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HOMOGENIZATION OF A NONLINEAR STRONGLY COUPLED MODEL OF MAGNETORHEOLOGICAL FLUIDS*

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Abstract. This paper concerns the rigorous periodic homogenization for a nonlinear strongly coupled system, which models a suspension of magnetizable rigid particles in a nonconducting carrier viscous Newtonian fluid. The fluid drags the particles and thus alters the magnetic field. Vice versa, the magnetic field acts on the particles, which in turn affect the fluid via the no-slip boundary condition. As the size of the particles approaches zero, it is shown that the suspension's behavior is governed by a generalized magnetohydrodynamic system, where the fluid is modeled by a stationary Navier–Stokes system, while the magnetic field is modeled by Maxwell equations. A corrector result from the theory of two-scale convergence allows us to obtain the limit of the product of several weakly convergent sequences, where the div-curl lemma, which is a typical tool in these types of problems, is not applicable.

Key words. homogenization, two-scale convergence, Navier–Stokes equation, strong coupling, magnetic particles

MSC codes. 35B27, 74E30, 74F10, 76M50, 78M40

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1. Introduction.

This paper is a counterpart of our previous works [15, 16], where we carried out the rigorous periodic homogenization of a *weakly (one-way) coupled* nonlinear system modeling a nondilute suspension of magnetizable particles in a viscous Newtonian fluid. In [15, 16], the fluid is assumed to be described by the stationary Stokes flow, and the particles are either paramagnetic or diamagnetic. The *one-way coupling* is understood as follows: the magnetic field alters the movement of the magnetizable particles, then the particles affect the fluid flow via a no-slip boundary assumption; however, the reverse effect is assumed to be negligible. For details and information about the manifestations of the one-way coupling (as well as of the full coupling), its applications, and further literature on the subject, we refer the reader to [15, 16] and references cited therein. In this paper, the *full (two-way) coupling* is considered, i.e., we also take into account the reverse effect: the fluid flow pushes the particles and thus generates an induced magnetic field that acts back on the original one. The mathematical formulation of the fully coupled model of the magnetic nondilute suspension is given in section 2 below.

Starting with the seminal work of Einstein on the effective viscosity of a suspension [24], there have been numerous studies on this subject, ranging from formal asymptotic

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analysis such as [43, 44, 45] to rigorous analysis, e.g., [7, 8, 18, 20, 21, 22, 23, 29, 33, 38, 41, 42, 46, 49] and references cited therein. The coupling between the velocity and the magnetic fields distinguishes our paper from the previously cited. In this paper, *we propose a nonlinear system to model the two-way coupling in the magnetorheological fluid and derive, and rigorously justify, the corresponding effective system.*

To overview the literature on this topic, we start with the phenomenological models proposed in, e.g., [25, 34, 47, 51, 53], whose well-posedness was studied in, e.g., [6, 28, 35, 52, 54]. A coupling mechanism, similar to the one discussed in this paper, was also considered in [50], where a different model describing fluids was used. The authors in [50] though obtained the results using *formal asymptotic analysis*. Although similar models in different contexts were also studied in [26, 57], to the best of our knowledge, this paper is the first one to deal with the fully coupled model for magnetorheological fluids using the *rigorous homogenization approach*. Last, we mention that the *rigorous homogenization* for the system described by one-way fluid-particle coupling was solved in [15, 16] with a fairly general assumption on the smoothness of the coefficients.

In what follows below, after a nonlinear model for the magnetorheological fluid is established, we obtain the well-posedness and a priori estimates for its solution by adapting the general functional analysis framework of stationary magnetohydrodynamics; cf. [30, 36, 37, 55] and references therein. Then, the two-scale convergence method (cf. [3, 9, 14, 48]) is utilized to obtain the effective, or *homogenized*, system. The main difficulty lies in the nonlinearity of the system (cf. (2.7a) and (2.7f)) and the full coupling mechanism captured by (2.7a), (2.7f), and (2.11) that make the choice of suitable oscillating test functions in the energy method by Tartar [56], which is a typical tool in homogenization problems, become extremely tricky. To overcome this difficulty, we rely on the corrector result from the two-scale convergence method (see Theorem 2.7). The results obtained in this paper can be extended to the stochastic setting, thanks to the work on stochastic two-scale convergence; cf. [11, 39, 40, 58] and references cited therein.

This paper is organized as follows. In section 2, the main notation is introduced and the formulation of the fine-scale problem is discussed. Our main result is stated in Theorem 3.1, and the conclusions are given in section 4.

2. Formulation.

2.1. Notation. Throughout this paper, the scalar-valued functions, such as the pressure p, are written in usual typefaces, while vector-valued or tensor-valued functions, such as the velocity $\mathbf{u}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$, are written in bold. Sequences are indexed by superscripts (ϕ^i), while elements of vectors or tensors are indexed by numeric subscripts (x_i). Finally, the Einstein summation convention is used whenever applicable; δ_{ij} is the Kronecker delta, and ϵ_{ijk} is the Levi–Civita permutation symbol.

2.2. Set up of the problem. Consider $\Omega \subset \mathbb{R}^d$ for $d \geq 2$ a simply connected and bounded domain of class $C^{1,1}$, and let $Y := (0,1)^d$ be the unit cell in $\mathbb{R}^d$. The unit cell Y is decomposed into

$$Y = Y_s \cup Y_f \cup \Gamma,$$

where Y_s, representing the magnetic inclusion, and Y_f, representing the fluid domain, are open sets in $\mathbb{R}^d$, and Γ is the closed $C^{1,1}$ interface that separates them. Let

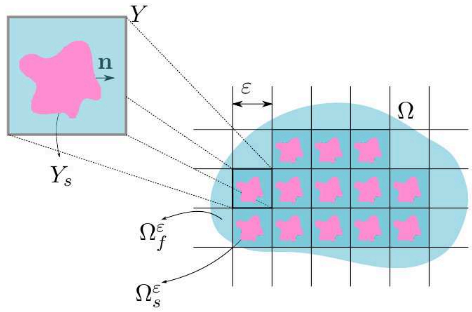

FIG. 1. *Reference cell Y and domain Ω.*

$i = (i_1, \ldots, i_d) \in \mathbb{Z}^d$ be a vector of indices and $\{\mathrm{e}^1, \ldots, \mathrm{e}^d\}$ be the canonical basis of $\mathbb{R}^d$. For a fixed small $\varepsilon > 0$, we define the dilated sets:

$$Y_i^\varepsilon := \varepsilon(Y + i),\ Y_{i,s}^\varepsilon := \varepsilon(Y_s + i),\ Y_{i,f}^\varepsilon := \varepsilon(Y_f + i),\ \Gamma_i^\varepsilon := \partial Y_{i,s}^\varepsilon.$$

Typically, in homogenization theory, the positive number $\varepsilon \ll 1$ is referred to as the *size of the microstructure*. The effective or homogenized response of the given suspension corresponds to the case $\varepsilon = 0$, whose derivation and justification is the main focus of this paper.

We denote by $\mathbf{n}_i$, $\mathbf{n}_\Gamma$, and $\mathbf{n}_{\partial\Omega}$ the unit normal vectors to Γ_i^ε pointing outward $Y_{i,s}^\varepsilon$, on Γ pointing outward Y_s, and on $\partial\Omega$ pointing outward, respectively; and also, we denote by $\mathrm{d}\mathcal{H}^{d-1}$ the $(d-1)$-dimensional Hausdorff measure. In addition, we define the sets:

$$I^\varepsilon := \{i \in \mathbb{Z}^d \colon Y_i^\varepsilon \subset \Omega\},\ \Omega_s^\varepsilon := \bigcup_{i \in I^\varepsilon} Y_{i,s}^\varepsilon,\ \Omega_f^\varepsilon := \Omega \setminus \Omega_s^\varepsilon,\ \Gamma^\varepsilon := \bigcup_{i \in I^\varepsilon} \Gamma_i^\varepsilon;$$

see Figure 1.

2.3. The model. Denote by $\rho_f, \rho_s, \nu_{es}, \mu$, and $\mathbf{g}$ the (mass) density of fluid, the density of inclusions, the electric conductivity of inclusions, the magnetic permeability, and the external force field, respectively. The unknowns include the fluid velocity $\mathbf{u}^\varepsilon$, the fluid pressure p^ε, and the magnetic field $\mathbf{B}^\varepsilon$ (which in turn determines the magnetizing field $\mathbf{H}^\varepsilon$). For simplicity, we assume that the magnetic permeability is piecewise-constant and given by

$$\mu(x) = \begin{cases} \mu_f & \text{if } x \in \Omega_f^\varepsilon, \\ \mu_s & \text{if } x \in \Omega_s^\varepsilon, \end{cases}$$

where μ_f, $\mu_s > 0$.

We consider the following nonlinear system modeling a suspension of rigid inclusions in a nonconducting carrier fluid:

$$\rho_f \left[\frac{\partial \mathbf{u}^\varepsilon}{\partial t} + (\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon \right] - \operatorname{div} \boldsymbol{\sigma}^\varepsilon = \rho_f \mathbf{g} \qquad \text{in } \Omega_f^\varepsilon, \tag{2.1a}$$

$$\operatorname{div} \mathbf{u}^\varepsilon = 0 \qquad \text{in } \Omega_f^\varepsilon, \tag{2.1b}$$

$$\mathbb{D}(\mathbf{u}^\varepsilon) = 0 \qquad \text{in } \Omega_s^\varepsilon, \tag{2.1c}$$

$$\text{curl } \mathbf{H}^\varepsilon = 0 \qquad \text{in } \Omega_f^\varepsilon, \tag{2.1d}$$

$$\frac{\partial \mathbf{B}^\varepsilon}{\partial t} + \frac{1}{\nu_{es}} \text{curl curl } \mathbf{H}^\varepsilon = \text{curl } (\mathbf{u}^\varepsilon \times \mathbf{B}^\varepsilon) \qquad \text{in } \Omega_s^\varepsilon, \tag{2.1e}$$

$$\text{div } \mathbf{B}^\varepsilon = 0 \qquad \text{in } \Omega, \tag{2.1f}$$

$$\mathbf{B}^\varepsilon = \mu \mathbf{H}^\varepsilon \qquad \text{in } \Omega. \tag{2.1g}$$

Suppose that the induced electric field is negligible; then the Lorentz force can be written as

$$\mathbf{f}^{\text{mag}} = \text{curl } \mathbf{H}^\varepsilon \times \mathbf{B}^\varepsilon = \begin{cases} 0 & \text{in } \Omega_f^\varepsilon, \\ \text{curl } \mathbf{H}^\varepsilon \times \mathbf{B}^\varepsilon & \text{in } \Omega_s^\varepsilon. \end{cases} \tag{2.2}$$

Thus, by the law of inertia [32, Axiom 5.2, p. 171], we obtain the balance equations of force and torque:

$$\int_{Y_{i,s}} \rho_s \dot{\mathbf{u}^\varepsilon} \mathrm{d}x = \int_{\Gamma_i} \boldsymbol{\sigma}^\varepsilon(\mathbf{u}^\varepsilon, p^\varepsilon)\mathbf{n} \mathrm{d}\Gamma + \int_{Y_{i,s}} \text{curl } \mathbf{H}^\varepsilon \times \mathbf{B}^\varepsilon \mathrm{d}x + \int_{Y_{i,s}} \rho_s \mathbf{g} \mathrm{d}x, \tag{2.3a}$$

$$\begin{aligned} \int_{Y_{i,s}} \rho_s \,(x - \mathrm{G}_i) \times \dot{\mathbf{u}^\varepsilon} \mathrm{d}x = & \int_{\Gamma_i} (x - \mathrm{G}_i) \times \boldsymbol{\sigma}^\varepsilon(\mathbf{u}^\varepsilon, p^\varepsilon)\mathbf{n} \mathrm{d}\Gamma \\ & + \int_{Y_{i,s}} (x - \mathrm{G}_i) \times (\text{curl } \mathbf{H}^\varepsilon \times \mathbf{B}^\varepsilon) \mathrm{d}x \\ & + \int_{Y_{i,s}} \rho_s \,(x - \mathrm{G}_i) \times \mathbf{g} \mathrm{d}x, \end{aligned} \tag{2.3b}$$

where $\dot{\mathbf{u}^\varepsilon} := \frac{\partial \mathbf{u}^\varepsilon}{\partial t} + (\mathbf{u}^\varepsilon \cdot \nabla)\, \mathbf{u}^\varepsilon$ is the convective derivative, and G_i is the center of mass of the particle $Y_{i,s}$. The (outer) boundary conditions on the external boundary $\partial\Omega$ are

$$\mathbf{u}^\varepsilon = 0, \quad \text{curl } \mathbf{H}^\varepsilon \times \mathbf{n} = 0, \quad \mathbf{B}^\varepsilon \cdot \mathbf{n} = q, \tag{2.4}$$

where $\boldsymbol{\sigma}^\varepsilon(\mathbf{u}^\varepsilon, p^\varepsilon) := 2\eta \mathbb{D}(\mathbf{u}^\varepsilon) - p^\varepsilon I$, $\mathbb{D}(\mathbf{u}^\varepsilon) := \frac{\nabla \mathbf{u}^\varepsilon + (\nabla \mathbf{u}^\varepsilon)^\top}{2}$, and $q \in H^{1/2}(\partial\Omega)$, satisfying the compatibility condition $\int_{\partial\Omega} q \mathrm{d}\mathcal{H}^{d-1} = 0$. Since μ is piecewise-constant, from now on, we write

$$\mathbf{H}^\varepsilon = \begin{cases} \dfrac{\mathbf{B}^\varepsilon}{\mu_f} & \text{if } x \in \Omega_f^\varepsilon, \\[2ex] \dfrac{\mathbf{B}^\varepsilon}{\mu_s} & \text{if } x \in \Omega_s^\varepsilon. \end{cases}$$

Remark 2.1. The physical meaning of the system (2.1) is as follows. Model (2.1a) is the momentum equation, which is the Navier–Stokes equation. The effect of the magnetic field on the fluid is expressed in the two balance equations (2.3). The incompressibility of the fluid is described by (2.1b). Equation (2.1c) describes the rigid body motion of the particles and (2.1g) provides the linear constitutive relation between the magnetic field $\mathbf{B}^\varepsilon$ and the magnetic induction field $\mathbf{H}^\varepsilon$, an assumption used in this paper.

Equations (2.1d), (2.1e), and (2.2) are derived from the Maxwell's equations as follows. In our applications, the electric conductivity constants of fluid phase and solid phase are 0 and ν_{es}, respectively. Then, by Ohm's law, the free current satisfies

$$\mathbf{J}^\varepsilon_{\text{free}} = 0 \text{ in } \Omega^\varepsilon_f \quad \text{and} \tag{2.5a}$$

$$\mathbf{J}^\varepsilon_{\text{free}} = \nu_{es}\left(\mathbf{E}^\varepsilon + \mathbf{u}^\varepsilon \times \mathbf{B}^\varepsilon\right) \text{ in } \Omega^\varepsilon_s. \tag{2.5b}$$

In other words, the induced electric field $\mathbf{E}^\varepsilon$ satisfies $\mathbf{E}^\varepsilon = \frac{\mathbf{J}^\varepsilon_{\text{free}}}{\nu_{es}} - \mathbf{u}^\varepsilon \times \mathbf{B}^\varepsilon$ in Ω^ε_s. Substituting into Faraday's law of induction, curl $\mathbf{E}^\varepsilon = -\frac{\partial \mathbf{B}^\varepsilon}{\partial t}$, we obtain

$$\frac{\partial \mathbf{B}^\varepsilon}{\partial t} + \text{curl }\left(\frac{\mathbf{J}^\varepsilon_{\text{free}}}{\nu_{es}} - \mathbf{u}^\varepsilon \times \mathbf{B}^\varepsilon\right) = 0 \text{ in } \Omega^\varepsilon_s. \tag{2.6}$$

Let $\mathbf{D}^\varepsilon$ denote the displacement field. In the current context, we may assume that the Maxwell displacement term in Ampère's Law, curl $\mathbf{H}^\varepsilon = \mathbf{J}^\varepsilon_{\text{free}} + \frac{\partial \mathbf{D}^\varepsilon}{\partial t}$, is negligible (cf. [17, Chapter 2]), so curl $\mathbf{H}^\varepsilon = \mathbf{J}^\varepsilon_{\text{free}}$ in Ω, which, together with (2.5a) and (2.6), implies (2.1d) and (2.1e). Moreover, if the induced electric field $\mathbf{E}^\varepsilon$ is negligible, and then the Lorentz force is $\mathbf{f}^{\text{mag}} = \mathbf{J}^\varepsilon_{\text{free}} \times \mathbf{B}^\varepsilon =$ curl $\mathbf{H}^\varepsilon \times \mathbf{B}^\varepsilon$, yielding (2.2).

2.4. Dimensional analysis. Let L, U, B, and μ_s be the characteristic scales corresponding to length, velocity, magnetic field, and magnetic permeability, respectively. The characteristic time T and body density force F are defined by $T = \frac{L}{U}$ and $F = \frac{U^2}{L}$.

Let $x^* := \frac{x}{L}, \mathbf{u}^{\varepsilon *} := \frac{\mathbf{u}}{U}, p^{\varepsilon *} := \frac{pL}{\eta U}, \mathbf{g}^* := \frac{\mathbf{g}L}{U^2}$, and $\mu^* := \frac{\mu}{\mu_s}$. The dimensionless quantities that appear are the *hydrodynamic Reynolds number* $R_e = \frac{\rho_f U L}{\eta}$, the *magnetic Reynolds number* $R_m = \mu_s \nu_{es} U L$, the *Alfven number* $A_l = \frac{B^2 L}{\eta \mu_s U}$, and the *density ratio,* which, for simplicity, is assumed to satisfy $\frac{\rho_s}{\rho_f} = 1$. Also, in what follows, we drop the star to lighten the notation. The dimensionless versions of (2.1), (2.3), and (2.4) are

$$R_e\left[\frac{\partial \mathbf{u}^\varepsilon}{\partial t} + (\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon\right] - \text{div } \boldsymbol{\sigma}^\varepsilon(\mathbf{u}^\varepsilon, p^\varepsilon) = R_e \mathbf{g} \quad \text{in } \Omega^\varepsilon_f, \tag{2.7a}$$

$$\text{div } \mathbf{u}^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon_f, \tag{2.7b}$$

$$\mathbb{D}(\mathbf{u}^\varepsilon) = 0 \quad \text{in } \Omega^\varepsilon_s, \tag{2.7c}$$

$$\text{curl } \mathbf{B}^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon_f, \tag{2.7d}$$

$$\frac{\partial \mathbf{B}^\varepsilon}{\partial t} + \frac{1}{R_m}\text{curl curl } \mathbf{B}^\varepsilon = \text{curl }\left(\mathbf{u}^\varepsilon \times \mathbf{B}^\varepsilon\right) \quad \text{in } \Omega^\varepsilon_s, \tag{2.7e}$$

$$\text{div } \mathbf{B}^\varepsilon = 0 \quad \text{in } \Omega, \tag{2.7f}$$

equipped with the balance equations

$$R_e \int_{Y^\varepsilon_{i,s}} \dot{\mathbf{u}^\varepsilon} \mathrm{d}x = \int_{\Gamma^\varepsilon_i} \boldsymbol{\sigma}^\varepsilon(\mathbf{u}^\varepsilon, p^\varepsilon)\mathbf{n}\mathrm{d}\mathcal{H}^{d-1} + A_l \int_{Y^\varepsilon_{i,s}} \text{curl } \mathbf{B}^\varepsilon \times \mathbf{B}^\varepsilon \mathrm{d}x + R_e \int_{Y^\varepsilon_{i,s}} \mathbf{g}\mathrm{d}x, \tag{2.8a}$$

$$R_e \int_{Y^\varepsilon_{i,s}} (x - \mathrm{G}^\varepsilon_i) \times \dot{\mathbf{u}^\varepsilon} \mathrm{d}x = \int_{\Gamma^\varepsilon_i} (x - \mathrm{G}^\varepsilon_i) \times \boldsymbol{\sigma}^\varepsilon(\mathbf{u}^\varepsilon, p^\varepsilon)\mathbf{n}\mathrm{d}\mathcal{H}^{d-1} + A_l \int_{Y^\varepsilon_{i,s}} (x - \mathrm{G}^\varepsilon_i) \times (\text{curl } \mathbf{B}^\varepsilon \times \mathbf{B}^\varepsilon)\mathrm{d}x + R_e \int_{Y^\varepsilon_{i,s}} (x - \mathrm{G}^\varepsilon_i) \times \mathbf{g}\mathrm{d}x \tag{2.8b}$$

and the boundary conditions

$$\mathbf{u}^\varepsilon = 0, \quad \text{curl } \mathbf{B}^\varepsilon \times \mathbf{n} = 0, \quad \mathbf{B}^\varepsilon \cdot \mathbf{n} = q, \tag{2.9}$$

where now $\boldsymbol{\sigma}^\varepsilon(\mathbf{u}^\varepsilon, p^\varepsilon) := 2\mathbb{D}(\mathbf{u}^\varepsilon) - p^\varepsilon\mathbf{I}$.

Hereafter, we consider the stationary flow, i.e., the time derivative is ignored,

$$R_e(\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon - \text{div } \boldsymbol{\sigma}^\varepsilon(\mathbf{u}^\varepsilon, p^\varepsilon) = R_e\mathbf{g} \quad \text{in } \Omega^\varepsilon_f, \tag{2.10a}$$

$$\text{div } \mathbf{u}^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon_f, \tag{2.10b}$$

$$\mathbb{D}(\mathbf{u}^\varepsilon) = 0 \quad \text{in } \Omega^\varepsilon_s, \tag{2.10c}$$

$$\text{curl } \mathbf{B}^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon_f, \tag{2.10d}$$

$$\frac{1}{R_m}\text{curl curl } \mathbf{B}^\varepsilon - \text{curl } (\mathbf{u}^\varepsilon \times \mathbf{B}^\varepsilon) = \mathbf{h} \quad \text{in } \Omega^\varepsilon_s, \tag{2.10e}$$

$$\text{div } \mathbf{B}^\varepsilon = 0 \quad \text{in } \Omega, \tag{2.10f}$$

equipped with the balance equations

$$R_e \int_{Y^\varepsilon_{i,s}} (\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon = \int_{\Gamma^\varepsilon_i} \boldsymbol{\sigma}^\varepsilon\mathbf{n}\mathrm{d}\mathcal{H}^{d-1} + A_l \int_{Y^\varepsilon_{i,s}} \text{curl } \mathbf{B}^\varepsilon \times \mathbf{B}^\varepsilon \mathrm{d}x + R_e \int_{Y^\varepsilon_{i,s}} \mathbf{g}\mathrm{d}x, \tag{2.11a}$$

$$R_e \int_{Y^\varepsilon_{i,s}} (x - \mathrm{G}^\varepsilon_i) \times (\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon = \int_{\Gamma^\varepsilon_i} (x - \mathrm{G}^\varepsilon_i) \times \boldsymbol{\sigma}^\varepsilon\mathbf{n}\mathrm{d}\mathcal{H}^{d-1} + A_l \int_{Y^\varepsilon_{i,s}} (x - \mathrm{G}^\varepsilon_i) \times (\text{curl } \mathbf{B}^\varepsilon \times \mathbf{B}^\varepsilon)\, \mathrm{d}x + R_e \int_{Y^\varepsilon_{i,s}} (x - \mathrm{G}^\varepsilon_i) \times \mathbf{g}\mathrm{d}x, \tag{2.11b}$$

the boundary conditions

$$\mathbf{u}^\varepsilon = 0, \quad \operatorname{curl} \mathbf{B}^\varepsilon \times \mathbf{n} = 0, \quad \mathbf{B}^\varepsilon \cdot \mathbf{n} = 0, \tag{2.12}$$

and the compatibility condition

$$\int_{\Omega_s} \mathbf{h} \cdot \nabla\psi \mathrm{d}x = 0 \text{ for all } \nabla\psi \in H^1_n(\Omega, \mathbb{R}^d). \tag{2.13}$$

We note that the function $\mathbf{h} \in L^2(\Omega, \mathbb{R}^d)$ appears in (2.10e) due to a *lifting* of the nonhomogeneous magnetic condition (2.4) to the homogeneous condition (2.12), i.e., substracting $\mathbf{B}^\varepsilon$ by a suitable function (see [37] and [30, section 3.8]. Here, $H^1_n(\Omega, \mathbb{R}^d)$ is the set of weakly differentiable functions from Ω to $\mathbb{R}^d$ with vanishing normal trace; see subsection 2.5.1 below.

2.5. Useful results from functional analysis. In this section, we collect some background results from functional analysis used in what follows. We separate the functional spaces and theorems of the two-scale convergence method from the ones of saddle point problems to make it easier to keep track.

2.5.1. Abstract framework for our nonlinear problem. The results for linear saddle point problems date back to the seminal works by Babuška and Brezzi (cf. [4, 10]). They are then adapted to the nonlinear cases such as the Navier–Stokes equations and magnetorhydrodynamic equations (cf. [30, 31, 36, 37, 55]). We summarize here the results used in our paper and refer readers to the works cited above for their proofs.

Let X and P be two real Hilbert spaces, and $f \in X$. Let $a(\,\cdot\,;\cdot,\cdot)\colon X \times X \times X \to \mathbb{R}$ be a nonlinear form such that for any $w \in X$, $a(w;\cdot,\cdot)$ is a bilinear continuous form on $X \times X$. Let $b\colon X \times P \to \mathbb{R}$ be a continuous bilinear form. Consider the following nonlinear problem:

Find $(u,p) \in X \times P$, *such that for all* $(v,q) \in X \times P$,

$$a(u;u,v) + b(v,p) = \langle f, v\rangle\,, \tag{2.14a}$$

$$b(u,q) = 0, \tag{2.14b}$$

where $\langle\cdot,\cdot\rangle$ is the dual pairing. The unknown p can be regarded as the Lagrange multiplier associated with the constraint (2.14b). The idea is to embed the constraint (2.14b) into X by introducing the space

$$M = \{u \in X \colon b(u,q) = 0 \text{ for all } q \in P\}$$

and consider a simpler problem that reads as follows: *Find* $u \in M$ *such that for all* $v \in M$,

$$a(u;u,v) = \langle f, v\rangle\,. \tag{2.15}$$

The continuity of b implies that M is a closed linear subspace of X, and, thus, M is also a Hilbert space.

Theorem 2.2 (existence and uniqueness of solution of (2.15)). *If the following conditions hold,*

(i) *there exists* $\alpha > 0$ *such that for all* $v \in M$,

$$a(v;v,v) \geq \alpha \left\|v\right\|^2_X\,, \tag{2.16}$$

(ii) *the space M is separable and such that for any sequence v_n that weakly converges to v in M, $a(v_n; v_n, w)$ converges to $a(v; v, w)$ for all $w \in M$,*

then there exists at least one solution of problem (2.15): $u \in M$. *If in addition, we assume that*

(iii) *the elliptic property* (i) *holds uniformly with respect to the first variable, i.e., there exists $\alpha > 0$ such that for all $v, w \in M$,*

$$a(w; v, v) \geq \alpha \|v\|_X^2 , \tag{2.17}$$

(iv) *there exists a constant $\gamma > 0$ such that, for all $u_1, u_2, v, w \in M$,*

$$|a(u_2; v, w) - a(u_1; v, w)| \leq \gamma \|u_2 - u_1\|_X \|v\|_X \|w\|_X , \tag{2.18}$$

then problem (2.15) *has a unique solution $u \in M$, provided that*

$$\frac{\gamma \|w\|_M}{\alpha^2} < 1, \tag{2.19}$$

where $w \in M$ is such that $\langle f|_M, v\rangle = (w, v)_X$ for all *$v \in M$, with $f|_M$ being the restriction of f on M.*

Theorem 2.2 allows us to establish the existence and uniqueness of the solution u of (2.15). To recover the unknown p that solves (2.14), we need to introduce the following definition.

Definition 2.3. *The following is called the inf-sup condition or the Babuška–Brezzi condition or the Ladyzhenskaya–Babuška–Brezzi condition:*

$$\exists\, \beta > 0 \quad \text{such that} \quad \inf_{q \in P\setminus\{0\}} \sup_{v \in X\setminus\{0\}} \frac{b(v, q)}{\|v\|_X \|q\|_P} \geq \beta. \tag{2.20}$$

If the bilinear form b in (2.14) satisfies the inf-sup condition (2.20), then, by the Riesz representation theorem and the closed range theorem [13], the existence and uniqueness of the solution u of (2.15) implies the existence and uniqueness of the solution (u, p) of (2.14).

The inf-sup condition can be verified by the following.

Proposition 2.4. *Let $B\colon X \to P$ be the continuous linear operator associated to the continuous bilinear form b by $(Bv, q)_P = b(v, q)$ for all $(v, q) \in X \times P$. (Here we use the Riesz representation theorem). Then the following statements are equivalent:*

(i) *The inf-sup condition* (2.20) *holds.*

(ii) *$B^\top : P \to X$ is injective and $B^\top$ has a closed range. Here $B^\top$ is the transpose of B, i.e., $(v, B^\top q)_X = (Bv, q)_P$ for all $(v, q) \in X \times P$.*

(iii) *$B\colon X \to P$ is surjective.*

2.5.2. The two-scale convergence method. Two-scale convergence was invented by Nguetseng and further developed by Allaire. We collect here the important notions and results relevant to this paper, whose proofs can be found in [11, 39, 40, 58]. The following spaces are used in the paper below:

- $C_{\mathrm{per}}(Y)$–the subspace of $C(\mathbb{R}^d)$ of Y-periodic functions;
- $C^\infty_{\mathrm{per}}(Y)$–the subspace of $C^\infty(\mathbb{R}^d)$ of Y-periodic functions;
- $H^1_{\mathrm{per}}(Y)$–the closure of $C^\infty_{\mathrm{per}}(Y)$ in the H^1-norm;
- $\mathcal{D}(\Omega, X)$, where X is a Banach space–the space infinitely differentiable functions from Ω to X, whose support is a compact set of $\mathbb{R}^d$ contained in Ω;

- $L^p(\Omega, X)$, where X is a Banach space and $1 \leq p \leq \infty$–the space of measurable functions $w\colon x \in \Omega \mapsto w(x) \in X$ such that $\|w\|_{L^p(\Omega,X)} := \left(\int_\Omega \|w(x)\|_X^p \,\mathrm{d}x\right)^{\frac{1}{p}} < \infty$;
- $L^p_{\text{per}}(Y, C(\bar{\Omega}))$–the space of measurable functions $w\colon y \in Y \mapsto w(\cdot, y) \in C(\bar{\Omega})$, such that w is periodic with respect to y and $\int_Y (\sup_{x\in\bar{\Omega}} |w(x,y)|)^p \mathrm{d}y < \infty$.

DEFINITION 2.5 (L^p-admissible test function). *Let $1 \leq p < \infty$. A function $\psi \in L^p(\Omega \times Y)$, Y-periodic in the second component, is called an L^p- admissible test function if for all $\varepsilon > 0$, $\psi\left(\cdot, \frac{\cdot}{\varepsilon}\right)$ is measurable and*

$$\lim_{\varepsilon\to 0} \int_\Omega \left|\psi\left(x, \frac{x}{\varepsilon}\right)\right|^p \mathrm{d}x = \frac{1}{|Y|} \int_\Omega \int_Y |\psi(x,y)|^p \,\mathrm{d}y\mathrm{d}x. \tag{2.21}$$

It is known that functions belonging to the spaces $\mathcal{D}(\Omega, C^\infty_{\text{per}}(Y))$, $C(\bar{\Omega}, C_{\text{per}}(Y))$, $L^p_{\text{per}}(Y, C(\bar{\Omega}))$, or $L^p(\Omega, C_{\text{per}}(Y))$ are admissible [3], but the precise characterization of those admissible test functions is still an open question.

DEFINITION 2.6. *A sequence $\{v^\varepsilon\}_{\varepsilon>0}$ in $L^2(\Omega)$ is said to two-scale converge to $v = v(x,y)$, with $v \in L^2(\Omega \times Y)$, and we write $v^\varepsilon \xrightarrow{2} v$, if and only if*

$$\lim_{\varepsilon\to 0} \int_\Omega v^\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \frac{1}{|Y|} \int_\Omega \int_Y v(x,y)\psi(x,y)\mathrm{d}y\mathrm{d}x \tag{2.22}$$

for any test function $\psi = \psi(x,y)$ with $\psi \in \mathcal{D}(\Omega, C^\infty_{\text{per}}(Y))$.

In (2.22), we can choose ψ be any (L^2-)admissible test function. Any bounded sequence $v^\varepsilon \in L^2(\Omega)$ has a subsequence that two-scale converges to a limit $v^0 \in L^2(\Omega \times Y)$. Moreover, from [3, Theorem 1.8, Remark 1.10, and Corollary 5.4], we have the following.

THEOREM 2.7 (corrector result). *Let u^ε be a sequence of functions in $L^2(\Omega)$ that two-scale converges to a limit $u^0(x,y) \in L^2(\Omega \times Y)$. Assume that*

$$\lim_{\varepsilon\to 0} \|u^\varepsilon\|_{L^2(\Omega)} = \left\|u^0\right\|_{L^2(\Omega\times Y)}. \tag{2.23}$$

Then for any sequence v^ε in $L^2(\Omega)$ that two-scale converges to $v^0 \in L^2(\Omega \times Y)$, one has

$$u^\varepsilon v^\varepsilon \rightharpoonup \frac{1}{|Y|} \int_Y u^0(x,y)v^0(x,y)\mathrm{d}x\mathrm{d}y \text{ in } \mathcal{D}'(\Omega). \tag{2.24}$$

Furthermore, if $u^0(x,y)$ belongs to $L^2(\Omega, C_{\text{per}}(Y))$ or $L^2_{\text{per}}(Y, C(\Omega))$, then

$$\lim_{\varepsilon\to 0} \left\|u^\varepsilon(x) - u^0\left(x, \frac{x}{\varepsilon}\right)\right\|_{L^2(\Omega)} = 0. \tag{2.25}$$

In fact, the smoothness assumption on u^0 in (2.25) is needed only for $u^0\left(x, \frac{x}{\varepsilon}\right)$ to be measurable and to belong to $L^2(\Omega)$. Finally, we recall that if $\psi \in L^2(\Omega \times Y)$ is a Carathéodory function, then $\psi\left(\cdot, \frac{\cdot}{\varepsilon}\right)$ is measurable. This fact is used later on to prove that $\mathbb{1}_{\Omega\times Y_s}$ is an admissible test function.

3. Main results. We now define the admissible spaces on the $C^{1,1}-$domain Ω for the fluid velocity $\mathbf{u}^\varepsilon$, the magnetic field $\mathbf{B}^\varepsilon$, and the fluid pressure p^ε. Let

$$
\begin{aligned}
H^1_n(\Omega,\mathbb{R}^d) &:= \left\{\mathbf{C}\in H^1(\Omega,\mathbb{R}^d)\colon \mathbf{C}\cdot\mathbf{n}_{\partial\Omega}=0\right\},\\
L^2_0(\Omega) &:= \left\{q\in L^2(\Omega)\colon \int_\Omega q\mathrm{d}x = 0\right\},\\
\mathfrak{H} &:= \left\{\mathbf{C}\in L^2(\Omega,\mathbb{R}^d)\colon \text{curl }\mathbf{C}\in L^2(\Omega,\mathbb{R}^d), \text{div }\mathbf{C}\in L^2(\Omega), \mathbf{C}\cdot\mathbf{n}_{\partial\Omega}=0\right\},\\
\mathfrak{V}^\varepsilon &:= \left\{\mathbf{v}\in H^1_0(\Omega,\mathbb{R}^d)\colon \mathbb{D}(\mathbf{v})=0 \text{ in } \Omega^\varepsilon_s\right\},\\
\mathfrak{P}^\varepsilon &:= \text{div }(\mathfrak{V}^\varepsilon) = \left\{q\in L^2_0(\Omega)\colon \exists\mathbf{v}\in\mathfrak{V}^\varepsilon \text{ such that } q=\text{div }\mathbf{v}\right\},\\
\mathfrak{U}^\varepsilon &:= \left\{\mathbf{v}\in H^1_0(\Omega,\mathbb{R}^d)\colon \mathbb{D}(\mathbf{v})=0 \text{ in } \Omega^\varepsilon_s, \text{ div }\mathbf{v}=0 \text{ in } \Omega^\varepsilon_f\right\},\\
\mathfrak{X}^\varepsilon &:= \mathfrak{V}^\varepsilon\times\left\{\mathbf{C}\in H^1_n(\Omega,\mathbb{R}^d)\colon \text{curl }\mathbf{C}=0 \text{ in } \Omega^\varepsilon_f\right\},\\
\mathfrak{Y}^\varepsilon &:= \mathfrak{U}^\varepsilon\times\left\{\mathbf{C}\in H^1_n(\Omega,\mathbb{R}^d)\colon \text{curl }\mathbf{C}=0 \text{ in } \Omega^\varepsilon_f\right\}.
\end{aligned}
$$

These spaces are equipped with natural Sobolev norms. Moreover, given normed spaces A and B, the norm of its product space $A\times B$ is defined by $\|(a,b)\|^2_{A\times B} := \|a\|^2_A + \|b\|^2_B$ for $a\in A, b\in B$.

As we will see later, to utilize the framework presented in subsection 2.5.1, we choose $X=\mathfrak{X}^\varepsilon$, $M=\mathfrak{Y}^\varepsilon$, and $P=\mathfrak{P}^\varepsilon$.

In addition, let κ^{-1}_{GR}, be the norm of the embedding $\mathfrak{H}\to H^1$, κ_S the norm of the Sobolev embedding $H^1\to L^4$, and κ^{-1}_K the constant in Korn's inequality, respectively. Then the main result of this paper is summarized in the following theorem.

Theorem 3.1. *Suppose the data* $\mathbf{g}$ *and* $\mathbf{h}$ *are small enough such that*

$$
R_e\,\|\mathbf{g}\|_{L^2} + A_l\,\|\mathbf{h}\|_{L^2} \le \frac{(\min\{\frac{A_l}{R_m}\kappa_{GR},\kappa_K\})^2}{\kappa_S\max\{R_e,2A_l\}}. \tag{3.1}
$$

Then, for $\varepsilon>0$, *the system* (2.10) *has a unique solution* $\mathbf{u}^\varepsilon\in H^1_0(\Omega,\mathbb{R}^d)$, $p^\varepsilon\in L^2_0(\Omega)$, $\mathbf{B}^\varepsilon\in H^1_n(\Omega,\mathbb{R}^d)$. *Moreover, there exist a constant, symmetric, and elliptic fourth-rank tensor* $\mathcal{N}$ *and two constant, symmetric, and elliptic matrices* $\mathcal{M},\mathcal{E}$ *such that*

$$
\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u}^0 \text{ in } H^1(\Omega,\mathbb{R}^d),\quad \mathbf{B}^\varepsilon \rightharpoonup \mathbf{B}^0 \text{ in } H^1(\Omega,\mathbb{R}^d),\quad p^\varepsilon \rightharpoonup \Pi \text{ in } L^2_0(\Omega), \tag{3.2}
$$

where $\mathbf{u}^0\in H^1_0(\Omega,\mathbb{R}^d)$, $\Pi\in L^2_0(\Omega)$, *and* $\mathbf{B}^0\in H^1_n(\Omega,\mathbb{R}^d)$ *satisfy the following effective system of equations, all defined on the domain* Ω:

$$
\begin{aligned}
&\text{div }\mathbf{u}^0 = \text{div }\mathbf{B}^0 = 0,\\
&\left(\mathbf{u}^0\cdot\nabla\right)\mathbf{u}^0 - \text{div }\left(2\mathcal{N}_{ijmn}\left[\mathbb{D}(\mathbf{u}^0)\right]_{ij}\mathrm{e}^m\otimes\mathrm{e}^n - \Pi\,\mathbf{I}\right) = R_e\mathbf{g} + A_l\frac{|Y_s|}{|Y|}\text{curl }\mathbf{B}^0\times\mathbf{B}^0,\\
&\frac{1}{R_m}\text{curl }\left(\mathcal{M}_{jn}\epsilon_{ijk}\frac{\partial B^0_i}{\partial x_k}\mathrm{e}^n\right) - \text{curl }\left(\mathcal{E}_{kn}\epsilon_{ijk}u^0_iB^0_j\mathrm{e}^n\right) = \frac{|Y_s|}{|Y|}\mathbf{h}.
\end{aligned} \tag{3.3}
$$

Remark 3.2. The solution mentioned in Theorem 3.1 is understood in the weak sense, which will be clarified in subsection 3.1. In that weak formulation, the pressure p^ε is extended to the entire domain Ω by setting it equal to a constant, e.g., zero, on each $Y^\varepsilon_{i,s}$. This explains why p^ε is defined in the entire domain Ω in Theorem 3.1.

The road map of the proof of Theorem 3.1 goes as follows:

- First, we present the variational formulation for problem (2.10)–(2.13) and prove their equivalence in subsection 3.1.

- Second, the existence and a priori estimates for the fine-scale velocity $\mathbf{u}^\varepsilon$ and the magnetic field $\mathbf{B}^\varepsilon$ are established in subsection 3.2, thanks to Theorem 2.2. The first two steps are adapted from the classical theory of magnetohydrodynamics (cf. [30, 31, 36, 37, 55]). In particular, the presentation of those two steps is inspired by [30, 37, 55].
- Third, in subsection 3.3, the existence and a priori estimate for the fine-scale pressure p^ε are recovered by an inf-sup condition. A construction based on the Bogovskiĭ map allows us to control the norm of the pressure p^ε uniformly with respect to ε (cf. [1, 2, 5, 19, 21, 22, 42]).
- Next, the two-scale homogenized problem is derived in subsection 3.4. Here, a corrector result of two-scale convergence [3] is crucial for passing to the limit of several integrals over a changing domain.
- Finally, the local and homogenized problems are recovered in subsections 3.5 and 3.6. Explicit formulas for the *effective viscosity* $\mathcal{N}$, the *effective magnetic reluctivity* $\mathcal{M}$, and the *effective electric conductivity* $\mathcal{E}$ are provided in (3.61).

3.1. Variational formulation. We define bilinear, trilinear, and linear forms $\mathcal{A}^\varepsilon(\cdot,\cdot) : \mathfrak{X}^\varepsilon \times \mathfrak{X}^\varepsilon \to \mathbb{R}$, $\mathcal{B}^\varepsilon(\cdot,\cdot) : \mathfrak{X}^\varepsilon \times \mathfrak{P}^\varepsilon \to \mathbb{R}$, and $\mathcal{C}^\varepsilon(\cdot,\cdot,\cdot) : \mathfrak{X}^\varepsilon \times \mathfrak{X}^\varepsilon \times \mathfrak{X}^\varepsilon \to \mathbb{R}$, $\mathcal{L}^\varepsilon(\cdot) : \mathfrak{X}^\varepsilon \to \mathbb{R}$ by

$$
\begin{aligned}
\mathcal{A}^\varepsilon\left((\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{C})\right) &:= 2\int_{\Omega_f^\varepsilon} \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\mathrm{d}x + \frac{A_l}{R_m}\left[\int_\Omega \operatorname{div}\mathbf{B}\cdot\operatorname{div}\mathbf{C}\,\mathrm{d}x\right.\\
&\qquad \left. + \int_{\Omega_s^\varepsilon} \operatorname{curl}\mathbf{B}\cdot\operatorname{curl}\mathbf{C}\,\mathrm{d}x\right],\\
\mathcal{B}^\varepsilon\left((\mathbf{v},\mathbf{C}),p\right) &:= \int_\Omega p\operatorname{div}\ \mathbf{v}\,\mathrm{d}x,\\
\mathcal{C}^\varepsilon\left((\mathbf{u}_1,\mathbf{C}_1),(\mathbf{u}_2,\mathbf{C}_2),(\mathbf{u}_3,\mathbf{C}_3)\right) &:= R_e\int_\Omega \left(\mathbf{u}_1\cdot\nabla\right)\mathbf{u}_2\cdot\mathbf{u}_3\,\mathrm{d}x\\
&\qquad - A_l\int_{\Omega_s^\varepsilon}\left[(\operatorname{curl}\mathbf{C}_2\times\mathbf{C}_1)\cdot\mathbf{u}_3\right.\\
&\qquad\quad \left. +\ (\mathbf{u}_2\times\mathbf{C}_1)\cdot\operatorname{curl}\mathbf{C}_3\right]\mathrm{d}x,\\
\mathcal{L}^\varepsilon(\mathbf{v},\mathbf{C}) &:= R_e\int_\Omega \mathbf{g}\cdot\mathbf{v}\,\mathrm{d}x + A_l\int_{\Omega_s^\varepsilon}\mathbf{h}\cdot\mathbf{C}\,\mathrm{d}x.
\end{aligned}
$$

We consider the weak formulation of problem (2.10):

Find $((\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon),p^\varepsilon) \in \mathfrak{X}^\varepsilon \times \mathfrak{P}^\varepsilon$ *such that for all* $((\mathbf{v},\mathbf{C}),q) \in \mathfrak{X}^\varepsilon \times \mathfrak{P}^\varepsilon$,

(3.4)
$$
\begin{aligned}
\mathcal{A}^\varepsilon\left((\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon),(\mathbf{v},\mathbf{C})\right) + \mathcal{B}^\varepsilon\left((\mathbf{v},\mathbf{C}),p^\varepsilon\right) + \mathcal{C}^\varepsilon\left((\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon),(\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon),(\mathbf{v},\mathbf{C})\right) &= \mathcal{L}^\varepsilon(\mathbf{v},\mathbf{C}),\\
\mathcal{B}^\varepsilon\left((\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon),q\right) &= 0.
\end{aligned}
$$

Before showing that the weak formulation (3.4) is equivalent to the strong formulation (2.10), we recall the following.

LEMMA 3.3 ([30, Lemma 3.17]). *If* $\mathbf{B} \in H^1_n(\Omega,\mathbb{R}^d)$, *then there exists* $\psi \in H^2(\Omega)$ *such that*

$$
\begin{cases}
-\Delta\psi &= \operatorname{div}\mathbf{B} \quad \text{in } \Omega,\\
\dfrac{\partial\psi}{\partial n} &= 0 \qquad\quad \text{on } \partial\Omega.
\end{cases} \tag{3.5}
$$

In particular, $\nabla\psi \in H^1_n(\Omega,\mathbb{R}^d)$.

PROPOSITION 3.4. *Suppose that* $\mathbf{h}$ *satisfies* (2.13). *Then* $((\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon), p^\varepsilon) \in \mathfrak{X}^\varepsilon \times \mathfrak{P}^\varepsilon$ *is a weak solution of* (3.4) *if and only if it is a solution of* (2.10)–(2.12).

Proof. The incompressibility condition (2.10b) is straightforward from the second equation of (3.4). We rewrite the first equation of (3.4) as

(3.6)
$$
\begin{aligned}
&2\int_{\Omega_f^\varepsilon} \mathbb{D}(\mathbf{u}^\varepsilon) : \mathbb{D}(\mathbf{v})\ \mathrm{d}x + \frac{A_l}{R_m}\left[\int_\Omega \operatorname{div}\ \mathbf{B}^\varepsilon \cdot \operatorname{div}\ \mathbf{C}\ \mathrm{d}x + \int_{\Omega_s^\varepsilon} \operatorname{curl} \mathbf{B}^\varepsilon \cdot \operatorname{curl} \mathbf{C}\ \mathrm{d}x\right]\\
&+ \int_\Omega p^\varepsilon \operatorname{div}\ \mathbf{v}\ \mathrm{d}x + R_e \int_\Omega (\mathbf{u}^\varepsilon \cdot \nabla)\,\mathbf{u}^\varepsilon \cdot \mathbf{v}\ \mathrm{d}x\\
&- A_l \int_{\Omega_s^\varepsilon} (\operatorname{curl} \mathbf{B}^\varepsilon \times \mathbf{B}^\varepsilon) \cdot \mathbf{v}\ \mathrm{d}x - A_l \int_{\Omega_s^\varepsilon} (\mathbf{u}^\varepsilon \times \mathbf{B}^\varepsilon) \cdot \operatorname{curl} \mathbf{C}\ \mathrm{d}x\\
&= R_e \int_\Omega \mathbf{g} \cdot \mathbf{v}\ \mathrm{d}x + A_l \int_{\Omega_s^\varepsilon} \mathbf{h} \cdot \mathbf{C}\ \mathrm{d}x.
\end{aligned}
$$

Letting $\mathbf{C} = 0$ and choosing $\mathbf{v} \in C_c^\infty(\Omega_f^\varepsilon, \mathbb{R}^d)$, we have

$$
\begin{aligned}
&2\int_{\Omega_f^\varepsilon} \mathbb{D}(\mathbf{u}^\varepsilon) : \mathbb{D}(\mathbf{v})\ \mathrm{d}x + \int_{\Omega_f^\varepsilon} p^\varepsilon \operatorname{div}\ \mathbf{v}\ \mathrm{d}x + R_e \int_{\Omega_f^\varepsilon} (\mathbf{u}^\varepsilon \cdot \nabla)\,\mathbf{u}^\varepsilon \cdot \mathbf{v}\mathrm{d}x\\
&= R_e \int_{\Omega_f^\varepsilon} \mathbf{g} \cdot \mathbf{v}\mathrm{d}x.
\end{aligned}
$$

Then using integration by parts, we obtain (2.10a). In (3.6), setting $\mathbf{C} = 0$ again and choosing $\mathbf{v} \in H_0^1(\Omega, \mathbb{R}^d)$ such that $\mathbb{D}(\mathbf{v}) = 0$ on Ω_s^ε, we obtain

$$
\begin{aligned}
&2\int_{\Omega_f^\varepsilon} \mathbb{D}(\mathbf{u}^\varepsilon) : \mathbb{D}(\mathbf{v})\ \mathrm{d}x + \int_\Omega p^\varepsilon \operatorname{div}\ \mathbf{v}\ \mathrm{d}x + R_e \int_\Omega (\mathbf{u}^\varepsilon \cdot \nabla)\,\mathbf{u}^\varepsilon \cdot \mathbf{v}\mathrm{d}x\\
&- A_l \int_{\Omega_s^\varepsilon} (\operatorname{curl} \mathbf{B}^\varepsilon \times \mathbf{B}^\varepsilon) \cdot \mathbf{v}\mathrm{d}x\\
&= R_e \int_\Omega \mathbf{g} \cdot \mathbf{v}\ \mathrm{d}x.
\end{aligned}
$$

Using integration by parts and the fact that $p^\varepsilon = 0$ in Ω_s^ε (because $p^\varepsilon \in \mathfrak{P}^\varepsilon$), we have

$$
\begin{aligned}
&\int_{\Omega_f^\varepsilon} (R_e(\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon - \operatorname{div}\ \boldsymbol{\sigma}^\varepsilon(\mathbf{u}^\varepsilon, p^\varepsilon)) \cdot \mathbf{v}\ \mathrm{d}x + \sum_i \int_{\Gamma_i^\varepsilon} \boldsymbol{\sigma}^\varepsilon(\mathbf{u}^\varepsilon, p^\varepsilon)\mathbf{n} \cdot \mathbf{v}\ \mathrm{d}\mathcal{H}^{d-1}\\
&+ R_e \int_{\Omega_s^\varepsilon} (\mathbf{u}^\varepsilon \cdot \nabla)\,\mathbf{u}^\varepsilon \cdot \mathbf{v}\ \mathrm{d}x - A_l \int_{\Omega_s^\varepsilon} (\operatorname{curl} \mathbf{B}^\varepsilon \times \mathbf{B}^\varepsilon) \cdot \mathbf{v}\ \mathrm{d}x\\
&= R_e \int_\Omega \mathbf{g} \cdot \mathbf{v}\mathrm{d}x.
\end{aligned}
$$

By (2.10a), this equation reduces to

$$
\begin{aligned}
&\sum_i \int_{\Gamma_i^\varepsilon} \boldsymbol{\sigma}^\varepsilon(\mathbf{u}^\varepsilon, p^\varepsilon)\mathbf{n} \cdot \mathbf{v}\ \mathrm{d}\mathcal{H}^{d-1}\\
&+ R_e \int_{\Omega_s^\varepsilon} (\mathbf{u}^\varepsilon \cdot \nabla)\,\mathbf{u}^\varepsilon \cdot \mathbf{v}\ \mathrm{d}x - A_l \int_{\Omega_s^\varepsilon} (\operatorname{curl} \mathbf{B}^\varepsilon \times \mathbf{B}^\varepsilon) \cdot \mathbf{v}\ \mathrm{d}x\\
&= R_e \int_{\Omega_s^\varepsilon} \mathbf{g} \cdot \mathbf{v}\ \mathrm{d}x.
\end{aligned}
$$

The rigid body motion equation $\mathbb{D}(\mathbf{v}) = 0$ in $Y^\varepsilon_{i,s}$ is equivalent to the existence of $\boldsymbol{U}_i, \boldsymbol{\omega}_i \in \mathbb{R}^d$, and $R_i \in [0,\infty)$ such that $\mathbf{v} = \boldsymbol{U}_i + R_i\boldsymbol{\omega}_i \times \mathbf{n}$ in $Y^\varepsilon_{i,s}$. This together with the last equation yields the balance laws (2.11).

Next, choosing $\mathbf{v} = 0$ in (3.6) results in

$$\begin{aligned}\frac{A_l}{R_m}\left[\int_\Omega \operatorname{div}\ \mathbf{B}^\varepsilon \cdot \operatorname{div}\ \mathbf{C}\mathrm{d}x + \int_{\Omega_s} \operatorname{curl} \mathbf{B}^\varepsilon \cdot \operatorname{curl} \mathbf{C}\mathrm{d}x\right] \\ -A_l\int_{\Omega_s}(\mathbf{u}^\varepsilon \times \mathbf{B}^\varepsilon)\cdot \operatorname{curl}\ \mathbf{C}\mathrm{d}x = A_l\int_{\Omega_s}\mathbf{h}\cdot\mathbf{C}\mathrm{d}x.\end{aligned} \tag{3.7}$$

Let ψ as in Lemma 3.3 and select $\mathbf{C} = \nabla\psi$ in (3.7); then by (2.13),

$$-\frac{A_l}{R_m}\int_\Omega (\operatorname{div}\ \mathbf{B}^\varepsilon)^2\mathrm{d}x = 0$$

so we obtain (2.10f). Therefore, (3.7) is simplified to

$$\frac{1}{R_m}\int_{\Omega_s}\operatorname{curl}\ \mathbf{B}^\varepsilon\cdot\operatorname{curl}\ \mathbf{C}\mathrm{d}x - \int_{\Omega_s}(\mathbf{u}^\varepsilon\times\mathbf{B}^\varepsilon)\cdot\operatorname{curl}\ \mathbf{C}\mathrm{d}x = \int_{\Omega_s}\mathbf{h}\cdot\mathbf{C}\mathrm{d}x.$$

Choosing $\mathbf{C} \in C_c^\infty(\Omega_s, \mathbb{R}^d)$ and integrating by parts, this implies (2.10e). ∎

3.2. Existence and a priori estimates for the fine-scale velocity and the magnetic field. First, we recall an important estimate for proving ellipticity (2.17).

PROPOSITION 3.5 ([31, Theorem 3.8]). *There exists $\kappa_{GR} > 0$ such that, for any $\mathbf{B} \in \mathfrak{H}$,*

$$\kappa_{GR}\,\|\mathbf{B}\|^2_{H^1(\Omega,\mathbb{R}^d)} \le \|\operatorname{curl}\ \mathbf{B}\|^2_{L^2(\Omega,\mathbb{R}^d)} + \|\operatorname{div}\ \mathbf{B}\|^2_{L^2(\Omega)}\,. \tag{3.8}$$

LEMMA 3.6. *The form $\mathcal{A}^\varepsilon$ is continuous and coercive on $\mathfrak{X}^\varepsilon \times \mathfrak{X}^\varepsilon$, with coercivity constant α independent of ε. In fact, $\alpha = \min\{\frac{A_l}{R_m}\kappa_{GR}, \kappa_K\} > 0$, where κ_{GR} is the constant in* (3.8) *and κ_K^{-1} is the constant in Korn's inequality.*

Proof. For any $((\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{C}))$ in $\mathfrak{X}^\varepsilon\times\mathfrak{X}^\varepsilon$, we have

$$\begin{aligned}|\mathcal{A}^\varepsilon((\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{C}))| &\le 2\,\|\mathbb{D}(\mathbf{u})\|_{L^2(\Omega,\mathbb{R}^{d\times d})}\,\|\mathbb{D}(\mathbf{v})\|_{L^2(\Omega,\mathbb{R}^{d\times d})} \\ &\quad + \frac{A_l}{R_m}\Big[\|\operatorname{div}\ \mathbf{B}\|_{L^2(\Omega)}\,\|\operatorname{div}\ \mathbf{C}\|_{L^2(\Omega)} \\ &\quad + \ \|\operatorname{curl}\ \mathbf{B}\|_{L^2(\Omega,\mathbb{R}^d)}\,\|\operatorname{curl}\ \mathbf{C}\|_{L^2(\Omega,\mathbb{R}^d)}\Big] \\ &\le C\left(\Omega,\frac{A_l}{R_m}\right)\|(\mathbf{u},\mathbf{B})\|_{\mathfrak{X}^\varepsilon}\,\|(\mathbf{v},\mathbf{C})\|_{\mathfrak{X}^\varepsilon}\,.\end{aligned}$$

Therefore, $\mathcal{A}^\varepsilon$ is continuous. Moreover, by (3.8) and Korn's inequality,

$$\begin{aligned}\mathcal{A}^\varepsilon((\mathbf{u},\mathbf{B}),(\mathbf{u},\mathbf{B})) &\ge \int_{\Omega_f}|\mathbb{D}(\mathbf{u})|^2\,\mathrm{d}x + \frac{A_l}{R_m}\left[\int_\Omega|\operatorname{div}\ \mathbf{B}|^2\,\mathrm{d}x + \int_{\Omega_s}|\operatorname{curl}\ \mathbf{B}|^2\,\mathrm{d}x\right] \\ &= \int_\Omega|\mathbb{D}(\mathbf{u})|^2\,\mathrm{d}x + \frac{A_l}{R_m}\left[\int_\Omega|\operatorname{div}\ \mathbf{B}|^2\,\mathrm{d}x + \int_\Omega|\operatorname{curl}\ \mathbf{B}|^2\,\mathrm{d}x\right] \\ &\ge \alpha\,\|(\mathbf{u},\mathbf{B})\|^2_{\mathfrak{X}^\varepsilon}\,.\end{aligned}$$ ∎

LEMMA 3.7. *The trilinear form $\mathcal{C}^\varepsilon$ is continuous on $\mathfrak{X}^\varepsilon\times\mathfrak{X}^\varepsilon\times\mathfrak{X}^\varepsilon$. Moreover, suppose $\rho_f = \rho_s$; then for all $((\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{C}),(\mathbf{w},\mathbf{D}))\in\mathfrak{X}^\varepsilon\times\mathfrak{X}^\varepsilon\times\mathfrak{X}^\varepsilon$ with* $\operatorname{div}\,\mathbf{u} = 0$, *one has*

$$\mathcal{C}^\varepsilon((\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{C}),(\mathbf{w},\mathbf{D})) = -\mathcal{C}^\varepsilon((\mathbf{u},\mathbf{B}),(\mathbf{w},\mathbf{D}),(\mathbf{v},\mathbf{C}))\,.$$

Proof. We write

$$
\begin{aligned}
&|\mathcal{C}^\varepsilon((\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{C}),(\mathbf{w},\mathbf{D}))| \\
&\quad \le C\left(\|\mathbf{u}\|_{H^1}\|\mathbf{v}\|_{H^1}\|\mathbf{w}\|_{H^1}+\|\mathbf{C}\|_{H^1}\|\mathbf{B}\|_{H^1}\|\mathbf{w}\|_{H^1}+\|\mathbf{v}\|_{H^1}\|\mathbf{B}\|_{H^1}\|\mathbf{D}\|_{H^1}\right) \\
&\quad \le C\,\|(\mathbf{u},\mathbf{B})\|_{\mathfrak{X}^\varepsilon}\|(\mathbf{v},\mathbf{C})\|_{\mathfrak{X}^\varepsilon}\|(\mathbf{w},\mathbf{D})\|_{\mathfrak{X}^\varepsilon}.
\end{aligned}
$$

The second part is a consequence of the following identities:

$$
\begin{aligned}
&(\mathbf{B}\times \operatorname{curl}\mathbf{C})\cdot\mathbf{v} = (\mathbf{v}\times\mathbf{B})\cdot\operatorname{curl}\mathbf{C},\\
&\int_U(\mathbf{u}\cdot\nabla)\,\mathbf{v}\cdot\mathbf{v}\,\mathrm{d}x = -\frac{1}{2}\int_U|\mathbf{v}|^2\operatorname{div}\mathbf{u}\,\mathrm{d}x+\frac{1}{2}\int_{\partial U}|\mathbf{v}|^2\,\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1}
\end{aligned}
$$

for $U=\Omega_f^\varepsilon$ or $U=\Omega_s^\varepsilon$.

Indeed, from the above identities and the definition of $\mathcal{C}^\varepsilon$, one has

$$\mathcal{C}^\varepsilon((\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{C}),(\mathbf{v},\mathbf{C}))=0$$

for all $(\mathbf{v},\mathbf{C})\in\mathfrak{X}^\varepsilon$; therefore,

$$
\begin{aligned}
0 &= \mathcal{C}^\varepsilon((\mathbf{u},\mathbf{B}),(\mathbf{v}-\mathbf{w},\mathbf{C}-\mathbf{D}),(\mathbf{v}-\mathbf{w},\mathbf{C}-\mathbf{D}))\\
&= \mathcal{C}^\varepsilon((\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{C}),(\mathbf{v}-\mathbf{w},\mathbf{C}-\mathbf{D}))-\mathcal{C}^\varepsilon((\mathbf{u},\mathbf{B}),(\mathbf{w},\mathbf{D}),(\mathbf{v}-\mathbf{w},\mathbf{C}-\mathbf{D}))\\
&= \mathcal{C}^\varepsilon((\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{C}),(\mathbf{v},\mathbf{C}))-\mathcal{C}^\varepsilon((\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{C}),(\mathbf{w},\mathbf{D}))\\
&\quad -\{\mathcal{C}^\varepsilon((\mathbf{u},\mathbf{B}),(\mathbf{w},\mathbf{D}),(\mathbf{v},\mathbf{C}))+\mathcal{C}^\varepsilon((\mathbf{u},\mathbf{B}),(\mathbf{w},\mathbf{D}),(\mathbf{w},\mathbf{D}))\}\\
&= -\mathcal{C}^\varepsilon((\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{C}),(\mathbf{w},\mathbf{D}))-\mathcal{C}^\varepsilon((\mathbf{u},\mathbf{B}),(\mathbf{w},\mathbf{D}),(\mathbf{v},\mathbf{C})).
\end{aligned} \tag{3.9}
$$

∎

We now define

$$a^\varepsilon((\mathbf{u},\mathbf{B});(\mathbf{v},\mathbf{C}),(\mathbf{w},\mathbf{D})) := \mathcal{A}^\varepsilon((\mathbf{v},\mathbf{C}),(\mathbf{w},\mathbf{D}))+\mathcal{C}^\varepsilon((\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{C}),(\mathbf{w},\mathbf{D})). \tag{3.10}$$

LEMMA 3.8. *The following properties hold:*

(i) *For any* $(\mathbf{v},\mathbf{C})$ *in* $\mathfrak{Y}^\varepsilon$, *we have*

$$a^\varepsilon((\mathbf{v},\mathbf{C});(\mathbf{v},\mathbf{C}),(\mathbf{v},\mathbf{C}))\ge\alpha\,\|(\mathbf{v},\mathbf{C})\|^2_{\mathfrak{X}^\varepsilon}. \tag{3.11}$$

Here α *is the coercivity constant of* $\mathcal{A}^\varepsilon$ *in Lemma* 3.6.

(ii) *If* $(\mathbf{u}_n,\mathbf{B}_n)$ *weakly converges to* $(\mathbf{u},\mathbf{B})$ *in* $\mathfrak{Y}^\varepsilon$, *then for all* $(\mathbf{v},\mathbf{C})$ *in* $\mathfrak{X}^\varepsilon$ *we have*

$$\lim_{n\to\infty}a^\varepsilon((\mathbf{u}_n,\mathbf{B}_n);(\mathbf{u}_n,\mathbf{B}_n),(\mathbf{v},\mathbf{C}))=a^\varepsilon((\mathbf{u},\mathbf{B});(\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{C})). \tag{3.12}$$

(iii) *For all* $(\mathbf{u}_1,\mathbf{B}_1),(\mathbf{u}_2,\mathbf{B}_2),(\mathbf{v},\mathbf{C})$ *and* $(\mathbf{w},\mathbf{D})$ *in* $\mathfrak{X}^\varepsilon$, *we have*

$$
\begin{aligned}
&|a^\varepsilon((\mathbf{u}_1,\mathbf{B}_1);(\mathbf{v},\mathbf{C}),(\mathbf{w},\mathbf{D}))-a^\varepsilon((\mathbf{u}_2,\mathbf{B}_2);(\mathbf{v},\mathbf{C}),(\mathbf{w},\mathbf{D}))|\\
&\le\kappa_S\max\{1,2A_l\}\,\|(\mathbf{u}_1,\mathbf{B}_1)-(\mathbf{u}_2,\mathbf{B}_2)\|_{\mathfrak{X}^\varepsilon}\|(\mathbf{v},\mathbf{C})\|_{\mathfrak{X}^\varepsilon}\|(\mathbf{w},\mathbf{D})\|_{\mathfrak{X}^\varepsilon},
\end{aligned} \tag{3.13}
$$

where $\kappa_S=\kappa_S(d,\Omega)$ *is the norm of the Sobolev embedding* H^1 *to* L^4.

Proof.

(i) This is a direct consequence of (3.10) and Lemmas 3.6 and 3.7.

(ii) Suppose $(\mathbf{u}_n, \mathbf{B}_n) \rightharpoonup (\mathbf{u}, \mathbf{B})$ in $\mathfrak{Y}^\varepsilon$. Write

$$\begin{aligned}
&|a^\varepsilon((\mathbf{u}_n,\mathbf{B}_n);(\mathbf{u}_n,\mathbf{B}_n),(\mathbf{v},\mathbf{C}) - a^\varepsilon((\mathbf{u},\mathbf{B});(\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{C})))| \\
&\quad \le |\mathcal{A}^\varepsilon((\mathbf{u}_n-\mathbf{u},\mathbf{B}_n-\mathbf{B}),(\mathbf{v},\mathbf{C}))| \\
&\qquad + |\mathcal{C}^\varepsilon((\mathbf{u}_n-\mathbf{u},\mathbf{B}_n-\mathbf{B}),(\mathbf{u}_n,\mathbf{B}_n),(\mathbf{v},\mathbf{C}))| \\
&\qquad + |\mathcal{C}^\varepsilon((\mathbf{u},\mathbf{B}),(\mathbf{u}_n-\mathbf{u},\mathbf{B}_n-\mathbf{B}),(\mathbf{v},\mathbf{C}))|.
\end{aligned} \tag{3.14}$$

Next, we have

$$\begin{aligned}
&\mathcal{A}^\varepsilon(\mathbf{u}_n-\mathbf{u},\mathbf{B}_n-\mathbf{B}),(\mathbf{v},\mathbf{C})) \\
&\quad = 2\int_{\Omega_f^\varepsilon} \mathbb{D}(\mathbf{u}_n-\mathbf{u}) : \mathbb{D}(\mathbf{v})\mathrm{d}x \\
&\qquad + \frac{A_l}{R_m}\left[\int_\Omega \mathrm{div}\,(\mathbf{B}_n-\mathbf{B})\cdot \mathrm{div}\,\mathbf{C}\mathrm{d}x + \int_{\Omega_s^\varepsilon} \mathrm{curl}\,(\mathbf{B}_n-\mathbf{B})\cdot \mathrm{curl}\,\mathbf{C}\mathrm{d}x\right] \\
&\quad = 2\int_{\Omega} \mathbb{D}(\mathbf{u}_n-\mathbf{u}) : \mathbb{D}(\mathbf{v})\mathrm{d}x \\
&\qquad + \frac{A_l}{R_m}\left[\int_\Omega \mathrm{div}\,(\mathbf{B}_n-\mathbf{B})\cdot \mathrm{div}\,\mathbf{C}\mathrm{d}x + \int_{\Omega_s^\varepsilon} \mathrm{curl}\,(\mathbf{B}_n-\mathbf{B})\cdot \mathrm{curl}\,\mathbf{C}\mathrm{d}x\right],
\end{aligned}$$

and thus, for each fixed $\varepsilon > 0$, the right hand side converges to 0 as $n \to \infty$. For the second term on the right hand side of (3.14), we have by Hölder's inequality

$$\begin{aligned}
&|\mathcal{C}^\varepsilon((\mathbf{u}_n-\mathbf{u},\mathbf{B}_n-\mathbf{B}),(\mathbf{u}_n,\mathbf{B}_n),(\mathbf{v},\mathbf{C}))| \\
&:= \Bigg| R_e\int_\Omega ((\mathbf{u}_n-\mathbf{u})\cdot\nabla)\,\mathbf{u}_n\cdot\mathbf{v}\mathrm{d}x \\
&\qquad - A_l\int_{\Omega_s^\varepsilon}\left[(\mathrm{curl}\,\mathbf{B}_n\times(\mathbf{B}_n-\mathbf{B}))\cdot\mathbf{v} + (\mathbf{u}_n\times(\mathbf{B}_n-\mathbf{B}))\cdot\mathrm{curl}\,\mathbf{C}\right]\mathrm{d}x\Bigg|, \\
&\le R_e\|\mathbf{u}_n-\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}_n\|_{L^2}\|\mathbf{v}\|_{L^4} \\
&\qquad + 2A_l\left[\|\nabla\mathbf{B}_n\|_{L^2}\|\mathbf{B}_n-\mathbf{B}\|_{L^4}\|\mathbf{v}\|_{L^4} + \|\mathbf{u}_n\|_{L^4}\|\mathbf{B}_n-\mathbf{B}\|_{L^4}\|\mathrm{curl}\,\mathbf{C}\|_{L^2}\right].
\end{aligned}$$

By the Rellich–Kondrachov theorem, we have that, up to a subsequence, $(\mathbf{u}_n, \mathbf{B}_n)$ strongly converges to $(\mathbf{u}, \mathbf{B})$ in $L^4(\Omega, \mathbb{R}^d) \times L^4(\Omega, \mathbb{R}^d)$. Therefore, the estimate above shows that the second term on the right hand side of (3.14) also converges to 0 as $n \to \infty$.

The last term on the right hand side of (3.14) is

$$\begin{aligned}
|\mathcal{C}^\varepsilon((\mathbf{u},\mathbf{B}),(\mathbf{u}_n-\mathbf{u},\mathbf{B}_n-\mathbf{B}),(\mathbf{v},\mathbf{C}))| &:= \Bigg| R_e\int_\Omega (\mathbf{u}\cdot\nabla)\,(\mathbf{u}_n-\mathbf{u})\cdot\mathbf{v}\mathrm{d}x \\
- A_l\int_{\Omega_s^\varepsilon}(\mathrm{curl}\,(\mathbf{B}_n-\mathbf{B})\times\mathbf{B})\cdot\mathbf{v}\mathrm{d}x &- \int_{\Omega_s^\varepsilon}((\mathbf{u}_n-\mathbf{u})\times\mathbf{B})\cdot\mathrm{curl}\,\mathbf{C}\mathrm{d}x\Bigg|.
\end{aligned}$$

The first and the last integrals converge to 0 by a similar argument as above. The middle one converges to 0 due to the weak convergence $\mathbf{B}_n \rightharpoonup \mathbf{B}$ in $H^1(\Omega, \mathbb{R}^d)$.

(iii) From definition (3.10) and the Sobolev embedding H^1 to L^4, where the norm of the embedding is denoted by $\kappa_S(d,\Omega)$, we obtain

$$
\begin{aligned}
&|a^\varepsilon((\mathbf{u}_1,\mathbf{B}_1);(\mathbf{v},\mathbf{C}),(\mathbf{w},\mathbf{D})) - a^\varepsilon((\mathbf{u}_2,\mathbf{B}_2);(\mathbf{v},\mathbf{C}),(\mathbf{w},\mathbf{D}))|\\
&= |\mathcal{C}^\varepsilon((\mathbf{u}_1,\mathbf{B}_1);(\mathbf{v},\mathbf{C}),(\mathbf{w},\mathbf{D})) - \mathcal{C}^\varepsilon((\mathbf{u}_2,\mathbf{B}_2);(\mathbf{v},\mathbf{C}),(\mathbf{w},\mathbf{D}))|\\
&= |\mathcal{C}^\varepsilon((\mathbf{u}_1-\mathbf{u}_2,\mathbf{B}_1-\mathbf{B}_2);(\mathbf{v},\mathbf{C}),(\mathbf{w},\mathbf{D}))|\\
&\le R_e\|\mathbf{u}_1-\mathbf{u}_2\|_{L^4}\|\nabla\mathbf{v}\|_{L^2}\|\mathbf{w}\|_{L^4} + 2A_l\|\nabla\mathbf{C}\|_{L^2}\|\mathbf{B}_1-\mathbf{B}_2\|_{L^4}\|\mathbf{w}\|_{L^4}\\
&\qquad + 2A_l\mu_s\|\mathbf{v}\|_{L^4}\|\mathbf{B}_1-\mathbf{B}_2\|_{L^4}\|\nabla\mathbf{D}\|_{L^2}\\
&\le \kappa_S(d,\Omega)\max\{R_e,2A_l\}\,[\|\mathbf{u}_1-\mathbf{u}_2\|_{H^1}\|\mathbf{v}\|_{H^1}\|\mathbf{w}\|_{H^1}\\
&\qquad + \|\mathbf{C}\|_{H^1}\|\mathbf{B}_1-\mathbf{B}_2\|_{H^1}\|\mathbf{w}\|_{H^1} + \|\mathbf{v}\|_{H^1}\|\mathbf{B}_1-\mathbf{B}_2\|_{H^1}\|\nabla\mathbf{D}\|_{H^1}]\\
&\le \kappa_S(d,\Omega)\max\{R_e,2A_l\}\,\|(\mathbf{u}_1,\mathbf{B}_1)-(\mathbf{u}_2,\mathbf{B}_2)\|_{\mathfrak{X}^\varepsilon}\|(\mathbf{v},\mathbf{C})\|_{\mathfrak{X}^\varepsilon}\|(\mathbf{w},\mathbf{D})\|_{\mathfrak{X}^\varepsilon}. \qquad \square
\end{aligned}
$$

From Theorem 2.2 and Lemmas 3.6, 3.7, and 3.8, we conclude the following.

PROPOSITION 3.9. *Let* $\alpha = \min\{\frac{A_l}{R_m}\kappa_{GR},\kappa_K\}$ *be the coercivity constant of* $\mathcal{A}^\varepsilon$ *in Lemma* 3.6 *and* $\kappa_S = \kappa_S(d,\Omega)$ *be the norm of the Sobolev embedding* H^1 *to* L^4. *Then the variational problem* (3.4) *has a solution* $((\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon),p^\varepsilon) \in \mathfrak{X}^\varepsilon \times \mathfrak{P}^\varepsilon$ *such that*

$$\|(\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon)\|_{\mathfrak{X}^\varepsilon} \le \frac{\|\mathcal{L}^\varepsilon\|_{(\mathfrak{Y}^\varepsilon)'}}{\alpha}. \tag{3.15}$$

Moreover, if

$$\kappa_S\max\{R_e,2A_l\}\,\|\mathcal{L}^\varepsilon\|_{(\mathfrak{Y}^\varepsilon)'} \le \alpha^2, \tag{3.16}$$

then the solution is unique.

By Hölder's inequality,

$$
\begin{aligned}
|\mathcal{L}^\varepsilon(\mathbf{v},\mathbf{C})| &\le R_e\|\mathbf{g}\|_{L^2}\|\mathbf{v}\|_{L^2} + A_l\|\mathbf{h}\|_{L^2}\|\mathbf{C}\|_{L^2}\\
&\le 2\left(R_e\|\mathbf{g}\|_{L^2} + A_l\|\mathbf{h}\|_{L^2}\right)\|(\mathbf{v},\mathbf{C})\|_{\mathfrak{X}^\varepsilon}.
\end{aligned}
$$

Thus, from (3.15), we obtain the following a priori estimate:

$$\|(\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon)\|_{\mathfrak{X}^\varepsilon} \le \frac{2}{\alpha}\left(R_e\|\mathbf{g}\|_{L^2} + A_l\|\mathbf{h}\|_{L^2}\right), \tag{3.17}$$

where the right hand side is surely independent of ε.

3.3. Existence and a priori estimate for the fine-scale pressure. The following result is adapted from [2, Theorem 4.1] (see also [1, Theorem 2.6], and [27, Theorem III.3.1]),

THEOREM 3.10. *Let* $\Omega \subset \mathbb{R}^d$ *be a Lipschitz domain with Lipschitz constant* ℓ. *Then, there exists a bounded linear operator* Bog: $L_0^2(\Omega) \to H_0^1(\Omega,\mathbb{R}^d)$, $f \mapsto \mathrm{Bog}f$, *called the Bogovskiĭ map, such that, for all* $f \in L_0^2(\Omega)$,

$$\mathrm{div}\ \mathrm{Bog}f = f. \tag{3.18}$$

Moreover, the norm $\|\mathrm{Bog}\|$ *depends only on* d,ℓ, *and* $\mathrm{diam}(\Omega)$.

For $p \in \mathfrak{P}^\varepsilon$, there exists $\mathbf{v} \in \mathfrak{V}^\varepsilon$ such that $p = \mathrm{div}\,\mathbf{v}$. Thus $p = 0$ in Ω_s^ε since $\mathbb{D}(\mathbf{v}) = 0$ in Ω_s^ε. Adapting the construction in [22, Step 1, Proof of Lemma 3.3] (see also [21, Step 4, Proof of Proposition 2.1], [19, Lemma 3.2], [5, Theorem 2.1], and [42, Lemma 4.8]) and using Theorem 3.10, we obtain the following.

LEMMA 3.11. *For each* $p \in \mathfrak{P}^\varepsilon$, *there exists* $\mathbf{v} \in H_0^1(\Omega, \mathbb{R}^d)$ *such that the following hold:*

1. $\mathbf{v}$ *is constant on* $Y_{i,s}^\varepsilon$ *for all* i *(and thus* $\mathbf{v} \in \mathfrak{V}^\varepsilon$*).*
2. $\text{div } \mathbf{v} = p$.
3. $\|\mathbf{v}\|_{H_0^1} \le \|\text{Bog}\| \|p\|_{L^2}$.

Note that we don't necessarily have $\mathbf{v} = \text{Bog}p$. Actually, $\mathbf{v}$ is obtained by modifying Bog$p$ so that 1 and 3 are satisfied.

LEMMA 3.12. *The space* $\mathfrak{P}^\varepsilon$ *defined in section* 3 *is a Hilbert space with respect to the* L^2-*inner product.*

Proof. Let the space $\mathfrak{P}^\varepsilon$ be equipped with the L^2-inner product. It is well-known that $L_0^2(\Omega)$ is a Hilbert space with respect to this inner product (see [12, Lemma IV.1.9]). Since $\mathfrak{P}^\varepsilon$ is a subset of $L_0^2(\Omega)$ closed under addition and scalar multiplication, we only need to show that $\mathfrak{P}^\varepsilon$ is closed. For that, let $\mathfrak{P}^\varepsilon \ni q_n \to q_0 \in L_0^2(\Omega)$, and we will prove that $q_0 \in \mathfrak{P}^\varepsilon$.

Since $q_n \in \mathfrak{P}^\varepsilon = \text{div } (\mathfrak{V}^\varepsilon)$, by Lemma 3.11, we have $q_n = \text{div } \mathbf{v}_n$ for some $\mathbf{v}_n \in \mathfrak{V}^\varepsilon$ and

$$\|\mathbf{v}_n\|_{H_0^1(\Omega,\mathbb{R}^d)} \le \|\text{Bog}\| \, \|q_n\|_{L_0^2(\Omega)} .$$

Since q_n converges to q_0 in $L_0^2(\Omega)$, it is bounded in $L_0^2(\Omega)$, which implies that $\mathbf{v}_n$ is also bounded in $H_0^1(\Omega, \mathbb{R}^d)$. On the one hand, since $H_0^1(\Omega, \mathbb{R}^d)$ is reflexive, the Eberlain–Šmulian theorem states that, up to a subsequence, there exists a $\mathbf{v}_0 \in H_0^1(\Omega, \mathbb{R}^d)$ such that $\mathbf{v}_n \rightharpoonup \mathbf{v}_0$ weakly in $H_0^1(\Omega, \mathbb{R}^d)$. Testing this convergence with $\mathcal{Q} \in C_c^\infty(\Omega, \mathbb{R}^{d\times d})$, with supp$\mathcal{Q} \subset \Omega_s^\varepsilon$, shows that $\mathbf{v}_0 \in \mathfrak{V}^\varepsilon$. On the other hand, by letting $\psi \in C_c^\infty(\Omega)$, we observe that

$$\int_\Omega (q_0 - \text{div } \mathbf{v}_0)\psi \mathrm{d}x = \int_\Omega (q_0 - q_n)\psi \mathrm{d}x + \int_\Omega (\text{div } \mathbf{v}_n - \text{div } \mathbf{v}_0)\psi \mathrm{d}x \qquad \xrightarrow{n\to\infty} 0.$$

Therefore, $q_0 = \text{div } \mathbf{v}_0$, which means that $q_0 \in \mathfrak{P}^\varepsilon$. ∎

LEMMA 3.13. *The bilinear form* $\mathcal{B}^\varepsilon$ *is continuous on* $\mathfrak{X}^\varepsilon \times \mathfrak{P}^\varepsilon$ *and satisfies the inf-sup condition*

$$\text{(3.19)} \qquad \exists \beta > 0 \quad \text{such that} \quad \inf_{q\in\mathfrak{P}^\varepsilon\setminus\{0\}} \sup_{(\mathbf{v},\mathbf{C})\in\mathfrak{X}^\varepsilon\setminus\{(\mathbf{0},\mathbf{0})\}} \frac{\mathcal{B}^\varepsilon((\mathbf{v},\mathbf{C}),q)}{\|(\mathbf{v},\mathbf{C})\|_{\mathfrak{X}^\varepsilon} \|q\|_{\mathfrak{P}^\varepsilon}} \ge \beta.$$

Moreover, the constant β *is independent of* ε. *In particular, one can choose* $\beta = \|\text{Bog}\|^{-1}$, *where* Bog *is the Bogovskiĭ map defined in Theorem* 3.10.

Proof. Recall that $\mathfrak{P}^\varepsilon$ inherits the L^2-norm from $L_0^2(\Omega)$. We have

$$|\mathcal{B}^\varepsilon((\mathbf{v},\mathbf{C}),q)| \le C \|q\|_{L^2} \|\text{div } \mathbf{v}\|_{L^2} \le C \|q\|_{\mathfrak{P}^\varepsilon} \|(\mathbf{v},\mathbf{C})\|_{\mathfrak{X}^\varepsilon} ,$$

so $\mathcal{B}^\varepsilon$ is continuous on $\mathfrak{X}^\varepsilon \times \mathfrak{P}^\varepsilon$.

Since $\mathfrak{P}^\varepsilon$ is a Hilbert space by Lemma 3.12, there exists a Riesz isomorphism $\imath_R\colon \mathfrak{P}^\varepsilon \to (\mathfrak{P}^\varepsilon)'$. Let $B := \imath_R \circ \text{div}$ and then B is a continuous surjective map from $\mathfrak{V}^\varepsilon$ to $(\mathfrak{P}^\varepsilon)'$. Moreover, for $\mathbf{v} \in \mathfrak{V}^\varepsilon$ and $q \in \mathfrak{P}^\varepsilon$,

$$\text{(3.20)} \qquad {}_{(\mathfrak{P}^\varepsilon)'}\langle B\mathbf{v}, p\rangle_{\mathfrak{V}^\varepsilon} = {}_{(\mathfrak{P}^\varepsilon)'}\langle \imath_R(\text{div } \mathbf{v}), q\rangle_{\mathfrak{V}^\varepsilon} = (\text{div } \mathbf{v}, q)_{L^2} = \mathcal{B}^\varepsilon((\mathbf{v},\mathbf{C}),q),$$

so B is the operator associated to $\mathcal{B}^\varepsilon$. Therefore, the inf-sup condition follows by Proposition 2.4.

Fix a function $q \in \mathfrak{P}^\varepsilon$ and denote by $\mathbf{v}_q$ the corresponding field obtained from Lemma 3.11. We have

$$\begin{aligned}\sup_{(\mathbf{v},\mathbf{C})\in\mathfrak{X}^\varepsilon\setminus\{(\mathbf{0},\mathbf{0})\}} \frac{\mathcal{B}^\varepsilon((\mathbf{v},\mathbf{C}),q)}{\|(\mathbf{v},\mathbf{C})\|_{\mathfrak{X}^\varepsilon}\|q\|_{\mathfrak{P}^\varepsilon}} &\geq \sup_{\substack{(\mathbf{v},\mathbf{C})\in\mathfrak{X}^\varepsilon\setminus\{(\mathbf{0},\mathbf{0})\}\\ \mathbf{C}=\mathbf{0}}} \frac{\mathcal{B}^\varepsilon((\mathbf{v},\mathbf{C}),q)}{\|(\mathbf{v},\mathbf{C})\|_{\mathfrak{X}^\varepsilon}\|q\|_{\mathfrak{P}^\varepsilon}} \\ &= \sup_{\mathbf{v}\in\mathfrak{V}^\varepsilon} \frac{\int_\Omega q \operatorname{div}\mathbf{v}\mathrm{d}x}{\|\mathbf{v}\|_{H_0^1}\|q\|_{L^2}} \geq \frac{\int_\Omega q \operatorname{div}\mathbf{v}_q\mathrm{d}x}{\|\mathbf{v}_q\|_{H_0^1}\|q\|_{L^2}} = \frac{\|q\|_{L^2}}{\|\mathbf{v}_q\|_{H_0^1}} \geq \frac{1}{\|\mathrm{Bog}\|}.\end{aligned} \tag{3.21}$$

Therefore, we choose $\beta = \|\mathrm{Bog}\|^{-1}$, which is independent of ε. ∎

Proposition 3.9 and Lemma 3.13 imply the existence and uniqueness of the fine-scale pressure p^ε. Moreover, from (3.19) (with $\beta = \|\mathrm{Bog}\|^{-1}$), (3.4), and (3.17), we have

$$\begin{aligned}\|p^\varepsilon\|_{L^2} &\leq \|\mathrm{Bog}\| \sup_{(\mathbf{v},\mathbf{C})\in\mathfrak{X}^\varepsilon\setminus\{(\mathbf{0},\mathbf{0})\}} \frac{\mathcal{B}^\varepsilon((\mathbf{v},\mathbf{C}),p^\varepsilon)}{\|(\mathbf{v},\mathbf{C})\|_{\mathfrak{X}^\varepsilon}} \\ &\leq \|\mathrm{Bog}\| \sup_{(\mathbf{v},\mathbf{C})\in\mathfrak{X}^\varepsilon\setminus\{(\mathbf{0},\mathbf{0})\}} \frac{1}{\|(\mathbf{v},\mathbf{C})\|_{\mathfrak{X}^\varepsilon}} \{|\mathcal{A}^\varepsilon((\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon),(\mathbf{v},\mathbf{C}))| \\ &\qquad + |\mathcal{C}^\varepsilon((\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon),(\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon),(\mathbf{v},\mathbf{C}))| + |\mathcal{L}^\varepsilon(\mathbf{v},\mathbf{C})|\} \\ &\leq C\left(\|(\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon)\|_{\mathfrak{X}^\varepsilon} + \|(\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon)\|^2_{\mathfrak{X}^\varepsilon} + R_e\|\mathbf{g}\|_{L^2} + A_l\|\mathbf{h}\|_{L^2}\right).\end{aligned}$$

In particular, by (3.17), we obtain

$$\|p^\varepsilon\|_{L^2} \leq C\left(R_e\|\mathbf{g}\|_{L^2} + A_l\|\mathbf{h}\|_{L^2} + 1\right)^2, \tag{3.22}$$

where C is independent of ε.

3.4. The two-scale homogenized problem. By (3.17) and (3.22), there exist $\mathbf{u}^0 \in H_0^1(\Omega,\mathbb{R}^d)$, $\mathbf{B}^0 \in H_n^1(\Omega,\mathbb{R}^d)$, $\mathbf{u}^1 \in L^2(\Omega, H^1_{\mathrm{per}}(Y,\mathbb{R}^d)/\mathbb{R})$, $\mathbf{B}^1 \in L^2(\Omega, H^1_{\mathrm{per}}(Y,\mathbb{R}^d)/\mathbb{R})$, and $p^0 \in L_0^2(\Omega\times Y)$ such that, up to a subsequence,

$$\begin{aligned}&\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u}^0,\ \mathbf{B}^\varepsilon \rightharpoonup \mathbf{B}^0 \text{ weakly in } H^1(\Omega,\mathbb{R}^d), \\ &\mathbf{u}^\varepsilon \overset{2}{\rightharpoonup} \mathbf{u}^0,\ \mathbf{B}^\varepsilon \overset{2}{\rightharpoonup} \mathbf{B}^0 \text{ two-scale}, \\ &\nabla\mathbf{u}^\varepsilon \overset{2}{\rightharpoonup} \nabla\mathbf{u}^0(x) + \nabla_y\mathbf{u}^1(x,y),\ \nabla\mathbf{B}^\varepsilon \overset{2}{\rightharpoonup} \nabla\mathbf{B}^0(x) + \nabla_y\mathbf{B}^1(x,y) \text{ two-scale}, \\ &p^\varepsilon \rightharpoonup [2]p^0 \text{ two-scale}.\end{aligned} \tag{3.23}$$

Let $\mathbf{v} = \mathbf{v}^0(\cdot) + \varepsilon\mathbf{v}^1\left(\cdot,\frac{\cdot}{\varepsilon}\right)$ and $\mathbf{C} = \mathbf{C}^0(\cdot) + \varepsilon\mathbf{C}^1\left(\cdot,\frac{\cdot}{\varepsilon}\right)$ with $\mathbf{v}^0,\mathbf{C}^0 \in \mathcal{D}(\Omega,\mathbb{R}^d)$ and $\mathbf{v}^1,\mathbf{C}^1 \in \mathcal{D}(\Omega, C^\infty_{\mathrm{per}}(Y,\mathbb{R}^d))$. Let $q = q^0(\cdot) + \varepsilon q^1\left(\cdot,\frac{\cdot}{\varepsilon}\right)$ with $q^0 \in \mathcal{D}(\Omega)$ and $q^1 \in \mathcal{D}(\Omega, C^\infty_{\mathrm{per}}(Y))$.

The effective form corresponding to $\mathcal{A}^\varepsilon$.

By definition and (2.7c),

$$\begin{aligned}&\mathcal{A}^\varepsilon((\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon),(\mathbf{v},\mathbf{C})) \\ &\quad = 2\int_{\Omega_f^\varepsilon}\mathbb{D}(\mathbf{u}^\varepsilon):\mathbb{D}(\mathbf{v})\mathrm{d}x + \frac{A_l}{R_m}\left(\int_\Omega \operatorname{div}\mathbf{B}^\varepsilon\cdot\operatorname{div}\mathbf{C}\mathrm{d}x + \int_{\Omega_s^\varepsilon}\operatorname{curl}\mathbf{B}^\varepsilon\cdot\operatorname{curl}\mathbf{C}\mathrm{d}x\right) \\ &\quad = 2\int_{\Omega}\mathbb{D}(\mathbf{u}^\varepsilon):\mathbb{D}(\mathbf{v})\mathrm{d}x + \frac{A_l}{R_m}\left(\int_\Omega \operatorname{div}\mathbf{B}^\varepsilon\cdot\operatorname{div}\mathbf{C}\mathrm{d}x + \int_{\Omega_s^\varepsilon}\operatorname{curl}\mathbf{B}^\varepsilon\cdot\operatorname{curl}\mathbf{C}\mathrm{d}x\right) \\ &\quad =: 2Q_1 + \frac{A_l}{R_m}(Q_2+Q_3).\end{aligned}$$

Then (3.23) implies

$$\begin{aligned}\lim_{\varepsilon\to 0} Q_1 &= \lim_{\varepsilon\to 0}\int_\Omega \mathbb{D}(\mathbf{u}^\varepsilon):\mathbb{D}(\mathbf{v})\mathrm{d}x\\ &= \lim_{\varepsilon\to 0}\int_\Omega \mathbb{D}(\mathbf{u}^\varepsilon):\left[\mathbb{D}(\mathbf{v}^0)(x)+\varepsilon\mathbb{D}\left(\mathbf{v}^1\right)\left(x,\frac{x}{\varepsilon}\right)+\mathbb{D}_y(\mathbf{v}^1)\left(x,\frac{x}{\varepsilon}\right)\right]\mathrm{d}x\\ &= \frac{1}{|Y|}\int_\Omega\int_Y\left[\mathbb{D}(\mathbf{u}^0)+\mathbb{D}_y(\mathbf{u}^1)\right]:\left[\mathbb{D}(\mathbf{v}^0)+\mathbb{D}_y(\mathbf{v}^1)\right]\mathrm{d}y\mathrm{d}x.\end{aligned} \tag{3.24}$$

Similarly, we have

$$\begin{aligned}\lim_{\varepsilon\to 0} Q_2 &= \lim_{\varepsilon\to 0}\int_\Omega \operatorname{div}\mathbf{B}^\varepsilon\cdot\operatorname{div}\mathbf{C}\mathrm{d}x\\ &= \frac{1}{|Y|}\int_\Omega\int_Y\left(\operatorname{div}\mathbf{B}^0+\operatorname{div}{}_y\mathbf{B}^1\right)\cdot\left(\operatorname{div}\mathbf{C}^0+\operatorname{div}{}_y\mathbf{C}^1\right)\mathrm{d}y\mathrm{d}x.\end{aligned} \tag{3.25}$$

To compute the limit of the integral Q_3, we make use of the following limiting behaviors of the domain Ω_s^ε, which varies as ε goes to 0. Clearly,

$$\mathbb{1}_{\Omega_s^\varepsilon}\xrightarrow{2}\mathbb{1}_{\Omega\times Y_s}\quad\text{and}\quad\lim_{\varepsilon\to 0}\left\|\mathbb{1}_{\Omega_s^\varepsilon}\right\|_{L^2(\Omega)}=\left\|\mathbb{1}_{\Omega\times Y_s}\right\|_{L^2(\Omega\times Y)}. \tag{3.26}$$

Since $\mathbb{1}_{\Omega\times Y_s}\in L^2_{\text{per}}(Y,C(\Omega))$, we obtain from Theorem 2.7 that

$$\lim_{\varepsilon\to 0}\left\|\mathbb{1}_{\Omega_s^\varepsilon}(x)-\mathbb{1}_{\Omega\times Y_s}\left(x,\frac{x}{\varepsilon}\right)\right\|_{L^2(\Omega)}=0. \tag{3.27}$$

Now we write

$$\begin{aligned}Q_3 = &\int_{\Omega_s^\varepsilon}\operatorname{curl}\mathbf{B}^\varepsilon(x)\cdot\operatorname{curl}\mathbf{C}^0(x)\mathrm{d}x+\varepsilon\int_{\Omega_s^\varepsilon}\operatorname{curl}\mathbf{B}^\varepsilon(x)\cdot\operatorname{curl}\mathbf{C}^1\left(x,\frac{x}{\varepsilon}\right)\mathrm{d}x\\ &+\int_{\Omega_s^\varepsilon}\operatorname{curl}\mathbf{B}^\varepsilon\cdot\operatorname{curl}_y\mathbf{C}^1\left(x,\frac{x}{\varepsilon}\right)\mathrm{d}x\\ =&: L_1+L_2+L_3.\end{aligned}$$

Clearly, $\lim_{\varepsilon\to 0} L_2 = 0$. By (3.26), (3.23), and (2.24) of Theorem 2.7 we have

$$\begin{aligned}\lim_{\varepsilon\to 0} L_1 &= \lim_{\varepsilon\to 0}\int_\Omega\mathbb{1}_{\Omega_s^\varepsilon}(x)\operatorname{curl}\mathbf{B}^\varepsilon(x)\cdot\operatorname{curl}\mathbf{C}^0(x)\mathrm{d}x\\ &= \frac{1}{|Y|}\int_\Omega\int_{Y_s}\left(\operatorname{curl}\mathbf{B}^0(x)+\operatorname{curl}_y\mathbf{B}^1(x,y)\right)\cdot\operatorname{curl}\mathbf{C}^0(x)\mathrm{d}y\mathrm{d}x.\end{aligned}$$

And finally, for L_3, we have

$$\begin{aligned}L_3 = &\int_\Omega\operatorname{curl}\mathbf{B}^\varepsilon(x)\cdot\left(\mathbb{1}_{\Omega_s^\varepsilon}(x)-\mathbb{1}_{\Omega\times Y_s}\left(x,\frac{x}{\varepsilon}\right)\right)\operatorname{curl}_y\mathbf{C}^1\left(x,\frac{x}{\varepsilon}\right)\mathrm{d}x\\ &+\int_\Omega\operatorname{curl}\mathbf{B}^\varepsilon(x)\cdot\mathbb{1}_{\Omega\times Y_s}\left(x,\frac{x}{\varepsilon}\right)\operatorname{curl}_y\mathbf{C}^1\left(x,\frac{x}{\varepsilon}\right)\mathrm{d}x.\end{aligned}$$

For the first integral above, we obtain

$$\begin{aligned}&\left|\int_\Omega\operatorname{curl}\mathbf{B}^\varepsilon(x)\cdot\left(\mathbb{1}_{\Omega_s^\varepsilon}(x)-\mathbb{1}_{\Omega\times Y}\left(x,\frac{x}{\varepsilon}\right)\right)\operatorname{curl}_y\mathbf{C}^1\left(x,\frac{x}{\varepsilon}\right)\mathrm{d}x\right|\\ &\quad\le C\left\|\nabla_y\mathbf{C}^1\right\|_{L^\infty}\|\nabla\mathbf{B}^\varepsilon\|_{L^2}\left\|\mathbb{1}_{\Omega_s^\varepsilon}(x)-\mathbb{1}_{\Omega\times Y_s}\left(x,\frac{x}{\varepsilon}\right)\right\|_{L^2}\longrightarrow 0\end{aligned}$$

as $\varepsilon \to 0$ due to (3.27) and Hölder's inequality. By the latter and (3.23), we have

$$\lim_{\varepsilon\to 0} L_3 = \lim_{\varepsilon\to 0}\int_\Omega \text{curl }\mathbf{B}^\varepsilon(x)\cdot \mathbb{1}_{\Omega\times Y_s}\text{curl}_y\mathbf{C}^1\left(x,\frac{x}{\varepsilon}\right)\text{d}x$$
$$= \frac{1}{|Y|}\int_\Omega\int_{Y_s}\left(\text{curl }\mathbf{B}^0(x)+\text{curl}_y\mathbf{B}^1(x,y)\right)\cdot\text{curl}_y\mathbf{C}^1(x,y)\text{d}y\text{d}x.$$

In conclusion, we have

$$\lim_{\varepsilon\to 0} Q_3 = \frac{1}{|Y|}\int_\Omega\int_{Y_s}\left(\text{curl }\mathbf{B}^0+\text{curl}_y\mathbf{B}^1\right)\cdot\left(\text{curl }\mathbf{C}^0+\text{curl}_y\mathbf{C}^1\right)\text{d}y\text{d}x. \tag{3.28}$$

From (3.24), (3.25), and (3.28), the effective form $\mathcal{A}^0$, corresponding to the limit as $\varepsilon\to 0$ of $\mathcal{A}^\varepsilon$, is given by

$$\begin{aligned}\mathcal{A}^0 &:= \frac{2}{|Y|}\int_\Omega\int_Y\left(\mathbb{D}(\mathbf{u}^0)+\mathbb{D}_y(\mathbf{u}^1)\right):\left[\mathbb{D}(\mathbf{v}^0)+\mathbb{D}_y(\mathbf{v}^1)\right]\text{d}y\text{d}x\\ &\quad+\frac{A_l}{R_m}\left\{\frac{1}{|Y|}\int_\Omega\int_Y\left(\text{div }\mathbf{B}^0+\text{div }_y\mathbf{B}^1\right)\cdot\left(\text{div }\mathbf{C}^0+\text{div }_y\mathbf{C}^1\right)\text{d}y\text{d}x\right.\\ &\quad\left.+\frac{1}{|Y|}\int_\Omega\int_{Y_s}\left(\text{curl }\mathbf{B}^0+\text{curl}_y\mathbf{B}^1\right)\cdot\left(\text{curl }\mathbf{C}^0+\text{curl}_y\mathbf{C}^1\right)\text{d}y\text{d}x\right\}.\end{aligned} \tag{3.29}$$

The effective forms corresponding to $\mathcal{B}^\varepsilon$ *and* $\mathcal{L}^\varepsilon$. From the last limit of (3.23), we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathcal{B}^\varepsilon\left((\mathbf{v},\mathbf{C}),p^\varepsilon\right) &= \lim_{\varepsilon\to 0}\int_\Omega p^\varepsilon\text{div }\mathbf{v}\text{d}x\\ &= \lim_{\varepsilon\to 0}\int_\Omega p^\varepsilon(x)\left(\text{div }\mathbf{v}^0(x)+\varepsilon\text{div }\mathbf{v}^1\left(x,\frac{x}{\varepsilon}\right)+\text{div }_y\mathbf{v}^1\left(x,\frac{x}{\varepsilon}\right)\right)\text{d}x\\ &= \frac{1}{|Y|}\int_\Omega\int_Y p^0\left(\text{div }\mathbf{v}^0+\text{div }_y\mathbf{v}^1\right)\text{d}y\text{d}x.\end{aligned} \tag{3.30}$$

Moreover,

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathcal{B}^\varepsilon\left((\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon),q\right) &= \lim_{\varepsilon\to 0}\int_\Omega q\text{div }\mathbf{u}^\varepsilon\text{d}x\\ &= \frac{1}{|Y|}\int_\Omega\int_Y q^0(x)\left(\text{div }\mathbf{u}^0(x)+\text{div }_y\mathbf{u}^1(x,y)\right)\text{d}y\text{d}x.\end{aligned} \tag{3.31}$$

From (3.26), we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathcal{L}^\varepsilon(\mathbf{v},\mathbf{C}) &= \lim_{\varepsilon\to 0}\left(R_e\int_\Omega\mathbf{g}\cdot\mathbf{v}\text{d}x+A_l\int_{\Omega_s^\varepsilon}\mathbf{h}\cdot\mathbf{C}\text{d}x\right)\\ &= R_e\int_\Omega\mathbf{g}\cdot\mathbf{v}^0\text{d}x+A_l\frac{|Y_s|}{|Y|}\int_\Omega\mathbf{h}\cdot\mathbf{C}^0\text{d}x.\end{aligned} \tag{3.32}$$

The effective form corresponding to $\mathcal{C}^\varepsilon$. Recall that

$$\begin{aligned}&\mathcal{C}^\varepsilon\left((\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon),(\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon),(\mathbf{v},\mathbf{C})\right)\\ &\quad= R_e\int_\Omega\left(\mathbf{u}^\varepsilon\cdot\nabla\right)\mathbf{u}^\varepsilon\cdot\mathbf{v}\text{d}x\\ &\qquad-A_l\int_{\Omega_s^\varepsilon}\left(\text{curl }\mathbf{B}^\varepsilon\times\mathbf{B}^\varepsilon\right)\cdot\mathbf{v}\text{d}x-A_l\int_{\Omega_s^\varepsilon}\left(\mathbf{u}^\varepsilon\times\mathbf{B}^\varepsilon\right)\cdot\text{curl }\mathbf{C}\text{d}x\\ &\quad=: R_eI_1-A_lI_2-A_lI_3.\end{aligned}$$

- To obtain $\lim_{\varepsilon\to 0} I_1$, we split

$$\begin{aligned} I_1 &= \int_\Omega (\mathbf{u}^\varepsilon \cdot \nabla)\, \mathbf{u}^\varepsilon \cdot \mathbf{v} \mathrm{d}x \\ &= \int_\Omega \left((\mathbf{u}^\varepsilon - \mathbf{u}^0)\cdot \nabla\right) \mathbf{u}^\varepsilon \cdot \mathbf{v}\mathrm{d}x + \int_\Omega \left(\mathbf{u}^0 \cdot \nabla\right) \mathbf{u}^\varepsilon \cdot \mathbf{v} \mathrm{d}x \\ &=: J_1 + J_2. \end{aligned} \tag{3.33}$$

From (3.17) and (3.23), we have

$$\begin{aligned} \lim_{\varepsilon\to 0} |J_1| &\le \lim_{\varepsilon\to 0} \left\|\mathbf{u}^\varepsilon - \mathbf{u}^0\right\|_{L^2} \|\nabla \mathbf{u}^\varepsilon\|_{L^2} \|\mathbf{v}\|_{L^\infty} \\ &\le \lim_{\varepsilon\to 0} \left\|\mathbf{u}^\varepsilon - \mathbf{u}^0\right\|_{L^2} \frac{1}{\alpha} \left(R_e \|\mathbf{g}\|_{L^2} + A_l \|\mathbf{h}\|_{L^2}\right) \|\mathbf{v}\|_{L^\infty} = 0. \end{aligned}$$

From the above and (3.23), and since $(\mathbf{u}^\varepsilon \cdot \nabla)\, \mathbf{u}^\varepsilon = u_i^\varepsilon \frac{\partial}{\partial x_i} \mathbf{u}^\varepsilon$, we obtain

$$\begin{aligned} \lim_{\varepsilon\to 0} I_1 = \lim_{\varepsilon\to 0} J_2 &= \lim_{\varepsilon\to 0} \int_\Omega u_i^\varepsilon(x) \frac{\partial}{\partial x_i} \mathbf{u}^\varepsilon(x) \cdot \left(\mathbf{v}^0(x) + \varepsilon \mathbf{v}^1\left(x, \frac{x}{\varepsilon}\right)\right) \mathrm{d}x \\ &= \frac{1}{|Y|} \int_\Omega \int_Y u_i^0(x) \left(\frac{\partial}{\partial x_i} \mathbf{u}^0(x) + \frac{\partial}{\partial y_i} \mathbf{u}^1(x,y)\right) \cdot \mathbf{v}^0(x) \mathrm{d}y \mathrm{d}x. \end{aligned} \tag{3.34}$$

- Similarly, to obtain $\lim_{\varepsilon\to 0} I_2$, we split

$$\begin{aligned} I_2 &= \int_{\Omega_s^\varepsilon} (\operatorname{curl} \mathbf{B}^\varepsilon \times \mathbf{B}^\varepsilon) \cdot \mathbf{v} \mathrm{d}x \\ &= \int_{\Omega_s^\varepsilon} \left[\operatorname{curl} \mathbf{B}^\varepsilon \times \left(\mathbf{B}^\varepsilon - \mathbf{B}^0\right)\right] \cdot \mathbf{v} \mathrm{d}x + \int_{\Omega_s^\varepsilon} \left(\operatorname{curl} \mathbf{B}^\varepsilon \times \mathbf{B}^0\right) \cdot \mathbf{v} \mathrm{d}x \\ &=: K_1 + K_2. \end{aligned}$$

From (3.17) and (3.23), we have

$$\begin{aligned} \lim_{\varepsilon\to 0} |K_1| &= \lim_{\varepsilon\to 0} \|\text{curl } \mathbf{B}^\varepsilon\|_{L^2} \left\|\mathbf{B}^\varepsilon - \mathbf{B}^0\right\|_{L^2} \|\mathbf{v}\|_{L^\infty} \\ &\le \lim_{\varepsilon\to 0} 2 \|\nabla \mathbf{B}^\varepsilon\|_{L^2} \left\|\mathbf{B}^\varepsilon - \mathbf{B}^0\right\|_{L^2} \|\mathbf{v}\|_{L^\infty} \\ &\le \lim_{\varepsilon\to 0} 2 \frac{1}{\alpha} \left(R_e \|\mathbf{g}\|_{L^2} + A_l \|\mathbf{h}\|_{L^2}\right) \left\|\mathbf{B}^\varepsilon - \mathbf{B}^0\right\|_{L^2} \|\mathbf{v}\|_{L^\infty} = 0. \end{aligned}$$

From the above, (3.23), (3.26), and (2.24) of Theorem 2.7, we obtain

$$\begin{aligned} \lim_{\varepsilon\to 0} I_2 &= \lim_{\varepsilon\to 0} K_2 \\ &= \lim_{\varepsilon\to 0} \int_\Omega \mathbb{1}_{\Omega_s^\varepsilon}(x) \left(\text{curl } \mathbf{B}^\varepsilon(x) \times \mathbf{B}^0(x)\right) \cdot \left(\mathbf{v}^0(x) + \varepsilon \mathbf{v}^1\left(x, \frac{x}{\varepsilon}\right)\right) \mathrm{d}x \\ &= \frac{1}{|Y|} \int_\Omega \int_{Y_s} \left(\text{curl } \mathbf{B}^0(x) + \text{curl}_y \mathbf{B}^1(x,y)\right) \times \mathbf{B}^0(x) \cdot \mathbf{v}^0(x) \mathrm{d}y \mathrm{d}x. \end{aligned}$$

- Finally, to obtain $\lim_{\varepsilon\to 0} I_3$,

$$\begin{aligned}
I_3 &= \int_{\Omega_s^\varepsilon} (\mathbf{u}^\varepsilon(x)\times\mathbf{B}^\varepsilon(x))\cdot\Big(\operatorname{curl}\mathbf{C}^0(x)+\varepsilon\operatorname{curl}\mathbf{C}^1\Big(x,\frac{x}{\varepsilon}\Big)\\
&\quad + \operatorname{curl}_y\mathbf{C}^1\Big(x,\frac{x}{\varepsilon}\Big)\Big)\,\mathrm{d}x\\
&= \int_\Omega \mathbb{1}_{\Omega_s^\varepsilon}(\mathbf{u}^\varepsilon(x)\times\mathbf{B}^\varepsilon(x))\cdot\operatorname{curl}\mathbf{C}^0(x)\mathrm{d}x\\
&\quad + \int_\Omega \mathbb{1}_{\Omega_s^\varepsilon}(\mathbf{u}^\varepsilon(x)\times\mathbf{B}^\varepsilon(x))\cdot\operatorname{curl}_y\mathbf{C}^1\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x\\
&\quad + \varepsilon\int_\Omega \mathbb{1}_{\Omega_s^\varepsilon}(\mathbf{u}^\varepsilon(x)\times\mathbf{B}^\varepsilon(x))\cdot\operatorname{curl}\mathbf{C}^1\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x\\
&=: M_1+M_2+\varepsilon\int_\Omega \mathbb{1}_{\Omega_s^\varepsilon}(\mathbf{u}^\varepsilon(x)\times\mathbf{B}^\varepsilon(x))\cdot\operatorname{curl}\mathbf{C}^1\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x.
\end{aligned}$$

The last term converges to zero as $\varepsilon\to 0$ since $\mathbf{u}^\varepsilon$ and $\mathbf{B}^\varepsilon$ are bounded in the H^1-norm, and $\mathbf{C}^1$ is continuous and compactly supported in $\Omega\times Y$. By the Rellich–Kondrachov theorem, we have that $\mathbf{u}^\varepsilon$ and $\mathbf{B}^\varepsilon$ strongly converge to $\mathbf{u}^0$ and $\mathbf{B}^0$ in $L^4(\Omega,\mathbb{R}^d)$, respectively. Moreover, since $\mathbf{C}^0$ is smooth, we have that $(\mathbf{u}^\varepsilon\times\mathbf{B}^\varepsilon)\cdot\operatorname{curl}\mathbf{C}^0$ strongly converges to $(\mathbf{u}^0\times\mathbf{B}^0)\cdot\operatorname{curl}\mathbf{C}^0$ in L^2. Also $\mathbb{1}_{\Omega_s^\varepsilon}\rightharpoonup\frac{1}{|Y|}\int_Y\mathbb{1}_{\Omega\times Y_s}\mathrm{d}y$ in L^2, so

$$\lim_{\varepsilon\to 0} M_1 = \frac{1}{|Y|}\int_\Omega\int_{Y_s}\big(\mathbf{u}^0(x)\times\mathbf{B}^0(x)\big)\cdot\operatorname{curl}\mathbf{C}^0(x)\mathrm{d}y\mathrm{d}x.$$

Next, rewrite M_2 as

$$\begin{aligned}
M_2 &= \int_\Omega \mathbb{1}_{\Omega_s^\varepsilon}\big(\mathbf{u}^\varepsilon(x)\times\mathbf{B}^\varepsilon(x)-\mathbf{u}^0(x)\times\mathbf{B}^0(x)\big)\cdot\operatorname{curl}_y\mathbf{C}^1\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x\\
&\quad + \int_\Omega \mathbb{1}_{\Omega_s^\varepsilon}\big(\mathbf{u}^0(x)\times\mathbf{B}^0(x)\big)\cdot\operatorname{curl}_y\mathbf{C}^1\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x,
\end{aligned}$$

Since $\mathbf{u}^\varepsilon\times\mathbf{B}^\varepsilon$ strongly converges to $\mathbf{u}^0\times\mathbf{B}^0$ in L^2, we have

$$\begin{aligned}
&\left|\int_\Omega \mathbb{1}_{\Omega_s^\varepsilon}\big(\mathbf{u}^\varepsilon(x)\times\mathbf{B}^\varepsilon(x)-\mathbf{u}^0(x)\times\mathbf{B}^0(x)\big)\cdot\operatorname{curl}_y\mathbf{C}^1\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x\right|\\
&\qquad\le \big\|\mathbf{u}^\varepsilon\times\mathbf{B}^\varepsilon-\mathbf{u}^0\times\mathbf{B}^0\big\|_{L^2}\big\|\operatorname{curl}_y\mathbf{C}^1\big\|_{L^\infty}\longrightarrow 0\quad\text{as}\quad\varepsilon\to 0.
\end{aligned}$$

Thus, using $\mathbb{1}_{\Omega_s^\varepsilon}\xrightharpoonup{2}\mathbb{1}_{\Omega\times Y_s}$, we obtain

$$\lim_{\varepsilon\to 0} M_2 = \frac{1}{|Y|}\int_\Omega\int_{Y_s}\big(\mathbf{u}^0(x)\times\mathbf{B}^0(x)\big)\cdot\operatorname{curl}_y\mathbf{C}^1(x,y)\mathrm{d}y\mathrm{d}x.$$

Therefore,

$$\lim_{\varepsilon\to 0} I_3 = \frac{1}{|Y|}\int_\Omega\int_{Y_s}\big(\mathbf{u}^0(x)\times\mathbf{B}^0(x)\big)\cdot\big(\operatorname{curl}\mathbf{C}^0(x)+\operatorname{curl}_y\mathbf{C}^1(x,y)\big)\,\mathrm{d}y\mathrm{d}x. \tag{3.35}$$

Summary. We now collect all relevant results obtained above in order to derive the two-scale homogenized system. In the weak formulation (3.4), we choose $\mathbf{v}=\mathbf{v}^0(\cdot)+\varepsilon\mathbf{v}^1\left(\cdot,\frac{\cdot}{\varepsilon}\right)$, $\mathbf{C}=\mathbf{C}^0(\cdot)+\varepsilon\mathbf{C}^1\left(\cdot,\frac{\cdot}{\varepsilon}\right)$, and $q=q^0(\cdot)+q^1\left(\cdot,\frac{\cdot}{\varepsilon}\right)$ with $\mathbf{v}^0,\mathbf{C}^0\in\mathcal{D}(\Omega,\mathbb{R}^d)$,

$q^0 \in \mathcal{D}(\Omega)$, and $\mathbf{v}^1, \mathbf{C}^1 \in \mathcal{D}(\Omega, C^\infty_{\mathrm{per}}(Y, \mathbb{R}^d))$, $q^1 \in \mathcal{D}(\Omega, C^\infty_{\mathrm{per}}(Y))$. Then, letting $\varepsilon \to 0$, we obtain

(3.36)
$$\begin{aligned}
&\frac{2}{|Y|}\int_\Omega\int_Y \big(\mathbb{D}(\mathbf{u}^0(x)) + \mathbb{D}_y(\mathbf{u}^1(x,y))\big) : \big(\mathbb{D}(\mathbf{v}^0(x)) + \mathbb{D}_y(\mathbf{v}^1(x,y))\big)\,\mathrm{d}y\mathrm{d}x \\
&\quad + \frac{A_l}{R_m}\Big\{\frac{1}{|Y|}\int_\Omega\int_Y \big(\mathrm{div}\ \mathbf{B}^0(x) + \mathrm{div}\ _y\mathbf{B}^1(x,y)\big)\cdot\big(\mathrm{div}\ \mathbf{C}^0(x) + \mathrm{div}\ _y\mathbf{C}^1(x,y)\big)\,\mathrm{d}y\mathrm{d}x \\
&\quad + \frac{1}{|Y|}\int_\Omega\int_{Y_s} \big(\mathrm{curl}\ \mathbf{B}^0(x) + \mathrm{curl}_y\mathbf{B}^1(x,y)\big)\cdot\big(\mathrm{curl}\ \mathbf{C}^0(x) + \mathrm{curl}_y\mathbf{C}^1(x,y)\big)\,\mathrm{d}y\mathrm{d}x\Big\} \\
&\quad + \frac{1}{|Y|}\int_\Omega\int_Y p^0(x,y)\big(\mathrm{div}\ \mathbf{v}^0(x) + \mathrm{div}\ _y\mathbf{v}^1(x,y)\big)\,\mathrm{d}y\mathrm{d}x \\
&\quad + R_e\frac{1}{|Y|}\int_\Omega\int_Y u_i^0(x)\left(\frac{\partial}{\partial x_i}\mathbf{u}^0(x) + \frac{\partial}{\partial y_i}\mathbf{u}^1(x,y)\right)\cdot\mathbf{v}^0(x)\mathrm{d}y\mathrm{d}x \\
&\quad - A_l\frac{1}{|Y|}\int_\Omega\int_{Y_s} \big(\mathrm{curl}\ \mathbf{B}^0(x) + \mathrm{curl}_y\mathbf{B}^1(x,y)\big)\times\mathbf{B}^0(x)\cdot\mathbf{v}^0(x)\mathrm{d}y\mathrm{d}x \\
&\quad - A_l\frac{1}{|Y|}\int_\Omega\int_{Y_s} \big(\mathbf{u}^0(x)\times\mathbf{B}^0(x)\big)\cdot\big(\mathrm{curl}\ \mathbf{C}^0(x) + \mathrm{curl}_y\mathbf{C}^1(x,y)\big)\,\mathrm{d}y\mathrm{d}x \\
&\quad = R_e\int_\Omega \mathbf{g}(x)\cdot\mathbf{v}^0(x)\mathrm{d}x + A_l\frac{|Y_s|}{|Y|}\int_\Omega \mathbf{h}(x)\cdot\mathbf{C}^0(x)\mathrm{d}x,
\end{aligned}$$

and

$$\frac{1}{|Y|}\int_\Omega\int_Y q^0(x)\big(\mathrm{div}\ \mathbf{u}^0(x) + \mathrm{div}\ _y\mathbf{u}^1(x,y)\big)\,\mathrm{d}y\mathrm{d}x = 0. \tag{3.37}$$

Finally, testing (2.10b), (2.10c), (2.10d), and (2.10f) with suitable test functions and applying (3.23), we obtain

$$\begin{aligned}
&\mathrm{div}\ \mathbf{u}^0 = 0 \text{ in } \Omega, && \mathrm{div}\ _y\mathbf{u}^1 = 0 \text{ in } \Omega\times Y, \\
& && \mathbb{D}\left(\mathbf{u}^0\right) + \mathbb{D}_y\left(\mathbf{u}^1\right) = 0 \text{ in } \Omega\times Y_s, \\
&\mathrm{div}\ \mathbf{B}^0 = 0 \text{ in } \Omega, && \mathrm{div}\ _y\mathbf{B}^1 = 0 \text{ in } \Omega\times Y, \\
& && \mathrm{curl}\ \mathbf{B}^0 + \mathrm{curl}_y\mathbf{B}^1 = 0 \text{ in } \Omega\times Y_f.
\end{aligned} \tag{3.38}$$

These identities allow us to simplify (3.36)–(3.37) in later calculations.

3.5. The local problem. The local problem is derived from (3.36)–(3.37) by letting $\mathbf{v}^0 = \mathbf{C}^0 = \mathbf{0}$ and $q^0 = 0$,

$$\begin{aligned}
&\frac{2}{|Y|}\int_\Omega\int_Y \big(\mathbb{D}(\mathbf{u}^0(x)) + \mathbb{D}_y(\mathbf{u}^1(x,y))\big) : \mathbb{D}_y(\mathbf{v}^1(x,y))\mathrm{d}y\mathrm{d}x \\
&\quad + \frac{A_l}{R_m}\Big\{\frac{1}{|Y|}\int_\Omega\int_Y \big(\mathrm{div}\ \mathbf{B}^0(x) + \mathrm{div}\ _y\mathbf{B}^1(x,y)\big)\cdot\mathrm{div}\ _y\mathbf{C}^1(x,y)\mathrm{d}y\mathrm{d}x \\
&\quad + \frac{1}{|Y|}\int_\Omega\int_{Y_s} \big(\mathrm{curl}\ \mathbf{B}^0(x) + \mathrm{curl}_y\mathbf{B}^1(x,y)\big)\cdot\mathrm{curl}_y\mathbf{C}^1(x,y)\mathrm{d}y\mathrm{d}x\Big\} \\
&\quad + \frac{1}{|Y|}\int_\Omega\int_Y p^0(x,y)\mathrm{div}\ _y\mathbf{v}^1(x,y)\mathrm{d}y\mathrm{d}x \\
&\quad - A_l\frac{1}{|Y|}\int_\Omega\int_{Y_s} \big(\mathbf{u}^0(x)\times\mathbf{B}^0(x)\big)\cdot\mathrm{curl}_y\mathbf{C}^1(x,y)\mathrm{d}y\mathrm{d}x \\
&\quad = 0.
\end{aligned} \tag{3.39}$$

Letting $\mathbf{v}^1(x,y) = \mathbf{w}(y)\varphi(x)$ and $\mathbf{C}^1(x,y) = \mathbf{G}(y)\varphi(x)$ for $\mathbf{w}, \mathbf{G} \in H^1_{\text{per}}(Y,\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\Omega)$, we deduce from (3.39) that, for a.e. $x \in \Omega$,

$$\begin{aligned}
&2\int_Y \big(\mathbb{D}(\mathbf{u}^0(x)) + \mathbb{D}_y(\mathbf{u}^1(x,y))\big) : \mathbb{D}_y(\mathbf{w}(y))\mathrm{d}y \\
&\quad + \frac{A_l}{R_m}\left\{\int_Y \big(\text{div } \mathbf{B}^0(x) + \text{div }_y\mathbf{B}^1(x,y)\big)\cdot \text{div }_y\mathbf{G}(y)\mathrm{d}y\right. \\
&\quad \left. + \int_{Y_s}\big(\text{curl } \mathbf{B}^0(x) + \text{curl}_y\mathbf{B}^1(x,y)\big)\cdot \text{curl}_y\mathbf{G}(y)\mathrm{d}y\right\} \\
&\quad + \int_Y p^0(x,y)\text{div }_y\mathbf{w}(y)\mathrm{d}y - A_l\int_{Y_s}\big(\mathbf{u}^0(x)\times\mathbf{B}^0(x)\big)\cdot\text{curl}_y\mathbf{G}(y)\mathrm{d}y \\
&\quad = 0.
\end{aligned} \tag{3.40}$$

Define

$$\mathfrak{X}_Y := \left\{ (\boldsymbol{\omega},\boldsymbol{\Theta}) \in H^1_{\text{per}}(Y,\mathbb{R}^d)\times H^1_{\text{per}}(Y,\mathbb{R}^d) \;\middle|\; \begin{array}{c} \text{div }_y\boldsymbol{\omega} = 0 \text{ in } Y \\ \mathbb{D}_y(\boldsymbol{\omega}) = 0 \text{ in } Y_s \\ \text{curl}_y\boldsymbol{\Theta} = 0 \text{ in } Y_f \end{array} \right\}.$$

So for $(\mathbf{w},\mathbf{G}) \in \mathfrak{X}_Y$, from (3.40) the following holds a.e. $x \in \Omega$,:

$$\begin{aligned}
&2\int_Y \big(\mathbb{D}(\mathbf{u}^0(x)) + \mathbb{D}_y(\mathbf{u}^1(x,y))\big) : \mathbb{D}_y(\mathbf{w}(y))\mathrm{d}y \\
&\quad + \frac{A_l}{R_m}\left\{\int_Y \big(\text{div } \mathbf{B}^0(x) + \text{div }_y\mathbf{B}^1(x,y)\big)\cdot \text{div }_y\mathbf{G}(y)\mathrm{d}y\right. \\
&\quad \left. + \int_{Y_s}\big(\text{curl } \mathbf{B}^0(x) + \text{curl}_y\mathbf{B}^1(x,y)\big)\cdot \text{curl}_y\mathbf{G}(y)\mathrm{d}y\right\} \\
&\quad - A_l\int_{Y_s}\big(\mathbf{u}^0(x)\times\mathbf{B}^0(x)\big)\cdot\text{curl}_y\mathbf{G}(y)\mathrm{d}y \\
&\quad = 0,
\end{aligned} \tag{3.41}$$

or equivalently,

$$\begin{aligned}
&2\int_Y \mathbb{D}_y(\mathbf{u}^1):\mathbb{D}_y(\mathbf{w})\mathrm{d}y + \frac{A_l}{R_m}\left\{\int_Y \text{div }_y\mathbf{B}^1\cdot\text{div }_y\mathbf{G}\mathrm{d}y + \int_{Y_s}\text{curl}_y\mathbf{B}^1\cdot\text{curl}_y\mathbf{G}\mathrm{d}y\right\} \\
&\quad = -\int_Y \mathbb{D}(\mathbf{u}^0):\mathbb{D}_y(\mathbf{w})\mathrm{d}y \\
&\qquad - \frac{A_l}{R_m}\left\{\int_Y \text{div } \mathbf{B}^0\cdot\text{div }_y\mathbf{G}\mathrm{d}y + \int_{Y_s}\text{curl } \mathbf{B}^0\cdot\text{curl}_y\mathbf{G}\mathrm{d}y\right\} \\
&\qquad + A_l\int_{Y_s}\big(\mathbf{u}^0(x)\times\mathbf{B}^0(x)\big)\cdot\text{curl}_y\mathbf{G}(y)\mathrm{d}y.
\end{aligned} \tag{3.42}$$

Clearly, for fixed $x \in \Omega$, problem (3.42) has a unique solution $(\mathbf{u}^1(x,\cdot),\mathbf{B}^1(x,\cdot)) \in \mathfrak{X}_Y$, because the left hand side of (3.42) is coercive, which in turn comes from the inequality (3.8) (note that this estimate also holds for a convex polyhedron, which is why we can replace Ω by Y). Therefore, as long as $\mathbf{u}^0$ and $\mathbf{B}^0$ are well-defined, $\mathbf{u}^1$ and $\mathbf{B}^1$ are independent of the choice of subsequences $\mathbf{u}^\varepsilon$ and $\mathbf{B}^\varepsilon$ in (3.23). Finally, $p^0(x,\cdot) \in L^2_0(Y)$ is also unique due to the inf-sup condition (repeating the first part of the proof of Lemma 3.13).

First, we calculate $\mathbf{u}^1$ in terms of $\mathbf{u}^0$. In (3.41), let $\mathbf{G}=0$, and then

$$\int_Y \left(\mathbb{D}\left(\mathbf{u}^0(x)\right)+\mathbb{D}_y\left(\mathbf{u}^1(x,y)\right)\right):\mathbb{D}_y\left(\mathbf{w}(y)\right)\mathrm{d}y=0 \tag{3.43}$$

with $\mathbf{w}\in H^1_{\mathrm{per}}(Y,\mathbb{R}^d)$ satisfying div $_y\mathbf{w}=0$ in Y and $\mathbb{D}_y(\mathbf{w})=0$ in Y_s. For $1\le i,j\le d$, define the function $\mathbf{U}^{ij}:=y_j\delta_{ik}\mathrm{e}^k$; then, by direct calculation, $\mathbb{D}_y(\mathbf{U}^{ij})=\frac{1}{2}\left(\delta_{jm}\delta_{in}+\delta_{jn}\delta_{im}\right)\mathrm{e}^n\otimes\mathrm{e}^m$. Let $\boldsymbol{\omega}^{ij}\in H^1_{\mathrm{per}}(Y,\mathbb{R}^d)$ and $\pi^{ij}\in L^2_0(Y)$ be the solutions of

$$\begin{aligned}
\text{div }_y\left(\mathbb{D}_y\left(\mathbf{U}^{ij}-\vec{\omega}^{ij}\right)-\pi^{ij}\right)&=0 \text{ in } Y_f,\\
\text{div }_y\vec{\omega}^{ij}&=0 \text{ in } Y_f,\\
\mathbb{D}_y\left(\mathbf{U}^{ij}-\vec{\omega}^{ij}\right)&=0 \text{ in } Y_s,\\
\int_\Gamma\left(\mathbb{D}_y\left(\mathbf{U}^{ij}-\vec{\omega}^{ij}\right)-\pi^{ij}\mathbf{I}\right)\mathbf{n}_\Gamma\mathrm{d}\mathcal{H}^{d-1}&=0,\\
\int_\Gamma\left(\mathbb{D}_y\left(\mathbf{U}^{ij}-\vec{\omega}^{ij}\right)-\pi^{ij}\mathbf{I}\right)\mathbf{n}_\Gamma\times\mathbf{n}_\Gamma\mathrm{d}\mathcal{H}^{d-1}&=0.
\end{aligned} \tag{3.44}$$

Then, integrating by parts (3.43) and using (3.38) and (3.44), we see that $\mathbf{u}^1$ is given by

$$\mathbf{u}^1(x,y)=-\left[\mathbb{D}\left(\mathbf{u}^0(x)\right)\right]_{ij}\vec{\omega}^{ij}(y). \tag{3.45}$$

We now calculate $\mathbf{B}^1$ in terms of $\mathbf{B}^0$. In (3.41), let $\mathbf{w}=0$ and use (3.38) to obtain

$$\int_{Y_s}\left(\text{curl }\mathbf{B}^0(x)-R_m\mathbf{u}^0(x)\times\mathbf{B}^0(x)+\text{curl}_y\mathbf{B}^1(x,y)\right)\cdot\text{curl}_y\mathbf{G}(y)\mathrm{d}y=0 \tag{3.46}$$

with $\mathbf{G}\in H^1_{\mathrm{per}}(Y,\mathbb{R}^d)$ satisfying $\text{curl}_y\mathbf{G}=0$ in Y_f. For $1\le j\le d$, let $\vec{\Theta}^j\in H^1_{\mathrm{per}}(Y,\mathbb{R}^d)$ and $\vec{\Psi}^j\in H^1_{\mathrm{per}}(Y,\mathbb{R}^d)$ be the solutions of

$$\begin{aligned}
\text{curl}_y\text{curl}_y\left(\vec{\Theta}^j+\mathrm{e}^j\right)&=0 \text{ in } Y_s, & \text{curl}_y\left(\vec{\Theta}^j+\mathrm{e}^j\right)&=0 \text{ in } Y_f,\\
\left(\vec{\Theta}^j+\mathrm{e}^j\right)\cdot\mathbf{n}_\Gamma&=0 \text{ on } \Gamma, & \text{curl}_y\left(\vec{\Theta}^j+\mathrm{e}^j\right)\times\mathbf{n}_\Gamma&=0 \text{ on } \Gamma,\\
\text{div }_y\vec{\Theta}^j&=0 \text{ in } Y,
\end{aligned} \tag{3.47}$$

and

$$\begin{aligned}
\text{curl}_y\text{curl}_y\left(\vec{\Psi}^j+R_m\mathrm{e}^j\right)&=0 \text{ in } Y_s, & \text{curl}_y\vec{\Psi}^j&=0 \text{ in } Y_f,\\
\left(\vec{\Psi}^j+R_m\mathrm{e}^j\right)\cdot\mathbf{n}_\Gamma&=0 \text{ on } \Gamma, & \text{curl}_y\left(\vec{\Psi}^j+R_m\mathrm{e}^j\right)\times\mathbf{n}_\Gamma&=0 \text{ on } \Gamma,\\
\text{div }_y\vec{\Psi}^j&=0 \text{ in } Y,
\end{aligned} \tag{3.48}$$

respectively. Then, integrating by parts (3.46), we see that $\mathbf{B}^1$ is given by

$$\mathbf{B}^1(x,y)=\epsilon_{ijk}\frac{\partial B^0_i}{\partial x_k}(x)\vec{\Theta}^j(y)-\epsilon_{ikj}u^0_i(x)B^0_k(x)\vec{\Psi}^j(y), \tag{3.49}$$

and here ϵ_{ijk} is the (Levi-Civita) permutation symbol.

Now, we find a formula for $p^0\in L^2_0(\Omega\times Y)$. Suppose

$$p^0(x,y)=2\left[\mathbb{D}\left(\mathbf{u}^0(x)\right)\right]_{ij}\pi^{ij}(y)+\Pi(x,y) \tag{3.50}$$

for some $\Pi \in L_0^2(\Omega \times Y)$. We claim Π is independent of y. To see this, substitute (3.45) and (3.49) and use the local problems (3.44), (3.47), and (3.48) in (3.40) to obtain

$$\int_Y p^0(x,y)\text{div }_y\mathbf{w}(y)\mathrm{d}y = 0 \text{ for any } \mathbf{w} \in H^1_{\text{per}}(Y).$$

Substituting (3.50) into the above equation and integrating by parts, all terms cancel by periodicity, except

$$\int_Y \nabla_y \Pi^0(x,y)\cdot \mathbf{w}(y)\mathrm{d}y = 0 \text{ for any } \mathbf{w} \in H^1_{\text{per}}(Y).$$

Therefore, $\nabla_y \Pi(x,y) = \mathbf{0}$, i.e., Π is independent of y, and we write $\Pi(x,y) \equiv \Pi(x)$. Clearly, $p^\varepsilon \rightharpoonup \Pi$ in $L^2(\Omega)$.

3.6. The homogenized problem. The variational form of the homogenized equation is derived by letting $\mathbf{v}^1 = \mathbf{C}^1 = \mathbf{0}$ and $q^1 = 0$ in (3.36)–(3.37), and then simplifying it by using (3.38):

$$\begin{aligned}
&\frac{2}{|Y|}\int_\Omega\int_Y \big(\mathbb{D}(\mathbf{u}^0(x)) + \mathbb{D}_y(\mathbf{u}^1(x,y))\big) : \mathbb{D}(\mathbf{v}^0(x))\mathrm{d}y\mathrm{d}x \\
&\quad + \frac{A_l}{R_m}\frac{1}{|Y|}\int_\Omega\int_{Y_s}\big(\text{curl }\mathbf{B}^0(x) + \text{curl}_y\mathbf{B}^1(x,y)\big)\cdot \text{curl }\mathbf{C}^0(x)\mathrm{d}y\mathrm{d}x \\
&+ \frac{1}{|Y|}\int_\Omega\int_Y p^0(x,y)\text{div }\mathbf{v}^0(x)\mathrm{d}y\mathrm{d}x \\
&+ R_e\frac{1}{|Y|}\int_\Omega\int_Y u_i^0(x)\left(\frac{\partial}{\partial x_i}\mathbf{u}^0(x) + \frac{\partial}{\partial y_i}\mathbf{u}^1(x,y)\right)\cdot \mathbf{v}^0(x)\mathrm{d}y\mathrm{d}x \\
&\quad - A_l\frac{1}{|Y|}\int_\Omega\int_{Y_s}\text{curl }\mathbf{B}^0(x)\times\mathbf{B}^0(x)\cdot\mathbf{v}^0(x)\mathrm{d}y\mathrm{d}x \\
&\quad - A_l\frac{1}{|Y|}\int_\Omega\int_{Y_s}\big(\mathbf{u}^0(x)\times\mathbf{B}^0(x)\big)\cdot \text{curl }\mathbf{C}^0(x)\mathrm{d}y\mathrm{d}x \\
&= R_e\int_\Omega \mathbf{g}(x)\cdot\mathbf{v}^0(x)\mathrm{d}x + A_l\frac{|Y_s|}{|Y|}\int_\Omega \mathbf{h}(x)\cdot\mathbf{C}^0(x)\mathrm{d}x.
\end{aligned} \tag{3.51}$$

In (3.51), letting $\mathbf{C}^0 = 0$ and $\mathbf{v}^0 \in H_0^1(\Omega,\mathbb{R}^d)$, we obtain

$$\begin{aligned}
&\frac{2}{|Y|}\int_\Omega\int_Y \big(\mathbb{D}(\mathbf{u}^0(x)) + \mathbb{D}_y(\mathbf{u}^1(x,y))\big) : \mathbb{D}(\mathbf{v}^0(x))\mathrm{d}y\mathrm{d}x \\
&+ \frac{1}{|Y|}\int_\Omega\int_Y p^0(x,y)\text{div }\mathbf{v}^0(x)\mathrm{d}y\mathrm{d}x \\
&+ R_e\frac{1}{|Y|}\int_\Omega\int_Y u_i^0(x)\left(\frac{\partial}{\partial x_i}\mathbf{u}^0(x) + \frac{\partial}{\partial y_i}\mathbf{u}^1(x,y)\right)\cdot \mathbf{v}^0(x)\mathrm{d}y\mathrm{d}x \\
&\quad - A_l\frac{|Y_s|}{|Y|}\int_\Omega \text{curl }\mathbf{B}^0(x)\times\mathbf{B}^0(x)\cdot\mathbf{v}^0(x)\mathrm{d}x \\
&= R_e\int_\Omega \mathbf{g}(x)\cdot\mathbf{v}^0(x)\mathrm{d}x.
\end{aligned} \tag{3.52}$$

Define the *effective viscosity* $\mathcal{N}$, which is a fourth-rank tensor, by

$$\mathcal{N}_{ijmn} := \frac{1}{|Y|}\int_Y \left[\mathbb{D}_y\left(\mathbf{U}^{ij} - \vec{\omega}^{ij}\right)\right]_{mn}\mathrm{d}y. \tag{3.53}$$

Substituting (3.45) and (3.50) into (3.52), we obtain

$$\begin{aligned}2\int_\Omega \mathcal{N}_{ijmn}\left[\mathbb{D}\left(\mathbf{u}^0\right)\right]_{ij}\left[\mathbb{D}\left(\mathbf{v}^0\right)\right]_{mn}\mathrm{d}x+\int_\Omega\left(\mathbf{u}^0\cdot\nabla\right)\mathbf{u}^0\cdot\mathbf{v}^0\mathrm{d}x+\int_\Omega \Pi\mathrm{div}\ \mathbf{v}^0\mathrm{d}x\\ -A_l\frac{|Y_s|}{|Y|}\int_\Omega \mathrm{curl}\ \mathbf{B}^0\times\mathbf{B}^0\cdot\mathbf{v}^0\mathrm{d}x=R_e\int_\Omega \mathbf{g}\cdot\mathbf{v}^0\mathrm{d}x.\end{aligned}\tag{3.54}$$

Here, we use the fact that $\int_Y \frac{\partial}{\partial y_i}\vec{\omega}^{ij}\mathrm{d}y = 0$ due to periodicity and that $\int_Y \pi^{ij}\mathrm{d}y = 0$ because $\pi^{ij}\in L^2_0(Y)$. Integrating by parts (3.54), we have, on Ω, that

$$\left(\mathbf{u}^0\cdot\nabla\right)\mathbf{u}^0-\mathrm{div}\ \left(2\mathcal{N}_{ijmn}\left[\mathbb{D}(\mathbf{u}^0)\right]_{ij}\mathrm{e}^m\otimes\mathrm{e}^n-\Pi\,\mathbf{I}\right)=R_e\mathbf{g}+A_l\frac{|Y_s|}{|Y|}\mathrm{curl}\ \mathbf{B}^0\times\mathbf{B}^0.\tag{3.55}$$

In (3.51), letting $\mathbf{v}^0=0$ and $\mathbf{C}^0\in H^1_n(\Omega,\mathbb{R}^d)$, we obtain

$$\begin{aligned}&\frac{A_l}{R_m}\frac{1}{|Y|}\int_\Omega\int_{Y_s}\left(\mathrm{curl}\ \mathbf{B}^0(x)+\mathrm{curl}_y\mathbf{B}^1(x,y)\right)\cdot\mathrm{curl}\ \mathbf{C}^0(x)\mathrm{d}y\mathrm{d}x\\ &\qquad -A_l\frac{1}{|Y|}\int_\Omega\int_{Y_s}\left(\mathbf{u}^0(x)\times\mathbf{B}^0(x)\right)\cdot\mathrm{curl}\ \mathbf{C}^0(x)\mathrm{d}y\mathrm{d}x\\ &= A_l\frac{|Y_s|}{|Y|}\int_\Omega \mathbf{h}(x)\cdot\mathbf{C}^0(x)\mathrm{d}x.\end{aligned}\tag{3.56}$$

Define the matrices $\mathcal{M}$ and $\mathcal{E}$, which represent the *effective magnetic reluctivity* and the *effective electric conductivity*, respectively, by

$$\mathcal{M}_{jq}:=\frac{1}{|Y|}\int_{Y_s}\left[\left(\mathrm{curl}_y\vec{\Theta}^j+\mathrm{e}^j\right)\right]_q\mathrm{d}y,\qquad \mathcal{E}_{kq}:=\frac{1}{|Y|}\int_{Y_s}\left[\left(\mathrm{curl}_y\vec{\Psi}^k+\mathrm{e}^k\right)\right]_q\mathrm{d}y.\tag{3.57}$$

Then, by substituting (3.49) into (3.56), and using (3.57), we obtain

$$\frac{1}{R_m}\int_\Omega \mathcal{M}_{jq}\epsilon_{ijk}\epsilon_{pqr}\frac{\partial B^0_i}{\partial x_k}\frac{\partial C^0_p}{\partial x_r}\mathrm{d}x-\int_\Omega \mathcal{E}_{kq}\epsilon_{ijk}\epsilon_{pqr}u^0_iB^0_j\frac{\partial C^0_p}{\partial x_r}\mathrm{d}x=\frac{|Y_s|}{|Y|}\int_\Omega \mathbf{h}\cdot\mathbf{C}^0\mathrm{d}x.\tag{3.58}$$

Using integration by parts, with $\mathbf{C}^0\in H^1_n(\Omega,\mathbb{R}^d)$, we conclude that, on Ω,

$$\frac{1}{R_m}\mathrm{curl}\ \left(\mathcal{M}_{jn}\epsilon_{ijk}\frac{\partial B^0_i}{\partial x_k}\mathrm{e}^n\right)-\mathrm{curl}\ \left(\mathcal{E}_{kn}\epsilon_{ijk}u^0_iB^0_j\mathrm{e}^n\right)=\frac{|Y_s|}{|Y|}\mathbf{h}.\tag{3.59}$$

In summary, from (3.38), (3.55), and (3.59), we obtain the macroscopic system that is about finding $\mathbf{u}^0\in H^1_0(\Omega,\mathbb{R}^d)$, $\Pi\in L^2_0(\Omega)$, and $\mathbf{B}^0\in H^1_n(\Omega,\mathbb{R}^d)$ satisfying on Ω,

$$\begin{gathered}\mathrm{div}\ \mathbf{u}^0=\mathrm{div}\ \mathbf{B}^0=0,\\ \left(\mathbf{u}^0\cdot\nabla\right)\mathbf{u}^0-\mathrm{div}\ \left(2\mathcal{N}_{ijmn}\left[\mathbb{D}(\mathbf{u}^0)\right]_{ij}\mathrm{e}^m\otimes\mathrm{e}^n-\Pi\,\mathbf{I}\right)=R_e\mathbf{g}+A_l\frac{|Y_s|}{|Y|}\mathrm{curl}\ \mathbf{B}^0\times\mathbf{B}^0,\\ \frac{1}{R_m}\mathrm{curl}\ \left(\mathcal{M}_{jn}\epsilon_{ijk}\frac{\partial B^0_i}{\partial x_k}\mathrm{e}^n\right)-\mathrm{curl}\ \left(\mathcal{E}_{kn}\epsilon_{ijk}u^0_iB^0_j\mathrm{e}^n\right)=\frac{|Y_s|}{|Y|}\mathbf{h}.\end{gathered}\tag{3.60}$$

Here $\mathcal{N}$ and $\mathcal{M}$, $\mathcal{E}$ are defined in (3.53), and (3.57), respectively. It is worth mentioning that by using the variational formulation of the local problem (3.44) and (3.47)–(3.48), we have

$$\begin{aligned}\mathcal{N}_{ijmn} &= \frac{1}{|Y|}\int_Y \mathbb{D}_y\left(\mathbf{U}^{ij}-\vec{\omega}^{ij}\right):\mathbb{D}_y\left(\mathbf{U}^{mn}-\vec{\omega}^{mn}\right)\mathrm{d}y,\\ \mathcal{M}_{ij} &= \frac{1}{|Y|}\int_{Y_s}\mathrm{curl}_y\left(\vec{\Theta}^i+\mathrm{e}^i\right)\cdot\mathrm{curl}_y\left(\vec{\Theta}^j+\mathrm{e}^j\right)\mathrm{d}y,\\ \mathcal{E}_{ij} &= \frac{1}{|Y|}\int_{Y_s}\mathrm{curl}_y\left(\vec{\Psi}^i+\mathrm{e}^i\right)\cdot\mathrm{curl}_y\left(\vec{\Psi}^j+\mathrm{e}^j\right)\mathrm{d}y.\end{aligned} \tag{3.61}$$

Thus, the tensors are symmetric and elliptic. The well-posedness of system (3.60) now follows from the classical theory of one-fluid magnetohydrodynamics (cf. [30, 31, 36, 37, 55]). By the uniqueness of $\mathbf{u}^0, \mathbf{B}^0, \mathbf{u}^1, \mathbf{B}^1$, and p^0, we conclude that the limits in (3.23) hold for the full sequence. Theorem 3.1 is proved.

4. Conclusions. The results obtained in subsection 3.1 demonstrate the *effective* response of a viscous fluid with a locally periodic array of magnetic particles suspended in it. The original fine-scale problem is described by the system of equations (2.10)–(2.13), and the effective equations are given by (3.60), in subsection 3.6, with the effective coefficients defined by (3.61). As evident from the effective system obtained, these effective quantities depend on the instantaneous position of the particles, their geometry, and the magnetic and flow properties of the original suspension decoded in the cell problems (3.44) and (3.47)–(3.48). The effective medium is an *incompressible electromagnetic fluid* described by the coupled set of Navier–Stokes and Maxwell's equations. The effective Cauchy stress of the fluid is $2\mathcal{N}_{ijmn}[\mathbb{D}(\mathbf{u}^0)]_{ij}\mathrm{e}^m\otimes\mathrm{e}^n - \Pi\,\mathbf{I}$, where $\mathcal{N}$ is the effective viscosity, and the coupling between the homogenized fluid velocity $\mathbf{u}$ and the homogenized magnetic field $\mathbf{B}$ is given through the Lorentz force. The Maxwell's equations are represented by the combination of Ampère's law, Ohm's law, and Faraday's law, where the first two laws eliminate the electric field from the equation.

It is worth mentioning that this paper is not concerned with *modeling issues* for colloids with magnetizable particles, but rather focuses on the homogenization results. This study is the promised follow-up of the work in [16] by the authors, where they considered a one-way coupling mechanism between the viscous fluid and the magnetic particles that are suspended in a viscous fluid and described by the linear relation between the magnetic flux density $\mathbf{B}$ and the magnetic field strength $\mathbf{H}$. In contrast to [16], this paper focuses on a *nonlinear* model of the given magnetorheological fluid, where the two phases are interacting via the *full (two-way) coupling* mechanism. And, as in [16], the *rigorous justification* of the obtained effective system is derived. This is also differing from previous contributions on the topic [44, 50], which dealt only with formal asymptotics and did not consider the complicated nonlinear model discussed in this paper.

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