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On the dimension drop conjecture for diagonal flows on the space of lattices ☆

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Let $X = G/\Gamma$, where G is a Lie group and Γ is a lattice in G, let U be an open subset of X, and let $\{g_t\}$ be a one-parameter subgroup of G. Consider the set of points in X whose g_t-orbit misses U; it has measure zero if the flow is ergodic. It has been conjectured that this set has Hausdorff dimension strictly smaller than the dimension of X. This conjecture is proved when X is compact or when G is a simple Lie group of real rank 1. In this paper we prove this conjecture for the case $G = \mathrm{SL}_{m+n}(\mathbb{R})$, $\Gamma = \mathrm{SL}_{m+n}(\mathbb{Z})$ and $g_t = \mathrm{diag}(e^{nt}, \dots, e^{nt}, e^{-mt}, \dots, e^{-mt})$, in fact providing an effective estimate for the codimension. The proof uses exponential mixing of the flow together with the method of integral inequalities for height functions on $\mathrm{SL}_{m+n}(\mathbb{R})/\mathrm{SL}_{m+n}(\mathbb{Z})$. We also discuss an application to the problem of improving Dirichlet's theorem in simultaneous Diophantine approximation.

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1. Introduction

Let G be a Lie group, and let Γ be a lattice in G. Denote by X the homogeneous space G/Γ and by μ the G-invariant probability measure on X. For an unbounded subset F of G and a non-empty open subset U of X define the sets $E(F,U)$ and $\widetilde{E}(F,U)$ as follows:

$$\begin{aligned} & E(F,U) := \{x \in X : gx \notin U \ \forall\, g \in F\} \\ \subset\ & \widetilde{E}(F,U) := \{x \in X : \exists \text{ compact } Q \subset G \text{ such that } gx \notin U \ \forall\, g \in F \smallsetminus Q\} \\ & \qquad\quad = \bigcup_{\text{compact } Q \subset G} E(F \smallsetminus Q, U) \end{aligned} \tag{1.1}$$

of points in X whose F-trajectory always (resp., eventually) stays away from U. If F is a subgroup or a subsemigroup of G acting ergodically on (X,μ), then the trajectory Fx of x is dense for μ-almost all $x \in X$, in particular $\mu\big(\widetilde{E}(F,U)\big) = 0$ whenever U has non-empty interior.

The present paper studies the following natural question, asked several years ago by Mirzakhani (private communication): if $E(F,U)$ has measure zero, does it necessarily have less than full Hausdorff dimension? In fact it is reasonable to conjecture that the answer is always 'yes'; in other words, that the following 'Dimension Drop Conjecture' holds: *if $F \subset G$ is a subsemigroup and U is an open subset of X, then either $E(F,U)$ has positive measure, or its dimension is less than the dimension of X.* The same can be stated about $\widetilde{E}(F,U)$.

If X is compact, or, more generally, if the complement of U is compact, then the dimension drop conjecture follows from the uniqueness of the measure of maximal entropy, see e.g. [30, Theorem 9.7] and [28, Proposition 7.5]. In that case an explicit estimate for the codimension of $E(F,U)$ was recently obtained in [23]. When X is not compact, the situation is more complicated due to a possibility of the 'escape of mass'. The conjecture is known in the following cases:

- F consists of *quasiunipotemt* elements, that is, for each $g \in F$ all eigenvalues of $\operatorname{Ad} g$ have absolute value 1. This follows from Ratner's Measure Classification Theorem and the work of Dani and Margulis, see [36, Lemma 21.2] and [9, Proposition 2.1].
- G is a simple Lie group of real rank 1 [11].

Another example is contained in a recent paper by Guan and Shi [17]: extending a method developed earlier in [19], they proved that for an arbitrary one-parameter subgroup action on a finite-volume homogeneous space the set of points with divergent trajectories (that is, trajectories eventually leaving any compact subset of the space) has Hausdorff dimension strictly less than full. See also [2,31] for a related work.

In this paper we establish a special case of the aforementioned conjecture for a specific, and important for applications, non-compact homogeneous space of a higher rank Lie group, and for a special choice of diagonalizable elements of G. More specifically, we fix $m, n \in \mathbb{N}$, let

$$G = \mathrm{SL}_{m+n}(\mathbb{R}),\ \Gamma = \mathrm{SL}_{m+n}(\mathbb{Z}),\ X = G/\Gamma, \tag{1.2}$$

and set

$$F^+ := \{g_t : t \geq 0\},\ \text{where } g_t := \mathrm{diag}(e^{nt}, \ldots, e^{nt}, e^{-mt}, \ldots, e^{-mt}). \tag{1.3}$$

We will also choose $a > 0$ and consider a subsemigroup F_a^+ of F^+ generated by g_a, that is, let

$$F_a^+ := \big\{ \mathrm{diag}(e^{ant}, \ldots, e^{ant}, e^{-amt}, \ldots, e^{-amt}) : t \in \mathbb{Z}_+ \big\}. \tag{1.4}$$

An important role in the proof will be played by the *unstable horospherical subgroup* with respect to F^+, namely

$$H := \{h_s : s \in M_{m,n}\},\ \text{ where } h_s := \begin{bmatrix} I_m & s \\ 0 & I_n \end{bmatrix}. \tag{1.5}$$

Here and hereafter $M_{m,n}$ stands for the space of $m \times n$ matrices with real entries. It will be repeatedly used in the proof that the conjugation map $h_s \mapsto g_t h_s g_{-t}$ corresponds to a dilation of s by $e^{(m+n)t}$.

For the rest of this paper we let G, Γ, $X = G/\Gamma$, F_a^+ and H be as in (1.2)–(1.5). We are going to denote by $\|\cdot\|$ the Euclidean norm on $M_{m,n}$, and will choose a right-invariant Riemannian structure on G which agrees with the one induced by $\|\cdot\|$ on $M_{m,n} \cong \mathrm{Lie}(H)$. If P is a subgroup of G, we will denote by $B^P(r)$ the open ball of radius r centered at the identity element with respect to the metric on P coming from the Riemannian structure induced from G. Also, to simplify notation, $B(r)$ will stand for the Euclidean ball in $M_{m,n}$ centered at 0 with radius r, so that

$$B^H(r) = \{h_s : s \in M_{m,n},\ \|s\| < r\} = \{h_s : s \in B(r)\}.$$

We will denote by 'dist' the corresponding Riemannian metric on G and will use the same notation for the induced metric on X.

We need to introduce the following notation: for an open subset U of X and $r > 0$ denote by $\sigma_r U$ the *inner r-core* of U, defined as

$$\sigma_r U := \{x \in X : \mathrm{dist}(x, U^c) > r\}.$$

This is an open subset of U, whose measure is close to $\mu(U)$ for small enough values of r. The latter implies that the quantity

$$\theta_U := \sup\left\{0 < \theta \leq 1 : \mu(\sigma_{2\sqrt{mn}\theta} U) \geq \frac{1}{2}\mu(U)\right\} \tag{1.6}$$

is positive if $U \neq \varnothing$. Also, for a closed subset S of X denote by $\partial_r S$ the *r-neighborhood* of S, that is,

$$\partial_r S := \{x \in X : \operatorname{dist}(x, S) < r\}.$$

Note that we always have $\partial_r S \subset \big(\sigma_r(S^c)\big)^c$. In particular, for $z \in X$ we have $\partial_r\{z\} = B(z, r)$, the open ball in X of radius r centered at z.

We denote by $\dim E$ the Hausdorff dimension of the set E, and by $\operatorname{codim} E$ its Hausdorff codimension, i.e. the difference between the dimension of the ambient set and the Hausdorff dimension of E. The next theorem, which is the main result of the paper, establishes the Dimension Drop Conjecture for the case (1.2)–(1.4), and, moreover, does it in a quantitative way, giving an explicit estimate for the codimension of $\widetilde{E}(F_a^+, U)$ as a function of U and a. In what follows, the notation $A \gg B$, where A and B are quantities depending on certain parameters, will mean $A \geq CB$, with C being a constant dependent only on m and n.

Theorem 1.1. *There exist positive constants c, r_1 such that for any $a > 0$ and for any open subset U of X one has*

$$\operatorname{codim} \widetilde{E}(F_a^+, U) \gg \frac{\mu(U)}{\log \frac{1}{r(U,a)}}, \tag{1.7}$$

where

$$r(U, a) := \min\left(\mu(U), \theta_U, ce^{-a}, r_1\right). \tag{1.8}$$

In particular, if U is non-empty we always have $\dim \widetilde{E}(F_a^+, U) < \dim X$.

Similarly to previous papers on the subject, Theorem 1.1 is deduced by considering the intersection of $\widetilde{E}(F_a^+, U)$ with the orbits Hx of the group H.

Theorem 1.2. *There exist positive constants c, r_1 such that for any $a > 0$, any $x \in X$, and for any open subset U of X one has*

$$\operatorname{codim}\left(\{h \in H : hx \in \widetilde{E}(F_a^+, U)\}\right) \gg \frac{\mu(U)}{\log \frac{1}{r(U,a)}},$$

where $r(U, a)$ is as in (1.8).

As a special case of the two theorems above, in the next corollary the Hausdorff dimension of the set of points whose g_a-trajectory misses a small enough neighborhood of a smooth submanifold of X is estimated.

Corollary 1.3. *If $S \subset X$ is a k-dimensional embedded smooth submanifold, then there exist $\varepsilon_S, c_S, C_S > 0$ such that for any $a > 0$ and any positive $\varepsilon < \min(\varepsilon_S, c_S e^{-a})$ one has*

$$\operatorname{codim}\left(\{h \in H : hx \in \widetilde{E}(F_a^+, \partial_\varepsilon S)\}\right) \geq C_S \frac{\varepsilon^{\dim X - k}}{\log(1/\varepsilon)}. \tag{1.9}$$

In addition, if $k = 0$ and $S = \{z\}$, the constants c_S and C_S can be chosen independent of z; that is there exist $r_z, c_* > 0$ such that for any $a > 0$, any $z \in X$ and any $0 < \varepsilon < \min\left(r_z, c_* e^{-a}\right)$ one has

$$\operatorname{codim}\left(\{h \in H : hx \in \widetilde{E}\big(F_a^+, B(z,\varepsilon)\big)\}\right) \gg \frac{\mu\big(B(z,\varepsilon)\big)}{\log(1/\varepsilon)}. \tag{1.10}$$

Similar estimates hold for the codimension of $\widetilde{E}(F_a^+, \partial_r S)$ and $\widetilde{E}\big(F_a^+, B(z,r)\big)$ in X.

Remark 1.4. It is clear from (1.8) that Theorems 1.1 and 1.2, as well as Corollary 1.3, produce analogous results for the action of the one-parameter semigroup F^+: namely, by letting a tend to zero one sees that the codimensions of $\widetilde{E}(F^+, U)$ in X and $\{h \in H : hx \in \widetilde{E}(F^+, U)\}$ in H are bounded from below by $\frac{\mu(U)}{-\log \min(\mu(U), \theta_U, r_1)}$ times a constant dependent only on m, n.

Finally let us describe an application of Theorem 1.2 to simultaneous Diophantine approximation. Given $c \leq 1$, say that $s \in M_{m,n}$ is *c-Dirichlet improvable* if for all sufficiently large $N \in \mathbb{N}$

$$\begin{gathered} \text{there exists } \mathbf{p} \in \mathbb{Z}^m \text{ and } \mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\} \text{ such that} \\ \|s\mathbf{q} - \mathbf{p}\|_\infty < cN^{-n/m} \text{ and } 0 < \|\mathbf{q}\|_\infty < N. \end{gathered} \tag{1.11}$$

Here and in the proof of Theorem 1.5 $\|\cdot\|_\infty$ stands for the supremum norm on $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^{m+n}$. We let $\mathbf{DI}_{m,n}(c)$ be the set of c-Dirichlet improvable $s \in M_{m,n}$. It is easy to see that $s \in \mathbf{DI}_{m,n}(c)$ if and only if (1.11) holds for all sufficiently large $N > 0$, and that Dirichlet's theorem (see e.g. [33]) implies that $\mathbf{DI}_{m,n}(1) = M_{m,n}$. Davenport and Schmidt [10] proved that the Lebesgue measure of $\mathbf{DI}_{m,n}(c)$ is zero for any $c < 1$. On the other hand, they also showed that $\bigcup_{c<1} \mathbf{DI}_{m,n}(c)$ contains the set of badly approximable $m \times n$ matrices, which is known [32] to have full Hausdorff dimension; in other words, $\dim \mathbf{DI}(c) \to mn$ as $c \to 1$.

In recent years much attention has been directed to the set

$$\mathbf{Sing}_{m,n} := \bigcap_{c<1} \mathbf{DI}_{m,n}(c)$$

of *singular* matrices. In [19] its Hausdorff dimension was estimated from above by $mn\left(1 - \frac{1}{m+n}\right)$, and then in [7] this estimate was shown to be sharp for any m, n with

$\max(m,n) > 1$, verifying a conjecture made in [19]. The case $m = 1$ was settled previously in [5]. Moreover, it is shown there that for any integer $n \geq 2$ and any $\varepsilon > 0$ for small enough c it holds that

$$\frac{n^2}{n+1} + c^{n+\varepsilon} \leq \dim\big(\mathbf{DI}_{1,n}(c)\big) \leq \frac{n^2}{n+1} + c^{n/2-\varepsilon}$$

(see [5, Theorem 1.3 and Corollary 6.10] for a more precise estimate).

As a corollary from our main result, we deduce that for any $c < 1$ the codimension of $\mathbf{DI}_{m,n}(c)$ is positive:

Theorem 1.5. $\dim\big(\mathbf{DI}_{m,n}(c)\big) < mn$ *for any* $c < 1$.

The structure of the paper is as follows. Roughly speaking, the proof has two main ingredients. One deals with orbits staying inside a fixed compact subset of X, which are handled in §2 with the help of the exponential mixing of the g_t-action on X as in [23]. The other one (§§3–4) takes care of orbits venturing far away into the cusp of X; there we use the method of integral inequalities for height functions on X pioneered in [13] and thoroughly explored in [19]. The two ingredients are combined in §5 in the form of a covering result (Proposition 5.2). Then in §6 the results of the preceding sections are used to derive two separate dimension bounds (Theorem 6.1), which are then used in §7 to prove Theorem 1.2. After that we show how the latter implies Theorem 1.1, and use Theorems 1.1 and 1.2 to deduce Corollary 1.3 and Theorem 1.5.

We remark that the methods of this paper are applicable in much wider generality: in particular, with some modification of the argument the Dimension Drop Conjecture can be established for arbitrary Ad-diagonalizable flows on quotients of connected semisimple Lie groups without compact factors by irreducible lattices. This is going to be addressed in a forthcoming work [24]. In the last section of the paper we list some other generalizations and open questions.

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2. A covering result for orbits staying in compact subsets of X

For $N \in \mathbb{N}$, for any subset S of X, any $x \in X$ and any $t > 0$ let us define the following set:

$$A_x^N(t,r,S) = \big\{s \in B(r) : g_{it}h_s x \in S \;\; \forall\, i \in \{1,\dots,N\}\big\}. \tag{2.1}$$

For our dimension estimates it will be useful to have a bound on the number of cubes of sufficiently small side-length needed to cover the sets of the above form. In this section we will consider the case of S being compact, which was thoroughly studied in [23]. We are going to apply [23, Theorem 4.1], which was proved in the generality of $X = G/\Gamma$ being an arbitrary homogeneous space, and H being a subgroup of G with the Effective Equidistribution Property (EEP) with respect to F^+. The latter property was shown there to hold in the case (1.2)–(1.3), or, more generally, as long as H is the expanding horospherical subgroup relative to F^+, and the F^+-action on X is exponentially mixing. See also [20,22] for some earlier motivating work on the subject.

Here we need to introduce the notion of the injectivity radius of points and subsets of X. Given $x \in X$, let us denote by $r_0(x)$ the *injectivity radius* of x, defined as

$$\sup\left\{r > 0 : \text{the map } G \to X,\ g \mapsto gx \text{ is injective on } B^G(r)\right\}.$$

If $K \subset X$ is bounded, we will denote by

$$r_0(K) := \inf_{x\in K} r_0(x)$$

the *injectivity radius* of K.

The following theorem is an immediate corollary of [23, Theorem 4.1] applied to $P = H$, $L = \dim P = mn$ and $U = S^c$.

Theorem 2.1. *There exist constants*

$$0 < r_2 < \frac{1}{16\sqrt{mn}},\ b_0 \geq 2,\ b \geq 1,\ 0 < K_1 \leq 4,\ K_0 \geq 1,\ K_2,\ \lambda > 0$$

such that for any compact subset S of X, any $0 < r < \min\left(r_0(\partial_{1/2}S^c), r_2\right)$, any $x \in \partial_r S$, any $N \in \mathbb{N}$, and any $t \in \mathbb{R}$ satisfying

$$t > b_0 + b\log\frac{1}{r}, \tag{2.2}$$

the set $A_x^N\left(t, \frac{r}{16\sqrt{mn}}, S\right)$ can be covered with at most

$$K_0 e^{mn(m+n)Nt}\left(1 - K_1\mu\left(\sigma_r S^c\right) + \frac{K_2 e^{-\lambda t}}{r^{mn}}\right)^N$$

balls in $M_{m,n}$ of diameter $re^{-(m+n)Nt}$.

We are going to apply the above theorem to cover sets of type (2.1) with cubes of diameter substantially bigger than $re^{-(m+n)Nt}$. Namely we will work with cubes of side length $\theta e^{-(m+n)Nt}$, where $\theta \in \left[4r, \frac{1}{2\sqrt{mn}}\right]$.

Theorem 2.2. *Let r_2, b_0, b, K_0, K_1, K_2 and λ be as in Theorem 2.1. Then for any compact subset S of X, any $r > 0$ such that*

$$r < \min\big(r_0(\partial_1 S), r_2\big), \tag{2.3}$$

any t satisfying (2.2), any $\theta \in \left[4r, \frac{1}{2\sqrt{mn}}\right]$, any $x \in \partial_r S$ and any $N \in \mathbb{N}$, the set $A_x^N\left(t, \frac{r}{32\sqrt{mn}}, S\right)$ can be covered with at most

$$\left(\frac{4r}{\theta}\right)^{mn} K_0 e^{mn(m+n)Nt}\left(1 - K_1\mu\big(\sigma_{2\sqrt{mn}\theta}S^c\big) + \frac{K_2 e^{-\lambda t}}{r^{mn}}\right)^N$$

cubes in $M_{m,n}$ of side length $\theta e^{-(m+n)Nt}$.

Proof. Let S be a compact subset in X, let r, t and N be such that conditions (2.2) and (2.3) are satisfied, and let $\theta \in \left[4r, \frac{1}{2\sqrt{mn}}\right]$. Let $\mathcal{C}_N$ be a covering of $B\big(\frac{r}{32\sqrt{mn}}\big)$ with cubes of side-length $\theta e^{-(m+n)Nt}$ in $M_{m,n}$ whose interiors are disjoint and whose sides are parallel to the coordinate axes. Next, consider a covering $\mathcal{C}'_N$ of $\cup_{R\in\mathcal{C}_N} R$ with interior-disjoint cubes of side-length $re^{-(m+n)Nt}$ in $M_{m,n}$, also with sides parallel to the coordinate axes. Here and hereafter we will denote by Leb the Lebesgue measure on $M_{m,n}$.

Let $x \in X$. We need the following lemma.

Lemma 2.3. *For any cube R in $\mathcal{C}_N$ which has non-empty intersection with the set $A_x^N\left(t, \frac{r}{32\sqrt{mn}}, S\right)$ there exist at least $(\frac{\theta}{2r})^{mn}$ cubes in $\mathcal{C}'_N$ which lie in the interior of R. Moreover, all such cubes are subset of $A_x^N\left(t, \frac{r}{16\sqrt{mn}}, \partial_{\sqrt{mn}\theta} S\right)$.*

Proof. Observe that any cube in $\mathcal{C}'_N$ that contains a point of $\sigma_{re^{-(m+n)Nt}} R$ must lie in the interior of R. Therefore, the number of cubes in $\mathcal{C}'_N$ that lie in the interior of R is at least

$$\frac{\mathrm{Leb}\,(\sigma_{re^{-(m+n)Nt}}R)}{r^{mn}e^{-mn(m+n)Nt}} = \frac{(\theta-2r)^{mn}e^{-mn(m+n)Nt}}{r^{mn}e^{-mn(m+n)Nt}} \ge \left(\frac{\theta}{2r}\right)^{mn}.$$

Now let B be one of those cubes. The side-length of R is

$$\begin{aligned}\theta e^{-(m+n)Nt} &\underset{(2.2)}{<} \theta e^{-b_0(m+n)N}\cdot r^{b(m+n)N} \underset{(b_0\ge 2)}{\le} \frac{1}{2\sqrt{mn}}e^{-2(m+n)}\cdot r \\ &\le \frac{r}{32mn}(4\sqrt{m}e^{-2m})(4\sqrt{n}e^{-2n}) \le \frac{r}{32mn},\end{aligned}$$

hence its diameter is at most $\frac{r}{32\sqrt{mn}}$. Since R has non-empty intersection with $B\big(\frac{r}{32\sqrt{mn}}\big)$, we have $B \subset R \subset B\big(\frac{r}{16\sqrt{mn}}\big)$. Moreover, since by our assumption $R \cap A_x^N\left(t, \frac{r}{32\sqrt{mn}}, S\right) \neq \varnothing$, we can find $s \in R$ such that $g_{it}h_s x \in S$ for all $i \in \{1, \dots, N\}$.

To prove that $B \subset A_x^N\left(t, \frac{r}{16\sqrt{mn}}, \partial_{\sqrt{mn}\theta}S\right)$, we need to take any $s' \in B$ and any $i \in \{1, \ldots, N\}$ and show that

$$g_{it}h_{s'}x \in \partial_{\sqrt{mn}\theta}S. \tag{2.4}$$

Clearly

$$g_{it}h_{s'}x = (g_{it}h_{s'-s}g_{-it})g_{it}h_s x, \tag{2.5}$$

and, since both s and s' are in R, it follows that

$$\|s' - s\| \leq \sqrt{mn}e^{-(m+n)Nt}\theta,$$

hence $g_{it}h_{s'-s}g_{-it} \in B^H(\sqrt{mn}\theta) \subset B^G(\sqrt{mn}\theta)$. Thus, since $g_{it}h_s x \in S$, from (2.5) we obtain (2.4), which finishes the proof of the lemma. $\square$

Now note that every ball of diameter $re^{-(m+n)Nt}$ in $M_{m,n}$ can be covered with at most 2^{mn} cubes of side-length $re^{-(m+n)Nt}$ in $\mathcal{C}_N$. Hence, by Lemma 2.3 and by Theorem 2.1 applied to S replaced with $\partial_{\sqrt{mn}\theta}S \subset \partial_{1/2}S$, for any $x \in \partial_r S \subset \partial_r(\partial_{\sqrt{mn}\theta}S)$ the set $A_x^N\left(t, \frac{r}{32\sqrt{mn}}, S\right)$ can be covered with at most

$$\begin{aligned}
&\left(\frac{2r}{\theta}\right)^{mn} 2^{mn} \cdot K_0 e^{mn(m+n)Nt}\left(1 - K_1\mu\big(\sigma_r(\sigma_{\sqrt{mn}\theta}S^c)\big) + \frac{K_2 e^{-\lambda t}}{r^{mn}}\right)^N \\
&\leq \left(\frac{4r}{\theta}\right)^{mn} K_0 e^{mn(m+n)Nt}\left(1 - K_1\mu\big(\sigma_{2\sqrt{mn}\theta}S^c\big) + \frac{K_2 e^{-\lambda t}}{r^{mn}}\right)^N
\end{aligned}$$

cubes in $M_{m,n}$ of side-length $\theta e^{-(m+n)Nt}$. This finishes the proof. $\square$

3. Height functions and non-escape of mass

In the next two sections we describe trajectories which venture outside of large compact subsets of X. The method we are using, based on integral inequalities for height functions, also known as *Margulis functions* (see [14] for a detailed survey), was introduced in a breakthrough paper of Eskin, Margulis and Mozes [13], and later adapted in [19]. Our argument basically follows the scheme developed in the latter paper, with minor modifications. See also [17,31,35] for other recent related results obtained with the technique of Margulis functions.

Let $x \in X$ be a lattice in $\mathbb{R}^{m+n}$. Following [13], say that a subspace L of $\mathbb{R}^{m+n}$ is x-rational if $L \cap x$ is a lattice in L, and for any x-rational subspace L, denote by $d_x(L)$ the volume of $L/(L\cap x)$. Equivalently, let us denote by $\|\cdot\|$ the extension of the Euclidean norm on $\mathbb{R}^{m+n}$ to $\bigwedge(\mathbb{R}^{m+n})$; then

$$d_x(L) = \|v_1 \wedge \cdots \wedge v_i\|, \text{ where } \{v_1, \ldots, v_i\} \text{ is a } \mathbb{Z}\text{-basis for } L \cap x. \tag{3.1}$$

For any $i = 1, \ldots, m + n$ and any $x \in X$ we let $F_i(x)$ denote the set of i-dimensional x-rational subspaces of $\mathbb{R}^{m+n}$.

Now for $1 \le i \le m + n$ define

$$\alpha_i(x) := \sup \left\{ \frac{1}{d_x(L)} : L \in F_i(x) \right\}.$$

Clearly $\alpha_{m+n}(x) \equiv 1$, and for convenience we also set $\alpha_0(x) \equiv 1$ for all $x \in X$. Also, note that $\alpha_1(x)$ is precisely the reciprocal of the norm of the shortest vector in x. Functions $\alpha_1, \ldots, \alpha_{m+n-1}$ can be thought of as height functions on X in the following sense:

Lemma 3.1. *A sequence of points x_j diverges in X (leaves every compact subset) if and only if $\lim_{j\to\infty} \alpha_i(x_j) = \infty$ for some (equivalently, for all) $i = 1, \ldots, m + n - 1$.*

Proof. We refer the reader to [4] for basic facts in geometry of numbers. By Mahler's Compactness Criterion, a sequence of lattices x_j diverges in X if and only if $\lim_{j\to\infty} \alpha_1(x_j) = \infty$. Thus to prove the lemma it suffices to show that, for any sequence $(x_j) \subset X$, $\lim_{j\to\infty} \alpha_1(x_j) = \infty$ if and only if $\lim_{j\to\infty} \alpha_i(x_j) = \infty$ for all $i = 1, \ldots, m + n - 1$.

We first prove the reverse implication. Assume that for some $x \in X$, $\varepsilon > 0$, and $1 \le i \le m+n-1$, we have $\alpha_i(x) > \frac{1}{\varepsilon}$. Then, by definition there must exist a i-dimensional x-rational subspace L such that $d_x(L) < \varepsilon$. It is easy to see that, by applying Minkowski's Convex Body Lemma to $L \cap x$, one can find a vector of length $\ll \varepsilon^{1/i}$ in x, which implies that $\alpha_1(x) \gg \frac{1}{\varepsilon^i}$.

To prove the forward implication, assume that $x \in X$, and denote the shortest vector in x by v_1. Note that by definition $\alpha_1(x) = \frac{1}{\|v_1\|}$. Extend $\{v_1\}$ to get a reduced basis $\{v_1, \ldots, v_{m+n}\}$ of x. By Minkowski's Second Theorem, the product $\prod_{k=1}^{m+n} \|v_k\|$ is bounded from both sides by uniform constants. Also, by definition of reduced lattice, we have $\|v_{i+1}\| \ge \|v_i\|$ for $1 \le i < m + n$. Hence, whenever $\|v_1\|$ is sufficiently small, for any $1 \le i < m + n$ the product $\prod_{k=1}^{i} \|v_k\|$ can be made arbitrarily small. Moreover, by Hadamard's inequality for any $i \in \mathbb{N}$ we have $\|v_1 \wedge \cdots \wedge v_i\| \le \prod_{k=1}^{i} \|v_k\|$. Hence, we conclude that whenever $\|v_1\|$ is sufficiently small (equivalently, $\alpha_1(x)$ is sufficiently large), for any $1 \le i < m+n$, $\|v_1 \wedge \cdots \wedge v_i\|$ can be made arbitrarily small, which implies that $\alpha_i(x)$ can be made arbitrarily large. This finishes the proof. □

As in [19], we will approximate the Lebesgue measure on a neighborhood of identity in H by the Gaussian distribution on $M_{m,n}$. Namely, we will let ρ_{σ^2} denote the Gaussian probability measure on $M_{m,n}$ where each component is i.i.d. with mean 0 and variance σ^2.

In the following theorem, which is a simplified version of [19, Corollary 3.6], we push forward the probability measure ρ_1 from $M_{m,n}$ to the orbit Hx, where $x \in X$, and then translate it by g_t. Let us use the following notation: for $x \in X$, $t > 0$ and a measurable function f on X define

$$I_{x,t}(f) := \int_{M_{m,n}} f(g_t h_s x)\, d\rho_1(s).$$

Theorem 3.2. *There exists $c_0 \geq 1$ depending only on m, n with the following property: for any $t \geq 1$, any $x \in X$, and for any $i \in \{1, \dots, m+n-1\}$ one has*

$$I_{x,t}\left(\alpha_i^{1/2}\right) \leq c_0 \left(e^{-t/2}\alpha_i(x)^{1/2} + e^{mnt} \max_{0<j\leq \min(m+n-i,i)} \sqrt{\alpha_{i+j}(x)^{1/2}\alpha_{i-j}(x)^{1/2}}\right). \tag{3.2}$$

To make the paper self-contained, we include all the details of the proof. The first step, an analogue of [19, Proposition 3.1], is to obtain an estimate similar to (3.2), but replace the height functions α_i with $\frac{1}{d_x(L)}$, where $L \in F_i(x)$ is fixed, and instead of the Gaussian measure ρ_1 use the probability measure dk on the maximal compact subgroup $K = \mathrm{SO}(m+n)$ of G. Note that in the argument below all the implicit constants depend only on m, n.

Proposition 3.3. *For any $t \geq 1$, any $i \in \{1, \dots, m+n-1\}$, and any decomposable $v = v_1 \wedge \cdots \wedge v_i \in \bigwedge^i(\mathbb{R}^{m+n})$ we have:*

$$\int_K \|g_t k v\|^{-1/2}\, dk \ll e^{-t/2}\|v\|^{-1/2}.$$

Proof. Notice that K acts transitively on the set of decomposable $v \in \bigwedge^i(\mathbb{R}^{m+n})$ with a fixed norm. Therefore $\int_K \|g_t k v\|^{-1/2}\, dk$ is a function of $\|v\|$, and from its homogeneity it follows that

$$\int_K \|g_t k v\|^{-1/2}\, dk = C(t)\|v\|^{-1/2}$$

for some function $C : \mathbb{R}_+ \to \mathbb{R}_+$. Now choose $x_1, \dots, x_i$ to be independent standard Gaussian $\mathbb{R}^{m+n}$-valued random variables. Then we have

$$\mathbb{E}\left(\int_K \|g_t k(x_1 \wedge \cdots \wedge x_i)\|^{-1/2}\, dk\right) = C(t)\mathbb{E}(\|x_1 \wedge \cdots \wedge x_i\|^{-1/2}),$$

where the right hand side is finite in view of [19, Lemma 3.2]. On the other hand, using the K-invariance of $x_1, \dots, x_i$ we get

$$\mathbb{E}\left(\int_K \|g_t k(x_1 \wedge \cdots \wedge x_i)\|^{-1/2}\, dk\right) = \mathbb{E}\|g_t(x_1 \wedge \cdots \wedge x_i)\|^{-1/2}.$$

Thus to prove the proposition, it suffices to show that

$$\mathbb{E}\big(\|g_t(x_1 \wedge \cdots \wedge x_i)\|^{-1/2}\big) \ll e^{-t/2}.$$

Let $V^+ \subset \mathbb{R}^{m+n}$ denote the m-dimensional subspace spanned by $e_1, \dots, e_m$ and let V^- be the complementary subspace, so that

$$\|g_t v\| = e^{nt}\|v\|,\ \ \|g_t w\| = e^{-mt}\|w\|$$

for $v \in V^+$ and $w \in V^-$. In particular, for any $v \in \bigwedge^i(V^+)$ we have $\|g_t v\| = e^{int}\|v\|$. Let $\pi_u^{(i)} : \bigwedge^i(\mathbb{R}^{m+n}) \to \bigwedge^i(V^+)$ be the natural (orthogonal) projection. Clearly, we have:

$$\pi_u^{(i)}(x_1 \wedge \cdots \wedge x_i) = \pi_u^{(1)}(x_1) \wedge \cdots \wedge \pi_u^{(1)}(x_i),$$

where each of $\pi_u^{(1)}(x_j)$ is a standard Gaussian random variable in m dimensions.

We first assume that $i \le m$. Then we have:

$$\|g_t(x_1 \wedge \cdots \wedge x_i)\| \ge \|\pi_u^{(i)} g_t(x_1 \wedge \cdots \wedge x_i)\| = \|g_t \pi_u^{(i)}(x_1 \wedge \cdots \wedge x_i)\| = e^{int}\|\pi_u^{(i)}(x_1 \wedge \cdots \wedge x_i)\|,$$

hence

$$\mathbb{E}\big(\|g_t(x_1 \wedge \cdots \wedge x_i)\|^{-1/2}\big) \le e^{\frac{-int}{2}}\,\mathbb{E}\big(\|\pi_u^{(i)}(x_1 \wedge \cdots \wedge x_i)\|^{-1/2}\big) \ll e^{-\frac{t}{2}},$$

where in the last inequality we are again using [19, Lemma 3.2], i.e. the finiteness of $\mathbb{E}(\|x_1 \wedge \cdots \wedge x_i\|^{-1/2})$. This finishes the proof for $i \le m$. The case $m < i \le n$ can be handled by duality, following the lines of the proof of [19, Proposition 3.1]. □

Let us introduce the following notation: if $h \in G$, we will denote by $\|h\|_\infty$ the norm of h viewed as an operator on $\bigwedge(\mathbb{R}^{m+n})$. We note that $\|h\|_\infty = \|h^{-1}\|_\infty$ for any $h \in H$, since $h = h_s$ and $h^{-1} = h_{-s}$ are conjugate by $\begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}$. That is,

$$\|h_s\|_\infty^{-1}\|v\| \le \|h_s v\| \le \|h_s\|_\infty \|v\| \quad \text{for any } s \in M_{m,n} \text{ and } v \in \textstyle\bigwedge(\mathbb{R}^{m+n}). \tag{3.3}$$

Note that $\|h_s\|_\infty$ grows polynomially in s: more precisely,

$$\|h_s\|_\infty \ll \|s\|^{\min(m,n)}. \tag{3.4}$$

We will also use a norm estimate similar to (3.3) but for the g_t-action:

$$e^{-mnt}\|v\| \le \|g_t v\| \le e^{mnt}\|v\| \quad \text{for any } t \ge 1 \text{ and } v \in \textstyle\bigwedge(\mathbb{R}^{m+n}). \tag{3.5}$$

The next lemma, which is a special case '$\beta = 1/2$' of [19, Lemma 3.5], shows that Proposition 3.3 will remain valid if integration over K is replaced with integration over a bounded subset of $M_{m,n}$.

Lemma 3.4. *There exists a neighborhood W of 0 in $M_{m,n}$ such that for any $s_0 \in M_{m,n}$, $t \geq 1$, $i \in \{1, \ldots, m+n-1\}$, and decomposable $v \in \bigwedge^i(\mathbb{R}^{m+n})$ we have*

$$\int\limits_{s_0+W} \|g_t h_s v\|^{-1/2}\, ds \ll \|h_{s_0}\|_\infty^{1/2} \int\limits_K \|g_t k v\|^{-1/2}\, dk.$$

Proof of Theorem 3.2. Fix $x \in X$ and $i \in \{1, \ldots, m+n-1\}$. Let $L_0 \in F_i(x)$ be such that

$$\alpha_i(x) = \frac{1}{d_x(L_0)}. \tag{3.6}$$

Note that in view of (3.3) and (3.5) we have

$$\begin{aligned} \alpha_i(g_t h x) &\leq \frac{1}{d_x(g_t h L_0)} \leq e^{mnt} \frac{1}{d_x(h L_0)} \\ &\leq e^{mnt} \|h\|_\infty \frac{1}{d_x(L_0)} \leq e^{mnt} \|h\|_\infty \alpha_i(x). \end{aligned} \tag{3.7}$$

We shall consider two cases.

Case 1. The subspace L_0 is an outlier, that is, $d_x(L_0)$ is much smaller than $d_x(L)$ for any $L \in F_i(x)$ different from L_0. Namely,

$$d_x(L) \geq e^{2mnt} d_x(L_0) \quad \forall\, L \in F_i(x) \smallsetminus \{L_0\}.$$

Then for any $L \in F_i(x) \smallsetminus \{L_0\}$ and $h \in H$ in view of (3.3) and (3.5) we have

$$d_x(hL_0) \leq \|h\|_\infty d_x(L_0) \leq e^{-2mnt} \|h\|_\infty d_x(L) \leq e^{-2mnt} \|h\|_\infty^2 d_x(hL),$$

hence

$$d_x(g_t h L_0) \leq e^{mnt} d_x(hL_0) \leq e^{-mnt} \|h\|_\infty^2 d_x(hL) \leq \|h\|_\infty^2 d_x(g_t h L).$$

Therefore $\alpha_i(g_t h x) \leq \dfrac{\|h\|_\infty^2}{d_x(g_t h L_0)}$ and

$$I_{x,t}\left(\alpha_i^{1/2}\right) \leq \int\limits_{M_{m,n}} \|h_s\|_\infty d_x(g_t h_s L_0)^{-1/2}\, d\rho_1(s). \tag{3.8}$$

Take $W \subset M_{m,n}$ as in Lemma 3.4. Clearly, for any $s' \in M_{m,n}$

$$\begin{aligned} &\int_{s'+W} \|h_s\|_\infty d_x(g_t h_s L_0)^{-1/2}\, d\rho_1(s) \\ &\ll \left(\max_{s \in s'+W} \|h_s\|_\infty e^{-\frac{\|s\|^2}{2}} \right) \int_{h_0 W} d_x(g_t h_s L_0)^{-1/2}\, ds \\ &\underset{(3.4)}{\le} e^{-\frac{\|s'\|^2}{2} + O(\|s'\|)} \int_{h_0 W} d_x(g_t h_s L)^{-1/2}\, ds, \end{aligned} \tag{3.9}$$

where the implied constant is independent of s'. Summing over a lattice Λ in $M_{m,n}$ sufficiently fine so that $M_{m,n} = W + \Lambda$, we conclude that

$$\begin{aligned} \int_{M_{m,n}} \|h_s\|_\infty d_x(g_t h_s L_0)^{-1/2}\, d\rho_1(s) &\le \sum_{s' \in \Lambda} \int_{s'+W} \|h_s\|_\infty d_x(g_t h_s L_0)^{-1/2}\, d\rho_1(s) \\ &\underset{(3.9)}{\ll} \sum_{s' \in \Lambda} e^{-\frac{\|s'\|^2}{2} + O(\|s'\|)} \int_{s'+W} d_x(g_t h_s L_0)^{-1/2}\, d\rho_1(s) \\ \text{(by Lemma 3.4)} &\ll \sum_{s' \in \Lambda} \|h_{s'}\|_\infty^{1/2} e^{-\frac{\|s'\|^2}{2} + O(\|s'\|)} \int_K d_x(g_t k L_0)^{-1/2}\, dk \\ &\ll \int_K d_x(g_t k L_0)^{-1/2}\, dk. \end{aligned}$$

Thus, (3.8) and Proposition 3.3 give

$$I_{x,t}\left(\alpha_i^{1/2}\right) \ll e^{-t/2} d_x(L_0)^{-1/2} \underset{(3.6)}{=} e^{-t/2} \alpha_i(x)^{1/2}.$$

Case 2. There exists $L \in F_i(x)$ different from L_0 such that

$$d_x(L) < e^{2mnt} d_x(L_0). \tag{3.10}$$

Let j be the dimension of $L/(L \cap L_0) \cong (L + L_0)/L_0$; then the dimension of $L + L_0$ is equal to $i + j$. Note that we have

$$d_x(L) d_x(L_0) \ge d_x(L \cap L_0) d_x(L + L_0), \tag{3.11}$$

see [13, Lemma 5.6]. Then for any $h \in H$ we can write

$$\alpha_i(g_t h x) \underset{(3.7)}{\le} e^{mnt} \|h\|_\infty \alpha_i(x) \underset{(3.6)}{=} \frac{e^{mnt} \|h\|_\infty}{d_x(L_0)} \underset{(3.10)}{<} \frac{e^{2mnt} \|h\|_\infty}{\sqrt{d_x(L) d_x(L_0)}}$$

$$\underset{(3.11)}{\leq} \frac{e^{2mnt}\|h\|_\infty}{\sqrt{d_x(L\cap L_0)d_x(L+L_0)}} \leq e^{2mnt}\|h\|_\infty\sqrt{\alpha_{i+j}(x)\alpha_{i-j}(x)}.$$

Hence

$$I_{x,t}\left(\alpha_i^{1/2}\right) \leq e^{mnt} \max_{0<j\leq \max(m+n-i,i)} \left(\alpha_{i+j}(x)\alpha_{i-j}(x)\right)^{1/4} \int\limits_{M_{m,n}} \|h_s\|_\infty^{1/2}\, d\rho_1(s).$$

It follows from (3.4) that

$$\int\limits_{M_{m,n}} \|h_s\|_\infty^{1/2}\, d\rho_1(s) \ll 1,$$

hence combining the above two cases establishes (3.2) with some uniform c_0. □

An immediate application of Theorem 3.2 is obtained via the 'convexity trick' introduced in [13] and formalized in [19]: from (3.2) and [19, Proposition 4.1] with $\beta_i = 1/2$ for each i it follows that for any $t \geq 1$ there exist positive constants $\omega_0 = \omega_0(t), \dots, \omega_{m+n} = \omega_{m+n}(t)$ and C_0 such that the linear combination

$$\tilde{\alpha} := \sum_{i=0}^{m+n} \omega_i {\alpha_i}^{1/2} \tag{3.12}$$

satisfies

$$I_{x,t}(\tilde{\alpha}) \leq 2c_0 e^{-t/2}\tilde{\alpha}(x) + C_0$$

for all $x \in X$. However, for our purposes it will be necessary to get precise expressions for the constants $\omega_0, \dots, \omega_{m+n}$ and C_0. This forces us to go through the argument from [13] and [19] adapted for this special case. Namely, take

$$\varepsilon = \varepsilon(t) = \frac{e^{-(mn+1/2)t}}{m+n-1}, \tag{3.13}$$

for $i \in \{0, \dots, m+n\}$ define $p(i) := i(m+n-i)$, and let

$$\omega_i(t) := \varepsilon^{p(i)} = \frac{e^{-(mn+\frac{1}{2})i(m+n-i)t}}{(m+n-1)^{i(m+n-i)}}.$$

This gives rise to the height function of the form (3.12) which we are going to use in the later sections. Since it depends on the (fixed) parameter t, with some abuse of notation we will denote it by

$$\tilde{\alpha}^t := \sum_{i=0}^{m+n} \omega_i(t){\alpha_i}^{1/2} = \sum_{i=0}^{m+n} \frac{e^{-(mn+1/2)i(m+n-i)t}}{(m+n-1)^{i(m+n-i)}}{\alpha_i}^{1/2}. \tag{3.14}$$

A key role in our proof will be played by subsets X consisting of points x with large (resp., not so large) values of $\tilde{\alpha}^t(x)$. Namely, for $M > 0$ let us define

$$X^t_{>M} := \{x \in X : \tilde{\alpha}^t(x) > M\} \text{ and } X^t_{\leq M} := \{x \in X : \tilde{\alpha}^t(x) \leq M\}. \tag{3.15}$$

Since $\tilde{\alpha}^t$ is proper, the sets $X^t_{\leq M}$ are compact, and $X^t_{>M}$ are 'cusp neighborhoods' with compact complements.

Observe that for any i, j such that $0 < j \leq \min\{i, m+n-i\}$ we have

$$2p(i)-p(i+j)-p(i-j) = 2i(m+n-i)-(i+j)(m+n-i-j)-(i-j)(m+n-i+j) = 2j^2.$$

Then for each $i \in \{1, \dots, m+n-1\}$ the inequality (3.2) implies

$$\begin{aligned} I_{x,t}\left(\omega_i \alpha_i^{1/2}\right) &\leq c_0 \varepsilon^{p(i)} \left(e^{-t/2}\alpha_i(x)^{1/2} + e^{mnt} \max_{0<j\leq \min(m+n-i,i)} \sqrt{\alpha_{i+j}(x)^{1/2}\alpha_{i-j}(x)^{1/2}}\right) \\ &= c_0\varepsilon^{p(i)} e^{-t/2}\alpha_i(x)^{1/2} + c_0 \varepsilon^{j^2} e^{mnt} \max_{0<j\leq \min(m+n-i,i)} \sqrt{\varepsilon^{p(i+j)}\alpha_{i+j}(x)^{1/2}\varepsilon^{p(i-j)}\alpha_{i-j}(x)^{1/2}} \\ &\leq c_0\omega_i e^{-t/2}\alpha_i(x)^{1/2} + c_0\varepsilon e^{mnt} \max_{0<j\leq \min(m+n-i,i)} \sqrt{\omega_{i+j}\alpha_{i+j}(x)^{1/2}\omega_{i-j}\alpha_{i-j}(x)^{1/2}}. \end{aligned}$$

Since both $\omega_{i+j}\alpha_{i+j}(x)^{1/2}$ and $\omega_{i-j}\alpha_{i-j}(x)^{1/2}$ are not greater than $\tilde{\alpha}^t(x)$, we obtain

$$\begin{aligned} I_{x,t}(\tilde{\alpha}^t) =\ & I_{x,t}\left(2 + \sum_{i=1}^{m+n-1} \omega_i {\alpha_i}^{1/2}\right) \leq 2 + \sum_{i=1}^{m+n-1} I_{x,t}\left(\omega_i {\alpha_i}^{1/2}\right) \\ =\ & 2 + c_0 e^{-t/2}\tilde{\alpha}^t(x) + (m+n-1)c_0\varepsilon(t)e^{mnt}\tilde{\alpha}^t(x). \end{aligned} \tag{3.16}$$

Thereby we have arrived at

Proposition 3.5. *Let $\tilde{\alpha}^t$ be defined by (3.14), and let c_0 be as in Theorem 3.2. Then:*

(a) *For any $t \geq 1$ any $x \in X$ one has*

$$I_{x,t}(\tilde{\alpha}^t) \leq 2 + 2c_0 e^{-t/2}\tilde{\alpha}^t(x). \tag{3.17}$$

(b) *For any $t \geq 1$ and any $x \in X^t_{>e^{t/2}/c_0}$ we have:*

$$I_{x,t}(\tilde{\alpha}^t) \leq 4c_0 e^{-t/2}\tilde{\alpha}^t(x). \tag{3.18}$$

Proof. (3.17) is obtained from (3.16) via the substitution (3.13). Part (b) is immediate from (a) since $\tilde{\alpha}^t(x) \geq \frac{e^{t/2}}{c_0}$ is equivalent to $2 \leq 2c_0 e^{-t/2}\tilde{\alpha}^t(x)$. $\square$

Remark 3.6. Note that it follows from (3.1) and the definition of functions α_i that for any $i = 0, \dots, m+n$, $h \in G$ and $x \in X$ one has

$$\frac{1}{\|h\|_\infty}\alpha_i(x) \le \alpha_i(hx) \le \|h^{-1}\|_\infty \alpha_i(x).$$

Since $\tilde{\alpha}^t$ is a linear combination of functions $\alpha_i^{1/2}$, it satisfies similar inequalities. Specifically, in what follows we are going to take h from the ball $B(2)$ of radius 2 in G. Let us define

$$C_\alpha := \sup_{h \in B(2)} \max\left(\|h\|_\infty, \|h^{-1}\|_\infty\right)^{1/2};$$

then it is clear that for any $h \in B(2)$ and any $x \in X$ we have:

$$C_\alpha^{-1}\tilde{\alpha}^t(x) \le \tilde{\alpha}^t(hx) \le C_\alpha \tilde{\alpha}^t(x). \tag{3.19}$$

4. Covering results for the orbits visiting non-compact part of X

In the following proposition, which is the main result of this section, we will fix $x \in X$, $k, N \in \mathbb{N}$ and $t, M > 0$, and will work with the set

$$\begin{aligned} A_x^N\left(kt, 1, g_t X^t_{>C_\alpha M}\right) &= \left\{s \in B(1) : g_{ikt} h_s x \in g_t X^t_{>C_\alpha M} \quad \forall\, i \in \{1, \dots, N\}\right\} \\ &= \left\{s \in B(1) : \tilde{\alpha}^t(g_{(ik-1)t} h_s x) > C_\alpha M \quad \forall\, i \in \{1, \dots, N\}\right\}, \end{aligned} \tag{4.1}$$

where C_α is as in Remark 3.6.

Proposition 4.1. *There exists $C_1 \ge 1$ such that for any $2 \le k \in \mathbb{N}$, any $t \ge 2$, any $N \in \mathbb{N}$, any $x \in X$, and for any $M \ge C_\alpha e^{\frac{mnt}{2}}$ we have*

$$\int\limits_{A_x^N\left(kt,1,g_t X^t_{>C_\alpha M}\right)} \tilde{\alpha}^t(g_{Nkt} h_s x)\, ds \le \left((k-1)C_1^k e^{-\frac{t}{2}}\right)^N \max\left(\tilde{\alpha}^t(x), 1\right).$$

Proof. Let us fix x, k, t, N and M as in the statement of the proposition; the sets defined in the course of the proof will depend on these parameters. Define

$$Z_M := \left\{(s_1, \dots, s_k) \in B(1)^k : \tilde{\alpha}^t(g_t h_{s_{k-1}} \cdots g_t h_{s_1} x) > M\right\}.$$

Then we can write

$$\begin{aligned}
&\int \cdots \int_{Z_{C_\alpha^{-1}M}} \tilde{\alpha}^t(g_t h_{s_k} \cdots g_t h_{s_1} x)\, d\rho_1(s_k) \cdots d\rho_1(s_1) \\
&= \int \cdots \int_{(M_{m,n})^{k-1}} 1_{X^t_{>C_\alpha^{-1}M}}(g_t h_{s_{k-1}} \cdots g_t h_{s_1} x) \cdot I_{g_t h_{s_{k-1}} \cdots g_t h_{s_1} x, t}(\tilde{\alpha}^t)\, d\rho_1(s_{k-1}) \cdots d\rho_1(s_1) \\
&\underset{(3.18)}{\leq} 4c_0 e^{-\frac{t}{2}} \int \cdots \int_{(M_{m,n})^{k-1}} \tilde{\alpha}^t(g_t h_{s_{k-1}} \cdots g_t h_{s_1} x)\, d\rho_1(s_{k-1}) \cdots d\rho_1(s_1),
\end{aligned} \tag{4.2}$$

where c_0 is as in Theorem 3.2. Note that the use of Proposition 3.5 in the last step is justified since $C_\alpha^{-1}M \geq e^{\frac{mnt}{2}} \geq e^{t/2}/c_0$. Next, by using (3.17) $(k-1)$ times we get:

$$\begin{aligned}
&\int \cdots \int_{(M_{m,n})^{k-1}} \tilde{\alpha}^t(g_t h_{s_{k-1}} \cdots g_t h_{s_1} x)\, d\rho_1(s_{k-1}) \cdots d\rho_1(s_1) \\
&\leq (2c_0 e^{-\frac{t}{2}})^{k-2} \tilde{\alpha}^t(x) + 2\big((2c_0 e^{-\frac{t}{2}})^{k-2} + \cdots + 1\big) \\
&\leq (2c_0)^{k-2} \tilde{\alpha}^t(x) + 2(k-2)(2c_0)^{k-2} \leq 4(k-1)(2c_0)^{k-2} \max\big(\tilde{\alpha}^t(x), 1\big).
\end{aligned} \tag{4.3}$$

So by combining (4.2) and (4.3) we have:

$$\int \cdots \int_{Z_{C_\alpha^{-1}M}} \tilde{\alpha}^t(g_t h_{s_k} \cdots g_t h_{s_1} x)\, d\rho_1(s_k) \cdots d\rho_1(s_1) \leq 8(k-1)(2c_0)^{k-1} e^{-\frac{t}{2}} \max\big(\tilde{\alpha}^t(x), 1\big).$$

Now define the function $\phi : B(1)^k \to M_{m,n}$ by

$$\phi(s_1, \ldots, s_k) := \sum_{j=1}^{k} e^{-(m+n)(j-1)t} s_j. \tag{4.4}$$

Note that

$$g_t h_{s_k} \cdots g_t h_{s_1} = g_{kt} h_{\phi(s_1, \ldots, s_k)}. \tag{4.5}$$

We will need the following observation:

Lemma 4.2. *For any $M > 0$, $\phi^{-1}\big(\phi(Z_M)\big) \subset Z_{C_\alpha^{-1}M}$.*

Proof. Let $(s_1, \ldots, s_k) \in B(1)^k$ be such that $\phi(s_1, \ldots, s_k) \in \phi(Z_M)$. Then there exists $(s'_1, \ldots, s'_k) \in Z_M$ such that $\phi(s_1, \ldots, s_k) = \phi(s'_1, \ldots, s'_k)$. Hence, using (4.5) we get

$$g_t h_{s_k} \cdots g_t h_{s_1} = g_t h_{s'_k} \cdots g_t h_{s'_1},$$

which implies

$$g_t h_{s_{k-1}} \cdots g_t h_{s_1} = h_{s'_k - s_k} g_t h_{s'_{k-1}} \cdots g_t h_{s'_1}.$$

Note that $h_{s'_k - s_k} \in B^H(2)$. Therefore, by (3.19) we have

$$\tilde{\alpha}^t(g_t h_{s_{k-1}} \cdots g_t h_{s_1} x) \geq C_\alpha^{-1} \tilde{\alpha}^t(g_t h_{s'_{k-1}} \cdots g_t h_{s'_1} x) > C_\alpha^{-1} M.$$

Hence, $(s_1, \dots, s_k) \in Z_{C_\alpha^{-1} M}$, which finishes the proof of the lemma. $\square$

Using the above lemma we obtain

$$\begin{aligned}
&\int \cdots \int_{B(1)^k} 1_{\phi(Z_M)}\big(\phi(s_1, \dots, s_k)\big) \tilde{\alpha}^t(g_{kt} h_{\phi(s_1,\dots,s_k)x})\, d\rho_1(s_k) \cdots d\rho_1(s_1) \\
&\leq \int \cdots \int_{Z_{C_\alpha^{-1} M}} \tilde{\alpha}^t(g_t h_{s_k} \cdots g_t h_{s_1} x)\, d\rho_1(s_k) \cdots d\rho_1(s_1) \qquad (4.6) \\
&\leq 8(k-1)(2c_0)^{k-1} e^{-\frac{t}{2}} \max(\tilde{\alpha}^t(x), 1).
\end{aligned}$$

To convert the above multiple integral to a single integral, we will use the following

Lemma 4.3. *There exists $0 < \Xi < 1$ such that for any positive measurable function f on $M_{m,n}$ and any*

$$0 < \varepsilon \leq \frac{1}{8},\ 0 \leq \delta < 1 \tag{4.7}$$

we have

$$\iint_{B(1)^2} f(\varepsilon x + y)\, d\rho_{1+\delta^2}(x) d\rho_1(y) \geq \Xi \cdot \int_{B(1)} f(z)\, d\rho_{1+\varepsilon^2(1+\delta^2)}(z).$$

Proof. Let ε and δ be as in (4.7). For convenience denote $\sigma := \sqrt{1+\delta^2}$. Consider the change of variables

$$(z, v) := \left(\varepsilon x + y, \frac{x}{\sigma} - \varepsilon\sigma y\right),$$

or, equivalently

$$x = \frac{\sigma(v + \varepsilon\sigma z)}{1 + \varepsilon^2\sigma^2}, \quad y = \frac{z - \varepsilon\sigma v}{1 + \varepsilon^2\sigma^2}. \tag{4.8}$$

It is easy to verify that

$$\left|\frac{\partial(z, v)}{\partial(x, y)}\right| = \left(\frac{1 + \varepsilon^2\sigma^2}{\sigma}\right)^{mn} \tag{4.9}$$

and

$$\frac{\|x\|^2}{\sigma^2} + \|y\|^2 = \frac{\|z\|^2 + \|v\|^2}{1+\varepsilon^2\sigma^2}. \tag{4.10}$$

Denote

$$\mathcal{D} := \left\{(z,v) \in (M_{m,n})^2 : \ \|z\| \le 1, \ \|v\| \le 1/4, \ z_{ij}v_{ij} \ge 0 \ \forall\, i \in \{1,\dots,m\}, \ j \in \{1,\dots,n\}\right\}.$$

It readily follows from (4.8) that

$$(z,v) \in \mathcal{D} \quad \Longrightarrow \quad \|x\| \le 1 \text{ and } \|y\| \le 1. \tag{4.11}$$

Therefore for any f one has

$$\begin{aligned}
\iint\limits_{B(1)^2} f(\varepsilon x + y)\, d\rho_{1+\delta^2}(x) d\rho_1(y) &= \frac{1}{(2\pi\sigma)^{mn}} \iint\limits_{B(1)^2} f(\varepsilon x + y) e^{-\left(\frac{\|x\|^2}{2\sigma^2} + \frac{\|y\|^2}{2}\right)}\, dx\, dy \\
&\underset{(4.9),\,(4.10),\,(4.11)}{\ge} \frac{1}{(2\pi(1+\varepsilon^2\sigma^2))^{mn}} \iint\limits_{\mathcal{D}} f(z) e^{-\frac{\|z\|^2 + \|v\|^2}{2(1+\varepsilon^2\sigma^2)}}\, dz\, dv \\
&\ge \rho_{1+\varepsilon^2\sigma^2}\left(\left[0, \frac{1}{4\sqrt{mn}}\right]^{mn}\right) \cdot \int\limits_{B(1)} f(z)\, d\rho_{1+\varepsilon^2\sigma^2}(z) \\
&\underset{(4.7)}{\ge} \rho_{33/32}\left(\left[0, \frac{1}{4\sqrt{mn}}\right]^{mn}\right) \cdot \int\limits_{B(1)} f(z)\, d\rho_{1+\varepsilon^2\sigma^2}(z). \qquad \square
\end{aligned}$$

Define $\sigma_i(t) := \sqrt{\sum_{j=1}^{i-1} e^{-2(m+n)jt}}$ for any $i \in \mathbb{N}$. Since $e^{-(m+n)t} \le \frac{1}{8}$ because of the assumption $t \ge 2$, for any $i \in \mathbb{N}$ we have $\sigma_i(t) < 1$. Hence, by using Lemma 4.3 $(k-1)$ times with $\varepsilon = e^{-(m+n)t}$ and $\delta = \sigma_1(t), \dots, \sigma_{k-1}(t)$ respectively we get

$$\begin{aligned}
&\Xi^{k-1} \int\limits_{B(1)} 1_{\phi(Z_M)}(s) \tilde{\alpha}^t(g_{kt} h_s x)\, d\rho_{1+\sigma_k(t)^2}(s) \\
&= \Xi^{k-1} \int\limits_{B(1)} 1_{\phi(Z_M)}(s) \tilde{\alpha}^t(g_{kt} h_s x)\, d\rho_{1+\varepsilon^2(1+\sigma_{k-1}(t)^2)}(s) \\
&\le \int \cdots \int\limits_{B(1)^k} 1_{\phi(Z_M)}\big(\phi(s_1,\dots,s_k)\big) \tilde{\alpha}^t(g_{kt} h_{\phi(s_1,\dots,s_k)})\, d\rho_1(s_k) \cdots d\rho_1(s_1) \\
&\underset{(4.6)}{\le} 8(k-1)(2c_0)^{k-1} e^{-\frac{t}{2}} \max\left(\tilde{\alpha}^t(x), 1\right).
\end{aligned}$$

Hence,

$$\int\limits_{B(1)} 1_{\phi(Z_M)}(s)\tilde{\alpha}^t(g_{kt}h_s x)\, d\rho_{1+\sigma_k(t)^2}(s) \le \frac{8(k-1)(2c_0)^{k-1}}{\Xi^{k-1}} e^{-\frac{t}{2}} \max\big(\tilde{\alpha}^t(x), 1\big). \quad (4.12)$$

Also, since $1+\sigma_k(t)^2 \in [1,2]$, $d\rho_1$ is absolutely continuous with respect to $d\rho_{1+\sigma_k(t)^2}$ with a uniform (over $B(1)$) bound on the Radon-Nikodym derivative. Thus, we can find $c_1 \ge 1$ such that (4.12) takes the form:

$$\int\limits_{B(1)} 1_{\phi(Z_M)}(s)\tilde{\alpha}^t(g_{kt}h_s x)\, d\rho_1(s) \le \frac{8c_1(k-1)(2c_0)^{k-1}}{\Xi^{k-1}} e^{-\frac{t}{2}} \max\big(\tilde{\alpha}^t(x), 1\big). \quad (4.13)$$

Now consider the set

$$A_x^1\left(tk, 1, g_t X^t_{>M}\right) = \left\{s \in B(1) : \tilde{\alpha}^t(g_{(k-1)t}h_s x) > M\right\}.$$

It is easy to see that if $s \in A_x^1\left(tk, 1, g_t X^t_{>M}\right)$, then

$$s = \phi(s, 0, \dots, 0) \text{ and } (s, 0, \dots, 0) \in Z_M,$$

where 0 is the zero matrix. Hence, (4.13) implies

$$\int\limits_{A_x^1(kt,1,g_t X^t_{>M})} \tilde{\alpha}^t(g_{kt}h_s x)\, d\rho_1(s) \le \frac{8c_1(k-1)(2c_0)^{k-1}}{\Xi^{k-1}} e^{-\frac{t}{2}} \max\big(\tilde{\alpha}^t(x), 1\big). \quad (4.14)$$

Next, given $M > 0$ and $i \in \mathbb{N}$, let us define:

$$\begin{aligned} Z'_{M,i} := \Big\{&(s_1, \dots, s_i) \in \left(M^1_{m,n}\right)^i : \\ &\tilde{\alpha}^t(g_{(k-1)t}h_{s_j}g_{kt}h_{s_{j-1}}\cdots g_{kt}h_{s_1}x) > M \ \ \forall\, j \in \{1, \dots, i\}\Big\}. \end{aligned}$$

Note that

$$Z'_{M,1} = A_x^1\left(tk, 1, g_t X^t_{>M}\right). \quad (4.15)$$

Since $M \ge e^{\frac{mnt}{2}}$, in view of (3.5) for any $y \in X$ one has

$$\tilde{\alpha}^t(g_{(k-1)t}y) > M \quad \Longrightarrow \quad \tilde{\alpha}^t(g_{kt}y) > 1. \quad (4.16)$$

Then for any $2 \le i \in \mathbb{N}$, we obtain the following:

$$
\begin{aligned}
&\int \cdots \int\limits_{Z'_{M,i}} \tilde{\alpha}^t(g_{kt}h_{s_i} \cdots g_{kt}h_{s_1}x)\, d\rho_1(s_i) \cdots d\rho_1(s_1) \\
&= \int \cdots \int\limits_{Z'_{M,i-1}} \int\limits_{A_{g_{kt}h_{s_{i-1}} \cdots g_{kt}h_{s_1}x}(tk,1,1,g_t X^t_{>M})} \tilde{\alpha}^t(g_{kt}h_{s_i} \cdots g_{kt}h_{s_1}x)\, d\rho_1(s_i) \cdots d\rho_1(s_1) \\
&\underset{(4.14)}{\leq} \int \cdots \int\limits_{Z'_{M,i-1}} \frac{8c_1(k-1)(2c_0)^{k-1}}{\Xi^{k-1}} e^{-\frac{t}{2}} \cdot \max\left(\tilde{\alpha}^t(g_{kt}h_{s_{i-1}} \cdots g_{kt}h_{s_1}x), 1\right) d\rho_1(s_{i-1}) \cdots d\rho_1(s_1) \\
&\underset{(4.16)}{=} \frac{8c_1(k-1)(2c_0)^{k-1}}{\Xi^{k-1}} e^{-\frac{t}{2}} \int \cdots \int\limits_{Z'_{M,i-1}} \tilde{\alpha}^t(g_{kt}h_{s_{i-1}} \cdots g_{kt}h_{s_1}x)\, d\rho_1(s_{i-1}) \cdots d\rho_1(s_1).
\end{aligned}
\tag{4.17}
$$

Thus, by using (4.17) repeatedly we get for any $N \in \mathbb{N}$

$$
\begin{aligned}
&\int \cdots \int\limits_{Z'_{M,N}} \tilde{\alpha}^t(g_{kt}h_{s_N} \cdots g_{kt}h_{s_1}x)\, d\rho_1(s_N) \cdots d\rho_1(s_1) \\
&\leq \left(\frac{8c_1(k-1)(2c_0)^{k-1}}{\Xi^{k-1}}\right)^{(N-1)} e^{-\frac{(N-1)t}{2}} \int\limits_{Z'_{M,1}} \tilde{\alpha}^t(g_{kt}h_{s_1}x)\, d\rho_1(s_1) \\
&\underset{(4.14),(4.15)}{\leq} \left(\frac{8c_1(k-1)(2c_0)^{k-1}}{\Xi^{k-1}}\right)^{N} e^{-\frac{Nt}{2}} \max\left(\tilde{\alpha}^t(x), 1\right).
\end{aligned}
\tag{4.18}
$$

Now, similarly to (4.4), define the function $\psi : B(1)^N \to M_{m,n}$ by

$$
\psi(s_1, \dots, s_N) := \sum_{j=1}^{N} e^{-(m+n)(j-1)kt} s_j,
$$

so that

$$
g_{kt}h_{s_N} \cdots g_{kt}h_{s_1} = g_{Nkt}h_{\psi(s_1,\dots,s_N)}. \tag{4.19}
$$

The following lemma is a modification of Lemma 4.2 applicable to the sets $Z'_{M,N}$:

Lemma 4.4. *For any $M > 0$, $\psi^{-1}\left(\psi(Z'_{M,N})\right) \subset Z'_{C_\alpha M,N}$.*

Proof. Let $(s_1, \dots, s_N) \in B(1)^N$ be such that $\psi(s_1, \dots, s_N) \in \psi(Z'_{C_\alpha M,N})$. Then for some $(s'_1, \dots, s'_N) \in Z'_{C_\alpha M,N}$ we have:

$$
\psi(s_1, \dots, s_N) = \psi(s'_1, \dots, s'_N).
$$

Hence, by using (4.19) we get:

$$g_{kt}h_{s_N}\dots g_{kt}h_{s_1} = g_{kt}h_{s'_N}\dots g_{kt}h_{s'_1}.$$

Thus, it is easy to see that for any $1 \le i \le N$

$$g_{kt}h_{s_i}\cdots g_{kt}h_{s_1} = h_{\psi_i(-s_{i+1},\dots,-s_N)+\psi_i(s'_{i+1},\dots,s'_N)}\big(g_{kt}h_{s'_i}\cdots g_{kt}h_{s'_1}\big), \tag{4.20}$$

where for any $(w_{i+1},\dots,w_N) \in B(1)^{N-i}$ we put

$$\psi_i(w_{i+1},\dots,w_N) := \sum_{j=i+1}^{N} e^{-(m+n)(j-i)kt}w_j.$$

Note that since $t \ge 2$, one has $\psi_i(w_{i+1},\dots,w_N) \in B(1)$ for any $(w_{i+1},\dots,w_N) \in B(1)^{N-i}$. Hence, in view of (4.20), for any $1 \le i \le N$ we have

$$g_{kt}h_{s_N}\cdots g_{kt}h_{s_1} \in B^H(2)g_{kt}h_{s'_i}\cdots g_{kt}h_{s'_1},$$

which, since $(s'_1,\dots,s'_N) \in Z'_{C_\alpha M,N}$, implies $(s_1,\dots,s_N) \in Z'_{M,N}$. This finishes the proof of the lemma. □

Now by combining (4.18) and Lemma 4.4 we get:

$$\begin{aligned}&\int\cdots\int_{B(1)^N} 1_{\psi(Z'_{C_\alpha M,N})}(\psi(s_1,\dots,s_N))\tilde{\alpha}^t(g_{Nkt}h_{\psi(s_1,\dots,s_N)}x)\,d\rho_1(s_N)\cdots d\rho_1(s_1)\\ &\le \left(\frac{8c_1(k-1)(2c_0)^{k-1}}{\Xi^{k-1}}\right)^N e^{-\frac{Nt}{2}}\max\big(\tilde{\alpha}^t(x),1\big).\end{aligned} \tag{4.21}$$

Then, as before, one can use Lemma 4.3 $(N-1)$ times with $\varepsilon = e^{-(m+n)kt}$ and $\delta = \sigma_1(kt),\dots,\sigma_{N-1}(kt)$ respectively and obtain:

$$\begin{aligned}&\Xi^{N-1}\int_{B(1)} 1_{\psi(Z'_{C_\alpha M,N})}(s)\tilde{\alpha}^t(g_{Nkt}h_sx)d\rho_{1+\sigma_N(kt)^2}(s)\\ &= \Xi^{N-1}\int_{B(1)} 1_{\psi(Z'_{C_\alpha M,N})}(s)\tilde{\alpha}^t(g_{Nkt}h_sx)d\rho_{1+\varepsilon^2(1+\sigma_{N-1}(kt)2)}(s)\\ &\le \int\cdots\int_{B(1)^N} 1_{\psi(Z'_{C_\alpha M,N})}(\psi(s_1,\cdots,s_N))\tilde{\alpha}^t(g_{Nkt}h_{\psi(s_1,\dots,s_N)}x)\,d\rho_1(s_N)\cdots d\rho_1(s_1)\\ &\underset{(4.21)}{\le} \left(\frac{8c_1(k-1)(2c_0)^{k-1}}{\Xi^{k-1}}\right)^N e^{-\frac{Nt}{2}}\max(\tilde{\alpha}^t(x),1).\end{aligned} \tag{4.22}$$

Thus, we get

$$\int_{B(1)} 1_{\psi(Z'_{C_\alpha M})}(s)\tilde{\alpha}^t(g_{Nkt}h_s x)\,d\rho_{1+\sigma_N(kt)^2}(s)$$
$$\leq \frac{\left(8c_1(k-1)(2c_0)^{k-1}\right)^N}{\Xi^{kN-1}} e^{-\frac{Nt}{2}} \max\left(\tilde{\alpha}^t(x), 1\right).$$

Now observe that, in view of (4.1), if $s \in A_x^N\left(kt, 1, g_t X^t_{>C_\alpha M}\right)$, then

$$s = \psi(s, 0, \ldots, 0) \text{ and } (s, 0, \ldots, 0) \in Z'_{C_\alpha M, N}.$$

Thus, (4.22) can be written as

$$\begin{aligned}\int_{A_x^N\left(kt,1,g_t X^t_{>C_\alpha M}\right)} &\tilde{\alpha}^t(g_{Nkt}h_s x)\,d\rho_{1+\sigma_N(kt)^2}(s)\\ &\leq \frac{\left(8c_1(k-1)(2c_0)^{k-1}\right)^N}{\Xi^{kN-1}} e^{-\frac{Nt}{2}} \max\left(\tilde{\alpha}^t(x), 1\right).\end{aligned} \tag{4.23}$$

Again, since $1 + \sigma_N(kt)^2 \in [1, 2]$, ds is absolutely continuous with respect to $d\rho_{1+\sigma_N(kt)^2}$ with a uniform (over $B(1)$) bound on the Radon-Nikodym derivative. Thus, we can find $c_2 \geq 1$ such that (4.23) takes the form

$$\int_{A_x^N\left(kt,1,g_t X^t_{>C_\alpha M}\right)} \tilde{\alpha}^t(g_{Nkt}h_s x)\,ds \leq \frac{c_2\left(8c_1(k-1)(2c_0)^{k-1}\right)^N}{\Xi^{kN-1}} e^{-\frac{Nt}{2}} \max\left(\tilde{\alpha}^t(x), 1\right).$$

Now define $C_1 := 16c_0c_1c_2/\Xi$. Then by the above inequality we have:

$$\int_{A_x^N\left(kt,1,g_t X^t_{>C_\alpha M}\right)} \tilde{\alpha}^t(g_{Nkt}h_s x)\,ds \leq \left((k-1)C_1^k e^{-\frac{t}{2}}\right)^N \max\left(\tilde{\alpha}^t(x), 1\right).$$

This ends the proof of the proposition. $\square$

As a corollary we get the following covering result:

Corollary 4.5. *There exists $C_1 \geq 1$ such that for any $\theta \in (0, \frac{1}{\sqrt{mn}}]$, any $2 \leq k \in \mathbb{N}$, any $t \geq 2$, any $M \geq C_\alpha^3 e^{mnt}$, any $N \in \mathbb{N}$, and any $x \in X$, the set*

$$A_x^N\left(kt, 1/2, X^t_{>M}\right) = \left\{s \in B(1/2) : \tilde{\alpha}^t(g_{ikt}h_s x) > M \ \ \forall\, i \in \{1, \ldots, N\}\right\}$$

can be covered with at most

$$\frac{C_\alpha}{\theta^{mn}}\left((k-1)C_1^k e^{(mn(m+n)k-\frac{1}{2})t}\right)^N \frac{\max\left(\tilde{\alpha}^t(x), 1\right)}{M}$$

cubes of side-length $\theta e^{-(m+n)Nkt}$ *in* $M_{m,n}$.

Proof. Let x, θ, M, N, t and k be as above, and take C_1 as in Proposition 4.1. Applying the latter with M replaced with $C_\alpha^{-2} M e^{-\frac{mnt}{2}}$, we have:

$$\int\limits_{A_x^N\left(kt,1,g_t X^t_{>C_\alpha^{-1} M e^{-mnt/2}}\right)} \tilde{\alpha}^t(g_{Nkt} h_s x)\, ds \le \left((k-1) C_1^k e^{-\frac{t}{2}}\right)^N \max(\tilde{\alpha}^t(x), 1). \tag{4.24}$$

In view of (3.5) we have $X^t_{>C_\alpha^{-1} M} \subset g_t X^t_{>C_\alpha^{-1} M e^{-mnt/2}}$, hence

$$\begin{aligned} C_\alpha^{-1} M \cdot \mathrm{Leb}\Big(A_x^N\left(kt, 1, X^t_{>C_\alpha^{-1} M}\right)\Big) &\le \int\limits_{A_x^N\left(kt,1,X^t_{>C_\alpha^{-1} M}\right)} \tilde{\alpha}^t(g_{Nkt} h_s x)\, ds \\ &\le \int\limits_{A_x^N\left(kt,1,g_t X^t_{>C_\alpha^{-1} M e^{-mnt/2}}\right)} \tilde{\alpha}^t(g_{Nkt} h_s x) ds. \end{aligned} \tag{4.25}$$

Thus, using (4.24) and (4.25) we obtain

$$\mathrm{Leb}\Big(A_x^N\left(kt, 1, X^t_{>C_\alpha^{-1} M}\right)\Big) \le C_\alpha \left((k-1) C_1^k e^{-\frac{t}{2}}\right)^N \cdot \frac{\max(\tilde{\alpha}^t(x), 1)}{M}. \tag{4.26}$$

Take a covering of $B(1/2)$ with interior-disjoint cubes of side-length $\theta e^{-(m+n)Nkt}$ in $M_{m,n}$ whose sides are parallel to the coordinate axes. Now let R be one of the cubes in this cover which has non-empty intersection with $A_x^N\left(kt, 1/2, X^t_{>M}\right)$. Note that since $R \cap B(1/2) \ne \varnothing$, we must have

$$R \subset B\left(\frac{1}{2} + \sqrt{mn}\,\theta e^{-(m+n)Nkt}\right) \underset{\theta \le \frac{1}{\sqrt{mn}}}{\subset} B(1). \tag{4.27}$$

Now let $s \in R \cap A_x^N\left(kt, 1/2, X^t_{>M}\right)$. Then

$$\tilde{\alpha}^t(g_{ikt} h_s x) > M \text{ for all } 1 \le i \le N.$$

On the other hand, for any $s' \in B$ and any $1 \le i \le N$ one has

$$\begin{aligned} g_{ikt} h_{s'} x = (g_{ikt} h_{s'-s} g_{-ikt})\, g_{ikt} h_s x &\in B^H(\sqrt{mn}\theta) g_{ikt} h_s x \\ &\subset B^H(1) g_{ikt} h_s x \subset B(1) g_{ikt} h_s x. \end{aligned} \tag{4.28}$$

Hence, in view of (4.27) and (4.28) we conclude that

$$R \subset A_x^N\left(kt, 1, X^t_{>M}\right). \tag{4.29}$$

Thus, by (4.26) and (4.29), the set $A_x^N\left(kt, 1/2, X^t_{>M}\right)$ can be covered with at most

$$\frac{\mathrm{Leb}\Big(A_x^N\left(kt, 1, X^t_{>M}\right)\Big)}{\left(\theta e^{-(m+n)Nkt}\right)^{mn}} \le \frac{C_\alpha}{\theta^{mn}}\left((k-1)C_1^k e^{(mn(m+n)k-\frac{1}{2})t}\right)^N \cdot \frac{\max\left(\tilde{\alpha}^t(x), 1\right)}{M}$$

cubes of side-length $\theta e^{-(m+n)Nkt}$ in $M_{m,n}$. This finishes the proof. □

5. The main covering result

For any $t > 0$, let us define the compact subset Q_t of X as follows:

$$Q_t := X^t_{\le C_\alpha^3 e^{mnt}}. \tag{5.1}$$

In the following lemma we obtain a lower bound for the injectivity radius of the set $\partial_1 Q_t$.

Lemma 5.1. *There exist $0 < C_2 \le 1$ and $p \ge m+n$ independent of t such that for any $t > 0$:*

$$r_0(\partial_1 Q_t) \ge C_2 e^{-pt}.$$

Proof. Let $t > 0$. Note that in view of (3.19) we have

$$\partial_1 Q_t \subset X^t_{\le C_\alpha^4 e^{mnt}}; \tag{5.2}$$

then, using (3.14) we can write

$$\begin{aligned} X^t_{\le C_\alpha^4 e^{mnt}} &\subset \left\{x \in X : \alpha_1(x) \le \frac{e^{-2(mn+\frac{1}{2})(m+n-1)t}}{(m+n-1)^{2(m+n-1)}} C_\alpha^8 e^{2mnt}\right\} \\ &= \left\{x : \frac{1}{\alpha_1(x)} \ge C_4 e^{-(2(mn+\frac{1}{2})(m+n-1)+2mn)t}\right\}, \end{aligned}$$

where $C_4 = \frac{1}{C_\alpha^8 (m+n-1)^{2(m+n-1)}}$. Recall that $\frac{1}{\alpha_1(x)}$ is equal to the norm of the shortest vector in the lattice x; therefore by [23, Lemma 7.2], $r_0\left(X^t_{\le C_\alpha^4 e^{mnt}}\right)$ is at least $C_2 e^{-pt}$, where

$$p = \left((m+n)^2 - 1\right) \cdot \left(2(mn+1/2)(m+n-1) + 2mn\right) \ge m+n$$

and $0 < C_2 \le 1$ is only dependent on m and n. Thus we have $r_0(\partial_1 Q_t) \ge C_2 e^{-pt}$, which finishes the proof. □

The following proposition is our most important covering result.

Proposition 5.2. *There exist constants*

$$p \geq m+n, 0 < r_2 < \frac{1}{16\sqrt{mn}}, b_0 \geq 2,\ b \geq 1,\ 0 < C_2 \leq 1,\ C_0, C_3, K_1, K_2, \lambda > 0$$

such that for any open subset U of X and all integers N and $k \geq 2$ the following holds: for all $t \geq 2$ and all $0 < r < 1$ satisfying

$$e^{\frac{b_0 - kt}{b}} \leq r \leq \min(C_2 e^{-pt}, r_2), \tag{5.3}$$

all $\theta \in \left[4r, \frac{1}{2\sqrt{mn}}\right]$, and for all $x \in \partial_r (Q_t \cap U^c)$, the set $A_x^N \left(kt, \frac{r}{32\sqrt{mn}}, U^c\right)$ can be covered with at most

$$\frac{C_0}{\theta^{2mn}} e^{mn(m+n)Nkt} \left(1 - K_1 \mu(\sigma_{2\sqrt{mn}\theta} U) + \frac{K_2 e^{-\lambda kt}}{r^{mn}} + \frac{k-1}{\theta^{mn}} C_3^k e^{-\frac{t}{4}}\right)^N$$

cubes of side-length $\theta e^{-(m+n)Nkt}$ in $M_{m,n}$.

Proof. The strategy of the proof consists of combining Theorem 2.2 with Corollary 4.5. Recall that the former estimates the number of cubes needed to cover the set of points whose trajectories visit a given compact set S, while the latter does the same for trajectories visiting the set $X^t_{>M}$ which is the complement of a large compact subset of X. Our goal now is to have a similar result for points whose trajectories visit the set U^c, which is not compact and may have a tiny complement. This is done by an inductive procedure which is inspired by the methods introduced in [19].

Take $t \geq 2$ and let C_2 and p be as in Lemma 5.1. Let $0 < r < 1$ and $2 \leq k \in \mathbb{N}$ be such that (5.3) is satisfied, where b_0, b, r_2 are as in Theorem 2.2.

Now let $x \in \partial_r (Q_t \cap U^c)$, $N \in \mathbb{N}$, and $\theta \in \left[4r, \frac{1}{2\sqrt{mn}}\right]$. Recall that

$$A_x^N \left(kt, \frac{r}{32\sqrt{mn}}, U^c\right) = \left\{s \in B\left(\frac{r}{32\sqrt{mn}}\right) : g_{\ell kt} h_s x \in U^c\ \forall \ell \in \{1, \ldots, N\}\right\}.$$

Our goal is to cover $A_x^N \left(kt, \frac{r}{32\sqrt{mn}}, U^c\right)$ with cubes of side-length $\theta e^{-(m+n)Nkt}$ in $M_{m,n}$. For any $s \in A_x^N \left(kt, \frac{r}{32\sqrt{mn}}, U^c\right)$, let us define:

$$J_s := \left\{j \in \{1, \ldots, N\} : g_{jkt} h_s x \in Q_t^c\right\},$$

and for any $J \subset \{1, \ldots, N\}$, set:

$$Z(J) := \left\{s \in A_x^N \left(kt, \frac{r}{32\sqrt{mn}}, U^c\right) : J_s = J\right\}.$$

Note that

$$A_x^N\left(kt, \frac{r}{32\sqrt{mn}}, U^c\right) = \bigcup_{J\subset\{1,\ldots,N\}} Z(J) \tag{5.4}$$

Now, set

$$D_1 := 1 - K_1\mu(\sigma_{2\sqrt{mn}\theta}U) + \frac{K_2 e^{-\lambda kt}}{r^{mn}}. \tag{5.5}$$

and

$$D_2 := (k-1)C_1^k e^{-t/2}, \tag{5.6}$$

where K_1, K_2, λ are as in Theorem 2.2 and C_1 is as in Corollary 4.5.

Let J be a subset of $\{1,\ldots,N\}$. We can decompose J and $I := \{1,\ldots,N\} \smallsetminus J$ into subintervals of maximal size $J_1,\ldots,J_q \subset J$ and $I_1,\ldots,I_{q'} \subset I$ (here and hereafter by a *subinterval* we mean a set of the form $\mathbb{N}\cap[a,b]$ where $0<a<b$) so that

$$J = \bigsqcup_{j=1}^{q} J_j \text{ and } I = \bigsqcup_{i=1}^{q'} I_i.$$

Hence, we get a partition of $\{1,\ldots,N\}$ as follows:

$$\{1,\ldots,N\} = \bigsqcup_{j=1}^{q} J_j \sqcup \bigsqcup_{i=1}^{q'} I_i.$$

Now we inductively prove the following

Claim 5.3. *For any integer $L \le N$, if*

$$\{1,\ldots,L\} = \bigsqcup_{j=1}^{\ell} J_j \sqcup \bigsqcup_{i=1}^{\ell'} I_i, \tag{5.7}$$

then the set $Z(J)$ can be covered with at most:

$$\left(\frac{C_\alpha^2}{\theta^{mn}}\right)^{d'_{J,L}+1} \left((2^9 mn)^{mn} K_0\right)^{d_{J,L}+1} e^{mn(m+n)Lkt} D_1^{\sum_{i=1}^{\ell'}|I_i| - d_{J,L}} D_2^{\sum_{j=1}^{\ell}|J_j|} \tag{5.8}$$

cubes of side-length $\theta e^{-(m+n)Lkt}$ in $M_{m,n}$, where K_0 is as in Theorem 2.2, and $d_{J,L}$, $d'_{J,L}$ are defined as follows:

$$d_{J,L} := \#\{i \in \{1,\ldots,L\} :\ i < L,\ i \in J \text{ and } i+1 \in I\},$$
$$d'_{J,L} := \#\{i \in \{1,\ldots,L\} :\ i < L,\ i \in I \text{ and } i+1 \in J\}.$$

Note that equivalently one can define

$$d_{J,L} = \begin{cases} \ell & \text{if } L \notin J \\ \ell - 1 & \text{if } L \in J \end{cases}$$

as the number of intervals in $J \cap \{1, \ldots, L\}$ with right endpoints $< L$, and, likewise,

$$d'_{J,L} = \begin{cases} \ell' & \text{if } L \notin I \\ \ell' - 1 & \text{if } L \in I \end{cases}$$

as the number of intervals in $I \cap \{1, \ldots, L\}$ with right endpoints $< L$.

Proof of Claim 5.3. We argue by induction on $\ell + \ell'$. When $\ell + \ell' = 1$, we have $d_{J,L} = d'_{J,L} = 0$, and there are two cases: either $\ell = 1$ and $\{1, \ldots, L\} = J_1$, or $\ell' = 1$ and $\{1, \ldots, L\} = I_1$. In the first case

$$\begin{aligned} Z(J) &\subset \left\{ s \in A_x^N \left(kt, \frac{r}{32\sqrt{mn}}, U^c \right) : g_{ikt} h_s x \in Q_t^c \ \ \forall i \in \{1, \ldots, L\} \right\} \\ &\subset A_x^L \left(kt, \frac{r}{32\sqrt{mn}}, Q_t^c \right) \subset A_x^L \left(kt, 1/2, X^t_{>C_\alpha^3 e^{mnt}} \right), \end{aligned}$$

where the last step is due to the bound (5.3) on r. Therefore, Corollary 4.5 applied with $M = C_\alpha^3 e^{mnt}$ and $N = L$ shows that this set can be covered with at most

$$\begin{aligned} & \frac{C_\alpha}{\theta^{mn}} \left((k-1) C_1^k e^{(mn(m+n)k - \frac{1}{2})t} \right)^L \frac{\tilde{\alpha}^t(x)}{C_\alpha^3 e^{mnt}} \\ \underset{(5.2)}{\leq} & \frac{C_\alpha}{\theta^{mn}} \left((k-1) C_1^k e^{(mn(m+n)k - \frac{1}{2})t} \right)^L \frac{C_\alpha^4 e^{mnt}}{C_\alpha^3 e^{mnt}} \\ = & \frac{C_\alpha^2}{\theta^{mn}} \left((k-1) C_1^k e^{(mn(m+n)k - \frac{1}{2})t} \right)^L \end{aligned}$$

cubes of side-length $\theta e^{-(m+n)Lkt}$ in $M_{m,n}$. Clearly this number is bounded from above by (5.8), which takes the form

$$\frac{C_\alpha^2}{\theta^{mn}} (2^9 mn)^{mn} K_0 e^{mn(m+n)Lkt} \left((k-1) C_1^k e^{-t/2} \right)^L.$$

In the second case

$$\begin{aligned} Z(J) &\subset \left\{ s \in A_x^N \left(kt, \frac{r}{32\sqrt{mn}}, U^c \right) : g_{ikt} h_s x \in Q_t \ \ \forall i \in \{1, \ldots, L\} \right\} \\ &\subset A_x^L \left(kt, \frac{r}{32\sqrt{mn}}, U^c \cap Q_t \right). \end{aligned}$$

By Lemma 5.1, for any $U \subset X$ we have

$$r_0\big(\partial_1(U^c \cap Q_t)\big) \geq r_0(\partial_1 Q_t) \geq C_2 e^{-pt}.$$

So it is easy to see that since condition (5.3) is satisfied, condition (2.2) with t replaced by kt and condition (2.3) with S replaced by $U^c \cap Q_t$ are satisfied as well. Hence we can apply Theorem 2.2 with S replaced by $U^c \cap Q_t$, N replaced by L, and t replaced with kt. This produces a covering of $A_x^N\left(kt, \frac{r}{32\sqrt{mn}}, U^c\right)$ by

$$\begin{aligned}&\left(\frac{4r}{\theta}\right)^{mn} K_0 e^{mn(m+n)Lt}\left(1 - K_1\mu\big(\sigma_{2\sqrt{mn}\theta}(U \cup Q_t^c)\big) + \frac{K_2 e^{-\lambda t}}{r^{mn}}\right)^L \\ &\leq \frac{C_\alpha^2}{\theta^{mn}}(2^9 mn)^{mn} K_0 e^{mn(m+n)Lkt}\left(1 - K_1\mu\big(\sigma_{2\sqrt{mn}\theta}(U)\big) + \frac{K_2 e^{-\lambda t}}{r^{mn}}\right)^L\end{aligned}$$

cubes of side-length $\theta e^{-(m+n)Lkt}$, finishing the proof of the base of the induction.

In the inductive step, let $L' > L$ be the next integer for which an equation similar to (5.7) is satisfied. We have two cases. Either

$$\{1, \ldots, L'\} = \{1, \ldots, L\} \sqcup I_{\ell'+1} \tag{5.9}$$

or

$$\{1, \ldots, L'\} = \{1, \ldots, L\} \sqcup J_{\ell+1}. \tag{5.10}$$

We start with the case (5.9). Note that in this case we have

$$d_{J,L'} = d_{J,L} + 1 \text{ and } d'_{J,L'} = d'_{J,L}. \tag{5.11}$$

Also, it is easy to see that every cube of side-length $\theta e^{-(m+n)Lkt}$ in $M_{m,n}$ can be covered with at most $2^{mn} e^{mn(m+n)kt}$ cubes of side-length $\theta e^{-(m+n)(L+1)kt}$. Therefore, by using the induction hypothesis and in view of (5.8), we can cover $Z(J)$ with at most

$$2^{mn}\left(\frac{C_\alpha^2}{\theta^{mn}}\right)^{d'_{J,L}+1}\left((2^9 mn)^{mn} K_0\right)^{d_{J,L}+1} e^{mn(m+n)(L+1)kt} \cdot D_1^{\sum_{i=1}^{\ell'}|I_i| - d_{J,L}} D_2^{\sum_{j=1}^{\ell}|J_j|} \tag{5.12}$$

cubes of side-length $\theta e^{-(m+n)(L+1)kt}$. Now let B be one of the cubes of side-length $\theta e^{-(m+n)(L+1)kt}$ in the aforementioned cover such that $B \cap Z(J) \neq \varnothing$. Clearly

$$B \text{ can be covered by } \left(\frac{2\theta}{\frac{r}{32mn}}\right)^{mn} \text{ cubes of side-length } \frac{re^{-(m+n)(L+1)kt}}{32mn}. \tag{5.13}$$

Let B_r be one of such cubes that has non-empty intersection with $Z(J)$, and let $s \in B_r \cap Z(J)$. Since $s \in Z(J)$, it follows that $g_{(L+1)kt}h_s x \in U^c \cap Q_t$. Therefore, if we denote the center of B_r by s_0, we have

$$g_{(L+1)kt}h_{s_0}x \in B^H\left(\frac{r}{32\sqrt{mn}}\right)(U^c \cap Q_t) \subset \partial_r(U^c \cap Q_t). \tag{5.14}$$

Moreover, for any $s' \in B_r$ and any positive integer $1 \leq i \leq L' - (L+1)$ we have:

$$\begin{aligned} g_{(L+1+i)kt}h_{s'}x &= g_{ikt}(g_{(L+1)kt}h_{s'-s_0}g_{-(L+1)kt})(g_{(L+1)kt}h_{s_0}x) \\ &= g_{ikt}h_{e^{(m+n)(L+1)kt}(s'-s_0)}(g_{(L+1)kt}h_{s_0}x). \end{aligned} \tag{5.15}$$

It is easy to see that the map $s' \to e^{(m+n)(L+1)kt}(s' - s_0)$ maps B_r into $B\left(\frac{r}{32\sqrt{mn}}\right)$. Hence, by (5.15)

$$\begin{aligned} &\left\{s' \in B_r : g_{(L+1+i)kt}h_{s'}x \in U^c \cap Q_t \ \ \forall i \in \{1, \cdots, L' - (L+1)\}\right\} \\ &\subset e^{-(m+n)(L+1)kt}A^{L'-(L+1)}_{g_{(L+1)kt}h_{s_0}x}\left(kt, \frac{r}{32\sqrt{mn}}, U^c \cap Q_t\right) + s_0. \end{aligned}$$

So, in view of the above inclusion and (5.14), we can go through the same procedure and apply Theorem 2.2 with t replaced with kt, S replaced with $U^c \cap Q_t$, N replaced with $|I_{\ell'+1}| - 1 = L' - (L+1)$, and x replaced with $g_{(L+1)kt}h_{s_0}x$, and conclude that $B_r \cap Z(J)$ can be covered with at most

$$\left(\frac{4r}{\theta}\right)^{mn} K_0 e^{mn(m+n)(|I_{\ell'+1}|-1)kt} D_1^{|I_{\ell'+1}|-1}$$

cubes of side-length $\theta e^{-(m+n)L'kt}$. Therefore, in view of (5.13), the set $B \cap Z(J)$ can be covered with at most

$$\begin{aligned} &2^{mn}\left(\frac{\theta}{\frac{r}{32mn}}\right)^{mn}\left(\frac{4r}{\theta}\right)^{mn} K_0 e^{mn(m+n)(|I_{\ell'+1}|-1)kt} D_1^{|I_{\ell'+1}|-1} \\ &= K_0\left(2^8 mn\right)^{mn} e^{mn(m+n)(|I_{\ell'+1}|-1)kt} D_1^{|I_{\ell'+1}|-1} \end{aligned}$$

cubes of side-length $\theta e^{-(m+n)L'kt}$. This, combined with (5.12) which is an upper bound for the number of cubes of side-length $\theta e^{-(m+n)(L+1)kt}$ in $M_{m,n}$ needed to cover $Z(J)$, implies that $Z(J)$ can be covered with at most

$$K_0\left(2^8 mn\right)^{mn} e^{mn(m+n)(|I_{\ell'+1}|-1)kt} D_1^{|I_{\ell'+1}|-1}.$$

$$2^{mn}\left(\frac{C_\alpha^2}{\theta^{mn}}\right)^{d'_{J,L}+1}\left((2^9 mn)^{mn}K_0\right)^{d_{J,L}+1} e^{mn(m+n)(L+1)kt} D_1^{\sum_{i=1}^{\ell'}|I_i| - d_{J,L'}} D_2^{\sum_{j=1}^{\ell}|J_j|}$$

$$\underset{(5.11)}{=} \left(\frac{C_\alpha^2}{\theta^{mn}}\right)^{d'_{J,L'}+1} \left((2^9 mn)^{mn} K_0\right)^{d_{J,L'}+1} e^{mn(m+n)Lkt} D_1^{\sum_{i=1}^{\ell'} |I_i| - d_{J,L'}} D_2^{\sum_{j=1}^{\ell} |J_j|}$$

cubes of side-length $\theta e^{-(m+n)L'kt}$. This ends the proof of the claim in this case.

Next assume that (5.10) holds. Note that in this case

$$d_{J,L'} = d_{J,L} \text{ and } d'_{J,L'} = d'_{J,L} + 1. \tag{5.16}$$

Take a covering of $Z(J)$ with cubes of side-length $\theta e^{-(m+n)kLt}$ in $M_{m,n}$, suppose B' is one of the cubes in the cover such that $B' \cap Z(J) \neq \varnothing$, and let s_1 be the center of B'. Then, since $\sqrt{mn}\theta < 1$, it is easy to see that:

$$g_{Lkt} h_{s_1} x \in B^H(\sqrt{mn}\theta)(U^c \cap Q_t) \subset \partial_1 Q_t. \tag{5.17}$$

On the other hand, for any $s \in B'$ and any positive integer $1 \le i \le L' - L$ we have:

$$\begin{aligned} g_{(L+i)kt} h_s x &= g_{ikt}(g_{Lkt} h_{s-s_1} g_{-Lkt})(g_{Lkt} h_{s_1} x) \\ &= g_{ikt} h_{e^{(m+n)Lkt}(s-s_1)}(g_{Lkt} h_{s_1} x). \end{aligned} \tag{5.18}$$

Note that the map $s \to e^{(m+n)Lkt}(s - s_1)$ maps B' into $B(1/2)$. Thus, by (5.18)

$$\begin{aligned} &\left\{ s \in B' : g_{(L+i)kt} h_s x \in Q_t^c \ \ \forall\, i \in \{1, \cdots, L' - L\} \right\} \\ &\subset e^{-(m+n)Lkt} A^{L'-L}_{g_{Lkt} h_{s_1} x}\left(kt, 1/2, Q_t^c\right) + s_1 \\ &= e^{-(m+n)Lkt} A^{L'-L}_{g_{Lkt} h_{s_1} x}\left(kt, 1/2, X_{> C_\alpha^3 e^{mnt}}\right) + s_1. \end{aligned}$$

So in view of the above inclusion and (5.17), we can apply Corollary 4.5 with $M = C_\alpha^3 e^{mnt}$, $g_{Lkt} h_{s_1} x$ in place of x, and $|J_{\ell+1}| = L' - L$ in place of N. This way, we get that the set $B' \cap Z(J)$ can be covered with at most

$$\begin{aligned} &\frac{C_\alpha}{\theta^{mn}} D_2^{|J_{\ell+1}|} \cdot e^{mn(m+n)k(|J_{\ell+1}|)t} \cdot \frac{\max\left(\tilde{\alpha}^t(g_{Lkt} h_{s_1} x), 1\right)}{M} \\ &\underset{(5.2),(5.17)}{\le} \frac{C_\alpha^2}{\theta^{mn}} D_2^{|J_{\ell+1}|} \cdot e^{mn(m+n)k(|J_{\ell+1}|)t} \end{aligned}$$

cubes of side-length $\theta e^{-(m+n)kL't}$. From this, combined with the induction hypothesis, we conclude that $Z(J)$ can be covered with at most

$$\begin{aligned} &\frac{C_\alpha^2}{\theta^{mn}} D_2^{|J_{\ell+1}|} e^{mn(m+n)(|J_{\ell+1}|)kt} \left(\frac{C_\alpha^2}{\theta^{mn}}\right)^{d'_{J,L}+1} \cdot \\ &\left((2^9 mn)^{mn} K_0\right)^{d_{J,L'}+1} e^{mn(m+n)Lkt} D_1^{\sum_{i=1}^{\ell'} |I_i| - d_{J,L}} \cdot D_2^{\sum_{j=1}^{\ell} |J_j|} \\ &\underset{(5.16)}{=} \left(\frac{C_\alpha^2}{\theta^{mn}}\right)^{d'_{J,L'}+1} \left((2^9 mn)^{mn} K_0\right)^{d_{J,L'}+1} e^{mn(m+n)L'kt} D_1^{\sum_{i=1}^{\ell'} |I_i| - d_{J,L}} D_2^{\sum_{j=1}^{\ell+1} |J_j|} \end{aligned}$$

cubes of side-length $\theta e^{-(m+n)L'kt}$, finishing the proof of the claim. $\square$

Now by letting $L = N$, we conclude that $Z(J)$ can be covered with at most

$$\left(\frac{C_\alpha^2}{\theta^{mn}}\right)^{d'_{J,N}+1} \left((2^9 mn)^{mn} K_0\right)^{d_{J,L'}+1} e^{mn(m+n)Nkt} D_1^{|I|-d_{J,N}} D_2^{|J|} \tag{5.19}$$

cubes of side-length $\theta e^{-(m+n)Nkt}$ in $M_{m,n}$.

Clearly

$$d'_{J,N} \leq d_{J,N} + 1. \tag{5.20}$$

Also, note that since $d_{J,N} \leq \max(|I|, |J|)$, the exponents $|I| - d_{J,N}, |J| - d_{J,N}$ in (5.19) are non-negative integers. So, in view of (5.4) and (5.19), the set $A_x^N\left(kt, \frac{r}{32\sqrt{mn}}, U^c\right)$ can be covered with at most:

$$\begin{aligned}
&\sum_{J\subset\{1,\ldots,N\}} \left(\frac{C_\alpha^2}{\theta^{mn}}\right)^{d'_{J,N}+1} \left((2^9 mn)^{mn} K_0\right)^{d_{J,N}+1} e^{mn(m+n)Nkt} D_1^{|I|-d_{J,N}} \cdot D_2^{|J|} \\
&\underset{(5.20)}{\leq} e^{mn(m+n)Nkt} \sum_{J\subset\{1,\ldots,N\}} \left(\frac{C_\alpha^2}{\theta^{mn}}\right)^{d_{J,N}+2} \left((2^9 mn)^{mn} K_0\right)^{d_{J,N}+1} D_1^{|I|-d_{J,N}} D_2^{|J|} \\
&\leq \frac{C_0}{\theta^{2mn}} e^{mn(m+n)Nkt} \sum_{J\subset\{1,\ldots,N\}} D_1^{|I|-d_{J,N}} D_2^{|J|} \cdot \left(\frac{C_0}{\theta^{mn}}\right)^{d_{J,N}} \\
&= \frac{C_0}{\theta^{2mn}} e^{mn(m+n)Nkt} \sum_{J\subset\{1,\ldots,N\}} D_1^{N-|J|-d_{J,N}} D_2^{|J|-d_{J,N}} \cdot \left(\sqrt{\frac{C_0 D_2}{\theta^{mn}}}\right)^{2d_{J,N}}
\end{aligned}$$

cubes of side-length $\theta e^{-(m+n)Nkt}$ in $M_{m,n}$, where $C_0 := C_\alpha^4 (2^9 mn)^{mn} K_0 \geq 1$.

To simplify the last expression we will use an auxiliary

Lemma 5.4. *For any $n_1, n_2, n_3 > 0$ it holds that*

$$\sum_{J\subset\{1,\ldots,N\}} n_1^{N-|J|-d_{J,N}} n_2^{|J|-d_{J,N}} n_3^{2d_{J,N}} \leq (n_1 + n_2 + n_3)^N .$$

Proof. Define the map $\phi : \{1, \ldots, N\} \to \{n_1, n_2, n_3\}^N$ by

$$\phi(J) = (x_1, \ldots, x_N),$$

where for any $i \in \{1, \ldots, N\}$, x_i is defined as follows:

$$x_i := \begin{cases} n_1 & \text{if } i \in I \text{ and } (i-1 \in I \text{ or } i = 1); \\ n_2 & \text{if } i \in J \text{ and } (i+1 \in J \text{ or } i = N); \\ n_3 & \text{otherwise.} \end{cases}$$

It is easy to see that ϕ is one to one; moreover for any $J \subset \{1, \dots, N\}$, the number of $i \in \{1, \dots, N\}$ such that $x_i = n_1$ is $|I| - d_{J,N} = N - |J| - d_{J,N}$, and the number of $i \in \{1, \dots, N\}$ such that $x_i = n_2$ is $|J| - d_{J,N}$. Therefore for any $J \subset \{1, \dots, N\}$, $\phi(J)$ corresponds to one of the terms of the form $n_1^{N-|J|-d_{J,N}} n_2^{|J|-d_{J,N}} n_3^{2d_{J,N}}$ in the multinomial expansion of $(n_1+n_2+n_3)^N$. Since ϕ is injective and there exists a one to one correspondence between $\{n_1, n_2, n_3\}^N$ and the terms in the expansion of $(n_1+n_2+n_3)^N$, we conclude that

$$\sum_{J \subset \{1,\dots,N\}} n_1^{N-|J|-d_{J,N}} n_2^{|J|-d_{J,N}} n_3^{2d_{J,N}} \leq (n_1 + n_2 + n_3)^N$$

and the proof is finished. □

Applying the above lemma with $n_1 = D_1$, $n_2 = D_2$, and $n_3 = \sqrt{\frac{C_0 D_2}{\theta^{mn}}}$, we conclude that $A_x^N\left(kt, \frac{r}{32\sqrt{mn}}, U^c\right)$ can be covered with at most

$$\begin{aligned} &\frac{C_0}{\theta^{2mn}} e^{mn(m+n)Nkt} \left(D_1 + D_2 + \sqrt{\frac{C_0 D_2}{\theta^{mn}}}\right)^N \\ \underset{(5.5),(5.6)}{=} &\frac{C_0}{\theta^{2mn}} e^{mn(m+n)Nkt} \left(1 - K_1 \mu(\sigma_{2\sqrt{mn}\theta} U) + \frac{K_2 e^{-\lambda kt}}{r^{mn}} + (k-1) C_1^k e^{-\frac{t}{2}} \right. \\ &\left. + \sqrt{\frac{(k-1) C_0 C_1^k}{\theta^{mn}}} e^{-\frac{t}{4}} \right)^N \\ \leq &\frac{C_0}{\theta^{2mn}} e^{mn(m+n)Nkt} \left(1 - K_1 \mu(\sigma_{2\sqrt{mn}\theta} U) + \frac{K_2 e^{-\lambda kt}}{r^{mn}} + \frac{k-1}{\theta^{mn}} C_3^k e^{-\frac{t}{4}}\right)^N \end{aligned}$$

cubes of side-length $\theta e^{-(m+n)Nkt}$ in $M_{m,n}$, where $C_3 := 2C_1C_0$. The proof of Proposition 5.2 is now complete. □

6. An intermediate dimension bound

Recall that we are given $a > 0$ and a non-empty open $U \subset X$, and our goal is to estimate the Hausdorff dimension of $E(F_a^+, U)$ from above. The following technical theorem shows how to express $E(F_a^+, U)$ as the union of two sets, taking into account the behavior of trajectories with respect to the family $\{Q_t\}$ constructed in the previous section, and estimate their dimension separately.

Theorem 6.1. *Let $\{Q_t\}_{t>0}$ of X be as in (5.1). Then:*

(1) *There exists $C_1 \geq 1$ such that for all $t > 2$ and for all $2 \leq k \in \mathbb{N}$, the set*

$$S(k,t,x) := \{h \in H : g_{Nkt}hx \in Q_t^c \ \ \forall N \in \mathbb{N}\} \tag{6.1}$$

satisfies

$$\operatorname{codim} S(k,t,x) \geq \frac{1}{(m+n)k}\left(\frac{1}{2} - \frac{\log\left((k-1)C_1^k\right)}{t}\right). \tag{6.2}$$

(2) *There exist $p \geq m+n$, $0 < r_2 < \frac{1}{16\sqrt{mn}}$, $0 < C_2 \leq 1$ and $b_0, b, K_1, K_2, C_3, \lambda > 0$ such that for all $t \in a\mathbb{N}$ with $t > 2$, all $2 \leq k \in \mathbb{N}$, all r satisfying*

$$e^{\frac{b_0 - kt}{b}} \leq r \leq \min(C_2 e^{-pt}, r_2), \tag{6.3}$$

all $\theta \in \left[4r, \frac{1}{2\sqrt{mn}}\right]$, all $x \in X$, and for all open subsets U of X we have

$$\begin{aligned} &\operatorname{codim}\left(\{h \in H \smallsetminus S(k,t,x) : hx \in E(F_a^+, U)\}\right) \\ &\geq \frac{K_1\mu\left(\sigma_{2\sqrt{mn}\theta}U\right) - \frac{K_2 e^{-\lambda kt}}{r^{mn}} - \frac{k-1}{\theta^{mn}}C_3^k e^{-t/4}}{kt(m+n)}. \end{aligned} \tag{6.4}$$

Informally speaking, $S(k,t,x)$ is the set of $h \in H$ such that along some arithmetic sequence (of times which are multiples of kt) the orbit of hx visits complements of large compact subsets of G. The dimension of $S(k,t,x)$ and the dimension of the set $\{h \in H \smallsetminus S(k,t,x) : hx \in E(F_a^+, U)\}$ are estimated separately.

Proof of Theorem 6.1. Take $\{Q_t\}_{t>0}$ as in (5.1), and let U be an open subset of X.

Proof of (1): Let $t > 2$, and take $2 \leq k \in \mathbb{N}$ and $x \in X$. Our goal is to find an upper bound for the Hausdorff dimension of the set $S(k,t,x)$ defined in (6.1); equivalently,

$$\dim S(k,t,x) = \dim\left\{s \in M_{m,n} : g_{Nkt}h_s x \in Q_t^c \ \ \forall N \in \mathbb{N}\right\}.$$

In view of the countable stability of Hausdorff dimension it suffices to estimate the dimension of

$$\left\{s \in B(1/2) : g_{Nkt}h_s x \in Q_t^c \ \ \forall N \in \mathbb{N}\right\},$$

which, due to (5.1), coincides with $\bigcap_{N\in\mathbb{N}} A_x^N\left(kt, 1/2, X^t_{>C_\alpha^3 e^{mnt}}\right)$.

Applying Corollary 4.5 with $M = C_\alpha^3 e^{2mnt}$, we get for any $x \in X$ and for any $0 < \theta \leq \frac{1}{\sqrt{mn}}$:

$$\begin{aligned}
&\dim \bigcap_{N\in\mathbb{N}} A_x^N\left(kt, 1/2, X^t_{>C_\alpha^3 e^{mnt}}\right) \\
&\leq \lim_{N\to\infty} \frac{\log \frac{C_\alpha}{\theta^{mn}}(k-1)^N C_1^{kN} e^{(mn(m+n)Nk-\frac{N}{2})t}\cdot \frac{\max(\tilde{\alpha}^t(x),1)}{M}}{-\log\theta e^{-(m+n)Nkt}} \\
&= \frac{\log(k-1)C_1^k e^{(mn(m+n)k-\frac{1}{2})t}}{kt(m+n)} \\
&= mn - \frac{1}{(m+n)}\cdot\left(\frac{1}{2k} - \frac{\log(k-1)}{kt} - \frac{\log C_1}{t}\right) \\
&= mn - \frac{1}{k(m+n)}\cdot\left(\frac{1}{2} - \frac{\log\left(C_1^k(k-1)\right)}{t}\right),
\end{aligned}$$

where C_1 is as in Corollary 4.5.

Proof of (2): Let $a > 0$, $2 \leq k \in \mathbb{N}$, $x \in X$, and let $t = \ell a$ for some $\ell \in \mathbb{N}$. Our goal is to find an upper bound for the Hausdorff dimension of the set

$$\{h \in H \smallsetminus S(k,t,x) : hx \in E(F_a^+, U)\}.$$

Recall that

$$S(k,t,x)^c = \{h \in H : g_{Nkt}hx \in Q_t \text{ for some } N \in \mathbb{N}\}.$$

Therefore

$$\begin{aligned}
&\{h \in H \smallsetminus S(k,t,x) : hx \in E(F_a^+, U)\} \\
&= \left\{h \in H : hx \in E(F_a^+, U) \bigcap \left(\bigcup_{N\in\mathbb{N}} g_{-Nkt}Q_t\right)\right\} \\
&\subset \left\{h \in H : hx \in \bigcup_{N\in\mathbb{N}} g_{-Nkt}\big(Q_t \cap E(F_a^+, U)\big)\right\}.
\end{aligned}$$

Now suppose that $t \geq 2$, and let $N \in \mathbb{N}$ and $r > 0$ be such that (6.3) is satisfied, where b_0, b, C_2, r_2 are as in Lemma 5.2. Similar to the proof of part (1) and in view of countable stability of Hausdorff dimension it suffices to find an upper bound for the dimension of the set

$$E'_{N,x,r} := \left\{s \in B\left(\frac{re^{-(m+n)Nkt}}{32\sqrt{mn}}\right) : h_s x \in g_{-Nkt}\big(Q_t \cap E(F_a^+, U)\big)\right\}$$

for any $x \in X$. Now let $x \in X$ and $s \in E'_{N,x,r}$. Then

$$\begin{aligned}
g_{ikt}g_{Nkt}h_s x &= g_{ikt}(g_{Nkt}h_s g_{-Nkt})g_{Nkt}x \\
&= g_{ikt}h_{e^{(m+n)Nkt}s}(g_{Nkt}x) \in U^c \quad \forall i \in \mathbb{N},
\end{aligned}$$

and at the same time $e^{(m+n)Nkt}s \in B\left(\frac{r}{32\sqrt{mn}}\right)$. It follows that

$$E'_{N,x,r} \subset e^{-(m+n)Nkt}\left(\bigcap_{i\in\mathbb{N}} A^i_{g_{Nkt}x}\left(kt, \frac{r}{32\sqrt{mn}}, U^c\right)\right). \tag{6.5}$$

It is easy to see that if $E'_{N,x,r}$ is non-empty, then $g_{Nkt}x \in \partial_{\frac{r}{32\sqrt{mn}}}(Q_t \cap U^c)$. Now take K_1, K_2, C_0, C_3, λ as in Lemma 5.2. By Lemma 5.2 applied to x replaced with $g_{Nkt}x$, and using the fact that the Hausdorff dimension is preserved by homotheties, we have for any $\theta \in \left[4r, \frac{1}{2\sqrt{mn}}\right]$:

$$\begin{aligned}
\dim E'_{N,x,r} &\underset{(6.5)}{\leq} \dim\left(e^{-(m+n)Nkt}\left(\bigcap_{i\in\mathbb{N}} A^i_{g_{Nkt}x}\left(kt, \frac{r}{32\sqrt{mn}}, U^c\right)\right)\right) \\
&= \dim\left(\bigcap_{i\in\mathbb{N}} A^i_{g_{Nkt}x}\left(kt, \frac{r}{32\sqrt{mn}}, U^c\right)\right) \\
&\leq \lim_{i\to\infty} \frac{\log\left(\frac{C_0}{\theta^{2mn}}e^{mn(m+n)Nkt}\left(1 - K_1\mu\big(\sigma_{2\sqrt{mn}\theta}U\big) + \frac{K_2e^{-\lambda kt}}{r^{mn}} + \frac{k-1}{\theta^{mn}}C_3^k e^{-\frac{t}{4}}\right)^i\right)}{-\log\theta e^{-(m+n)ikt}} \\
&\leq mn - \frac{-\log\left(1 - K_1\mu\big(\sigma_{2\sqrt{mn}\theta}(U)\big) + \frac{K_2e^{-\lambda kt}}{r^{mn}} + \frac{k-1}{\theta^{mn}}C_3^k e^{-\frac{t}{4}}\right)}{(m+n)kt} \\
&\leq mn - \frac{K_1\mu\big(\sigma_{2\sqrt{mn}\theta}(U)\big) - \frac{K_2e^{-\lambda kt}}{r^{mn}} - \frac{k-1}{\theta^{mn}}C_3^k e^{-\frac{t}{4}}}{(m+n)kt}.
\end{aligned}$$

This finishes the proof. □

7. Theorem 6.1 ⇒ Theorem 1.2 ⇒ Theorem 1.1 ⇒ applications

We begin with a remark that

$$\widetilde{E}(F_a^+, U) = \bigcup_{j\in N} g_{-aj}E(F_a^+, U),$$

hence if an upper estimate for $\dim E(F_a^+, U)$ is proved, the same estimate holds for $\widetilde{E}(F_a^+, U)$ because of the countable stability of Hausdorff dimension and its invariance under diffeomorphisms. The same argument applies to

$$\{h \in H : hx \in \widetilde{E}(F_a^+, U)\} = \bigcup_{j\in N} g_{-aj}\{h \in H : hg_{aj}x \in \widetilde{E}(F_a^+, U)\}g_{aj}.$$

Therefore it is enough to prove Theorems 1.1 and 1.2 with $E(F_a^+, U)$ in place of $\widetilde{E}(F_a^+, U)$.

We now show how the two parts of Theorem 6.1 are put together.

Proof of Theorem 1.2. Let $x \in X$ and $a > 0$. Recall that we are given the constants $p, r_2, b, K_1, K_2, C_1, C_2, C_2, \lambda$ and a family of compact sets $\{Q_t\}_{t>0}$ such that statements (1) and (2) of Theorem 6.1 hold. To apply the theorem we need to choose $k \in \mathbb{N}$ and $t \in a\mathbb{N}$. Here is how to do it. First define

$$k := \left\lceil \max\left(\frac{4p}{m+n}, \frac{2p(mn+2)}{\lambda}, 4bp\right)\right\rceil \tag{7.1}$$

(note that $k \geq 4$ since $p \geq m+n$), and then choose $t_1 := \max\big(K_1, 4\log\big((k-1)C_1^k\big)\big)$. We remark that $t_1 \geq 4\log 3 > 4$, since $C_1 \geq 1$ and $k \geq 4$. Statement (1) of Theorem 6.1 readily implies that

$$\operatorname{codim} S(k,t,x) \geq \frac{1}{4k(m+n)} \tag{7.2}$$

whenever $t \geq t_1$. Now let

$$c := C_2, \tag{7.3}$$

$$\begin{aligned} r_3 := \min\Big(&c^2 e^{-b_0/b},\ c^{mn+2}\frac{K_1}{8K_2}, \\ &c^3\left(\frac{K_1}{8(k-1)C_3^k}\right)^{24p}, ce^{-2pt_1}, \big(\frac{1}{2\sqrt{mn}}\big)^{24pmn}, r_2\Big) \end{aligned} \tag{7.4}$$

$$r_1 := r_3^{\frac{1}{24pmn}}, \tag{7.5}$$

and set

$$r := r(U,a)^{24pmn} \quad \text{and} \quad t := a\left\lceil \frac{1}{2ap}\log\frac{c}{r}\right\rceil, \tag{7.6}$$

where $r(U,a)$ is defined by (1.8). Note that in view of (1.8), (7.5) and (7.6) one has

$$r \leq r_3. \tag{7.7}$$

Also, it follows from (7.6) that

$$ce^{-2pt} \leq r \leq ce^{-2p(t-a)}. \tag{7.8}$$

Moreover,

$$t \underset{(7.6)}{\geq} \frac{1}{2p}\log\frac{c}{r} \underset{(7.7)}{\geq} \frac{1}{2p}\log\frac{c}{r_3} \underset{(7.4)}{\geq} t_1,$$

and

$$t \underset{(7.6)}{\geq} \frac{1}{2p} \log \frac{c}{r} \underset{(1.8),(7.6)}{\geq} \frac{1}{2p} \log \frac{c}{ce^{-24apmn}} = 12amn. \tag{7.9}$$

We now claim that the inequalities (6.3) are satisfied. Indeed, the second inequality $r \leq \min(C_2 e^{-pt}, r_2)$ follows immediately because

- $r \leq r_3$ by (7.7), and $r_3 \leq r_2$ by (7.4);
- $r \leq C_2 e^{-2p(t-a)}$ by (7.3) and (7.8), and $t \geq 4a$ by (7.9).

Furthermore, we have

$$e^{\frac{b_0 - kt}{b}} \underset{(7.1)}{\leq} e^{\frac{b_0}{b} - 4pt} \underset{(7.8)}{\leq} \frac{e^{b_0/b}}{c^2} \cdot r^2 \underset{(7.7)}{\leq} \frac{e^{b_0/b} r_3}{c^2} \cdot r \underset{(7.4)}{\leq} r,$$

so the claim follows. We therefore can apply (6.4) to any $\theta \in \left[4r, \frac{1}{2\sqrt{mn}}\right]$. We put $\theta := \min(\theta_U, \frac{1}{2\sqrt{mn}})$, which is not greater than $\frac{1}{2\sqrt{mn}}$ by definition. To show that it not less than $4r$, write

$$\begin{aligned} \theta &\underset{(1.8),(7.7)}{\geq} \min\left(r^{\frac{1}{24pmn}}, \frac{1}{2\sqrt{mn}}\right) \underset{(7.4),(7.7)}{=} r^{\frac{1}{24pmn}} \\ &= \frac{r}{r^{1-\frac{1}{24pmn}}} \underset{(7.7)}{\geq} \frac{r}{r_3^{1-\frac{1}{24pmn}}} \geq \frac{r}{r_3^{1/2}} \underset{(7.4)}{\geq} 4r. \end{aligned}$$

Thus we can conclude that

$$\begin{aligned} &\operatorname{codim}\left(\{h \in H \smallsetminus S(k,t,x) : hx \in E(F_a^+, U)\}\right) \\ \geq\ &\frac{K_1 \mu\left(\sigma_{2\sqrt{mn}\theta} U\right) - \frac{K_2 e^{-\lambda kt}}{r^{mn}} - \frac{k-1}{\theta^{mn}} C_3^k e^{-t/4}}{kt(m+n)}. \end{aligned} \tag{7.10}$$

Observe that since $\theta \leq \theta_U$, $\mu\left(\sigma_{2\sqrt{mn}\theta} U\right)$ is not less than $\mu(U)/2$ by definition of θ_U, see (1.6). We now claim that the numerator in the right hand side of (7.10) is not less than $K_1 \mu(U)/4$. Indeed, we can write

$$\begin{aligned} \frac{k-1}{\theta^{mn}} C_3^k e^{-\frac{t}{4}} &= \frac{k-1}{\theta^{mn}} C_3^k \left(e^{-6pt}\right)^{\frac{1}{24p}} \underset{(7.8)}{\leq} \frac{k-1}{\theta^{mn}} C_3^k \left(\frac{r^3}{c^3}\right)^{\frac{1}{24p}} \\ &\underset{(7.3)}{=} (k-1) C_3^k \left(\frac{r}{c^3}\right)^{\frac{1}{24p}} \cdot \left(\frac{r^{\frac{1}{24pmn}}}{\theta}\right)^{mn} \cdot r^{\frac{1}{24p}} \\ &\underset{(1.8),(7.4),(7.6),(7.7)}{\leq} (k-1) C_3^k \left(\frac{r_3}{c^3}\right)^{\frac{1}{24p}} \cdot 1 \cdot \mu(U) \underset{(7.4)}{\leq} \frac{K_1}{8} \mu(U) \end{aligned}$$

and

$$\begin{aligned}\frac{K_2 e^{-\lambda k t}}{r^{mn}} &\underset{(7.1)}{\leq} \frac{K_2 e^{-\lambda\cdot\frac{2p(mn+2)}{\lambda}t}}{r^{mn}} = \frac{K_2 (e^{-2pt})^{mn+2}}{r^{mn}} \underset{(7.8)}{\leq} \frac{K_2\left(\frac{r}{c}\right)^{mn+2}}{r^{mn}} \\ &= K_2 \frac{r}{c^{mn+2}}\cdot r \underset{(1.8),\,(7.4),\,(7.6),\,(7.7)}{\leq} K_2\cdot\frac{r_3}{c^{mn+2}}\cdot\mu(U)^{24pmn} \\ &\underset{(7.4)}{\leq} K_2\cdot\frac{K_1}{8K_2}\cdot\mu(U)^{24pmn} \leq \frac{K_1}{8}\mu(U).\end{aligned}$$

Thus (7.10) implies

$$\operatorname{codim}\left(\{h \in H \smallsetminus S(k,t,x) : hx \in E(F_a^+, U)\}\right) \geq \frac{K_1\mu(U)}{4kt(m+n)} \underset{(6.3)}{\geq} \frac{K_1\mu(U)}{4k(m+n)\cdot\frac{1}{p}\log\frac{c}{r}};$$

hence, using (7.2), we get

$$\operatorname{codim}\left(\{h \in H : hx \in E(F_a^+, U)\}\right) \geq \frac{1}{4k(m+n)}\min\left(1, \frac{pK_1\mu(U)}{\log\frac{c}{r}}\right).$$

Finally, we claim that the minimum in the right hand side of the above inequality is equal to $\frac{pK_1\mu(U)}{\log\frac{c}{r}}$. Indeed,

$$r \underset{(6.3)}{\leq} ce^{-pt} \leq ce^{-pt_1} < ce^{-pK_1} \implies \log\frac{c}{r} \geq K_1 p \implies \frac{pK_1\mu(U)}{\log\frac{c}{r}} < 1.$$

Therefore

$$\begin{aligned}\operatorname{codim}\left(\{h \in H : hx \in E(F_a^+, U)\}\right) &\geq \frac{pK_1\mu(U)}{4k(m+n)\cdot\log\frac{c}{r}} \\ &\underset{(c=C_2\leq 1)}{\geq} \frac{pK_1}{4k(m+n)}\cdot\frac{\mu(U)}{\log\frac{1}{r}} \\ &\underset{(7.6)}{=} \frac{K_1}{96kmn(m+n)}\cdot\frac{\mu(U)}{\log\frac{1}{r(U,a)}}.\end{aligned}$$

This finishes the proof. $\square$

Proof of Theorem 1.1. Denote by $\tilde{H}$ the weak stable horospherical subgroup with respect to F^+ defined by

$$\tilde{H} := \left\{\begin{bmatrix} s' & 0 \\ s & s''\end{bmatrix} : s \in M_{n,m},\ s' \in M_{m,m},\ s'' \in M_{n,n},\ \det(s')\det(s'') = 1\right\}.$$

Let U be an open subset of X. Choose $\eta > 0$ sufficiently small so that for any $0 < r < \eta$ the following conditions are satisfied:

$$\begin{aligned}&\mu\big(\sigma_{r/2}U\big) \geq \mu\big(U\big)/2,\\ &\theta_{\sigma_{r/2}U} \geq \frac{1}{2}\theta_U,\end{aligned} \tag{7.11}$$

where θ_U is as in (1.6). We choose $r' > 0$ and $0 < r < \eta$ sufficiently small such that the following properties are satisfied:

(1) Every $g \in B^G(r')$ can be written as $g = h'h$, where $h' \in B^{\tilde{H}}(r/4)$ and $h \in B^H(r/4)$.

(2)
$$g_t B^{\tilde{H}}(r) g_{-t} \subset B^{\tilde{H}}(2r) \text{ for any } 0 < r < \eta \text{ and } t \geq 0 \tag{7.12}$$

(this can be done since for any $t \geq 0$ the restriction of the map $g \to g_t g g_{-t}$ to $\tilde{H}$ is non-expanding).

For $x \in X$ denote

$$E_{x,r'} := \big\{g \in B^G(r') : gx \in E(F_a^+, U)\big\}.$$

Clearly $E(F_a^+, U)$ can be covered by countably many sets of type $\{gx : g \in E_{x,r'}\}$. Thus, in view of the countable stability of Hausdorff dimension, in order to prove the theorem it suffices to show that for any $x \in X$,

$$\operatorname{codim} E_{x,r'} \gg \frac{\mu(U)}{\log \frac{1}{r(U,a)}},$$

where $r(U, a)$ is as in (1.8) and c, r_1 are as in Theorem 1.2.

Now let $g \in B^G(r')$ and suppose $g = h'h$, where $h' \in B^{\tilde{H}}(r/4)$ and $h \in B^H(r/4)$, then for any $y \in X$ and any $t > 0$ we can write

$$\begin{aligned}\operatorname{dist}(g_t g x, y) &\leq \operatorname{dist}(g_t h' h x, g_t h x) + \operatorname{dist}(g_t h x, y)\\ &= \operatorname{dist}\big(g_t h' g_{-t} g_t h x, g_t h x\big) + \operatorname{dist}(g_t h x, y) \underset{(7.12)}{\leq} r/2 + \operatorname{dist}(g_t h x, y).\end{aligned}$$

Hence $g \in E_{x,r'}$ implies that hx belongs to $E(F_a^+, \sigma_{r/2}U)$, and by using Wegmann's Product Theorem [37] we conclude that:

$$\begin{aligned}\dim E_{x,r'} &\leq \dim\Big(\{h \in B^H(r/4) : hx \in E(F_a^+, \sigma_{r/2}U)\} \times B^{\tilde{H}}(r/4)\Big)\\ &\leq \dim\big(\{h \in B^H(r/4) : hx \in E(F_a^+, \sigma_{r/2}U)\}\big) + \dim \tilde{H}\\ &\leq \dim\big(\{h \in H : hx \in E(F_a^+, \sigma_{r/2}U)\}\big) + \dim \tilde{H}\end{aligned} \tag{7.13}$$

Note that by (7.11) we have:

$$\begin{aligned} r(\sigma_{r/2}U, a) &= \min\left(\mu(\sigma_{r/2}U), \theta_{\sigma_{r/2}U}, ce^{-a}, r_1\right) \\ &\geq \min\left(\mu(U)/2, \frac{\theta_U}{2}, ce^{-a}, r_1\right) \geq \frac{1}{2}\cdot r(U,a). \end{aligned} \tag{7.14}$$

Therefore, by Theorem 1.2 applied U replaced by $\sigma_{r/2}U$ and in view of (7.14) and (7.13) we get

$$\begin{aligned} \operatorname{codim} E_{x,r'} &\geq \operatorname{codim}\left(\{h \in H : hx \in E(F_a^+, \sigma_{r/2}U)\}\right) \\ &\gg \frac{\mu(\sigma_{r/2}U)}{\log \frac{1}{r(\sigma_{r/2}U,a)}} \geq \frac{\frac{1}{2}\mu(U)}{\log \frac{2}{r(U,a)}} \underset{r(U,a) \leq r_1 \leq 1/2}{\geq} \frac{1}{4}\cdot \frac{\mu(U)}{\log \frac{1}{r(U,a)}} \end{aligned}$$

This ends the proof of the theorem. □

Proof of Corollary 1.3. Let S be a k-dimensional smooth embedded submanifold of X, which we can assume to be compact. Then it is easy to see that one can find $\varepsilon_0, \varkappa_1, \varkappa_2 > 0$ such that

$$\mu\big(\partial_\varepsilon S\big) \geq \varkappa_1 \varepsilon^{\dim X - k}$$

and

$$\theta_{\partial_\varepsilon S} \geq \varkappa_2 \varepsilon$$

for any $0 < \varepsilon < \varepsilon_0$. Hence, in view of (1.8),

$$r(\partial_\varepsilon S, a) \geq \min\left(r_1, \varkappa_1 \varepsilon^{\dim X - k}, \varkappa_2\varepsilon, ce^{-a}\right),$$

where r_1, c are as in Theorem 1.1. Therefore, if we denote

$$\varkappa_0 := \min\left(\varkappa_1^{\dim X - k}, \varkappa_2\right) \text{ and } p_0 = \max\left(\dim X - k, 1\right),$$

we will have $r(\partial_\varepsilon S, a) \geq \varkappa_0 \varepsilon^{p_0}$ as long as $\varkappa_0 \varepsilon^{p_0} < \min\left(r_1, ce^{-a}\right)$. By Theorem 1.1 applied with $U = \partial_\varepsilon S$ for ε as above we have

$$\operatorname{codim} E(F_a^+, \partial_\varepsilon S) \gg \frac{\mu(\partial_\varepsilon S)}{\log\left(\frac{1}{r(\partial_\varepsilon S, a)}\right)} \geq \frac{\varkappa_2 \varepsilon^{\dim X - k}}{\log\left(\frac{1}{\varkappa_0 \varepsilon^{p_0}}\right)},$$

which implies (1.9) for a suitable choice of ε_S, c_S, and C_S. The 'in addition' part is proved along similar lines and is left to the reader. □

Proof of Theorem 1.5. Recall that X can be identified with the space of unimodular lattices in $\mathbb{R}^{m+n}$. It was essentially observed by Davenport and Schmidt in [10] (see also [6,21,27,29] for other instances of the so-called *Dani Correspondence*) that the c-Dirichlet improvability of $s \in M_{m,n}$ can be restated in terms of a certain property of the F^+-trajectory of the lattice $h_s\mathbb{Z}^{m+n}$. Specifically, suppose that (1.11) holds for some $N \geq 1$ and $0 < c < 1$, and take t such that

$$e^{mt}(cQ^{-n/m}) = e^{-nt}Q; \tag{7.15}$$

then both sides of (7.15) are equal to $c^{\frac{m}{m+n}}$, and hence, in view of (1.3), $s \in \mathbf{DI}_{m,n}(c)$ implies that for all large enough $t > 0$ the lattice $g_t h_s \mathbb{Z}^{m+n}$ has a non-zero vector of supremum norm $< c^{\frac{m}{m+n}}$. This in turn implies that $h_s\mathbb{Z}^{m+n} \in \widetilde{E}(F^+, U_c)$, where

$$U_c = \left\{x \in X : \|\mathbf{v}\|_\infty > c^{\frac{m}{m+n}} \text{ for all } \mathbf{v} \in x \smallsetminus \{0\}\right\}.$$

The latter is an open subset of X which, for example, contains a small enough neighborhood of the standard lattice $\mathbb{Z}^{m+n}$. An application of Theorem 1.2 shows that the codimension of $\mathbf{DI}_{m,n}(c)$ in $M_{m,n}$ is positive. □

We remark that, as explained in [26, Remark 6], a combination of the methods from the present paper with measure estimates obtained in [26] can produce an effective estimate for the codimension of $\mathbf{DI}_{m,n}(c)$. We refer the reader to [3] for some recent results on the set of Dirichlet improvable vectors, and to [25] for an extension of the problem of improving Dirichlet's theorem to the set-up of arbitrary norms on $\mathbb{R}^{m+n}$.

8. Concluding remarks

8.1. *More precise estimates for* $\dim E(F^+, U)$

Studying trajectories missing a given open subset has been a notable theme in ergodic theory. Such a set-up is often referred to as 'open dynamics' or 'systems with holes', see e.g. [15,16] and references therein. In particular, [15, Theorem 1.2] considers a conformal repeller supporting a Gibbs measure and gives an asymptotic formula for the set of points missing a ball of radius ε, showing the codimension to be asymptotically (as $\varepsilon \to 0$) proportional to the measure of the ball. A similar formula was obtained by Hensley [18] in the setting of continued fractions. See also [8] for a modern treatment of the subject.

In view of these results one can expect that in our set-up the codimension of $E(F^+, U)$ should also be asymptotically (as $\mu(U) \to 0$) proportional to the measure of U. In other words, conjecturally there should not be any logarithmic term in the right side of (1.7). However it is not clear how to improve our upper bound, as well as how to obtain a complimentary lower bound for $\dim E(F^+, U)$ using the exponential mixing of the action or any other method. The only known result supporting this conjecture in a partially hyperbolic setting is a theorem of Simmons [34] which establishes the asymptotics for the codimension of $E(F^+, U)$ in the set-up (1.2)–(1.3) and with U being a complement of a large compact subset of X.

8.2. Large deviations in homogeneous spaces

Let $X = G/\Gamma$ be an arbitrary finite volume homogeneous space, let μ be a G-invariant probability measure on X, and let $F^+ = \{g_t\}_{t\geq 0}$ be a one-parameter subsemigroup of G acting ergodically on (X, μ). Given an open subset U of X and $0 < \delta \leq 1$, let us say that a point $x \in X$ *δ-escapes U on average with respect to F^+* if x belongs to

$$E_\delta(F^+, U) := \left\{ x \in X : \limsup_{T\to\infty} \frac{1}{T} \int_0^T 1_{U^c}(g_t x)\, dt \geq \delta \right\},$$

that is, to the set of points in X whose orbit spends at least δ-proportion of time in U^c. Note that for any $0 < \delta \leq 1$ we have

$$E(F^+, U) \subset E_\delta(F^+, U), \tag{8.1}$$

which means that the sets $E_\delta(F^+, U)$ are larger compared to $E(F^+, U)$; hence their dimension is greater than or equal to dimension of $E(F^+, U)$. Birkhoff's Ergodic theorem implies

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T 1_{U^c}(g_t x) dt = \mu(U^c) \text{ for almost all } x \in X.$$

Hence, the set $E_\delta(F^+, U)$ has full measure for any $0 < \delta \leq \mu(U^c)$, and has zero measure whenever $\mu(U^c) < \delta \leq 1$. This motivates estimating the Hausdorff dimension of $E_\delta(F^+, U)$ for $\mu(U^c) < \delta \leq 1$.

Now let F^+ be Ad-diagonalizable, and let H be a subgroup of G with the Effective Equidistribution Property (EEP) with respect to F^+. In a forthcoming work, by obtaining an explicit upper bound for $\dim E_\delta(F^+, U)$, we plan to prove that for any non-empty open subset U of X there exists $\delta_U \in [\mu(U^c), 1)$ such that for any $\delta_U < \delta \leq 1$ we have $\dim E_\delta(F^+, U) < \dim X$. This, in view of (8.1), strengthens the main result of [23]. A similar result was proved in [19] in the set-up (1.2)–(1.3) for trajectories divergent on average; see also [1,31] for extensions.

8.3. Dimension drop conjecture for arbitrary homogeneous spaces and arbitrary flows

As we saw in this paper, height functions on the space of lattices provide a powerful tool for studying orbits which spend a large proportion of time in the cusp neighborhoods. The construction of such functions for arbitrary homogeneous spaces was given by Eskin and Margulis in [12], see also [14] for a survey. This can be used to control geodesic excursions into cusps in any homogeneous space. For example, Guan and Shi in [17] used a generalized version of the Eskin-Margulis function to extend the methods employed in [19] to arbitrary homogeneous spaces and show that the set of points with divergent on average trajectories has less than full Hausdorff dimension. By taking a similar approach, and by combining the methods of this paper with those of [12] and [17], one can solve the Dimension Drop Conjecture in much wider generality. This is the subject of the follow-up paper [24].

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