DYNAMICAL BOREL–CANTELLI LEMMA FOR RECURRENCE UNDER LIPSCHITZ TWISTS

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ABSTRACT. In the study of some dynamical systems the limsup set of a sequence of measurable sets is often of interest. The shrinking targets and recurrence are two of the most commonly studied problems that concern limsup sets. However, the zero-one laws for the shrinking targets and recurrence are usually treated separately and proved differently. In this paper, we introduce a generalized definition that can specialize into the shrinking targets and recurrence; our approach gives a unified proof of the zero-one laws for the two problems.

1. INTRODUCTION

Throughout the paper, let (X, d) be a separable and compact metric space, and let (X, μ, T) be a probability measure preserving system. One of the most fundamental results in ergodic theory is the Poincaré Recurrence Theorem, see e.g. [EW, Theorem 2.11], which asserts that almost all points in measurable dynamical systems return close to themselves under a measure-preserving map; namely, that

$$\mu(R_T) = 1, \tag{1.1}$$

where R_T is the *set of recurrence* for T:

$$R_T := \{x \in X : \liminf_{n\to\infty} d(T^n x, x) = 0\}.$$

One of the first results concerning the speed of recurrence is due to Boshernitzan in [B]. Namely, assume that the α-dimensional Hausdorff measure of X is zero for some $\alpha > 0$. Then

$$\liminf_{n\to\infty} n^{1/\alpha} d(T^n x, x) = 0$$

for μ-almost every $x \in X$. In other words, for a function $\psi : \mathbb{N} \to \mathbb{R}^+$ let us define the following set:

$$R_T(\psi) := \{x \in X : d(T^n x, x) < \psi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

Then the Poincaré Recurrence Theorem says that the set

$$R_T = \bigcap_{\varepsilon>0} R_T(\varepsilon 1_{\mathbb{N}})$$

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has full measure, and, with the notation

$$\psi_s(x) := x^{-s}, \tag{1.2}$$

Bosherniztan's result says that $R_T(\varepsilon\psi_{1/\alpha})$ has full measure for any $\varepsilon > 0$ and for any α such that $\mathcal{H}^\alpha(X) = 0$.

It is a natural problem to find necessary and sufficient conditions on ψ to guarantee that the set $R_T(\psi)$ has measure zero or one. In fact, under some additional assumptions one expects this condition to be the convergence/divergence of the sum of measures of the sets

$$A_T(n,\psi) := \{x \in X : d(T^n x, x) < \psi(n)\}. \tag{1.3}$$

And indeed this was proved in several special cases such as [BF, CWW, HLSW]; see also [KKP, DFL, Pe] for similar results.

Note that a topic closely related to recurrence is the so-called *shrinking target problem*, which is concerned with determining the speed at which the orbit of a μ-typical point accumulates near a fixed point $y \in X$. More precisely, for $y \in X$ one can define the set

$$R_T^y := \left\{x \in X : \liminf_{n\to\infty} d(T^n x, y) = 0\right\},$$

and, more generally, for a function $\psi : \mathbb{N} \to \mathbb{R}^+$ define

$$R_T^y(\psi) := \{x \in X : d(T^n x, y) < \psi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

Equivalently, letting $B(x,r)$ stand for the open ball in X centered in x of radius r, we can write $R_T^y(\psi) = \limsup A^y(n,\psi)$, where

$$A_T^y(n,\psi) := \{x \in X : d(T^n x, y) < \psi(n)\} = T^{-n}B\big(y,\psi(n)\big). \tag{1.4}$$

Clearly

$$\mu(R_T^y) = 1 \text{ for any } y \in \operatorname{supp}\mu \quad \text{if } T \text{ is ergodic;} \tag{1.5}$$

furthermore, there have been plenty of results in the literature giving 0–1 laws for $\mu\big(R_T^y(\psi)\big)$. In fact, one can often use mixing properties of T to conclude that $\mu\big(R_T^y(\psi)\big)$ is equal to zero/one if and only if the series

$$\sum_{n=1}^{\infty} \mu\big(A_T^y(n,\psi)\big) = \sum_{n=1}^{\infty} \mu\Big(B\big(y,\psi(n)\big)\Big)$$

converges/diverges. See [Ph, CK, KM, FMP, HNPV] and many other references.

The goal of the current paper is to study a property unifying these two settings, and to prove a zero–one law applying to both. Namely, for a Borel measurable function $f : X \to X$ define R_T^f, the set of *f-twisted recurrent* points for T, by

$$R_T^f := \left\{x \in X : \liminf_{n\to\infty} d\big(T^n x, f(x)\big) = 0\right\}.$$

The two previous settings correspond to f being the identity and constant functions respectively. We will show in the next section that $\mu(R_T^f) = 1$ for any measurable f if T is ergodic and μ has full support. Furthermore, one can study the rate of twisted recurrence as follows: for $\psi : \mathbb{N} \to \mathbb{R}^+$ define

$$R_T^f(\psi) := \left\{ x \in X \,\middle|\, \begin{array}{c} d\big(T^n x, f(x)\big) < \psi(n) \\ \text{for infinitely many } n \in \mathbb{N} \end{array} \right\}, \tag{1.6}$$

so that $R_T^f = \bigcap_{\varepsilon>0} R_T^f(\varepsilon 1_X)$. In general the rate of twisted recurrence can be arbitrary slow, see §2 for examples. The main goal of the paper is to prove, under assumptions similar to those of [HLSW], a zero–one law for the sets $R_T^f(\psi)$ for a large class of functions f.

To state the main result of the paper, we need to adapt and modify the settings and assumptions from [HLSW]. Throughout the paper we write $a \lesssim b$ if $a \le Cb$ for some constant $C > 0$, and $a \asymp b$ if $a \lesssim b$ and $b \lesssim a$.

Our main assumption is that there exist at most countably many pairwise disjoint open subsets X_i, $i \in \mathcal{I}$, of X such that $T|_{X_i}$ is continuous and injective for each i, and $\mu(X \smallsetminus \cup_i X_i) = 0$. Those will be called *cylinders of order* 1. Then for any $m \in \mathbb{N}$ one can define

$$\mathcal{F}_m := \Big\{ X_{i_1} \cap T^{-1} X_{i_2} \cap \cdots \cap T^{-(m-1)} X_{i_m} : i_1, \ldots, i_m \in \mathcal{I} \Big\} \tag{1.7}$$

to be the collection of *cylinders of order* m. Note that for $J \in \mathcal{F}_m$ and $x, y \in J$, the points $T^n x$ and $T^n y$ are in the same partition set X_i for $0 \le n < m$, and hence $T, \ldots, T^m$ are injective on J. Also, since T is continuous, each cylinder in $\mathcal{F}_m$ is open.

Now let us list our assumptions on the measure μ. The first one is Ahlfors regularity of dimension $\delta > 0$; namely, that there exist positive real numbers η_1, η_2, r_0 such that

$$\eta_1 r^\delta \le \mu(B(x,r)) \le \eta_2 r^\delta \text{ for any ball } B(x,r) \subset X \text{ with } 0 < r < r_0. \tag{1.8}$$

As a consequence, since μ was assumed to be a probability measure, the space X has finite diameter.

Next, we assume that (X, μ, T) is uniformly mixing (a property introduced in [FMP]), that is: there exist a summable sequence of positive real numbers $(a_n)_{n\in\mathbb{N}}$ such such that

$$\begin{array}{c} \big|\mu(E \cap T^{-n}F) - \mu(E)\mu(F)\big| \le a_n \mu(F) \\ \text{for any balls } E, F \subset X \text{ and for all } n \ge 1. \end{array} \tag{1.9}$$

Note that it was proved in [FMP] that under the aforementioned mixing assumption, for any $y \in X$ and any ψ the set $R_T^y(\psi)$ is null (resp., conull) if the series

$$\sum_{n=1}^{\infty} \mu\big(B\big(y, \psi(n)\big)\big) \underset{(1.8)}{\asymp} \sum_{n=1}^{\infty} \psi(n)^\delta$$

converges (resp., diverges). However, in order to similarly treat the sets $R_T^f(\psi)$ for more general functions f we will require some more information on the expanding properties of T. For a m-cylinder J, we define

$$K_J := \inf_{x,y\in J,\, x\neq y} \frac{d(T^m x, T^m y)}{d(x,y)},$$

and impose the following additional assumptions:

- Bounded distortion: There exists a constant $K_1 > 0$ such that
$$K_1^{-1} \le \frac{d(T^m x, T^m y)/d(x,y)}{d(T^m x, T^m z)/d(x,z)} \le K_1 \tag{1.10}$$
for all $m \in \mathbb{N}$ and $x, y, z \in J \in \mathcal{F}_m$ with $x \neq y$ and $x \neq z$.
- Expanding properties:
$$\inf_{J\in\mathcal{F}_m} K_J \to \infty \text{ as } m \to \infty \tag{1.11}$$
and
$$\sup_{m\in\mathbb{N}} \sum_{J\in\mathcal{F}_m} K_J^{-\delta} < \infty. \tag{1.12}$$
- Conformality: There exists a constant $K_2 \ge 1$ such that
$$\begin{gathered} B(T^m x, K_2^{-1} K_J r) \subset T^m B(x,r) \subset B(T^m x, K_2 K_J r) \\ \text{for any } m \in \mathbb{N} \text{ and any ball } B(x,r) \subset J \in \mathcal{F}_m. \end{gathered} \tag{1.13}$$

Remark 1.1. Notice that the bounded distortion condition (1.10) implies the second inclusion in (1.13) with K_2 replaced by K_1. However the first inclusion there does not automatically follow from (1.10), hence the need for an additional condition.

Remark 1.2. We note that conditions (1.8)–(1.13) are essentially equivalent to Conditions I–V from [HLSW]. Namely:

- (1.8) is a slightly weaker version of [HLSW, Condition I].
- (1.9) replaces [HLSW, Condition II] where the rate of mixing was assumed to be exponential.
- As for (1.10)–(1.13), in [HLSW] the standing assumption was that the restriction of T to X_i for every i is differentiable and expanding, namely it was assumed that
$$\|D_x(T^{-1})\|^{-1} > 1 \text{ for any } x \in \cup_i X_i. \tag{1.14}$$
The role of (1.10) was played there by [HLSW, Condition III] stated as follows: there exists a constant $K_1 > 0$ such that
$$K_1^{-1} \le \frac{d(T^m x, T^m y)}{d(x,y)\|D_x T^m\|} \le K_1 \ \forall\, m \in \mathbb{N} \text{ and } \forall\, x, y \in J \in \mathcal{F}_m \text{ with } x \neq y.$$
- Similarly, the constant K_J for $J \in \mathcal{F}_m$ was defined in [HLSW] by $K_J := \inf_{x\in J} \|D_x T^m\|$, and the role of (1.11) was played by $\inf_{J\in\mathcal{F}_m} K_J > 1$ for some $m \in \mathbb{N}$, which, in view of (1.14), is easily seen to be equivalent to $\inf_{J\in\mathcal{F}_m} K_J \to \infty$ as $m \to \infty$. Conditions IV and V of [HLSW] are identical to (1.12) and (1.13) respectively.

Examples of dynamical systems satisfying conditions (1.8)–(1.13) include, as mentioned in [HLSW], β-transformations

$$M_\beta : x \mapsto \beta x \mod 1$$

of the unit interval, where $\beta \in \mathbb{R}_{>1}$, as well as the Gauss map. In §7 we add another example to the list: expanding maps defined by systems of contracting similarities with the open set condition.

Let us now specify the class of functions f which we can treat by our technique. Say that $f : X \to X$ is *Lipschitz* if

$$\sup_{x,y\in X,\, x\neq y} \frac{d\big(f(x), f(y)\big)}{d(x,y)} < \infty,$$

and that f is *piecewise Lipschitz* if there exist at most countably many measurable subsets Y_i of X and Lipschitz functions $f_i : X \to X$, $i \in \mathcal{I}$, such that $\mu(X \smallsetminus \cup_i Y_i) = 0$ and $f|_{Y_i} = f_i$ for each i. An example: when $X = [0,1]$, the function $f(x) = \sqrt{x}$ is piecewise Lipschitz but not Lipschitz.

Now we are ready to state our main theorems.

Theorem 1.3. *Assume that* (X, μ, T) *satisfies conditions* (1.8)–(1.13)*. Then for any function* $\psi : \mathbb{N} \to \mathbb{R}_+$ *with* $\lim_{n\to\infty} \psi(n) = 0$ *and any piecewise Lipschitz function* $f : X \to X$*, the set* $R_T^f(\psi)$ *is null if and only if the series*

$$\sum_{n=1}^{\infty} \psi(n)^\delta \tag{1.15}$$

converges.

It is natural to expect that Theorem 1.3 can be strengthened to the full measure of $R_T^f(\psi)$ in the case when the series (1.15) diverges. This was done in [HLSW] in the case $f = \mathrm{Id}_X$. Unfortunately for an arbitrary Lipschitz function f the full measure conclusion is outside of our reach. In the following theorems we handle several special cases. First, employing an argument from [HLSW], we prove

Theorem 1.4. *Let* (X, μ, T)*,* $\psi : \mathbb{N} \to \mathbb{R}$ *and* $f : X \to X$ *be as in Theorem 1.3. Furthermore, assume that*

$$T \circ f = f \circ T. \tag{1.16}$$

Then $\mu\big(R_T^f(\psi)\big) = 1$ *whenever the series* (1.15) *diverges.*

Clearly (1.16) holds when $f = \text{const}$ or $f = \mathrm{Id}_X$, but not in general. Next we present an alternative approach to upgrading Theorem 1.3 to a full measure result, requiring introducing additional assumptions on (X, μ, T).

Namely, let $\{X_i\}_{i\in\mathcal{I}}$ be as defined above. We say the partition $\{X_i\}_{i\in\mathcal{I}}$ is *pseudo-Markov* with respect to T if

- for all $i, j \in \mathcal{I}$, TX_i is measurable;
- $TX_i \cap X_j \neq \varnothing$ implies $X_j \subset TX_i$ for any $i, j \in \mathcal{I}$;
- there exists $\tau > 0$ such that $\mu(TX_i) \geq \tau\mu(X)$ for any $i \in \mathcal{I}$.

Theorem 1.5. *Let (X,μ,T), $\psi:\mathbb{N}\to\mathbb{R}$ and $f:X\to X$ be as in Theorem 1.3. Furthermore, assume that $\{X_i\}_{i\in\mathcal{I}}$ is pseudo-Markov. Then $\mu\big(R^f_T(\psi)\big)=1$ whenever the series* (1.15) *diverges.*

Examples of systems with pseudo-Markov (in fact, truly Markov) partitions include the Gauss map, the multiplication map M_b where $b\geq 2$ is an integer, and, more generally, conformal expanders described in §7. One can also show that β-transformations for some specific β admit pseudo-Markov partitions. This is however not true for arbitrary β. Yet, the twisted recurrence set-up was recently considered in [LWW] for $T=M_\beta$, where $\beta>1$ is arbitrary, establishing the conclusion of Theorems 1.4 and 1.5 in that case. Namely they prove

Theorem 1.6. *Let $X=[0,1]$, $T=M_\beta$, μ the M_β-invariant probability measure on $[0,1]$, and let $\psi:\mathbb{N}\to\mathbb{R}$ and $f:X\to X$ be as in Theorem 1.3. Then $\mu\big(R^f_T(\psi)\big)$ is equal to 0 (resp., 1) whenever the series* (1.15) *converges (resp., diverges).*

In §8 we show how our methods can be modified to yield an an independent proof of the above theorem.

We also remark that the paper [DFL] suggests an even more general set-up: there the authors consider a uniformly Lipschitz function $\Phi:X\times X\to\mathbb{R}$ and under certain assumptions recover zero–one laws for sets of the form

$$\left\{x\in X\;\middle|\;\begin{array}{l}\phi_1(n)\leq\Phi(x,T^nx)\leq\phi_2(n)\\ \text{for infinitely many } n\in\mathbb{N}\end{array}\right\}.$$

Our set-up corresponds to $\phi_1=0$, $\phi_2=\psi$ and $\Phi(x,y)=d\big(f(x),y\big)$. It would be interesting to see if the methods of our paper can be applied to the generalized setting of [DFL].

The structure of the paper is as follows. In §2 we discuss several basic properties of f-twisted recurrence sets and some examples of such sets. In §3 we prove the convergence part of Theorem 1.3. In §4 we study quasi-independence properties of the sequence of measurable sets whose limsup set is given by (1.6). In §§5–6 we consider the divergence case and complete the proof of Theorems 1.3, 1.4 and 1.5. In §7 we discuss examples of dynamical systems to which our theorems apply. The final section contains a separate discussion of β-transformations and results in proving Theorem 1.6.

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2. More about f-twisted recurrence

We start with several elementary observations concerning sets of f-twisted recurrence.

Lemma 2.1. *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be an arbitrary function, and let $f : X \to X$ be such that there exist at most countably many measurable subsets Y_i of X and functions $f_i : X \to X$, $i \in \mathcal{I}$, such that $\mu(X \setminus \cup_i Y_i) = 0$,*

$$f|_{Y_i} = f_i \text{ and } \mu\big(R_T^{f_i}(\psi)\big) = 1 \text{ for each } i \in \mathcal{I}. \tag{2.1}$$

Then $\mu\big(R_T^f(\psi)\big) = 1$.

Proof. Indeed, it follows from (1.6) and (2.1) that

$$\mu\big(R_T^f(\psi) \cap Y_i\big) = \mu\big(R_T^{f_i}(\psi) \cap Y_i\big) = \mu(Y_i)$$

for each $i \in \mathcal{I}$. □

Let us say that a function is *simple* if it takes at most countably many values.

Corollary 2.2. *Suppose T is ergodic and* $\operatorname{supp} \mu$ *is dense in X. Then $\mu(R_T^f) = 1$ for any simple function $f : X \to X$.*

Proof. Immediate from Lemma 2.1 and (1.5). □

Lemma 2.3. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions $X \to X$ such that $\mu(R_T^{f_n}) = 1$ for each n. Suppose that $f_n \to f$ uniformly on a set of full measure. Then $\mu(R_T^f) = 1$.*

Proof. Since $\bigcap_n R_T^{f_n}$ has full measure, for almost every $x \in X$ and each $n \in \mathbb{N}$ one has

$$\liminf_{k \to \infty} d\big(T^k x, f_n(x)\big) = 0.$$

Fix $\varepsilon > 0$; then there exists N so that for all $n > N$, $d\big(f_n(x), f(x)\big) < \frac{\varepsilon}{2}$ for almost every $x \in X$; on the other hand, for almost every $x \in X$ such that $d\big(f_n(x), f(x)\big) < \frac{\varepsilon}{2}$, $d\big(T^k x, f_n(x)\big) < \frac{\varepsilon}{2}$ for infinitely many k. This implies $d\big(T^k x, f(x)\big) < \varepsilon$ for infinitely many k. Since ε is chosen arbitrarily, we have $\liminf_{k \to \infty} d\big(T^k x, f(x)\big) = 0$. □

Corollary 2.4. *Suppose that T is ergodic and* $\operatorname{supp} \mu$ *is dense in X. Then $\mu(R_T^f) = 1$ for any Borel-measurable $f : X \to X$.*

Proof. Let $\{x_n\}_{n=1}^\infty$ be a dense subset of X. Let $\varepsilon > 0$ and $f : X \to X$ be a Borel-measurable function. Then $\{B(x_n, \varepsilon)\}_{n=1}^\infty$ covers X. Define

$$g_\varepsilon(x) = x_n \text{ where } n = \inf_m \{m : f(x) \in B(x_m, \varepsilon)\} \tag{2.2}$$

Then g_ε is simple and $||g_\varepsilon - f||_\infty \leq \varepsilon$. Since ε is chosen arbitrarily, f is a uniform limit of simple functions. By Corollary 2.2 and Lemma 2.3, x is f-recurrent for almost every $x \in X$. □

Next, let us observe that the properties of sets $R_T^f(\psi)$ could be strikingly different from the conclusion of Theorem 1.3 if the assumptions of that theorem are not imposed. Let us start with the simplest possible non-trivial[1] example of an ergodic dynamical system: an irrational circle rotation $X = \mathbb{R}/\mathbb{Z}$, $\mu =$ Lebesgue measure, $T_\alpha(x) = x + \alpha \bmod \mathbb{Z}$ where $\alpha \in \mathbb{R}\smallsetminus\mathbb{Q}$. Then the condition defining the recurrence set

$$R_{T_\alpha}(\psi) = \{x : |n\alpha - m| < \psi(n) \text{ for infinitely many } n \in \mathbb{N} \text{ and some } m \in \mathbb{Z}\}$$

is independent of x; hence $R_{T_\alpha}(\psi)$ is either X or $\varnothing$, and this dichotomy is different for different α. More precisely, Dirichlet's Theorem implies that $R_{T_\alpha}(\psi_1) = X$ for any α (see (1.2) for this notation), and the same is true for ψ_1 replaced with $\frac{1}{\sqrt{5}}\psi_1$, but not with $c\psi_1$ for $c < \frac{1}{\sqrt{5}}$. In particular, α is badly approximable if and only if $R_{T_\alpha}(c\psi_1) = \varnothing$ for some $c > 0$. On the other hand, the theory of continued fractions shows that for any positive non-increasing ψ (decaying arbitrarily fast) there exists α such that $R_{T_\alpha}(\psi)$ contains 0 (and hence coincides with $\mathbb{R}/\mathbb{Z}$).

Likewise, studying targets shrinking to $y \in X$ for the above system reduces to inhomogeneous Diophantine approximation:

$$R^y_{T_\alpha}(\psi) = \{x : \operatorname{dist}(n\alpha, y - x) < \psi(n) \text{ for infinitely many } n \in \mathbb{N}\}$$

According to Minkowski's theorem [C, Chapter III, Theorem II], for any irrational α and any $y \in \mathbb{R}/\mathbb{Z}$, the complement of $R^y_{T_\alpha}(\frac{1}{4}\psi_1)$ is at most countable. A precise zero-one law for sets $R^y_{T_\alpha}(\psi)$ again depends on the Diophantine properties of α. For example, it is a theorem of Kurzweil [K] that α is badly approximable if and only if the following statement holds: for any non-increasing ψ, the set $R^y_{T_\alpha}(\psi)$ is null/conull if $\sum_{k=1}^\infty \psi(k)$ converges/diverges. However, well approximable α come with their own convergence/divergence condition on ψ guaranteeing that $R^y_{T_\alpha}(\psi)$ is null or conull; see [FK] for the most general statement.

Clearly the set-up of f-twisted recurrence can be similarly and straightforwardly restated in a Diophantine approximation language:

$$R^f_{T_\alpha}(\psi) = \big\{x \in X : \operatorname{dist}\big(n\alpha, f(x) - x\big) < \psi(n) \text{ for infinitely many } n \in \mathbb{N}\big\}\,.$$

Thus if $f(x) = x + \beta \bmod \mathbb{Z}$ for a fixed β, then $R_{T_\alpha}(\psi)$ is either X or $\varnothing$; alternatively, if the pushforward of Lebesgue measure by the map $x \mapsto f(x) - x$ is absolutely continuous with respect to Lebesgue, then the zero/one law for the sets $R^f_{T_\alpha}(\psi)$ depends on the Diophantine properties of α as described in [FK].

The situation is even trickier if one considers irrational rotations of higher-dimensional tori. Namely, if we let $X = \mathbb{R}^d/\mathbb{Z}^d$ and $\mu =$ Lebesgue measure, then it is shown in [GP] that for any (arbitrarily slowly decaying) non-increasing function ψ with $\lim_{t\to\infty}\psi(t) = 0$ there exists an ergodic translation $T_\alpha : x \mapsto x + \alpha \bmod \mathbb{Z}^d$ such that $\mu\big(R^y_{T_\alpha}(\psi)\big) = 0$ for any $y \in X$. Moreover, by suitably reparametrizing the aforementioned example one can

[1]For us ergodic self-maps T of finite sets X will be trivial: indeed, since those are transitive, it easily follows that $R^f_T(\psi) = X$ for any f and any positive ψ.

construct a smooth mixing transformation on the three dimensional torus with the same property. Thus some conditions on the speed of mixing is crucial for a zero-one law as in Theorem 1.3.

3. The convergence part

In the next two sections we prove Theorem 1.3, thereby assuming that (X, μ, T) satisfies conditions (1.8)–(1.13) and fixing $\psi : \mathbb{N} \to \mathbb{R}^+$ with $\lim_{n\to\infty} \psi(n) = 0$. Similarly to (1.3) and (1.4), for an arbitrary $f : X \to X$ let us define

$$A_n = A_T^f(n, \psi) := \{x \in X : d(T^n x, f(x)) < \psi(n)\} . \tag{3.1}$$

Clearly $R_T^f(\psi) = \limsup A_n$.

Unlike the shrinking target case, corresponding to constant functions f, the sets A_n cannot be expressed in the form $T^{-n}B_n$ for some balls B_n. Our strategy is to consider the intersection of A_n with $f^{-1}B(x_0, r)$, where $x_0 \in X$ and $r > 0$, and approximate this intersection by the preimages of some balls under T.

Lemma 3.1. *For any $x_0 \in X$, any $r > 0$ and any subset E of $f^{-1}B(x_0, r)$,*

$$E \cap A_n \subset E \cap T^{-n}B(x_0, \psi(n) + r). \tag{3.2}$$

Furthermore, if $r < \psi(n)$, then

$$E \cap T^{-n}B(x_0, \psi(n) - r) \subset E \cap A_n. \tag{3.3}$$

Proof. Fix a point $x \in E \cap A_n$. Then

$$d(f(x), x_0) < r \text{ and } d(T^n x, f(x)) < \psi(n),$$

which implies that

$$d(T^n x, x_0) < d(T^n x, f(x)) + d(f(x), x_0) < \psi(n) + r.$$

Hence $E \cap A_n \subset E \cap T^{-n}B(x_0, \psi(n) + r)$.

On the other hand, fix $x \in E \cap T^{-n}B(x_0, \psi(n) - r)$. Then $d(f(x), x_0) < r$ and $d(T^n x, x_0) < \psi(n) - r$. Hence

$$d(T^n x, f(x)) \le d(T^n x, x_0) + d(x_0, f(x)) < \psi(n),$$

thus $E \cap T^{-n}B(x_0, \psi(n) - r) \subset E \cap A_n$. □

Choose $n_0 \in \mathbb{N}$ such that $5\psi(n) < r_0$ for all $n > n_0$, where r_0 is as in (1.8); the next several statements in this section will be proved for $n > n_0$.

Lemma 3.2. *Let $B = B(x_0, \psi(n)/2)$ for some $x_0 \in X$ and $n > n_0$. Then for any open ball E contained in $f^{-1}B$,*

$$2^{-\delta}(\eta_1 \mu(E) - \eta_2 a_n)\psi(n)^\delta \le \mu(E \cap A_n) \le \eta_2 (3/2)^\delta (\mu(E) + a_n)\psi(n)^\delta,$$

with δ, η_1, η_2 as in (1.8) and $(a_n)_{n\in\mathbb{N}}$ as in (1.9).

Proof. Let $r = \psi(n)/2$, and let E be an open ball contained in $f^{-1}B(x_0, r)$. Combining (3.2) with (1.9), we get

$$\begin{aligned}
\mu(E \cap A_n) &\geq \mu\Big(E \cap T^{-n}B\big(x_0, \psi(n) - r\big)\Big) \\
&\geq \mu(E)\mu\big(T^{-n}B(x_0, \psi(n) - r)\big) - a_n\mu\big(E \cap T^{-n}B(x_0, \psi(n) - r)\big) \\
&= \mu(E)\mu\Big(T^{-n}B\big(x_0, \psi(n)/2\big)\Big) - a_n\mu\Big(T^{-n}B\big(x_0, \psi(n)/2\big)\Big) \\
&= \mu(E)\mu\Big(B\big(x_0, \psi(n)/2\big)\Big) - a_n\mu\Big(B\big(x_0, \psi(n)/2\big)\Big) \\
&\underset{(1.8)}{\geq} \big(\eta_1\mu(E) - \eta_2 a_n\big)2^{-\delta}\psi(n)^\delta
\end{aligned}$$

and

$$\begin{aligned}
\mu(E \cap A_n) &\leq \mu\Big(E \cap T^{-n}B\big(x_0, \psi(n) + r\big)\Big) \\
&\leq \mu(E)\mu\big(T^{-n}B(x_0, \psi(n) + r)\big) + a_n\mu\big(E \cap T^{-n}B(x_0, \psi(n) + r)\big) \\
&= \mu(E)\mu\Big(T^{-n}B\big(x_0, 3\psi(n)/2\big)\Big) + a_n\mu\Big(T^{-n}B\big(x_0, 3\psi(n)/2\big)\Big) \\
&\leq \mu(E)\mu\Big(B\big(x_0, 3\psi(n)/2\big)\Big) + a_n\mu\Big(B\big(x_0, 3\psi(n)/2\big)\Big) \\
&\underset{(1.8)}{\leq} \big(\mu(E) + a_n\big)\eta_2(3/2)^\delta\psi(n)^\delta,
\end{aligned}$$

establishing the claim. □

To prove Theorems 1.3–1.6, in view of Lemma 2.1 it is enough to assume that f is Lipschitz. Thus for the rest of the paper we let $f : X \to X$ be a p-Lipschitz function for some $p > 0$.

The next lemma estimates the measure of the sets A_n.

Lemma 3.3. *For $n > n_0$,*

$$\eta_2^{-1}\eta_1^2 10^{-\delta}\psi(n)^\delta - (p/5)^\delta a_n \leq \mu(A_n) \leq \eta_1^{-1}\eta_2(3/2)^\delta\big(\eta_2 5^\delta\psi(n)^\delta + (2p)^\delta a_n\big).$$

Proof. Take $x \in X$, $y \in f^{-1}\{x\}$ and $z \in B\left(y, \frac{\psi(n)}{2p}\right)$. Then by the p-Lipshitz condition, $d\big(x, f(z)\big) \leq pd(y, z) < \psi(n)/2$. Thus

$$B\big(y, \psi(n)/p\big) \subset f^{-1}B\big(x, \psi(n)/2\big).$$

We have an open covering

$$\big\{B\big(y, \psi(n)/p\big) : y \in X\big\}$$

with each $B\left(y, \frac{\psi(n)}{2p}\right) \subset f^{-1}B\big(x, \psi(n)/2\big)$ for some $x \in X$.

By Vitali's covering theorem ($5r$-covering lemma), we can find countably many disjoint balls $\left\{B\left(y_j, \frac{\psi(n)}{2p}\right)\right\}_{j\in\mathcal{J}}$ such that

$$X \subset \bigcup_{j\in\mathcal{J}} B\left(y_j, \frac{5\psi(n)}{2p}\right). \tag{3.4}$$

By the disjointness of $\left\{B\left(y_j, \frac{\psi(n)}{2p}\right)\right\}_{j\in\mathcal{J}}$, we have

$$\sum_{j\in\mathcal{J}} \eta_1 \left(\frac{\psi(n)}{2p}\right)^\delta \leq \sum_{j\in\mathcal{J}} \mu\left(B\Big(y_j, \frac{\psi(n)}{2p}\Big)\right) \leq \mu(X) = 1.$$

Hence $|\mathcal{J}| \leq \eta_1^{-1}\left(\frac{2p}{\psi(n)}\right)^\delta$. On the other hand, by (3.4) we have

$$\sum_{j\in\mathcal{J}} \eta_2 \left(\frac{5\psi(n)}{2p}\right)^\delta \geq \sum_{j\in\mathcal{J}} \mu\left(B\Big(y_j, \frac{5\psi(n)}{2p}\Big)\right) \geq \mu(X) = 1;$$

hence $|\mathcal{J}| \geq \eta_2^{-1}\left(\frac{2p}{5\psi(n)}\right)^\delta$.

By Lemma 3.2, since for each j we have $B\left(y_j, \frac{\psi(n)}{2p}\right) \subset f^{-1}B\big(x_j, \psi(n)/2\big)$ and $B\left(y_j, \frac{5\psi(n)}{2p}\right) \subset f^{-1}B\big(x_j, 5\psi(n)/2\big)$, it follows that

$$\begin{aligned}
\mu(A_n) &\leq \sum_{j\in J} \mu\left(B\Big(y_j, \frac{5\psi(n)}{2p}\Big) \cap A_n\right) \\
&\leq \sum_{j\in J} \eta_2 (3/2)^\delta \left[\mu\left(B\Big(y_j, \frac{5\psi(n)}{2p}\Big)\right) + a_n\right]\psi(n)^\delta \\
&\leq \eta_1^{-1}\left(\frac{2p}{\psi(n)}\right)^\delta \eta_2(3/2)^\delta \left[\Big(\frac{5\psi(n)}{2p}\Big)^\delta \eta_2 + a_n\right]\psi(n)^\delta \\
&= \eta_1^{-1}\eta_2(3/2)^\delta\big(\eta_2 5^\delta \psi(n)^\delta + (2p)^\delta a_n\big)
\end{aligned}$$

and

$$\begin{aligned}
\mu(A_n) &\geq \sum_{j\in\mathcal{J}} \mu\left(B\Big(y_j, \frac{\psi(n)}{2p}\Big) \cap A_n\right) \\
&\geq \sum_{j\in J} \left[\eta_1 2^{-\delta}\mu\left(B\Big(y_j, \frac{\psi(n)}{2p}\Big)\right) - \eta_2 2^{-\delta} a_n\right]\psi(n)^\delta \\
&\geq \eta_2^{-1}\left(\frac{2p}{5\psi(n)}\right)^\delta \left[\eta_1 2^{-\delta}\eta_1\Big(\frac{\psi(n)}{2p}\Big)^\delta - \eta_2 2^{-\delta} a_n\right]\psi(n)^\delta \\
&= \eta_2^{-1}\eta_1^2 10^{-\delta}\psi(n)^\delta - \eta_2^{-1}(2p/5)^\delta \eta_2 2^{-\delta} a_n,
\end{aligned}$$

finishing the proof of the lemma. □

Proposition 3.4.

$$\sum_{n=1}^{\infty} \psi(n)^\delta = \infty \iff \sum_{n=1}^{\infty} \mu(A_n) = \infty \tag{3.5}$$

Proof. By Lemma 3.3 we know that

$$\begin{aligned}&\eta_2^{-1}\eta_1^2 10^{-\delta}\sum_{n>n_0}\psi(n)^\delta-(p/5)^\delta\sum_{n>n_0}a_n\\ \le&\sum_{n>n_0}\mu(A_n)\\ \le&\,\eta_1^{-1}\eta_2(3/2)^\delta\left(\eta_2 5^\delta\sum_{n>n_0}\psi(n)^\delta+(2p)^\delta\sum_{n>n_0}a_n\right).\end{aligned}$$

Since $\{a_n\}$ is summable, (3.5) holds. □

Remark 3.5. Note that Proposition 3.4 and the Borel–Cantelli Lemma immediately imply the convergence case of Theorem 1.3: if $\sum_{n=1}^\infty\psi(n)<\infty$, then $\mu\big(R_T^f(\psi)\big)=\mu(\limsup_n A_n)=0$. Note also that or this conclusion one only needs the first two conditions of Theorem 1.3, that is, (1.8) and (1.9); the remaining conditions (1.10)–(1.13) will be used in the proof of the divergence case.

4. A quasi-independence estimate

Now let us make use of assumptions (1.10)–(1.13). The following lemma was stated and used in [HLSW]; we prove it here since our set-up is slightly different.

Lemma 4.1. *For $m\in\mathbb{N}$, J a cylinder in $\mathcal{F}_m$ and for any open set U contained in J, $\mu(T^mU)\asymp K_J^\delta\mu(U)$.*

Proof. By (1.8) and (1.13), we know that for all open balls $B\subset J$ with radius smaller than r_0, it holds that $\mu(T^mB)\asymp K_J^\delta\mu(B)$. Let $U\subset J$ be an open subset. Consider the cover

$$\mathcal{S}=\big\{B(x,r):x\in U,\ B(x,5r)\subset U,\ r<r_0\big\}$$

of U. By Vitali's covering theorem, $\mathcal{S}$ has a countable sub-collection $\mathcal{B}$ of disjoint balls so that

$$\bigcup_{B(x,r)\in\mathcal{B}}B(x,r)\subset U\subset\bigcup_{B(x,r)\in\mathcal{B}}B(x,5r).$$

Since T^m is injective on J,

$$\bigcup_{B(x,r)\in\mathcal{B}}T^mB(x,r)\subset T^mU\subset\bigcup_{B(x,r)\in\mathcal{B}}T^mB(x,5r).$$

Hence

$$K_J^\delta\sum_{B(x,r)\in\mathcal{B}}\mu\big(B(x,r)\big)\asymp\mu(T^mU).$$

On the other hand, $\mu(U)\asymp\sum_{B(x,r)\in\mathcal{B}}\mu\big(B(x,r)\big)$, and the lemma is proved. □

Now recall that we were working with the sets A_n defined in (3.1). The next lemma shows that the intersection of a cylinder of high enough level with A_n is contained in a small ball. Namely, let $m_0 \geq n_0$ be such that $K_J > \max\left\{\frac{K_1 \operatorname{diam}(X)}{r_0}, 2p\right\}$ for all $m > m_0$ and $J \in \mathcal{F}_m$ (which is possible in view of (1.11)).

Lemma 4.2. *For $m > m_0$, for every cylinder $J \in \mathcal{F}_m$ and any $z \in J \cap A_m$ there exists a ball of radius*

$$r = \frac{2\psi(m)}{K_J - p}, \tag{4.1}$$

say $B(z, r)$, such that

$$J \cap A_m \subset B(z, r) \cap J. \tag{4.2}$$

Proof. Choose any $x, z \in J \cap A_m$. Since $J \in \mathcal{F}_m$, in view of (1.10) we have

$$d(T^m x, T^m z) K_J^{-1} \geq d(x, z).$$

On the other hand,

$$\begin{aligned} d(T^m x, T^m z) &\leq d\big(T^m x, f(x)\big) + d\big(f(x), f(z)\big) + d\big(T^m z, f(z)\big) \\ &\leq 2\psi(m) + p d(x, z). \end{aligned}$$

Then $K_J d(x, z) < 2\psi(m) + p d(x, z)$, i.e. $d(x, z) < \frac{2\psi(m)}{K_J - p}$. □

Now let us prove a quasi-independence property of the sequence $\{A_n\}_n$. For any $m \in \mathbb{N}$ and $J \in \mathcal{F}_m$ define

$$J^* := B(z, r) \cap J, \tag{4.3}$$

where r and z are defined in (4.1) and (4.2).

Lemma 4.3. *For all $n > m > m_0$ and for each $J \in \mathcal{F}_m$,*

$$\mu(J \cap A_m \cap A_n) \lesssim K_J^{-\delta} \left[\psi(m)^\delta \psi(n)^\delta + a_{n-m} \psi(n)^\delta + a_n \psi(m)^\delta\right].$$

Proof. Let $J \in \mathcal{F}_m$. By Lemma 4.2,

$$J \cap A_m \cap A_n \subset J^* \cap A_n$$

where J^* is defined in (4.3). Now let us estimate $\mu(J^* \cap A_n)$.

Case (i): $pr = \frac{2p\psi(m)}{K_J - p} \leq \psi(n)$.

Note that for all $x \in B(z, r)$ we have $d\big(f(x), f(z)\big) < p d(x, z) < pr$, therefore $B(z, r) \subset f^{-1} B\big(f(z), pr\big)$. Thus J^* is a subset of $f^{-1} B\big(f(z), pr\big)$, and we can apply Lemma 3.1 to J^* and obtain

$$J^* \cap A_m \subset J^* \cap T^{-n} B\big(f(z), \psi(n) + pr\big) \subset J^* \cap T^{-n} B\big(f(z), 2\psi(n)\big). \tag{4.4}$$

Then apply Lemma 4.1 to $J^* \cap T^{-n} B\big(f(z), 2\psi(n)\big)$, getting

$$\begin{aligned} \mu\Big(J^* \cap T^{-n} B\big(f(z), 2\psi(n)\big)\Big) &\lesssim K_J^{-\delta} \Big(T^m J^* \cap T^{-(n-m)} B\big(f(z), 2\psi(n)\big)\Big) \\ &\leq K_J^{-\delta} \mu\Big(T^m B(z, r) \cap T^{-(n-m)} B\big(f(z), 2\psi(n)\big)\Big). \end{aligned}$$

Since $m > m_0$, we have $\inf_{J\in\mathcal{F}_m} K_J > 2p$. Then by the conformality assumption (1.12), we have

$$\begin{aligned} T^m B(z,r) &= T^m B\left(z, \frac{2\psi(m)}{K_J - p}\right) \\ &\subset B\left(T^m z, K_2 K_J \frac{2\psi(m)}{K_J - p}\right) \subset B\big(T^m z, K_2 4\psi(m)\big), \end{aligned}$$

where K_2 is defined in (1.12). Thus

$$\mu(J^* \cap A_n) \lesssim K_J^{-\delta} \mu\Big(B\big(T^m z, 4K_2\psi(m)\big) \cap T^{-(n-m)} B\big(f(z), 2\psi(n)\big)\Big).$$

By the mixing property (1.8),

$$\begin{aligned} &\mu(J^* \cap A_n) \lesssim K_J^{-\delta} \mu\Big(T^m B\big(T^m z, 4K_2\psi(m)\big) \cap T^{-(n-m)} B\big(f(z), 2\psi(n)\big)\Big) \\ &\lesssim K_J^{-\delta} \Big[\mu\Big(B\big(T^m z, 4K_2\psi(m)\big)\Big)\mu\Big(B\big(z, 2\psi(n)\big)\Big) + a_{n-m}\mu\Big(B\big(z, 2\psi(n)\big)\Big)\Big] \\ &\lesssim K_J^{-\delta} \Big[\psi(m)^\delta \psi(n)^\delta + a_{n-m}\psi(n)^\delta\Big]. \end{aligned}$$

<u>Case (ii)</u>: $pr = \frac{2p\psi(m)}{K_J - p} > \psi(n)$.

We replace the ball $B\big(f(z), pr\big)$ by a collection of balls of radius $\psi(n)$. Choose a maximal $\frac{3}{2}\psi(n)$-separated set in $B\big(f(z), pr\big)$, denoted by $\{z_i\}_{1\le i\le N_{m,n}}$. Then

$$B\big(f(z), pr\big) \subset \bigcup_{i=1}^{N_{m,n}} B\big(z_i, \psi(n)\big) \subset B\big(f(z), 2pr\big).$$

Since μ is Ahlfors regular and $m > n_0$,

$$N_{m,n} \asymp \left(\frac{p\psi(m)}{(K_J - p)\psi(n)}\right)^\delta.$$

Since $B\big(z_i, \psi(n)\big) \subset f^{-1} B\big(f(z_i), p\psi(n)\big)$, we can apply Lemma 3.1 to each ball $B\big(z_i, \psi(n)\big)$ with $1 \le i \le N_{m,n}$ and obtain

$$\begin{aligned} \mu\Big(B\big(z_i, \psi(n)\big) \cap A_n\Big) &\underset{(3.2)}{\le} \mu\Big(B\big(z_i, \psi(n)\big) \cap T^{-n} B\big(z_i, (1+p)\psi(n)\big)\Big) \\ &\underset{(1.9)}{\le} \mu\Big(B\big(z_i, \psi(n)\big)\Big)\mu\Big(B\big(z_i, (1+p)\psi(n)\big)\Big) + a_n \mu\Big(B\big(z_i, (1+p)\psi(n)\big)\Big) \\ &\underset{(1.8)}{\lesssim} \Big[\psi(n)^\delta + a_n\Big]\psi(n)^\delta. \end{aligned}$$

Now summing over $1 \le i \le N_{m,n}$, we have

$$\begin{aligned}\mu(J^* \cap A_n) &\le \sum_{i=1}^{N_{m,n}} \mu\Big(B\big(z_i, \psi(n)\big) \cap A_n\Big) \\ &\lesssim \left(\frac{p\psi(m)}{K_J - p}\right)^{\delta} \Big[\psi(n)^{\delta} + a_n\Big] \\ &\lesssim K_J^{-\delta}[\psi(m)^{\delta}\psi(n)^{\delta} + a_n\psi(m)^{\delta}].\end{aligned}$$

Combining the two cases we obtain the desired conclusion. □

Proposition 4.4. *For $n > m > m_0$,*

$$\mu(A_m \cap A_n) \lesssim \psi(m)^{\delta}\psi(n)^{\delta} + a_{n-m}\psi(n)^{\delta} + a_n\psi(m)^{\delta}.$$

Proof. This follows directly from the previous lemma and (1.12), since

$$\begin{aligned}\mu(A_m \cap A_n) &\lesssim \sum_{J \in \mathcal{F}_m} K_J^{-\delta}\Big[\psi(m)^{\delta}\psi(n)^{\delta} + a_{n-m}\psi(n)^{\delta} + a_n\psi(m)^{\delta}\Big] \\ &\underset{(1.12)}{\lesssim} \psi(m)^{\delta}\psi(n)^{\delta} + a_{n-m}\psi(n)^{\delta} + a_n\psi(m)^{\delta}.\end{aligned}$$

□

5. Proof of Theorems 1.3 and 1.4

To prove the divergence case of Theorem 1.3, let us recall

Lemma 5.1 (Chung–Erdös inequality, [CE])**.** *Let (X, μ) be a probability space, and let $\{E_n\}_n$ be a sequence of events such that $\sum_{n=1}^{\infty} \mu(E_n) = \infty$. Then*

$$\mu\left(\limsup_{n\to\infty} E_n\right) \ge \limsup_{N\to\infty} \frac{\left(\sum_{n=1}^{N} \mu(E_n)\right)^2}{\sum_{n,m=1}^{N} \mu(E_n \cap E_m)}.$$

The next lemma is based on the above inequality.

Lemma 5.2. *Let (X, μ) be a probability space. Let $(E_k)_k$ be a sequence of measurable subsets of X, and let $(a_n)_n, (b_n)_n$ be sequences of positive numbers such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} b_n = \infty$. Assume that for some $s_1, s_2, s_3 > 0$ it holds that*

$$s_1(b_n - a_n) \le \mu(E_n) \le s_3(b_n + a_n) \quad \textit{for all } n \in \mathbb{N}$$

and

$$\mu(E_m \cap E_n) \le s_2(b_m b_n + a_{n-m}b_n + a_n b_m) \quad \textit{for all } m < n.$$

Then $\mu(\limsup_{n\to\infty} E_n) \ge \frac{s_1^2}{2s_2}$.

Proof. Let us denote $\sum_{k=1}^{\infty} a_k = S$. Choose $\varepsilon > 0$. On the one hand,

$$\sum_{n=1}^{N} \mu(E_n) \ge s_1 \sum_{n=1}^{N}(b_n - a_n) \ge s_1 \left(\sum_{n=1}^{N} b_n - S\right) \ge (s_1 - \varepsilon) \sum_{n=1}^{N} b_n$$

when N is sufficiently large, because $\sum_{n=1}^{\infty} b_n = \infty$. On the other hand,

$$\begin{aligned}\sum_{m,n=1}^{N} \mu(E_m \cap E_n) &= \sum_{n=1}^{N} \mu(E_n) + 2 \sum_{1\le m<n\le N} \mu(E_m \cap E_n) \\ &\le s_3 \sum_{n=1}^{N}(b_n + a_n) + 2s_2 \sum_{m=1}^{N} \sum_{n=m+1}^{N} (b_m b_n + a_{n-m} b_n + a_n b_m) \\ &\le s_3 \sum_{n=1}^{N} b_n + s_3 S + 2s_2 \sum_{m=1}^{N} \sum_{n=m+1}^{N} b_m b_n + 2s_2 S \sum_{n=1}^{n} b_n + 2s_2 S \sum_{m=1}^{n} b_m \\ &\le 2s_2 \left(\sum_{n=1}^{N} b_n\right)^2 + (s_3 + 4s_2 S) \sum_{n=1}^{N} b_n + s_3 S \le (2s_2 + \varepsilon) \left(\sum_{n=1}^{N} b_n\right)^2\end{aligned}$$

when N is sufficiently large, again because $\sum_{n=1}^{\infty} b_n = \infty$. Hence by Lemma 5.1

$$\mu\left(\limsup_{n\to\infty} E_n\right) \ge \limsup_{N\to\infty} \frac{\left((s_1 - \varepsilon) \sum_{n=1}^{N} b_n\right)^2}{(2s_2 + \varepsilon)\left(\sum_{n=1}^{N} b_n\right)^2} = \frac{(s_1 - \varepsilon)^2}{2s_2 + \varepsilon},$$

and, since ε was arbitrary, the conclusion follows. □

Proof of Theorem 1.3, the divergence part. Take $E_n = A_{m_0+n}$, where m_0 is chosen prior to Lemma 4.2. By Lemma 3.3, there exist $s_1, s_3 > 0$ such that

$$s_1\big(\psi(m_0 + n)^{\delta} - a_{m_0+n}\big) \le \mu(E_n) \le s_3\big(\psi(m_0 + n)^{\delta} + a_{m_0+n}\big)$$

for any $n \in \mathbb{N}$. Also by Proposition 4.4 there exists $s_2 > 0$ such that

$$\mu(E_m \cap E_n) \le s_2\big(\psi^{\delta}(m_0+m)\psi^{\delta}(m_0+n) + a_{n-m}\psi^{\delta}(m_0+n) + a_{m_0+n}\psi^{\delta}(m_0+m)\big)$$

for $n > m$. By taking $b_n = \psi(m_0 + n)^{\delta}$, the above lemma implies that $\mu\big(R_T^f(\psi)\big) = \mu(\limsup_{n\to\infty} E_n) \ge \frac{s_1^2}{2s_2} > 0$. □

We conclude the section with the proof of Theorem 1.4, that is, a passage from positive measure to full measure under the assumption that f and T commute. This proof is adapted from [HLSW].

Proof of Theorem 1.4. Suppose that (1.16) holds. Consider the set

$$R'(\psi) := \left\{x \in X : \liminf_{n\to\infty} \psi(n)^{-1} d\big(T^n x, f(x)\big) < \infty\right\}$$

Take a point $x \in R'(\psi) \cap f^{-1} X_i$. By definition, there exist $c(x) > 0$ and $\{n_k\}_k \subset \mathbb{N}$ so that

$$\psi(n_k)^{-1} d\big(T^{n_k} x, f(x)\big) < c(x) \quad \forall\, k \ge 1$$

Let $s(x)$ be a positive real number such that $B\big(f(x), s(x)\big) \subset X_i$. Since $\psi(n) \to 0$, take $N \in \mathbb{N}$ such that for all $k \ge N$,

$$c(x)\psi(n_k) < s(x)$$

Then $d\big(T^{n_k}x, f(x)\big) < c(x)\psi(n_k) < s(x)$, therefore $T^{n_k}x \in X_i$ for all $k \geq N$. Hence for all $k \geq N$,

$$d\Big(T^{n_k}(Tx), \big(f(Tx)\big)\Big) \underset{(1.16)}{=} d\Big(T^{n_k}(Tx), T\big(f(x)\big)\Big) = d\Big(T(T^{n_k}x), T\big(f(x)\big)\Big) \leq d\big(T^{n_k}x, f(x)\big) < c(x)\psi(n_k).$$

This implies $R'(\psi)\cap(\bigcup_i f^{-1}X_i) \subset T^{-1}R'(\psi)$, thus $\mu\big(R'(\psi)\smallsetminus T^{-1}R'(\psi)\big) = 0$. But $R_T^f(\psi) \subset R'(\psi)$, hence $\mu\big(R'(\psi)\big) > 0$, and by the ergodicity of T, $\mu\big(R'(\psi)\big) = 1$.

Now we show that $\mu\big(R_T^f(\psi)\big) = 1$. Take a sequence of positive numbers $\{\ell(n) : n \geq 1\}$ such that

$$\sum_{n=1}^{\infty} \frac{\psi(n)}{\ell(n)} = \infty \quad \text{and} \quad \lim_{n\to\infty} \ell(n) = \infty.$$

Consider $\widetilde{\psi}(n) := \psi(n)/\ell(n)$; then $R'(\widetilde{\psi})$ has full measure, i.e. for μ-almost every $x \in X$,

$$\liminf_{n\to\infty} \widetilde{\psi}(n)^{-1} d\big(T^n x, f(x)\big) < \infty.$$

By Egorov's theorem, for any $\varepsilon > 0$ there exists $M > 0$ such that the set

$$R_M := \left\{ x \in X : \frac{\ell(n)}{\psi(n)} d\big(T^n x, f(x)\big) < M \text{ for infinitely many } n \in \mathbb{N} \right\}$$

is of measure at least $1-\varepsilon$. Then $R_M \subset R_T^f(\psi)$, by letting $\ell(n) \to \infty$. Since ε is arbitrary, it implies $\mu\big(R_T^f(\psi)\big) = 1$. □

6. Proof of Theorem 1.5

We first prove a local version of Lemma 3.3; i.e., fix a ball B with sufficiently small radius and estimate $\mu(A_n \cap B)$ for n sufficiently large.

Lemma 6.1. *For any $r < r_0/2$ there exists $n_r \in \mathbb{N}$ such that for any open ball $B = B(x,r)$ in X and all $n > n_r$,*

$$\frac{\eta_1}{\eta_2 20^\delta}\mu(B)\left[\frac{\eta_1^2}{\eta_2}\psi(n)^\delta - (2p)^\delta a_n\right] \leq \mu(A_n \cap B) \leq \frac{\eta_2^2 3^\delta}{\eta_1^2}\mu(B)\left[\eta_2 5^\delta \psi(n)^\delta + a_n(2p)^\delta\right]$$

Proof. Let $n_r \in \mathbb{N}$ be such that

$$\frac{5\psi(n)}{p} < \frac{r}{2} \tag{6.1}$$

for all $n > n_r$. As in the proof of Lemma 3.3, we have an open covering

$$\left\{ B\Big(y, \frac{\psi(n)}{2p}\Big) : y \in B \right\}$$

of B, with each $B\Big(y, \frac{\psi(n)}{2p}\Big) \subset f^{-1}B\big(x, \psi(n)/2\big)$ for some $x \in X$. Again by Vitali's Covering Theorem, we can find countable sub-collection of disjoint balls $\big\{B\big(y_j, \psi(n)/2p\big)\big\}_{j\in\mathcal{J}}$ such that

$$B \subset \bigcup_{j\in\mathcal{J}} B\big(y_j, 5\psi(n)/2p\big). \tag{6.2}$$

Then we have

$$\sum_{j\in\mathcal{J}} \eta_1\big(\psi(n)/2p\big)^\delta \le \sum_{j\in\mathcal{J}} \mu\Big(B\big(y_j, \psi(n)/2p\big)\Big) \underset{(6.1)}{\le} \mu\big(B(x, 2r)\big) \le 2^\delta \frac{\eta_2}{\eta_1}\mu(B),$$

hence

$$|\mathcal{J}| \le \frac{\eta_2(4p)^\delta \mu(B)}{\eta_1^2 \psi(n)^\delta}. \tag{6.3}$$

On the other hand,

$$\sum_{\substack{j\in\mathcal{J} \\ B\left(y, \frac{5\psi(n)}{2p}\right)\cap B(x,r/2)\neq\varnothing}} \eta_2\big(5\psi(n)/2p\big)^\delta \ge \sum_{\substack{j\in\mathcal{J} \\ B\left(y, \frac{5\psi(n)}{2p}\right)\cap B(x,r/2)\neq\varnothing}} \mu\Big(B\big(y_j, 5\psi(n)/2p\big)\Big)$$
$$\underset{(6.2)}{\ge} \mu\big(B(x, r/2)\big) \ge \frac{\eta_1}{2^\delta \eta_2}\mu(B),$$

therefore

$$\begin{aligned} &\left|\left\{j \in \mathcal{J} : B\Big(y_j, \frac{\psi(n)}{2p}\Big) \subset B\right\}\right| \\ \underset{(6.1)}{\ge} &\left|\left\{j \in \mathcal{J} : B\Big(y_j, \frac{\psi(n)}{2p}\Big) \cap B(x, r/2) \neq \varnothing\right\}\right| \ge \frac{\eta_1 p^\delta \mu(B)}{\eta_2^2\big(5\psi(n)\big)^\delta}. \end{aligned} \tag{6.4}$$

Since for each $j \in \mathcal{J}$ we have $B\big(y_j, \psi(n)/2p\big) \subset f^{-1}B\big(x_j, \psi(n)/2\big)$ and $B\big(y_j, 5\psi(n)/2p\big) \subset f^{-1}B\big(x_j, 5\psi(n)/2\big)$, similarly to the proof of Lemma 3.3 we can write

$$\begin{aligned} \mu(A_n \cap B) &\le \sum_{j\in\mathcal{J}} \mu\big(B(y_j, 5\psi(n)/2p) \cap A_n\big) \\ &\underset{\text{Lemma 3.2}}{\le} \sum_{j\in\mathcal{J}} \eta_2(3/2)^\delta \left[\mu\Big(B\big(y_j, 5\psi(n)/2p\big)\Big) + a_n\right] \psi(n)^\delta \\ &\underset{(6.3),(1.8)}{\le} \frac{\eta_2(4p)^\delta \mu(B)}{\eta_1^2 \psi(n)^\delta} \eta_2(3/2)^\delta \left[\eta_2\big(5\psi(n)/2p\big)^\delta \psi(n)^\delta + a_n \psi(n)^\delta\right] \\ &= \frac{\eta_2^2 3^\delta}{\eta_1^2}\mu(B)\left[\eta_2 5^\delta \psi(n)^\delta + a_n(2p)^\delta\right] \end{aligned}$$

and

$$\begin{aligned}
\mu(A_n\cap B) &\geq \sum_{\substack{j\in\mathcal{J}\\ B(y_j,\psi(n)/2p)\subset B}} \mu\big(B(y_j,\psi(n)/2p)\cap A_n\big)\\
&\underset{\text{Lemma 3.2}}{\geq} \sum_{\substack{j\in\mathcal{J}\\ B(y_j,\psi(n)/2p)\subset B}} 2^{-\delta}\left[\eta_1\mu\big(B(y_j,\psi(n)/2p)\big)-\eta_2 a_n\right]\psi(n)^\delta\\
&\underset{(6.4),(1.8)}{\geq} \frac{\eta_1 p^\delta\mu(B)}{\eta_2^2\big(5\psi(n)\big)^\delta}\,2^{-\delta}\left[\eta_1^2\big(\psi(n)/2p)\big)^\delta-\eta_2 a_n\right]\psi(n)^\delta\\
&= \frac{\eta_1}{\eta_2 20^\delta}\mu(B)\left[\frac{\eta_1^2}{\eta_2}\psi(n)^\delta-(2p)^\delta a_n\right],
\end{aligned}$$

finishing the proof. □

Lemma 6.2. *Let $m > m_0$, and let J be a cylinder in $\mathcal{F}_m$. Then* $\operatorname{diam}(J)\leq K_1\operatorname{diam}(X)K_J^{-1}$.

Proof. Let $x,y\in J$. By (1.10) and the definition of K_J,

$$d(x,y)\leq\frac{K_1 d(T^m x,T^m y)}{K_J},$$

hence by (1.10)

$$d(x,y)\leq K_1\operatorname{diam}(X)K_J^{-1},$$

which implies the needed upper bound on $\operatorname{diam}(J)$. □

For the rest of the section, let us assume that $\{X_i\}_{i\in\mathcal{I}}$ is pseudo-Markov. Let $0<\tau<1$ be such that $\mu(TX_i)\geq\tau\mu(X)$ for all $i\in\mathcal{I}$.

Lemma 6.3. *For all $n\in\mathbb{N}$ and for a nonempty cylinder*

$$J = X_{i_0}\cap T^{-1}X_{i_1}\cap\cdots\cap T^{-n+1}X_{i_n}\in\mathcal{F}_n,$$

one has $T^nJ = TX_{i_n}$.

Proof. For each $0\leq j<n$, we have $T^{-j+1}X_{i_j}\cap T^{-j}X_{i_{j+1}}\neq\varnothing$, and then $TX_{i_j}\cap X_{i_{j+1}}\neq\varnothing$, so by the pseudo-Markov condition, $TX_{i_j}\supset X_{i_{j+1}}$. Now let $x\in X_{i_n}$, then since $TX_{i_{n-1}}\supset X_{i_n}$, there exists some $x_{n-1}\in X_{i_{n-1}}$ so that $Tx_{n-1}=x$. Similarly, there exists some $x_{n-2}\in X_{i_{n-2}}$ such that $Tx_{n-2}=x_{n-1}$. Continue this process until we find such $x_0\in X_{i_0}$. Then $T^{j-1}x_0\in X_{i_j}$ for each $0<j<n$, so in particular $x_0\in J$ and $T^{n-1}x_0=x$. Hence $T^{n-1}J\supset X_{i_n}$.

For the reverse containment, since $J\subset T^{-n+1}X_{i_n}$, it follows that $T^{n-1}J\subset X_{i_n}$. It remains to apply T to both sides of $T^{n-1}J=X_{i_n}$ to conclude the lemma. □

Lemma 6.4. *Let $m>m_0$, and let J be a cylinder in $\mathcal{F}_m$. Then* $\mu(J)\gtrsim K_J^{-\delta}$.

Proof. Write J in the form

$$J = X_{i_1} \cap T^{-1} X_{i_2} \cap \cdots T^{-m+1} X_{i_m};$$

then

$$K_J^{\delta} \mu(J) \underset{\text{Lemma 4.1}}{\asymp} \mu(T^m J) \underset{\text{Lemma 6.3}}{=} \mu(T X_{i_m}) \geq \tau \mu(X),$$

thus $\mu(J) \gtrsim K_J^{-\delta}$. □

We now prove a local estimate for the quasi-independence of the intersection of sets $\{A_n\}_n$ with balls.

Corollary 6.5. *For any open ball $B = B(x,r)$ in X with $2r < r_0$, there exists $m_r \in \mathbb{N}$ so that for all $n > m > m_r$,*

$$\mu(B \cap A_m \cap A_n) \lesssim \mu(B) \left[\psi(m)^{\delta} \psi(n)^{\delta} + a_{n-m} \psi(n)^{\delta} + a_n \psi(m)^{\delta} \right],$$

where the implicit constant in the above inequality is independent of B.

Proof. Let $q > m_0$ and let $I \in \mathcal{F}_q$. For all $n > m \geq q$, by Lemma 4.3 we get

$$\begin{aligned} \mu(I \cap A_m \cap A_n) &= \sum_{\substack{J \in \mathcal{F}_m \\ J \subset I}} \mu(J \cap A_m \cap A_n) \lesssim \sum_{\substack{J \in \mathcal{F}_m \\ J \subset I}} \mu(J^* \cap A_n) \\ &\lesssim \sum_{\substack{J \in \mathcal{F}_m \\ J \subset I}} K_J^{-\delta} \Big[[\psi(m)^{\delta} \psi(n)^{\delta} + a_{n-m} \psi(n)^{\delta}] + [\psi(m)^{\delta} \psi(n)^{\delta} + a_n \psi(m)^{\delta}] \Big] \\ &\underset{\text{Lemma 6.4}}{\lesssim} K_I^{-\delta} \left[\psi(m)^{\delta} \psi(n)^{\delta} + a_{n-m} \psi(n)^{\delta} + a_n \psi(m)^{\delta} \right]. \end{aligned} \tag{6.5}$$

Now let $B(x,r)$ be an open ball in X with $r < r_0/2$. By the expanding property (1.11), we can take some $m_r > m_0$ so that for all $q > m_r$,

$$\inf_{J \in \mathcal{F}_q} K_J > r^{-1} K_1 \operatorname{diam}(X);$$

Let $q > m_r$. Then by Lemma 6.2 for all $J \in \mathcal{F}_q$, $\operatorname{diam}(J) < r$, so

$$B(x,r) \subset \bigsqcup_{\substack{J \in \mathcal{F}_q, \\ J \cap B(x,r) \neq \varnothing}} J \subset B(x, 2r).$$

Then

$$\mu\big(B(x,r)\big) \leq \sum_{\substack{J \in \mathcal{F}_q, \\ J \cap B(x,r) \neq \varnothing}} \mu(J) \leq \mu\big(B(x,2r)\big).$$

Therefore by (6.5),

$$\begin{aligned}
\mu\big(B(x,r)\cap A_n\cap A_m\big) &\le \sum_{\substack{J\in\mathcal{F}_q,\\ J\cap B(x,r)\neq\varnothing}} \mu(J\cap A_n\cap A_m)\\
&\underset{(6.5)}{\lesssim} \sum_{\substack{J\in\mathcal{F}_q,\\ J\cap B(x,r)\neq\varnothing}} K_I^{-\delta}\left[\psi(m)^\delta\psi(n)^\delta + a_{n-m}\psi(n)^\delta + a_n\psi(m)^\delta\right]\\
&\underset{\text{Lemma 6.4}}{\lesssim} \sum_{\substack{J\in\mathcal{F}_q,\\ J\cap B(x,r)\neq\varnothing}} \mu(J)\left[\psi(m)^\delta\psi(n)^\delta + a_{n-m}\psi(n)^\delta + a_n\psi(m)^\delta\right]\\
&\le \mu\big(B(x,2r)\big)\left[\psi(m)^\delta\psi(n)^\delta + a_{n-m}\psi(n)^\delta + a_n\psi(m)^\delta\right]\\
&\le 2^\delta\frac{\eta_2}{\eta_1}\mu\big(B(x,r)\big)\left[\psi(m)^\delta\psi(n)^\delta + a_{n-m}\psi(n)^\delta + a_n\psi(m)^\delta\right].
\end{aligned}$$

□

Finally let us apply the following generalization of the Lebesgue Density Theorem to finish the proof of Theorem 1.5. Recall that a probability measure μ on X is called *doubling* if there exists a constant $D > 0$ so that for any $x \in X$ and $r > 0$,

$$\mu(B(x,2r)) \le D\mu(B(x,r)).$$

It is clear that Ahlfors regular measures are doubling.

Theorem 6.6 (Lebesgue Density Theorem)**.** *Let (X,d) be a metric space with a Borel doubling probability measure μ, and let E be a Borel subset of X. Suppose there exist constants $C > 0$ and $r_0 > 0$ so that for all balls $B \subset X$ with radii less than r_0, we have*

$$\mu(E\cap B) \ge C\mu(B).$$

Then $\mu(E) = 1$.

For a proof, see [BDV, §8, Lemma 7].

Proof of Theorem 1.5. By Lemma 6.1 and Corollary 6.5, there exist positive constants s_1, s_2, s_3 so that for all $B = B(x,r)$ in X with $r < r_0/2$ and for all $n > \max\{n_r, m_r\}$,

$$\mu(B)s_1\left(\psi(n)^\delta - a_n\right) \le \mu(B\cap A_n) \le \mu(B)s_3\left(\psi(n)^\delta + a_n\right).$$

$$\mu(B\cap A_n\cap A_m) \le \mu(B)s_2\left(\psi(m)^\delta\psi(n)^\delta + a_{n-m}\psi(n)^\delta + a_n\psi(m)^\delta\right).$$

Now take $b_n = \psi(n)^\delta$ and

$$E_k = \begin{cases} A_k\cap B & \text{if } k > \max\{n_r, m_r\}\\ \varnothing & \text{otherwise.}\end{cases}$$

$$\mu(\limsup_{n\to\infty} A_n \cap B) = \mu(\limsup_{k\to\infty} E_k) \geq \frac{\big(\mu(B)s_1\big)^2}{2s_2\mu(B)} = \frac{s_1^2}{2s_2}\mu(B).$$

Then by the Lebesgue Density Theorem (Theorem 6.6),

$$\mu\big(R_T^f(\psi)\big) = \mu(\limsup_{n\to\infty} A_n) = 1.$$

□

7. Examples

Here we list several examples of dynamical systems to which our theorems apply. The first two come from the paper [HLSW]:

- $X = [0,1]$, $T : x \mapsto \beta x \mod 1$, where $\beta > 1$, and μ is the T-invariant probability measure absolutely continuous with respect to Lebesgue measure, namely (see [R])

$$\mu(E) = \begin{cases} \text{Leb}(E) & \text{if } \beta \text{ is an integer,} \\ \frac{1}{\sum_{k=0}^{\infty} \frac{\{\beta^k\}}{\beta^k}} \sum_{k=0}^{\infty} \frac{\text{Leb}\big(E\cap[0,\{\beta^k\}]\big)}{\beta^k} & \text{if } \beta \text{ is not an integer,} \end{cases}$$

  where $\{x\}$ denotes the fractional part of x;
- $X = [0,1]$, $T : x \mapsto \frac{1}{x} \mod 1$, and μ is the Gauss measure given by $d\mu = \frac{dx}{(\log 2)(1+x)}$.

Sections 3.1–3.2 of [HLSW] together with Remark 1.2 show that in both cases the assumptions of Theorem 1.3 are satisfied. In fact, in both cases uniform mixing with exponential rate was first exhibited in [Ph], together with a quantitative shrinking target property of these systems.

The pseudo-Markov property holds for the Gauss map but only for some special β-transformations. We will prove the full measure in the divergence case for arbitrary β-transformations in §8.

Our last example deals with self-similar sets. Let

$$\Theta := \{\theta_i(x) : \mathbb{R} \to \mathbb{R}\}_{i=1}^{L}$$

be a system of similarities with

$$|\theta_i(x) - \theta_i(y)| = r_i|x-y| \text{ for all } x, y \in \mathbb{R},\ i = 1, \dots, L,$$

where $0 < r_i < 1$ for all i. Then by [H, Theorem 3.1.(3)] there exists a unique nonempty compact set $X \subset \mathbb{R}$, called the *attractor* of the system, such that

$$X = \bigcup_{i=1}^{L} \theta_i(X).$$

Furthermore, we assume that Θ satisfies the *open set condition*: that is, there exists a non-empty bounded open set $U \subset \mathbb{R}$ such that

$$\bigcup_{i=1}^{L} \theta_i(U) \subset U \text{ and } \theta_i(U) \cap \theta_j(U) = \varnothing \text{ for all } i \neq j.$$

Then it is known that the Hausdorff dimension of X is equal to the unique solution $\delta \in [0,1]$ of the equation $\sum_{i=1}^{L} r_i^\delta = 1$ (see [F, Theorem 9.3]). Furthermore, the normalized restriction μ of the δ-dimensional Hausdorff measure to X is positive, finite and satisfies

$$\mu = \sum_{i=1}^{L} r_i^\delta \cdot (\theta_i)_* \mu. \tag{7.1}$$

(For a proof, see [H, Theorem 4.4.(1)].)

To define the corresponding expanding map and construct the cylinders, we consider the following lemma from [Sc, Gr]:

Lemma 7.1 ([Sc], Theorem 2.2; [Gr], Lemma 3.3)**.** *Let $\{\theta_i : \mathbb{R}^n \to \mathbb{R}^n\}_{i=1}^L$ be a system of similarities satisfying the open set condition, X its attractor, and μ the self-similar measure given by* (7.1)*. Then there exists a nonempty compact set A with*

(i) *$\theta_i(A) \subset A$ for all $i = 1, \ldots, L$;*
(ii) *$\theta_i\big(\mathsf{int}(A)\big) \cap \theta_j\big(\mathsf{int}(A)\big) = \varnothing$ for each $i \neq j$;*
(iii) *$\mu\big(\mathsf{int}(A) \cap X\big) = 1$.*

We remark that parts (i) and (ii) are stated in [Sc, Theorem 2.2], and part (iii) follows from the proof of [Gr, Lemma 3.3], where it is shown that $\mu\big((A \smallsetminus \mathsf{int}(A)\big) = 0$ and $X = \operatorname{supp}\mu \subset A$.

Now define

$$X_i := \theta_i\big(\mathsf{int}(A) \cap X\big). \tag{7.2}$$

Each X_i is open in X because θ_i is an open map. The disjointness of X_i and X_j for $i \neq j$ follows from Lemma 7.1(ii). Finally, one can write

$$\begin{aligned} \mu\left(\bigcup_{i=1}^{L} X_i\right) &= \sum_{i=1}^{L} \mu\big(\theta_i(\mathsf{int}(A) \cap X)\big) = \sum_{i=1}^{L} r_i^\delta \mu(\mathsf{int}(A) \cap X) \\ \text{(by Lemma 7.1(iii))} \ &= \sum_{i=1}^{L} r_i^\delta = 1 = \mu(X). \end{aligned}$$

Hence one can define the map $T : X \to X$ μ-almost everywhere by setting $T|_{X_i} = \theta_i^{-1}|_{X_i}$. It follows from (7.1) that (X, μ, T) is a measure-preserving system. Clearly $T|_{X_i}$ is continuous and injective for every i. Therefore the collection $\{X_i\}_{i=1}^L$ satisfies our assumption for being cylinders of order 1.

For $\mathbf{i} = (i_1, \ldots, i_m) \in \{1, \ldots, L\}^m$ let us define

$$\theta_{\mathbf{i}} := \theta_{i_1} \circ \cdots \circ \theta_{i_m} \quad \text{and} \quad r_{\mathbf{i}} := \prod_{k=1}^{m} r_{i_k}.$$

Using the definition (1.7) of cylinders of order m together with (7.2), it is easy to see that the set $\mathcal{F}_m$ of cylinders of order m is precisely

$$\big\{X_{\mathbf{i}} := \theta_{\mathbf{i}}\big(\mathsf{int}(A) \cap X\big) : \mathbf{i} \in \{1, \ldots, L\}^m\big\},$$

and the restriction of T^m onto $X_{\mathbf{i}} \in \mathcal{F}_m$ coincides with $\theta_{\mathbf{i}}^{-1}$. This, in particular, implies that

$$\mu(X_{\mathbf{i}}) = r_{\mathbf{i}}^{\delta} \tag{7.3}$$

and

$$K_J = \frac{|T^m x - T^m y|}{|x-y|} = r_{\mathbf{i}}^{-1} \quad \text{for all } x, y \in J = X_{\mathbf{i}} \in \mathcal{F}_m. \tag{7.4}$$

Denote

$$r_{\max} := \max_{i=1,\dots,L} r_i.$$

Let us now verify assumptions (1.8)–(1.13) of Theorem 1.3.

(1.8): By [H, Theorem 5.3(1)(i)],

$$\gamma_1 \le \liminf_{r\to 0} \frac{\mu\big(B(x,r)\big)}{r^\delta} \le \limsup_{r\to 0} \frac{\mu\big(B(x,r)\big)}{r^\delta} \le \gamma_2$$

for some $0 < \gamma_1 < \gamma_2 < \infty$ and all $x \in X$. Clearly it implies that μ is Ahlfors regular of dimension δ.

(1.9): (T is uniformly mixing) Let E be a non-empty open ball in X, let F be a measurable set in X, and let $m \in \mathbb{N}$. Note that for all cylinders $J = X_{\mathbf{i}} \in \mathcal{F}_m$, where $\mathbf{i} \in \{1,\dots,L\}^m$ one can write

$$\mu(J \cap T^{-m}F) = r_{\mathbf{i}}^{\delta}\mu(T^m J \cap F) = r_{\mathbf{i}}^{\delta}\mu\big(\mathsf{int}(A) \cap F\big) \underset{(7.3)}{=} \mu(J)\mu(F).$$

Note that since E is an interval, we can (up to a set of measure zero) write E as a disjoint union of cylinders of order m and at most two balls contained in cylinders of order m; i.e.,

$$E = \Big(\bigcup_{J\in\mathcal{F}_m,\, J\subset E} J\Big) \cup E_1 \cup E_2$$

where the unions above are disjoint, and E_1, E_2 are contained in some cylinders $J_1, J_2 \in \mathcal{F}_m$ respectively, hence have measure not greater than $r_{\max}^{m\delta}$. Then

$$\begin{aligned}
&\big|\mu(E\cap T^{-m}F) - \mu(E)\mu(F)\big| \\
&= \Big|\sum_{\substack{J\in\mathcal{F}_m,\\ J\subset E}} (J\cap T^{-m}F) - \sum_{\substack{J\in\mathcal{F}_m,\\ J\subset E}} \mu(J)\mu(F) + \mu(E_1\cap T^{-m}F) - \mu(E_1)\mu(F) \\
&+ \mu(E_2\cap T^{-m}F) - \mu(E_2)\mu(F)\Big| \\
&= \Big|\mu(E_1\cap T^{-m}F) - \mu(E_1)\mu(F) + \mu(E_2\cap T^{-m}F) - \mu(E_2)\mu(F)\Big| \\
&\le \mu(J_1\cap T^{-m}F) + \mu(J_1)\mu(F) + \mu(J_2\cap T^{-m}F) + \mu(J_2)\mu(F) \\
&= 4r_{\max}^{m\delta}\mu(F),
\end{aligned}$$

and (1.9) follows with $a_n = 4r_{\max}^{n\delta}$.

(1.10): Follows from (7.4) with $K_1 = 1$.

(1.11): Again from (7.4), for all $m \in \mathbb{N}$ we have

$$\inf_{J\in\mathcal{F}_m} K_J = \inf_{\mathbf{i}\in\{1,\dots,L\}^m} r_{\mathbf{i}}^{-1} = r_{\max}^{-m},$$

which goes to ∞ as $m \to \infty$.

(1.12): In view of (7.4), for all $m \in \mathbb{N}$ one can write

$$\sum_{J\in\mathcal{F}_m} K_J^{-\delta} = \sum_{\mathbf{i}\in\{1,\dots,L\}^m} r_{\mathbf{i}}^{\delta} = (r_1^{\delta} + \cdots + r_m^{\delta})^m = 1.$$

(1.13): Also follows from (7.4) with $K_2 = 1$.

It is clear that $\{X_i\}_{i\in\mathcal{I}}$ is pseudo-Markov. Thus, by Theorem 1.5, for any function $\psi : \mathbb{N} \to \mathbb{R}^+$ and any Lipshitz function $f : X \to X$, the f-twisted recurrence set $R_T^f(\psi)$ satisfies

- $\mu\big(R_T^f(\psi)\big) = 0$ if $\sum_{n=1}^{\infty} \psi(n)^{\delta} < \infty$;
- $\mu\big(R_T^f(\psi)\big) = 1$ if $\sum_{n=1}^{\infty} \psi(n)^{\delta} = \infty$.

8. Proof of Theorem 1.6

Let $\beta > 1$ be a real number and suppose $\sum_{n=1}^{\infty} \psi(n) = \infty$. In this section, we will consider the system

$$X = [0, 1], \quad T = M_\beta, \quad \mu \text{ the } M_\beta\text{-invariant measure,}$$

and the partition

$$\left\{X_i = \left(\frac{i}{\beta}, \frac{i+1}{\beta}\right) : i = 0, 1, \dots, \lfloor\beta\rfloor - 1\right\} \cup \left\{X_{\lfloor\beta\rfloor} = \left(\frac{\lfloor\beta\rfloor}{\beta}, 1\right)\right\}. \quad (8.1)$$

If β is an integer, then (8.1) is pseudo-Markov. Furthermore, if β satisfies

$$\beta^2 - k\beta - \ell = 0, \quad \text{for some } k \geq \ell \in \mathbb{N}^*$$

then

$$TX_i = X \ \ \forall i < \lfloor\beta\rfloor \quad \text{and} \quad TX_{\lfloor\beta\rfloor} = \left(0, \frac{\ell}{\beta}\right),$$

and hence (8.1) is also pseudo-Markov. In both cases, we have $\mu\left(R_T^f(\psi)\right) = 1$. by Theorem 1.5. We now prove the general case, which is also proved in [LWW].

We will apply Lemmas 6.1 and 4.3. We remark that analogous results are also proved in [LWW]; however our lemmas are proved in a more general abstract setting, do not depend on the actual arithmetic and symbolic coding of the systems, and have a much weaker assumption on the regularity of the cylinders.

We first state some facts about (X, μ, T).

Lemma 8.1 ([R]). *For (X, μ, T) defined above, μ is equivalent to* Leb *and* $\mu(E) = \frac{1}{\sum_{k=0}^{\infty} \frac{\{\beta^k\}}{\beta^k}} \sum_{k=0}^{\infty} \frac{\mathrm{Leb}\left(E\cap[0,\{\beta^k\}]\right)}{\beta^k}$ *for all Lebesgue-measurable set E.*

For a proof, see [R, Theorem 1].

To state the next lemma, we define the *lexicographical order* $\prec$. For two words $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$, we write $\alpha \prec \beta$ if there exists $k \in \mathbb{N}$ so that $\alpha_i = \beta_i$ for all $i \leq k$ and $\alpha_k < \beta_k$. We write $\alpha \preceq \beta$ if $\alpha \prec \beta$ or $\alpha = \beta$. Denote the β-expansion of 1 by

$$1 = \sum_{n=1}^{\infty} \frac{\xi_n}{\beta^n}.$$

If $\{\xi_n\}_n$ is not eventually zero, then we define $\xi_n^* = \xi_n$ for all $n \in \mathbb{N}$. If $\{\xi_n\}_n$ is eventually zero, then denote $M_\xi := \max\{n \in \mathbb{N} : \xi_n \neq 0\}$ and define

$$\xi_n^* = \begin{cases} \xi_k & \text{if } n \equiv k \mod M_\xi,\ k > 0 \\ \xi_{M_\xi} - 1 & \text{if } n \equiv 0 \mod M_\xi. \end{cases}$$

A classical result says that the right-most cylinder has the maximal coding in lexicographical order, in the following sense:

Lemma 8.2 ([P])**.** *Let (X, μ, T) and $\{X_i\}_{i \in \mathcal{I}}$ be defined above, and let*

$$J = X_{i_0} \cap T^{-1} X_{i_1} \cap \cdots \cap T^{-n+1} X_{i_{n-1}}.$$

Then J is nonempty if and only if

$$(i_j, i_{j+1}, \ldots, i_{n-1}) \prec (\xi_1^*, \xi_2^*, \ldots), \quad \text{for each } j = 0, \ldots, n-1.$$

For a proof, see [P, Theorem 3].

For a cylinder $J \in \mathcal{F}_n$ we always have $\mathrm{Leb}(J) \leq \beta^{-n}$. Let us call J *full* if the upper bound is reached; i.e., $\mathrm{Leb}(J) = \beta^{-n}$.

Lemma 8.3 ([FW])**.** *Let J be as in Lemma 8.2. If J is nonempty, then*

$$X_{i_0} \cap T^{-1} X_{i_1} \cap \cdots \cap T^{-n+1} X_{j_{n-1}}$$

is full for all $j_{n-1} < i_{n-1}$.

For a proof, see [FW, Lemma 3.2(1)].

Proposition 8.4. *For all $n \in \mathbb{N}$ and for all $J \in \mathcal{F}_n$ with $J \neq \varnothing$, there exists some full cylinder $I \in \mathcal{F}_m$ with*

$$I \subset J \quad \textit{and} \quad \mathrm{Leb}(I) \geq \frac{\mathrm{Leb}(J)}{\beta}.$$

A similar inequality is proved in [LWW].

Proof. Let $J \in \mathcal{F}_n$ with $J \neq \varnothing$. By Lemma 8.2, the set

$$\mathcal{N}_J := \left\{ k \in \mathbb{N} : J \cap \bigcap_{j=0}^{k} T^{-n-j+1} X_0 \cap T^{-n-k} X_1 \neq \varnothing \right\}$$

is nonempty. Let $k_J = \inf \mathcal{N}_J$. If $k_J = 0$, then trivially we have $J \cap \bigcap_{j=0}^{k_J} T^{-n-j+1} X_0 = J$. Now suppose $k_J > 0$. Assume

$$J \supsetneq J \cap \bigcap_{j=0}^{k_J} T^{-n-j+1} X_0.$$

Then there exists some $1 \leq i < n-1$ such that

$$J \cap \bigcap_{j=0}^{k_J-1} T^{-n-j+1} X_0 \cap T^{-n-k_J+1} X_i \neq \varnothing. \tag{8.2}$$

But by Lemma 8.3, $i = 1$ must satisfy (8.2), which contradicts the minimality of k_J in $\mathcal{N}_J$, so

$$J \cap \bigcap_{j=0}^{k_J} T^{-n-j+1} X_0 = J.$$

By Lemma 8.3

$$\mathrm{Leb}\left(J \cap \bigcap_{j=0}^{k_J+1} T^{-n-j+1} X_0 \right) = \beta^{-n-k-1}.$$

On the other hand,

$$\mathrm{Leb}\left(J \cap \bigcap_{j=0}^{k_J} T^{-n-j+1} X_0 \right) \leq \beta^{-n-k},$$

hence

$$\mathrm{Leb}(J) \leq \beta \, \mathrm{Leb}\left(J \cap \bigcap_{j=0}^{k_J+1} T^{-n-j+1} X_0 \right).$$

Taking $I = J \cap \bigcap_{j=0}^{k_J+1} T^{-n-j+1} X_0$, we finish the proof. □

Note that $K_J = \beta^n$ for all $J \in \mathcal{F}_n$. Now we can prove Theorem 1.6.

Proof of Theorem 1.6. Sections 3.1–3.2 of [HLSW] together with Remark 1.2 show that the system satisfies (1.8)–(1.13). By Lemma 6.1, there exists positive constants s_1, s_3 so that for all $B = (x, r)$ with $r < \frac{1}{2}$ and for all $n > n_0$,

$$s_1 \mu(B) \left(\psi(n)^\delta - a_n \right) \leq \mu(B \cap A_n) \leq s_3 \mu(B) \left(\psi(n)^\delta + a_n \right). \tag{8.3}$$

By Lemma 8.1, there exist a constant $\alpha > 0$ such that

$$\frac{1}{\alpha} \mathrm{Leb}(E) \leq \mu(E) \leq \alpha \, \mathrm{Leb}(E), \quad \text{for all Lebesgue-measurable } E \subset X.$$

Suppose $I \in \mathcal{F}_q$ and $\mathrm{Leb}(I) = \beta^{-q}$. Then by Lemma 4.3, for all $n > m > q > m_0$,

$$\begin{aligned}\mu(I \cap A_m \cap A_n) &\lesssim \sum_{\substack{J\in\mathcal{F}_m \\ J\subset I}} \beta^{-m} \left(\psi(m)^\delta \psi(n)^\delta + a_{n-m}\psi(n)^\delta + a_n\psi(m)^\delta\right) \\ &\leq \mathrm{Leb}(I) \left(\psi(m)^\delta \psi(n)^\delta + a_{n-m}\psi(n)^\delta + a_n\psi(m)^\delta\right) \\ &\leq \alpha\mu(I) \left(\psi(m)^\delta \psi(n)^\delta + a_{n-m}\psi(n)^\delta + a_n\psi(m)^\delta\right),\end{aligned}$$

so there exists a constant s_2 such that

$$\mu(I \cap A_m \cap A_n) \leq s_2\mu(I) \left(\psi(m)^\delta \psi(n)^\delta + a_{n-m}\psi(n)^\delta + a_n\psi(m)^\delta\right). \quad (8.4)$$

Note that in this system all cylinders are intervals, so by (8.3) and (8.4) together with Lemma 5.2, there exists a constant $\gamma > 0$ so that for all $q > m_0$ and for all cylinders $I \in \mathcal{F}_q$ with $\mathrm{Leb}(I) = \beta^{-q}$, we have

$$\mu\left(I \cap R_f^T(\psi)\right) \geq \gamma\mu(I).$$

By Proposition 8.4, for all cylinders $J \in \mathcal{F}_n$, there exists a cylinder $I \subset J$ with $I \in \mathcal{F}_q$, $\mathrm{Leb}(I) = \beta^{-q}$, and $\mathrm{Leb}(I) \geq \frac{1}{\beta}\mathrm{Leb}(J)$. If $n > m_0$, then

$$\mu\left(J \cap R_f^T(\psi)\right) \geq \mu\left(I \cap R_f^T(\psi)\right) \geq \gamma\mu(I) \geq \frac{\gamma}{\alpha^2\beta}\mu(J).$$

Now take $B(x,r) \subset X$ with $r < 1$. Take $n \in \mathbb{N}$ with $n > -\log_\beta \frac{r}{4}$. Then for all $J \in \mathcal{F}_n$, $\mathrm{diam}(J) \leq 2\beta^{-n} < r/2$, and

$$B\left(x, r/2\right) \subset \bigsqcup_{\substack{J\in\mathcal{F}_n \\ J\cap B(x,r/2)\neq\varnothing}} J \subset B(x,r).$$

Hence

$$\begin{aligned}\mu\left(B(x,r) \cap R_f^T(\psi)\right) &\geq \mu\left(\bigsqcup_{\substack{J\in\mathcal{F}_n \\ J\cap B(x,r/2)\neq\varnothing}} J \cap R_f^T(\psi)\right) \geq \sum_{\substack{J\in\mathcal{F}_n \\ J\cap B(x,r/2)\neq\varnothing}} \frac{\gamma}{\alpha^2\beta}\mu(J) \\ &\geq \frac{\gamma}{\alpha^2\beta}\mu\big(B(x,r/2)\big) \geq \frac{\gamma}{2\alpha^4\beta}\mu\big(B(x,r)\big).\end{aligned}$$

Thus it follows from the Lebesgue Density Theorem (Theorem 6.6) that $\mu\big(R_T^f(\psi)\big) = 1$. □

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