MEASURE THEORETIC LAWS FOR LIMSUP SETS DEFINED BY RECTANGLES

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Abstract. In this paper, we present a general principle for the Lebesgue measure theory of limsup sets defined by rectangles under the hypothesis of *ubiquity for rectangles*. Applications are given for simultaneous Diophantine approximation, Khintchine-Groshev theorem for rectangles and higher dimensional shrinking target problems.

1. Introduction

Metric Diophantine approximation originated from the study of how well a real number can be approximated by rationals in the sense of measure and dimension. The pioneer works of Khintchine [29], Jarník [28], Groshev [22], Sprindzuk [38] and Schmidt [36] address the metric theory of sets in classic Diophantine approximation. A key development is the introduction of the terminology of *regular systems* by Baker & Schmidt [3] in 1970, which has opened up a possibility about general principles for the metric theory of limsup sets including the Lebesgue measure theory and Hausdorff measure/dimension theory.

Definition 1.1 (Regular system). *Let I be an interval in $\mathbb{R}$ and let $\Gamma = \{\gamma_n\}_{n\geq 1}$ be a sequence of real numbers in I, together with a positive function $\mathcal{N} : \Gamma \to \mathbb{R}_+$. Call $(\Gamma, \mathcal{N})$ a regular system, if for any subinterval J of length $|J|$ there exists an integer K_J such that for any $K \geq K_J$, there exist elements $\gamma_{n_1}, \dots, \gamma_{n_t}$ in $\Gamma \cap J$ such that*

$$t \geq c(\Gamma)\cdot |J| \cdot K, \quad \text{and} \quad \mathcal{N}(\gamma_{n_i}) \leq K, \ |\gamma_{n_i} - \gamma_{n_j}| \geq K^{-1} \text{ for all } 1 \leq i \neq j \leq t,$$

where $c(\Gamma)$ is an absolute constant.

Equipped with the assumption of $(\Gamma, \mathcal{N})$ being a regular system, one is able to set up a complete metric theory for sets of the form

$$\big\{x \in I : |x - \gamma| < \psi\big(\mathcal{N}(\gamma)\big) \text{ for infinitely many } \gamma \in \Gamma\big\}.$$

See Baker & Schmidt [3] and Beresnevich [4] where Lebesgue and Hausdorff measures of those sets were computed. Special cases include approximation of real numbers by rational and algebraic numbers.

The notion of regular systems in $\mathbb{R}$ was extended to ubiquitous systems in $\mathbb{R}^d$ by Dodson, Rynne & Vickers [20] in 1990 to establish a general principle for the study of Hausdorff measures of limsup sets in higher dimensional spaces. This includes Diophantine approximation for systems of linear forms and beyond. In 2006 the notion of ubiquitous systems in $\mathbb{R}^d$ was further generalized to the setting of abstract metric spaces by Beresnevich, Dickinson & Velani [7]. Here we name it as *ubiquity for balls*.

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Definition 1.2 (Ubiquity for balls)**.** *Let X be a locally compact metric space with a finite Borel measure μ. Let $\{\mathfrak{R}_\alpha : \alpha \in J\}$ be a sequence of subsets in X and $\beta : J \to \mathbb{R}_+$ be a positive function attaching a weight to $\alpha \in J$. Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be a function (henceforth referred to as a ubiquitous function), and let $\{\ell_n, u_n : n \geq 1\}$ be two sequences of positive numbers with $u_n \geq \ell_n \to \infty$ as $n \to \infty$. Call $\big(\{\mathfrak{R}_\alpha\}_{\alpha\in J}, \rho\big)$ a ubiquitous system with respect to $\{\ell_n, u_n : n \geq 1\}$ if for any ball $B \subset X$ there exists $n_o(B) \in \mathbb{N}$ such that for all $n \geq n_o(B)$*

$$\mu\left(B \cap \bigcup_{\alpha\in J:\ell_n\leq\beta_\alpha\leq u_n} \Delta\big(\mathfrak{R}_\alpha, \rho(u_n)\big)\right) \geq c(J)\cdot\mu(B), \tag{1.1}$$

where $\Delta(\mathfrak{R}_\alpha, \epsilon)$ denotes the ϵ-neighborhood of $\mathfrak{R}_\alpha$, and $c(J)$ is an absolute constant.

Equipped with the assumption of ubiquity, Beresnevich, Dickinson & Velani [7] established a complete metric theory on the set of the form

$$\mathcal{W}(\psi) := \Big\{x \in X : x \in \Delta\big(\mathfrak{R}_\alpha, \psi(\beta_\alpha)\big) \text{ for infinitely many } \alpha \in J\Big\}.$$

This included Hausdorff measures as well as other measures satisfying some mild and natural conditions. Instead of citing the full generality of the measure theory in [7], we state the following special case (all the necessary notation will be explained in §2.1).

Theorem 1.3 (Beresnevich, Dickinson & Velani [7])**.** *Assume the δ-Ahlfors regularity for a measure μ on X, the κ-scaling property for every resonant set $\mathfrak{R}_\alpha$ with $\alpha \in J$, and the ubiquity for balls with respect the function ρ and the sequences $\{\ell_n, u_n : n \geq 1\}$. Assume that ψ is decreasing and that either ψ or ρ is λ-regular for some $0 < \lambda < 1$. Then*

$$\mu\big(\mathcal{W}(\psi)\big) = \mu(X) \quad \textit{if} \quad \sum_{n\geq1}\left(\frac{\psi(u_n)}{\rho(u_n)}\right)^{\delta(1-\kappa)} = \infty.$$

Amongst many of applications of Theorem 1.3 in [7], we cite one application to the classical Diophantine approximation, that is the Khintchine-Groshev theorem.

Corollary 1.4. [7, p.66] *Let $\psi : \mathbb{N} \to \mathbb{R}_+$ be a non-increasing positive function. Then the set*

$$\Big\{A \in [0,1]^{dh} : \|A_i\mathbf{q}\| < \psi(|\mathbf{q}|),\ \forall\ 1 \leq i \leq d,\ \textit{for infinitely many } \mathbf{q} \in \mathbb{Z}^h\Big\}$$

is of Lebesgue measure zero or one according to

$$\sum_{q=1}^{\infty} q^{h-1}\psi(q)^d < \infty \ \textit{or} \ = \infty.$$

Here A_i are the rows of A, $|\mathbf{q}| = \max_{1\leq k\leq h}|q_k|$, and $\|x\|$ stands for the distance of a real number x from integers. See also [9] for the most advanced progress on Khintchine-Groshev theorem, where the approximation function ψ can be extended to multi-variable case.

Proof. The divergence part follows from Theorem 1.3 directly, while the convergence part can be obtained easily by the convergence part of Borel-Cantelli lemma. The same situation for convergence part applies to all other examples given in this paper. □

These general principles have become fundamental in metric number theory. In addition to the wide usage in classical Diophantine approximation (for example [6, 10, 11, 12, 16, 18, 19, 32, 34]), they are also used in Diophantine approximation of p-adic fields and formal power fields [17, 31], as well as Diophantine approximation on manifolds (for example [2, 5, 8, 14]).

It should be noted that all these principles are designed to attack the metric theory of limsup sets defined by balls, that is, starting from Dirichlet's theorem in simultaneous Diophantine approximation. However, more generally one can take Minkowski's theorem, see [35, 37], as a starting point. Here is a special case:

For any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, any non-negative vector $(\hat{a}_1, \dots, \hat{a}_d)$ with $\hat{a}_1 + \cdots + \hat{a}_d = 1$, and any $Q \in \mathbb{N}$, there exists an integer $1 \le q \le Q$ such that

$$\|qx_i\| < Q^{-\hat{a}_i},\ 1 \le i \le d.$$

Consequently, by letting $a_i = 1 + \hat{a}_i$, $1 \le i \le d$, there exist infinitely many integer vectors $(p_1, \dots, p_d, q)$ such that

$$|x_i - p_i/q| < q^{-a_i},\ 1 \le i \le d. \tag{1.2}$$

This means that all real vectors will fall into a sequence of rectangles centered at rational vectors infinitely often.

Minkowski's theorem provides a more profound understanding on the distribution of rational vectors, which works sufficiently well in high dimensional Diophantine approximation compared with Dirichlet's theorem (for example, Minkowski's theorem intervenes as an essential tool in Diophantine approximation on manifolds, see e.g. [14, 8]). So it should be valuable to study the improvement based on Minkowski's theorem, or more precisely, to consider the metric theory of limsup sets defined by rectangles. However, besides some specific examples found in the work of Sprindzuk [38], Schmidt [36], Gallagher [21] and Hussain & Yusupova [25, 26], Thorn [39], Fischler, Hussain, Kristensen & Levesley [23] on linear forms, no general principles for determining Lebesgue measure of those limsup sets have been put forth. So the study of limsup sets defined by rectangles lags much behind the study of limsup sets defined by balls. In this paper we hope to push this forward by presenting a general principle for the measure theory of limsup sets defined by rectangles.

The Organization of the Paper. In §2 we present the setting, our main result (Theorem 2.5) and applications (Theorem 2.6, Theorem 2.7, Corollary 2.8). We prove Theorem 2.5 in §3, and in the next three sections discuss its applications. In §4 we establish Theorem 2.6 and in fact prove a more general statement, where $[0, 1]^d$ is replaced by the product of Cantor sets defined by digit restrictions. Theorem 2.7 is proved in §6, and before that in §5 we present a streamlined proof of Corollary 2.8.

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2. The Framework and Main Results

In this section, we describe our general framework and state the main result of the paper. In fact, one of the major tasks is to find suitable assumptions which could possibly catch the nature for the metric theory of limsup sets defined by rectangles.

2.1. **The Framework.** Throughout, fix an integer $d \ge 1$. Let $(X_i, \mathrm{dist}_i, \mu_i)$ be a bounded locally compact metric space with μ_i a Borel probability measure and dist_i a metric on X_i for each $1 \le i \le d$. We consider the product space (X, dist, μ), where

$$X = \prod_{i=1}^{d} X_i;\ \mu = \prod_{i=1}^{d} \mu_i;\ \ \mathrm{dist}(\mathbf{x}, \mathbf{y}) = \max_{1 \le i \le d} \mathrm{dist}_i(x_i, y_i),$$

for $\mathbf{x} = (x_1, \dots, x_d)$, $\mathbf{y} = (y_1, \dots, y_d)$ in X. A rectangle $R(\mathbf{x}, \mathbf{r})$ in X is the product of balls in $\{X_i\}_{1\le i\le d}$, i.e.

$$R(\mathbf{x},\mathbf{r}) = \prod_{i=1}^{d} B(x_i, r_i), \quad \text{for } \mathbf{x} = (x_1, \dots, x_d),\ \mathbf{r} = (r_1, \cdots, r_d).$$

- Let J be an infinite countable index set and let $\beta : J \to \mathbb{R}_+$ be a positive function such that for any $M > 1$, $\{\alpha \in J : |\beta(\alpha)| < M\}$ is finite;
- Let $\{\ell_n, u_n : n \ge 1\}$ be two sequences of positive numbers such that $u_n \ge \ell_n \to \infty$ as $n \to \infty$, and define
$$J_n = \{\alpha \in J : \ell_n \le \beta(\alpha) \le u_n\}.$$
- Let $\boldsymbol{\rho} = (\rho_1, \dots, \rho_d)$ be a d-tuple of functions with $\rho_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $\rho_i(u) \to 0$ as $u \to \infty$ for each $1 \le i \le d$.

For each $1 \le i \le d$, let $\{\mathfrak{R}_{\alpha,i} : \alpha \in J\}$ be a sequence of subsets of X_i. The resonant sets in X we are considering are

$$\Big\{\mathfrak{R}_\alpha = \prod_{i=1}^{d} \mathfrak{R}_{\alpha,i}, \quad \alpha \in J\Big\}.$$

For any $\mathbf{r} = (r_1, \dots, r_d)$, denote a rectangle-like set as

$$\Delta(\mathfrak{R}_\alpha, \mathbf{r}) = \prod_{i=1}^{d} \Delta(\mathfrak{R}_{\alpha,i}, r_i),$$

where $\Delta(\mathfrak{R}_{\alpha,i}, r_i)$ is the r_i-neighborhood of $\mathfrak{R}_{\alpha,i}$ in X_i.

Let $\Psi = (\psi_1, \dots, \psi_d)$ be a d-tuple of positive functions defined on $\mathbb{R}_+$. The set we would like to describe is:

$$\mathcal{W}(\Psi) = \Big\{\mathbf{x} \in X : \mathbf{x} \in \prod_{i=1}^{d} \Delta\Big(\mathfrak{R}_{\alpha,i}, \psi_i\big(\beta(\alpha)\big)\Big), \text{ i.m. } \alpha \in J\Big\},$$

i.e. the set of points which fall into the '*rectangle*' $\prod_{i=1}^{d} \Delta(\mathfrak{R}_{\alpha,i}, \psi_i(\beta_\alpha))$ for infinitely many $\alpha \in J$. For abbreviation, *i.m.* denotes *infinitely many.*

Next we impose some regularity assumptions on the measures μ_i and the resonant sets $\mathfrak{R}_{\alpha,i}$.

Definition 2.1 (δ-Alhfors regularity). *Let Ω be a complete metric space and ν be a Borel probability measure. Call ν Ahlfors regular with exponent δ if there exist constants $c > 0$ and $r_0 > 0$ such that for any $x \in \Omega$ and $r < r_0$,*

$$c^{-1} r^\delta \le \nu\big(B(x, r)\big) \le c r^\delta.$$

In the following, we will assume the measure μ_i to be δ_i-Ahlfors regular for each $1 \le i \le d$. We will also require the resonant sets to have a special form generalizing the Euclidean space set-up where the resonant sets are points or, more generally, affine subspaces.

Definition 2.2 (κ_i-scaling property). *Let $1 \le i \le d$ and $0 \le \kappa_i < 1$. Say that $\{\mathfrak{R}_{\alpha,i}\}_{\alpha\in J}$ has a κ_i-scaling property if for any $\alpha \in J$ and any ball $B(x_i, r)$ in X_i with center $x_i \in \mathfrak{R}_{\alpha,i}$ and $0 < \epsilon < r$, one has*

$$c^{-1} \cdot r^{\delta_i \kappa_i} \cdot \epsilon^{\delta_i(1-\kappa_i)} \le \mu_i\big(B(x_i, r) \cap \Delta(\mathfrak{R}_{\alpha,i}, \epsilon)\big) \le c \cdot r^{\delta_i \kappa_i} \cdot \epsilon^{\delta_i(1-\kappa_i)}$$

for some absolute constant $c > 0$.

We list some examples for which the κ_i-scaling property holds.

(1) For each $\alpha \in J$, the ith coordinate $\mathfrak{R}_{\alpha,i}$ is a point in X_i, so $\kappa_i = 0$.

(2) Let $X_i = \mathbb{R}^n$. For each $\alpha \in J$, the ith coordinate $\mathfrak{R}_{\alpha,i}$ is an l-dimensional affine subspace in X_i, so $\delta_i = n$ and $\kappa_i = l/n$.

(3) Let $X_i = \mathbb{R}^n$, and for all $\alpha \in J$ let $\mathfrak{R}_{\alpha,i}$ be an l-dimensional smooth compact manifold embedded in X_i. Then $\delta_i = n$ and $\kappa_i = l/n$.

(4) Let $X_i = \mathbb{R}^n$ and for all $\alpha \in J$ let $\mathfrak{R}_{\alpha,i}$ be a self-similar set of Hausdorff dimension l satisfying the open set condition. Then $\delta_i = n$ and $\kappa_i = l/n$.

For a proof of the scaling property in the last two examples, one is referred to Allen & Baker [1].

Definition 2.3 (λ-regularity). *Let $0 < \lambda < 1$. A function φ is said to be λ-regular with respect to the sequence $\{u_n\}_{n\geq 1}$ if*

$$\varphi(u_{n+1}) \leq \lambda \cdot \varphi(u_n) \quad \text{for all} \quad n \gg 1.$$

2.2. **Ubiquitous systems for rectangles.** The ubiquity condition for balls (1.1) is mainly rooted in Dirichlet's theorem in Diophantine approximation [7]. Thus, as far as the metric theory of limsup sets defined by rectangles is concerned, it is reasonable to expect that Minkowski's theorem should intervene in some form. The following notion of *"ubiquity for rectangles"* is designed to catch the nature of the rectangles inspired by Minkowski's theorem. It first appeared in the previous work of the second-named author with Xu and Wu [40, 41], where the Hausdorff measures of limsup sets defined by rectangles were investigated.

Definition 2.4 (Ubiquity for rectangles). *Call $(\{\mathfrak{R}_\alpha\}_{\alpha\in J}, \beta)$ a ubiquitous system for rectangles with respect to the function $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_d)$ and the sequences $\{\ell_n, u_n : n \geq 1\}$ if there exist a constant $c > 0$ such that for any ball B in X*

$$\mu\left(B \cap \bigcup_{\alpha\in J_n} \prod_{i=1}^{d} \Delta\big(\mathfrak{R}_{\alpha,i}, \rho_i(u_n)\big)\right) \geq c \cdot \mu(B) \quad \text{for all} \quad n \geq n_o(B). \tag{2.1}$$

Our main result is the following general principle for the measure theory of limsup sets defined by rectangles, which together with the study of Hausdorff measures developed in [41] provides a rather complete metric theory for this set-up (under the ubiquity hypothesis). For ease of notation, for two d-tuples of functions $\boldsymbol{\rho}$ and Ψ, we write

$$\begin{aligned} \boldsymbol{\rho} \text{ is } \lambda\text{-regular} &\iff \rho_i \text{ is } \lambda\text{-regular for all } 1 \leq i \leq d; \\ \Psi(u) \leq \boldsymbol{\rho}(u) &\iff \psi_i(u) \leq \rho_i(u) \text{ for all } 1 \leq i \leq d. \end{aligned}$$

Theorem 2.5 (Measure theory). *Let $(\{\mathfrak{R}_\alpha\}_{\alpha\in J}, \beta)$ be a ubiquitous system for rectangles with respect to $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_d)$ and $\{\ell_n, u_n : n \geq 1\}$. Assume the δ_i-Ahlfors regularity for μ_i for every $1 \leq i \leq d$, the κ_i-scaling property for every $\mathfrak{R}_{\alpha,i}$ with $\alpha \in J$ and $1 \leq i \leq d$. Also assume that Ψ is decreasing, that either Ψ or $\boldsymbol{\rho}$ is λ-regular with respect to $\{u_n\}_{n\geq 1}$, and that $\Psi(u_n) \leq \boldsymbol{\rho}(u_n)$ for all $n \gg 1$. Then*

$$\mu\big(\mathcal{W}(\Psi)\big) = \mu(X) \quad \text{if} \quad \sum_{n\geq 1} \prod_{i=1}^{d} \left(\frac{\psi_i(u_n)}{\rho_i(u_n)}\right)^{\delta_i(1-\kappa_i)} = \infty.$$

The first application of our result is a solution to a simple shrinking target problem. It can be proved by the exponential mixing property of the underlying dynamics, however with the help of Theorem 2.5 one can see that the proof uses only very basic arithmetic features of the system.

Theorem 2.6. *Let $b_1, \ldots, b_d \geq 2$ be integers, and let $T_i(x) = b_i x \pmod 1$. Then for any $\mathbf{x}_o \in [0,1]^d$ and a d-tuple Ψ of positive*[1] *functions $\psi_1, \ldots, \psi_d : \mathbb{N} \to \mathbb{R}_+$, the Lebesgue measure of the set*

$$\mathfrak{S}(\Psi) = \Big\{ \mathbf{x} \in [0,1]^d : |T_i^n x_i - x_{o,i}| < \psi_i(n), \ \forall\, 1 \leq i \leq d, \ \textit{i.m. } n \in \mathbb{N} \Big\}$$

is zero or one according to

$$\sum_{n=1}^{\infty} \prod_{i=1}^{d} \psi_i(n) < \infty \ \textit{or} \ = \infty.$$

We next apply our result to systems of linear forms mainly to illustrate the way for choosing the ubiquity function. Let $\varphi = \{\varphi_i\}_{1 \leq i \leq d}$ be a d-tuple of non-increasing positive functions defined on $\mathbb{N}$ with

$$\varphi_i(q) \to 0, \ \text{as } q \to \infty,$$

and let $\Phi = \{\Phi_k\}_{1 \leq k \leq h}$ be an h-tuple of non-decreasing positive functions defined on $\mathbb{N}$ with

$$\Phi_k(u) \to \infty \text{ as } u \to \infty.$$

Consider the following set:

$$\begin{aligned} W(\varphi, \Phi) : = \Big\{ A \in [0,1]^{dh} : \text{the system} & \begin{cases} \|A_i \mathbf{q}\| < \varphi_i(u), 1 \leq i \leq d, \\ |q_k| \leq \Phi_k(u), \quad 1 \leq k \leq h, \end{cases} \\ & \text{has a solution in } \mathbf{q} \in \mathbb{Z}^h \smallsetminus \{0\} \text{ for infinitely many } u \in \mathbb{N} \Big\} \\ = \Big\{ A \in [0,1]^{dh} : \|A_i \mathbf{q}\| < \varphi_i\big(\max\{\Phi_1^{-1}(|q_1|), \ldots, \Phi_h^{-1}(|q_h|)\}\big), & \ \forall\, 1 \leq i \leq d, \\ & \text{for infinitely many } \mathbf{q} \in \mathbb{Z}^h \Big\}. \end{aligned} \tag{2.2}$$

Here $\Phi_k^{-1}(|q_k|)$ is defined as the least integer u_k such that $\Phi_k(u_k) \geq |q_k|$ to avoid the situation when the inverse of Φ_k may not be well defined. An important example is given by $\varphi_i(u) = u^{-r_i}$ and $\Phi_k(u) = u^{s_k}$ for positive numbers $r_1, \ldots, r_h$ and $s_1, \ldots, s_h$; this corresponds to *Diophantine approximation with weights*. Also in what follows we will denote Lebesgue measure on Euclidean spaces by $\mathcal{L}$.

Theorem 2.7. *Assume that there exists an integer $M > 1$ such that for all $n \gg 1$,*

$$c_1 \Phi_k(M^n) \leq \Phi_k(M^{n+1}) \leq c_2 \Phi_k(M^n), \ 1 \leq k \leq h,$$

for some absolute constants $c_1, c_2 > 1$. Then $\mathcal{L}\big(W(\varphi, \Phi)\big)$ is zero or one according to

$$\sum_{q=1}^{\infty} q^{-1} \cdot \prod_{i=1}^{d} \varphi_i(q) \cdot \prod_{k=1}^{h} \Phi_k(q) < \infty \ \textit{or} \ = \infty.$$

It should be mentioned that Sprindžuk [38] established a metric result for systems of linear forms which goes beyond the set-up involving rectangles. Though in Sprindžuk's result only primitive vectors $\mathbf{q} \in \mathbb{Z}^h$ are involved, Theorem 2.7 can be obtained from [38, Chapter 1, Theorem 13] with the help of an elementary calculation. See also Remark 6.3 for a further discussion.

It is instructive to state the special case $h = 1$ of the above theorem. Then one can take $\Phi_1(q) = q$ and thus study the set

$$W(\varphi) := \big\{ \mathbf{x} \in [0,1]^d : \|q x_i\| < \varphi_i(q), \ \forall\, 1 \leq i \leq d, \quad \text{i.m. } q \in \mathbb{N} \big\}.$$

[1] Note that here we do not assume monotonicity of the approximating functions; see the beginning of the proof of Theorem 2.5 for an explanation.

Theorem 2.7 immediately implies

Corollary 2.8. *Let $\varphi = \{\varphi_i\}_{1\le i\le d}$ be as above; then $\mathcal{L}\big(W(\varphi)\big)$ is zero or one according to*

$$\sum_{q=1}^{\infty}\prod_{i=1}^{d}\varphi_i(q) < \infty \ \text{or}\ = \infty.$$

Remark 2.9. The necessity of the ubiquity assumption in Theorem 2.5 can be justified by alluding to a result of Boshernitzan & Chaika [13] about Borel-Cantelli sequences. According to [13, Theorem 2], if $\{x_n : n \in \mathbb{N}\} \subset [0,1]$ is a sequence such that for some ball $B \subset [0,1]$ and for any $\epsilon > 0$ there exists N_ϵ such that

$$\mathcal{L}\Big(B \cap \bigcup_{n\le N_\epsilon} B(x_n, N_\epsilon^{-1})\Big) \le \epsilon\cdot\mathcal{L}(B), \tag{2.3}$$

then there exists a non-increasing positive function ψ such that

$$\sum_{n=1}^{\infty}\psi(n) = \infty,\ \text{but}\ \ \mathcal{L}\big(\{x\in B : |x - x_n| < \psi(n),\ \text{i.m. } n\in\mathbb{N}\}\big) = 0.$$

Though the negation of (2.3) is slightly weaker than the regularity or ubiquity of the corresponding system, at least to some extent it verifies the necessity of the ubiquity assumption in our main result.

At the end of this section, we fix some notation.

- $\mathfrak{R}_\alpha$: a resonant set.
- $\widetilde{R}$: big rectangles; R: small rectangles.
- $5B$ or $5R$: a ball B or a rectangle R scaled by a factor of 5, that is, the ball/rectangle with the same center but with radius or side lengths multiplied by 5.
- c, c_i: absolute constants;
- $a \ll b$: $a \le cb$ for an unspecified constant $c > 0$ which is fixed throughout the argument; $a \asymp b$: $a \ll b$ and $b \ll a$;
- r_B: the radius of a ball B.

3. Proof of the Main Result

To estimate the measure of a limsup set from below, the following Chung-Erdös inequality [15] is widely used.

Lemma 3.1 (Chung-Erdös inequality [15], see also [30])**.** *Let $(\Omega, \mathcal{B}, \nu)$ be a a finite measure space, and let $\{E_n\}_{n\ge1}$ be a sequence of measurable sets. If $\sum_{n\ge1}\nu(E_n) = \infty$, then for any $N_0\in\mathbb{N}$,*

$$\nu\left(\limsup_{n\to\infty} E_n\right) \ge \limsup_{N\to\infty}\frac{\Big(\sum_{N_0\le n\le N}\nu(E_n)\Big)^2}{\sum_{N_0\le i\ne j\le N}\nu(E_i\cap E_j)}.$$

The Chung-Erdös lemma enables one to conclude the positivity of the measure of a set. In applications, to arrive at a full measure result, one often uses the Chung-Erdös lemma restricted to a local set and then applies the following proposition, which is a generalization of the Lebesgue density theorem. We recall that a Borel measure ν on a metric space Ω is called *doubling* if $\exists\, c > 0$ such that for any $x\in\Omega$ and $r > 0$, $\nu\big(B(x, 2r)\big) \le c\cdot\nu\big(B(x,r)\big)$.

Lemma 3.2 ([7])**.** *Let Ω be a metric space, and let ν be a finite doubling Borel measure on Ω. Let E be a Borel subset of Ω. Assume that there are constants r_0 and $c > 0$ such that*

$$\nu(E\cap B) \ge c\cdot\nu(B)\ \ \text{for any ball}\ \ B\subset\Omega \text{ with } r_B < r_0.$$

Then E has full measure in Ω, i.e. $\nu(\Omega \smallsetminus E) = 0$.

The following $5r$-covering lemma for rectangles will be used frequently later. Generally speaking, there are no such covering lemmas for arbitrary rectangles compared with the ones for balls; some additional assumptions on the rectangles are needed. In the product space $(X, |\cdot|)$, we say that two rectangles

$$\prod_{i=1}^{d} B(x_i, r_i), \quad \prod_{i=1}^{d} B(y_i, \epsilon_i)$$

are uniform in size if

$$r_i \geq \epsilon_i \text{ for some } 1 \leq i \leq d \Longrightarrow r_i \geq \epsilon_i, \text{ for all } 1 \leq i \leq d.$$

Lemma 3.3. *Let (X, dist) be the product of the metric spaces (X_i, dist_i) for $1 \leq i \leq d$. Every family $\mathcal{G}$ of rectangles which are uniform in size and have bounded diameters in X contains a disjoint subfamily $\mathcal{F}$ such that*

$$\bigcup_{R \in \mathcal{G}} R \subset \bigcup_{R \in \mathcal{F}} 5R.$$

The proof applies the same idea for the classical $5r$-covering lemma for balls, so we omit the proof here. For a proof of the $5r$-covering lemma for balls one is referred to [24, Theorem 1.2] for the case of general metric spaces or to [33, Theorem 2.1] for a constructive proof.

Proof of Theorem 2.5. We will apply the Chung-Erdös inequality to $\mathcal{W}(\Psi)$ restricted to an arbitrary ball. Fix a ball $B \subset X$. By the monotonicity of Ψ, it is trivial that

$$\begin{aligned} \mathcal{W}(\Psi) \cap B &= \limsup_{n \to \infty} \left(B \cap \bigcup_{\alpha \in J_n} \prod_{i=1}^{d} \Delta\Big(\mathfrak{R}_{\alpha,i}, \psi_i(\beta_\alpha)\Big) \right) \\ &\supset \limsup_{n \to \infty} \left(B \cap \bigcup_{\alpha \in J_n} \prod_{i=1}^{d} \Delta\Big(\mathfrak{R}_{\alpha,i}, \psi_i(u_n)\Big) \right). \end{aligned} \tag{3.1}$$

We note that this is the only point where monotonicity of Ψ is used. That is why we do not require monotonicity in Theorem 2.6, since (3.1) will be an equality in that case.

Without loss of generality, we can assume that $\rho_i(u_n) \geq 5\psi_i(u_n)$ for all $n \geq 1$ and $1 \leq i \leq d$, since otherwise we just replace ψ_i by $\frac{1}{5}\psi_i$. Recall the λ-regularity. Let N_0 be an integer such that

$$\boldsymbol{\rho}(u_{n+1}) \leq \lambda \boldsymbol{\rho}(u_n), \text{ for all } n \geq N_0, \text{ or } \Psi(u_{n+1}) \leq \lambda \Psi(u_n) \text{ for all } n \geq N_0.$$

Step 1. For each n, cover the intersection

$$\frac{1}{2} B \cap \bigcup_{\alpha \in J_n} \prod_{i=1}^{d} \Delta\Big(\mathfrak{R}_{\alpha,i}, \rho_i(u_n)\Big)$$

by the collection of rectangles in X of the following form:

$$\left\{ \widetilde{\mathcal{R}} = \prod_{i=1}^{d} B\big(x_i, \rho_i(u_n)\big) : \mathbf{x} = (x_1, \ldots, x_d) \in \mathfrak{R}_\alpha, \ \alpha \in J_n \right\}.$$

Then one can use the $5r$-covering lemma for these rectangles (it is clear that the uniformity in size condition is satisfied) to choose a certain subfamily $\mathcal{F}_n$ of those rectangles $\widetilde{\mathcal{R}}$. Denoting by $\mathcal{A}_n$ the collection of their centers, we can guarantee that these rectangles satisfy the following two assumptions:

- (1) Disjointness

$$5\widetilde{\mathcal{R}}\cap 5\widetilde{\mathcal{R}}'=\left(\prod_{i=1}^d B\big(x_i,5\rho_i(u_n)\big)\right)\cap\left(\prod_{i=1}^d B\big(x_i',5\rho_i(u_n)\big)\right)=\emptyset,\quad \text{for any } \mathbf{x}\neq\mathbf{x}'\in\mathcal{A}_n;$$

- (2) Almost packing

$$\frac{1}{2}B\cap\bigcup_{\alpha\in J_n}\prod_{i=1}^d\Delta\big(R_{\alpha,i},\rho_i(u_n)\big)\subset\bigcup_{\widetilde{\mathcal{R}}\in\mathcal{F}_n}5\widetilde{\mathcal{R}}=\bigcup_{\mathbf{x}\in\mathcal{A}_n}\prod_{i=1}^d B\big(x_i,5\rho_i(u_n)\big)\subset B.$$

  Thus by a measure computation argument, together with the ubiquity property applied to $\frac{1}{2}B$, there is an integer $n_o(B)$ such that for all $n\geq n_o(B)$ one has

$$\#\mathcal{A}_n\asymp\prod_{i=1}^d\left(\frac{r_B}{\rho_i(u_n)}\right)^{\delta_i}.$$

We will refer to the rectangles $\widetilde{\mathcal{R}}$ in $\mathcal{F}_n$ as *big rectangles* of level n.

Step 2. We intend to construct a subset of $\mathcal{W}(\Psi)$. Fix a rectangle $\widetilde{\mathcal{R}}\in\mathcal{F}_n$ centered at $\mathbf{x}\in\mathcal{A}_n$. Let $\mathfrak{R}_\alpha$ be a resonant set containing $\mathbf{x}$ for some $\alpha\in J_n$ (if there are multiple of these α, we only choose and fix one). Then we consider the set

$$\widetilde{\mathcal{R}}\cap\prod_{i=1}^d\Delta\big(\mathfrak{R}_{\alpha,i},\psi_i(u_n)\big)=\prod_{i=1}^d\Big[B\big(x_i,\rho_i(u_n)\big)\cap\Delta\big(\mathfrak{R}_{\alpha,i},\psi_i(u_n)\big)\Big].$$

We can cover it by rectangles of the form

$$\mathcal{R}=\prod_{i=1}^d B\big(z_i,\psi_i(u_n)\big)$$

with centers $\mathbf{z}=(z_1,\ldots,z_d)$ in $\mathfrak{R}_\alpha$. Again applying the $5r$-covering lemma, we get a certain subfamily $\mathcal{C}(\widetilde{\mathcal{R}})$ of rectangles. Denoting the collection of their centers by $\mathcal{C}(\mathbf{x})$, we see that these rectangles satisfy the following two conditions:

- (1) Disjointness

$$\prod_{i=1}^d B\big(z_i,5\psi_i(u_n)\big)\cap\prod_{i=1}^d B\big(z_i',5\psi_i(u_n)\big)=\emptyset\quad\text{for any } \mathbf{z}\neq\mathbf{z}'\in\mathcal{C}(\mathbf{x});$$

- (2) Almost packing

$$\frac{1}{2}\widetilde{\mathcal{R}}\cap\prod_{i=1}^d\Delta\big(\mathfrak{R}_{\alpha,i},\psi_i(u_n)\big)\subset\bigcup_{\mathbf{z}\in\mathcal{C}(\mathbf{x})}\prod_{i=1}^d B\big(z_i,5\psi_i(u_n)\big)\subset\widetilde{\mathcal{R}}\cap\prod_{i=1}^d\Delta\big(\mathfrak{R}_{\alpha,i},5\psi_i(u_n)\big).$$

  Recall the κ_i-scaling property of $\mathfrak{R}_{\alpha,i}$ for each $1\leq i\leq d$, so still by a measure argument, one has

$$\#\mathcal{C}(\mathbf{x})\asymp\prod_{i=1}^d\left(\frac{\rho_i(u_n)}{\psi_i(u_n)}\right)^{\delta_i\kappa_i}.$$

We will refer to these small rectangles $\mathcal{R}$ as to *shrunk rectangles* of level n. Then define

$$\mathcal{E}_n=\Big\{\mathcal{R}\in\mathcal{C}(\widetilde{\mathcal{R}}):\widetilde{\mathcal{R}}\in\mathcal{F}_n\Big\}=\Big\{\prod_{i=1}^d B\big(z_i,\psi_i(u_n)\big):\mathbf{z}\in\mathcal{C}(\mathbf{x}),\mathbf{x}\in\mathcal{A}_n\Big\}$$

and

$$E_n := \bigcup_{\mathcal{R}\in\mathcal{E}_n} \mathcal{R} = \bigcup_{\widetilde{\mathcal{R}}\in\mathcal{F}_n} \bigcup_{\mathcal{R}\in\mathcal{C}(\widetilde{\mathcal{R}})} \mathcal{R} = \bigcup_{\mathbf{x}\in\mathcal{A}_n} \bigcup_{\mathbf{z}\in\mathcal{C}(\mathbf{x})} \prod_{i=1}^{d} B\big(z_i, \psi_i(u_n)\big).$$

The process of the construction of E_n can be outlined as follows: for a given ball B,

$$\frac{1}{2}B \overset{\text{ubiquity}}{--\longrightarrow} \mathcal{F}_n \text{ or } \mathcal{A}_n : \text{ big rectangles } \widetilde{\mathcal{R}} = \prod_{i=1}^{d} B\big(x_i, \rho_i(u_n)\big)$$

$$\overset{\text{intersect with } \Delta(\mathfrak{R}_\alpha, \psi_i(u_n))}{-----------\longrightarrow} \mathcal{C}(\widetilde{\mathcal{R}}) \text{ or } \mathcal{C}(\mathbf{x}) : \text{ shrunk rectangles } \mathcal{R} = \prod_{i=1}^{d} B\big(z_i, \psi_i(u_n)\big).$$

Clearly

$$B \cap \limsup_{n\to\infty} \left(\bigcup_{\alpha\in J_n} \prod_{i=1}^{d} \Delta\big(\mathfrak{R}_{\alpha,i}, \psi_i(\beta_\alpha)\big) \right) \supset \limsup_{n\to\infty} E_n.$$

The limsup set in the right hand side is the one to which we will apply the Chung-Erdös lemma.

At first, it is easy to see that the measure of E_n can be estimated as follows:

$$\begin{aligned}\mu(E_n) &= \sum_{\mathbf{x}\in\mathcal{A}_n} \#\mathcal{C}(\mathbf{x}) \cdot \prod_{i=1}^{d} \psi_i(u_n)^{\delta_i} = \prod_{i=1}^{d} \left(\frac{r_B}{\rho_i(u_n)}\right)^{\delta_i} \cdot \prod_{i=1}^{d} \left(\frac{\rho_i(u_n)}{\psi_i(u_n)}\right)^{\delta_i \kappa_i} \cdot \prod_{i=1}^{d} \psi_i(u_n)^{\delta_i} \\ &\asymp \mu(B) \cdot \prod_{i=1}^{d} \left(\frac{\psi_i(u_n)}{\rho_i(u_n)}\right)^{\delta_i(1-\kappa_i)}.\end{aligned}$$

So

$$\sum_{n=1}^{\infty} \mu(E_n) = \infty,$$

and then the first condition in the Chung-Erdös lemma is satisfied.

Step 3. We estimate the measure of $E_m \cap E_n$ for $N_0 \le m < n$. Notice that

$$\begin{aligned}\mu(E_m \cap E_n) = \sum_{\mathcal{R}\in\mathcal{E}_m} \mu(\mathcal{R}\cap E_n) &= \sum_{\mathbf{x}\in\mathcal{A}_m} \sum_{\mathbf{z}\in\mathcal{C}(\mathbf{x})} \mu\left(\prod_{i=1}^{d} B\big(z_i, \psi_i(u_m)\big) \cap E_n \right) \\ &= \sum_{\mathbf{x}\in\mathcal{A}_m} \sum_{\mathbf{z}\in\mathcal{C}(\mathbf{x})} \mu\left(\prod_{i=1}^{d} B\big(z_i, \psi_i(u_m)\big) \cap \bigcup_{\mathbf{x}'\in\mathcal{A}_n} \bigcup_{\mathbf{z}'\in\mathcal{C}(\mathbf{x}')} \prod_{i=1}^{d} B\big(z_i', \psi_i(u_n)\big) \right).\end{aligned}$$

Since all the rectangles in $\mathcal{E}_n$ are of the same size, we need only estimate the number of elements in $\mathcal{E}_n$ which can intersect a given element in $\mathcal{E}_m$. So fix an arbitrary rectangle $\mathcal{R}$ in $\mathcal{E}_m$ which is of the form

$$\mathcal{R} = \prod_{i=1}^{d} B\big(z_i, \psi_i(u_m)\big).$$

Recall the construction of E_n. At first, we estimate the number of big rectangles in $\mathcal{F}_n$ which can intersect $\mathcal{R}$. Remember that all the big rectangles in $\mathcal{F}_n$ are of the same side lengths $\big(\rho_1(u_n), \dots, \rho_d(u_n)\big)$. Let

$$I_1 := \big\{1 \le i \le d : \psi_i(u_m) \ge \rho_i(u_n)\big\}, \quad I_2 := \big\{1 \le i \le d : \psi_i(u_m) < \rho_i(u_n)\big\}.$$

Define an enlarged body of the rectangle $\mathcal{R}$:

$$H := \prod_{i=1}^{d} B(z_i, 3\epsilon_i), \text{ where } \epsilon_i := \begin{cases} \psi_i(u_m), & \text{for } i \in I_1; \\ \rho_i(u_n), & \text{for } i \in I_2. \end{cases}$$

Thus all the big rectangles in $\mathcal{F}_n$ which can intersect $\mathcal{R}$ are contained in H. Since these big rectangles in $\mathcal{F}_n$ are disjoint, a measure argument gives that the number of big rectangles in $\mathcal{F}_n$ which can possibly intersect the rectangle $\mathcal{R}$ is bounded from above by

$$\prod_{i\in I_1} \left(\frac{\psi_i(u_m)}{\rho_i(u_n)}\right)^{\delta_i}.$$

Secondly, fix a center $\mathbf{x}' \in \mathcal{A}_n$ or, equivalently, a big rectangle $\widetilde{\mathcal{R}}_n = \prod_{i=1}^{d} B\big(x_i', \rho_i(u_n)\big)$ in $\mathcal{F}_n$ which has non-empty intersection with the rectangle $\mathcal{R}$. We consider the number L of shrunk rectangles in $\mathcal{E}_n$ which can intersect the set

$$\mathcal{R} \cap \widetilde{\mathcal{R}}_n = \prod_{i=1}^{d} B\big(z_i, \psi_i(u_m)\big) \cap \prod_{i=1}^{d} B\big(x_i', \rho_i(u_n)\big).$$

Clearly all these L shrunk rectangles are contained in

$$\prod_{i=1}^{d} B\big(z_i, 2\psi_i(u_m)\big) \cap \prod_{i=1}^{d} B\big(x_i', 2\rho_i(u_n)\big) \cap \prod_{i=1}^{d} \Delta\big(\mathfrak{R}_{\alpha,i}, \psi_i(u_n)\big), \tag{3.2}$$

where $\alpha \in J_n$ is the index for which $\mathbf{x}' = (x_1', \ldots, x_d')$ sits. Then by a measure argument, the number L can be estimated as

$$L \le \frac{\text{the measure of the set (3.2)}}{\text{the measure of a shrunk rectangle}}.$$

It is clear that the intersection of the first two rectangles in (3.2) is contained in a rectangle, say $\overline{R} = \prod_{i=1}^{d} B(y_i, r_i)$, with

$$r_i = \rho_i(u_n), \text{ for } i \in I_1; \text{ and } r_i = \psi_i(u_m), \text{ for } i \notin I_1.$$

Thus the measure of the set in (3.2) is bounded from above by the measure of

$$\overline{R} \cap \prod_{i=1}^{d} \Delta\big(\mathfrak{R}_{\alpha,i}, \psi_i(u_n)\big). \tag{3.3}$$

Then applying the κ-scaling property of $\mathfrak{R}_\alpha$ to the set in (3.3), it follows that

$$\begin{aligned} L &\ll \frac{\prod_{i\in I_1} \rho_i(u_n)^{\delta_i\kappa_i} \cdot \psi_i(u_n)^{\delta_i(1-\kappa_i)} \cdot \prod_{i\in I_2} \psi_i(u_m)^{\delta_i\kappa_i} \cdot \psi_i(u_n)^{\delta_i(1-\kappa_i)}}{\prod_{i=1}^{d} \psi_i(u_n)^{\delta_i}} \\ &= \prod_{i\in I_1} \left(\frac{\rho_i(u_n)}{\psi_i(u_n)}\right)^{\delta_i\kappa_i} \cdot \prod_{i\in I_2} \left(\frac{\psi_i(u_m)}{\psi_i(u_n)}\right)^{\delta_i\kappa_i}. \end{aligned}$$

At last, we can estimate the measure of $E_m \cap E_n$. More precisely,

$$\mu(E_m \cap E_n) \leq \sum_{\mathcal{R}\in\mathcal{E}_m} \prod_{i\in I_1} \left(\frac{\psi_i(u_m)}{\rho_i(u_n)}\right)^{\delta_i} \cdot L \cdot \prod_{i=1}^{d} \psi_i(u_n)^{\delta_i}$$

$$\ll \sum_{\mathcal{R}\in\mathcal{E}_m} \prod_{i\in I_1} \left(\frac{\psi_i(u_m)}{\rho_i(u_n)}\right)^{\delta_i} \cdot \prod_{i\in I_1} \left(\rho_i(u_n)^{\delta_i\kappa_i} \cdot \psi_i(u_n)^{\delta_i(1-\kappa_i)}\right) \cdot \prod_{i\in I_2} \left(\psi_i(u_m)^{\delta_i\kappa_i} \cdot \psi_i(u_n)^{\delta_i(1-\kappa_i)}\right).$$

Recall the number of the elements in $\mathcal{E}_m$. It follows that

$$\mu(E_m \cap E_n) \ll \left[\prod_{i=1}^{d} \left(\frac{r_B}{\rho_i(u_m)}\right)^{\delta_i} \cdot \prod_{i=1}^{d} \left(\frac{\rho_i(u_m)}{\psi_i(u_m)}\right)^{\delta_i\kappa_i}\right] \cdot \prod_{i\in I_1} \left(\frac{\psi_i(u_m)}{\rho_i(u_n)}\right)^{\delta_i} \cdot$$

$$\prod_{i\in I_1} \left(\rho_i(u_n)^{\delta_i\kappa_i} \cdot \psi_i(u_n)^{\delta_i(1-\kappa_i)}\right) \cdot \prod_{i\in I_2} \left(\psi_i(u_m)^{\delta_i\kappa_i} \cdot \psi_i(u_n)^{\delta_i(1-\kappa_i)}\right)$$

(3.4)
$$= \mu(B) \cdot \prod_{i\in I_1} \left(\frac{\psi_i(u_m)}{\rho_i(u_m)}\right)^{\delta_i(1-\kappa_i)} \cdot \prod_{i\in I_2} \left(\frac{\rho_i(u_n)}{\rho_i(u_m)}\right)^{\delta_i(1-\kappa_i)} \cdot \prod_{i=1}^{d} \left(\frac{\psi_i(u_n)}{\rho_i(u_n)}\right)^{\delta_i(1-\kappa_i)}$$

(3.5)
$$= \mu(B) \cdot \prod_{i=1}^{d} \left(\frac{\psi_i(u_m)}{\rho_i(u_m)}\right)^{\delta_i(1-\kappa_i)} \cdot \prod_{i\in I_2} \left(\frac{\psi_i(u_n)}{\psi_i(u_m)}\right)^{\delta_i(1-\kappa_i)} \cdot \prod_{i\in I_1} \left(\frac{\psi_i(u_n)}{\rho_i(u_n)}\right)^{\delta_i(1-\kappa_i)}.$$

When $I_2 = \emptyset$,

$$\mu(E_m \cap E_n) \ll \mu(B) \cdot \prod_{i\in I_1} \left(\frac{\psi_i(u_m)}{\rho_i(u_m)}\right)^{\delta_i(1-\kappa_i)} \cdot \left(\prod_{i=1}^{d} \frac{\psi_i(u_n)}{\rho_i(u_n)}\right)^{\delta_i(1-\kappa_i)} \asymp \mu(E_m) \cdot \mu(E_n) \cdot \mu(B)^{-1}.$$

When $I_2 \neq \emptyset$,

- if $\boldsymbol{\rho}$ is λ-regular, then by (3.4) it follows that

$$\mu(E_m \cap E_n) \ll \mu(B) \cdot \prod_{i\in I_1} \left(\frac{\psi_i(u_m)}{\rho_i(u_m)}\right)^{\delta_i(1-\kappa_i)} \cdot \prod_{i\in I_2} \lambda^{(n-m)\delta_i(1-\kappa_i)} \cdot \left(\prod_{i=1}^{d} \frac{\psi_i(u_n)}{\rho_i(u_n)}\right)^{\delta_i(1-\kappa_i)}$$

$$\leq \mu(B) \cdot \lambda^{(n-m)\epsilon} \cdot \left(\prod_{i=1}^{d} \frac{\psi_i(u_n)}{\rho_i(u_n)}\right)^{\delta_i(1-\kappa_i)} \qquad (\text{by } \psi_i(u_m) \leq \rho_i(u_m))$$

$$\asymp \mu(E_n) \cdot \lambda^{(n-m)\epsilon}.$$

- if Ψ is λ-regular, then by (3.5) it follows that

$$\mu(E_m \cap E_n) \ll \mu(B) \cdot \prod_{i=1}^{d} \left(\frac{\psi_i(u_m)}{\rho_i(u_m)}\right)^{\delta_i(1-\kappa_i)} \cdot \prod_{i\in I_2} \lambda^{(n-m)\delta_i(1-\kappa_i)} \cdot \left(\prod_{i\in I_1} \frac{\psi_i(u_n)}{\rho_i(u_n)}\right)^{\delta_i(1-\kappa_i)}$$

$$\leq \mu(B) \cdot \left(\prod_{i=1}^{d} \frac{\psi_i(u_m)}{\rho_i(u_m)}\right)^{\delta_i(1-\kappa_i)} \cdot \lambda^{(n-m)\epsilon}$$

$$\asymp \mu(E_m) \cdot \lambda^{(n-m)\epsilon}.$$

Step 4. Finally, to apply the Chung-Erdös lemma, we calculate the correlations.

- if $\boldsymbol{\rho}$ is λ-regular,

$$\begin{aligned}\sum_{N_0\le m<n\le N}\mu(E_m\cap E_n)&=\sum_{n=N_0}^{N}\sum_{m=N_0}^{n-1}\mu(E_m\cap E_n)\\&\ll\sum_{n=N_0}^{N}\sum_{m=N_0}^{n-1}\left(\frac{1}{\mu(B)}\cdot\mu(E_m)\cdot\mu(E_n)+\mu(E_n)\cdot\lambda^{(n-m)\epsilon}\right)\\&\ll\frac{1}{\mu(B)}\cdot\left(\sum_{n=N_0}^{N}\mu(E_n)\right)^2+\sum_{n=N_0}^{N}\mu(E_n).\end{aligned}$$

- if Ψ is λ-regular,

$$\begin{aligned}\sum_{N_0\le m<n\le N}\mu(E_m\cap E_n)&=\sum_{m=N_0}^{N}\sum_{n=m+1}^{N}\mu(E_m\cap E_n)\\&\ll\sum_{m=N_0}^{N}\sum_{n=m+1}^{N}\left(\frac{1}{\mu(B)}\cdot\mu(E_m)\cdot\mu(E_n)+\mu(E_m)\cdot\lambda^{(n-m)\epsilon}\right)\\&\ll\frac{1}{\mu(B)}\left(\sum_{m=N_0}^{N}\mu(E_m)\right)^2+\sum_{m=N_0}^{N}\mu(E_m).\end{aligned}$$

In a summary, we have shown

$$\sum_{N_0\le m<n\le N}\mu(E_m\cap E_n)\ll\mu(B)^{-1}\left(\sum_{N_0\le n\le N}\mu(E_n)\right)^2+\sum_{N_0\le n\le N}\mu(E_n).$$

By the Chung-Erdös lemma, it follows that

$$\mu\big(\mathcal{W}(\Psi)\cap B\big)\gg\mu(B),$$

where the constant implied in $\gg$ depends only on λ and the constants implied in the Ahlfors regularity property of the measure μ_i, but is independent on B.

Clearly, the measure μ is Ahlfors regular, hence doubling. Thus by Lemma 3.2, one concludes that $\mathcal{W}(\Psi)$ is of full measure. □

4. A Shrinking Target Problem

Here we are going to prove a statement slightly more general than Theorem 2.6. Let $b_1,\dots,b_d\ge 2$ be a d-tuple of integers. Let

$$\Lambda_i\subset\{0,1,\dots,b_i-1\},\ \text{with}\ \ \#\Lambda_i\ge 2\ \ \text{for}\ \ 1\le i\le d.$$

Then let $\mathcal{C}_i$ be the Cantor sets defined by the iterated function systems

$$\Big\{g_{b_i,k}(x)=\frac{x+k}{b_i},\ x\in[0,1],\ k\in\Lambda_i\Big\}.$$

The natural Cantor measure μ_i supported on $\mathcal{C}_i$ is Ahlfors regular [27] with exponent $\delta_i=\frac{\log\#\Lambda_i}{\log b_i}$.

For d positive functions $\psi_i:\mathbb{R}_+\to\mathbb{R}_+$ $(1\le i\le d)$ and $(x_{o,1},\cdots,x_{o,d})\in\prod_{i=1}^d\mathcal{C}_i$, define

$$M_c(\psi):=\Big\{(x_1,\dots,x_d)\in\prod_{i=1}^{d}\mathcal{C}_i:\|b_i^nx_i-x_{o,i}\|<\psi_i(n),\ 1\le i\le d,\ \text{i.m. }n\in\mathbb{N}\Big\}.$$

We use the symbolic representations of the points x_i in $\mathcal{C}_i$. For each $\mathbf{w}_i = (\epsilon_1, \ldots, \epsilon_n) \in \Lambda_i^n$ with $n \geq 1$, write

$$I_{n,b_i}(\mathbf{w}_i) := g_{b_i,\epsilon_1} \circ g_{b_i,\epsilon_2} \circ \cdots \circ g_{b_i,\epsilon_n}[0,1], \quad x_i(\mathbf{w}_i) = \frac{\epsilon_1}{b_i} + \cdots + \frac{\epsilon_n + x_{o,i}}{b_i^n},$$

in other words $I_{n,b_i}(\mathbf{w}_i)$ is an nth order cylinder with respect to $\mathcal{C}_i$, and $x_i(\mathbf{w}_i)$ is the nth inverse image of $x_{o,i}$ in $I_{n,b_i}(\mathbf{w}_i)$. Note that for any $\mathbf{w}_i$ there is an inverse image of $x_{o,i}$ in $I_{n,b_i}(\mathbf{w}_i)$ and the length of $I_{n,b_i}(\mathbf{w}_i)$ is b_i^{-n}.

Clearly the set $M_c(\psi)$ can be rewritten as

$$M_c(\psi) = \Big\{ x \in \prod_{i=1}^d \mathcal{C}_i : \big| x_i - x_i(\mathbf{w}_i) \big| < \frac{\psi_i(n)}{b_i^n}, \ \mathbf{w}_i \in \Lambda_i^n, 1 \leq i \leq d, \text{ i.m. } n \in \mathbb{N} \Big\}$$

Thus one has

- the index set J:
$$J = \Big\{ \alpha = (\mathbf{w}_1, \ldots, \mathbf{w}_d) \in \prod_{i=1}^d \Lambda_i^n : n \geq 1 \Big\};$$
- the resonant sets $\mathfrak{R}_\alpha$:
$$\mathfrak{R}_\alpha = \big(x_1(\mathbf{w}_1), \ldots, x_d(\mathbf{w}_d) \big) \quad \text{for } \alpha = (\mathbf{w}_1, \ldots, \mathbf{w}_d);$$
- the weight function β_α:
$$\beta_\alpha = n, \quad \text{for } \alpha = (\mathbf{w}_1, \ldots, \mathbf{w}_d) \in \prod_{i=1}^d \Lambda_i^n;$$
- the ubiquitous function ρ_i:
$$\rho_i : \mathbb{R}_+ \to \mathbb{R}_+ : n \to b_i^{-n};$$
- the sequences
$$\ell_n = u_n = n, \ n \geq 1.$$

Proposition 4.1. *The pair* $(\{\mathfrak{R}_\alpha\}_{\alpha \in J}, \beta)$ *is a ubiquitous system for rectangles with respect to the function* $\boldsymbol{\rho}$ *and the sequences* $\{\ell_n, u_n\}_{n \geq 1}$. *Meanwhile, the* κ_i*-scaling property holds with* $\kappa_i = 0$ *for all* $1 \leq i \leq d$.

Proof. This is rather simple since

$$\bigcup_{\ell_n \leq \beta_\alpha \leq u_n} \prod_{i=1}^d \Delta\big(\mathfrak{R}_{\alpha,i}, \rho_i(u_n)\big) = \bigcup_{\mathbf{w}_i \in \Lambda_i^n, 1 \leq i \leq d} \prod_{i=1}^d B\big(x_i(\mathbf{w}_i), b_i^{-n}\big) = \prod_{i=1}^d \mathcal{C}_i.$$

□

It is trivial that ρ_i is b_i^{-1}-regular for each $1 \leq i \leq d$. Then by Theorem 2.5, it follows that

$$\mu\big(M_c(\psi)\big) = 1, \text{ or } 0 \Longleftrightarrow \sum_{n=1}^\infty \prod_{i=1}^d \psi_i(n)^{\delta_i} = \infty \text{ or } < \infty,$$

where the convergence part follows easily from the Borel-Cantelli lemma.

If we choose $\Lambda_i = \{0, \ldots, b_i - 1\}$ for all i, then one has $M_c(\psi) = \mathfrak{S}(\psi)$ and μ is the Lebesgue measure. Then Theorem 2.6 follows.

5. Simultaneous Diophantine approximation

In this section we apply Theorem 2.5 to simultaneous Diophantine approximation, establishing Corollary 2.8 as a warm-up before proving Theorem 2.7.

When equipped with Theorem 2.5, the next task in applications is to search for the right ubiquitous function. A general idea to do so is to enlarge the approximation function φ_i a little bit to the ubiquitous function ρ_i, where the enlarged convex body is of certain volume (for example, of volume 2^d in $\mathbb{R}^d$ to apply Minkowski's theorem). For example see the ubiquitous functions given in (5.3) and (6.5) below.

Recall that we are given a d-tuple $\varphi = \{\varphi_i\}_{1\le i\le d}$ of non-increasing positive functions defined on $\mathbb{N}$ with

$$\varphi_i(q) \to 0, \text{ as } q \to \infty,$$

and our goal is to show that the Lebesgue measure of

$$W(\varphi) = \big\{x \in [0,1]^d : \|qx_i\| < \varphi_i(q),\ 1 \le i \le d, \ \text{ i.m. } q \in \mathbb{N}\big\}.$$

is zero or one according to

$$\sum_{q=1}^{\infty}\prod_{i=1}^{d}\varphi_i(q) < \infty \text{ or } = \infty.$$

Proof of Corollary 2.8. First observe that the following conditions can be assumed without loss of generality:

- for all $q \gg 1$,

$$q\prod_{i=1}^{d}\varphi_i(q) \le 1 \tag{5.1}$$

  (otherwise by Minkowski's convex body theorem, $W(\varphi) = [0,1]^d$);
- for all $1 \le i \le d$,

$$\varphi_i(q) \ge q^{-1-\frac{1}{2d}}, \text{ and so } q^{d+1}\cdot\prod_{i=1}^{d}\varphi_i(q) \ge q^{1/2} \to \infty. \tag{5.2}$$

  Otherwise, we define

$$\overline{\varphi}_i(q) = \max\{\varphi_i(q), q^{-1-\frac{1}{2d}}\},$$

  and consider the set $W(\overline{\varphi})$. For any $x \in W(\overline{\varphi}) \setminus W(\varphi)$, there exists an index $1 \le i \le d$ such that

$$\|qx_i\| < q^{-1-\frac{1}{2d}}, \text{ for infinitely many } q \in \mathbb{N}.$$

  Thus, by the Borel-Cantelli Lemma, the above set is Lebesgue null, and it follows that

$$\mathcal{L}\big(W(\overline{\varphi})\big) = \mathcal{L}\big(W(\varphi)\big).$$

Now we will check that all the conditions in Theorem 2.5 are satisfied by a suitable choice of the ubiquitous function $\boldsymbol{\rho}$.

- the index and resonant sets:

$$J = \big\{(q, p_1, \dots, p_d) : q \in \mathbb{N},\ 0 \le p_i \le q,\ 1 \le i \le d\big\},$$
$$\mathfrak{R}_\alpha = \left(\frac{p_1}{q}, \dots, \frac{p_d}{q}\right) \quad \text{and} \quad \beta_\alpha = q, \quad \text{for} \quad \alpha = (q, p_1, \dots, p_d);$$

- $\mu_i = \mathcal{L}$, which is Ahlfors regular with $\delta_i = 1$;
- κ_i-scaling: $\kappa_i = 0$ since $\mathfrak{R}_{\alpha,i}$ are points for all $1 \le i \le d$ and $\alpha \in J$;

- the approximating function:

$$\psi_i(q) = \frac{\varphi_i(q)}{q}, \quad 1 \le i \le d;$$

- the ubiquitous fucntion: let

$$\rho_i(q) = \frac{\varphi_i(q)}{q} \cdot \left(q \prod_{i=1}^{d} \varphi_i(q) \right)^{-1/d}, \ 1 \le i \le d; \tag{5.3}$$

- $u_n = M^n$ and $\ell_n = M^{n-1}$ with $M \ge 2^{2d+3}$.

Lemma 5.1 (Ubiquity for rectangles)**.** *With the notation above, the pair* $(\{\mathfrak{R}_\alpha\}_{\alpha \in J}, \beta)$ *is a ubiquitous system with respect to the function* $\boldsymbol{\rho}$ *and the sequences* $\{\ell_n, u_n\}_{n \ge 1}$*.*

Proof. We will give a detailed proof for the case of linear forms later, see Lemma 6.2. Then Lemma 5.1 follows by taking $h = 1$ and $\Phi(q) = q$ there. □

Lemma 5.2. *For all* $1 \le i \le d$*,*

$$\psi_i(q) \le \rho_i(q), \quad \rho_i(q) \to 0, \quad \psi_i \textit{ is } M^{-1}\textit{-regular with respect to } \{M^n\}_{n \ge 1}.$$

Proof. The first inequality is clear by (5.1) and (5.3). For the third condition, by the monotonicity of φ_i, one has

$$\psi_i(M^{n+1}) = \frac{\varphi_i(M^{n+1})}{M^{n+1}} \le \frac{\varphi_i(M^n)}{M^{n+1}} = \frac{1}{M} \cdot \psi_i(M^n).$$

For the second one, replacing $\varphi_i(q)$ by 1 in (5.3), it suffices to show that

$$q^{d+1} \prod_{i=1}^{d} \varphi_i(q) \to \infty, \text{ as } q \to \infty,$$

which follows from (5.2). □

At last, we notice that

$$\sum_{n=1}^{\infty} \prod_{i=1}^{d} \frac{\psi_i(u_n)}{\rho_i(u_n)} = \sum_{n=1}^{\infty} M^n \prod_{i=1}^{d} \varphi_i(M^n) \asymp \sum_{q=1}^{\infty} \prod_{i=1}^{d} \varphi_i(q),$$

where the monotonicity of φ_i is used for each $1 \le i \le d$. Thus all the conditions in Theorem 2.5 are satisfied, and then it yields that

$$\mathcal{L}\big(W(\Phi)\big) = 1 \text{ if } \sum_{q=1}^{\infty} \prod_{i=1}^{d} \varphi(q) = \infty.$$

The convergence part of Corollary 2.8 follows from the convergence part of the Borel-Cantelli Lemma, which finishes the proof of the corollary. □

6. Systems of Linear Forms

In this section, we prove Theorem 2.7 by applying Theorem 2.5. The main task is to find the suitable ubiquitous function. Recall that $\{\varphi_i\}_{1 \le i \le d}$ are d non-increasing positive functions defined on $\mathbb{N}$ with

$$\varphi_i(u) \to 0, \text{ as } u \to \infty,$$

$\{\Phi_k\}_{1 \le k \le h}$ are h non-decreasing positive functions with

$$\Phi_k : \mathbb{N} \to \mathbb{R}_+, \quad \Phi_k(u) \to \infty, \text{ as } u \to \infty,$$

and we are considering the set (2.2).

We begin with a technical lemma which enables us to choose the ubiquitous functions fulfilling the conditions that $\rho_i(u) \to 0$ and $\psi_i \le \rho_i$.

Lemma 6.1. *Assume that there exists an integer $M > 1$ such that for all $n \gg 1$,*

$$c_1\Phi_k(M^n) \le \Phi_k(M^{n+1}) \le c_2\Phi_k(M^n),\ 1 \le k \le h$$

for some absolute constants $c_1, c_2 > 1$. As far as the measure of $W(\varphi, \Phi)$ is concerned, we can assume, without loss of generality, that

$$\Big(\max_{1\le k\le h} \Phi_k(u)\Big)^d \cdot \prod_{k=1}^h \Phi_k(u) \cdot \prod_{i=1}^d \varphi_i(u) \to \infty \text{ when } u \to \infty. \tag{6.1}$$

Proof. We define a new collection of functions $\widetilde{\varphi}_i$ for $1 \le i \le d$ satisfying the condition (6.1) and

$$\mathcal{L}\big(W(\widetilde{\varphi}, \Phi)\big) = \mathcal{L}\big(W(\varphi, \Phi)\big).$$

Fix an increasing function f, say $f(u) = \big(\max_{1\le k\le h} \Phi_k(u)\big)^\epsilon$ for example, which tends to infinity with a slow speed as $u \to \infty$, and check that

$$\frac{f(u)}{\Big(\max_{1\le k\le h} \Phi_k(u)\Big)^d \cdot \prod_{k=1}^h \Phi_k(u)} \text{ is decreasing with respect to } u.$$

Partition the integers $\mathbb{N}$ into two classes:

$$\mathcal{N}_1 = \Big\{u \in \mathbb{N} : \Big(\max_{1\le k\le h} \Phi_k(u)\Big)^d \cdot \prod_{k=1}^h \Phi_k(u) \cdot \prod_{i=1}^d \varphi_i(u) \ge f(u)\Big\}$$

and its complement.

We can assume $\mathcal{N}_1$ to be non-empty by redefining $\varphi_i(1)$ and $\Phi_i(1)$ so that they are large enough. Let u_0 be the smallest element in $\mathcal{N}_1$. We define a new collection of functions $\widetilde{\varphi}_i$ for $u \ge u_0$ with $1 \le i \le d$ inductively as follows:

(1) For $u = u_0$, define

$$\widetilde{\varphi}_i(u) = \varphi_i(u), \quad \text{for all } 1 \le i \le d;$$

(2) Let $u = u_0 + 1$.

- if $u \in \mathcal{N}_1$, define

$$\widetilde{\varphi}_i(u) = \varphi_i(u), \quad \text{for all } 1 \le i \le d;$$

- if $u \notin \mathcal{N}_1$, we increase the value of $\varphi_i(u)$ as follows. Define a function

$$G(t) = \prod_{i=1}^d \Big(t\widetilde{\varphi}_i(u_0) + (1-t)\varphi_i(u)\Big), \quad \text{for } t \in [0,1].$$

Then

$$\begin{aligned} G(1) = \prod_{i=1}^d \widetilde{\varphi}_i(u_0) &\ge \frac{f(u_0)}{\Big(\max_{1\le k\le h} \Phi_k(u_0)\Big)^d \cdot \prod_{k=1}^h \Phi_k(u_0)} \\ &\ge \frac{f(u)}{\Big(\max_{1\le k\le h} \Phi_k(u)\Big)^d \cdot \prod_{k=1}^h \Phi_k(u)} > \prod_{i=1}^d \varphi_i(u) = G(0), \end{aligned}$$

where the last inequality holds because $u \notin \mathcal{N}_1$. So there exists some $t^* \in [0,1]$ such that

$$G(t^*) = \frac{f(u)}{\Big(\max_{1\le k\le h}\Phi_k(u)\Big)^d \cdot \prod_{k=1}^h \Phi_k(u)}.$$

Thus define

$$\widetilde{\varphi}_i(u) = t^*\widetilde{\varphi}_i(u_0) + (1-t^*)\varphi_i(u), \quad 1\le i\le d.$$

It is clear that

$$\widetilde{\varphi}_i(u_0) \ge \widetilde{\varphi}_i(u) \ge \varphi_i(u). \tag{6.2}$$

(3) Assume that $\widetilde{\varphi}_i(u')$ for all $1\le i\le d$ have been defined. Then for $u''=u'+1$ the process is the same with the role of $\widetilde{\varphi}_i(u_0)$ and $\varphi_i(u)$ replaced by $\widetilde{\varphi}_i(u')$ and $\varphi_i(u'')$ respectively.

To summarize, for the new functions $\widetilde{\varphi}_i(u)$ for $1\le i\le d$ one has:

- by (6.2),
$$\widetilde{\varphi}_i \ \text{ is decreasing, and,} \quad \widetilde{\varphi}_i \ge \varphi_i;$$
- for $u\in\mathcal{N}_1$,
$$\widetilde{\varphi}_i(u) = \varphi_i(u); \tag{6.3}$$
- for $u\notin\mathcal{N}_1$,
$$\Big(\max_{1\le k\le h}\Phi_k(u)\Big)^d \cdot \prod_{k=1}^h \Phi_k(u)\cdot \prod_{i=1}^d \widetilde{\varphi}_i(u) = f(u).$$

Finally, we consider the measure of the set $W(\widetilde{\varphi},\Phi)$. Write

$$\mathcal{M}_1 = \big\{\mathbf{q}=(q_1,\ldots,q_h)\in\mathbb{Z}^h : u=\max\{\Phi_1^{-1}(|q_1|),\ldots,\Phi_h^{-1}(|q_h|)\}\in\mathcal{N}_1\big\}.$$

By (6.3), one sees that

$$\begin{aligned} W(\widetilde{\varphi},\Phi) &= \limsup_{\mathbf{q}\in\mathcal{M}_1} E_{\mathbf{q}}(\widetilde{\varphi},\Phi)\cup \limsup_{\mathbf{q}\notin\mathcal{M}_1} E_{\mathbf{q}}(\widetilde{\varphi},\Phi) \\ &\subset W(\varphi,\Phi)\cup \limsup_{\mathbf{q}\notin\mathcal{M}_1} E_{\mathbf{q}}(\widetilde{\varphi},\Phi) \end{aligned}$$

We claim that the second set is of measure zero by the convergence part of the Borel-Cantelli lemma, which results in

$$\mathcal{L}\big(W(\widetilde{\varphi},\Phi)\big) = \mathcal{L}\big(W(\varphi,\Phi)\big)$$

as wanted. More precisely,

$$\begin{aligned} \sum_{\mathbf{q}\notin\mathcal{M}_1}\mathcal{L}\big(E_{\mathbf{q}}(\widetilde{\varphi},\Phi)\big) &= \sum_{u\notin\mathcal{N}_1}\ \sum_{\mathbf{q}\in\mathbb{Z}^h:u=\max\{\Phi_1^{-1}(|q_1|),\ldots,\Phi_h^{-1}(|q_h|)\}} \mathcal{L}\big(E_{\mathbf{q}}(\widetilde{\varphi},\Phi)\big) \\ &= \sum_{u\notin\mathcal{N}_1}\ \sum_{\mathbf{q}\in\mathbb{Z}^h:u=\max\{\Phi_1^{-1}(|q_1|),\ldots,\Phi_h^{-1}(|q_h|)\}} \prod_{i=1}^d \widetilde{\varphi}_i(u) \\ &\le \sum_{u\in\mathbb{N}}\ \sum_{\mathbf{q}\in\mathbb{Z}^h:u=\max\{\Phi_1^{-1}(|q_1|),\ldots,\Phi_h^{-1}(|q_h|)\}} \frac{f(u)}{\Big(\max_{1\le k\le t}\Phi_k(u)\Big)^d\cdot\prod_{k=1}^h\Phi_k(u)}. \end{aligned}$$

By the monotonicity of the terms in the summation and by dividing the integers $u \in \mathbb{N}$ into M-adic blocks, one has

$$\sum_{\mathbf{q}\notin\mathcal{M}_1} \mathcal{L}\big(E_{\mathbf{q}}(\widetilde{\varphi},\Phi)\big) \le 2^h \sum_{t=0}^{\infty}\prod_{k=1}^{h}\Phi_k(M^{t+1})\cdot \frac{1}{\big(\max_{1\le k\le h}\Phi_k(M^t)\big)^{d-\epsilon}\cdot\prod_{k=1}^{h}\Phi_k(M^t)}$$

$$\le 2^h\cdot c_2^h\cdot\sum_{t=0}^{\infty}\big(c_1^{-t}\big)^{d-\epsilon}<\infty,$$

where the second inequality uses the assumptions posed on Φ. Thus, by the Borel-Cantelli lemma, $\limsup_{\mathbf{q}\notin\mathcal{M}_1}\mathcal{L}\big(E_{\mathbf{q}}(\widetilde{\varphi},\Phi)\big)$ is of Lebesgue measure zero. □

We are now ready to proceed with Theorem 2.7.

Proof of Theorem 2.7. By Lemma 6.1, we can assume that the functions φ,Φ satisfy the conclusion given there. Moreover, without loss of generality we can assume that

$$\prod_{i=1}^{d}\varphi_i(u)\cdot\prod_{k=1}^{h}\Phi_k(u)\le 1,\ \text{for all } u\gg 1 \tag{6.4}$$

otherwise, by Minkowski's convex body theorem, it is trivial that $W(\varphi,\Phi)$ is of full measure.

Now let us check that all the conditions in Theorem 2.5 are satisfied.

- The index set J:
$$\alpha=(q_1,\dots,q_h,p_1,\dots,p_d):\mathbf{q}\in\mathbb{Z}^h,\ |p_i|\le h\cdot\max_{1\le k\le h}|q_k|.$$
- The weight function:
$$\beta_\alpha=\max\big\{\Phi_1^{-1}(|q_1|),\dots,\Phi_h^{-1}(|q_h|)\big\},\ \text{for } \alpha=(q_1,\dots,q_h,p_1,\dots,p_d).$$
- Resonant sets:
$$\mathfrak{R}_\alpha=\prod_{i=1}^{d}\big\{A_i:A_i\mathbf{q}=p_i\big\},\quad\text{for } \alpha=(q_1,\dots,q_h,p_1,\dots,p_d).$$
- $u_n=M^n$ for some integer $M\ge 2^{2d+3}$, and ℓ_n is defined later in (6.8).
- The ubiquitous function:
$$\rho_i(u)=\frac{M\cdot\varphi_i(u)}{\max_{1\le k\le h}\Phi_k(u)}\cdot\left(\prod_{i=1}^{d}\varphi_i(u)\cdot\prod_{k=1}^{h}\Phi_k(u)\right)^{-1/d},\ \text{for all } 1\le i\le d. \tag{6.5}$$
We will show in Lemma 6.2 that $(\{\mathfrak{R}_\alpha\}_{\alpha\in J},\beta)$ forms a ubiquitous system with respect to the function $\boldsymbol{\rho}=(\rho_1,\cdots,\rho_d)$ and the sequences $\{\ell_n,u_n\}_{n\ge1}$.
- The approximating function:
$$\psi_i(u)=\frac{1}{h}\cdot\frac{\varphi_i(u)}{\max_{1\le k\le h}\Phi_k(u)},\ \text{for all } 1\le i\le d. \tag{6.6}$$
Note that for any A such that
$$A\in\prod_{i=1}^{d}\Delta\left(\mathfrak{R}_{\alpha,i},\frac{1}{h}\cdot\frac{\varphi_i(u)}{\max_{1\le k\le h}\Phi_k(u)}\right)\quad\text{with }\ \alpha=(q_1,\dots,q_h,p_1,\dots,p_d),\ \beta_\alpha=u,$$
one has
$$|A_i\mathbf{q}-p_i|<|\mathbf{q}|\cdot\frac{\varphi_i(u)}{\max_{1\le k\le h}\Phi_k(u)}\le\varphi_i(u),\ \text{for } 1\le i\le d.$$

In other words,

$$\bigcup_{\alpha\in J_n}\{A:|A_i\mathbf{q}-p_i|<\varphi_i(u),\ 1\le i\le d\}\supset\bigcup_{\alpha\in J_n}\prod_{i=1}^{d}\Delta\big(\mathfrak{R}_{\alpha,i},\psi_i(u_n)\big).$$

Therefore

$$W(\varphi,\Phi)\supset\limsup_{n\to\infty}\bigcup_{\alpha\in J_n}\prod_{i=1}^{d}\Delta\big(\mathfrak{R}_{\alpha,i},\psi_i(u_n)\big).$$

Now let

$$\widetilde{\rho}_i(u)=\varphi_i(u)\cdot\left(\prod_{i=1}^{d}\varphi_i(u)\cdot\prod_{k=1}^{h}\Phi_k(u)\right)^{-1/d},\quad 1\le i\le d.$$

Lemma 6.2 (Ubiquity for rectangles)**.** *With the notation above, the pair* $(\{\mathfrak{R}_\alpha\}_{\alpha\in J},\beta)$ *is a ubiquitous system with respect to the function* $\boldsymbol{\rho}$ *and the sequences* $\{\ell_n,u_n\}_{n\ge1}$*.*

Proof. For any $u\in\mathbb{N}$, by Minkowski's convex body theorem, for any fixed matrix $A\in[0,1]^{dh}$ there exists a non-zero integer vector $(q_1,\dots,q_h,p_1,\dots,p_d)$ such that

$$\begin{cases}|A_i\mathbf{q}-p_i|<\widetilde{\rho}_i(u), & 1\le i\le d;\\ |q_k|\le\Phi_k(u) & 1\le k\le h.\end{cases}$$

In other words, for any $u\in\mathbb{N}$ and $A\in[0,1]^{dh}$, there exists $\alpha\in J$ with $\beta_\alpha\le u$ such that

$$A\in\prod_{i=1}^{d}\Delta\left(\mathfrak{R}_{\alpha,i},\frac{\widetilde{\rho}_i(u)}{\max_{1\le k\le h}|q_k|}\right).\tag{6.7}$$

Recall $u_n=M^n$ and choose ℓ_n small enough such that

$$\begin{aligned}J_n&=\{\alpha:\ell_n\le\beta_\alpha\le u_n\}\\&\supset\left\{(q_1,\dots,q_h,p_1,\dots,p_d):\frac{1}{M}\Phi_k(M^n)\le|q_k|\le\Phi_k(M^n),1\le k\le h\right\}:=\widetilde{J}_n.\end{aligned}\tag{6.8}$$

Thus

$$\begin{aligned}\Big\{\alpha\in J:\beta_\alpha\le u_n\Big\}&=\widetilde{J}_n\cup\bigcup_{j=1}^{h}\left\{\alpha\in J:|q_j|\le\frac{\Phi_j(M^n)}{M},\ |q_k|\le\Phi_k(M^n),\ \text{for all } k\ne j\right\}\\&:=\widetilde{J}_n\cup\bigcup_{j=1}^{h}J_{n,j}.\end{aligned}$$

Let $B=\prod_{i=1}^{d}B(x_i,r)$ be a ball in $[0,1]^{dh}$. Taking $u=u_n$ in (6.7), one has

$$\begin{aligned}B&=B\cap\bigcup_{\alpha:\beta_\alpha\le u_n}\prod_{i=1}^{d}\Delta\left(\mathfrak{R}_{\alpha,i},\frac{\widetilde{\rho}_i(u_n)}{\max_{1\le k\le h}|q_k|}\right)\\&=\left(B\cap\bigcup_{\alpha\in\widetilde{J}_n}\prod_{i=1}^{d}\Delta\left(\mathfrak{R}_{\alpha,i},\frac{\widetilde{\rho}_i(u_n)}{\max_{1\le k\le h}|q_k|}\right)\right)\bigcup\left(B\cap\bigcup_{j=1}^{h}\bigcup_{\alpha\in J_{n,j}}\prod_{i=1}^{d}\Delta\left(\mathfrak{R}_{\alpha,i},\frac{\widetilde{\rho}_i(u_n)}{\max_{1\le k\le h}|q_k|}\right)\right)\\&=I_1\cup I_2.\end{aligned}$$

We give an upper bound estimation on the measure of I_2:

$$\mathcal{L}(I_2) \le \sum_{j=1}^{h} \sum_{|q_j| \le \frac{\Phi_j(u_n)}{M};|q_k|\le \Phi_k(u_n), k\neq j} \sum_{p_1,\dots,p_d} \prod_{i=1}^{d} \mathcal{L}\left(B(x_i, r) \cap \Delta\left(\mathfrak{R}_{\alpha,i}, \frac{\widetilde{\rho}_i(u_n)}{\max_{1\le k\le h}|q_k|}\right)\right)$$

For any fixed $(q_1,\dots,q_h)$, the number of p_i such that the intersection in the product is non-empty is at most $2r\cdot\max_{1\le k\le h}|q_k|+2$. Thus it follows that

$$\begin{aligned}\mathcal{L}(I_2) &\le \sum_{j=1}^{h} \sum_{|q_j| \le \frac{\Phi_j(u_n)}{M};|q_k|\le \Phi_k(u_n), k\neq j} \left(2r\cdot \max_{1\le k\le h}|q_k|+2\right)^d \cdot \prod_{i=1}^{d} \frac{\widetilde{\rho}_i(u_n)\cdot r^{h-1}}{\max_{1\le k\le h}|q_k|} \\ &= \sum_{j=1}^{h} \sum_{|q_j| \le \frac{\Phi_j(u_n)}{M},|q_k|\le \Phi_k(u_n),\ k\neq j} \left(2r\cdot \max_{1\le k\le h}|q_k|+2\right)^d \cdot \left(\max_{1\le k\le h}|q_k|\right)^{-d} \cdot \left(\prod_{k=1}^{h}\Phi_k(u_n)\right)^{-1} \cdot r^{d(h-1)}.\end{aligned}$$

Then using a simple inequality that $(a+b)^d \le (2a)^d + (2b)^d$, it follows that

$$\mathcal{L}(I_2) \le \frac{2^{2d}\cdot r^{dh}}{M} + \frac{h\cdot 2^{2d}\cdot r^{d(h-1)}\log\Phi_1(u_n)}{\Phi_1(u_n)} \le \frac{1}{2}\cdot\mathcal{L}(B)$$

whenever $M > 2^{2d+3}$ and n is large enough. Thus one gets

$$\mathcal{L}\left(B \cap \bigcup_{\alpha\in\widetilde{J}_n}\prod_{i=1}^{d}\Delta\left(\mathfrak{R}_{\alpha,i}, \frac{\widetilde{\rho}_i(u_n)}{\max_{1\le k\le h}|q_k|}\right)\right) \ge \frac{1}{2}\cdot\mathcal{L}(B).$$

Note also that for any $\alpha\in\widetilde{J}_n$,

$$\max_{1\le k\le h}|q_k| \ge \frac{1}{M}\cdot \max_{1\le k\le h}\Phi_k(u_k),$$

which implies that

$$\begin{aligned}&\mathcal{L}\left(B \cap \bigcup_{\alpha\in J_n}\prod_{i=1}^{d}\Delta\left(\mathfrak{R}_{\alpha,i}, \frac{M\cdot\widetilde{\rho}_i(u_n)}{\max_{1\le k\le h}\Phi_k(u_n)}\right)\right) \ge \mathcal{L}\left(B \cap \bigcup_{\alpha\in \widetilde{J}_n}\prod_{i=1}^{d}\Delta\left(\mathfrak{R}_{\alpha,i}, \frac{M\cdot\widetilde{\rho}_i(u_n)}{\max_{1\le k\le h}\Phi_k(u_n)}\right)\right) \\ &\ge \mathcal{L}\left(B \cap \bigcup_{\alpha\in\widetilde{J}_n}\prod_{i=1}^{d}\Delta\left(\mathfrak{R}_{\alpha,i}, \frac{\widetilde{\rho}_i(u_n)}{\max_{1\le k\le h}|q_k|}\right)\right) \ge \frac{1}{2}\cdot\mathcal{L}(B).\end{aligned}$$

This shows the ubiquity property with the ubiquitous function

$$\rho_i(u) = \frac{M\cdot\widetilde{\rho}_i(u)}{\max_{1\le k\le h}\Phi_k(u)},\ 1\le i\le d.$$

□

To summarize, we have

- the ubiquitous system $(\{\mathfrak{R}_\alpha\}_{\alpha\in J}, \beta)$ with respect to $\boldsymbol{\rho}$: by Lemma 6.2.
- the λ-regularity property: by the monotonicity of φ_i and the condition assumed on Φ_k, namely

$$\psi_i(u_{n+1}) = \frac{\varphi_i(u_{n+1})}{\max_{1\le k\le h}\Phi_k(u_{n+1})} \le \frac{1}{c_1}\cdot\frac{\varphi_i(u_n)}{\max_{1\le k\le h}\Phi_k(u_n)} = \frac{1}{c_1}\cdot\psi_i(u_n).$$

Recall the definitions of ρ_i and ψ_i:

$$\rho_i(u) = \frac{M\cdot\varphi_i(u)}{\max_{1\le k\le h}\Phi_k(u)}\cdot\left(\prod_{i=1}^d \varphi_i(u)\cdot\prod_{k=1}^h \Phi_k(u)\right)^{-1/d}, \quad \psi_i(u) = \frac{1}{h}\cdot\frac{\varphi_i(u)}{\max_{1\le k\le h}\Phi_k(u)}.$$

Then one has

- $\rho_i(u) \ge \psi_i(u)$: by (6.4);
- $\rho_i(u)\to 0$ as $u\to\infty$: by Lemma 6.1, since the denominator in $\rho_i(u)$ tends to infinity.

Thus all the conditions in Theorem 2.5 are satisfied, and then we apply it to arrive at the desired result, i.e.

$$\mathcal{L}\big(W(\varphi,\Phi)\big) = 1 \iff \sum_{t=0}^\infty \prod_{i=1}^d \varphi_i(M^t)\prod_{k=1}^h \Phi_k(M^t) = \infty.$$

Finally, by monotonicity of φ and the assumption on Φ, one has

$$\begin{aligned}\infty = \sum_{q=1}^\infty q^{-1}\cdot\prod_{i=1}^d\varphi_i(q)\cdot\prod_{k=1}^h\Phi_k(q) &= \sum_{t=0}^\infty\sum_{M^t\le q<M^{t+1}} q^{-1}\cdot\prod_{i=1}^d\varphi_i(q)\cdot\prod_{k=1}^h\Phi_k(q)\\ &\le (M-1)c_2^h\cdot\sum_{t=0}^\infty\prod_{i=1}^d\varphi_i(M^t)\prod_{k=1}^h\Phi_k(M^t).\end{aligned}$$

This proves the divergence part of Theorem 2.7, while the convergence part follows easily from the Borel-Cantelli lemma. □

Remark 6.3. It is natural to ask whether Theorem 2.7 can be extended to the settings where the approximating functions are multivariable, that is, if the functions

$$\varphi_i\big(\max\{\Phi_1^{-1}(|q_1|),\dots,\Phi_h^{-1}(|q_h|)\}\big)$$

in (2.2) are replaced with $\varphi_i:\mathbb{Z}^h\to\mathbb{R}_{\ge 0}$. In other words, for $\varphi=(\varphi_1,\dots,\varphi_d):\mathbb{Z}^h\to\mathbb{R}^d$ one can define the set

$$W(\varphi) := \big\{A\in[0,1]^{dh} : \|A_i\mathbf{q}\| < \varphi_i(\mathbf{q}),\ 1\le i\le d,\ \text{i.m. } \mathbf{q}\in\mathbb{Z}^h\big\}.$$

In [9], Beresnevich & Velani presented the measure theory of $W(\varphi)$ in the case when $\varphi_1=\dots=\varphi_d$. However, it is unclear to the authors whether in the most general case the measure of $W(\varphi)$ can be deduced from our main result (Theorem 2.5), although Theorem 2.5 can indeed be extended to a setting with multi-variable approximating functions.

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