemma 3.4 and Lemma 4.8, for each $m = 0, 1, 2, 3$, we find an $(m + \frac{1}{2})$ -homogeneous harmonic function $h_{m+\frac{1}{2}}$ such that

$$\|\hat{u} - (h_{\frac{1}{2}} + h_{\frac{3}{2}} + h_{\frac{5}{2}} + h_{\frac{7}{2}})\|_{L^1(B_{2\rho_0})} = C\rho_0^{\frac{15}{2}} + o(1). \quad (5.34)$$

We also have that

$$\|[(\hat{u})_{(1/2)} - \hat{u}] - (7h_{\frac{1}{2}} + 3h_{\frac{3}{2}} + h_{\frac{5}{2}})\|_{L^2(\partial B_{\rho_0})} \leq C\rho_0^{\frac{11}{2}} + o(1).$$

Recall that $f_{(1/2)}$ denotes the rescaling of f as in (2.5). With the definitions of $\hat{u}$, Φ_1 and (5.30), this implies

$$\left\| \frac{1}{d + \kappa_1 + \kappa_2} \left[c\varphi_1 \left(\frac{x}{|x|} \right) |x|^{\frac{7}{2}} + (\Phi_2)_{(1/2)} - \Phi_2 \right] - (7h_{\frac{1}{2}} + 3h_{\frac{3}{2}} + h_{\frac{5}{2}}) \right\|_{L^2(\partial B_{\rho_0})} \leq C\rho_0^{\frac{9}{2}}(c_0 + \rho_0) + o(1).$$

Now with (5.33), we have $\|\Phi_2\|_{L^\infty(B_1)} = d \cdot o(1)$. The previous estimate leads to

$$\left\| \frac{c}{d + \kappa_1 + \kappa_2} \varphi_1 \left(\frac{x}{|x|} \right) |x|^{\frac{7}{2}} - (7h_{\frac{1}{2}} + 3h_{\frac{3}{2}} + h_{\frac{5}{2}}) \right\|_{L^2(\partial B_{\rho_0})} \leq C\rho_0^{\frac{9}{2}}(c_0 + \rho_0) + o(1).$$

From here, we apply orthogonality and Lemma 5.5 to conclude that

$$\kappa_1 \leq Cd(c_0 + \rho_0 + o(1)). \quad (5.35)$$

Consequently,

$$\|\Phi_1\|_{L^1(B_{2\rho_0})} \leq Cd[(c_0 + \rho_0)\rho_0^{\frac{13}{2}} |\log \rho_0| + o(1)].$$

Putting this into (5.34), we have that

$$\|u - \bar{p} - (d + \kappa_1 + \kappa_2)h_{\frac{7}{2}}\|_{L^1(B_{2\rho_0})} \leq Cd[(c_0 + \rho_0)\rho_0^{\frac{13}{2}} |\log \rho_0| + o(1)].$$

With Lemma 5.4 and Lemma 5.7, if we take $p' = p + (d + \kappa_1 + \kappa_2)h_{\frac{7}{2}}$, then

$$\|u - \bar{p}'\|_{L^\infty(B_{\rho_0})} \leq Cd[(c_0 + \rho_0)\rho_0^{\frac{7}{2}} |\log \rho_0| + o(1)].$$

Note that we used (5.32) to absorb the $O(\varepsilon^6)$ error from Lemma 5.4.

Choosing c_0, ρ_0 universally small and n large, we have that

$$u \in \mathcal{S}(p', \frac{1}{4}d, \rho_0).$$

With (5.35), we apply Corollary 5.1 with $a = \frac{1}{4}\sigma$ to get that

$$\kappa_{1,p'} \leq \kappa_1 + \frac{1}{4}\sigma d + C_\sigma \varepsilon d \leq \frac{1}{4}\sigma d + Cd(c_0 + \rho_0 + o(1)).$$

Choosing now c_0, ρ_0 small, depending on σ , we have $\kappa_{1,p'} < \frac{1}{2}\sigma d$, which implies

$$\kappa_{p'} < \sigma d$$

for large n by (5.33), contradicting (5.31). $\square$

5.4. Almost symmetric solutions

Based on Lemma 5.5, the term $f_1 dH^1|_{\partial\mathcal{L}_\eta}$ from (5.15) has a significant projection into $\mathcal{H}_{\frac{7}{2}}$. When $\kappa_1 \geq \varepsilon^5$, we can apply Lemma 5.5 to see that $\kappa_1 \gg \kappa_2$. In this case, κ_2 is negligible, and the lower bound on the projection is what leads to possibilities (1) and (2) in Lemma 5.1.

When $\kappa_1 \ll \varepsilon^5$, this argument no longer works. In this case, the extended profile p_{ext} from Definition 5.1 is ‘almost continuous’ along the big circle $\{x_1 = 0\} \cap \mathbb{S}^2$ since $\kappa_1 \sim |b_1^+ - b_1^-|\varepsilon^4 + |b_2^+ - b_2^-|\varepsilon^5$ by Lemma 5.5. This leads to information about the profile by our analysis of the problem in $\mathbb{R}^2$.

Lemma 5.9. *For a_j, ε satisfying (5.3), we set p_{ext} be as in Definition 5.1, $b_j^\pm$ as in Remark 5.4, and $\bar{p}$ as in Definition 5.2.*

Given any $\gamma > 0$, we can find two small constants, σ_γ and ε_γ , depending on γ , such that

if

$$\varepsilon < \varepsilon_\gamma \text{ and } |b_1^+ - b_1^-| + |b_2^+ - b_2^-|\varepsilon < \sigma_\gamma \cdot \varepsilon,$$

then

$$\|\bar{p} - U_{\tau\varepsilon}(\bar{q})\|_{L^\infty(B_1)} \leq \gamma\varepsilon^5 \quad (5.36)$$

for

$$q = a'_0 u_{\frac{7}{2}} + a'_1 \varepsilon v_{\frac{5}{2}} + a'_2 \varepsilon^2 v_{\frac{3}{2}} + a'_3 \varepsilon^3 v_{\frac{1}{2}}$$

satisfying

$$|a'_0 - a_0| < C\varepsilon^2, \quad |a'_j| \leq C, \quad |\tau| \leq C \text{ for a universal } C,$$

and

$$a'_2 > -\gamma, \quad |a'_3| < \gamma, \text{ and } (a'_1/a'_0)^2 < \frac{85}{24} \cdot a'_2/a'_0 + \gamma. \quad (5.37)$$

Recall the rotation operator U from (2.4).

Compare with Remark 5.3, we see from (5.37) that q ‘almost solves’ the thin obstacle problem up to an error of size γ .

Proof. Let $\delta > 0$ be a small constant to be chosen, depending on γ .

With Lemma 5.3, we have

$$\left| b_1^+ - a_0 \cdot b_1^{\mathbb{R}^2} \left[\frac{a_1}{a_0}, \frac{a_2}{a_0}, \frac{a_3}{a_0} \right] \right| < \omega(\varepsilon), \text{ and } \left| b_1^- - a_0 \cdot b_1^{\mathbb{R}^2} \left[-\frac{a_1}{a_0}, \frac{a_2}{a_0}, -\frac{a_3}{a_0} \right] \right| < \omega(\varepsilon).$$

With our assumption on $|b_1^+ - b_1^-|$ and $a_0 \in [\frac{1}{2}, 2]$, this implies

$$\left| b_1^{\mathbb{R}^2} \left[\frac{a_1}{a_0}, \frac{a_2}{a_0}, \frac{a_3}{a_0} \right] - b_1^{\mathbb{R}^2} \left[-\frac{a_1}{a_0}, \frac{a_2}{a_0}, -\frac{a_3}{a_0} \right] \right| < 4(\omega(\varepsilon) + \sigma).$$

Similarly, we have that

$$\left| b_2^{\mathbb{R}^2} \left[\frac{a_1}{a_0}, \frac{a_2}{a_0}, \frac{a_3}{a_0} \right] + b_2^{\mathbb{R}^2} \left[-\frac{a_1}{a_0}, \frac{a_2}{a_0}, -\frac{a_3}{a_0} \right] \right| < 4(\omega(\varepsilon) + \sigma).$$

The change of sign in front of $b_2^{\mathbb{R}^2}$ is due to the odd symmetry of $v_{-\frac{3}{2}}$ with respect to $\{x_1 = 0\}$.

We perform a change of basis (see (2.16) and (2.19))

$$\begin{aligned} p_{ext} &= a_0 u_{\frac{7}{2}} + \cdots + b_1^{\pm} \varepsilon^4 \chi_{\{\pm x_1 > 0\}} v_{-\frac{1}{2}} + b_2^{\pm} \varepsilon^5 \chi_{\{\pm x_1 > 0\}} v_{-\frac{3}{2}} \\ &= \tilde{a}_0 u_{\frac{7}{2}} + \tilde{a}_1 \varepsilon w_{\frac{5}{2}} + \tilde{a}_2 \varepsilon^2 w_{\frac{3}{2}} + \tilde{a}_3 \varepsilon^3 w_{\frac{1}{2}} + \tilde{b}_1^{\pm} \varepsilon^4 \chi_{\{\pm x_1 > 0\}} w_{-\frac{1}{2}} \\ &\quad + \tilde{b}_2^{\pm} \varepsilon^5 \chi_{\{\pm x_1 > 0\}} w_{-\frac{3}{2}}. \end{aligned}$$

With Corollary B.2, if ε and σ are small, depending on δ , then we find universally bounded τ , α_j and β_j satisfying that

$$|\alpha_0 - \tilde{a}_0| \leq C\varepsilon^2, \quad \alpha_2 \geq -\delta, \quad |\alpha_3| < \delta, \quad \text{and} \quad \alpha_1^2 \leq \frac{12}{5}\alpha_2 + \delta, \quad (5.38)$$

and

$$|p_{ext} - U_{-\tau\varepsilon}(f)| \leq C_{\rho_0} \delta \varepsilon^5 \text{ in } \{r > \rho_0\} \cap \{x_1 > C|\tau|\varepsilon\}, \quad (5.39)$$

where $\rho_0 > 0$ is a small parameter to be chosen, and

$$f = \alpha_0 u_{\frac{7}{2}} + \alpha_1 \varepsilon w_{\frac{5}{2}} + \alpha_2 \varepsilon^2 w_{\frac{3}{2}} + \alpha_3 \varepsilon^3 w_{\frac{1}{2}} + \beta_1 \varepsilon^4 w_{-\frac{1}{2}} + \beta_2 \varepsilon^5 w_{-\frac{3}{2}}.$$

If we take $q = \alpha_0 u_{\frac{7}{2}} + \alpha_1 \varepsilon w_{\frac{5}{2}} + \alpha_2 \varepsilon^2 w_{\frac{3}{2}} + \alpha_3 \varepsilon^3 w_{\frac{1}{2}}$, and suppose

$$q_{ext} = q + \gamma_1 \varepsilon^4 w_{-\frac{1}{2}} + \gamma_2 \varepsilon^5 w_{-\frac{3}{2}},$$

then $\bar{p}$ and $U_{-\tau\varepsilon}(\bar{q})$ both solve (5.5) in $\mathcal{C}_{\eta-C|\tau|\varepsilon}^+$. With (5.39) and

$$\begin{aligned} \bar{p} - U_{-\tau\varepsilon}(\bar{q}) &= (\bar{p} - p_{ext}) + (p_{ext} - U_{-\tau\varepsilon}(f)) + (U_{-\tau\varepsilon}(f - q_{ext})) \\ &\quad + (U_{-\tau\varepsilon}(q_{ext} - \bar{q})), \end{aligned}$$

we can apply the same argument as in Step 1 from the proof for Lemma 5.4 to conclude

$$|\gamma_1 - \beta_1| \varepsilon^4 + |\gamma_2 - \beta_2| \varepsilon^5 \leq C_{\rho_0} \delta \varepsilon^5$$

if ρ_0 is fixed small enough.

With (5.39), this implies

$$|p_{ext} - U_{-\tau\varepsilon}(q_{ext})| \leq C_{\rho_0} \delta \varepsilon^5 \text{ in } \{r > \rho_0\} \cap \{x_1 > C|\tau|\varepsilon\}.$$

An application of the maximum principle leads to

$$|\bar{p} - U_{-\tau\varepsilon}(\bar{q})| \leq C_{\rho_0} \delta \varepsilon^5 \text{ in } \{x_1 > C|\tau|\varepsilon\}.$$

A similar argument gives $|\bar{p} - U_{-\tau\varepsilon}(\bar{q})| \leq C_{\rho_0} \delta \varepsilon^5$ in $\{x_1 < -C|\tau|\varepsilon\}$. Together with the estimate in $\{x_1 > C|\tau|\varepsilon\}$, we can apply the maximum principle in $\mathcal{L}_\eta$ to get

$$|\bar{p} - U_{-\tau\varepsilon}(\bar{q})| < C_{\rho_0} \delta \varepsilon^5 \text{ in } \mathbb{S}^2.$$

Choosing δ small, depending on γ , we have (5.36). The constraint (5.37) follows from (5.38). $\square$

Remark 5.6. For $q = a'_0 u_{\frac{7}{2}} + a'_1 \varepsilon v_{\frac{5}{2}} + a'_2 \varepsilon^2 v_{\frac{3}{2}} + a'_3 \varepsilon^3 v_{\frac{1}{2}}$ as in the statement of Lemma 5.9, if we assume

$$a'_2 \geq \frac{1}{24},$$

then we can use the same argument as in Step 3 of the proof for Lemma 3.4 to find τ and $\hat{a}_j$ with $|\tau| = O(\gamma)$, and $|\hat{a}_j - a'_j| = O(\gamma)$ such that

$$\hat{q} = a'_0 u_{\frac{7}{2}} + \hat{a}_1 \varepsilon v_{\frac{5}{2}} + \hat{a}_2 \varepsilon^2 v_{\frac{3}{2}}$$

satisfies

$$\hat{q}|_{\{x_3=0\}} \geq 0, \quad \Delta \hat{q}|_{\{x_3=0\}} \leq 0,$$

and

$$|q - U_{\tau\varepsilon}(\hat{q})| \leq C\gamma\varepsilon^3 \text{ in } \{r > \eta/2\} \cap B_1.$$

Moreover, if γ and ε are universally small, then we have $\hat{a}_2 \geq \frac{1}{24} - O(\gamma) \geq 0$, and

$$\begin{aligned} \left(\frac{\hat{a}_1}{a'_0}\right)^2 - 4\frac{\hat{a}_2}{a'_0} \left(1 - \frac{1}{5}\frac{\hat{a}_2\varepsilon^2}{a'_0}\right) &\leq \left(\frac{a'_1}{a'_0}\right)^2 - 4\frac{a'_2}{a'_0} + O(\gamma + \varepsilon) \\ &\leq \frac{-11}{24}\frac{a'_2}{a'_0} + O(\gamma + \varepsilon) \\ &\leq -\left(\frac{1}{24}\right)^2. \end{aligned}$$

Note that we used $a'_2 \geq \frac{1}{24}$ together with (5.37).

Consequently, if $\hat{a}_1 \geq 0$, then *small perturbations* from $\hat{q}$ solve the thin obstacle problem outside a cone of $O(\varepsilon)$ -opening near $\{r = 0, x_1 > 0\}$ (see Remark 5.3).

5.5. One-sided replacement

For profile of the form $\hat{q}$ as in Remark 5.6, that is,

$$p = a_0 u_{\frac{7}{2}} + a_1 \varepsilon v_{\frac{5}{2}} + a_2 \varepsilon^2 v_{\frac{3}{2}}$$

with

$$\begin{aligned} a_0 &\in \left[\frac{1}{2}, 2 \right], \quad a_1 \geq 0, \quad a_2 \geq \frac{1}{24}, \quad \text{and} \\ \left(\frac{a_1}{a_0} \right)^2 - 4 \frac{a_2}{a_0} \left(1 - \frac{1}{5} \frac{a_2 \varepsilon^2}{a_0} \right) &\leq - \left(\frac{1}{24} \right)^2, \end{aligned} \quad (5.40)$$

we only need to replace it in a small spherical cap near $\{r = 0, x_1 > 0\}$.

To this end, we solve in $\mathcal{C}_\eta^+$ from (5.2) the following:

$$\begin{cases} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})v_p \leq 0 & \text{in } \mathcal{C}_\eta^+, \\ v_p \geq 0 & \text{in } \mathcal{C}_\eta^+ \cap \{x_3 = 0\}, \\ (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})v_p = 0 & \text{in } \mathcal{C}_\eta^+ \cap (\{v_p > 0\} \cup \{x_3 \neq 0\}), \\ v_p = p & \text{along } \partial \mathcal{C}_\eta^+. \end{cases} \quad (5.41)$$

With this, we define the replacement of p as

Definition 5.4. For p satisfying (5.40), its *replacement*, $\tilde{p}$, is defined as

$$\tilde{p} = \begin{cases} v_p & \text{in } \mathcal{C}_\eta^+, \\ p & \text{in } \mathbb{S}^2 \setminus \mathcal{C}_\eta^+. \end{cases}$$

Equivalently, the replacement $\tilde{p}$ is the minimizer of the energy

$$v \mapsto \int_{\mathbb{S}^2} |\nabla_{\mathbb{S}^2} v|^2 - \lambda_{\frac{7}{2}} v^2$$

over $\{v : v = p \text{ outside } \mathcal{C}_\eta^+, \text{ and } v \geq 0 \text{ on } \{x_3 = 0\}\}$.

This replacement satisfies

$$\begin{cases} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})\tilde{p} = f_p dH^1|_{\partial \mathcal{C}_\eta^+} + g_p dH^1|_{\mathcal{C}_\eta^+ \cap \{x_3=0\}} & \text{in } \widehat{\mathbb{S}^2} \cup \mathcal{C}_\eta^+, \\ \tilde{p} = 0 & \text{in } \widetilde{\mathbb{S}^2} \setminus \mathcal{C}_\eta^+. \end{cases} \quad (5.42)$$

Similar to Subsection 4.1, we define some auxiliary functions.

The projection of f_p into $\mathcal{H}_{\frac{7}{2}}$ (see (2.12)) is denoted by φ_p .

The function $H_p : \mathbb{S}^2 \rightarrow \mathbb{R}$ is the unique solution to

$$\begin{cases} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})H_p = f_p dH^1|_{\partial \mathcal{C}_\eta^+} - \varphi_p & \text{on } \widehat{\mathbb{S}^2}, \\ H_p = 0 & \text{on } \widetilde{\mathbb{S}^2}. \end{cases}$$

Corresponding to these, we define

$$\Phi_p := H_{1,p}\left(\frac{x}{|x|}\right)|x|^{\frac{7}{2}} + \frac{1}{8}\varphi_{1,p}\left(\frac{x}{|x|}\right)|x|^{\frac{7}{2}}\log(|x|),$$

which satisfies

$$\begin{cases} \Delta \Phi_p = f_p dH^2|_{\partial C_\eta^+} & \text{in } \widehat{\mathbb{R}^3}, \\ \Phi_p = 0 & \text{on } \widetilde{\mathbb{R}^3}. \end{cases}$$

Finally, let us denote

$$\kappa_p := \|\Phi_p\|_{L^\infty(B_1)}.$$

With a similar argument as to that in Subsection 4.1, we have the following properties:

Lemma 5.10. *For p satisfying (5.40), we have that*

$$\tilde{p} = 0 \text{ in } (C_\eta^+ \setminus C_{C\varepsilon}^+)',$$

where C is a universal constant.

Lemma 5.11. *Suppose that p satisfies (5.40), and take $q = \alpha_0 u_{\frac{7}{2}} + \alpha_1 v_{\frac{5}{2}} + \alpha_2 v_{\frac{3}{2}} + \alpha_3 v_{\frac{1}{2}}$ with $|\alpha_j| \leq 1$.*

Then we can find a modulus of continuity, $\omega(\cdot)$, such that

$$\|p + \widetilde{dq} - (\tilde{p} + dq)\|_{L^\infty(\mathbb{S}^2)} \leq \omega(\varepsilon + d) \cdot d.$$

Corollary 5.2. *Under the same assumption as in Lemma 5.11, we have that*

$$\kappa_{p+dq} \leq \kappa_p + \omega(\varepsilon + d) \cdot d.$$

By directly computing the inner product of f and $v_{\frac{1}{2}}$, we have that

Lemma 5.12. *Suppose that p satisfies (5.40). Then*

$$\|\varphi\|_{L^\infty(\mathbb{S}^2)} \geq c\kappa$$

for a universal $c > 0$.

Note that $p \geq 0$ in $C_\eta^+ \cap \{x_3 = 0\}$, thus p is admissible in the minimization problem in Definition 5.4, we have

Lemma 5.13. *Suppose that p satisfies (5.40), then $W_{\frac{7}{2}}(\tilde{p}; 1) \leq 0$.*

This implies

Lemma 5.14. *Suppose that p satisfies (5.40). Let u be a solution to (1.1) in B_1 , then*

$$\|u - \tilde{p}\|_{L^\infty(B_{1/2})} + \|u - \tilde{p}\|_{H^1(B_{1/2})} \leq C(\|u - \tilde{p}\|_{L^1(B_1)} + \kappa)$$

and

$$W_{\frac{7}{2}}(u; \frac{1}{2}) \leq C(\|u - \tilde{p}\|_{L^1(B_1)}^2 + \kappa^2).$$

Similarly to Subsection 4.2, we have that

Definition 5.5. Suppose that the coefficients of p satisfy (5.40).

For $d, \rho \in (0, 1]$, we say that u is a *solution d -approximated by p at scale ρ* if u solves the thin obstacle problem (1.1) in B_ρ , and

$$|u - \tilde{p}| \leq d\rho^{\frac{7}{2}} \text{ in } B_\rho.$$

In this case, we write

$$u \in \mathcal{S}(p, d, \rho).$$

We can localize the contact set of well-approximated solutions

Lemma 5.15. Suppose that $u \in \mathcal{S}(p, d, 1)$ with d small, and that p satisfies (5.40).

We have that

$$\Delta u = 0 \text{ in } \widehat{B}_1 \cap \{r > Cd^{\frac{2}{7}}\},$$

and

$$u = 0 \text{ in } \widetilde{B}_{\frac{7}{8}} \cap \{r > C(d + \varepsilon)^\alpha\}$$

for universal small $\alpha > 0$ and big $C > 0$.

With these preparations, we have the following dichotomy similar to the one in Subsection 4.3:

Lemma 5.16. Suppose that

$$u \in \mathcal{S}(p, d, 1)$$

for some $p = a_0 u_{\frac{7}{2}} + a_1 \varepsilon v_{\frac{5}{2}} + a_2 \varepsilon^2 v_{\frac{3}{2}}$ satisfying (5.40).

There is a universal small constant $\tilde{\varepsilon} > 0$, such that if $\varepsilon < \tilde{\varepsilon}$ and $d \leq \varepsilon^3$, then we have the following dichotomy:

1. either

$$W_{\frac{7}{2}}(u; 1) - W_{\frac{7}{2}}(u; \rho_0) \geq c_0^2 d^2$$

and

$$u \in \mathcal{S}(p, Cd, \rho_0);$$

2. or

$$u \in \mathcal{S}(p', \frac{1}{2}d, \rho_0)$$

for some

$$p' = U_{\tau\varepsilon}[a'_0 u_{\frac{7}{2}} + a'_1 \varepsilon v_{\frac{5}{2}} + a'_2 \varepsilon^2 v_{\frac{3}{2}}]$$

with $\kappa_{p'} < d$, and

$$|a'_0 - a_0| \leq Cd, \quad |\tau| + |a'_1 - a_1| + |a'_2 - a_2| \leq Cd/\varepsilon^3.$$

The constants c_0 , ρ_0 and C are universal.

Proof. For c_0 and ρ_0 to be chosen, suppose the lemma is false, we find a sequence $(u_n, p_n, d_n, \varepsilon_n)$ satisfying the hypothesis of the lemma with $d_n, \varepsilon_n \rightarrow 0$, but neither of the two alternatives happens.

This allows us to find

$$q = p + d(\alpha_0 u_{\frac{7}{2}} + \alpha_1 w_{\frac{5}{2}} + \alpha_2 w_{\frac{3}{2}} + \alpha_3 w_{\frac{1}{2}})$$

such that

$$\|u - \tilde{q}\|_{L^1(B_{2\rho_0})} \leq Cd\rho_0^{\frac{13}{2}} [(c_0 + \rho_0)|\log \rho_0| + o(1)]. \quad (5.43)$$

With $a_2 \geq \frac{1}{24}$ as in (5.40), there are τ and a'_j satisfying

$$a'_0 = a_0 + d\alpha_0, \quad |\tau| \leq Cd/\varepsilon^3, \quad \text{and } |a'_j - a_j| \leq Cd/\varepsilon^3 \text{ for } j = 1, 2$$

such that

$$\|q - U_{-\tau\varepsilon}(a'_0 u_{\frac{7}{2}} + a'_1 \varepsilon v_{\frac{5}{2}} + a'_2 \varepsilon^2 v_{\frac{3}{2}})\|_{L^\infty(\mathbb{S}^2 \setminus \mathcal{C}_{\eta/2}^\pm)} \leq C\tau^2 \varepsilon^4 \leq C\varepsilon d.$$

Note that we used our assumption that $d \leq \varepsilon^3$ for the last comparison. If we denote by $p' = a'_0 u_{\frac{7}{2}} + a'_1 \varepsilon v_{\frac{5}{2}} + a'_2 \varepsilon^2 v_{\frac{3}{2}}$, then the maximum principle gives $\|\tilde{q} - \tilde{p}'\|_{L^\infty(\mathbb{S}^2)} \leq C\varepsilon d$. Together with (5.43), we get that

$$\|u - \tilde{p}'\|_{L^1(B_{2\rho_0})} \leq Cd\rho_0^{\frac{13}{2}} [(c_0 + \rho_0)|\log \rho_0| + o(1)].$$

From here the remaining of the argument is similar to the proof of Lemma 4.8. $\square$

6. Proof of Main Results

In this section, we prove our main results, Theorem 1.1, Theorem 1.2 and Theorem 1.3.

We begin with some preparatory propositions.

Proposition 6.1. *For a solution u to the thin obstacle problem (1.1) in $B_1 \subset \mathbb{R}^3$, suppose that its frequency at 0 is $\frac{7}{2}$, and that*

$$|u - u_{\frac{7}{2}}| \leq d \text{ in } B_1.$$

There is a small universal $\tilde{d} > 0$, such that if $d < \tilde{d}$, then up to a normalization

1. *either*

$$|u - u_{\frac{7}{2}}| \leq O(|x|^{\frac{7}{2}} |\log |x||^{-c_0}),$$

2. or

$$|u - p| \leq O(|x|^{\frac{7}{2}+c_0})$$

for some $p \in \mathcal{F}_1 \setminus \{u_{\frac{7}{2}}\}$.

Here $c_0 > 0$ is universal.

Recall the notion of normalization from Remark 1.1. The family of normalized solutions, $\mathcal{F}_1$, is given in (1.7).

Proof. The proof is based on iterations of the trichotomy in Lemma 5.1 and the dichotomy in Lemma 5.16.

To begin with, let γ denote a small universal constant to be chosen, and let σ_γ denote the constant from Lemma 5.9 corresponding to this γ . Choose σ such that $\sigma/\sigma_\gamma \ll 1$. This choice of σ fixes $\tilde{\delta}$, c_0 and ρ_0 as in the statement of Lemma 5.1.

Step 1: Initiation.

For u satisfying the conditions in the proposition, let

$$u_0 := u_{(1/2)}, \quad p_0 := u_{\frac{7}{2}}, \quad d_0 := d, \quad \text{and} \quad w_0 = W_{\frac{7}{2}}(u_0; 1).$$

Recall the notation for rescaling from (2.5), and the Weiss energy functional from (2.6).

Let ε_p be as in (5.1), then $\varepsilon_{p_0} = 0$. According to definitions in Subsection 5.2, we have

$$\kappa_{p_0} = 0,$$

and

$$u_0 \in \mathcal{S}(p_0, d_0, 1).$$

Consequently, if $\tilde{d}$ is small, then $d_0 < \tilde{\delta}$ and $\varepsilon_{p_0} < \tilde{\delta}$ as in Lemma 5.1. Moreover, by Lemma 5.7, we have

$$w_0 \leq C d_0^2. \tag{6.1}$$

Step 2: Induction.

Suppose for $k = 0, \dots, n-1$, we have found (u_k, p_k, d_k) such that $u_k \in \mathcal{S}(p_k, d_k, 1)$, $\varepsilon_{p_k} < \tilde{\delta}$ and $d_k < \tilde{\delta}$. We can apply the trichotomy in Lemma 5.1 to u_{n-1} .

If possibility (1) happens, we let

$$p_n := p_{n-1}, \quad \text{and} \quad d_n = C d_{n-1}.$$

If possibility (2) happens, we let

$$p_n := p', \quad \text{and} \quad d_n = \frac{1}{2} d_{n-1}.$$

In both cases, we let

$$u_n := (u_{n-1})_{(\rho_0)}, \quad \text{and} \quad w_n := W_{\frac{7}{2}}(u_n; 1).$$

We claim that until possibility (3) happens, Lemma 5.1 remains applicable. To be precise, we have

$$\begin{aligned} \text{Claim: Until } d_n \leq \varepsilon_{p_n}^5, \text{ we have } \kappa_{p_k} \leq d_k, \quad w_k \leq C d_k^{6/5}, \\ \text{and } d_k < \tilde{\delta} \text{ for all } k \leq n. \end{aligned} \quad (6.2)$$

To see this claim, we first notice that all three comparisons are true when $k = 0$. It suffices to show that they stay true in the iteration.

Note that each time possibility (1) happens, $\kappa_{p_k} = \kappa_{p_{k-1}}$ and $d_k > d_{k-1}$. Each time possibility (2) happens, we have $\kappa_{p_k} \leq \sigma d_{k-1} = 2\sigma d_k$. We see that if $\sigma < \frac{1}{2}$, then the comparison between κ_{p_k} and d_k stays true.

With this, we can apply Lemma 5.7 to see that

$$w_k \leq C(d_{k-1}^2 + \kappa_{p_{k-1}}^2 + \varepsilon_{p_{k-1}}^6) \leq C(d_{k-1}^2 + \varepsilon_{p_{k-1}}^6).$$

With $d_{k-1} \geq \varepsilon_{p_{k-1}}^5$, we have that

$$w_k \leq C d_{k-1}^{6/5} \leq C d_k^{6/5}.$$

It remains to see the comparison between d_k and $\tilde{\delta}$. Note that d_k decreases if possibility (2) happens, thus we only need to prove the comparison when possibility (1) happens. In this case, using Lemma 2.1 and our assumption that 0 is a point with frequency $\frac{7}{2}$, we have that

$$w_0 \geq w_{k-1} - w_k \geq c_0^2 d_{k-1}^2.$$

With (6.1), this implies that

$$d_k = C d_{k-1} \leq C d_0 \leq C \tilde{d}.$$

Consequently, d_k stays below $\tilde{\delta}$ if $\tilde{d}$ is chosen small.

In summary, the claim (6.2) holds.

From here we see that until $d_n \leq \varepsilon_{p_n}^5$, the double sequence (w_k, d_k) satisfies the conditions in Lemma 2.2 with $\gamma = \frac{1}{5}$. In particular, if $\tilde{d}$ is chosen small, then $\sum d_k$ is small.

Recall that the deviation in the coefficients of p_k is comparable to $\sum d_k$, and that $p_0 = u_{\frac{7}{2}}$. If we denote

$$p_k = a_0^k u_{\frac{7}{2}} + a_1^k v_{\frac{5}{2}} + a_2^k v_{\frac{3}{2}} + a_3^k v_{\frac{1}{2}},$$

then a_0^k stays in $[\frac{1}{2}, 2]$. By choosing $\tilde{d}$ smaller if necessary, we ensure that ε_{p_k} , as defined in (5.1), stays below $\tilde{\delta}$.

Therefore, until $d_n \leq \varepsilon_{p_n}^5$, the conditions in Lemma 5.1 are satisfied, and we can iterate this lemma to continue the sequence (u_n, p_n, d_n) .

In particular, if d_n stays above $\varepsilon_{p_n}^5$ indefinitely, we can apply the same argument in Section 5 of [18] to conclude

$$|u - u_{\frac{7}{2}}| \leq O(|x|^{\frac{7}{2}} |\log |x||^{-c_0})$$

up to a normalization.

We now analyze the case when d_n drops below $\varepsilon_{p_n}^5$.

Step 3: Adjustment when $d_n \leq \varepsilon_{p_n}^5$.

Suppose this happens for the first time at step n in the iteration, then possibility (2) from Lemma 5.1 happens at this step. In particular, we have

$$d_n = \frac{1}{2}d_{n-1}, \quad \kappa_{p_n} \leq \sigma d_{n-1} \leq 2\sigma d_n,$$

and

$$\|u_n - \bar{p}_n\|_{L^\infty(B_1)} \leq \frac{1}{4}d_{n-1}.$$

As a result, we have that

$$\varepsilon_{p_{n-1}}^5 \leq d_{n-1} \leq 2\varepsilon_{p_n}^5.$$

Also note that in this case, the coefficients of p_n deviates from those of p_{n-1} by $O(d_{n-1})$, the definition of ε_p in (5.1) gives

$$\varepsilon_{p_n}^5 \leq \varepsilon_{p_{n-1}}^5 + Cd_{n-1}^{7/5} \leq d_{n-1}(1 + Cd_{n-1}^{2/5}) \leq 2d_{n-1} \quad (6.3)$$

if $\tilde{d}$ is small.

If we denote by

$$p_{n,ext} = p_n + b_1^{\pm,n} \varepsilon_{p_n}^4 \chi_{\{\pm x_1 > 0\}} v_{-\frac{1}{2}} + b_2^{\pm,n} \varepsilon_{p_n}^5 \chi_{\{\pm x_1 > 0\}} v_{-\frac{3}{2}},$$

where the coefficients $b_j^\pm$ are from Remark 5.4, then Lemma 5.5 gives

$$|b_1^{+,n} - b_1^{-,n}| \varepsilon_{p_n}^4 + |b_2^{+,n} - b_2^{-,n}| \varepsilon_{p_n}^5 \leq C\kappa_{p_n} \leq C\sigma d_{n-1} \leq C\sigma \varepsilon_{p_n}^5 < \sigma_\gamma \varepsilon_{p_n}^5$$

by our choice of $\sigma/\sigma_\gamma \ll 1$ before Step 1.

This allows us to apply Lemma 5.9, leading to

$$q = a'_0 u_{\frac{7}{2}} + a'_1 \varepsilon_{p_n} v_{\frac{5}{2}} + a'_2 \varepsilon_{p_n}^2 v_{\frac{3}{2}} + a'_3 \varepsilon_{p_n}^3 v_{\frac{1}{2}} \quad (6.4)$$

such that

$$\|\bar{p}_n - U_{\tau \varepsilon_{p_n}}(\bar{q})\|_{L^\infty(B_1)} \leq \gamma \varepsilon_{p_n}^5. \quad (6.5)$$

Depending on the size of a'_2 , we divide the discussion into two cases.

Step 4: The case when $a'_2 \leq \frac{1}{16}$ each time the adjustment in Step 3 is made.

In this case, we define $p'_n = q$, then up to a rotation, we have

$$\|u_n - \bar{p}'_n\|_{L^\infty(B_1)} \leq \frac{1}{4}d_{n-1} + \gamma \varepsilon_{p_n}^5 \leq \frac{1}{4}d_{n-1} + 2\gamma d_{n-1} \leq \frac{1}{2}d_{n-1}$$

if γ is small. Note that we used (6.3) and (6.5).

In particular, with $d_n = \frac{1}{2}d_{n-1}$ we still have that

$$u_n \in \mathcal{S}(p'_n, d_n, 1).$$

Meanwhile, in this case, Lemma 5.9 gives that

$$|a'_1|^2 \leq \frac{84}{25}|a'_2| + \gamma \leq \frac{1}{4}, \quad |a'_2| \leq \frac{1}{16}, \quad \text{and} \quad |a'_3| \leq \gamma.$$

By (6.4) and (5.1), we have that

$$\varepsilon_{p'_n} \leq \frac{1}{2} \varepsilon_{p_n}.$$

Consequently, if $a'_2 \leq \frac{1}{16}$ each time the adjustment happens, after this adjustment, we have that

$$d_n = \frac{1}{2} d_{n-1} \geq \frac{1}{4} \varepsilon_{p_n}^5 \geq 8 \varepsilon_{p'_n}^5,$$

and the induction in Step 2 can be continued, leading to

$$|u - u_{\frac{7}{2}}| \leq O(|x|^{\frac{7}{2}} |\log |x||^{-c_0})$$

with the argument in Section 5 of [18].

Step 5: The case when $a'_2 \geq \frac{1}{16}$ at one time the adjustment in Step 3 is made.

Suppose after the adjustment described in Step 3, we have that

$$q = a'_0 u_{\frac{7}{2}} + a'_1 \varepsilon_{p_n} v_{\frac{5}{2}} + a'_2 \varepsilon_{p_n}^2 v_{\frac{3}{2}} + a'_3 \varepsilon_{p_n}^3 v_{\frac{1}{2}}$$

with $a'_2 \geq \frac{1}{16}$.

In this case, with the reasoning in Remark 5.6, we find a solution to the thin obstacle problem $\hat{q}$ such that

$$|q - U_{\tau \varepsilon_{p_n}}(\hat{q})| \leq C \gamma \varepsilon_{p_n}^3 \text{ in } \{r > \eta/2\} \cap B_1. \quad (6.6)$$

With (6.5), (6.6), and $d_n \leq \varepsilon_{p_n}^5$, we have, up to a rotation

$$|u_n - \hat{q}| \leq C \gamma \varepsilon_{p_n}^3 \text{ in } B_1.$$

From here, we fix the parameter ε by

$$\varepsilon = \varepsilon_{p_n},$$

and relabeling our sequence

$$u_0 := u_n, \quad p_0 := \hat{q}, \quad \text{and} \quad d_0 := C \gamma \varepsilon^3.$$

Then

$$u_0 \in \mathcal{S}(p_0, d_0, 1),$$

where the class $\mathcal{S}$ is defined in Definition 5.5. Moreover, with p_0 solving the thin obstacle problem, we can apply Lemma 5.14 to get that

$$w_0 := W(u_0; 1) \leq C d_0^2 \leq C \gamma^2 \varepsilon^6. \quad (6.7)$$

If $\tilde{d}$ is chosen small enough, then $\varepsilon < \tilde{\varepsilon}$ from Lemma 5.16. Similar to Step 2, we apply Lemma 5.16 iteratively and obtain a sequence (u_n, p_n, d_n) with

$$u_n \in \mathcal{S}(p_n, d_n, 1).$$

Note that if alternative (1) in Lemma 5.16 happens, we apply (6.7) to get that

$$c_0^2 d_n^2 \leq w_{n-1} - w_n \leq w_0 \leq C\gamma^2 \varepsilon^6.$$

With a similar argument for comparison between d and $\tilde{d}$ in claim (6.2), this implies that

$$d_n \leq C\gamma \varepsilon^3 \text{ for all } n.$$

With a similar argument for comparison between κ and d in claim (6.2), we have that

$$\kappa p_n \leq d_n \text{ for all } n.$$

Consequently, with Lemma 5.14, we have that

$$w_n = W_{\frac{7}{2}}(u_n; 1) \leq C d_n^2.$$

Therefore, the double sequence (w_n, d_n) satisfies the condition in Lemma 2.2 for $\gamma = 1$, and we have that

$$\sum d_n \leq C(d_0^2 + w_0)^{1/2} \leq C\gamma \varepsilon^3.$$

In particular, the deviation for coefficients of p_n is of order $C\gamma$. Consequently, if γ is universally small, the conditions on the coefficients from Lemma 5.16 are satisfied, and Lemma 5.16 can be applied indefinitely. With the same argument in Section 5 of [18], we conclude that

$$|u - p| \leq O(|x|^{\frac{7}{2}+c_0})$$

up to a normalization for some $p \in \mathcal{F}_1 \setminus \{u_{\frac{7}{2}}\}$. $\square$

With a similar argument, we can apply Lemma 3.4 and Lemma 4.9 to establish the following:

Proposition 6.2. *For a solution u to the thin obstacle problem (1.1) in $B_1 \subset \mathbb{R}^3$, suppose that its frequency at 0 is $\frac{7}{2}$, and that*

$$|u - p| \leq d \text{ in } B_1$$

for some $p \in \mathcal{F}_1$ and $|p - u_{\frac{7}{2}}| \geq \frac{1}{2}\tilde{d}$, where $\tilde{d}$ is the universal constant from Proposition 6.1.

There is a small universal $\bar{d} > 0$, such that if $d < \bar{d}$, then up to a normalization

$$|u - p'| \leq O(|x|^{\frac{7}{2}+c_0})$$

for some $p' \in \mathcal{F}_1 \setminus \{u_{\frac{7}{2}}\}$.

Here $c_0 > 0$ is universal.

With these two propositions in hand, we sketch the proofs for the main results.

Proof of Theorem 1.1. For a $\frac{7}{2}$ -homogeneous solution u as in Theorem 1.1, we see that if $d < \min\{\tilde{d}, \bar{d}\}$ with $\tilde{d}$ and $\bar{d}$ from Proposition 6.1 and Proposition 6.2 respectively, then

$$|u - p| = o(|x|^{\frac{7}{2}}) \text{ as } x \rightarrow 0$$

for some $p \in \mathcal{F}_1$.

From here the homogeneity of u leads to $u = p$. $\square$

Proof of Theorem 1.2. Suppose that $p \in \mathcal{F}_1$ is a blow-up profile of u at 0. Then up to a rescaling we have that

$$|u - p| \leq \min\{\tilde{d}, \bar{d}\} \text{ in } B_1$$

for $\tilde{d}$ and $\bar{d}$ from Proposition 6.1 and Proposition 6.2, which leads to

$$|u - p'| = o(|x|^{\frac{7}{2}}) \text{ as } x \rightarrow 0$$

for some $p' \in \mathcal{F}_1$ up to a normalization.

However, with p being a blow-up profile, this forces $p' = p$. From here we either apply Proposition 6.1 or Proposition 6.2 to get the desired rate of convergence. $\square$

With this rate of convergence, the stratification in Theorem 1.3 follows from the Whitney extension lemma; see, for instance, [4, 13]. We sketch the proof for the more precise Remark 1.3:

Proof of Remark 1.3. Case 1: the solution blows up to $u_{\frac{7}{2}}$ at 0.

With the rate of convergence in Theorem 1.2 and Lemma 5.8, we see that in a ball of radius r , the free boundary $\partial_{\mathbb{R}^{n-1}} \Lambda(u)$ is trapped between two parallel lines with distance $r|\log(r)|^{-c_0}$. This is the desired $C^{1,\log}$ -regularity of the free boundary at a point where $u_{\frac{7}{2}}$ is the blow-up profile.

Case 2: the solution blows up to $p \in \mathcal{F}_1 \setminus \{u_{\frac{7}{2}}\}$ at 0.

Now suppose $p \in \mathcal{F}_1 \setminus \{u_{\frac{7}{2}}\}$ is a blow-up profile at 0, we need to find $\rho > 0$ such that $\Lambda_{\frac{7}{2}}(u) \cap B_\rho = \{0\}$. If the conditions $p|_{\{x_3=0\}} \geq 0$ and $\Delta p|_{\{x_3=0\}} \leq 0$ are not degenerate, the conclusion follows from a standard blow-up argument.

We give the proof when $p = p_{dc}$, the doubly critical profile from (4.4).

Suppose, on the contrary, that there is a sequence $x_k \in \Lambda_{\frac{7}{2}}(u) \rightarrow 0$. With the notation from (2.5), we define

$$u_k := u_{(|x_k|)}.$$

With the Hölder rate of convergence from Theorem 1.2 and Lemma 4.6, we have that $x_k/|x_k|$ converges to the two rays of degeneracies $R^\pm$ from (4.1) or $\{r = 0\}$. On the other hand, for p_{dc} , points on $R^\pm \cup \{r = 0\}$ have frequencies in $\{1, \frac{3}{2}, 2\}$, all bounded away from $\frac{7}{2}$. This implies that $x_k/|x_k|$ has frequency bounded away from $\frac{7}{2}$, a contradiction. $\square$

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Appendix A. Fourier expansion in spherical caps

In this appendix, we study the decay of a harmonic function in a slit domain near the boundary of a spherical cap if some of its Fourier coefficients vanish along a smaller cap.

Recall that we use (x_1, r, θ) as the coordinate system for $\mathbb{R}^3$, where $r \geq 0$ and $\theta \in (-\pi, \pi]$ are the polar coordinates for the (x_2, x_3) -plane. For small $r_0 > 0$, the r_0 -spherical cap is defined as

$$C_{r_0} := \{r < r_0, x_1 > 0\} \cap \mathbb{S}^2.$$

The main result of this appendix is

Lemma A.1. *For two small parameters η, ε with $\varepsilon \ll \eta$, suppose that v is a bounded solution to*

$$\begin{cases} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})v = 0 & \text{in } C_\eta \setminus \widetilde{C}_\varepsilon, \\ v = 0 & \text{on } \partial C_\eta \cup C_\eta \setminus \widetilde{C}_\varepsilon. \end{cases}$$

If we have, for $n = 0, 1, \dots, m-1$,

$$\int_{\partial C_\varepsilon} v \cdot \cos((n + \frac{1}{2})\theta) = 0,$$

then

$$\sup_{C_\eta \setminus C_{\eta/2}} |v| \leq C^m \varepsilon^{m+\frac{1}{2}} \cdot \sup_{\partial C_\varepsilon} |v|$$

for a constant C depending only on η .

Recall the notations for slit domains and homogeneous harmonic functions in slit domains from (2.10) and (2.13).

Proof. With the functions from (2.17), we define, for $n = 0, 1, \dots$,

$$f_n(x_1, r, \theta) := u_{-(n+\frac{1}{2})} \cdot \sum_{0 \leq k \leq k^*} a_k x_1^{n+4-2k} r^{2k}$$

where k^* satisfies $(n - 2k^* + 4)(n - 2k^* + 3) = 0$, $a_0 = 1$, and

$$a_k(n - 2k + 4)(n - 2k + 3) = a_{k+1}(2n - 2k - 1)(2k + 2). \quad (\text{A.1})$$

It is elementary to verify that f_n is $\frac{7}{2}$ -homogeneous and harmonic in $\widehat{\mathbb{R}^3}$.

By the iterative relation (A.1), we can find a universal large constant M such that

$$|a_k| \leq M^n \text{ for } k = 0, 1, \dots, k^*.$$

As a result, by taking M larger if necessary, we have that

$$|f_n| \leq r^{-n-\frac{1}{2}}[1 + r^2 M^n] \text{ in } \mathcal{C}_\eta.$$

On the other hand, we have $f_n(x_1, r, 0) \geq r^{-n-\frac{1}{2}}[1 - r^2 M^n]$ in $\mathcal{C}_\eta$, which gives that

$$f_n(x_1, \varepsilon, 0) \geq \frac{1}{2}\varepsilon^{-n-\frac{1}{2}}$$

if ε is small and $n \leq 5$. For $n \geq 6$, the same comparison follows directly from the fact that $a_k \geq 0$ for all k if $n \geq 6$.

Consequently, the ratio $f_n(r, \theta)/f_n(\varepsilon, 0)$ satisfies that

$$f_n(\varepsilon, \theta)/f_n(\varepsilon, 0) = \cos((n + \frac{1}{2})\theta)$$

and

$$|f_n(r, \theta)/f_n(\varepsilon, 0)| \leq \varepsilon^{n+\frac{1}{2}}r^{-n-\frac{1}{2}}M^n \text{ in } \mathcal{C}_\eta$$

by choosing M larger if necessary.

For each n , let φ_n denote the solution to

$$\begin{cases} (\Delta_{\mathbb{S}^2} + \lambda_{\frac{7}{2}})\varphi_n = 0 & \text{in } \mathcal{C}_\eta \setminus \widehat{\mathcal{C}_\varepsilon}, \\ \varphi_n = \cos((n + \frac{1}{2})\theta) & \text{along } \partial\mathcal{C}_\varepsilon, \\ \varphi_n = 0 & \text{along } \partial\mathcal{C}_\eta \cup \widetilde{\mathcal{C}_\eta \setminus \mathcal{C}_\varepsilon}. \end{cases}$$

With the maximum principle, we have that

$$|\varphi_n - \frac{f_n(r, \theta)}{f_n(\varepsilon, 0)}| \leq \varepsilon^{n+\frac{1}{2}}\eta^{-n-\frac{1}{2}}M^n \text{ in } \mathcal{C}_\eta \setminus \mathcal{C}_\varepsilon,$$

which implies that

$$|\varphi_n| \leq C\varepsilon^{n+\frac{1}{2}}r^{-n-\frac{1}{2}}M^n \text{ in } \mathcal{C}_\eta \setminus \mathcal{C}_\varepsilon.$$

Now with $\{\cos((n + \frac{1}{2})\theta)\}$ being a basis for $L^2(\partial\mathcal{C}_\varepsilon)$, for v as in the statement of the lemma, we can write $v = \sum c_n \varphi_n$, where

$$c_n = \frac{\int_{\partial\mathcal{C}_\varepsilon} v \cdot \cos((n + \frac{1}{2})\theta)}{\int_{\partial\mathcal{C}_\varepsilon} \cos^2((n + \frac{1}{2})\theta)}.$$

For $r \geq \eta/2$, this implies that

$$|v(r, \theta)| \leq \left(\sum c_n^2 \right)^{\frac{1}{2}} \left(\sum_{c_n \neq 0} \varphi_n(r, \theta)^2 \right)^{\frac{1}{2}} \leq C \sup_{\partial C_\varepsilon} |v| \cdot \left(\sum_{c_n \neq 0} \varepsilon^{2n+1} M^{2n} \right)^{1/2}$$

for a constant C depending on η .

With our assumption on v , we have $c_n = 0$ for $n \leq m - 1$. The conclusion follows by observing that

$$\sum_{n \geq m} \varepsilon^{2n+1} M^{2n} \leq C \varepsilon^{2m+1} M^{2m}.$$

□

Appendix B. The thin obstacle problem in $\mathbb{R}^2$

Our treatment of solutions near $u_{\frac{7}{2}} = r^{\frac{7}{2}} \cos(\frac{7}{2}\theta)$ relies on a fine analysis of the thin obstacle problem in tiny spherical caps around $\mathbb{S}^2 \cap \{r = 0\}$. In the limit, this problem leads to the thin obstacle problem in $\mathbb{R}^2$ with prescribed expansion at infinity.

In this section, we use (r, θ) to denote the polar coordinates of $\mathbb{R}^2 = \{(x_1, x_2)\}$. The notations for slit domains from (2.9) and (2.10) carry over with straightforward modifications. We will also take advantage of the functions from (1.6) and (2.17). Similarly to the functions in (2.16), in this appendix, we denote the derivatives of $u_{\frac{7}{2}}$ by the following:

$$w_{\frac{5}{2}} := \frac{\partial}{\partial x_1} u_{\frac{7}{2}}, \quad w_{\frac{3}{2}} := \frac{\partial}{\partial x_1} w_{\frac{5}{2}}, \quad \text{and} \quad w_{\frac{1}{2}} := \frac{\partial}{\partial x_1} w_{\frac{3}{2}}.^5$$

The following two derivatives are singular near $\{r = 0\}$:

$$w_{-\frac{1}{2}} := \frac{\partial}{\partial x_1} w_{\frac{1}{2}}, \quad \text{and} \quad w_{-\frac{3}{2}} := \frac{\partial}{\partial x_1} w_{-\frac{1}{2}}. \quad (\text{B.1})$$

Let $p = u_{\frac{7}{2}} + a_1 u_{\frac{5}{2}} + a_2 u_{\frac{3}{2}} + a_3 u_{\frac{1}{2}} = u_{\frac{7}{2}} + \tilde{a}_1 w_{\frac{5}{2}} + \tilde{a}_2 w_{\frac{3}{2}} + \tilde{a}_3 w_{\frac{1}{2}}$, then p solves the thin obstacle problem in $\mathbb{R}^2$ if and only if

$$a_2 \geq 0, \quad a_3 = 0, \quad \text{and} \quad a_1^2 \leq \frac{84}{25} a_2; \quad \text{equivalently,} \quad \tilde{a}_2 \geq 0, \quad \tilde{a}_3 = 0, \quad \text{and} \quad \tilde{a}_1^2 \leq \frac{12}{5} \tilde{a}_2. \quad (\text{B.2})$$

⁵ The two bases $\{u_{\frac{7}{2}}, u_{\frac{5}{2}}, u_{\frac{3}{2}}, u_{\frac{1}{2}}\}$ and $\{u_{\frac{7}{2}}, w_{\frac{5}{2}}, w_{\frac{3}{2}}, w_{\frac{1}{2}}\}$ are related by

$$w_{\frac{5}{2}} = \frac{7}{2} u_{\frac{5}{2}}, \quad w_{\frac{3}{2}} = \frac{35}{4} u_{\frac{3}{2}}, \quad \text{and} \quad w_{\frac{1}{2}} = \frac{105}{8} u_{\frac{1}{2}}.$$

For $\tau \in \mathbb{R}$, the *translation operator* U_τ is defined by its action on points, sets, and functions in the following manner:

$$U_\tau(x_1, x_2) = (x_1 + \tau, x_2), \quad U_\tau(E) = \{x : U_{-\tau}x \in E\}, \quad U_\tau(f)(x) = f(U_{-\tau}x).$$

In this appendix, for $p = u_{\frac{7}{2}} + a_1 u_{\frac{5}{2}} + a_2 u_{\frac{3}{2}} + a_3 u_{\frac{1}{2}}$, we study solutions to the *thin obstacle problem in $\mathbb{R}^2$ with data p at infinity*:

$$\begin{cases} u \text{ solves (1.1) in } \mathbb{R}^2, \\ \sup_{\mathbb{R}^2} |u - p| < +\infty. \end{cases} \quad (\text{B.3})$$

The starting point is the following proposition:

Proposition B.1. *For $|a_j| \leq 1$, there is a unique solution to (B.3).*

For this solution, there is a universal constant $A > 0$ such that

$$\sup_{\mathbb{R}^2} |u - p| \leq A; \quad \Delta u = 0 \text{ in } \{r \widehat{>} A\}; \text{ and } u = 0 \text{ in } \{r \widetilde{>} A\}.$$

Moreover, we can find b_1, b_2 satisfying $|b_j| \leq A$ such that

$$|u - (p + b_1 u_{-\frac{1}{2}} + b_2 u_{-\frac{3}{2}})| \leq A|x|^{-2} u_{-\frac{1}{2}} \text{ for all } x \in \mathbb{R}^2.$$

Recall the harmonic functions with negative homogeneities from (2.17).

Remark B.1. For simplicity, we will denote the coefficients b_j by $b_j^{\mathbb{R}^2}[a_1, a_2, a_3]$ or simply $b_j[a_1, a_2, a_3]$ when there is no ambiguity.

Proof. *Step 1: Uniqueness.*

Suppose that u_1 and u_2 are two solutions to (1.1) in $\mathbb{R}^2$ with $\sup |u_j - p| < +\infty$. With a similar argument as in Lemma 3.1, we find $R > 0$ such that

$$\Delta u_j = 0 \text{ in } \{r \widehat{>} R\}, \text{ and } u_j = 0 \text{ in } \{r \widetilde{>} R\}.$$

Let $w(x) := (u_1 - u_2)(R^2 x / |x|^2)$ be the Kelvin transform of $(u_1 - u_2)$ with respect to ∂B_R . Then w is a harmonic function in the slit domain $\widehat{B}_R$, as defined in (2.11). Applying Theorem 2.1, we have that $|w| \leq C u_{\frac{1}{2}}$ in B_R , which implies that

$$|u_1 - u_2| \leq C u_{-\frac{1}{2}} \text{ in } \mathbb{R}^2.$$

From here we have $u_1 = u_2$ by the maximum principle.

Step 2: A barrier function.

Rewrite p in the basis $\{u_{\frac{7}{2}}, w_{\frac{5}{2}}, w_{\frac{3}{2}}, w_{\frac{1}{2}}\}$ as $p = u_{\frac{7}{2}} + \tilde{a}_1 w_{\frac{5}{2}} + \tilde{a}_2 w_{\frac{3}{2}} + \tilde{a}_3 w_{\frac{1}{2}}$. For $\tau > 0$ to be chosen, if we let (α_1, α_2) denote the solution to

$$\alpha_1 + \tau = \tilde{a}_1, \text{ and } \alpha_2 + \alpha_1 \tau + \frac{1}{2} \tau^2 = \tilde{a}_2,$$

and define

$$q = u_{\frac{7}{2}} + \alpha_1 w_{\frac{5}{2}} + \alpha_2 w_{\frac{3}{2}},$$

then Taylor's Theorem gives that

$$U_{-\tau}(q) - p \geq \left(\frac{1}{6}\tau^3 - \frac{1}{2}\tilde{a}_1\tau^2 + \tilde{a}_2\tau \right) u_{\frac{1}{2}} - C\tau^4 u_{-\frac{1}{2}}.$$

Choosing τ large universally,

$$U_{-\tau}(q) - p \geq 0 \text{ on } \{r \geq A\}$$

for a universal large A .

By choosing τ larger, if necessary, it is elementary to verify that (α_1, α_2) satisfies condition (B.2), and consequently, $Q := U_{-\tau} q$ solves the thin obstacle problem in $\mathbb{R}^2$.

Step 3: Existence, universal boundedness, and localization of contact set.

For large $n \in \mathbb{N}$, let u_n be the solution to the thin obstacle problem (1.1) in B_n with $u_n = p$ along ∂B_n .

By the maximum principle, we have that

$$u_n \geq p \text{ in } B_n, \text{ and } u_n \leq Q \text{ in } B_n \quad (\text{B.4})$$

if n is large. Consequently, this family $\{u_n\}$ is locally uniformly bounded. Therefore, we can extract a subsequence converging to some u_∞ locally uniformly on $\mathbb{R}^2$. This limit u_∞ solves the thin obstacle problem in $\mathbb{R}^2$.

With (B.4), we have $u_n = 0$ in $B_n \cap \{x_1 \leq -A, x_2 = 0\}$ and $u_n \geq 1$ in $B_n \cap \{x_1 \geq A, x_2 = 0\}$ for a universal $A > 0$. Thus we have that

$$\Delta u_\infty = 0 \text{ in } \{r \gtrsim A\}; \text{ and } u_\infty = 0 \text{ in } \{r \lesssim A\}.$$

Along $\{r = A\}$, we have $0 \leq u_\infty - p \leq Q - p \leq C$. Thus the maximum principle, applied in the domain $\{r > A\}$, gives that

$$|u_\infty - p| \leq C$$

for a universal constant C . In particular, u_∞ is the unique solution to (B.3), according to Step 1.

Step 4: Finer expansion.

Let $w(x) := (u - p)(A^2 x / |x|^2)$ be the Kelvin transform of $(u - p)$ with respect to ∂B_A . Results from the previous step implies that w is a harmonic function in the slit domain $\widehat{B}_A$. An application of Theorem 2.1 gives universally bounded b_1 and b_2 such that

$$|w - (b_1 u_{\frac{1}{2}} + b_2 u_{\frac{3}{2}})| \leq C|x|^2 u_{\frac{1}{2}} \text{ in } B_A.$$

Inverting the Kelvin transform, we have that

$$|u - (p + b_1 u_{-\frac{1}{2}} + b_2 u_{-\frac{3}{2}})| \leq C|x|^{-2} u_{-\frac{1}{2}} \text{ in } \mathbb{R}^2.$$

□

For the solution from the previous proposition, we have precise information on its first two Fourier coefficients along big circles.

Corollary B.1. *With the same assumptions and notations from Proposition B.1, we have that*

$$\int_{\partial B_R} [u - (p + b_1 u_{-\frac{1}{2}} + b_2 u_{-\frac{3}{2}})] \cdot \cos\left(\frac{1}{2}\theta\right) = 0$$

and

$$\int_{\partial B_R} [u - (p + b_1 u_{-\frac{1}{2}} + b_2 u_{-\frac{3}{2}})] \cdot \cos\left(\frac{3}{2}\theta\right) = 0$$

for all $R \geq A$.

Proof. For simplicity, let us denote that

$$p_{ext} := p + b_1 u_{-\frac{1}{2}} + b_2 u_{-\frac{3}{2}}.$$

With Proposition B.1, we have $\Delta(u - p_{ext}) = 0$ in $\{r \gtrsim A\}$, and $u - p_{ext} = 0$ in $\{r \gtrsim A\}$.

For $R > A$, define $v := (r^{\frac{1}{2}} - Rr^{-\frac{1}{2}}) \cos(\frac{1}{2}\theta)$. Then

$$\Delta v = 0 \text{ in } \widehat{\mathbb{R}^2}, \text{ and } v = 0 \text{ along } \{r \gtrsim 0\}.$$

With these properties, we have, for $L > R$, that

$$\begin{aligned} 0 &= \int_{B_L \setminus B_R} (u - p_{ext}) \cdot \Delta v - \Delta(u - p_{ext}) \\ &\quad \cdot v = \int_{\partial(B_L \setminus B_R)} (u - p_{ext})_v \cdot v - (u - p_{ext}) \cdot v_v. \end{aligned}$$

Along ∂B_L , we have $|u - p_{ext}| = O(L^{-\frac{5}{2}})$, $|(u - p_{ext})_v| = O(L^{-\frac{7}{2}})$, $|v| = O(L^{\frac{1}{2}})$ and $|v_v| = O(L^{-\frac{1}{2}})$, thus

$$\int_{\partial B_L} (u - p_{ext})_v \cdot v - (u - p_{ext}) \cdot v_v = O(L^{-2}).$$

Along ∂B_R , we have $v = 0$ and $v_v = -R^{-\frac{1}{2}} \cos(\frac{1}{2}\theta)$. Combining all of these, we have that

$$\int_{\partial B_R} (u - p_{ext}) \cdot \cos\left(\frac{1}{2}\theta\right) = O(R^{\frac{1}{2}} L^{-2}).$$

Sending $L \rightarrow \infty$ gives the first conclusion. The second follows from a similar argument. $\square$

The following lemma is one of the main reasons for the restriction to 3d in the main part of this work:

Lemma B.1. *Given functions*

$$p = u_{\frac{7}{2}} + a_1 u_{\frac{5}{2}} + a_2 u_{\frac{3}{2}} + a_3 u_{\frac{1}{2}}, \text{ and } q = u_{\frac{7}{2}} - a_1 u_{\frac{5}{2}} + a_2 u_{\frac{3}{2}} - a_3 u_{\frac{1}{2}}$$

with $|a_j| \leq 1$, suppose that u and v are solutions to (B.3) with p and q as data at infinity, respectively.

Assume $b_1[a_1, a_2, a_3] = b_1[-a_1, a_2, -a_3]$ and $b_2[a_1, a_2, a_3] = -b_2[-a_1, a_2, -a_3]$, then we can find universally bounded constants α_1, α_2 and τ such that

$$u = U_{\tau}(u_{\frac{7}{2}} + \alpha_1 u_{\frac{5}{2}} + \alpha_2 u_{\frac{3}{2}}), \text{ and } v = U_{-\tau}(u_{\frac{7}{2}} - \alpha_1 u_{\frac{5}{2}} + \alpha_2 u_{\frac{3}{2}}).$$

Recall the definition of b_j 's from Remark B.1.

Proof. *Step 1: Two auxiliary polynomials.*

For simplicity, let us define $b_j = b_j[a_1, a_2, a_3]$ for $j = 1, 2$, and

$$p_{ext} := p + b_1 u_{-\frac{1}{2}} + b_2 u_{-\frac{3}{2}}, \text{ and } q_{ext} := q + b_1 u_{-\frac{1}{2}} - b_2 u_{-\frac{3}{2}}.$$

With Proposition B.1, we have that

$$|u - p_{ext}| + |v - q_{ext}| \leq A|x|^{-\frac{5}{2}} \text{ in } \mathbb{R}^2,$$

which implies that

$$|\nabla u - \nabla p_{ext}| + |\nabla v - \nabla q_{ext}| \leq C|x|^{-\frac{7}{2}} \text{ for } |x| \geq 1. \quad (\text{B.5})$$

Since u is an entire solution to the thin obstacle problem of order $O(|x|^{\frac{7}{2}})$ at infinity, we see that $(\partial_{x_1} u - i \partial_{x_2} u)^2$ is a polynomial of degree 5. Meanwhile, a direct computation gives that

$$(\partial_{x_1} p_{ext} - i \partial_{x_2} p_{ext})^2 = \mathcal{P}(x_1 + ix_2) + \sum_{k=1}^5 \mathcal{R}_k(x_1 + ix_2),$$

where $\mathcal{P}$ is a polynomial of degree 5, and $\mathcal{R}_k$ is a $(-k)$ -homogeneous rational function for $k = 1, 2, \dots, 5$.

With (B.5), it follows that

$$(\partial_{x_1} u - i \partial_{x_2} u)^2 = \mathcal{P} \text{ in } \mathbb{R}^2.$$

If we define

$$P(t) := \text{Re}(\partial_{x_1} u - i \partial_{x_2} u)^2(t, 0) = [(\partial_{x_1} u)^2 - (\partial_{x_2} u)^2](t, 0) = \text{Re } \mathcal{P}(t),$$

then P is a real polynomial of degree 5.

Similarly, corresponding to v and q_{ext} , we have that

$$(\partial_{x_1} q_{ext} - i \partial_{x_2} q_{ext})^2 = \mathcal{Q}(x_1 + ix_2) + \sum_{k=1}^5 \mathcal{S}_k(x_1 + ix_2),$$



Fig. 4. P and Q along the x_1 -axis

where Q is a polynomial of degree 5, and S_k is a $(-k)$ -homogeneous rational function for $k = 1, 2, \dots, 5$. Moreover, we have that

$$Q(t) := \operatorname{Re}(\partial_{x_1} v - i \partial_{x_2} v)^2(t, 0) = [(\partial_{x_1} v)^2 - (\partial_{x_2} v)^2](t, 0) = \operatorname{Re} \mathcal{Q}(t),$$

also a real polynomial of degree 5.

With $b_1[a_1, a_2, a_3] = b_1[-a_1, a_2, -a_3]$ and $b_2[a_1, a_2, a_3] = -b_2[-a_1, a_2, -a_3]$, a direct computation gives that

$$P(t) = -Q(-t). \quad (\text{B.6})$$

Step 2: Half-space solutions.

With (B.6), we show that up to a translation, u must be a half-space solution. Since $u = 0$ in $\{r \succ A\}$ according to Proposition B.1, it suffices to show that $\operatorname{spt}(\Delta u)$ has only one component.

Suppose, on the contrary, that

$$(-\infty, a] \cup [b, +\infty) \supset \operatorname{spt}(\Delta u) \supset (-\infty, a] \cup [b, c] \text{ with } b > a,$$

Note that the second component has to terminate in finite length since $\Delta u = 0$ in $\{r \succ A\}$.

On $(-\infty, a] \cup [b, c]$, we have $\partial_{x_1} u = 0$. Thus $P(t) = -(\partial_{x_2} u)^2 \leq 0$ for $t \in (-\infty, a] \cup [b, c]$. On the contrary, on (a, b) , $\partial_{x_2} u = 0$ and $P(t) = (\partial_{x_1} u)^2 \geq 0$. Moreover, since $u(a) = u(b) = 0$ and $u > 0$ on (a, b) , we must have $\partial_{x_1} u(d) = 0$ at some point $d \in (a, b)$. Thus $P(d) = 0$. Note that d is a root of multiplicity at least 2. Together with the roots a, b, c , this implies that P cannot have other roots; see Fig. 4.

With the symmetry described in (B.6), if we let $b' = -b$ and $c' = -c$, then $Q(b') = Q(c') = 0$ while $Q > 0$ on (b', c') . This implies that $v > 0$ on (b', c') , while $v(b') = v(c') = 0$. However, this implies that $\partial_{x_1} v$ must vanish at some point on (b', c') , and so does Q . This is a contradiction.

As a result, $\operatorname{spt}(\Delta u)$ must be a half line. A similar result holds for $\operatorname{spt}(\Delta v)$. With (B.6), we see that if $\operatorname{spt}(\Delta u) = (-\infty, a]$, then $\operatorname{spt}(\Delta v) = (-\infty, -a]$.

Step 4: Conclusion.

After the previous step, we can apply Theorem 2.1 to get that

$$U_{-a} u = \alpha'_0 u_{\frac{7}{2}} + \alpha_1 u_{\frac{5}{2}} + \alpha_2 u_{\frac{3}{2}} + \alpha'_3 u_{\frac{1}{2}}.$$

We must have $a'_3 = 0$ by (B.2). With $|u - (u_{\frac{7}{2}} + a_1 u_{\frac{5}{2}} + a_2 u_{\frac{3}{2}})|$ being bounded in $\mathbb{R}^2$, we conclude $a'_0 = 1$. Therefore,

$$U_{-a} u = u_{\frac{7}{2}} + \alpha_1 u_{\frac{5}{2}} + \alpha_2 u_{\frac{3}{2}}.$$

Similarly, we have that

$$U_a v = u_{\frac{7}{2}} + \beta_1 u_{\frac{5}{2}} + \beta_2 u_{\frac{3}{2}}.$$

From here, we use (B.6) to conclude $\alpha_1 = \beta_1$ and $\alpha_2 = -\beta_2$. The conclusion follows. $\square$

A perturbation of the previous lemma leads to the following corollary (recall notations from (B.1) and Remark B.1):

Corollary B.2. *Given $p = u_{\frac{7}{2}} + a_1 u_{\frac{5}{2}} + a_2 u_{\frac{3}{2}} + a_3 u_{\frac{1}{2}} = u_{\frac{7}{2}} + \tilde{a}_1 w_{\frac{5}{2}} + \tilde{a}_2 w_{\frac{3}{2}} + \tilde{a}_3 w_{\frac{1}{2}}$ with $|a_j| \leq 1$, we set*

$$b_j^+ := b_j[a_1, a_2, a_3], \quad b_j^- := b_j[-a_1, a_2, -a_3] \text{ for } j = 1, 2,$$

and

$$p_{ext} = p + b_1^+ u_{-\frac{1}{2}} + b_2^+ u_{-\frac{3}{2}} = p + \tilde{b}_1^+ w_{-\frac{1}{2}} + \tilde{b}_2^+ w_{-\frac{3}{2}}.$$

Then there is a universal modulus of continuity, ω , such that

$$\begin{aligned} & |\tilde{a}_1 - (\alpha_1 + \tau)| + \left| \tilde{a}_2 - \left(\alpha_2 + \alpha_1 \tau + \frac{1}{2} \tau^2 \right) \right| + \left| \tilde{a}_3 - \left(\alpha_2 \tau + \frac{1}{2} \alpha_1 \tau^2 + \frac{1}{6} \tau^3 \right) \right| \\ & + \left| \tilde{b}_1^+ - \left(\frac{1}{2} \alpha_2 \tau^2 + \frac{1}{6} \alpha_1 \tau^3 + \frac{1}{24} \tau^4 \right) \right| + \left| \tilde{b}_2^+ - \left(\frac{1}{6} \alpha_2 \tau^3 + \frac{1}{24} \alpha_1 \tau^4 + \frac{1}{120} \tau^5 \right) \right| \\ & \leq \omega(|b_1^+ - b_1^-| + |b_2^+ + b_2^-|) \end{aligned}$$

for universally bounded α_j and τ satisfying

$$\alpha_2 \geq 0 \text{ and } \alpha_1^2 \leq \frac{12}{5} \alpha_2. \quad (\text{B.7})$$

Proof. Suppose there is no such ω , we find a sequence (a_j^n) such that the corresponding $(b_j^{\pm, n})$ satisfy

$$|b_1^{+, n} - b_1^{-, n}| + |b_2^{+, n} + b_2^{-, n}| \rightarrow 0, \quad (\text{B.8})$$

but for any bounded α_j and τ satisfying (B.7), we have that

$$\begin{aligned} & |\tilde{a}_1^n - (\alpha_1 + \tau)| + \left| \tilde{a}_2^n - \left(\alpha_2 + \alpha_1 \tau + \frac{1}{2} \tau^2 \right) \right| + \left| \tilde{a}_3^n - \left(\alpha_2 \tau + \frac{1}{2} \alpha_1 \tau^2 + \frac{1}{6} \tau^3 \right) \right| \\ & + \left| \tilde{b}_1^{+, n} - \left(\frac{1}{2} \alpha_2 \tau^2 + \frac{1}{6} \alpha_1 \tau^3 + \frac{1}{24} \tau^4 \right) \right| + \left| \tilde{b}_2^{+, n} - \left(\frac{1}{6} \alpha_2 \tau^3 + \frac{1}{24} \alpha_1 \tau^4 + \frac{1}{120} \tau^5 \right) \right| \\ & \geq \varepsilon > 0 \end{aligned} \quad (\text{B.9})$$

Up to a subsequence, we have that

$$a_j^n \rightarrow a_j^\infty \text{ and } b_j^{\pm, n} \rightarrow b_j^{\pm, \infty}.$$

If we take that

$$p_n^+ = u_{\frac{7}{2}} + a_1^n u_{\frac{5}{2}} + a_2^n u_{\frac{3}{2}} + a_3^n u_{\frac{1}{2}} = u_{\frac{7}{2}} + \tilde{a}_1^n w_{\frac{5}{2}} + \tilde{a}_2^n w_{\frac{3}{2}} + \tilde{a}_3^n w_{\frac{1}{2}}$$

and denote by u_n^+ the solution to (B.3) with data p_n^+ at infinity, then, by Proposition B.1, we have that

$$|u_n^+ - p_n^+| \leq A \text{ in } \mathbb{R}^2.$$

Up to a subsequence, we have u_n^+ locally uniformly converge to u_∞^+ , a solution to the thin obstacle problem in $\mathbb{R}^2$. Moreover, we have

$$|u_\infty^+ - [u_{\frac{7}{2}} + a_1^\infty u_{\frac{5}{2}} + a_2^\infty u_{\frac{3}{2}} + a_3^\infty u_{\frac{1}{2}}]| \leq A \text{ in } \mathbb{R}^2.$$

Thus u_∞^+ is the solution to (B.3) with data $p_\infty^+ = u_{\frac{7}{2}} + a_1^\infty u_{\frac{5}{2}} + a_2^\infty u_{\frac{3}{2}} + a_3^\infty u_{\frac{1}{2}}$ at infinity.

With Corollary B.1, we see that $b_j^{+, \infty} := b_j[a_j^\infty] = \lim b_j^{+, n}$. A similar argument applied to $p_n^- = u_{\frac{7}{2}} - a_1^n u_{\frac{5}{2}} + a_2^n u_{\frac{3}{2}} - a_3^n u_{\frac{1}{2}}$ leads to $b_j^{-, \infty} := b_j[-a^\infty, a_2^\infty, -a_3^\infty] = \lim b_j^{-, n}$. With (B.8), we conclude that

$$\begin{aligned} b_1[a_1^\infty, a_2^\infty, a_3^\infty] &= b_1[-a_1^\infty, a_2^\infty, -a_3^\infty] \text{ and } b_2[a_1^\infty, a_2^\infty, a_3^\infty] \\ &= -b_2[-a_1^\infty, a_2^\infty, -a_3^\infty]. \end{aligned}$$

Lemma B.1 gives that

$$u_\infty^+ = U_\tau(u_{\frac{7}{2}} + \alpha_1 w_{\frac{5}{2}} + \alpha_2 w_{\frac{3}{2}})$$

for α_j satisfying (B.7).

Consequently, we have that

$$\begin{aligned} &\left| \tilde{a}_1^\infty - (\alpha_1 + \tau) \right| + \left| \tilde{a}_2^\infty - \left(\alpha_2 + \alpha_1 \tau + \frac{1}{2} \tau^2 \right) \right| + \left| \tilde{a}_3^\infty - \left(\alpha_2 \tau + \frac{1}{2} \alpha_1 \tau^2 + \frac{1}{6} \tau^3 \right) \right| \\ &+ \left| \tilde{b}_1^{\infty, +} - \left(\frac{1}{2} \alpha_2 \tau^2 + \frac{1}{6} \alpha_1 \tau^3 + \frac{1}{24} \tau^4 \right) \right| \\ &+ \left| \tilde{b}_2^{\infty, +} - \left(\frac{1}{6} \alpha_2 \tau^3 + \frac{1}{24} \alpha_1 \tau^4 + \frac{1}{120} \tau^5 \right) \right| = 0. \end{aligned}$$

With convergence of $\tilde{a}_j^n \rightarrow \tilde{a}_j^\infty$ and $\tilde{b}_j^{+, n} \rightarrow \tilde{b}_j^{+, \infty}$, this contradicts (B.9). $\square$

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