Parallel Repetition for All 3-Player Games over Binary Alphabet*

Uma Girish
Princeton University
USA
ugirish@cs.princeton.edu

Justin Holmgren
NTT Research
USA
justin.holmgren@ntt-research.com

Kunal Mittal
Princeton University
USA
kmittal@cs.princeton.edu

Ran Raz
Princeton University
USA
ranr@cs.princeton.edu

Wei Zhan
Princeton University
USA
weizhan@cs.princeton.edu

ABSTRACT

We prove that for every 3-player (3-prover) game, with binary questions and answers and value < 1, the value of the n-fold parallel repetition of the game decays polynomially fast to 0. That is, for every such game, there exists a constant $c > 0$, such that the value of the n-fold parallel repetition of the game is at most n^{-c}.

Along the way to proving this theorem, we prove two additional parallel repetition theorems for multiplayer (multiprover) games, that may be of independent interest:

Playerwise Connected Games (with any number of players and any Alphabet size): We identify a large class of multiplayer games and prove that for every game with value < 1 in that class, the value of the n-fold parallel repetition of the game decays polynomially fast to 0.

More precisely, our result applies for *playerwise connected games*, with any number of players and any alphabet size: For each player i, we define the graph G_i, whose vertices are the possible questions for that player and two questions x, x' are connected by an edge if there exists a vector y of questions for all other players, such that both (x, y) and (x', y) are asked by the referee with non-zero probability. We say that the game is *playerwise connected* if for every i, the graph G_i is connected.

Our class of playerwise connected games is strictly larger than the class of *connected* games that was defined by Dinur, Harsha, Venkat and Yuen (ITCS 2017) and for which they proved exponential decay bounds on the value of parallel repetition. For playerwise connected games that are not connected, only inverse Ackermann decay bounds were previously known (Verbitsky 1996).

Exponential Bounds for the Anti-Correlation Game: In the *3-player anti-correlation game*, two out of three players are given 1 as input, and the remaining player is given 0. The two players who were given 1 must produce different outputs in $\{0, 1\}$. We prove that the value of the n-fold parallel repetition of that game decays exponentially fast to 0. That is, there exists a constant $c > 0$, such that the value of the n-fold parallel repetition of the game is at most 2^{-cn}. Only inverse Ackermann decay bounds were previously known (Verbitsky 1996).

The 3-player anti-correlation game was studied and motivated in several previous works. In particular, Holmgren and Yang (STOC 2019) gave it as an example for a 3-player game whose non-signaling value (is smaller than 1 and yet) does not decrease at all under parallel repetition.

CCS CONCEPTS

• **Theory of computation → Computational complexity and cryptography**.

KEYWORDS

Parallel Repetition, Multiplayer Games

ACM Reference Format:
Uma Girish, Justin Holmgren, Kunal Mittal, Ran Raz, and Wei Zhan. 2022. Parallel Repetition for All 3-Player Games over Binary Alphabet. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing (STOC '22), June 20–24, 2022, Rome, Italy*. ACM, New York, NY, USA, 12 pages. https://doi.org/10.1145/3519935.3520071

1 INTRODUCTION

We study multiplayer games and their behavior under parallel repetition. In a k-player game $\mathcal{G}$, a referee samples questions $x = (x^1, \ldots, x^k)$ from some distribution Q. Then, for each $j \in [k]$, the j^{th} player is given the question x^j, based on which they give back an answer a^j. The referee then declares if the players win or not based on the evaluation of a predicate $V(x^1, \ldots, x^k, a^1, \ldots, a^k)$. The value $\text{val}(\mathcal{G})$ of the game $\mathcal{G}$ is defined to be the maximum winning probability for the players, where the maximum is over all possible strategies (functions mapping questions to answers) of the players.

A very basic operation on a game $\mathcal{G}$ is to consider its parallel repetition, in which the players are asked to play many independent copies of the game in parallel. More formally, in the n-fold parallel repetition $\mathcal{G}^{\otimes n}$, the referee draws questions $(x_i^1, \ldots, x_i^k)$ from Q, independently for each $i \in [n]$. Then, for each $j \in [k]$, the j^{th} player is given the questions $(x_1^j, \ldots, x_n^j)$, based on which they

*Uma Girish, Kunal Mittal, Ran Raz, and Wei Zhan were supported by the Simons Collaboration on Algorithms and Geometry, by a Simons Investigator Award, and by the National Science Foundation grants No. CCF-1714779, CCF-2007462. Uma Girish was also supported by the IBM PhD Fellowship.

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. Copyrights for components of this work owned by others than the author(s) must be honored. Abstracting with credit is permitted. To copy otherwise, or republish, to post on servers or to redistribute to lists, requires prior specific permission and/or a fee. Request permissions from permissions@acm.org.
STOC '22, June 20–24, 2022, Rome, Italy
© 2022 Copyright held by the owner/author(s). Publication rights licensed to ACM.
ACM ISBN 978-1-4503-9264-8/22/06…$15.00
https://doi.org/10.1145/3519935.3520071

answer back $(a_1^j, \ldots, a_n^j)$. The referee says that the players win if for every $i \in [n]$, the predicate $V(x_i^1, \ldots, x_i^k, a_i^1, \ldots, a_i^k)$ evaluates to win.

A natural question is to study how the value of the game $\mathcal{G}^{\otimes n}$ behaves as a function of n, the number of parallel repetitions [19]. It is not hard to see that $\mathrm{val}(\mathcal{G}^{\otimes n}) \geq \mathrm{val}(\mathcal{G})^n$, since the players can achieve value $\mathrm{val}(\mathcal{G})^n$ in the game $\mathcal{G}^{\otimes n}$ by simply repeating an optimal strategy for the game $\mathcal{G}$ independently in all the n coordinates. It also seems that this should be optimal, and that $\mathrm{val}(\mathcal{G}^{\otimes n}) \leq \mathrm{val}(\mathcal{G})^n$. However, this turns out not to be the case, and there are games such that $\mathrm{val}(\mathcal{G}^{\otimes n})$ is exponentially larger than $\mathrm{val}(\mathcal{G})^n$ [12, 17, 18, 38]. Hence, it is interesting to study the behavior of $\mathrm{val}(\mathcal{G}^{\otimes n})$ for games $\mathcal{G}$ with $\mathrm{val}(\mathcal{G}) < 1$.

The special case of 2-player games is very well understood, and it was proven by Raz [36] that if $\mathrm{val}(\mathcal{G}) < 1$, then the value of $\mathcal{G}^{\otimes n}$ decays exponentially in n; that is, $\mathrm{val}(\mathcal{G}^{\otimes n}) \leq 2^{-\Omega(n)}$, with the constants depending on the base game $\mathcal{G}$. There have been improvements in the constants [3, 27, 35, 39], and we even know tight results based on the value of the initial game [6, 11]. These results on 2-player games have found many applications, in particular in the theory of interactive proofs [5], PCPs and hardness of approximation [4, 13, 25], geometry of foams [1, 15, 30], quantum information [9], and communication complexity [2, 7, 33]. The reader is referred to this survey [37] for more details.

The case of general k-player multiplayer games is still open. The only general result, by Verbitsky [40], says that if $\mathrm{val}(\mathcal{G}) < 1$, then $\mathrm{val}(\mathcal{G}^{\otimes n}) \to 0$ as $n \to \infty$. This result uses the density Hales-Jewett theorem as a black box, and gives bounds of the form $\frac{1}{\alpha(n)}$, where α is an inverse Ackermann function [20, 34]. Apart from being interesting in its own right, studying parallel repetition of multiplayer games has some applications. For example, it is known that a strong parallel repetition theorem for a particular class of multiplayer games implies super-linear lower bounds for Turing machines in the non-uniform model [31]. Also (as mentioned by [10]), the technical limitations that arise when analyzing games with more than two players seem very similar to the ones we encounter when studying direct sum and direct product questions for multiparty number-on-forehead communication complexity (which is related to lower bounds in circuit complexity). Therefore, studying parallel repetition for multiplayer games may lead to progress in these areas.

Although we know very little about general multiplayer games, there has been some recent progress on special classes of multiplayer games:

(1) Dinur, Harsha, Venkat and Yuen [10] extend the two player techniques of [27, 36] and show that any *connected game* satisfies an exponentially small bound on the value of parallel repetition (and this includes all games for which exponentially small bounds were previously known). The class of connected games is defined as follows: Define the graph $\mathcal{H}_{\mathcal{G}}$, whose vertices are the ordered k-tuples of questions to the k-players, and there is an edge between questions x and x' if they differ in the question to exactly one of the k players, and are the same for the remaining $k-1$ players. The game is said to be *connected* if the graph $\mathcal{H}_{\mathcal{G}}$ is connected.

(2) The GHZ game [24] is defined as follows: The referee samples the questions (x^1, x^2, x^3) uniformly at random from $\{0,1\}^3$ such that $x^1 \oplus x^2 \oplus x^3 = 0$. The players answer back with $a^1, a^2, a^3 \in \{0,1\}$, and are said to win if $a^1 \oplus a^2 \oplus a^3 = x^1 \vee x^2 \vee x^3$. It has been shown that any game with the same distribution as the GHZ game satisfies an inverse polynomial bound on the value of parallel repetition [21, 28].

1.1 Our Results

We prove that for every 3-player game, with binary questions and answers and value < 1, the value of the n-fold parallel repetition of the game decays polynomially fast to 0.

Theorem 1.1. *Let $\mathcal{G}$ be a 3-player game such that $\mathrm{val}(\mathcal{G}) < 1$ and each question and answer is in $\{0,1\}$. Then, there exists a constant $c > 0$, such that $\mathrm{val}(\mathcal{G}^{\otimes n}) \leq n^{-c}$.*

In the proof of Theorem 1.1, we show that from the perspective of studying the behaviour of $\mathrm{val}(\mathcal{G}^{\otimes n})$ as a function of n, every 3-player game $\mathcal{G}$ with binary questions and answers, is equivalent to, or can be reduced to, a game in one of the following five classes:

(1) **2-Player Games:** As mentioned above, exponentially small bounds on the value of the parallel repetition of games in this class have been known for a long time.
(2) **Playerwise Connected Games:** This is a new class of games that we define and study in this work and we prove polynomially small bounds on the value of the parallel repetition of games in this class.
(3) **The GHZ Game:** (and other games with the same query distribution): As mentioned above, polynomially small bounds on the value of the parallel repetition of games in this class were recently proved.
(4) **The Anti-Correlation Game:** (and other games with the same query distribution and binary answers): The 3-player anti-correlation game is defined as follows: The referee samples the questions (x^1, x^2, x^3) uniformly at random from $\{0,1\}^3$ such that $x^1 + x^2 + x^3 = 2$ (that is, two out of three players are given 1 as input, and the remaining player is given 0). The two players who were given 1 must produce different outputs in $\{0,1\}$. We prove exponentially small bounds on the value of the parallel repetition of that game (and all other games with the same query distribution and binary answers).
(5) **Games over the Set of Questions** $\{(0,0,0), (0,1,0), (1,0,0), (1,1,1)\}$**:** We prove polynomially small bounds on the value of the parallel repetition of games in this class.

We note that the reduction to these five classes of games works more generally for all 3-player games with binary questions and arbitrary length of answers, except that we need to extend Class 4 so that it contains games with arbitrary length of answers. Note also that for all other classes, the bounds that we have hold more generally for games with arbitrary length of answers. This means that improving the bounds that we prove for Class 4 so that they hold for arbitrary length of answers (or even proving weaker, polynomially small, bounds for that case) would imply that Theorem 1.1 holds more generally, for games with arbitrary answer length.

We note also that the three new bounds that we prove in this work, the bounds for Class 2, Class 4 and Class 5, are each proved by a completely different proof method. Next we elaborate on each of these three classes.

1.1.1 The Anti-Correlation Game (Class 4). In the hilarious essay *"Test Your Telepathic Skills"*, Uri Feige tells the fictional story of the *"amazing Tachman family"*, who astonished the team at FEXI *(the Foolproof Experiments Institute)* with their telepathic skills, by playing incredibly well the 3-fold parallel repetition of the 3-player anti-correlation game [Uri Feige, 1995][1]. Feige showed that the value of the 3-player anti-correlation game, played in parallel 3 times, is $\frac{2}{3}$, exactly the same as the value of the original game.

More than two decades later, Holmgren and Yang proved that while the, so called, *non-signaling value*, of the 3-player anti-correlation game is strictly smaller than 1, it does not decrease at all under parallel repetition [29]. This gave a surprising first example for a total failure of parallel repetition in reducing the value of a game, in any model of multiplayer games.

Hazla, Holenstein and Rao studied games with the same query distribution as the anti-correlation game [26] and showed barriers on proving parallel repetition theorems for such games using a technique known as the *forbidden subgraph bounds* [17].

The anti-correlation game can also be presented as a "pigeonhole-principle" game, where 2 out of 3 pigeons are chosen randomly and each of them needs to choose 1 out of 2 pigeonholes, without communicating between them, so that the two chosen pigeons end in 2 different pigeonholes. This may occur in situations when 3 players share 2 identical resources (such as 2 communication channels to an external party): Two (randomly chosen) players (out of the three players) need to use one of the two resources each and there is no communication between the players. Another description of the game, the one that was presented by Feige, can be viewed as a matching game: The 3 players try to output 3 different answers X, Y, Z, where two of the players, chosen randomly, can only output Y or Z and the remaining player can only output X or Z.

Although the 3-player anti-correlation game has been around for more than two and a half decades, no bound on the value of its parallel repetition was previously known (other than Verbitsky's general inverse Ackermann bound on the value of the parallel repetition of every game [40]). In this work, we prove that the value of the n-fold parallel repetition of the 3-player anti-correlation game decays exponentially fast to 0. (We also extend this bound to all other games with the same query distribution and binary answers).

Theorem 1.2. *Let $\mathcal{G}$ be the 3-player anti-correlation game (or any other game with the same query distribution and binary answers). Then, there exists a constant $c > 0$, such that $\mathrm{val}(\mathcal{G}^{\otimes n}) \leq 2^{-cn}$.*

In light of the above mentioned result by Holmgren and Yang [29], Theorem 1.2 also implies an example for a 3-player game where the value of the parallel repetition of the game behaves completely differently for classical strategies versus non-signaling strategies. Namely, while parallel repetition doesn't decrease the non-signalling value of the game at all, it does decrease the classical value of the game exponentially fast to 0.

Techniques: The techniques that we use for the proof of Theorem 1.2 are, to the best of our knowledge, completely new in the context of parallel repetition and are different than the techniques used in all previous works. In particular, we don't use here the usual *embedding paradigm*, that is used in almost all previous works, where one tries to embed a copy of the original game in the set of success of the players on some set of coordinates. Instead, our proof shows a *local to global* property of the strategy of each player. Very roughly speaking, we prove that if the players win the parallel repetition game with sufficiently high probability, then there exists a fixed (large) set of coordinates and a fixed global constant strategy for each of the players, that doesn't depend on the input for the player at all, and such that the global strategies win the parallel repetition game with a sufficiently high probability, on almost all the coordinates in the fixed set of coordinates. This leads to a contradiction since fixed global strategies are, in particular, independent between the different coordinates. We note also that this is the first inverse exponential bound on the parallel repetition of any 3-player game that is not connected (in the sense of [10])[2].

1.1.2 Playerwise Connected Games (Class 2). We define the class of *playerwise connected games* as follows: For each player j, we define the graph $\mathcal{H}^j_{\mathcal{G}}$, whose vertices are the possible questions for player j, and two questions x^j and x'^j are connected by an edge if there exists a vector y of questions for all other players, such that both (x^j, y) and (x'^j, y) are asked by the referee with non-zero probability. We say that the game is *playerwise connected* if for every j, the graph $\mathcal{H}^j_{\mathcal{G}}$ is connected.

We prove polynomially small bounds on the value of the parallel repetition of any game in this class:

Theorem 1.3. *Let $\mathcal{G}$ be a* playerwise connected *game such that $\mathrm{val}(\mathcal{G}) < 1$ (with any number of players and any Alphabet size). Then, there exists a constant $c > 0$, such that $\mathrm{val}(\mathcal{G}^{\otimes n}) \leq n^{-c}$.*

Theorem 1.3 gives an inverse polynomial bound on the value of parallel repetition for many games for which the previously best known bound was inverse Ackerman.

Our class of playerwise connected games is related to the above mentioned class of connected games that was studied by Dinur, Harsha, Venkat and Yuen and for which exponentially small bounds were established [10]. Observe that every connected game is also playerwise connected (the graph $\mathcal{H}^j_{\mathcal{G}}$ is simply the *projection* of the graph $\mathcal{H}_{\mathcal{G}}$ in the j^{th} direction). The vice-versa is however not true:

Example 1.4. *The following 3-player game is playerwise connected, but not connected: The referee samples (x, y, z) uniformly from $\mathcal{S} = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$ and gives x, y, z to the three players respectively. The players give answers $a, b, c \in \{0,1\}$ respectively. The players win if the following condition holds: $a+b+c = 1 \iff x+y+z \neq 3$.*

We note that the set $\mathcal{S}$ of possible questions, from Example 1.4, is the only set with 3 players and binary questions that gives a game

[1] https://www.wisdom.weizmann.ac.il/~feige/tachman.html (Feige's description of the game is somewhat different than ours and is described below).

[2] or reduces to a connected game

that is playerwise connected but not connected (up to a change of names). When the number of players is larger than 3, or the question's Alphabet size is larger than 2, there are many additional examples.

Example 1.5. *Fix a random 3-CNF formula $\varphi = (C_1, \ldots, C_m)$, with m clauses, over d variables. This is generated by sampling m times independently and uniformly from the set of all $(2d)^3 = 8d^3$ possible clauses.*

A 3-player game $\mathcal{G}$ is defined by this formula φ as follows: The referee samples $r \in [m]$ uniformly and gives the variables corresponding to the literals in C_r to the 3 players respectively (with each player getting one variable). The players answer back values for the variables they get, and the referee declares that the players win if these values satisfy the clause C_r.

Then, it is not hard to show (see the full version for a proof) that with high probability:

1. *If $m = \omega(d)$, the value of the game is close to 7/8, and hence less than 1.*
2. *If $m = \omega(d^2 \log d)$, the graph $\mathcal{H}_{\mathcal{G}}$ is connected, and we get $\mathrm{val}(\mathcal{G}^{\otimes n}) = 2^{-\Omega(n)}$ by [10]. Furthermore, if $m = o(d^2)$, the graph $\mathcal{H}_{\mathcal{G}}$ is not connected, and [10] is not applicable.*
3. *If $m = \omega(d^{1.5}\sqrt{\log d})$, the game $\mathcal{G}$ is playerwise connected, and $\mathrm{val}(\mathcal{G}^{\otimes n}) = n^{-\Omega(1)}$ by Theorem 1.3. Furthermore, if $m = o(d^{1.5})$, the game $\mathcal{G}$ is not playerwise connected.*

Note that the ω and o bounds on m are with respect to $d \to \infty$. Once the formula φ is fixed, we think of m and d as constants and the Ω bounds on the value of parallel repetition are with respect to $n \to \infty$.

Remark. *The above example is also interesting when compared to the works on refutation of random 3-CNFs, where different regimes of the parameter m lead to different consequences. It is known that with high probability:*

1. *If $m = \Omega(d^{1.5})$, there is a polynomial time algorithm for refuting the random 3-CNFs [16].*
2. *If $m = \Omega(d^2/\log d)$, resolution provides polynomial size witnesses for refutation. On the other hand, it fails to provide short witnesses when $m = O(d^{1.5-\epsilon})$ [8].*
3. *If $m = \Omega(d^{1.4})$, there exist polynomial size witnesses for refutation, based on spectral approach [14].*

In both cases, there is a polynomial gap in d between the base assumption $m = \omega(d)$ and the regime of best known results.

Techniques: Our proof of Theorem 1.3 relies on information-theoretic techniques, extending the ideas of [10, 27, 36]. In particular, we use here the usual *embedding paradigm* and condition on a *dependency breaking event*, as in many previous works. However, these techniques heavily rely on the game being connected and thus the result of [10] applies only to connected games[3] and we are not aware of any previous work that applies these techniques to games that are not connected. We hence need to deviate from these techniques at a crucial point. Very roughly speaking, at a crucial place in the proof where the connectivity of the game is necessary, our key idea is to replace the distribution of the game with a connected distribution and we carefully analyze how this change affects the rest of the proof.

[3] or disjoint unions of connected games

1.1.3 Support $\{(\mathbf{0},\mathbf{0},\mathbf{0}), (\mathbf{0},\mathbf{1},\mathbf{0}), (\mathbf{1},\mathbf{0},\mathbf{0}), (\mathbf{1},\mathbf{1},\mathbf{1})\}$ (Class 5). We consider 3-player games where the set of possible questions for the 3 players is: $\{(0,0,0), (0,1,0), (1,0,0), (1,1,1)\}$, and we prove polynomially small bounds on the value of the parallel repetition of any game in this class:

Theorem 1.6. *Let $\mathcal{G}$ be a 3-player game where the possible questions for the 3 players are: $(0,0,0), (0,1,0), (1,0,0), (1,1,1)$, and such that $\mathrm{val}(\mathcal{G}) < 1$ (with any length of answers). Then, there exists a constant $c > 0$, such that $\mathrm{val}(\mathcal{G}^{\otimes n}) \le n^{-c}$.*

Theorem 1.6 is necessary for the proof of Theorem 1.1 and we believe that the proof technique is interesting and might be useful for other games.

Techniques: The techniques that we use for the proof of Theorem 1.6 are, to the best of our knowledge, new. Very roughly speaking, we consider the possible pairs of answers (a, b) by Player 1 and Player 2 on questions $(1, 1)$ for these two players. We distinguish between pairs (a, b) for which Player 3 has an answer c such that the referee accepts the answers (a, b, c) on questions $(1, 1, 1)$ and pairs (a, b) for which Player 3 has no answer c such that the referee accepts the answers (a, b, c) on questions $(1, 1, 1)$. Intuitively, if the pair (a, b) is of the second type, that is, no answer c causes the referee to accept on $(1, 1, 1)$, the pair (a, b) cannot be used too often by Player 1 and Player 2, and we are able to make this intuition precise by conditioning on a carefully and inductively defined, but possibly exponentially small, product event between the inputs of Player 1 and Player 2. Very roughly speaking, when all pairs (a, b) of the second type are used with negligible (polynomially small) probability when conditioning on our product event, we are able to essentially reduce the parallel repetition game to parallel repetition of a 2-player game with value < 1, played by Player 1 and Player 2 on a subset of the coordinates and conditioned on the product event between the inputs of Player 1 and Player 2 that we defined. We then rely on the fact that bounds for 2-player games also hold when conditioning the inputs of the two players on a product event between the two players. The final bound that we obtain is inverse polynomial, rather than inverse exponential, because we must take into account the answers (a, b) of the second type that are still used with polynomially small probability. We do that using a union bound and it's crucial here that the 2-player game that we reduce to is only played on a small subset of the coordinates so that we can apply a union bound over these coordinates.

Subsequent Work. In further work, [23] extend our result by proving a polynomial decay bound on the value of parallel repetition for all games with binary inputs and arbitrary length answers. This is done by proving a parallel repetition theorem for all games described in Definition 8.2, which was left as an open problem in this work.

Organization. In Section 2, we give an overview of the main technical results of the paper. In Section 3, we formally define the various concepts relevant to this paper, and in the sections following it, we formally state our main results and some important lemmas.

Most of the technical proofs are deferred to the full version of this paper [22].

2 OVERVIEW

2.1 Organization

For the problem of parallel repetition for all three-player games on binary alphabets, the following results were known prior to our work (Section 3.4).

(1) For three-player games in which there are some two players whose inputs are in a bijective correspondence, we may treat these players as identical, and thus reduce the problem to showing parallel repetition for two-player games. The Parallel Repetition Theorem of [36] shows that parallel repetition decreases the value of two-player games exponentially fast (Theorem 3.11).
(2) There is a class of games known as *connected* (or *expanding*) games for which [10] showed an exponential decay on the parallel repetition value (Theorem 3.12). A k-player game is said to be connected if the $(k-1)$-connection graph is connected. This graph is defined as follows: the vertices are the elements in the support of the query distribution and the edges are between every pair of elements that agree on the questions to all but one player (Definition 3.9).
(3) For any game (with value less than one) for which the query distribution has support $\{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}$, [21, 28] showed that parallel repetition decreases the value at least polynomially fast (Theorem 3.13).

In this work, we study all three-player binary-alphabet games that do not fall into the above categories. In Section 8, we classify all such games. It turns out that there are essentially three such classes of games.

(1) Games whose query distribution has support $\{(0,1,1),(1,0,1),(1,1,0)\}$. Of these games, the anti-correlation game is the most interesting one. In this game, the players who receive one need to output distinct bits (Definition 4.1). We prove an exponential decay on the parallel repetition value of this game in Section 4 (Theorem 4.2). In Section 7, we show that this implies a similar result for all games with binary outputs whose query distribution has the same support as the anti-correlation game (Theorem 7.1).
(2) Games whose query distribution has support $\{(0,0,0),(0,1,0),(1,0,0),(1,1,1)\}$. We refer to the uniform distribution on these four points as the four-point AND distribution. In Section 5, we show that parallel repetition for such games decreases the value at least polynomially fast (Theorem 5.1). We remark that our result holds even if the answers are from an arbitrary alphabet.
(3) Games whose query distribution has support $\{(0,0,0),(0,0,1),(0,1,0),(1,0,0),(1,1,1)\}$. Such games fall into an even more general class of games which we call *playerwise connected* games; these are k-player games in which the projection of the $(k-1)$-connection graph on every player is connected (Definition 3.10). We show in Section 6 that parallel repetition for this class of games decreases the value at least polynomially fast (Theorem 6.1). We remark that our result holds even if the answers are from an arbitrary alphabet.

2.2 The Anti-Correlation Game

In the three-player anti-correlation game $\mathcal{G}$, a random pair of players are given 1 as input, and the remaining player is given 0. To win, the two players who are given 1 must produce different outputs in $\{0,1\}$, and the output of the player who is given 0 does not matter. We present an overview of the proof of Theorem 4.2 which shows that the parallel repetition of the anti-correlation game decreases the value exponentially fast. The details are presented in Section 4.

Let Q denote the joint input distribution for all the players in the game $\mathcal{G}$, and let X, Y, and Z respectively denote the first, second, and third player's inputs in the game $\mathcal{G}^{\otimes n}$. Let $f, g, h : \{0,1\}^n \to \{0,1\}^n$ be any strategy that wins the n-wise repeated game $\mathcal{G}^{\otimes n}$ with probability $\alpha > 0$. Our goal is to prove that $\alpha \le e^{-\Omega(n)}$.

The players' inputs are fully determined by the inputs of any pair of players by the equation $X_i + Y_i + Z_i = 2$ for all $i \in [n]$. We will say "(f,g,h) wins on (x,y)" as short-hand for "(f,g,h) wins on (x,y,z) where $z_i = 2 - x_i - y_i$ for each i".

Winning Implies Self-Agreement on Correlated X, X': We first consider a distribution in which X, X', Y are random variables with both (X,Y) and (X',Y) distributed like $Q^{\otimes n}_{X,Y}$, and with X and X' conditionally independent given Y. More explicitly, this distribution is sampled as follows:

(1) Sample an n-bit string Y according to $Q^{\otimes n}_Y$. That is, for each $i \in [n]$ independently sample $Y_i = 1$ with probability $2/3$, and $Y_i = 0$ otherwise.
(2) Independently sample X and X' from the conditional distribution of the first player's input in $\mathcal{G}^{\otimes n}$ given that the second player's input is Y. That is, for each $i \in [n]$, if $Y_i = 0$, set $X_i = X'_i = 1$. Otherwise, independently sample $X_i, X'_i \leftarrow \{0,1\}$ uniformly at random.

By the assumption that (f,g,h) wins with probability α, we know that (f,g,h) wins on (X,Y) with probability α, and (f,g,h) wins on (X',Y) with probability α. Because of how (X,Y) and (X',Y) are correlated, we show that (f,g,h) must simultaneously win on both (X,Y) and (X',Y) with probability at least α^2. Thus,

$$\Pr_{X,X'}\left[\Pr_Y\left[(f,g,h) \text{ wins on } (X,Y) \text{ and on } (X',Y)\right] \ge \alpha^2/2\right] \ge \alpha^2/2.$$

Now suppose that X and X' are such that

$$\Pr_Y[(f,g,h) \text{ wins on } (X,Y) \text{ and on } (X',Y)] \ge \alpha^2/2.$$

Then, we have $\Pr[Y_i = 1 | X, X'] = 1/3$ for i such that $X_i = X'_i = 1$, and all such Y_i are conditionally independent given X, X'. Also, if $X_i = X'_i = 1$ and $f(X)_i \ne f(X')_i$, then the only way (f,g,h) can win on (X,Y) and on (X',Y) is if $Y_i = 0$, because if $Y_i = 1$ then the win conditions require that $f(X)_i \ne g(Y)_i \ne f(X')_i$. Combining these two facts implies that $f(X)_i \ne f(X')_i$ for at most $\log_{2/3}(\alpha^2/2) = O(\log 1/\alpha)$ coordinates i with $X_i = X'_i = 1$.

This shows that when X, X', and Y are sampled as above, it holds with $\mathrm{poly}(\alpha)$ probability that:

(1) (Approximate Self-Agreement): For all but at most $O(\log 1/\alpha)$ values of $i \in [n]$, if $X_i = X'_i = 1$ then $f(X)_i = f(X')_i$.

(2) (Winning): (f, g, h) wins on (X, Y) and on (X', Y).

A similar argument gives an analogous statement, where the Winning property is replaced by a property that we call Winning', requiring that (f, g, h) has probability at least $\alpha/2$ of winning in $\mathcal{G}^{\otimes n}$ conditioned on the first player's input being X'.

We say that (X, X') is *good* if it satisfies Approximate Self-Agreement and Winning'.

Constructing a strategy for $\mathcal{G}^{\otimes n'}$: We next use the fact that (X, X') is good with $\text{poly}(\alpha)$ probability to show that if there exists a strategy (f, g, h) that wins $\mathcal{G}^{\otimes n}$ with probability α, then there exists a strategy (f', g', h') for $\mathcal{G}^{\otimes n'}$ (with $n' \geq \Omega(n)$) such that:

- f' is a constant function, and
- (f', g', h') wins in all but $O(\log 1/\alpha)$ coordinates of $\mathcal{G}^{\otimes n'}$ with probability $\text{poly}(\alpha)$.

The main idea is that (X, X') can be equivalently sampled as follows:

(1) Sample each bit of X_i independently such that $X_i = 1$ with probability $2/3$.
(2) Sample a set $S \subseteq [n]$ by independently including each $i \in [n]$ with probability $1/4$.
(3) For $i \in S$, set $X'_i = X_i$. For all other i, sample X'_i such that $X'_i = 1$ with probability $2/3$.

The point of this alternative sampling process is that conditioned on any value of X and S, the distribution of X'_{-S} is $Q_X^{\otimes n-|S|}$. In contrast, the distribution of X' given X is not $Q_X^{\otimes n}$ because of the correlation between X and X'.

We will first condition on random values of X, S, and X'_T, where $T = S \cup \{i : X_i = 0\}$. This ensures that

$$\Pr_{X,S,X'_T}\left[\Pr_{X'_{-T}}\left[(X, X') \text{ is good}\right] \geq \text{poly}(\alpha)\right] \geq \text{poly}(\alpha).$$

Also, the conditional distribution of X'_{-T} given any values of X, S, and X'_T is just the first player's input distribution in $\mathcal{G}^{\otimes n-|T|}$. This means that we can view f as inducing a first-player strategy f' on $\mathcal{G}^{n'}$ for $n' = n - |T|$ by fixing the appropriate part of f's input to X'_T. Note that $n' = \Omega(n)$ with overwhelming probability.

Part of (X, X') being good means that (f, g, h) has $\text{poly}(\alpha)$ probability of winning on (X', Y') when Y' is sampled from the distribution of the second player's input in $\mathcal{G}^{\otimes n}$ conditioned on the first player's input being X'. We split the sampling of Y' into two parts: Y'_T and Y'_{-T}. We show that we can sample and fix Y'_T, and use it to define g' and h' analogously to f' (for h' implicitly defining Z'_T such that $(Z'_T)_i = 2 - (X'_T)_i - (Y'_T)_i$), such that with $\text{poly}(\alpha)$ probability over the choice of (X'_{-T}, Y'_{-T}):

- (f', g', h') wins on (X'_{-T}, Y'_{-T}), and
- (X, X') is good. In particular, since X_{-T} is the all-ones string, we have $f'(X'_{-T})_i = f(X')_i = f(X)_i$ for all but $O(\log 1/\alpha)$ values of $i \notin T$ for which $(X'_{-T})_i = 1$.

This implies that up to a difference in its outputs for $O(\log 1/\alpha)$ coordinates, f' might as well be the constant function that always outputs $f(X)_{-T}$.

We could in principle continue onwards, eventually finding a smaller n'' (still $\Omega(n)$) such that $\mathcal{G}^{\otimes n''}$ has a strategy (f'', g'', h'') that wins in all but $O(\log 1/\alpha)$ coordinates with probability $\text{poly}(\alpha)$, but consists only of constant functions. This is a contradiction unless $\alpha \leq e^{-\Omega(n)}$. For simplicity, however, we instead directly show that when f' is a constant function, the strategy (f', g', h') must lose in a constant fraction of the coordinates with all but exponentially small probability. This implies that α is $e^{-\Omega(n)}$.

2.3 Four-Point AND Distribution

We present a technical overview of the proof of Theorem 5.1 which shows an inverse polynomial bound for the parallel repetition value for the four-point AND distribution. The details can be found in Section 5.

We first note the following observation about the set of points $\mathcal{S} := \{(0,0,0), (0,1,0), (1,0,0), (1,1,1)\}$. Let $\mathcal{G}$ be any game on $\mathcal{S}$ with value less than one and fix any strategy (f, g, h) for the players for one copy of the game $\mathcal{G}$. Observe that whenever Charlie receives 1, he knows that the inputs of Alice and Bob are 1. Consider the following two cases.

- Case A: Suppose the answers of Alice and Bob on input 1 are such that there is *no answer* that Charlie can give to satisfy the predicate, then the strategy loses on the input $(1, 1, 1)$.
- Case B: On the other hand, suppose the answers of Alice and Bob on input 1 are such that *there exists an answer for Charlie* that satisfies the predicate, then we can assume that Charlie answers this on input 1. Since the value of the game is less than one and this strategy succeeds on $(1, 1, 1)$, the strategy must fail on one of the remaining three points $\{(0,0,0), (0,1,0), (1,0,0)\}$. For these points, Charlie's input is fixed to zero and in particular, his answer is also fixed. Therefore, the predicate V when restricted to these inputs, induces a predicate $\tilde{V}$ which depends only on the inputs and outputs of Alice and Bob. The game $\mathcal{G}$ thus defines a two-player game on the uniform distribution $\tilde{\mathcal{S}} := \{(0,0), (0,1), (1,0)\}$ with predicate $\tilde{V}$.

Next, we consider the n-fold parallel repetition of the game.

Pre-processing the game. Let Q be the uniform distribution over $\mathcal{S}$ and let $P = Q^{\otimes n}$. We will always maintain a product event of the form $E = E^1 \times E^2 \times \{0,1\}^n$ on the players inputs where E^1 is a subset of Alice's inputs and E^2 is a subset of Bob's inputs. We begin by showing that we can assume without loss of generality that conditioned on a large product event across Alice's and Bob's inputs, all coordinates satisfy a property similar to case B with high probability, otherwise, we would get exponential decay of the parallel repetition value. For each $i \in [n]$, let $\tilde{L}_i$ denote the event that Alice and Bob receive 1 in the i-th coordinate and their answers are such that *there is no answer for Charlie* that satisfies the predicate. Let $\tilde{W}_i$ denote the complement of $\tilde{L}_i$. For $S \subseteq [n]$, let W_S denote the event that the players win the game in coordinate i for all $i \in S$. Note that whenever $\tilde{L}_i$ happens, the players lose in the i-th coordinate. Thus,

$$\Pr[W_{[n]}] \leq \Pr[\tilde{W}_1] \cdot \Pr[\tilde{W}_2|\tilde{W}_1] \cdots \Pr[\tilde{W}_n|\tilde{W}_1, \ldots, \tilde{W}_{n-1}]. \quad (1)$$

While there exists a coordinate $i \in [n]$ such that $\Pr[\tilde{L}_i|E]$ is significant, i.e., $\Pr[\tilde{L}_i|E] \geq 1/n^{4\delta}$ for some small constant $\delta > 0$, we will try to *condition on* $\tilde{W}_i$ and proceed. (To ensure that we maintain a product event across Alice and Bob, we will also randomly fix their

inputs and answers in the i-th coordinate and update E based on this fixing and proceed.[4]) If this conditioning process happens more than $n^{5\delta}$ times, then Equation (1) implies that the probability of winning all coordinates is at most $(1-1/n^{4\delta})^{n^{5\delta}} \leq \exp(-\Omega(n^{\delta}))$. It suffices to study the other case, that is, this conditioning process happens at most $n^{5\delta}$ times.

Reduction to a two-player game. We are left with a product event E of the form $E^1 \times E^2 \times \{0,1\}^n$ of measure at least $\exp(-\Omega(n^{5\delta}))$ such that for all $i \in [n]$, we have $\Pr[\tilde{L}_i|E] \leq 1/n^{4\delta}$ and we will show that the probability of winning all coordinates when the inputs are drawn from $P|E$ is at most $n^{-\Omega(\delta)}$. Let $\mathcal{A}_i$ be the set of answers a_i such that with significant probability (namely, more than $1/n^{2\delta}$ probability over $P|E$), Alice's input in i-th coordinate is 1 and her output in the i-th coordinate is a_i. Define $\mathcal{B}_i$ for Bob similarly. Since Alice's and Bob's inputs are independent under P and E is a product event across Alice and Bob, and $\Pr[\tilde{L}_i|E] \leq 1/n^{4\delta}$, it follows that for every pair of answers of Alice and Bob in $\mathcal{A}_i \times \mathcal{B}_i$, there is an answer that Charlie can give so that the predicate is satisfied when all players get 1. Define a product event G_i across Alice and Bob which is true if and only if whenever Alice and Bob get input 1 in the i-th coordinate, they answer from $\mathcal{A}_i \times \mathcal{B}_i$. A union bound implies that

$$\Pr[G_i|E] \geq 1 - \frac{|\mathcal{A}|}{n^{2\delta}} - \frac{|\mathcal{B}|}{n^{2\delta}} \geq 1 - O(n^{-\delta}). \tag{2}$$

We now randomly fix an input $z \in \{0,1\}^n$ to Charlie. Let K denote the set of coordinates that are zero in z, and let m denote $|K|$. With all but exponentially small probability, $m = \Omega(n)$. We also randomly fix the inputs x_{-K}, y_{-K} to Alice and Bob in coordinates outside K. Pick a random subset $S \subseteq K$ of size m^{ϵ} for some constant $0 < \epsilon < \delta$. We have (in expectation over z, x_{-K}, y_{-K}):

$$\begin{aligned}\Pr[W_S|E] \leq{}& \Pr[\vee_{i\in S}\neg G_i|E] \\ &+ \Pr[\wedge_{i\in S} G_i \wedge W_S|E, z, x_{-K}, y_{-K}]\end{aligned} \tag{3}$$

The first term $\Pr[\vee_{i\in S}\neg G_i|E]$ is at most $O(n^{-\delta+\epsilon})$ by Equation (2) and a union bound. To analyze the second term, we will define a two-player game $\tilde{\mathcal{G}}$ such that the probability of winning the coordinates in S in the m-fold parallel repetition of $\tilde{\mathcal{G}}$ is exactly the second term in the R.H.S. of Equation (3). For now, we will define a game $\tilde{\mathcal{G}}_i$ for each $i \in [n]$. Although these games can be different for different $i \in [n]$, there are only finitely many possibilities for $\tilde{\mathcal{G}}_i$ and we simply restrict our attention to the game that appears in most number of coordinates. The query distribution for $\tilde{\mathcal{G}}_i$ is the uniform distribution on $\tilde{\mathcal{S}} = \{(0,0),(1,0),(1,1)\}$. The predicate $\tilde{V}_i$ is $\tilde{V}_i(x,y,a,b) := V(x,y,0,a,b,h(z)_i) \wedge (x = 1 \implies a \in \mathcal{A}_i) \wedge (y = 1 \implies b \in \mathcal{B}_i)$. Note that the value of $\tilde{\mathcal{G}}_i$ is less than one. To see this, given any strategy $(\tilde{f}, \tilde{g})$ for $\tilde{\mathcal{G}}$ with value one, there is a strategy $\tilde{h}$ for Charlie on input 1 such $(\tilde{f}, \tilde{g}, \tilde{h})$ succeeds on $(1,1,1)$, since when Alice answers from $\mathcal{A}_i$ on input 1 and Bob from $\mathcal{B}_i$ on input 1, there is an answer Charlie can give on input 1 to satisfy the predicate. Define the strategy $\tilde{h}$ to output $h(z)_i$ when Charlie receives 0. We know that $(\tilde{f}, \tilde{g}, \tilde{h})$ must fail on one of the remaining points in $\{(0,0,0),(0,1,0),(1,0,0)\}$. This implies that $(\tilde{f}, \tilde{g})$ falsifies the predicate $\tilde{V}_i$ at the corresponding point in $\tilde{\mathcal{S}}$. We use two-player parallel repetition techniques and show that for a random $S \subseteq K$ of size m^{ϵ}, the probability of winning the m-fold parallel repetition of the two-player game in coordinates in S is at most $\exp(-\Omega(m^{\epsilon}))$. We remark that we are able to apply two-player parallel repetition techniques even though the measure of E could be smaller than $\exp(-\Omega(|S|))$, and this is because the set S was chosen randomly. Thus, the second term in the R.H.S. of Equation (3) is bounded by $\exp(-\Omega(m^{\epsilon})) = \exp(-\Omega(n^{\epsilon}))$. This completes the proof overview.

[4]This can be done since $\tilde{W}_i$ and $\tilde{L}_i$ depend only on the inputs and answers to Alice and Bob in the i-th coordinate.

2.4 Playerwise Connected Games

We present an overview of the proof of Theorem 6.1 which shows an inverse polynomial bound on the parallel repetition value of all playerwise connected games. The details are in Section 6. We focus on the case of the uniform distribution Q over $\mathcal{S} := \{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$. Let $P = Q^{\otimes n}$. For a random variable W, we use P_W to denote the distribution of W where the probability space is P. We use (X,Y,Z) to denote inputs of the three players for the game $\mathcal{G}^{\otimes n}$.

Our proof builds on the framework of the Parallel Repetition Theorem from [10, 27, 36]. We now describe this framework. Let $\mathcal{G}$ be a game whose query distribution is Q and whose value is less than one. Consider its n-fold parallel repetition. Using an inductive argument, it suffices to show that for every large *product event* $E = E^1 \times E^2 \times E^3$ across the players' inputs to the n-fold game $\mathcal{G}^{\otimes n}$, when the inputs are drawn from the distribution $P|E$, there exists some *hard coordinate* $i \in [n]$, meaning that the probability of winning the game in the i-th coordinate is $1-\epsilon$ for some constant $\epsilon > 0$. Since the event E has large measure, it cannot reveal too much information about too many coordinates, and hence the distribution of the marginal of $P|E$ on the i-th coordinate is similar to the original distribution Q for most $i \in [n]$. It then suffices to show a way for the players to approximately embed the inputs they receive for the original game $\mathcal{G}$, into the i-th coordinate of the inputs to the n-fold game drawn according to the distribution $P|E$. In order to do this, they need to be able to sample the remaining $n-1$ coordinates of the inputs according to the correct distribution. To do this, as in [10, 27, 36], we define a dependency breaking random variable R as follows. The random variable R for each copy $i \in [n]$ of the game, independently, chooses two players uniformly at random and samples the inputs to those players according to the distribution induced by $P|E$. Let $(x_i, y_i, z_i) \in \mathrm{supp}(Q)$. If the players can jointly sample from $P_{R_{-i}|E,x_i,y_i,z_i}$, then since R breaks the correlation between the player's inputs, any player can independently sample the rest of their own input given R_{-i}. Note that each player only knows one of x_i, y_i and z_i and it is not evident how the players can jointly sample from $P_{R_{-i}|E,x_i,y_i,z_i}$. Ideally, one would like to show that $P_{R_{-i}|E,x_i,y_i,z_i}$ is close to some global distribution for all $(x_i, y_i, z_i) \in \mathrm{supp}(Q)$; in such a case, the global distribution would be $P_{R_{-i}|E}$. We denote this by $P_{R_{-i}|E} \approx P_{R_{-i}|E,x_i,y_i,z_i}$. This would mean that the players only need to sample from a global distribution $P_{R_{-i}|E}$ and this can be done using shared randomness. Prior works [10, 27, 36] showed that $P_{R_{-i}|E,x_i,y_i,z_i} \approx P_{R_{-i}|E,x_i,y_i}$, but it is not clear if this suffices to prove the desired result.

In our work, we show that for the query distribution Q, we have $P_{R_{-i}|E} \approx P_{R_{-i}|E,x_i,y_i,z_i}$ for all $(x_i, y_i, z_i) \in \mathrm{supp}(Q)$. (We prove a

similar result for all playerwise connected games.) By choosing parameters appropriately and by Bayes' Rule, it will suffice to show that

$$P_{X_i,Y_i,Z_i|r_{-i},E} \approx P_{X_i,Y_i,Z_i|E} \quad \text{for most } r_{-i} \sim P_{R_{-i}|E}. \tag{4}$$

The key idea we use is to modify the distribution of P in the i-th coordinate as follows. Let Γ be a product distribution over $\{0,1\}^3$ such that the marginal on each player's input agrees with that of Q. The idea is to consider the distribution $P_{-i}\Gamma$, which is a product of n independent distributions, where the distribution in the i-th coordinate is Γ and the distribution in every other coordinate is Q. We can recover the original distribution P from $P_{-i}\Gamma$ by conditioning on some event that depends on the i-th coordinate (and possibly on some additional randomness).

Note that $(P_{-i}\Gamma)_{X,Y,Z|r_{-i}}$ is a product distribution across the inputs of the players, since R is a correlation breaking random variable and since the distribution of $P_{-i}\Gamma$ in the i-th coordinate was a product distribution to begin with. Since $E = E^1 \times E^2 \times E^3$ is a product event, the i-th coordinate of $(P_{-i}\Gamma)_{X,Y,Z|r_{-i},E}$ has a product distribution across the players. Finally, since the marginal of Γ on any player's input agrees with that of Q, we have

$$\begin{aligned} &(P_{-i}\Gamma)_{X_i,Y_i,Z_i|r_{-i},E} \\ &= (P_{-i}\Gamma)_{X_i|r_{-i},E^1} \times (P_{-i}\Gamma)_{Y_i|r_{-i},E^2} \times (P_{-i}\Gamma)_{Z_i|r_{-i},E^3} \\ &= P_{X_i|r_{-i},E^1} \times P_{Y_i|r_{-i},E^2} \times P_{Z_i|r_{-i},E^3} \quad \text{for all } r_{-i}. \end{aligned} \tag{5}$$

To study the distribution $P_{X_i|r_{-i},E^1}$, we will study the distribution $P_{X_i,Y_i,Z_i|r_{-i},E^1}$ and show that it is close to Q. The support of the latter distribution is contained in $\mathcal{S}$. Observe that the distribution $P_{Y_i,Z_i|r_{-i},E^1,X_i=x_i}$ is exactly the uniform distribution over $\{(y_i,z_i) \in \{0,1\}^2 : (x_i,y_i,z_i) \in \mathrm{supp}(Q)\}$ for all $x_i \in \{0,1\}$.[5] This implies that the probabilities assigned by $P_{X_i,Y_i,Z_i|r_{-i},E^1}$ to all points in $\mathcal{S}$ of the form $(1,*,*)$ are identical, and similarly, the probabilities assigned to all points in $\mathcal{S}$ of the form $(0,*,*)$ are identical. To conclude that the probabilities assigned to $(0,0,0)$ and $(1,0,0)$ are close for most $r_{-i} \sim P_{R_{-i}|E^1}$, we use techniques similar to [10, 27, 36]. We thus show that the probabilities assigned to all points in $\mathcal{S}$ are similar and hence the distribution of $P_{X_i,Y_i,Z_i|r_{-i},E^1}$ is close to Q for most $r_{-i} \sim P_{R_{-i}|E^1}$. This along with a similar argument for the second and third terms in the R.H.S. of Equation (5) (and the fact that the marginals of Γ on any player agree with that of Q) implies that

$$\begin{aligned} P_{X_i|r_{-i},E^1} &\approx (P_{-i}\Gamma)_{X_i} \quad \text{for most } r_{-i} \sim P_{R_{-i}|E^1} \\ P_{Y_i|r_{-i},E^2} &\approx (P_{-i}\Gamma)_{Y_i} \quad \text{for most } r_{-i} \sim P_{R_{-i}|E^2} \\ P_{Z_i|r_{-i},E^3} &\approx (P_{-i}\Gamma)_{Z_i} \quad \text{for most } r_{-i} \sim P_{R_{-i}|E^3} \end{aligned} \tag{6}$$

Let us pretend for now that Equation (6) actually holds for most $r_{-i} \sim P_{R_{-i}|E}$. If so, this, along with Equation (5) would imply that

$$(P_{-i}\Gamma)_{X_i,Y_i,Z_i|r_{-i},E} \approx (P_{-i}\Gamma)_{X_i,Y_i,Z_i} \quad \text{for most } r_{-i} \sim P_{R_{-i}|E}.$$

We now condition both sides of the above equation on an event T_i such $\Gamma|T_i = Q$. Note that T_i depends only on the inputs in coordinate i and some additional shared randomness (see the full version for details.) This implies that

$$P_{X_i,Y_i,Z_i|r_{-i},E} \approx P_{X_i,Y_i,Z_i} \quad \text{for most } r_{-i} \sim P_{R_{-i}|E}.$$

This along with the fact that the distribution of $P_{X_i,Y_i,Z_i|E}$ is close to Q for most coordinates $i \in [n]$ completes the proof of Equation (4), under the *incorrect assumption* that the distribution of r_{-i} in Equation (6) was $P_{R_{-i}|E}$. To fix this, we use the property that for any random variable G, we have $P[G=g|E] \le P[G=g|E^1] \cdot \frac{P[E^1]}{P[E]} \le P[G=g|E^1] \cdot \frac{1}{P[E]}$. This allows us to connect probabilities over $P|E$ and probabilities over $P|E^1$, $P|E^2$ and $P|E^3$. This is the place where we incur a loss of $\frac{1}{P(E)}$ and as a result, our bound only holds for polynomially large events.

[5] To see this, note that the distribution $P_{XYZ|r_{-i}}$ is a product distribution across coordinates where the marginal in the i-th coordinate is simply the uniform distribution over $\mathcal{S}$. Once we condition on $X_i = x_i$ for any $x_i \in \{0,1\}$, the distribution $P_{Y,Z|r_{-i},X_i=x_i}$ is still a product distribution across coordinates and in the i-th coordinate is exactly the uniform distribution over $\{(y_i,z_i) \in \{0,1\}^2 : (x_i,y_i,z_i) \in \mathcal{S}\}$. If we further condition on E^1, it only affects the distribution of inputs in coordinates other than i.

3 PRELIMINARIES

Let $\mathbb{N} = \{1,2,\ldots\}$ be the set of all natural numbers. For each $n \in \mathbb{N}$, we use $[n]$ to denote the set $\{1,2,\ldots,n\}$.

We will mostly follow [10, 27, 28] for notation.

3.1 Probability Distributions

We will use calligraphic letters to denote sets, capital letters to denote random variables and small letters to denote values.

Let P be a distribution (with the underlying *finite* set clear from context). For a random variable X, we use P_X to denote the distribution of X, that is, $P_X(x) = P(X=x)$. For random variables X and Y, we use P_{XY} to denote the joint distribution of X and Y. For an event E with $P(E) > 0$, we use $P_{X|E}$ to denote the distribution of X conditioned on the event E, given by

$$P_{X|E}(x) = \frac{P(X=x \wedge E)}{P(E)}.$$

Suppose R is a random variable, and r is such that $P_R(r) > 0$. We will frequently use the shorthand $P_{X|r}$ to denote the distribution $P_{X|R=r}$.

Let P_X and Q_X be distributions over a set $\mathcal{X}$. The L^1-distance (or ℓ_1-norm) between P_X and Q_X is defined as $\|P_X - Q_X\|_1 = \sum_{x\in\mathcal{X}} |P_X(x) - Q_X(x)|$.

We will also be using the following facts:

Fact 3.1. *(Chernoff Bounds, see [32] for reference) Let $X_1,\ldots,X_n \in \{0,1\}$ be independent random variables each with mean μ, and let $X = \sum_{i=1}^n X_i$. Then, for all $\delta \in (0,1)$, it holds that*

$$\Pr[X \le (1-\delta)\mu n] \le e^{-\frac{\delta^2\mu n}{2}},$$

$$\Pr[X \ge (1+\delta)\mu n] \le e^{-\frac{\delta^2\mu n}{3}}.$$

$$\Pr[X - \mu n \ge \delta n] \le e^{-2\delta^2 n}.$$

Fact 3.2. *Let P_X and Q_X be probability distributions over a set $\mathcal{X}$, and let $W \subseteq \mathcal{X}$ be an event such that $P_X(W), Q_X(W) > 0$. Then,*

$$\left\|P_{X|W} - Q_{X|W}\right\|_1 \le \frac{2}{Q_X(W)} \cdot \|P_X - Q_X\|_1$$

Proof. The proof is deferred to the full version of this paper. □

Fact 3.3. *Let P be a probability distribution, and let X be a random variable over a set $\mathcal{X}$, and let E be any event with $P(E) > 0$. Let $\alpha > 0$ be arbitrary, and let $\mathcal{T} = \{x \in \mathcal{X} : P(E \mid x) \geq \alpha \cdot P(E)\}$. Then, it holds that*

$$P(X \in \mathcal{T} \mid E) > 1 - \alpha.$$

Proof. The proof is deferred to the full version of this paper. □

Fact 3.4. *Let P be a probability distribution, and let X be a random variable over a set $\mathcal{X}$, and let E be any event. Then, there exists $x \in \mathcal{X}$ such that $P(E \mid X = x) \geq P(E)$.*

3.2 Multiplayer Games

Definition 3.5. *(Multiplayer Game) A k-player game $\mathcal{G}$ is a tuple $\mathcal{G} = (\mathcal{X}, \mathcal{A}, Q, V)$, where the question set $\mathcal{X} = \mathcal{X}^1 \times \cdots \times \mathcal{X}^k$, and the answer set $\mathcal{A} = \mathcal{A}^1 \times \cdots \times \mathcal{A}^k$ are finite sets, Q is a probability distribution over $\mathcal{X}$, and $V : \mathcal{X} \times \mathcal{A} \to \{0, 1\}$ is a predicate.*

Definition 3.6. *(Game Value) Let $\mathcal{G} = (\mathcal{X}, \mathcal{A}, Q, V)$ be a k-player game.*

For a sequence $\left(f^j : \mathcal{X}^j \to \mathcal{A}^j\right)_{j \in [k]}$ of functions, define the function $f = f^1 \times \cdots \times f^k : \mathcal{X} \to \mathcal{A}$ by

$$f\left(x^1, \ldots, x^k\right) = \left(f^1(x^1), \ldots, f^k(x^k)\right).$$

We use the term product functions *to denote functions f defined in this manner.*

The value $\mathrm{val}(\mathcal{G})$ of the game $\mathcal{G}$ is defined as

$$\mathrm{val}(\mathcal{G}) = \max_{f = f^1 \times \cdots \times f^k} \Pr_{X \sim Q}\left[V(X, f(X)) = 1\right],$$

where the maximum is over all product functions $f = f^1 \times \cdots \times f^k$. The functions $(f^j)_{j \in [k]}$ are called player strategies.

Fact 3.7. *The value of the game is unchanged even if we allow the player strategies to be randomized, that is, we allow the strategies to depend on some additional shared and private randomness.*

Definition 3.8. *(Parallel Repetition of a game)*
Let $\mathcal{G} = (\mathcal{X}, \mathcal{A}, Q, V)$ be a k-player game. We define its n-fold repetition as $\mathcal{G}^{\otimes n} = (\mathcal{X}^{\otimes n}, \mathcal{A}^{\otimes n}, P, V^{\otimes n})$. The sets $\mathcal{X}^{\otimes n}$ and $\mathcal{A}^{\otimes n}$ are defined to be the n-fold product of the sets $\mathcal{X}$ and $\mathcal{A}$ with themselves respectively. The distribution P is the n-fold product of the distribution Q with itself, that is, $P(x) = \prod_{i=1}^n Q(x_i)$. The predicate $V^{\otimes n}$ is defined as $V^{\otimes n}(x, a) = \bigwedge_{i=1}^n V(x_i, a_i)$.

Note that we use the notation $\mathcal{X}^{\otimes n}$ instead of the standard notation $\mathcal{X}^n$ so as to avoid confusion with the sets $\mathcal{X}^1, \ldots, \mathcal{X}^k$.

Following the notation in [10], we use subscripts to denote the coordinates in the parallel repetition, and superscripts to denote the players. For example, for $i \in [n]$ and $j \in [k]$, we will use x_i^j to refer to the question to the j^{th} player in the i^{th} repetition of the game. Similarly, x_i will refer to the vector of questions to the k players in the i^{th} repetition, and x^j will refer to the vector of questions received by the j^{th} player over all repetitions. We use x^{-j} to refer to the questions to all players except the j^{th} player, and use x_{-i} to refer to the questions in all coordinates except the i^{th} coordinate.

When we are dealing with 3 player games, we will not be using superscripts to refer to different players, and rather use the notation $\mathcal{G} = (\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}, \mathcal{A} \times \mathcal{B} \times C, Q, V)$. That is, we use $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ in place of $\mathcal{X}^1, \mathcal{X}^2, \mathcal{X}^3$ and $\mathcal{A}, \mathcal{B}, C$ in place of $\mathcal{A}^1, \mathcal{A}^2, \mathcal{A}^3$.

We will use the notation $A \lesssim B$ (resp. $\gtrsim$) to mean that $A \leq c \cdot B$ (resp. $A \geq c \cdot B$) for some constant $c > 0$. For our purposes, we will allow the constant to depend on the size of the (initial) game being considered, but not on the number of repetitions.

3.3 Playerwise Connected Games

We will particularly be interested in a special class of games, which we refer to as playerwise connected games. Before we define this class, we recall the following definition:

Definition 3.9. *($(k-1)$-connection graph [10])*
Let $\mathcal{G} = (\mathcal{X}, \mathcal{A}, Q, V)$ be a k-player game, and let $\mathcal{S} \subseteq \mathcal{X}$ be the support of Q. We define its (undirected) $(k-1)$-connection graph $H_{\mathcal{G}}$ as follows. The vertex set of $H_{\mathcal{G}}$ is $\mathcal{S}$, and there is an edge between $x, y \in \mathcal{S}$ if and only if they differ in the question to exactly one of the players. That is, $\{x, y\}$ is an edge if and only if there exists $j \in [k]$ such that $x^{-j} = y^{-j}$ and $x^j \neq y^j$.

We say that a game $\mathcal{G}$ is connected *if the graph $\mathcal{H}_{\mathcal{G}}$ is connected.*

We will define a game to be *playerwise connected* if the projection of the above graph with respect to each of the players is connected. This is formally defined as follows:

Definition 3.10. *(Playerwise Connected Game)*
Let $\mathcal{G} = (\mathcal{X}, \mathcal{A}, Q, V)$ be a k-player game, and let $\mathcal{S} \subseteq \mathcal{X}$ be the support of Q. We assume that for all $j \in [k]$, and for all $x^j \in \mathcal{X}^j$, the j^{th} player is given the question x^j with positive probability.

For every $j \in [k]$, we define the graph $H_{\mathcal{G}}^j$ as the graph with vertex set $\mathcal{X}^j$, with an edge between $x^j, y^j \in \mathcal{X}^j$ if and only if there exists $x^{-j} \in \mathcal{X}^{-j}$ such that both $(x^{-j}, x^j) \in \mathcal{S}$ and $(x^{-j}, y^j) \in \mathcal{S}$. We say that the game $\mathcal{G}$ is playerwise connected *if $H_{\mathcal{G}}^j$ is connected for each $j \in [k]$.*

Note that the assumption on $\mathcal{G}$ in the above definition is without loss of generality, because we can simply remove each x^j which occurs with zero probability, without affecting the game in any meaningful way.

3.4 Previously Known Results

We state the known results on parallel repetition that will be useful for us.

Theorem 3.11. *(Parallel Repetition for 2-Player Games [36]) Let $\mathcal{G} = (\mathcal{X} \times \mathcal{Y}, \mathcal{A} \times \mathcal{B}, Q, V)$ be a 2-player game such that $\mathrm{val}(\mathcal{G}) < 1$. Then, there exists a constant $c = c(\mathcal{G}) > 0$ such that for every $n \in \mathbb{N}$, it holds that $\mathrm{val}(\mathcal{G}^{\otimes n}) \leq 2^{-cn}$.*

Theorem 3.12. *(Parallel Repetition for Connected Games [10]) Let $\mathcal{G}$ be a* connected *k-player game (see Definition 3.9) such that $\mathrm{val}(\mathcal{G}) < 1$. Then, there exists a constant $c = c(\mathcal{G}) > 0$ such that for every $n \in \mathbb{N}$, it holds that $\mathrm{val}(\mathcal{G}^{\otimes n}) \leq 2^{-cn}$.*

Theorem 3.13. *(Parallel Repetition for The GHZ Game [21, 28]) Let $\mathcal{G} = (\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}, \mathcal{A} \times \mathcal{B} \times C, Q, V)$ be a 3-player game with $\mathcal{X} = \mathcal{Y} = \mathcal{Z} = \{0, 1\}$, and Q the uniform distribution over $\mathcal{S} = \{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$, and such that $\mathrm{val}(\mathcal{G}) < 1$. Then, there exists a constant $c = c(\mathcal{G}) > 0$, such that for every $n \in \mathbb{N}$, it holds that $\mathrm{val}(\mathcal{G}^{\otimes n}) \leq n^{-c}$.*

3.5 Some Results on Multiplayer Games

3.5.1 Restriction to Uniform Distributions. We state a lemma from [17], which shows that it suffices to prove parallel repetition in the case when the game's distribution is the uniform distribution over its support. For the sake of completeness, we also include a short proof.

Lemma 3.14. *Let $\mathcal{G} = (\mathcal{X}, \mathcal{A}, Q, V)$ be a k-player game such that $\mathrm{val}(\mathcal{G}) < 1$, and let $\mathcal{S} \subseteq \mathcal{X}$ be the support of Q. Let $\tilde{\mathcal{G}} = (\mathcal{X}, \mathcal{A}, U, V)$, where U is the uniform distribution over $\mathcal{S}$. Let $v : \mathbb{N} \cup \{0\} \to [0,1]$ be the function defined by $v(n) = \mathrm{val}(\tilde{\mathcal{G}}^{\otimes n})$, for every $n \in \mathbb{N} \cup \{0\}$, with the convention $v(0) = 1$. Then,*

(a) $v(1) = \mathrm{val}(\tilde{\mathcal{G}}) < 1$.
(b) There exists a constant $\beta > 0$ such that $\mathrm{val}(\mathcal{G}^{\otimes n}) \leq 2v(\lfloor \beta n \rfloor)$ for all $n \in \mathbb{N}$.

Proof. The proof is deferred to the full version of this paper. □

3.5.2 A Restriction on Predicates. We show that to prove parallel repetition, it suffices to assume that the game has the following property: If some input y^j for the j^{th} player completely determines the input y to all the players, then on input y, the game's predicate does not depend on the answer a^j given by the j^{th} player.

A recursive application of the next lemma shows that we can assume the aforementioned property.

Lemma 3.15. *Let $\mathcal{G} = (\mathcal{X}, \mathcal{A}, Q, V)$ be a k-player game. Suppose $y \in \mathcal{X}, j \in [k]$ are such that y is the unique input with $Q(y) > 0$ that has y^j as the input to the j^{th} player. Then, there exists a predicate V' such that the game $\mathcal{G}' = (\mathcal{X}, \mathcal{A}, Q, V')$ satisfies:*

(a) For every $a, b \in \mathcal{A}$ with $a^{-j} = b^{-j}$, it holds that $V'(y, a) = V'(y, b)$.
(b) For every $n \in \mathbb{N}$, it holds that $\mathrm{val}(\mathcal{G}^{\otimes n}) \leq \mathrm{val}(\mathcal{G}'^{\otimes n})$.
(c) $\mathrm{val}(\mathcal{G}') = \mathrm{val}(\mathcal{G})$.

Proof. The proof is deferred to the full version of this paper. □

3.5.3 An Inductive Parallel Repetition Criterion. We state a parallel repetition criterion from [28, 36]. For the sake of completeness, we also include a proof.

Definition 3.16. *Let $\mathcal{G} = (\mathcal{X}, \mathcal{A}, Q, V)$ be a k-player game, and let Q' be some distribution over $\mathcal{X}$. We define $\mathcal{G} \mid Q'$ to be the game $\mathcal{G} \mid Q' = (\mathcal{X}, \mathcal{A}, Q', V)$.*

Definition 3.17. *Let $\mathcal{G} = (\mathcal{X}, \mathcal{A}, Q, V)$ be a k-player game, and $\mathcal{G}^{\otimes n} = (\mathcal{X}^{\otimes n}, \mathcal{A}^{\otimes n}, P, V^{\otimes n})$ be its n-fold repetition. For each $i \in [n]$, we define the value of the i^{th} coordinate of $\mathcal{G}^{\otimes n}$, denoted by $\mathrm{val}_i(\mathcal{G}^{\otimes n})$, to be the value of the game $(\mathcal{X}^{\otimes n}, \mathcal{A}^{\otimes n}, P, V')$, where $V'(x, a) = V(x_i, a_i)$.*

Lemma 3.18. *Let $\mathcal{G} = (\mathcal{X}, \mathcal{A}, Q, V)$ be a k-player game, and $\mathcal{G}^{\otimes n} = (\mathcal{X}^{\otimes n}, \mathcal{A}^{\otimes n}, P, V^{\otimes n})$ be its n-fold repetition. Suppose that there exists a constant $\epsilon > 0$, and a non-increasing function $\rho : \mathbb{N} \to [0,1]$ such that $\rho(n) \geq 2^{-O(n)}$, and for every $n \in \mathbb{N}$, and every product event $E = E^1 \times \cdots \times E^k \subseteq (\mathcal{X}^1)^{\otimes n} \times \cdots \times (\mathcal{X}^k)^{\otimes n} = \mathcal{X}^{\otimes n}$ with $P(E) \geq \rho(n)$, there exists a coordinate $i \in [n]$ such that $\mathrm{val}_i(\mathcal{G}^{\otimes n} \mid (P|E)) \leq 1 - \epsilon$. Then, there exists a constant $c > 0$ such that $\mathrm{val}(\mathcal{G}^{\otimes n}) \leq \rho(n)^c$ for every $n \in \mathbb{N}$.*

Proof. The proof is deferred to the full version of this paper. □

4 THE ANTI-CORRELATION GAME

In this section, we will focus on the following game.

Definition 4.1. *(The Anti-Correlation Game) The* anti-correlation *game, which we denote as $\mathcal{G} = (\{0,1\}^3, \{0,1\}^3, Q, V)$, is a 3-player game in which the query distribution Q is uniform over the set $\{(0,1,1), (1,0,1), (1,1,0)\}$ of strings of hamming-weight 2, and the win predicate $V : \{0,1\}^3 \times \{0,1\}^3 \to \{0,1\}$ is defined so that $V((x,y,z),(a,b,c)) = 1$ if and only if $\langle (x,y,z), (a,b,c) \rangle = xa + yb + zc = 1$.*

In words, a random pair of players receive 1, and these players must produce different bits.

If $V((x,y,z),(a,b,c)) = 1$, we say that (a,b,c) wins on (x,y,z).

We will denote this game by $\mathcal{G}$, and denote its query distribution by Q (i.e. the uniform distribution on $\{(0,1,1), (1,0,1), (1,1,0)\}$. Observe that the value of this game is $2/3$. We will show that parallel repetition exponentially decays the value of this game.

Theorem 4.2. *Let $\mathcal{G}$ be the anti-correlation game as in Definition 4.1. Then, there exists a constant $c > 0$ such that for every $n \in \mathbb{N}$, it holds that $\mathrm{val}(\mathcal{G}^{\otimes n}) \leq \exp(-c \cdot n)$.*

The proof of this theorem is deferred to the full version of this paper.

5 FOUR POINT AND DISTRIBUTION

This section is devoted to the proof of the following theorem:

Theorem 5.1. *Let $\mathcal{G} = (\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}, \mathcal{A} \times \mathcal{B} \times C, Q, V)$ be a 3-player game with $\mathcal{X} = \mathcal{Y} = \mathcal{Z} = \{0,1\}$, and Q the uniform distribution over*

$$\mathcal{S} = \{(0,0,0), (1,0,0), (0,1,0), (1,1,1)\} = \left\{(x,y,z) \in \{0,1\}^3 : z = x \wedge y\right\},$$

and such that $\mathrm{val}(\mathcal{G}) < 1$. Then, there exists a constant $c = c(\mathcal{G}) > 0$, such that for every $n \in \mathbb{N}$, it holds that $\mathrm{val}(\mathcal{G}^{\otimes n}) \leq n^{-c}$.

The proof of this theorem is deferred to the full version of this paper.

6 PLAYERWISE CONNECTED GAMES

In this section, we will prove the following theorem:

Theorem 6.1. *(Parallel Repetition for Playerwise Connected Games) Let $\mathcal{G}$ be a playerwise connected k-player game such that $\mathrm{val}(\mathcal{G}) < 1$. Then, there exists a constant $c = c(\mathcal{G}) > 0$ such that for every $n \in \mathbb{N}$, it holds that $\mathrm{val}(\mathcal{G}^{\otimes n}) \leq n^{-c}$.*

The proof of this theorem is deferred to the full version of this paper.

7 HAMMING WEIGHT ONE DISTRIBUTION WITH BINARY OUTPUTS

In this section, we analyze parallel repetition for three-player games with inputs drawn uniformly from the set $\mathcal{S} = \{(1,0,0), (0,1,0), (0,0,1)\}$ of hamming-weight one inputs, and having binary outputs. Formally, we prove the following theorem:

Theorem 7.1. *Let $\mathcal{G} = (\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}, \mathcal{A} \times \mathcal{B} \times C, Q, V)$ be a 3-player game with $\mathcal{X} = \mathcal{Y} = \mathcal{Z} = \mathcal{A} = \mathcal{B} = C = \{0,1\}$, and Q the*

uniform distribution over $\mathcal{S} = \{(1,0,0),(0,1,0),(0,0,1)\}$*, and such that* $\mathrm{val}(\mathcal{G}) < 1$*. Then, there exists a constant* $c = c(\mathcal{G}) > 0$*, such that for every* $n \in \mathbb{N}$*, it holds that* $\mathrm{val}(\mathcal{G}^{\otimes n}) \le 2^{-cn}$.

The proof of this theorem is deferred to the full version of this paper.

8 THREE PLAYER GAMES OVER BINARY ALPHABET

8.1 The Main Theorem

In this section, we prove the following main result:

Theorem 8.1. *Let* $\mathcal{G} = (\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}, \mathcal{A}\times\mathcal{B}\times C, Q, V)$ *be a 3-player game with* $\mathcal{X}=\mathcal{Y}=\mathcal{Z}=\mathcal{A}=\mathcal{B}=C=\{0,1\}$*, and such that* $\mathrm{val}(\mathcal{G}) < 1$*. Then, there exists a constant* $c = c(\mathcal{G}) > 0$*, such that for every* $n \in \mathbb{N}$*, it holds that* $\mathrm{val}(\mathcal{G}^{\otimes n}) \le n^{-c}$.

Proof. By Lemma 3.14, it suffices to only consider the case when the distribution Q is the uniform distribution over its support $\mathcal{S} \subseteq \{0,1\}^3$. Also notice that we only need to analyze $\mathcal{S}$ up to symmetry among the 3 players, and up to symmetry of the inputs 0 and 1 (that is, up to symmetries of the cube $\{0,1\}^3$).

When $\mathcal{G}$ is connected, Theorem 3.12 provides an inverse exponential bound $\mathrm{val}(\mathcal{G}^{\otimes n}) = 2^{-\Omega(n)}$. Therefore we only need to consider the case when $\mathcal{G}$ is not connected, or equivalently, the graph $\mathcal{H}_{\mathcal{G}}$ (see Definition 3.9) is not connected. Notice that the graph $\mathcal{H}_{\mathcal{G}}$ is the subgraph of the cubical graph $\{0,1\}^3$ induced by $\mathcal{S}$. Since $\mathcal{H}_{\mathcal{G}}$ is not connected, there must be a smallest connected component $\mathcal{S}' \subseteq \mathcal{S}$ in $\mathcal{H}_{\mathcal{G}}$ of size 1 or 2.

If $|\mathcal{S}'| = 2$, by symmetry assume that $S' = \{(1,1,0),(1,1,1)\}$. Then $S \setminus S'$ is contained in $\{(0,0,0),(0,0,1)\}$, which implies that the input (x,y,z) always satisfies $x = y$. This means that the game $\mathcal{G}$ is essentially a two-player game, where an inverse exponential decay bound is known by Theorem 3.11.

If $|\mathcal{S}'| = 1$, by symmetry assume that $S' = \{(1,1,1)\}$. Then $S \setminus S'$ is contained in $\{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\}$, and we perform a case analysis in below:

(1) $|S| \le 2$. Then $\mathcal{G}$ always degenerates to a two-player game, similar to the case of $|\mathcal{S}'| = 2$ above.
(2) $|S| = 3$. To avoid degeneracy it must hold that $(0,0,0) \notin S$, so by symmetry we only consider $S = \{(1,0,0),(0,1,0),(1,1,1)\}$, or equivalently, $S = \{(1,0,0),(0,1,0),(0,0,1)\}$. The specific game of interest, the anti-correlation game, was studied in Section 4. The general game with binary outputs was analyzed in Section 7, where we proved an inverse exponential decay bound (see Theorem 7.1).
(3) $|S| = 4$ and $(0,0,0) \in S$. By symmetry we consider $S = \{(0,0,0),(1,0,0),(0,1,0),(1,1,1)\}$. We proved an inverse polynomial decay bound for this four-point AND distribution in Section 5 (see Theorem 5.1).
(4) $|S| = 4$ and $(0,0,0) \notin S$:
Then, $S = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$. This is equivalent to the support of the GHZ game, and and an inverse polynomial decay bound is known (see Theorem 3.13)
(5) $|S| = 5$, that is, $S = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$. In particular, the game $\mathcal{G}$ is playerwise connected (see Definition 3.10), and we proved inverse polynomial decay bounds for all playerwise connected games in Section 6 (see Theorem 6.1). □

8.2 A General Game on Hamming Weight One Input

We observe that Theorem 8.1 works for arbitrary answer lengths in all cases except when the support $\mathcal{S}$ has $|\mathcal{S}| = 3$ with all disjoint points, for example, $\mathcal{S} = \{(1,0,0),(0,1,0),(0,0,1)\}$.

Next, we describe a very simple family of 3-player games $\{\mathcal{G}_k\}_{k\in\mathbb{N}}$, such that proving a bound on the value of parallel repetition for games in this family will extend Theorem 8.1 to all games with $\mathcal{X}=\mathcal{Y}=\mathcal{Z}=\{0,1\}$, and arbitrary answer sets $\mathcal{A},\mathcal{B},C$.

Definition 8.2. *For every* $k \in \mathbb{N}$*, we define a 3-player game* $\mathcal{G}_k = (\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}, \mathcal{A}_k\times\mathcal{B}_k\times C_k, Q, V_k)$*, with* $\mathcal{X}=\mathcal{Y}=\mathcal{Z}=\{0,1\}$*, and* Q *the uniform distribution over* $\mathcal{S} = \{(1,0,0),(0,1,0),(0,0,1)\}$*, as follows:*

(a) $\mathcal{A}_k = \mathcal{B}_k = \{0,1\}^k$*, and* $C_k = [k]$.
(b) For all $(x,y,z) \in \mathcal{S}$*, and* $(a,b,c) \in \mathcal{A}\times\mathcal{B}\times C$*,*

$$V_k((x,y,z),(a,b,c)) = \begin{cases} (a_i \wedge b_i)_{i\in[k]} = 0^k, & \text{if } (x,y,z) = (0,0,1) \\ a_c = 1, & \text{if } (x,y,z) = (0,1,0) \\ b_c = 1, & \text{if } (x,y,z) = (1,0,0) \end{cases}.$$

It is an easy check that $\mathrm{val}(\mathcal{G}_k) = 2/3$*. For every* $n \in \mathbb{N}$*, we define* $\rho_k(n) = \mathrm{val}(\mathcal{G}_k^{\otimes n})$.

Proposition 8.3. *Let* $\mathcal{G} = (\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}, \mathcal{A}\times\mathcal{B}\times C, Q, V)$ *be a 3-player game with* $\mathcal{X}=\mathcal{Y}=\mathcal{Z}=\{0,1\}$*, and* Q *the uniform distribution over* $\mathcal{S} = \{(1,0,0),(0,1,0),(0,0,1)\}$*, and such that* $\mathrm{val}(\mathcal{G}) < 1$*. Then, for every* $n \in \mathbb{N}$*, it holds that* $\mathrm{val}(\mathcal{G}^{\otimes n}) \le \rho_k(n)$*, where* $k = \max\{|\mathcal{A}|, |\mathcal{B}|, |C|\}$.

Proof. Let $\mathcal{G}$ be a game as specified. Without loss of generality, we assume that $\mathcal{A} = \mathcal{B} = C = [k]$. By Lemma 3.15, it also suffices to assume that the predicate V only depends on the answers of the two players that get input 0. We observe that since $\mathrm{val}(\mathcal{G}) < 1$, for every $a,b \in [k]$, if it holds that $V((0,0,1),(a,b,*)) = 1$, then for every $c \in [k]$, it holds that $V((0,1,0),(a,*,c)) \wedge V((1,0,0),(*,b,c)) = 0$

We consider $\mathcal{G}^{\otimes n} = ((\mathcal{X}\times\mathcal{Y}\times\mathcal{Z})^{\otimes n}, (\mathcal{A}\times\mathcal{B}\times C)^{\otimes n}, P, V^{\otimes n})$. Let $f,g,h : \{0,1\}^n \to [k]^n$ be optimal strategies for the game $\mathcal{G}^{\otimes n}$.

We define strategies $f_k : \{0,1\}^n \to \mathcal{A}_k^{\otimes n}$, $g_k : \{0,1\}^n \to \mathcal{B}_k^{\otimes n}$, $h_k : \{0,1\}^n \to C_k^{\otimes n}$, for the game $\mathcal{G}_k^{\otimes n}$, as follows: For every $x,y,z \in \{0,1\}^n$, and every $i \in [n]$, we define

$$f_k(x)_i = (V((0,1,0),(f(x)_i,*,c)))_{c\in[k]},$$

$$g_k(y)_i = (V((1,0,0),(*,g(y)_i,c)))_{c\in[k]},$$

$$h_k(z)_i = h(z)_i.$$

It is clear (from the above observation about $\mathcal{G}$) that if the strategies f,g,h win the game $\mathcal{G}^{\otimes n}$ on an input (x,y,z), then the strategies f_k,g_k,h_k also win the game $\mathcal{G}_k^{\otimes n}$ on input (x,y,z). This shows that $\rho_k(n) = \mathrm{val}(\mathcal{G}_k^{\otimes n}) \ge \mathrm{val}(\mathcal{G}^{\otimes n})$. □

REFERENCES

[1] Noga Alon and Bo'az Klartag. 2009. Economical toric spines via Cheeger's inequality. *J. Topol. Anal.* 1, 2 (2009), 101–111. https://doi.org/doi/10.1142/S1793525309000096

[2] Boaz Barak, Mark Braverman, Xi Chen, and Anup Rao. 2013. How to compress interactive communication. *SIAM J. Comput.* 42, 3 (2013), 1327–1363. https://doi.org/10.1137/100811969 (also in STOC 2010).

[3] Boaz Barak, Anup Rao, Ran Raz, Ricky Rosen, and Ronen Shaltiel. 2009. Strong parallel repetition theorem for free projection games. In *APPROX-RANDOM*. 352–365. https://doi.org/10.1007/978-3-642-03685-9_27

[4] Mihir Bellare, Oded Goldreich, and Madhu Sudan. 1998. Free bits, PCPs, and nonapproximability—towards tight results. *SIAM J. Comput.* 27, 3 (1998), 804–915. https://doi.org/10.1137/S0097539796302531 (also in FOCS 1995).

[5] Michael Ben-Or, Shafi Goldwasser, Joe Kilian, and Avi Wigderson. 1988. Multi-Prover Interactive Proofs: How to Remove Intractability Assumptions. In *STOC*. 113–131. https://doi.org/10.1145/62212.62223

[6] Mark Braverman and Ankit Garg. 2015. Small value parallel repetition for general games. In *STOC*. 335–340. https://doi.org/10.1145/2746539.2746565

[7] Mark Braverman, Anup Rao, Omri Weinstein, and Amir Yehudayoff. 2013. Direct products in communication complexity. In *FOCS*. 746–755. https://doi.org/10.1109/FOCS.2013.85

[8] Vašek Chvátal and Endre Szemerédi. 1988. Many hard examples for resolution. *J. Assoc. Comput. Mach.* 35, 4 (1988), 759–768. https://doi.org/10.1145/48014.48016

[9] Richard Cleve, Peter Høyer, Benjamin Toner, and John Watrous. 2004. Consequences and Limits of Nonlocal Strategies. In *CCC*. 236–249. https://doi.org/10.1109/CCC.2004.1313847

[10] Irit Dinur, Prahladh Harsha, Rakesh Venkat, and Henry Yuen. 2017. Multiplayer parallel repetition for expanding games. In *ITCS (LIPIcs, Vol. 67)*. Art. No. 37, 16. https://doi.org/10.4230/LIPIcs.ITCS.2017.37

[11] Irit Dinur and David Steurer. 2014. Analytical approach to parallel repetition. In *STOC*. 624–633. https://doi.org/10.1145/2591796.2591884

[12] Uriel Feige. 1991. On the Success Probability of the Two Provers in One-Round Proof Systems. In *Structure in Complexity Theory Conference*. 116–123. https://doi.org/10.1109/SCT.1991.160251

[13] Uriel Feige. 1998. A threshold of $\ln n$ for approximating set cover. *J. ACM* 45, 4 (1998), 634–652. https://doi.org/10.1145/285055.285059 (also in STOC 1996).

[14] Uriel Feige, Jeong Han Kim, and Eran Ofek. 2006. Witnesses for non-satisfiability of dense random 3CNF formulas. In *FOCS*. 497–508. https://doi.org/10.1109/FOCS.2006.78

[15] Uriel Feige, Guy Kindler, and Ryan O'Donnell. 2007. Understanding Parallel Repetition Requires Understanding Foams. In *CCC*. 179–192. https://doi.org/10.1109/CCC.2007.39

[16] Uriel Feige and Eran Ofek. 2007. Easily refutable subformulas of large random 3CNF formulas. *Theory Comput.* 3 (2007), 25–43. https://doi.org/10.4086/toc.2007.v003a002 (also in ICALP 2004).

[17] Uriel Feige and Oleg Verbitsky. 2002. Error reduction by parallel repetition—a negative result. *Combinatorica* 22, 4 (2002), 461–478. https://doi.org/10.1007/s00493-002-0001-0

[18] Lance Fortnow. 1989. *Complexity theoretic aspects of interactive proof systems*. Ph. D. Dissertation. MIT.

[19] Lance Fortnow, John Rompel, and Michael Sipser. 1994. On the power of multi-prover interactive protocols. *Theoret. Comput. Sci.* 134, 2 (1994), 545–557. https://doi.org/10.1016/0304-3975(94)90251-8

[20] H. Furstenberg and Y. Katznelson. 1991. A density version of the Hales-Jewett theorem. *J. Anal. Math.* 57 (1991), 64–119. https://doi.org/10.1007/BF03041066

[21] Uma Girish, Justin Holmgren, Kunal Mittal, Ran Raz, and Wei Zhan. 2021. Parallel Repetition for the GHZ Game: A Simpler Proof. In *APPROX-RANDOM*. 62:1–62:19. https://doi.org/10.4230/LIPIcs.APPROX/RANDOM.2021.62

[22] Uma Girish, Justin Holmgren, Kunal Mittal, Ran Raz, and Wei Zhan. 2022. Parallel Repetition For All 3-Player Games Over Binary Alphabet. *CoRR* abs/2202.06826 (2022). https://arxiv.org/abs/2202.06826

[23] Uma Girish, Kunal Mittal, Ran Raz, and Wei Zhan. 2022. Polynomial Bounds On Parallel Repetition For All 3-Player Games With Binary Inputs. *Electron. Colloquium Comput. Complex.* 43 (2022). https://eccc.weizmann.ac.il/report/2022/043/

[24] Daniel M. Greenberger, Michael A. Horne, and Anton Zeilinger. 1989. Going Beyond Bell's Theorem. In *Bell's Theorem, Quantum Theory and Conceptions of the Universe*. Springer Netherlands, 69–72. https://doi.org/10.1007/978-94-017-0849-4_10

[25] Johan Håstad. 2001. Some optimal inapproximability results. *J. ACM* 48, 4 (2001), 798–859. https://doi.org/doi/10.1145/502090.502098 (also in STOC 1997).

[26] Jan Hazla, Thomas Holenstein, and Anup Rao. 2016. Forbidden Subgraph Bounds for Parallel Repetition and the Density Hales-Jewett Theorem. *CoRR* abs/1604.05757 (2016). http://arxiv.org/abs/1604.05757

[27] Thomas Holenstein. 2009. Parallel repetition: simplifications and the no-signaling case. *Theory Comput.* 5 (2009), 141–172. https://doi.org/10.4086/toc.2009.v005a008 (also in STOC 2007).

[28] Justin Holmgren and Ran Raz. 2020. A Parallel Repetition Theorem for the GHZ Game. *CoRR* abs/2008.05059 (2020). https://arxiv.org/abs/2008.05059

[29] Justin Holmgren and Lisa Yang. 2019. The parallel repetition of non-signaling games: counterexamples and dichotomy. In *STOC*. 185–192. https://doi.org/10.1145/3313276.3316367

[30] Guy Kindler, Ryan O'Donnell, Anup Rao, and Avi Wigderson. 2008. Spherical Cubes and Rounding in High Dimensions. In *FOCS*. 189–198. https://doi.org/10.1109/FOCS.2008.50

[31] Kunal Mittal and Ran Raz. 2021. Block rigidity: strong multiplayer parallel repetition implies super-linear lower bounds for Turing machines. In *ITCS*. Art. No. 71, 15. https://doi.org/10.4230/LIPIcs.ITCS.2021.71

[32] Michael Mitzenmacher and Eli Upfal. 2005. *Probability and computing*. Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9780511813603 Randomized algorithms and probabilistic analysis.

[33] Itzhak Parnafes, Ran Raz, and Avi Wigderson. 1997. Direct product results and the GCD problem, in old and new communication models. In *STOC*. 363–372. https://doi.org/10.1145/258533.258620

[34] D. H. J. Polymath. 2012. A new proof of the density Hales-Jewett theorem. *Ann. of Math. (2)* 175, 3 (2012), 1283–1327. https://doi.org/10.4007/annals.2012.175.3.6

[35] Anup Rao. 2011. Parallel repetition in projection games and a concentration bound. *SIAM J. Comput.* 40, 6 (2011), 1871–1891. https://doi.org/10.1145/1374376.1374378 (also in STOC 2008).

[36] Ran Raz. 1998. A parallel repetition theorem. *SIAM J. Comput.* 27, 3 (1998), 763–803. https://doi.org/10.1137/S0097539795280895 (also in STOC 1995).

[37] Ran Raz. 2010. Parallel repetition of two prover games. In *CCC*. 3–6. https://doi.org/10.1109/CCC.2010.9

[38] Ran Raz. 2011. A counterexample to strong parallel repetition. *SIAM J. Comput.* 40, 3 (2011), 771–777. https://doi.org/doi/10.1137/090747270 (also in FOCS 2008).

[39] Ran Raz and Ricky Rosen. 2012. A strong parallel repetition theorem for projection games on expanders. In *CCC*. 247–257. https://doi.org/10.1109/CCC.2012.11

[40] Oleg Verbitsky. 1996. Towards the parallel repetition conjecture. *Theoret. Comput. Sci.* 157, 2 (1996), 277–282. https://doi.org/10.1016/0304-3975(95)00165-4