

Settling the Sharp Reconstruction Thresholds of Random Graph Matching

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Abstract—This paper studies the problem of recovering the hidden vertex correspondence between two edge-correlated random graphs. We focus on the Gaussian model where the two graphs are complete graphs with correlated Gaussian weights and the Erdős-Rényi model where the two graphs are subsampled from a common parent Erdős-Rényi graph $\mathcal{G}(n, p)$. For dense Erdős-Rényi graphs with $p = n^{-o(1)}$, we prove that there exists a sharp threshold, above which one can correctly match all but a vanishing fraction of vertices and below which correctly matching any positive fraction is impossible, a phenomenon known as the “all-or-nothing” phase transition. Even more strikingly, in the Gaussian setting, above the threshold all vertices can be exactly matched with high probability. In contrast, for sparse Erdős-Rényi graphs with $p = n^{-\Theta(1)}$, we show that the all-or-nothing phenomenon no longer holds and we determine the thresholds up to a constant factor. Along the way, we also derive the sharp threshold for exact recovery, sharpening the existing results in Erdős-Rényi graphs. The proof of the negative results builds upon a tight characterization of the mutual information based on the truncated second-moment computation and an “area theorem” that relates the mutual information to the integral of the reconstruction error. The positive results follows from a tight analysis of the maximum likelihood estimator that takes into account the cycle structure of the induced permutation on the edges.

Index Terms—Graph matching, Erdős-Rényi random graphs, Gaussian model, recovery threshold, mutual information, cycle decomposition of permutations.

I. INTRODUCTION

THE problem of graph matching (or network alignment) refers to finding the underlying vertex correspondence between two graphs on the sole basis of their network topologies. Going beyond the worst-case intractability of finding the optimal correspondence (quadratic assignment problem [2], [22]), an emerging line of research is devoted to the average-case analysis of graph matching under meaningful statistical models, focusing on either information-theoretic limits [4]–[6], [12], [16], [26] or computationally efficient

algorithms [1], [8]–[11], [13], [19]. Despite these recent advances, the sharp thresholds of graph matching remain not fully understood especially for approximate reconstruction. The current paper aims to close this gap.

Following [8] and [21], we consider the following probabilistic model for two random graphs correlated through a hidden vertex correspondence. Let the ground truth π be a uniformly random permutation on $[n]$. We generate two random weighted graphs on the common vertex set $[n]$ with (weighted) adjacency vectors $A = (A_{ij})_{1 \leq i < j \leq n}$ and $B = (B_{ij})_{1 \leq i < j \leq n}$ such that $(A_{\pi(i)\pi(j)}, B_{ij})$ are i.i.d. pairs of correlated random variables with a joint distribution P , which implicitly depends on n . Of particular interest are the following two special cases:

- (Gaussian model): $P = \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$ is the joint distribution of two standard Gaussian random variables with correlation coefficient $\rho > 0$. In this case, we have $B = \rho A^\pi + \sqrt{1 - \rho^2} Z$, where A and Z are independent standard normal vectors and $A_{ij}^\pi = A_{\pi(i)\pi(j)}$.
- (Erdős-Rényi random graph): P denotes the joint distribution of two correlated Bern(q) random variables X and Y such that $\mathbb{P}\{Y = 1 \mid X = 1\} = s$, where $q \leq s \leq 1$. In this case, A and B are the adjacency vectors of two Erdős-Rényi random graphs $G_1, G_2 \sim \mathcal{G}(n, q)$, where G_1^π (with the adjacency vector A^π) and G_2 are independently edge-subsampled from a common parent graph $G \sim \mathcal{G}(n, p)$ with $p = q/s$.

Given the observation A and B , the goal is to recover the latent vertex correspondence π as accurately as possible. More specifically, given two permutations $\pi, \hat{\pi}$ on $[n]$, denote the fraction of their overlap by

$$\text{overlap}(\pi, \hat{\pi}) \triangleq \frac{1}{n} |\{i \in [n] : \pi(i) = \hat{\pi}(i)\}|.$$

Definition 1: We say an estimator $\hat{\pi}$ of π achieves, as $n \rightarrow \infty$,

- *partial recovery*, if $\mathbb{P}\{\text{overlap}(\hat{\pi}, \pi) \geq \delta\} = 1 - o(1)$ for some constant $\delta \in (0, 1)$;
- *almost exact recovery*, if $\mathbb{P}\{\text{overlap}(\hat{\pi}, \pi) \geq \delta\} = 1 - o(1)$ for any constant $\delta \in (0, 1)$;
- *exact recovery*, if $\mathbb{P}\{\text{overlap}(\hat{\pi}, \pi) = 1\} = 1 - o(1)$.

The information-theoretic threshold of exact recovery has been determined for the Erdős-Rényi graph model [5] in the regime of $p = O(\log^{-3}(n))$ and more recently for the Gaussian model [12]; however, the results and proof

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techniques in [5] do not hold for denser graphs. In contrast, approximate recovery is far less well understood. Apart from the sharp condition for almost exact recovery in the sparse regime $p = n^{-\Omega(1)}$ [6], only upper and lower bounds are known for partial recovery [16]. See Section I-B for a detailed review of these previous results.

In this paper, we characterize the sharp reconstruction thresholds for both the Gaussian and dense Erdős-Rényi graphs with $p = n^{-o(1)}$. Specifically, we prove that there exists a sharp threshold, above which one can estimate all but a vanishing fraction of the latent permutation and below which recovering any positive fraction is impossible, a phenomenon known as the “all-or-nothing” phase transition [24]. This phenomenon is even more striking in the Gaussian model, in the sense that above the threshold the hidden permutation can be estimated error-free with high probability. In contrast, for sparse Erdős-Rényi graphs with $p = n^{-\Theta(1)}$, we show that the all-or-nothing phenomenon no longer holds. To this end, we determine the threshold for partial recovery up to a constant factor and show that it is order-wise smaller than the almost exact recovery threshold found in [6].

Along the way, we also derive a sharp threshold for exact recovery, sharpening existing results in [4], [5]. As a byproduct, the same technique yields an alternative proof of the result in [12] for the Gaussian model.

A. Main Results

Throughout the paper, we let $\epsilon > 0$ denote an arbitrarily small but fixed constant. Let $\hat{\pi}_{\text{ML}}$ denote the maximum likelihood estimator, which reduces to

$$\hat{\pi}_{\text{ML}} \in \arg \max_{\pi'} \langle A^{\pi'}, B \rangle. \quad (1)$$

Theorem 1 (Gaussian Model): If

$$\rho^2 \geq \frac{(4 + \epsilon) \log n}{n}, \quad (2)$$

then $\mathbb{P}\{\text{overlap}(\hat{\pi}_{\text{ML}}, \pi) = 1\} = 1 - o(1)$.

Conversely, if

$$\rho^2 \leq \frac{(4 - \epsilon) \log n}{n}, \quad (3)$$

then for any estimator $\hat{\pi}$ and any fixed constant $\delta > 0$, $\mathbb{P}\{\text{overlap}(\hat{\pi}, \pi) \leq \delta\} = 1 - o(1)$.

Theorem 1 implies that in the Gaussian setting, the recovery of π exhibits a sharp phase transition in terms of the limiting value of $\frac{n\rho^2}{\log n}$ at threshold 4, above which exact recovery is possible and below which even partial recovery is impossible. The positive part of Theorem 1 was first shown in [12]. Here we provide an alternative proof that does not rely on the Gaussian property and works for Erdős-Rényi graphs as well.

The next result determines the sharp threshold for the Erdős-Rényi model in terms of the key quantity nps^2 , the average degree of the intersection graph $G_1 \wedge G_2$ (whose edges are sampled by both G_1 and G_2).

Theorem 2 (Erdős-Rényi Graphs, Dense Regime): Assume p is bounded away from 1 and $p = n^{-o(1)}$. If

$$nps^2 \geq \frac{(2 + \epsilon) \log n}{\log \frac{1}{p} - 1 + p}, \quad (4)$$

then for any constant $\delta < 1$, $\mathbb{P}\{\text{overlap}(\hat{\pi}_{\text{ML}}, \pi) \geq \delta\} = 1 - o(1)$. Conversely, if

$$nps^2 \leq \frac{(2 - \epsilon) \log n}{\log \frac{1}{p} - 1 + p}, \quad (5)$$

then for any estimator $\hat{\pi}$ and any constant $\delta > 0$, $\mathbb{P}\{\text{overlap}(\hat{\pi}, \pi) \leq \delta\} = 1 - o(1)$.

Theorem 2 implies that analogous to the Gaussian model, in dense Erdős-Rényi graphs, the recovery of π exhibits an “all-or-nothing” phase transition in terms of the limiting value of $\frac{nps^2(\log \frac{1}{p} - 1 + p)}{\log n}$ at threshold 2, above which almost exact recovery is possible and below which even partial recovery is impossible. However, as we will see in Theorem 4, unlike the Gaussian model, this threshold differs from that of exact recovery.

Remark 1 (Entropy Interpretation of the Thresholds): The sharp thresholds in Theorem 1 and Theorem 2 can be interpreted from an information-theoretic perspective. Suppose an estimator $\hat{\pi} = \hat{\pi}(A, B)$ achieves almost exact recovery with $\mathbb{E}[\text{overlap}(\pi, \hat{\pi})] = 1 - o(1)$, which, by a rate-distortion computation, implies that $I(\pi; \hat{\pi})$ must be close to the full entropy of π , that is, $I(\pi; \hat{\pi}) = (1 - o(1))n \log n$. On the other hand, by the data processing inequality, we have $I(\pi; \hat{\pi}) \leq I(\pi; A, B)$. The latter can be bounded simply as (see Section I-C.2)

$$I(\pi; A, B) \leq \binom{n}{2} I(P), \quad (6)$$

where $I(P)$ denotes the mutual information between a pair of random variables with joint distribution P . For the Gaussian model, we have

$$I(P) = \frac{1}{2} \log \frac{1}{1 - \rho^2}. \quad (7)$$

For the correlated Erdős-Rényi graph,

$$I(P) = qd(s\|q) + (1 - q)d(\eta\|q), \quad (8)$$

where $q = ps$, $\eta \triangleq \frac{q(1-s)}{1-q}$, and $d(s\|q) \triangleq D(\text{Bern}(s)\|\text{Bern}(q))$ denotes the binary KL divergence. By Taylor expansion, we have $I(P) = s^2 p \left(p - 1 + \log \frac{1}{p} \right) (1 - o(1))$. Combining these with $\binom{n}{2} I(P) \geq (1 - o(1))n \log n$ shows the impossibility of almost exact recovery under the conditions (3) and (5). The fact that they are also necessary for partial recovery takes more effort to show, which we do in Section II.

Theorem 3 (Erdős-Rényi Graphs, Sparse Regime): Assume $p = n^{-\Omega(1)}$. If

$$nps^2 \geq (2 + \epsilon) \max \left\{ \frac{\log n}{\log(1/p)}, 2 \right\}, \quad (9)$$

then there exists a constant $\delta > 0$ such that $\mathbb{P}\{\text{overlap}(\hat{\pi}_{\text{ML}}, \pi) \geq \delta\} = 1 - o(1)$. Conversely, assuming $np = \omega(\log^2 n)$, if

$$nps^2 \leq 1 - \epsilon, \quad (10)$$

then for any estimator $\hat{\pi}$ and any constant $\delta > 0$, $\mathbb{P}\{\text{overlap}(\hat{\pi}, \pi) \leq \delta\} = 1 - o(1)$.

Theorem 3 implies that for sparse Erdős-Rényi graphs with $p = n^{-\alpha}$ for a constant $\alpha \in (0, 1)$, the information-theoretic thresholds for partial recovery is at $nps^2 \asymp 1$, which is much lower than the almost exact recovery threshold $nps^2 = \omega(1)$ as established in [6]. Hence, interestingly the all-or-nothing phenomenon no longer holds for sparse Erdős-Rényi graphs. Note that the conditions (9) and (10) differ by a constant factor. Determining the sharp threshold for partial recovery in the sparse regime remains an open question. Since the initial posting of this paper to arXiv, a subsequent work [14] shows that partial recovery is impossible when $nps^2 \leq 1$ in the sparse regime with a constant average degree $np = \Theta(1)$, using a different technique of analyzing the posterior distribution. Determining the sharp threshold for partial recovery in the sparse regime remains an open question.

Finally, we address the exact recovery threshold in the Erdős-Rényi graph model. For ease of notation, we consider a general correlated Erdős-Rényi graph model specified by the joint distribution $P = (p_{ab} : a, b \in \{0, 1\})$, so that $\mathbb{P}\{A_{\pi(i)\pi(j)} = a, B_{ij} = b\} = p_{ab}$ for $a, b \in \{0, 1\}$. In this general Erdős-Rényi model, $\hat{\pi}_{\text{ML}}$ is again given by the maximization problem (1) if $p_{11}p_{00} > p_{01}p_{10}$ (positive correlation) and changes to minimization if $p_{11}p_{00} < p_{01}p_{10}$ (negative correlation). The subsampling model is a special case with positive correlation, where

$$p_{11} = ps^2, \quad p_{10} = p_{01} = ps(1-s), \quad p_{00} = 1 - 2ps + ps^2. \quad (11)$$

Theorem 4 (Erdős-Rényi Graphs, Exact Recovery): Under the subsampling model (11), if

$$n(\sqrt{p_{00}p_{11}} - \sqrt{p_{01}p_{10}})^2 \geq (1+\epsilon)\log n, \quad (12)$$

then $\mathbb{P}\{\text{overlap}(\hat{\pi}_{\text{ML}}, \pi) = 1\} = 1 - o(1)$.

Conversely, if

$$n(\sqrt{p_{00}p_{11}} - \sqrt{p_{01}p_{10}})^2 \leq (1-\epsilon)\log n, \quad (13)$$

then for any estimator $\hat{\pi}$, $\mathbb{P}\{\text{overlap}(\hat{\pi}, \pi) = 1\} = o(1)$.

If p is bounded away from 1, Theorem 4 implies that the exact recovery threshold is given by $\lim_{n \rightarrow \infty} \frac{nps^2(1-\sqrt{p})^2}{\log n} = 1$. Since $\log \frac{1}{p} - 1 + p \geq 2(1-\sqrt{p})^2$, with equality if and only if $p = 1$, the threshold of exact recovery is strictly higher than that of almost exact recovery in the Erdős-Rényi graph model, unlike the Gaussian model. If $p = 1 - o(1)$, Theorem 4 implies that the exact recovery threshold is given by $\lim_{n \rightarrow \infty} \frac{n\rho^2}{\log n} = 4$, where $\rho \triangleq \frac{s(1-p)}{1-ps}$ denotes the correlation parameter between $A_{\pi(i)\pi(j)}$ and B_{ij} for any $i < j$ under the latent permutation π .

B. Comparisons to Prior Work

Exact recovery: The information-theoretic thresholds for exact recovery have been determined for the Gaussian model and the general Erdős-Rényi graph model in certain regimes. In particular, for the Gaussian model, it is shown in [12] that if $n\rho^2 \geq (4+\epsilon)\log n$ for any constant $\epsilon > 0$, then the MLE achieves exact recovery; if instead $n\rho^2 \leq (4-\epsilon)\log n$, then exact recovery is impossible. Theorem 1 significantly

strengthens this negative result by showing that under the same condition even partial recovery is impossible.

Analogously, for Erdős-Rényi random graphs, it is shown in [4] and [5] that the MLE achieve exact recovery when $nps^2 = \log n + \omega(1)$ under the additional restriction that $p = O(\log^{-3}(n))$.¹ Conversely, exact recovery is shown in [4] to be information-theoretically impossible provided that $nps^2 \leq \log n - \omega(1)$, based on the fact the intersection graph $G_1 \wedge G_2 \sim \mathcal{G}(n, ps^2)$ has many isolated nodes with high probability. These two results together imply that when $p = O(\log^{-3}(n))$, the exact recovery threshold is given by $\lim_{n \rightarrow \infty} \frac{nps^2}{\log n} = 1$, coinciding with the connectivity threshold of $G_1 \wedge G_2$. In comparison, Theorem 4 implies that if p is bounded away from 1, the precise exact recovery threshold is instead given by $\lim_{n \rightarrow \infty} \frac{nps^2(1-\sqrt{p})^2}{\log n} = 1$, strictly higher than the connectivity threshold. In particular, deriving the tight condition (13) requires more than eliminating isolated nodes. In fact, our results show that when p is bounded away from 1 and $\log n < nps^2 < \frac{\log n}{(1-\sqrt{p})^2}$, exact recovery still fails even when the intersection graph is asymmetric (no non-trivial automorphism) with high probability [25]. See the discussions in Section I-C.4 for more details.

Partial and almost exact recovery: Compared to exact recovery, the understanding of partial and almost exact recovery is less precise. It is shown in [6] that in the sparse regime $p = n^{-\Omega(1)}$, almost exact recovery is information-theoretically possible if and only if $nps^2 = \omega(1)$. The more recent work [16] further investigates partial recovery. It is shown that if $nps^2 \geq C(\delta) \max\left\{1, \frac{\log n}{\log(1/p)}\right\}$, then there exists an exponential-time estimator $\hat{\pi}$ that achieves $\text{overlap}(\hat{\pi}, \pi) \geq \delta$ with high probability, where $C(\delta)$ is some large constant that tends to ∞ as $\delta \rightarrow 1$; conversely, if $\frac{I(P)}{\delta} = o\left(\frac{\log(n)}{n}\right)$ with $I(P)$ given in (8), then no estimator can achieve $\text{overlap}(\hat{\pi}, \pi) \geq \delta$ with positive probability. These conditions do not match in general and are much looser than the results in Theorems 2 and 3.

C. Proof Techniques

We start by introducing some preliminary definitions associated with permutations (cf. [26, Section 3.1] for more details and examples).

1) Node Permutation, Edge Permutation, and Cycle Decomposition: Let $\mathcal{S}_n$ denote the set of permutations on the node set $[n]$. Each $\sigma \in \mathcal{S}_n$ induces a permutation σ^E on the edge set $\binom{[n]}{2}$ of unordered pairs, according to

$$\sigma^E((i, j)) \triangleq (\sigma(i), \sigma(j)). \quad (14)$$

We shall refer to σ and σ^E as a node permutation and edge permutation. Each permutation can be decomposed as disjoint cycles known as orbits. Orbits of σ (resp. σ^E) are referred as *node orbits* (resp. *edge orbits*). Let n_k (resp. N_k) denote the number of k -node (resp. k -edge) orbits in σ (resp. σ^E). The

¹In fact, [4], [5] studied exact recovery condition in a more general correlated Erdős-Rényi model with $\mathbb{P}\{A_{\pi(i)\pi(j)} = a, B_{ij} = b\} = p_{ab}$ for $a, b \in \{0, 1\}$, which will also be the setting in Section IV.

cycle structure of σ^E is determined by that of σ . For example, we have

$$N_1 = \binom{n_1}{2} + n_2, \quad (15)$$

because an edge (i, j) is a fixed point of σ^E if and only if either both i and j are fixed points of σ or (i, j) forms a 2-node orbit of σ . Let F be the set of fixed points of σ with $|F| = n_1$. Denote $\mathcal{O}_1 = \binom{F}{2} \subset \mathcal{O}$ as the subset of fixed points of edge permutation σ^E , where $\mathcal{O}$ denotes the collection of all edge orbits of σ^E .

2) *Negative Results on Partial Recovery*: Let $\mathcal{P}$ denote the joint distribution of A and B under the correlated model. To prove our negative results, we introduce an auxiliary null model $\mathcal{Q}$, under which A and B are independent with the same marginal as $\mathcal{P}$. In other words, under $\mathcal{Q}$, (A_{ij}, B_{ij}) are i.i.d. pairs of independent random variables with a joint distribution Q equal to the product of the marginals of P .

As the first step, we leverage the previous truncated second moment computation in [26] to conclude that the KL-divergence $\mathcal{D}(\mathcal{P} \parallel \mathcal{Q})$ is negligible under the desired conditions. By expressing the mutual information as $I(\pi; A, B) = \binom{n}{2} D(P \parallel Q) - D(\mathcal{P} \parallel \mathcal{Q})$ where $D(P \parallel Q) = I(P)$, this readily implies that $I(\pi; A, B) = \binom{n}{2} I(P)(1 + o(1))$. Next, we relate the mutual information $I(\pi; A, B)$ to the integral of the minimum mean-squared error (MMSE) of A^π , the weighted adjacency vector relabeled according to the ground truth. For the Gaussian model, this directly follows from the celebrated I-MMSE formula [15]. For correlated Erdős-Rényi graphs, we introduce an appropriate interpolating model and obtain an analogous but more involved “area theorem”, following [7], [20]. These two steps together imply that the MMSE of A^π given the observation (A, B) is asymptotically equal to the estimation error of the trivial estimator $\mathbb{E}[A^\pi]$. Finally, we connect the MMSE of A^π to the Hamming loss of reconstructing π , concluding the impossibility of the partial recovery.

Note that by the non-negativity of $\mathcal{D}(\mathcal{P} \parallel \mathcal{Q})$, we arrive at the simple upper bound (6), that is, $I(\pi; A, B) \leq \binom{n}{2} I(P)$. Interestingly, our proof relies on establishing an asymptotically matching lower bound to the mutual information $I(\pi; A, B)$. This significantly deviates from the existing results in [16] based on Fano’s inequality: $\mathbb{P}\{\text{overlap}(\hat{\pi}, \pi) \leq \delta\} \geq 1 - \frac{I(\pi; A, B) + 1}{\log(n! / m)}$ with $m = |\{\pi : \text{overlap}(\pi, \pi) \geq \delta\}|$, followed by applying the simple bound (6).

3) *Positive Results on Partial and Almost Exact Recovery*: Our positive results follow from a large deviation analysis of the maximum likelihood estimator (1). A crucial observation is that the difference between the objective function in (1) evaluated at a given permutation π' and that at the ground truth π can be decomposed across the edge orbits of $\sigma \triangleq \pi^{-1} \circ \pi'$ as

$$\langle A^{\pi'} - A^\pi, B \rangle = \sum_{O \in \mathcal{O} \setminus \mathcal{O}_1} X_O - \sum_{O \in \mathcal{O} \setminus \mathcal{O}_1} Y_O \triangleq X - Y,$$

where F is the set of fixed points of σ , $\mathcal{O}_1 = \binom{F}{2} \subset \mathcal{O}$ is a subset of fixed points of the edge permutation σ^E ,² $X_O \triangleq \sum_{(i,j) \in O} A_{\pi'(i)\pi'(j)} B_{ij}$, and $Y_O \triangleq \sum_{(i,j) \in O} A_{\pi(i)\pi(j)} B_{ij}$, are independent across edge orbits O . Crucially, Y depends on π' only through its fixed point set F , which has substantially fewer choices than π' itself when $n - |F| \asymp n$. Therefore, for the purpose of applying the union bound it is beneficial to separately control X and Y . Indeed, we show that Y is highly concentrated on its mean. Hence, it remains to analyze the large-deviation event of X exceeding $\mathbb{E}[Y]$, which is accomplished by a careful computation of the moment generation function (MGF) $M_{|O|} \triangleq \mathbb{E}[\exp(tX_O)]$ and proving that for any $t \geq 0$,

$$M_{|O|} \leq M_2^{|O|/2}, \text{ for } |O| \geq 2. \quad (16)$$

Intuitively, it means that the contribution of longer edge orbits can be effectively bounded by that of the 2-edge orbits. Capitalizing on this key finding and applying the Chernoff bound together with a union bound over π' yields the tight condition when $q = o(1)$. When $q = \Theta(1)$, it turns out that the correlation between X and Y can no longer be ignored so that separately controlling X and Y leads to suboptimal results. To remedy this issue, we need to center the adjacency vectors A and B . More precisely, define $\bar{X}_O \triangleq \sum_{(i,j) \in O} (A_{\pi'(i)\pi'(j)} - q)(B_{ij} - q)$, $\bar{Y}_O \triangleq \sum_{(i,j) \in O} (A_{\pi(i)\pi(j)} - q)(B_{ij} - q)$, $\bar{X} \triangleq \sum_{O \in \mathcal{O} \setminus \mathcal{O}_1} \bar{X}_O$, and $\bar{Y} \triangleq \sum_{O \in \mathcal{O} \setminus \mathcal{O}_1} \bar{Y}_O$. By definition, $X - Y = \bar{X} - \bar{Y}$. Crucially, it can be verified that $\bar{X}_O$ and $\bar{Y}_O$ (and hence $\bar{X}$ and $\bar{Y}$) are uncorrelated after the centering. Thus we can apply our aforementioned techniques to separately bound $\bar{X}$ and $\bar{Y}$, which yield the sharp conditions of recovery.

We remark that the partial recovery results in [16] are obtained by analyzing an estimator slightly different from the MLE and the same MGF bound (16) is used. However, there are two major differences that led to the looseness of their results. First, their analysis does not separately bound X and Y . Second, the tilting parameter in the Chernoff bound is suboptimal.

4) *Exact Recovery*: For exact recovery, we need to further consider π' that is close to π , i.e., $n - |F| = o(n)$. In this regime, the number of choices of F is comparable to that of π' . Hence, instead of separately bounding X and Y , it is more favorable to directly applying the Chernoff bound to the difference $X - Y$. Crucially, the moment generation function $\mathbb{E}[\exp(t(X_O - Y_O))]$ continues to satisfy the relation (16) and the bottleneck for exact recovery happens at $|F| = n - 2$, where π' differs from π from a 2-cycle (transposition).

Prompted by this observation, we prove a matching necessary condition of exact recovery by considering all possible permutations $\sigma \triangleq \pi^{-1} \circ \pi'$ that consists of $n - 2$ fixed points and a 2-node cycle (i, j) (transposition), in which case,

$$\begin{aligned} & \langle A^{\pi'} - A^\pi, B \rangle \\ &= - \sum_{k \in [n] \setminus \{i, j\}} (A_{ik}^\pi - A_{jk}^\pi) (B_{ik} - B_{jk}). \end{aligned} \quad (17)$$

²Note that $\mathcal{O}_1$ consists of fixed points of the edge permutation, while F denotes the set of fixed points of the node permutation. An edge pair $(i, j) \in \mathcal{O}_1$ if and only if both i and j belong to F .

There remain two key challenges to conclude the existence of many choices of (i, j) for which $\langle A^{\pi'}, B \rangle \geq \langle A^{\pi}, B \rangle$. First, to derive a tight impossibility condition, we need to obtain a tight large-deviation lower estimate for this event. Second, the RHS of (17) for different pairs (i, j) are correlated. This dependency is addressed by restricting the choices of (i, j) and applying a second moment computation.

Note that the impossibility proof of exact recovery for the Gaussian model in [12] also considers the permutations that consist of a single transposition. The difference is that their large-deviation lower estimate simply follows from Gaussian tail probability and the correlations among different pairs (i, j) is bounded by a second-moment calculation using densities of correlated Gaussians.

II. IMPOSSIBILITY OF PARTIAL RECOVERY

To start, we characterize the asymptotic value of the mutual information $I(A, B; \pi)$ – a key quantity that measures the amount of information about π provided by the observation (A, B) . By definition,

$$\begin{aligned} I(A, B; \pi) &\triangleq \mathbb{E} [D(\mathcal{P}_{A,B|\pi} \| \mathcal{P}_{A,B})] \\ &= \mathbb{E} [D(\mathcal{P}_{A,B|\pi} \| \mathcal{Q}_{A,B})] - D(\mathcal{P}_{A,B} \| \mathcal{Q}_{AB}) \end{aligned}$$

for any joint distribution $\mathcal{Q}_{A,B}$ of (A, B) such that $D(\mathcal{P}_{A,B} \| \mathcal{Q}_{AB}) < \infty$. Note that $\mathcal{P}_{A,B|\pi}$ factorizes into a product distribution $\prod_{i < j} \mathcal{P}_{A_{\pi(i)\pi(j)}, B_{ij}} = P^{\otimes \binom{n}{2}}$, where P is the joint distribution of $(A_{\pi(i)\pi(j)}, B_{ij})$. Thus, to exploit the tensorization property of the KL-divergence, we choose $\mathcal{Q}_{A,B}$ to be a product distribution under which A and B are independent and (A_{ij}, B_{ij}) are i.i.d. pairs of independent random variables with a joint distribution Q with the same marginals as P . (We shall refer to this $\mathcal{Q}_{A,B}$ as the null model.) In particular, for the Gaussian (resp. Erdős-Rényi) model, Q is the joint distribution of two independent standard normal (resp. Bern(q)) random variables. Under this choice, we have $D(\mathcal{P}_{A,B|\pi} \| \mathcal{Q}_{A,B}) = \binom{n}{2} D(P \| Q) = \binom{n}{2} I(P)$ and hence

$$I(A, B; \pi) = \binom{n}{2} I(P) - D(\mathcal{P}_{A,B} \| \mathcal{Q}_{AB}).$$

By the non-negativity of the KL divergence, we have $I(A, B; \pi) \leq \binom{n}{2} I(P)$. This bound turns out to be tight, as made precise by the following proposition.

Proposition 1: It holds that

$$I(A, B; \pi) = \binom{n}{2} I(P) - \zeta_n,$$

where

- $\zeta_n = o(1)$ in the Gaussian model with $\rho^2 \leq \frac{(4-\epsilon)\log n}{n}$;
- $\zeta_n = o(1)$ in the dense Erdős-Rényi graphs with $p = n^{-o(1)}$ and $nps^2(\log(1/p) - 1 + p) \leq (2 - \epsilon)\log(n)$;
- $\zeta_n = O(\log n)$ in the sparse Erdős-Rényi graphs with $p = n^{-\Omega(1)}$ and $np = \omega(1)$ and $nps^2 \leq 1 - \epsilon$;

for some arbitrarily small but fixed constant $\epsilon > 0$.

Given the tight characterization of the mutual information in Proposition 1, we now relate it to the Bayes risk. Using the chain rule, we have

$$I(A, B; \pi) = I(B; \pi | A) = I(B; A^{\pi} | A),$$

where the second equality follows from the fact that $A \rightarrow A^{\pi} \rightarrow B$ forms a Markov chain. The intuition is that conditioned on A , B is a noisy observation of A^{π} (which is random owing to π). In such a situation, the mutual information can typically be related to an integral of the reconstruction error of the signal A^{π} . To make this precise, we first introduce a parametric model $\mathcal{P}_{\theta}$ that interpolates between the planted model $\mathcal{P}$ and the null model $\mathcal{Q}$ as θ varies. We write $\mathbb{E}_{\theta}$ to indicate expectation taken with respect to the law $\mathcal{P}_{\theta}$.

For the Gaussian model, let $\mathcal{P}_{\theta}$ denote the model under which $B = \sqrt{\theta}A^{\pi} + \sqrt{1-\theta}Z$, where A, Z are two independent Gaussian matrices and $\theta \in [0, 1]$. Then $\theta = \rho^2$ corresponds to the planted model $\mathcal{P}$ while $\theta = 0$ corresponds to the null model $\mathcal{Q}$. As θ increases from 0 to ρ^2 , $\mathcal{P}_{\theta}$ interpolates between $\mathcal{Q}$ and $\mathcal{P}$. Let

$$\text{mmse}_{\theta}(A^{\pi}) \triangleq \mathbb{E}_{\theta}[\|A^{\pi} - \mathbb{E}_{\theta}[A^{\pi}|A, B]\|^2] \quad (18)$$

denote the minimum mean-squared error (MMSE) of estimating A^{π} based on (A, B) distributed according to $\mathcal{P}_{\theta}$. The following proposition follows from the celebrated I-MMSE formula [15].

Proposition 2 (Gaussian Model):

$$I(A, B; \pi) = \frac{1}{2} \int_0^{\rho^2} \frac{\text{mmse}_{\theta}(A^{\pi})}{(1-\theta)^2} d\theta.$$

The correlated Erdős-Rényi graph model requires more effort. Let us fix $q = ps$ and consider the following coupling P_{θ} between two Bern(q) random variables with joint probability mass function p_{θ} , where $p_{\theta}(11) = q\theta$, $p_{\theta}(01) = p_{\theta}(10) = q(1-\theta)$, and $p_{\theta}(00) = 1 - (2-\theta)q$, with $\theta \in [q, s]$. Let $\mathcal{P}_{\theta}$ denote the interpolated model under which $(A_{\pi(i)\pi(j)}, B_{ij})$ are i.i.d. pairs of correlated random variables with joint distribution P_{θ} . As θ increases from q to s , $\mathcal{P}_{\theta}$ interpolates between the null model $\mathcal{Q} = \mathcal{P}_q$ and the planted model $\mathcal{P} = \mathcal{P}_s$. We have the following area theorem that relates $I(A, B; \pi)$ to the MMSE of A^{π} .

Proposition 3 (Erdős-Rényi Random Graph): It holds that

$$\begin{aligned} I(A, B; \pi) &\leq \binom{n}{2} I(P) + \binom{n}{2} qs^2 \\ &\quad + \int_q^s \frac{\theta - q}{s(1-q)^2} \left(\text{mmse}_{\theta}(A^{\pi}) - \binom{n}{2} q(1-q) \right) d\theta. \end{aligned}$$

Finally, we relate the estimation error of A^{π} to that of π .

Proposition 4: In both the Gaussian and Erdős-Rényi graph model, if

$$\text{mmse}_{\theta}(A^{\pi}) \geq \mathbb{E}[\|A\|^2] (1 - \xi), \quad (19)$$

for some $\xi > 0$, then for any estimator $\hat{\pi} = \hat{\pi}(A, B)$,

$$\mathbb{E}_{\theta}[\text{overlap}(\hat{\pi}, \pi)] \leq O\left(\xi^{1/4} + \left(\frac{n \log n}{\mathbb{E}[\|A\|^2]}\right)^{1/4}\right).$$

Now, we are ready to prove the negative results on partial recovery. We start with the Gaussian case.

Proof of the negative part in Theorem 1: In the Gaussian model, we have

$$\begin{aligned} I(P) &= D\left(\mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right) \parallel \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\right) \\ &= \frac{1}{2} \log \frac{1}{1 - \rho^2}. \end{aligned}$$

Assume that $\rho^2 = (4 - \epsilon/2) \frac{\log n}{n}$. Fix some $\theta_0 \in (0, \rho^2)$ to be chosen. Then,

$$\begin{aligned} &\binom{n}{2} \frac{1}{2} \log \frac{1}{1 - \rho^2} - \zeta_n \\ &\stackrel{(a)}{=} I(A, B; \pi) \\ &\stackrel{(b)}{\leq} \frac{1}{2(1 - \rho^2)^2} \int_0^{\rho^2} \text{mmse}_{\theta}(A^{\pi}) d\theta \\ &\stackrel{(c)}{\leq} \frac{1}{2(1 - \rho^2)^2} \left(\binom{n}{2} \theta_0 + \text{mmse}_{\theta_0}(A^{\pi})(\rho^2 - \theta_0) \right) \\ &= \frac{1}{2(1 - \rho^2)^2} \left(\binom{n}{2} \rho^2 + \left(\text{mmse}_{\theta_0}(A^{\pi}) - \binom{n}{2} \right) (\rho^2 - \theta_0) \right), \end{aligned}$$

where $\zeta_n = o(1)$ and (a) holds by Proposition 1; (b) follows from the I-MMSE formula given in Proposition 2; (c) holds because $\text{mmse}_{\theta}(A^{\pi}) \leq \mathbb{E}[\|A\|_2^2] = \binom{n}{2}$ and the fact that $\text{mmse}_{\theta}(A^{\pi})$ is monotone decreasing in θ . Rearranging the terms in the last displayed equation, we get

$$\begin{aligned} &\text{mmse}_{\theta_0}(A^{\pi}) - \binom{n}{2} \\ &\geq \frac{(1 - \rho^2)^2}{\rho^2 - \theta_0} \left(\binom{n}{2} \left(\log \frac{1}{1 - \rho^2} - \frac{\rho^2}{(1 - \rho^2)^2} \right) - 2\zeta_n \right) \\ &\geq -\frac{(1 - \rho^2)^2}{\rho^2 - \theta_0} \left(\binom{n}{2} \frac{2\rho^4}{(1 - \rho^2)^2} + 2\zeta_n \right), \end{aligned}$$

where the last inequality holds because $\log(1+x) \geq x - x^2$ for $x \geq 0$. Choosing $\theta_0 = (4 - \epsilon) \frac{\log n}{n}$, we conclude that

$$\begin{aligned} \text{mmse}_{\theta_0}(A^{\pi}) &\geq \binom{n}{2} \left(1 - O\left(\rho^2 + \frac{\zeta_n}{n^2 \rho^2}\right) \right) \\ &= \binom{n}{2} \left(1 - O\left(\frac{\log n}{n}\right) \right), \end{aligned} \quad (20)$$

where the last equality holds because $\rho^2 = \Theta(\log(n)/n)$ and $\zeta_n = o(1)$. Since $\mathbb{E}[\|A\|_2^2] = \binom{n}{2}$, it follows from Proposition 4 that

$$\mathbb{E}_{\theta_0}[\text{overlap}(\hat{\pi}, \pi)] \leq O\left(\left(\frac{\log n}{n}\right)^{1/4}\right).$$

Finally, by Markov's inequality, for any $\delta_n = \omega\left(\left(\frac{\log n}{n}\right)^{1/4}\right)$ (in particular, a fixed constant $\delta_n > 0$), $\mathcal{P}_{\theta_0}\{\text{overlap}(\hat{\pi}, \pi) \geq \delta_n\} = o(1)$. Note that $\mathcal{P}_{\theta_0}$ corresponds to the Gaussian model with squared correlation coefficient equal to $\theta_0 = (4 - \epsilon) \frac{\log n}{n}$. By the arbitrariness of ϵ , this completes the proof of Theorem 1. $\square$

Next, we move to the Erdős-Rényi graph model.

Proof of the negative parts in Theorems 2 and 3: Let $s^2 = \frac{(2-\epsilon)\log n}{np(\log(1/p)-1+p)}$ in the dense regime and $s^2 = \frac{1-\epsilon}{np}$ in the sparse regime. Then we get that

$$\begin{aligned} &-\zeta_n - \binom{n}{2} qs^2 \\ &\stackrel{(a)}{\leq} \int_q^s \frac{\theta - q}{s(1-q)^2} \left(\text{mmse}_{\theta}(A^{\pi}) - \binom{n}{2} q(1-q) \right) d\theta \\ &\stackrel{(b)}{\leq} \int_q^s \frac{\theta - q}{s(1-q)^2} \left(\text{mmse}_{\theta \wedge (1-\epsilon)s}(A^{\pi}) - \binom{n}{2} q(1-q) \right) d\theta \\ &\stackrel{(c)}{\leq} \int_{(1-\epsilon)s}^s \frac{\theta - q}{s(1-q)^2} \left(\text{mmse}_{(1-\epsilon)s}(A^{\pi}) - \binom{n}{2} q(1-q) \right) d\theta \\ &= \underbrace{\frac{s^2(2\epsilon - \epsilon^2) - 2\epsilon sq}{2s(1-q)^2}}_{=\Theta(s)} \left(\text{mmse}_{(1-\epsilon)s}(A^{\pi}) - \binom{n}{2} q(1-q) \right) \end{aligned}$$

where $\zeta_n = o(1)$ in the dense regime and $\zeta_n = O(\log n)$ in the sparse regime; (a) follows from Proposition 1 and Proposition 3; (b) holds because $\text{mmse}_{\theta}(A^{\pi})$ is monotone decreasing in θ ; (c) holds because

$$\int_q^{(1-\epsilon)s} \frac{\theta - q}{s(1-q)^2} \left(\text{mmse}_{\theta \wedge (1-\epsilon)s}(A^{\pi}) - \binom{n}{2} q(1-q) \right) d\theta,$$

is bounded above by 0, in view of $\text{mmse}_{\theta}(A^{\pi}) \leq \mathbb{E}[\|A - \mathbb{E}[A]\|_2^2] = \binom{n}{2} q(1-q)$.³ Rearranging the terms in the last displayed equation, we conclude that

$$\begin{aligned} \text{mmse}_{(1-\epsilon)s}(A^{\pi}) &\geq \binom{n}{2} q(1-q) \left(1 - O\left(\frac{\zeta_n}{n^2 qs} + s\right) \right) \\ &\geq \binom{n}{2} q(1 - O(s)), \end{aligned}$$

where the last inequality holds because $q \leq s$, $\zeta_n = O(n^2 qs^2)$ and $s^2 \asymp \frac{\log n}{np \log(1/p)}$. Since $\mathbb{E}[\|A\|_2^2] = \binom{n}{2} q$, it follows from Proposition 4 that

$$\begin{aligned} \mathbb{E}_{(1-\epsilon)s}[\text{overlap}(\hat{\pi}, \pi)] &\leq O\left(s^{1/4} + \left(\frac{\log n}{nq}\right)^{1/4}\right) \\ &= O\left(\left(\frac{\log n}{nq}\right)^{1/4}\right), \end{aligned}$$

where the last equality holds because $nqs = nps^2 = O(\log n)$. Note that in the dense regime, since $s^2 \asymp \frac{\log n}{np(\log \frac{1}{p}-1+p)}$ and $p = n^{-o(1)}$, we have $nq = nps = \omega(\log n)$. This also holds in the sparse regime when $s^2 \asymp \frac{1}{np}$ under the extra assumption that $np = \omega(\log^2 n)$. Thus, by Markov's inequality, for any $\delta_n = \omega\left(\left(\frac{\log n}{nq}\right)^{1/4}\right)$, in particular, any fixed constant $\delta_n > 0$, we have $\mathcal{P}_{(1-\epsilon)s}\{\text{overlap}(\hat{\pi}, \pi) \geq \delta_n\} = o(1)$. In other words, we have shown the desired impossibility result under the distribution $\mathcal{P}_{(1-\epsilon)s}$, which corresponds to

³The fact that $\text{mmse}_{\theta}(A^{\pi})$ is monotone decreasing in θ follows from a simulation argument. Let $(A, B) \sim \mathcal{P}_{\theta}$. Fix θ' such that $q < \theta' < \theta < s$. Define $B' = (B'_{ij})$ by passing each B_{ij} independently through the same (asymmetric) channel W to obtain B'_{ij} , where $W(0|1) = \frac{(1-q)(\theta-\theta')}{\theta-q}$ and $W(1|0) = \frac{q(\theta-\theta')}{\theta-q}$ are well-defined. Then $(A, B') \sim \mathcal{P}_{\theta'}$.

the correlated Erdős-Rényi model with parameters $p' = \frac{p}{1-\epsilon}$ and $s' = s(1-\epsilon)$. By the arbitrariness of ϵ , this completes the proof. $\square$

A. Proof of Proposition 1

In this subsection, we prove Proposition 1, which reduces to bounding $D(\mathcal{P}_{A,B} \parallel \mathcal{Q}_{A,B})$. It is well-known that KL divergence can be bounded by the χ^2 -divergence (variance of the likelihood ratio). This method, however, is often too loose as the second moment can be derailed by rare events. Thus a more robust version is by means of *truncated second moment*, which has been carried out in [26] to bound TV $(\mathcal{P}_{A,B}, \mathcal{Q}_{A,B})$ for studying the hypothesis testing problem in graph matching. Here we leverage the same result to bound the KL divergence. To this end, we first present a general bound then specialize to our problem in both Gaussian and Erdős-Rényi models.

Lemma 1: Let $\mathcal{P}_{XY}$ denote the joint distribution of (X, Y) . Let $\mathcal{E}$ be an event independent of X such that $\mathcal{P}(\mathcal{E}) = 1 - \delta$. Let $\mathcal{Q}_Y$ be an auxiliary distribution such that $\mathcal{P}_{Y|X} \ll \mathcal{Q}_Y$ $\mathcal{P}_X$ -a.s. Then

$$\begin{aligned} D(\mathcal{P}_Y \parallel \mathcal{Q}_Y) &\leq \log(1 + \chi^2(\mathcal{P}_{Y|\mathcal{E}} \parallel \mathcal{Q}_Y)) \\ &\quad + \delta \left(\log \frac{1}{\delta} + \mathbb{E}[D(\mathcal{P}_{Y|X} \parallel \mathcal{Q}_Y)] \right) \\ &\quad + \sqrt{\delta \cdot \text{Var} \left(\log \frac{d\mathcal{P}_{Y|X}}{d\mathcal{Q}_Y} \right)}, \end{aligned} \quad (21)$$

where $\mathcal{P}_{Y|\mathcal{E}}$ denote the distribution of Y conditioned on $\mathcal{E}$, and the χ^2 -divergence is defined as $\chi^2(P \parallel Q) = \mathbb{E}_Q[(\frac{dP}{dQ} - 1)^2]$ if $P \ll Q$ and ∞ otherwise.

Proof: Note that $\mathcal{P}_Y = (1 - \delta)\mathcal{P}_{Y|\mathcal{E}} + \delta\mathcal{P}_{Y|\mathcal{E}^c}$. Thanks to the convexity of the KL divergence, Jensen's inequality yields

$$D(\mathcal{P}_Y \parallel \mathcal{Q}_Y) \leq (1 - \delta)D(\mathcal{P}_{Y|\mathcal{E}} \parallel \mathcal{Q}_Y) + \delta D(\mathcal{P}_{Y|\mathcal{E}^c} \parallel \mathcal{Q}_Y).$$

The first term can be bounded using the generic fact that $D(P \parallel Q) \leq \log \mathbb{E}_Q[(\frac{dP}{dQ})^2] = \log(1 + \chi^2(P \parallel Q))$. Let $g(X, Y) = \log \frac{d\mathcal{P}_{Y|X}}{d\mathcal{Q}_Y}$. Using the convexity of KL divergence and the independence of $\mathcal{E}$ and X , we bound the second term as follows:

$$\begin{aligned} D(\mathcal{P}_{Y|\mathcal{E}^c} \parallel \mathcal{Q}_Y) &\leq \mathbb{E}[D(\mathcal{P}_{Y|X, \mathcal{E}^c} \parallel \mathcal{Q}_Y)] \\ &= \mathbb{E} \left[\int d\mathcal{P}_{Y|X, \mathcal{E}^c} \log \left(\frac{d\mathcal{P}_{Y|X, \mathcal{E}^c}}{d\mathcal{Q}_Y} \right) \right] \\ &= \mathbb{E} \left[\int d\mathcal{P}_{Y|X} \frac{\mathbf{1}_{\{(X, Y) \in \mathcal{E}^c\}}}{\mathcal{P}(\mathcal{E}^c)} \log \left(\frac{d\mathcal{P}_{Y|X}}{d\mathcal{Q}_Y \mathcal{P}(\mathcal{E}^c)} \right) \right] \\ &= \log \frac{1}{\delta} + \frac{1}{\delta} \mathbb{E}[(g(X, Y) \mathbf{1}_{\{(X, Y) \in \mathcal{E}^c\}})] \\ &= \log \frac{1}{\delta} + \mathbb{E}[g(X, Y)] \\ &\quad + \frac{1}{\delta} \mathbb{E}[(g(X, Y) - \mathbb{E}[g(X, Y)]) \mathbf{1}_{\{(X, Y) \in \mathcal{E}^c\}}]. \end{aligned}$$

Applying Cauchy-Schwarz to the last term completes the proof. $\square$

Next we apply Lemma 1 in the context of the random graph matching by identifying X and Y with the latent π

and the observation (A, B) respectively. Let $\mathcal{E}$ be a certain high-probability event independent of π . Then

$$\mathcal{P}_{A,B|\mathcal{E}} = \frac{1}{\mathcal{P}(\mathcal{E}) n!} \sum_{\pi \in \mathcal{S}_n} \mathcal{P}_{A,B} \mathbf{1}_{\{(A,B,\pi) \in \mathcal{E}\}}.$$

Recall that the null model is chosen to be $\mathcal{Q}_{A,B} = \mathcal{P}_A \otimes \mathcal{P}_B$. As shown in [26], for both the Gaussian and Erdős-Rényi graph model, it is possible to construct a high-probability event $\mathcal{E}$ satisfying the symmetry condition $\mathcal{P}(\mathcal{E} \mid \pi) = \mathcal{P}(\mathcal{E})$, such that $\chi^2(\mathcal{P}_{A,B|\mathcal{E}} \parallel \mathcal{Q}_{A,B}) = o(1)$. This bounds the first term in (21). For the second term, since both A and B are individually independent of π , we have

$$\begin{aligned} \mathbb{E} \left[\log \frac{d\mathcal{P}_{A,B|\pi}}{d\mathcal{Q}_{A,B}} \right] &= I(A; B \mid \pi) \\ &= \binom{n}{2} D(P \parallel Q) = \binom{n}{2} I(P), \end{aligned} \quad (22)$$

where the last equality holds because Q is the product of the marginals of P . The third term in (21) can be computed explicitly. Next we give details on how to complete the proof of Proposition 1.

a) Gaussian model: It is shown in [26, Section 4.1, Lemma 1] that there exists an event $\mathcal{E}$ independent of π such that $\mathcal{P}(\mathcal{E}^c) = e^{-\Omega(n)}$ and $\chi^2(\mathcal{P}_{\mathcal{E}} \parallel \mathcal{Q}) = o(1)$, provided that $\rho^2 \leq \frac{(4-\epsilon) \log n}{n}$. Furthermore, by (22) and (7), we have $\mathbb{E} \left[\log \frac{d\mathcal{P}_{A,B|\pi}}{d\mathcal{Q}_{A,B}} \right] = \binom{n}{2} \frac{1}{2} \log \frac{1}{1-\rho^2} = O(n \log n)$. To compute the variance, note that

$$\log \frac{d\mathcal{P}_{A,B|\pi}}{d\mathcal{Q}_{A,B}} = -\frac{1}{2} \binom{n}{2} \log(1 - \rho^2) - \frac{h(A, B, \pi)}{4(1 - \rho^2)},$$

where

$$h(A, B, \pi) \triangleq \rho^2 \|A\|^2 + \rho^2 \|B\|^2 - 2\rho \langle A^\pi, B \rangle.$$

Thus $\text{Var}(\log \frac{d\mathcal{P}_{A,B|\pi}}{d\mathcal{Q}_{A,B}}) = \frac{1}{16(1-\rho^2)^2} \text{Var}(h(A, B, \pi))$. Write $B = \rho A^\pi + \sqrt{1-\rho^2} Z$ where Z is an independent copy of A , we have $h(A, B, \pi) = \rho^2(\|B\|^2 - \|A\|^2) - 2\rho \sqrt{1-\rho^2} \langle A^\pi, Z \rangle$. Here both $\|A\|^2$ and $\|B\|^2$ are distributed as $\chi^2_{\binom{n}{2}}$, with variance equal to $2\binom{n}{2}$. Furthermore, $\text{Var}(\langle A^\pi, Z \rangle) = \binom{n}{2}$. Thus $\text{Var}(h(A, B, \pi)) = O(n \log n)$. Applying Lemma 1, we conclude that $D(\mathcal{P}_{A,B} \parallel \mathcal{Q}_{A,B}) = o(1)$.

b) Erdős-Rényi Graphs: In the dense regime of $p = n^{-o(1)}$ and $p = 1 - \Omega(1)$, it is shown in [26, Section A.3, Lemma 6] that there exists an event $\mathcal{E}$ such that $\mathcal{P}(\mathcal{E}^c) = e^{-n^{1-o(1)}}$ and $\chi^2(\mathcal{P}_{\mathcal{E}} \parallel \mathcal{Q}) = o(1)$, provided that $nps^2(\log(1/p) - 1 + p) \leq (2 - \epsilon) \log(n)$. In the sparse regime (see [26, Section 5, Lemma 2]), it is possible to choose $\mathcal{E}$ such that $\mathcal{P}(\mathcal{E}^c) = O(\frac{1}{n})$ and $\chi^2(\mathcal{P}_{\mathcal{E}} \parallel \mathcal{Q}) = o(1)$, provided that $nps^2 \leq 1 - \epsilon$ and $np = \omega(1)$.

By (22) and (8), we have $\mathbb{E} \left[\log \frac{d\mathcal{P}_{A,B|\pi}}{d\mathcal{Q}_{A,B}} \right] = \binom{n}{2} I(P)$, where $I(P) = qd(s \parallel q) + (1 - q)d(\eta \parallel q)$, with $q = ps$ and $\eta = \frac{q(1-s)}{1-q}$. In both the dense and sparse regime, one can verify that

$$I(P) = s^2 p \left(p - 1 + \log \frac{1}{p} \right) (1 + o(1)).$$

As a result, we have $\mathbb{E} \left[\log \frac{d\mathcal{P}_{A,B|\pi}}{d\mathcal{Q}_{A,B}} \right] = O(n \log n)$ in both cases.

It remains to bound the variance in (21). Note that

$$\log \frac{\mathcal{P}(A, B | \pi)}{\mathcal{Q}(A, B)} = \binom{n}{2} \log \frac{1-\eta}{1-ps} + \frac{1}{2} h(A, B, \pi),$$

where

$$h(A, B, \pi) \triangleq \log \frac{1-s}{1-\eta} (\|A\|^2 + \|B\|^2) + \log \frac{s(1-\eta)}{\eta(1-s)} \langle A^\pi, B \rangle.$$

Since p is bounded away from 1 and $s = o(1)$ in both the dense regime ($p = n^{-o(1)}$) and sparse regime ($p = n^{-\Omega(1)}$ and $np = \omega(1)$), it follows that

$$\begin{aligned} \log \frac{1-\eta}{1-s} &= \log \left(1 + \frac{s(1-p)}{(1-s)(1-ps)} \right) \\ &= (1+o(1)) s(1-p) \\ \frac{s(1-\eta)}{\eta(1-s)} &= \frac{(1-\eta)(1-ps)}{p(1-s)^2} = \frac{1+o(1)}{p}. \end{aligned}$$

Note that $\|A\|^2, \|B\|^2 \sim \text{Binom}(\binom{n}{2}, ps)$ and $\langle A^\pi, B \rangle \sim \text{Binom}(\binom{n}{2}, ps^2)$. We have $\text{Var}(h) = O(n^2 ps^2 \log^2 \frac{1}{p})$. Consequently, $\sqrt{\text{Var}(h) \mathcal{P}(\mathcal{E}^c)} = o(1)$ and $O(\log n)$ in the dense and sparse case, respectively. Applying Lemma 1 yields the same upper bound on $D(\mathcal{P}_{A,B} \| \mathcal{Q}_{A,B})$.

B. Proof of Proposition 2

The proof follows from a simple application of the I-MMSE formula for the additive Gaussian channel. Note that under the interpolated model $\mathcal{P}_\theta$, we have

$$\frac{B}{\sqrt{1-\theta}} = \sqrt{\frac{\theta}{1-\theta}} A^\pi + Z,$$

which is the output of an additive Gaussian channel with input A^π and the standard Gaussian noise Z . Letting $I(\theta) = I(B; A^\pi | \pi) = I(A, B; \pi)$ and using the I-MMSE formula [15, Theorem 2], we have

$$\frac{dI(\theta)}{d(\theta/(1-\theta))} = \frac{1}{2} \text{mmse}_\theta(A^\pi). \quad (23)$$

Thus $\frac{dI(\theta)}{d\theta} = \frac{1}{2(1-\theta)^2} \text{mmse}_\theta(A^\pi)$. Integrating over θ from 0 to ρ^2 and noting $I(0) = 0$, we arrive at

$$I(\rho^2) = \int_0^{\rho^2} \frac{1}{2(1-\theta)^2} \text{mmse}_\theta(A^\pi) d\theta.$$

C. Proof of Proposition 3

Note that under the interpolated model $\mathcal{P}_\theta$, we have

$$p_\theta(y|x) = \begin{cases} \theta & x=1, y=1 \\ 1-\theta & x=1, y=0 \\ \eta & x=0, y=1 \\ 1-\eta & x=0, y=0 \end{cases}, \quad \eta = \frac{q(1-\theta)}{1-q}.$$

Let $g(\theta) \triangleq D(P_\theta \| Q) = q \cdot d(\theta \| q) + (1-q)d(\eta \| q)$. Then $g(s) = D(P \| Q)$ and $g(q) = 0$. Let $I(\theta) = I_\theta(A, B; \pi)$,

where the subscript θ indicates that A^π and B are distributed according to $\mathcal{P}_\theta$. Then

$$\begin{aligned} I_s(A^\pi; B|A) &= H_s(A^\pi|A) - H_s(A^\pi|B, A) \\ &= H_q(A^\pi|B, A) - H_s(A^\pi|B, A) \\ &= - \int_q^s \frac{d}{d\theta} H_\theta(A^\pi|B, A) d\theta, \end{aligned}$$

where the second equality holds because for a fixed q , $H_\theta(A^\pi|A)$ does not change with θ and when $\theta = q$, A^π and B are independent and hence $I_q(A^\pi; B|A) = 0$ so that $H_q(A^\pi|A) = H_q(A^\pi|B, A)$. By [7, Lemma 7.1], we have

$$\begin{aligned} \frac{d}{d\theta} H_\theta(A^\pi|B, A) &= (I) + \binom{n}{2} \frac{d}{d\theta} (h(q) - g(\theta)) \\ &= (I) - \binom{n}{2} g'(\theta), \end{aligned}$$

where $h(q) = -q \log q - (1-q) \log(1-q)$ is the binary entropy function,

$$(I) = \sum_{e \in \binom{[n]}{2}} \sum_{x_e, y_e} \frac{\partial p_\theta(y_e | x_e)}{\partial \theta} \mathbb{E} \left[\mu_e(x_e | B_{\setminus e}, A) \log \sum_{x'_e} p_\theta(y_e | x'_e) \mu_e(x'_e | B_{\setminus e}, A) \right],$$

$B_{\setminus e}$ denotes the adjacency vector B excluding B_e , $\mu_e(\cdot | B_{\setminus e}, A)$ is the distribution of A_e^π conditional on $(B_{\setminus e}, A)$ under $\mathcal{P}_\theta$. Note that the expectation in (I) is taken over the random variables $A, B_{\setminus e}$ which depend on θ . Since $g(q) = 0$, we have

$$\begin{aligned} I_s(A^\pi; B|\pi) &= - \int_q^s (I) + \binom{n}{2} g(s) \\ &= - \int_q^s (I) + \binom{n}{2} D(P \| Q). \end{aligned} \quad (24)$$

It remains to relate (I) to the reconstruction error. Note that for $x, y \in \{0, 1\}$,

$$p_\theta(y|x) = \alpha(x)y + (1-\alpha(x))(1-y), \quad \alpha(x) = \theta x + \eta(1-x) \quad (25)$$

and

$$\frac{\partial p_\theta(1|x)}{\partial \theta} = \frac{\partial \alpha(x)}{\partial \theta} = x + \frac{\partial \eta}{\partial \theta} (1-x) = - \frac{\partial p_\theta(0|x)}{\partial \theta}.$$

Thus for each x_e ,

$$\begin{aligned} \sum_{y_e=0,1} \frac{\partial p_\theta(y_e | x_e)}{\partial \theta} \mathbb{E} \left[\mu_e(x_e | B_{\setminus e}, A) \times \right. \\ \left. \log \sum_{x'_e=0,1} p_\theta(y_e | x'_e) \mu_e(x'_e | B_{\setminus e}, A) \right] \\ = \frac{\partial p_\theta(1|x_e)}{\partial \theta} \mathbb{E} \left[\mu_e(x_e | B_{\setminus e}, A) \times \right. \\ \left. \log \frac{\sum_{x'_e=0,1} p_\theta(1|x'_e) \mu_e(x'_e | B_{\setminus e}, A)}{1 - \sum_{x'_e=0,1} p_\theta(1|x'_e) \mu_e(x'_e | B_{\setminus e}, A)} \right] \\ = - \frac{\partial p_\theta(1|x_e)}{\partial \theta} \mathbb{E} [\mu_e(x_e | B_{\setminus e}, A) h'(y_e)], \end{aligned}$$

where we defined

$$y_e \equiv y_e(B_{\setminus e}, A) \triangleq \sum_{x_e=0,1} p_\theta(1|x_e)\mu_e(x_e|B_{\setminus e}, A).$$

and used $h'(x) = \log \frac{1-x}{x}$. Then we have

$$\begin{aligned} & \sum_{x_e=0,1} \sum_{y_e=0,1} \frac{\partial p_\theta(y_e|x_e)}{\partial \theta} \\ & \mathbb{E} \left[\mu_e(x_e|B_{\setminus e}, A) \log \sum_{x'_e=0,1} p_\theta(y_e|x'_e)\mu_e(x'_e|B_{\setminus e}, A) \right] \\ &= -\mathbb{E} \left[\frac{\partial y_e}{\partial \theta} h'(y_e) \right] \end{aligned}$$

Let $\tilde{x}_e = \mathbb{E}[A_e^\pi|B_{\setminus e}, A]$. Then $y_e = \theta\tilde{x}_e + \eta(1 - \tilde{x}_e)$. Let

$$\Delta_e = y_e - q = (\theta - \eta)(\tilde{x}_e - q) = \frac{\theta - q}{1 - q}(\tilde{x}_e - q).$$

Then $\mathbb{E}[\Delta_e] = 0$ and $\mathbb{E}[\Delta_e^2] = (\frac{\theta - q}{1 - q})^2 \text{Var}(\tilde{x}_e)$. Furthermore,

$$\frac{\partial y_e}{\partial \theta} = \frac{1}{1 - q}(\tilde{x}_e - q).$$

Using $h''(x) = -\frac{1}{x(1-x)}$, we get $h'(y_e) = h'(q) - \frac{1}{\xi(1-\xi)}\Delta_e$ for some ξ between y_e and q . Note that $\eta \leq q \leq \theta \leq s$ and $y_e \in [\eta, \theta]$. Thus $\eta \leq \xi \leq s$. So

$$\begin{aligned} & -\mathbb{E} \left[\frac{\partial y_e}{\partial \theta} h'(y_e) \right] \\ &= -\frac{h'(q)}{1 - q} \mathbb{E}[\tilde{x}_e - q] + \frac{\theta - q}{(1 - q)^2} \mathbb{E} \left[\frac{(\tilde{x}_e - q)^2}{\xi(1 - \xi)} \right] \\ &\geq \frac{\theta - q}{s(1 - q)^2} \text{Var}_\theta(\tilde{x}_e). \end{aligned}$$

Integrating over θ we get

$$\begin{aligned} \int_q^s (\text{I}) &= \sum_e \int_q^s d\theta \left(-\mathbb{E} \left[\frac{\partial y_e}{\partial \theta} h'(y_e) \right] \right) \\ &\geq \sum_e \int_q^s d\theta \frac{\theta - q}{s(1 - q)^2} \text{Var}_\theta(\tilde{x}_e) \end{aligned} \quad (26)$$

Finally, note that the above bound pertains to $\tilde{x}_e = \mathbb{E}[A_e^\pi|B_{\setminus e}, A]$, which we now relate to $\hat{x}_e = \mathbb{E}[A_e^\pi|B, A]$. Denote by $\mu_e(\cdot|B, A)$ the full posterior law of A_e^π . Note that

$$\hat{x}_e = \sum_{x_e} x_e \mu_e(x_e|B, A) = \frac{\sum_{x_e} x_e \mu_e(x_e|B_{\setminus e}, A) p_\theta(B_e|x_e)}{\sum_{x_e} \mu_e(x_e|B_{\setminus e}, A) p_\theta(B_e|x_e)},$$

where the last equality holds by the Bayes' theorem and the independence between B_e and $(A, B_{\setminus e})$ conditional on A_e^π , that is,

$$\begin{aligned} \mu_e(x_e|B, A) &= \mathbb{P}\{A_e^\pi = x_e|B, A\} \\ &\propto \mathbb{P}\{A_e^\pi = x_e|B_{\setminus e}, A\} \mathbb{P}\{B_e|A_e^\pi = x_e, B_{\setminus e}, A\} \\ &= \mu_e(x_e|B_{\setminus e}, A) p_\theta(B_e|x_e). \end{aligned}$$

By (25), we have

$$p_\theta(y|x) = 1 - y + (\theta x + \eta(1 - x))(2y - 1).$$

After some simplification, we have

$$\begin{aligned} \hat{x}_e &= \sum_{x_e} x_e \mu_e(x_e|B, A) \\ &= \frac{\tilde{x}_e(\theta B_e + (1 - \theta)(1 - B_e))}{1 - \eta - (1 - 2\eta)B_e + \tilde{x}_e(2B_e - 1)(\theta - \eta)} \\ &= \begin{cases} \frac{(1 - \theta)\tilde{x}_e}{1 - \eta - \tilde{x}_e(\theta - \eta)} & B_e = 0 \\ \frac{\theta\tilde{x}_e}{\eta + \tilde{x}_e(\theta - \eta)} & B_e = 1 \end{cases}. \end{aligned}$$

Since $\eta \leq q \leq \theta \leq s$, we have

$$\hat{x}_e \leq B_e \min\left(1, \frac{s}{\eta}\tilde{x}_e\right) + (1 - B_e)\tilde{x}_e$$

and hence

$$\mathbb{E}[\hat{x}_e^2] \leq \mathbb{E}\left[\min\left(1, \frac{s}{\eta}\tilde{x}_e\right)^2 B_e\right] + \mathbb{E}[\tilde{x}_e^2].$$

Note that

$$\begin{aligned} & \mathbb{E}\left[\min\left(1, \frac{s}{\eta}\tilde{x}_e\right)^2 B_e\right] \\ &\stackrel{(a)}{=} \mathbb{E}\left[\mathbb{E}\left[\min\left(1, \frac{s}{\eta}\tilde{x}_e\right)^2 \middle| A_e^\pi\right] \mathbb{E}[B_e | A_e^\pi]\right] \\ &\stackrel{(b)}{=} \mathbb{E}\left[\mathbb{E}\left[\min\left(1, \frac{s}{\eta}\tilde{x}_e\right)^2 \middle| A_e^\pi\right] (sA_e^\pi + \eta(1 - A_e^\pi))\right] \\ &\stackrel{(c)}{\leq} s\mathbb{E}[A_e^\pi] + s\mathbb{E}\left[\mathbb{E}[\tilde{x}_e | A_e^\pi] (1 - A_e^\pi)\right] \\ &\leq sq + s\mathbb{E}[\tilde{x}_e] = 2sq, \end{aligned}$$

where (a) follows from the conditional independence of $\tilde{x}_e$ (which depends on $(A, B_{\setminus e})$) and B_e given A_e^π ; (b) follows from (25); (c) follows by using $\min\left(1, \frac{s}{\eta}\tilde{x}_e\right)^2 \leq 1$ to get the first term and $\min\left(1, \frac{s}{\eta}\tilde{x}_e\right)^2 \leq \frac{s}{\eta}\tilde{x}_e$ to get the second term. Combining the previous two displays yields that

$$\mathbb{E}_\theta[\hat{x}_e^2] \leq 2sq + \mathbb{E}[\tilde{x}_e^2] \leq 2sq + q^2 + \text{Var}(\tilde{x}_e^2).$$

It follows that

$$\begin{aligned} & \text{mmse}(A^\pi) \\ &= \sum_e \mathbb{E}[(x_e - \hat{x}_e)^2] \\ &= \sum_e (\mathbb{E}[x_e^2] - \mathbb{E}[\hat{x}_e^2]) \\ &\geq \binom{n}{2} q(1 - q) - 2 \binom{n}{2} sq - \sum_e \text{Var}(\tilde{x}_e^2). \end{aligned} \quad (27)$$

Combining (26) with (27) yields that

$$\begin{aligned} & \int_q^s (\text{I}) \\ &\geq \int_q^s d\theta \frac{\theta - q}{s(1 - q)^2} \\ &\quad \left(\binom{n}{2} q(1 - q) - 2 \binom{n}{2} sq - \text{mmse}_\theta(A^\pi) \right) d\theta \\ &= \int_q^s d\theta \frac{\theta - q}{s(1 - q)^2} \left(\binom{n}{2} q(1 - q) - \text{mmse}_\theta(A^\pi) \right) d\theta \end{aligned}$$

$$\begin{aligned}
& - \binom{n}{2} \frac{(s-q)^2 q}{(1-q)^2} \\
& \geq \int_q^s d\theta \frac{\theta - q}{s(1-q)^2} \left(\binom{n}{2} q(1-q) - \text{mmse}_\theta(A^\pi) \right) d\theta \\
& - \binom{n}{2} q s^2
\end{aligned}$$

The conclusion follows by combining the last display with (24).

D. Proof of Proposition 4

In this section, we prove Proposition 4 by connecting the MSE of A^π to the Hamming risk of estimating π . In particular, assuming (19), that is, $\text{mmse}(A^\pi) \geq \mathbb{E}[\|A\|^2] (1-\xi)$, we aim to show that $\mathbb{E}[\text{overlap}(\pi, \hat{\pi})] = O\left(\xi^{1/4} + \left(\frac{n \log n}{\mathbb{E}[\|A\|^2]}\right)^{1/4}\right)$ for any estimator $\hat{\pi}(A, B)$. We first present a general program and then specialize the argument to the Gaussian and Erdős-Rényi graph model.

Recall that $\text{overlap}(\pi, \hat{\pi})$ denotes the fraction of fixed points of $\sigma \triangleq \pi^{-1} \circ \hat{\pi}$. Let $\alpha(\pi, \hat{\pi})$ denote the fraction of fixed points of the edge permutation σ^E induced by the node permutation σ (cf. Section I-C.1). The following simple lemma relates $\alpha(\pi, \hat{\pi})$ to $\text{overlap}(\pi, \hat{\pi})$.

Lemma 2: It holds that

$$\mathbb{E}[\text{overlap}(\pi, \hat{\pi})] \leq \sqrt{\mathbb{E}[\alpha(\pi, \hat{\pi})]} + \frac{1}{n}.$$

Proof: In view of (15),

$$\binom{n \text{overlap}(\pi, \hat{\pi})}{2} \leq \binom{n}{2} \alpha(\pi, \hat{\pi}).$$

By Jensen's inequality,

$$\begin{aligned}
\left(\frac{n \mathbb{E}[\text{overlap}(\pi, \hat{\pi})]}{2} \right) & \leq \mathbb{E} \left[\binom{n \text{overlap}(\pi, \hat{\pi})}{2} \right] \\
& \leq \binom{n}{2} \mathbb{E}[\alpha(\pi, \hat{\pi})].
\end{aligned}$$

The desired conclusion follows because for $x, y \geq 0$, $\binom{nx}{2} \leq \binom{n}{2} y \iff nx^2 - x - (n-1)y \leq 0 \implies x \leq \frac{1 + \sqrt{1 + 4n(n-1)y}}{2n} \leq \sqrt{y} + \frac{1}{n}$. $\square$

In view of Lemma 2 and the fact that $\mathbb{E}[\|A\|^2] \leq n^2$, it suffices to show $\mathbb{E}[\alpha(\pi, \hat{\pi})] = O\left(\xi^{1/2} + \left(\frac{n \log n}{\mathbb{E}[\|A\|^2]}\right)^{1/2}\right)$. Let $\alpha_0 = \mathbb{E}[\alpha(\pi, \hat{\pi})]$ and define an estimator of A^π by $\hat{A} = \alpha_0 A^{\hat{\pi}} + (1 - \alpha_0) \mathbb{E}[A]$. This is well-defined since α_0 is deterministic and $\hat{\pi}$ only depends on (A, B) . Intuitively, $\hat{A}$ can be viewed as an interpolation between the “plug-in” estimator $A^{\hat{\pi}}$ and the trivial estimator $\mathbb{E}[A^\pi] = \mathbb{E}[A]$. We remark that to derive the desired lower bound to α_0 , it is crucial to use the interpolated estimator $\hat{A}$ rather than the “plug-in” estimator, because we expect α_0 to be small and $\hat{\pi}$ is only slightly correlated with π .

On the one hand, by definition of the MMSE and the assumption (19),

$$\mathbb{E}[\|A^\pi - \hat{A}\|^2] \geq \text{mmse}(A^\pi) \geq \mathbb{E}[\|A\|^2] (1 - \xi). \quad (28)$$

On the other hand, we claim that in both the Gaussian and Erdős-Rényi model,

$$\begin{aligned}
\mathbb{E}[\langle A^\pi, A^{\hat{\pi}} \rangle] & \geq \mathbb{E}[\|A\|_2^2] \alpha_0 \\
& - O\left(\sqrt{\mathbb{E}[\|A\|_2^2] n \log n}\right), \quad (29)
\end{aligned}$$

so that

$$\begin{aligned}
\mathbb{E}[\|A^\pi - \hat{A}\|^2] & = \mathbb{E}[\|A^\pi\|^2] + \mathbb{E}[\|\hat{A}\|^2] - 2\mathbb{E}[\langle A^\pi, \hat{A} \rangle] \\
& \stackrel{(a)}{=} (1 + \alpha_0^2) \mathbb{E}[\|A\|^2] - (1 - \alpha_0)^2 \mathbb{E}[\|A\|^2] \\
& \quad - 2\alpha_0 \mathbb{E}[\langle A^\pi, A^{\hat{\pi}} \rangle] \\
& \leq (1 - \alpha_0^2) \mathbb{E}[\|A\|^2] \\
& \quad + O\left(\alpha_0 \sqrt{\mathbb{E}[\|A\|_2^2] n \log n}\right), \quad (30)
\end{aligned}$$

where in (a) we used the fact that $\mathbb{E}[A^\pi] = \mathbb{E}[A]$ is entrywise constant and $\langle \mathbb{E}[A], A^{\hat{\pi}} \rangle = \mathbb{E}[A_{12}] \sum_{i < j} A_{\hat{\pi}(i), \hat{\pi}(j)} = \mathbb{E}[A_{12}] \sum_{i < j} A_{i,j} = \langle \mathbb{E}[A], A \rangle$ so that $\langle \mathbb{E}[A], \mathbb{E}[A^{\hat{\pi}}] \rangle = \|\mathbb{E}[A]\|^2$. Combining (28) and (30) yields that

$$\begin{aligned}
\alpha_0^2 & \leq \xi + O\left(\alpha_0 \sqrt{\frac{n \log n}{\mathbb{E}[\|A\|^2]}}\right) \\
\implies \alpha_0 & = O\left(\xi^{1/2} + \sqrt{\frac{n \log n}{\mathbb{E}[\|A\|^2]}}\right). \quad (31)
\end{aligned}$$

To finish the proof, it remains to prove the claim (29).

Proof of Claim (29): Let C be a sufficiently large constant. For each permutation $\pi' \in \mathcal{S}_n$, define an event

$$\begin{aligned}
\mathcal{F}_{\pi'} & = \left\{ \langle A^\pi, A^{\pi'} \rangle \geq \mathbb{E}[\|A\|^2] \alpha(\pi, \pi') \right. \\
& \quad \left. - C \sqrt{\mathbb{E}[\|A\|^2] n \log n} \right\} \quad (32)
\end{aligned}$$

and set $\mathcal{F} \triangleq \cap_{\pi' \in \mathcal{S}_n} \mathcal{F}_{\pi'}$. It follows that

$$\begin{aligned}
& \mathbb{E}[\langle A^\pi, A^{\hat{\pi}} \rangle] \\
& = \mathbb{E}[\langle A^\pi, A^{\hat{\pi}} \rangle \mathbf{1}_{\{\mathcal{F}\}}] + \mathbb{E}[\langle A^\pi, A^{\hat{\pi}} \rangle \mathbf{1}_{\{\mathcal{F}^c\}}] \\
& \geq \mathbb{E}[\|A\|^2] \mathbb{E}[\alpha(\pi, \hat{\pi}) \mathbf{1}_{\{\mathcal{F}\}}] \\
& \quad - C \sqrt{\mathbb{E}[\|A\|^2] n \log n} - \mathbb{E}[\|A\|^2 \mathbf{1}_{\{\mathcal{F}^c\}}] \\
& \geq \mathbb{E}[\|A\|^2] (\alpha_0 - \mathbb{P}\{\mathcal{F}^c\}) \\
& \quad - C \sqrt{\mathbb{E}[\|A\|^2] n \log n} - \sqrt{\mathbb{E}[\|A\|^4] \mathbb{P}\{\mathcal{F}^c\}},
\end{aligned}$$

where the last inequality holds because

$$\begin{aligned}
\mathbb{E}[\alpha(\pi, \hat{\pi}) \mathbf{1}_{\{\mathcal{F}\}}] & = \mathbb{E}[\alpha(\pi, \hat{\pi})] - \mathbb{E}[\alpha(\pi, \hat{\pi}) \mathbf{1}_{\{\mathcal{F}^c\}}] \\
& \geq \alpha_0 - \mathbb{P}\{\mathcal{F}^c\},
\end{aligned}$$

and $\mathbb{E}[\|A\|^2 \mathbf{1}_{\{\mathcal{F}^c\}}] \leq \sqrt{\mathbb{E}[\|A\|^4] \mathbb{P}\{\mathcal{F}^c\}}$ by the Cauchy-Schwarz inequality. Note that $\mathbb{E}[\|A\|^4] = O(n^4)$, and $\mathbb{E}[\|A\|^2]$ is equal to $\binom{n}{2}$ in the Gaussian case and $\binom{n}{2} q$ in the Erdős-Rényi case (with $q \geq n^{-O(1)}$). To get (29), it suffices to prove $\mathbb{P}\{\mathcal{F}^c\} \leq e^{-n \log n}$, which, by union bound, further reduces to showing that $\mathbb{P}\{\mathcal{F}_{\pi'}^c\} \leq e^{-2n \log n}$ for any permutation $\pi' \in \mathcal{S}_n$. To this end, we consider the Gaussian and Erdős-Rényi graph model separately.

For the Gaussian Wigner model, let $M \in \{0, 1\}^{\binom{[n]}{2} \times \binom{[n]}{2}}$ denote the permutation matrix corresponding to the edge permutation σ^E induced by $\sigma = \pi^{-1} \circ \pi'$. Recalling that A denotes the weighted adjacency vector, we have $\langle A^\pi, A^{\pi'} \rangle = A^\top M A$. By Hanson-Wright inequality Lemma 9 in Appendix , with probability at least $1 - \delta$

$$A^\top M A \geq \text{Tr}(M) - C' \left(\|M\|_F^2 \sqrt{\log(1/\delta)} + \|M\|_2 \log(1/\delta) \right),$$

where $C' > 0$ is a universal constant. Since $\text{Tr}(M) = \binom{n}{2} \alpha(\pi, \pi')$, $\|M\|_F^2 = \binom{n}{2}$, and the spectral norm $\|M\|_2 = 1$, it follows that with probability at least $1 - \delta$,

$$A^\top M A \geq \binom{n}{2} \alpha(\pi, \pi') - C' \left(n \sqrt{\log(1/\delta)} + \log(1/\delta) \right).$$

Choosing $\delta = e^{-2n \log n}$ and C in (32) to be a large enough constant, we get that $\mathbb{P}\{\mathcal{F}_{\pi'}\} \geq 1 - e^{-2n \log n}$.

Next, we move to the Erdős-Rényi graph model. Fix any permutation $\pi' \in \mathcal{S}_n$. Let O_1 denote the set of fixed points of the edge permutation induced by $\pi' \circ \pi^{-1}$. By definition, $|O_1| = \binom{n}{2} \alpha(\pi, \pi')$ and

$$\langle A^\pi, A^{\pi'} \rangle \geq \sum_{(i,j) \in O_1} A_{ij} \sim \text{Binom} \left(\binom{n}{2} \alpha(\pi, \pi'), q \right).$$

By Bernstein's inequality, with probability at least $1 - \delta$,

$$\begin{aligned} & \langle A^\pi, A^{\pi'} \rangle \\ & \geq \binom{n}{2} \alpha(\pi, \pi') q \\ & \quad - C' \left(\sqrt{\binom{n}{2} \alpha(\pi, \pi') q \log(1/\delta)} + \log(1/\delta) \right) \\ & \geq \binom{n}{2} \alpha(\pi, \pi') q \\ & \quad - C' \left(\sqrt{n^2 q \log(1/\delta)} + \log(1/\delta) \right), \end{aligned}$$

where $C' > 0$ is a universal constant. Choosing $\delta = e^{-2n \log n}$ and C to be a large enough constant, we get that $\mathbb{P}\{\mathcal{F}_{\pi'}\} \geq 1 - e^{-2n \log n}$. $\square$

III. POSSIBILITY OF PARTIAL AND ALMOST EXACT RECOVERY

In this section, we prove the positive parts of Theorem 2 and Theorem 3.

We first argue that we can assume $ps \leq 1/2$ without loss of generality. Note that the existence/absence of an edge is a matter of representation and they are mathematically equivalent. As a consequence, by flipping 0 and 1, the model with parameter (n, p, s) is equivalent to that with parameter (n, p', s') for an appropriate choice of p' and s' such that $p's' = 1 - ps$, where

$$p' = \frac{(1 - ps)^2}{1 - 2ps + ps^2}, \quad s' = \frac{1 - 2ps + ps^2}{1 - ps}. \quad (33)$$

Therefore when $ps > \frac{1}{2}$, we can replace the model (n, p, s) with (n, p', s') so that $p's' = 1 - ps < \frac{1}{2}$.

For any two permutations $\pi, \pi' \in \mathcal{S}_n$, let $d(\pi, \pi')$ denote the number of non-fixed points in the $\pi' \circ \pi^{-1}$. The following proposition provides sufficient conditions for $\hat{\pi}_{\text{ML}}$ defined in (1) to achieve the partial recovery and almost exact recovery in Erdős-Rényi graphs.

Proposition 5: Let $ps \leq \frac{1}{2}$. If $p = 1 - o(1)$ and

$$\frac{ns^2(1-p)^2}{(1-ps)^2} \geq (4+\epsilon) \log n, \quad (34)$$

or if $p = 1 - \Omega(1)$ and

$$nps^2 \geq \begin{cases} \frac{(2+\epsilon) \log n}{\log(1/p)-1+p} & \text{if } p \geq n^{-\frac{1}{2}} \\ 4+\epsilon & \text{if } p < n^{-\frac{1}{2}} \end{cases}, \quad (35)$$

for any arbitrarily small constant $\epsilon > 0$, there exists a constant $0 < \delta < 1$ such that

$$\mathbb{P}\{d(\hat{\pi}_{\text{ML}}, \pi) < \delta n\} \geq 1 - n^{-1+o(1)},$$

that is, $\hat{\pi}_{\text{ML}}$ achieves partial recovery.

If in addition $nps^2(1-p)^2 = \omega(1)$, then for any constant $\delta > 0$,

$$\mathbb{P}\{d(\hat{\pi}_{\text{ML}}, \pi) < \delta n\} \geq 1 - n^{-1+o(1)},$$

that is, $\hat{\pi}_{\text{ML}}$ achieves almost exact recovery.

Remark 2: We explain how to prove the positive parts of Theorem 2 and Theorem 3 using Proposition 5.

- In the dense regime of $p = n^{-o(1)}$, either (34) or (35) already implies that $nps^2(1-p)^2 = \omega(1)$. By Proposition 5, the MLE achieves almost exact recovery under the condition (34) when $p = 1 - o(1)$ or $nps^2 \geq \frac{(2+\epsilon) \log n}{\log(1/p)-1+p}$ when $p = 1 - \Omega(1)$; this proves the positive part of Theorem 2.
- In the sparse regime of $p = n^{-\Omega(1)}$, condition (9) implies (35) and hence the MLE achieves partial recovery; this proves the positive part of Theorem 3. Furthermore, since $nps^2 = \omega(1)$ implies (35), the MLE achieves the almost exact recovery provided that $nps^2 = \omega(1)$, which is in fact needed for any estimator to succeed [6].

To prove Proposition 5, we need the following intermediate lemma, which bounds the probability that the ML estimator (1) makes a given number of errors.

Lemma 3: Let $\epsilon \in (0, 1)$ be an arbitrary constant and $ps \leq \frac{1}{2}$.

- For the Erdős-Rényi model, suppose that either (34) or (35) holds. Then there exists some constant $0 < \delta < 1$ such that for any $k \geq \delta n$,

$$\begin{aligned} \mathbb{P}\{d(\hat{\pi}_{\text{ML}}, \pi) = k\} & \leq 2 \exp \left(-nh \left(\frac{k}{n} \right) \right) \mathbf{1}_{\{k \leq n-1\}} \\ & \quad + e^{-2 \log n} \mathbf{1}_{\{k=n\}} + \exp \left(-\frac{1}{64} \epsilon k \log n \right), \end{aligned} \quad (36)$$

where $h(x) = -x \log x - (1-x) \log(1-x)$ is the binary entropy function.

If in addition $nps^2(1-p) = \omega(1)$, then (36) holds for any constant $0 < \delta < 1$ and all $k \geq \delta n$.

- For the Gaussian model, suppose that $n\rho^2 \geq (4+\epsilon) \log n$. Then (36) holds for any constant $0 < \delta < 1$ and all $k \geq \delta n$.

Note that Lemma 3 also includes the Gaussian case. In fact, analogous to Proposition 5, we can apply Lemma 3 to show that the MLE attains almost exact recovery when $n\rho^2 \geq (4+\epsilon)\log n$. We will not do it here; instead in the next section, we will directly prove a stronger result, showing that the MLE attains exact recovery under the same condition.

Now, we proceed to prove Proposition 5.

Proof of Proposition 5: Applying Lemma 3 with a union bound yields that

$$\begin{aligned} & \mathbb{P}\{d(\hat{\pi}_{\text{ML}}, \pi) \geq \delta n\} \\ & \leq \sum_{k \geq \delta n}^n \mathbb{P}\{d(\hat{\pi}_{\text{ML}}, \pi) = k\} \\ & \leq \exp(-2 \log n) + 2 \sum_{k \geq \delta n}^{n-1} \exp\left(-nh \left(\frac{k}{n}\right)\right) \\ & \quad + \sum_{k \geq \delta n} \exp\left(-\frac{1}{64} \epsilon k \log n\right) \\ & = n^{-1+o(1)}, \end{aligned} \quad (37)$$

where the last inequality follows from $\sum_{k \geq \delta n} \exp(-\frac{1}{64} \epsilon k \log n) \leq \frac{\exp(-\frac{\epsilon}{64} \delta n \log n)}{1 - \exp(-\frac{\epsilon}{64} \log n)} = n^{-\Omega(n)}$ for any fixed constant $\delta > 0$, and

$$\begin{aligned} & \sum_{k \geq 1}^{n-1} \exp\left(-nh \left(\frac{k}{n}\right)\right) \\ & \stackrel{(a)}{\leq} 2 \sum_{1 \leq k \leq n/2} \exp\left(-k \log \frac{n}{k}\right) \\ & \leq 2 \sum_{k=1}^{10 \log n} \exp\left(-k \log \frac{n}{k}\right) + 2 \sum_{10 \log n \leq k \leq n/2} 2^{-k} \\ & \leq 2e^{-\log n} \times 10 \log n + 4 \times 2^{-10 \log n} = n^{-1+o(1)}, \end{aligned}$$

where (a) follows from $h(x) = h(1-x)$ and $h(x) \geq x \log \frac{1}{x}$. $\square$

A. Proof of Lemma 3

Without loss of generality, we assume $\epsilon < 1$. Fix $k \in [n]$. Let $\mathcal{T}_k$ denote the set of permutations π' such that $d(\pi, \pi') = k$. Recall that F is the set of fixed points of $\sigma \triangleq \pi^{-1} \circ \pi'$ with $|F| = n - k$ and $\mathcal{O}_1 = \binom{F}{2}$ is a subset of fixed points of edge permutation σ^E . Let $\bar{A} = (\bar{A}_{ij})_{1 \leq i < j \leq n} = (A_{ij} - \mathbb{E}[A_{ij}])_{1 \leq i < j \leq n}$ and $\bar{B} = (\bar{B}_{ij})_{1 \leq i < j \leq n} = (B_{ij} - \mathbb{E}[B_{ij}])_{1 \leq i < j \leq n}$ denote the centered adjacency vectors of A and B respectively. It follows from the definition of MLE in (1) that for any $\tau \in \mathbb{R}$,

$$\begin{aligned} & \{d(\hat{\pi}_{\text{ML}}, \pi) = k\} \\ & \subset \left\{ \exists \pi' \in \mathcal{T}_k : \langle A^\pi, B \rangle - \langle A^{\pi'}, B \rangle \leq 0 \right\} \\ & \stackrel{(a)}{\subset} \left\{ \exists \pi' \in \mathcal{T}_k : \sum_{i < j} \bar{A}_{\pi(i)\pi(j)} \bar{B}_{ij} - \sum_{i < j} \bar{A}_{\pi'(i)\pi'(j)} \bar{B}_{ij} \leq 0 \right\} \\ & = \left\{ \exists \pi' \in \mathcal{T}_k : \sum_{(i,j) \notin \mathcal{O}_1} \bar{A}_{\pi(i)\pi(j)} \bar{B}_{ij} \right. \end{aligned}$$

$$\begin{aligned} & \left. - \sum_{(i,j) \notin \mathcal{O}_1} \bar{A}_{\pi'(i)\pi'(j)} \bar{B}_{ij} \leq 0 \right\} \\ & \subset \left\{ \exists \pi' \in \mathcal{T}_k : \sum_{(i,j) \notin \mathcal{O}_1} \bar{A}_{\pi(i)\pi(j)} \bar{B}_{ij} < \tau \right\} \\ & \cup \left\{ \exists \pi \in \mathcal{T}_k : \sum_{(i,j) \notin \mathcal{O}_1} \bar{A}_{\pi'(i)\pi'(j)} \bar{B}_{ij} \geq \tau \right\}, \end{aligned}$$

where (a) holds because $\langle A^\pi, B \rangle - \langle A^{\pi'}, B \rangle = \sum_{i < j} \bar{A}_{\pi(i)\pi(j)} \bar{B}_{ij} - \sum_{i < j} \bar{A}_{\pi'(i)\pi'(j)} \bar{B}_{ij}$. Note that

$$\begin{aligned} & \left\{ \exists \pi' \in \mathcal{T}_k : \sum_{(i,j) \notin \mathcal{O}_1} \bar{A}_{\pi(i)\pi(j)} \bar{B}_{ij} < \tau \right\} \\ & = \left\{ \exists F \subset [n] : |F| = n - k, \sum_{(i,j) \notin \binom{F}{2}} \bar{A}_{\pi(i)\pi(j)} \bar{B}_{ij} < \tau \right\}. \end{aligned}$$

Thus, by the union bound,

$$\begin{aligned} & \mathbb{P}\{d(\hat{\pi}_{\text{ML}}, \pi) = k\} \\ & \leq \sum_{F \subset [n] : |F| = n-k} \mathbb{P}\left\{ \sum_{(i,j) \notin \binom{F}{2}} \bar{A}_{\pi(i)\pi(j)} \bar{B}_{ij} < \tau \right\} \\ & \quad + \sum_{\pi' \in \mathcal{T}_k} \mathbb{P}\left\{ \sum_{(i,j) \notin \mathcal{O}_1} \bar{A}_{\pi'(i)\pi'(j)} \bar{B}_{ij} \geq \tau \right\}. \end{aligned}$$

Let $Y = \sum_{(i,j) \notin \mathcal{O}_1} \bar{A}_{\pi(i)\pi(j)} \bar{B}_{ij}$ and $X_{\pi'} = \sum_{(i,j) \notin \mathcal{O}_1} \bar{A}_{\pi'(i)\pi'(j)} \bar{B}_{ij}$. Then,

$$\begin{aligned} \mathbb{P}\{d(\hat{\pi}_{\text{ML}}, \pi) = k\} & \leq \binom{n}{k} \mathbb{P}\{Y \leq \tau\} \\ & \quad + \sum_{\pi' \in \mathcal{T}_k} \mathbb{P}\{X_{\pi'} \geq \tau\} \\ & \triangleq \text{(I)} + \text{(II)}. \end{aligned} \quad (38)$$

For (I), we claim that for both the Erdős-Rényi random graph and Gaussian model and appropriate choice of τ ,

$$\text{(I)} \leq \exp\left(-nh \left(\frac{k}{n}\right)\right) + \exp(-2 \log n) \mathbf{1}_{\{k=n\}}. \quad (39)$$

For (II), applying the Chernoff bound together with the union bound yields that for any $t > 0$,

$$\begin{aligned} \text{(II)} & \leq \sum_{\pi' \in \mathcal{T}_k} \mathbb{P}\{X_{\pi'} \geq \tau\} \\ & \leq n^k \exp(-t\tau) \mathbb{E}[\exp(tX_{\pi'})]. \end{aligned} \quad (40)$$

Note that

$$X_{\pi'} = \sum_{O \in \mathcal{O} \setminus \mathcal{O}_1} \underbrace{\sum_{(i,j) \in O} \bar{A}_{\pi'(i)\pi'(j)} \bar{B}_{ij}}_{\triangleq \bar{X}_O},$$

where $\mathcal{O}$ denotes the collection of all edge orbits of σ . Since edge orbits are disjoint, it follows that $\overline{X}_{\mathcal{O}}$ are mutually independent across different $\mathcal{O}$. Therefore,

$$\mathbb{E}[\exp(tX_{\pi'})] = \prod_{\mathcal{O} \in \mathcal{O} \setminus \mathcal{O}_1} \mathbb{E}[\exp(t\overline{X}_{\mathcal{O}})]. \quad (41)$$

It turns out that the MGF of $\overline{X}_{\mathcal{O}}$ can be explicitly computed for both Erdős-Rényi random graph and Gaussian model. In particular, applying Lemma 7 yields that $\mathbb{E}[e^{t\overline{X}_{\mathcal{O}}}] = M_{|\mathcal{O}|}$, and $M_{\ell} \leq M_2^{\ell/2}$ for $\ell \geq 2$. It follows that

$$\begin{aligned} \mathbb{E}[\exp(tX_{\pi'})] &\stackrel{(a)}{=} M_1^{n_2} \prod_{\ell=2}^{\binom{n}{2}} M_{\ell}^{N_{\ell}} \\ &\stackrel{(b)}{\leq} M_1^{n_2} \prod_{\ell=2}^{\binom{n}{2}} M_2^{\ell N_{\ell}/2} \\ &\stackrel{(c)}{=} \left(\frac{M_1^2}{M_2}\right)^{\frac{1}{2}n_2} M_2^{\frac{\binom{n}{2} - \binom{n_1}{2}}{2}} \\ &\stackrel{(d)}{=} \left(\frac{M_1^2}{M_2}\right)^{\frac{k}{2}} M_2^{\frac{m}{2}}, \end{aligned}$$

where (a) follows from (41) and $N_1 = \binom{n_1}{2} + n_2$ in view of (15); (b) follows from $M_{\ell} \leq M_2^{\ell/2}$ for $\ell \geq 2$; (c) follows from $\sum_{\ell=1}^{\binom{n}{2}} \ell N_{\ell} = \binom{n}{2}$; (d) follows from $n_2 \leq k$, $n_1 = n - k$, and $m \triangleq \binom{n}{2} - \binom{n-k}{2}$. Combining the last displayed equation with (40) yields that

$$\begin{aligned} \text{(II)} &\leq \exp\left(k \log n - t\tau + \frac{k}{2} \log \frac{M_1^2}{M_2} + \frac{m}{2} \log M_2\right) \\ &\stackrel{(a)}{\leq} \exp\left(k \log n - t\tau + \frac{k}{2} \log \frac{M_1^2}{M_2} + \frac{m}{2} (M_2 - 1)\right) \\ &\stackrel{(b)}{\leq} \exp\left(-\frac{\epsilon}{64} k \log n\right), \end{aligned}$$

where (a) holds because $\log(1+x) \leq x$ for any $x > -1$; (b) follows from the claim that

$$\begin{aligned} &\inf_{t \geq 0} \left\{ -t\tau + \frac{k}{2} \log \frac{M_1^2}{M_2} + \frac{m}{2} (M_2 - 1) \right\} \\ &\leq -\left(1 + \frac{\epsilon}{64}\right) k \log n. \end{aligned} \quad (42)$$

It remains to specify τ and verify (39) and (42) for Erdős-Rényi and Gaussian models separately. In the following, we will use the inequality

$$\frac{1}{2} \left(1 - \frac{1}{n}\right) kn \leq m \leq kn, \quad (43)$$

which follows from $m = \binom{n}{2} - \binom{n-k}{2} = \frac{1}{2}kn(2 - \frac{k+1}{n})$.

- For Erdős-Rényi random graphs, $\overline{A} = (A_{ij} - q)_{1 \leq i < j \leq n}$ and $\overline{B} = (B_{ij} - q)_{1 \leq i < j \leq n}$, where $q = ps$. Then for any $F \subset [n]$ with $|F| = n - k$,

$$Y = \sum_{(i,j) \notin \binom{F}{2}} (A_{\pi(i)\pi(j)} - q)(B_{ij} - q).$$

Note that $\mathbb{E}[(A_{\pi(i)\pi(j)} - q)(B_{ij} - q)] = ps^2(1-p)$ any $i < j$.

Let $\mu \triangleq \mathbb{E}[Y] = mps^2(1-p)$ and set $\tau = (1-\gamma)\mu$ where

$$\gamma \triangleq \begin{cases} \sqrt{\frac{16h(k/n)}{kps^2(1-p)^2}} & k \leq n-1 \\ \sqrt{\frac{16 \log n}{n(n-1)ps^2(1-p)^2}} & k = n \end{cases}.$$

We next choose a constant $0 < \delta < 1$ such that $h(\delta)/\delta \leq \frac{\epsilon^2}{2^{12}} nps^2(1-p)^2$. Since $h(x)/x$ is monotone decreasing in x and converges to 0 as $x \rightarrow 1$, under the condition (34) or (35), there exists some $0 < \delta < 1$ such that $h(\delta)/\delta \leq \frac{\epsilon^2}{2^{12}} nps^2(1-p)^2$. If further $nps^2(1-p)^2 = \omega(1)$ then for any constant $0 < \delta < 1$, $h(\delta)/\delta \leq \frac{\epsilon^2}{2^{12}} nps^2(1-p)^2$. Hence, for all $\delta n \leq k \leq n-1$, we have $\frac{n}{k} h(\frac{k}{n}) \leq h(\delta)/\delta \leq \frac{\epsilon^2}{2^{12}} nps^2(1-p)^2$, and then $\gamma \leq \sqrt{\frac{16h(\delta)/\delta}{nps^2(1-p)^2}} < \frac{\epsilon}{16}$; if $k = n$, since $nps^2(1-p)^2 = \Omega(1)$ under the condition (34) or (35), $\gamma < \frac{\epsilon}{16}$ for all sufficiently large n . In conclusion, we have $\gamma < \frac{\epsilon}{16}$.

Applying the Bernstein inequality yields

$$\begin{aligned} \mathbb{P}\{Y \leq \tau\} &\stackrel{(a)}{\leq} \exp\left(-\frac{\frac{1}{2}\gamma^2\mu^2}{mps^2 + \frac{1}{3}\gamma\mu}\right) \\ &\stackrel{(b)}{\leq} \exp\left(-\frac{1}{2 + \frac{\epsilon}{24}}\gamma^2\mu(1-p)\right) \\ &\leq \exp\left(-\frac{1}{4}\gamma^2\mu(1-p)\right), \end{aligned} \quad (44)$$

where (a) holds because for any $i < j$

$$\begin{aligned} &\mathbb{E}[(A_{\pi(i)\pi(j)} - q)^2(B_{ij} - q)^2] \\ &= ps^2(1 - 4ps + 2p^2s + 4p^2s^2 - 3p^3s^2) \\ &\leq ps^2(1 - 2ps + 4p^2s^2) \leq ps^2, \end{aligned}$$

in view of $ps \leq \frac{1}{2}$; (b) holds because $mps^2 = \frac{\mu}{1-p}$ and $\gamma < \frac{\epsilon}{16}$. By (43), we have

$$\gamma^2\mu(1-p) \geq \begin{cases} 8(n-1)h(k/n). & k \leq n-1 \\ 8 \log n & k = n \end{cases}.$$

Finally, applying (44) together with $\binom{n}{k} \leq e^{nh(\frac{k}{n})}$, we arrive at the desired claim (39).

Next, we verify (42) by choosing $t > 0$ appropriately. Recall that $q = ps$. In view of (64) in Lemma 7, M_2 depends on t and q . When q is small and p is bounded away from 1, we will choose $t \leq \log \frac{1}{p}$; otherwise, we will choose $t \leq \frac{1}{2^{10}}$. As such, we bound M_2 using its Taylor expansion around $q = 0$ and $t = 0$, respectively, resulting in the following lemma.

Lemma 4: Assume $ps \leq \frac{1}{2}$.

- If $t \leq \log \frac{1}{p}$, then $\frac{M_1^2}{M_2} \leq e^2$, and

$$\begin{aligned} M_2 &\leq 1 + q^2s^2 - 2q^2 + 10q^3t(1+t) \\ &\quad + 2e^tq^2(1-s^2) + e^{2t}q^2s^2. \end{aligned} \quad (45)$$

- If $t \leq \frac{1}{2^{10}}$, then $\frac{M_1^2}{M_2} \leq e^2$, and

$$M_2 \leq 1 + t^2\sigma^4(1 + \rho^2) + 8t^3qs, \quad (46)$$

where $\rho = \frac{s-q}{1-q}$ and $\sigma^2 = q(1-q)$.

The proof of Lemma 4 is deferred to Appendix . With Lemma 4, we separately consider two cases, depending on whether q is small and p is bounded away from 1. Pick q_0 and c_0 to be sufficiently small constants such that

$$c_0 \leq \frac{\epsilon}{2^{13}}, \quad q_0 \leq \frac{\epsilon}{480} \phi(1 - c_0),$$

where for any $x \in (0, 1)$, we define

$$\phi(x) \triangleq \frac{\log \frac{1}{x} - 1 + x}{x \left(\log \frac{1}{x} \right) (1 + \log \frac{1}{x})} \quad (47)$$

This function is monotonically decreasing on $(0, 1)$ (see Appendix for a proof).

Then we separately consider two cases.

Case 1: $q \leq q_0$ and $p \leq 1 - c_0$. Here we will pick $t \leq \log \frac{1}{p}$. Then, by (45) in Lemma 4,

$$\begin{aligned} & -t\tau + \frac{k}{2} \log \frac{M_1^2}{M_2} + \frac{m}{2} (M_2 - 1) \\ & \stackrel{(a)}{\leq} -t(1 - \gamma)mps^2(1 - p) + k + \frac{m}{2} (M_2 - 1) \\ & \stackrel{(b)}{\leq} k + \frac{1}{2}mps^2(f(t) + 10p^2st(1 + t)), \end{aligned}$$

where

$$f(t) \triangleq -2(1 - \gamma)t + e^{2t}ps^2 + 2e^tp(1 - s^2) + ps^2 - 2p;$$

(a) holds by $\frac{M_1^2}{M_2} \leq e^2$; (b) holds by (45). Note that for $t \leq \log \frac{1}{p}$,

$$\begin{aligned} 10p^2st(1 + t) & \leq 10qp \left(\log \frac{1}{p} \right) \left(1 + \log \frac{1}{p} \right) \\ & \leq \frac{\epsilon}{48} \left(\log \frac{1}{p} - 1 + p \right), \end{aligned} \quad (48)$$

where the last inequality holds because $q \leq q_0 \leq \frac{\epsilon}{480} \phi(1 - c_0) \leq \frac{\epsilon}{480} \phi(p)$, and $\phi(x)$ is a monotone decreasing function on $x \in (0, 1)$ by Lemma 10.

Therefore, to verify (42), it suffices to check

$$\begin{aligned} & \frac{1}{2}mps^2 \left(f(t) + \frac{\epsilon}{48} \left(\log \frac{1}{p} - 1 + p \right) \right) \\ & \leq - \left(1 + \frac{\epsilon}{64} \right) k \log n - k. \end{aligned} \quad (49)$$

To this end, we need the following simple claim.

Claim 1: Suppose $mps^2 \left(\log \frac{1}{p} - 1 + p \right) \geq (2 + \epsilon) \log n$. Then, for all sufficiently large n and all $0 < p \leq 1 - c_0$,

$$\gamma(1 - p) \leq \frac{\epsilon}{48} \left(\log \frac{1}{p} - 1 + p \right). \quad (50)$$

We defer the proof of Claim 1 and first finish the proof of (49), by choosing different values of $t \in [0, \log \frac{1}{p}]$ according to different cases.

– **Case 1(a):** $mps^2 (\log(1/p) - 1 + p) \geq (4 + \epsilon) \log n$. We pick $t = \frac{1}{2} \log \frac{1}{p}$. Then

$$\begin{aligned} f(t) & = -(1 - \gamma) \log \frac{1}{p} + s^2(\sqrt{p} - 1)^2 \\ & \quad + 2\sqrt{p} - 2p \end{aligned}$$

$$\begin{aligned} & \stackrel{(a)}{\leq} -(1 - \gamma) \left(\log \frac{1}{p} + 1 - p \right) + \gamma(1 - p) \\ & \stackrel{(b)}{\leq} - \left(1 - \gamma - \frac{\epsilon}{48} \right) \left(\log \frac{1}{p} - 1 + p \right), \end{aligned}$$

where (a) holds because $s^2 \leq 1$; (b) holds by (50) in Claim 1. Hence,

$$\begin{aligned} & \frac{1}{2}mps^2 \left(f(t) + \frac{\epsilon}{48} \left(\log \frac{1}{p} - 1 + p \right) \right) \\ & \leq - \frac{1}{2}mps^2 \left(1 - \gamma - \frac{\epsilon}{24} \right) \left(\log \frac{1}{p} - 1 + p \right) \\ & \leq - \left(1 + \frac{\epsilon}{4} \right) \left(1 - \frac{1}{n} \right) \left(1 - \gamma - \frac{\epsilon}{24} \right) k \log n. \end{aligned}$$

where the last inequality follows from (43) and our assumption $mps^2 (\log(1/p) - 1 + p) \geq (4 + \epsilon) \log n$.

– **Case 1(b):** $(2 + \epsilon) \log n \leq mps^2 (\log(1/p) - 1 + p) \leq (4 + \epsilon) \log n$ and $p \geq n^{-\frac{1}{2}}$. We pick $t = \log \frac{1}{p}$ and get

$$\begin{aligned} f(t) & \leq -2(1 - \gamma) \log \frac{1}{p} + \frac{s^2}{p} + 2(1 - s^2) \\ & \quad + ps^2 - 2p \\ & = -2(1 - \gamma) \left(\log \frac{1}{p} - 1 + p \right) + \frac{s^2}{p} \\ & \quad + 2\gamma(1 - p) + s^2(p - 2) \\ & \stackrel{(a)}{\leq} -2 \left(1 - \gamma - \frac{\epsilon}{48} \right) \left(\log \frac{1}{p} - 1 + p \right) \\ & \quad + 2\gamma(1 - p) \\ & \stackrel{(b)}{\leq} -2 \left(1 - \gamma - \frac{\epsilon}{24} \right) \left(\log \frac{1}{p} - 1 + p \right), \end{aligned}$$

where (a) holds because $\frac{s^2}{p} \leq \frac{(4 + \epsilon) \log n}{np^2(\log(1/p) - 1 + p)} \leq \frac{\epsilon}{48} (\log(1/p) - 1 + p)$ for all sufficiently large n when $n^{-\frac{1}{2}} \leq p \leq 1 - c_0$; (b) holds in view of (50) in Claim 1. Then under the assumption that $mps^2 (\log(1/p) - 1 + p) \geq (2 + \epsilon) \log n$, and by (43),

$$\begin{aligned} & \frac{1}{2}mps^2 \left(f(t) + \frac{\epsilon}{48} \left(\log \frac{1}{p} - 1 + p \right) \right) \\ & \leq - \left(1 + \frac{\epsilon}{2} \right) \left(1 - \frac{1}{n} \right) \left(1 - \gamma - \frac{\epsilon}{16} \right) k \log n. \end{aligned}$$

– **Case 1(c):** $mps^2 \geq 4 + \epsilon$ and $p < n^{-\frac{1}{2}}$. We pick $t = \frac{1}{2} \log \frac{1}{ps^2} \leq \log \frac{1}{p}$, where the inequality follows because $s^2 \geq p$, in view of $mps^2 \geq 4 + \epsilon$ and $p < n^{-\frac{1}{2}}$. Then

$$\begin{aligned} f(t) & = -(1 - \gamma) \log \frac{1}{ps^2} + 1 + 2\sqrt{\frac{p}{s^2}}(1 - s^2) \\ & \quad - 2p + ps^2 \\ & \leq -(1 - \gamma) \log \frac{1}{ps^2} + 1 + 2\sqrt{\frac{p}{s^2}} \\ & \leq - \left(1 - \gamma - \frac{\epsilon}{48} \right) \log \frac{1}{ps^2}, \end{aligned}$$

where the last inequality holds for n sufficiently large because $\frac{p}{s^2} \leq \frac{np^2}{4 + \epsilon} \leq \frac{1}{4 + \epsilon}$ and $\log \frac{1}{ps^2} \geq \frac{1}{2} \log n$

when $nps^2 \geq 4 + \epsilon$ and $p < n^{-\frac{1}{2}}$. Then,

$$\begin{aligned} & \frac{1}{2}mps^2 \left(f(t) + \frac{\epsilon}{48} \left(\log \frac{1}{p} - 1 + p \right) \right) \\ & \stackrel{(a)}{\leq} -\frac{1}{4} \left(1 - \frac{1}{n} \right) \left(1 - \gamma - \frac{\epsilon}{24} \right) knps^2 \log \frac{1}{ps^2} \\ & \stackrel{(b)}{\leq} -\left(1 + \frac{\epsilon}{4} \right) \left(1 - \frac{1}{n} \right) \left(1 - \gamma - \frac{\epsilon}{24} \right) k \log \frac{n}{4 + \epsilon}, \end{aligned}$$

where (a) holds by (43) and $\log \frac{1}{p} - 1 + p \leq \log \frac{1}{ps^2}$; (b) holds due to $nps^2 \geq 4 + \epsilon$.

Hence, in view of $\gamma < \frac{\epsilon}{16}$ for $\delta n \leq k \leq n$, combining the three cases, we get

$$\begin{aligned} & \frac{1}{2}mps^2 \inf_{0 \leq t \leq \log(1/ps^2)} f(t) \\ & < -\left(1 + \frac{\epsilon}{4} \right) \left(1 - \frac{1}{n} \right) \left(1 - \frac{\epsilon}{8} \right) k \log \frac{n}{4 + \epsilon} \\ & \leq -\left(1 + \frac{\epsilon}{64} \right) k \log n - k, \end{aligned}$$

where the last inequality holds for all sufficiently large n . Thus we arrive at the desired (49) which further implies (42).

Now we are left to prove Claim 1.

Proof of Claim 1: Note that for any $0 < p \leq 1 - c_0$,

$$\begin{aligned} \gamma(1-p) & \stackrel{(a)}{\leq} \sqrt{\frac{16h(\delta)/\delta}{nps^2}} \stackrel{(b)}{\leq} \sqrt{\frac{8h(\delta)/\delta \left(\log \frac{1}{p} - 1 + p \right)}{\log n}} \\ & \stackrel{(c)}{\leq} \sqrt{\frac{16h(\delta)/\delta}{\left(\log \frac{1}{1-c_0} - c_0 \right) \log n}} \left(\log \frac{1}{p} - 1 + p \right) \\ & \stackrel{(d)}{\leq} \frac{\epsilon}{48} \left(\log \frac{1}{p} - 1 + p \right), \end{aligned}$$

where (a) holds because if $\delta n \leq k \leq n - 1$, $h(x)/x$ is a monotone decreasing function on $0 < x \leq 1$ and $\frac{n}{k}h(\frac{k}{n}) \leq h(\delta)/\delta$, and if $k = n$, $\frac{\log n}{n-1} \leq h(\delta)/\delta$ for n sufficiently large; (b) holds by assumption $nps^2 \left(\log \frac{1}{p} - 1 + p \right) \geq (2 + \epsilon) \log n$; (c) holds because $\log \frac{1}{x} - 1 + x$ is monotone decreasing function on $x \in (0, 1)$ and we have $p \leq 1 - c_0$; (d) holds for n sufficiently large. Hence, the claim (50) follows. $\square$

Case 2: $q \geq q_0$ or $p \geq 1 - c_0$. Here we will pick $t \leq \frac{1}{2^{10}}$. Then, by Lemma 4, we get

$$\begin{aligned} & -t\tau + \frac{k}{2} \log \frac{M_1^2}{M_2} + \frac{m}{2} (M_2 - 1) \\ & \stackrel{(a)}{\leq} k - t\tau + \frac{m}{2} (M_2 - 1) \\ & \stackrel{(b)}{\leq} k + mf(t), \end{aligned}$$

where

$$f(t) \triangleq -t\rho\sigma^2 \left(1 - \frac{\epsilon}{16} \right) + \frac{1}{2} (M_2 - 1);$$

(a) holds by $\frac{M_1^2}{M_2} \leq e^2$, and (46) given $t \leq \frac{1}{2^{10}}$; (b) holds because $\tau \geq \rho\sigma^2 m \left(1 - \frac{\epsilon}{16} \right)$, in view of $\tau = (1 - \gamma)mnp^2(1-p)$, $\rho = \frac{s(1-p)}{1-ps}$, $\sigma^2 = ps(1-ps)$, and $\gamma \leq \frac{\epsilon}{16}$.

Therefore, to verify (42), it suffices to check

$$mf(t) \leq -\left(1 + \frac{\epsilon}{64} \right) k \log n - k. \quad (51)$$

– **Case 2(a):** $p > 1 - c_0$. We pick $t = \frac{\rho}{\sigma^2(1+\rho^2)} = \frac{(1-p)}{p(1-ps)(1+\rho^2)} \stackrel{(a)}{<} \frac{2c_0}{1-c_0} \leq \frac{1}{2^{10}}$, where (a) holds in view of $p > 1 - c_0$ and $ps \leq \frac{1}{2}$. By (46),

$$\begin{aligned} M_2 & \leq 1 + \frac{\rho^2}{1 + \rho^2} + \frac{8\rho^3qs}{\sigma^6(1 + \rho^2)^3} \\ & \leq 1 + \frac{\rho^2}{1 + \rho^2} \left(1 + \frac{\epsilon}{16} \right), \end{aligned}$$

where the last inequality holds because $\frac{8\rho^3qs}{\sigma^6(1+\rho^2)^2} = \frac{8(1-p)}{p^2(1-ps)^4(1+\rho^2)^2} \leq \frac{\epsilon}{32}$ given $1-p \leq c_0 \leq \frac{\epsilon}{2^{13}}$ and $ps \leq \frac{1}{2}$. Then, we have

$$\begin{aligned} f(t) & \leq -\frac{\rho^2}{1 + \rho^2} \left(1 - \frac{\epsilon}{16} - \frac{1}{2} \left(1 + \frac{\epsilon}{32} \right) \right) \\ & = -\frac{\rho^2}{1 + \rho^2} \left(\frac{1}{2} - \frac{5\epsilon}{64} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} mf(t) & \stackrel{(a)}{\leq} -\frac{1}{2} \left(1 - \frac{1}{n} \right) kn \left(\frac{\rho^2}{1 + \rho^2} \right) \left(\frac{1}{2} - \frac{5\epsilon}{64} \right) \\ & \stackrel{(b)}{\leq} -\left(1 + \frac{\epsilon}{64} \right) k \log n - k, \end{aligned}$$

where (a) holds by (43); (b) holds under the assumption that $\rho^2 = \frac{s^2(1-p)^2}{(1-ps)^2} \geq (4 + \epsilon) \frac{\log n}{n}$ in (34) for n sufficiently large.

– **Case 2(b):** $q \geq q_0$ and $p \leq 1 - c_0$. We pick $t = \frac{4 \log n}{\rho\sigma^2 n} = \frac{4 \log n}{ps^2(1-p)n} \leq \frac{1}{2^{10}}$, where the last inequality holds for n sufficiently large. By (46),

$$\begin{aligned} M_2 & \leq 1 + t^2\sigma^4 \left(1 + \rho^2 + \frac{8tqs}{\sigma^4(1 + \rho^2)} \right) \\ & \leq 1 + t^2\sigma^4(1 + \rho^2) \left(1 + \frac{\epsilon}{64} \right), \end{aligned}$$

where the last inequality holds because $\frac{8tqs}{\sigma^4(1+\rho^2)^2} = \frac{32 \log n}{(1-p)q^2(1-q)^2(1+\rho^2)^2n} \leq \frac{\epsilon}{64}$ for n sufficiently large, given $q_0 \leq q \leq \frac{1}{2}$ and $p \leq 1 - c_0$. Then, we have

$$\begin{aligned} f(t) & \leq -\frac{4 \log n}{n} \left(1 - \frac{\epsilon}{32} \right) \\ & \quad + \frac{8 \log^2 n}{n^2} \frac{1 + \rho^2}{\rho^2} \left(1 + \frac{\epsilon}{64} \right) \\ & \leq -\frac{4 \log n}{n} \left(1 - \frac{\epsilon}{32} \right) + \frac{32 \log^2 n}{\rho^2 n^2} \\ & \leq -\frac{3 \log n}{n}, \end{aligned}$$

where the last inequality holds for n sufficiently large because $\rho = \frac{s(1-p)}{1-ps} \geq \frac{q_0 c_0}{1-q_0}$. Therefore,

$$\begin{aligned} mf(t) & \stackrel{(a)}{\leq} -\frac{3}{2} \left(1 - \frac{1}{n} \right) k \log n \\ & \stackrel{(b)}{\leq} -\left(1 + \frac{\epsilon}{64} \right) k \log n - k, \end{aligned}$$

where (a) holds by (43); (b) holds for n sufficiently large.

- For Gaussian model, set $\tau = \rho m - a_k$, where

$$a_k = \begin{cases} C\sqrt{2h\left(\frac{k}{n}\right)nm} & k \leq n-1 \\ Cn\sqrt{\log n} & k = n \end{cases}$$

for some universal constant $C > 0$ to be specified later. Recall that $h(x)/x$ is monotone decreasing in x and converges to 0 as $x \rightarrow 1$, under the condition $n\rho^2 \geq (4 + \epsilon)\log n$, for any constant $0 < \delta < 1$, we have $h(\delta)/\delta \leq \frac{\epsilon^2}{2^{13}C^2}n\rho^2$ when n is sufficiently large. Therefore, for $\delta n \leq k \leq n-1$, $\frac{n}{k}h\left(\frac{k}{n}\right) \leq h(\delta)/\delta \leq \frac{\epsilon^2}{2^{13}C^2}n\rho^2$. Since $m = kn\left(1 - \frac{k+1}{2n}\right) \geq \frac{1}{4}kn$, we have $a_k \leq \frac{\epsilon}{64}\rho\sqrt{knm} \leq \frac{\epsilon}{32}\rho m$. For $k = n$, by the assumption that $n\rho^2 \geq (4 + \epsilon)\log n$, $a_k \leq \frac{\epsilon}{32}\rho m$. In conclusion, we have $\tau \geq \rho m\left(1 - \frac{\epsilon}{32}\right)$.

Note that $(A_{\pi(i)\pi(j)}, B_{ij})$ are i.i.d. pairs of standard normal random variables with zero mean and correlation coefficient ρ . Therefore, $\bar{A} = A$ and $\bar{B} = B$. It follows that $Y = \sum_{(i,j) \notin \binom{F}{2}} A_{\pi(i)\pi(j)} B_{ij}$.

First, we verify (39) using the Hanson-Wright inequality stated in Lemma 9. Pick $C = 2c$, where $c > 0$ is the universal constant in Lemma 9. If $k = n$, applying Lemma 9 with $M = I_m$ to $Y = \sum_{(i,j) \notin \binom{F}{2}} A_{\pi(i)\pi(j)} B_{ij}$ yields that $\mathcal{P}(Y \leq \tau) \leq e^{-2\log n}$. If $k \leq n-1$, applying Lemma 9 again yields that

$$\begin{aligned} \binom{n}{k} \mathcal{P}(Y \leq \tau) &\leq \binom{n}{k} \exp\left(-2nh\left(\frac{k}{n}\right)\right) \\ &\leq \exp\left(-nh\left(\frac{k}{n}\right)\right). \end{aligned}$$

Next, we check (42). In view of (65) in Lemma 7, $M_1 = \frac{1}{\lambda_2}$ and $M_2 = \frac{1}{\lambda_1\lambda_2}$, where $\lambda_1 = \sqrt{(1 + \rho t)^2 - t^2}$ and $\lambda_2 = \sqrt{(1 - \rho t)^2 - t^2}$ for $0 \leq t \leq \frac{1}{1+\rho}$. Thus for $0 \leq t \leq \frac{\rho}{1+\rho^2}$, $\frac{M_1}{M_2} = \lambda_1 \leq 1 + \rho^2 \leq 2$. It follows that

$$\begin{aligned} &\inf_{t \geq 0} \left(-t\tau + \frac{k}{2} \log \frac{M_1^2}{M_2} + \frac{m}{2} (M_2 - 1) \right) \\ &= \inf_{t \geq 0} \left(-t\tau + k \log \frac{M_1}{M_2} + \frac{m+k}{2} (M_2 - 1) \right) \\ &\leq k + \inf_{0 \leq t \leq \frac{\rho}{1+\rho^2}} \left(-t\tau + \frac{m+k}{2} (M_2 - 1) \right) \\ &\leq k + \frac{1}{2}k(n-1)\rho \inf_{0 \leq t \leq \frac{\rho}{1+\rho^2}} f(t), \end{aligned}$$

where

$$f(t) \triangleq -t\left(1 - \frac{\epsilon}{32}\right) + \frac{1}{2\rho} \left(1 + \frac{2}{n-1}\right) (M_2 - 1),$$

and the last inequality holds because $\tau \geq \rho m\left(1 - \frac{\epsilon}{32}\right)$ and (43).

Therefore, to verify (42), it suffices to check

$$\frac{1}{2}(n-1)\rho \inf_{0 \leq t \leq \frac{\rho}{1+\rho^2}} f(t)$$

$$\leq -\left(1 + \frac{\epsilon}{64}\right) \log n - 1. \quad (52)$$

Case 1: $\rho^2 \geq \frac{\epsilon^2}{256}$. We pick $t = \frac{\rho}{1+2\rho^2}$. Then,

$$\begin{aligned} M_2 &= \frac{(1 + 2\rho^2)^2}{\sqrt{1 + \rho^2 + \rho^4} \sqrt{1 + 5\rho^2 + 9\rho^4}} \\ &\leq \left(\frac{1 + 2\rho^2}{1 + \frac{3}{2}\rho^2} \right)^2 \leq 1 + \frac{2\rho^2}{2 + 3\rho^2}, \end{aligned}$$

where the first inequality holds because $(1 + \rho^2 + \rho^4)(1 + 5\rho^2 + 9\rho^4) \geq (1 + \frac{3}{2}\rho^2)^4$. Therefore,

$$\begin{aligned} f(t) &= -\frac{\rho}{1 + 2\rho^2} \left(1 - \frac{\epsilon}{32}\right) + \frac{\rho}{2 + 3\rho^2} \left(1 + \frac{2}{n-1}\right) \\ &= -\frac{\rho}{2 + 3\rho^2} \left(\left(2 - \frac{\rho^2}{1 + 2\rho^2}\right) \left(1 - \frac{\epsilon}{32}\right) - \left(1 + \frac{2}{n-1}\right) \right) \\ &\leq -\frac{\rho}{2 + 3\rho^2} \left(\frac{5}{3} \left(1 - \frac{\epsilon}{32}\right) - \left(1 + \frac{2}{n-1}\right) \right) \\ &\leq -\frac{\rho}{5} \left(\frac{2}{3} - \frac{5\epsilon}{96} - \frac{2}{n-1} \right), \end{aligned}$$

where the first inequality holds because $\frac{\rho^2}{1+2\rho^2} \leq \frac{1}{3}$ and the last inequality follows from $2 + 3\rho^2 \leq 5$. Hence,

$$\frac{1}{2}(n-1)\rho f(t) \leq -(n-1)\frac{\rho^2}{10} \left(\frac{2}{3} - \frac{5\epsilon}{96} - \frac{2}{n-1} \right).$$

Thus it follows from $\rho^2 \geq \frac{\epsilon^2}{256}$ that (52) holds for all sufficiently large n .

Case 2: $(4 + \epsilon)\frac{\log n}{n} \leq \rho^2 < \frac{\epsilon^2}{256}$. We pick $t = \frac{\rho}{1+\rho^2}$. Then,

$$\begin{aligned} M_2 &= \frac{(1 + \rho^2)^2}{\sqrt{1 + 3\rho^2 + 4\rho^4} \sqrt{1 - \rho^2}} \\ &\leq \frac{(1 + \rho^2)^2}{1 + \rho^2 - \rho^4} \leq 1 + \rho^2 + 2\rho^4, \end{aligned}$$

where the first inequality holds because $(1 + 3\rho^2 + 4\rho^4)(1 - \rho^2) - (1 + \rho^2 - \rho^4)^2 = \rho^4(2 - 2\rho^2 + \rho^4) \geq 0$ under the assumption that $\rho^2 \leq \epsilon^2/256$. Therefore,

$$\begin{aligned} f(t) &\leq -\frac{\rho}{1 + \rho^2} \left(1 - \frac{\epsilon}{32}\right) + \frac{\rho(1 + 2\rho^3)}{2} \left(1 + \frac{2}{n-1}\right) \\ &\leq -\rho \left(\frac{1}{2} - \frac{\epsilon}{16} \right) \left(1 - \frac{4}{n-1}\right), \end{aligned}$$

where the first inequality holds because $\rho^2 < \frac{\epsilon^2}{256}$. Hence,

$$\begin{aligned} &\frac{1}{2}(n-1)\rho f(t) \\ &\leq -\frac{1}{4}(n-1)\rho^2 \left(1 - \frac{\epsilon}{32}\right) \left(1 - \frac{4}{n-1}\right) \\ &\leq -\left(1 + \frac{\epsilon}{32}\right) \left(1 - \frac{1}{n}\right) \left(1 - \frac{4}{n-1}\right) \log n, \end{aligned}$$

where the last inequality follows from the assumption that $\rho^2 \geq (4 + \epsilon) \frac{\log n}{n}$. Thus (52) holds for all sufficiently large n .

IV. EXACT RECOVERY

A. Possibility of Exact Recovery

Building upon the almost exact recovery results in the preceding section, we now analyze the MLE for exact recovery. Improving upon Lemma 3, the following lemma gives a tighter bound on the probability that the MLE makes a small number of errors with respect to the general correlated Erdős-Rényi graph model specified by the joint distribution $P = (p_{ab} : a, b \in \{0, 1\})$.

Lemma 5: Suppose that for any constant $0 < \epsilon < 1$,

- for general Erdős-Rényi random graphs, if $n(\sqrt{p_{00}p_{11}} - \sqrt{p_{01}p_{10}})^2 \geq (1 + \epsilon) \log n$;
- for Gaussian model, if $n\rho^2 \geq (4 + \epsilon) \log n$;

then for any $k \in [n]$ such that $k \leq \frac{\epsilon}{16}n$,

$$\mathbb{P}\{d(\hat{\pi}_{\text{ML}}, \pi) = k\} \leq \exp\left(-\frac{\epsilon}{8}k \log n\right). \quad (53)$$

Note that Lemma 5 only holds when k/n is small but requires less stringent conditions than Lemma 3. Inspecting the proof of Lemma 5, one can see that if the condition is strengthened by a factor of 2 to $n(\sqrt{p_{00}p_{11}} - \sqrt{p_{01}p_{10}})^2 \geq (2 + \epsilon) \log n$ for the Erdős-Rényi graphs (which is the assumption of [4]), then (53) holds for any $k \in [n]$. Conversely, we will prove in Section IV-B that exact recovery is impossible if $n(\sqrt{p_{00}p_{11}} - \sqrt{p_{01}p_{10}})^2 \leq (1 - \epsilon) \log n$. Closing this gap of two for general Erdős-Rényi graphs is an open problem.

In the following, we apply Lemma 5 for small k and Lemma 3 (or Proposition 5) for large k to show the success of MLE in exact recovery.

Proof of positive parts in Theorem 1 and Theorem 4: Without loss of generality, we assume $\epsilon < 1$. For all $k \leq \frac{\epsilon}{16}n$, by Lemma 5, for both Erdős-Rényi random graphs and Gaussian model,

$$\begin{aligned} \sum_{k=1}^{\frac{\epsilon}{16}n} \mathbb{P}\{d(\hat{\pi}_{\text{ML}}, \pi) = k\} &\leq \sum_{k=1}^{\frac{\epsilon}{16}n} \exp\left(-\frac{\epsilon}{8}k \log n\right) \\ &\leq \frac{\exp\left(-\frac{\epsilon}{8} \log n\right)}{1 - \frac{\epsilon}{8} \log n} = n^{-\frac{\epsilon}{8} + o(1)}. \end{aligned}$$

Moreover,

- For the Gaussian model, pick $\delta = \frac{\epsilon}{16}$. Thus by (36) in Lemma 3 and (37), (57) follows.
- For the subsampling Erdős-Rényi random graphs, we divide the proof into two cases depending on whether $ps \leq 1/2$.

Case 1: $ps \leq \frac{1}{2}$. Note that condition (12) is equivalent to

$$\begin{aligned} n(\sqrt{p_{11}p_{00}} - \sqrt{p_{01}p_{10}})^2 &= nps^2 f(p, s) \\ &\geq (1 + \epsilon) \log n, \end{aligned} \quad (54)$$

where

$$f(p, s) \triangleq \left(\sqrt{1 - 2ps + ps^2} - \sqrt{p}(1 - s)\right)^2.$$

In the sequel, we aim to show that if (54) holds, then $nps^2(1 - p) = \omega(1)$, (34) holds for $p = 1 - o(1)$, and (35) holds for $p = 1 - \Omega(1)$. To this end, we will use the inequality

$$0 \leq \frac{\partial f(p, s)}{\partial s} \leq 2\sqrt{p}, \quad (55)$$

which follows from

$$\frac{\partial f(p, s)}{\partial s} = \frac{2\sqrt{p} \left(\sqrt{1 - 2ps + ps^2} - \sqrt{p}(1 - s)\right)^2}{\sqrt{1 - 2ps + ps^2}}.$$

- By (55), $f(p, s)$ is a monotone increasing on $s \in (0, 1)$. Therefore, $f(p, s) \leq 1 - p$. Hence, if (54) holds, then $nps^2(1 - p) = \omega(1)$.
- Suppose $p = 1 - o(1)$. We have

$$\begin{aligned} f(p, s) &= p(1 - s)^2 \left(\sqrt{1 + \frac{1 - p}{p(1 - s)^2}} - 1\right)^2 \\ &\leq \frac{(1 - p)^2}{4p(1 - s)^2}, \end{aligned}$$

where the last inequality holds because $\sqrt{1 + x} \leq 1 + \frac{x}{2}$. Thus, if (54) holds, then

$$\frac{n(1 - p)^2}{(1 - ps)^2} \geq \frac{n(1 - p)^2}{(1 + \frac{\epsilon}{2})p(1 - s)^2} \geq (4 + \epsilon) \log n,$$

where the first inequality holds because $p = 1 - o(1)$ and $ps \leq \frac{1}{2}$. Hence, (34) follows.

- Suppose $p = 1 - \Omega(1)$. By the mean value theorem and (55), we have

$$\begin{aligned} f(p, s) &\leq f(p, 0) + s \sup_{s: ps \leq 1/2} \frac{\partial f(p, s)}{\partial s} \\ &\leq (1 - \sqrt{p})^2 + 2\sqrt{p}s, \end{aligned}$$

Now, suppose $s = o(1)$ or $p = o(1)$. Then

$$(1 - \sqrt{p})^2 + 2\sqrt{p}s \leq (1 - \sqrt{p})^2 \left(1 + \frac{\epsilon}{4}\right),$$

for all sufficiently large n . Therefore, if (54) holds, then

$$nps^2(1 - \sqrt{p})^2 \geq \left(1 + \frac{\epsilon}{2}\right) \log n. \quad (56)$$

If instead $s = \Omega(1)$ and $p = \Omega(1)$, then (56) follows directly.

In conclusion, for both cases, since $\log \frac{1}{p} - 1 + p \geq 2(1 - \sqrt{p})^2$, we get that (35) holds.

Finally, by Proposition 5, we have that

$$\mathbb{P}\left\{d(\hat{\pi}_{\text{ML}}, \pi) > \frac{\epsilon}{16}n\right\} \leq n^{-1+o(1)}. \quad (57)$$

Case 2: $ps > \frac{1}{2}$. In this case, we consider the equivalent model (n, p', s') as specified in (33) with $p's' = 1 - ps \leq 1/2$ by flipping 0 and 1. Note that after flipping 0 and 1, both the value of $\sqrt{p_{11}p_{00}} - \sqrt{p_{01}p_{10}}$ and the MLE are clearly unchanged. Therefore, applying the Case 1 of Theorem 4 to the model (n, p', s') , we get that (57) holds for the model (n, p, s) .

Hence, for both the Erdős-Rényi random graph and Gaussian model,

$$\begin{aligned} \mathbb{P}\{d(\hat{\pi}_{\text{ML}}, \pi) > 0\} &\leq \sum_{k=1}^{\frac{\epsilon}{16}n} \mathbb{P}\{d(\hat{\pi}_{\text{ML}}, \pi) = k\} \\ &\quad + \mathbb{P}\left\{d(\hat{\pi}_{\text{ML}}, \pi) > \frac{\epsilon}{16}n\right\} \\ &\leq n^{-\frac{\epsilon}{8}+o(1)}. \end{aligned}$$

□

1) *Proof of Lemma 5:* In this proof we focus on the case with positive correlation, i.e., $p_{00}p_{11} \geq p_{01}p_{10}$ in the general Erdős-Rényi graphs and $\rho \geq 0$ in the Gaussian model. The case with negative correlation can be proved analogously by bounding the probability $\langle A^{\pi'} - A^{\pi}, B \rangle \leq 0$.

Fix $k \in [n]$ and let $\mathcal{T}_k$ denote the set of permutation π' such that $d(\pi, \pi') = k$. Let $\mathcal{O}'_1$ is the set of fixed points of edge permutation σ^E , where in view of (15),

$$|\mathcal{O}'_1| = N_1 = \binom{n_1}{2} + n_2 = \binom{n-k}{2} + n_2.$$

Then, applying the Chernoff bound together with the union bound yields that for any $t \geq 0$,

$$\begin{aligned} \mathbb{P}\{d(\hat{\pi}_{\text{ML}}, \pi) = k\} &\leq \sum_{\pi' \in \mathcal{T}_k} \mathbb{P}\{Z_{\pi'} \geq 0\} \\ &\leq n^k \mathbb{E}[\exp(tZ_{\pi'})], \end{aligned} \quad (58)$$

where

$$\begin{aligned} Z_{\pi'} &\triangleq \sum_{i < j} A_{\pi'(i)\pi'(j)} B_{ij} - \sum_{i < j} A_{\pi(i)\pi(j)} B_{ij} \\ &= \sum_{\mathcal{O} \setminus \mathcal{O}'_1} \left(\sum_{(i,j) \in \mathcal{O}} A_{\pi'(i)\pi'(j)} B_{i,j} - \sum_{(i,j) \in \mathcal{O}} A_{\pi(i)\pi(j)} B_{i,j} \right) \\ &\triangleq \sum_{\mathcal{O} \setminus \mathcal{O}'_1} Z_{\mathcal{O}}, \end{aligned}$$

where

$$Z_{\mathcal{O}} = X_{\mathcal{O}} - Y_{\mathcal{O}}.$$

Since edge orbits are disjoint, it follows that $Z_{\mathcal{O}}$ are mutually independent across different $\mathcal{O}$. Therefore, for any $t > 0$,

$$\mathbb{E} \left[\exp \left(t \sum_{\mathcal{O} \in \mathcal{O} \setminus \mathcal{O}'_1} Z_{\mathcal{O}} \right) \right] = \prod_{\mathcal{O} \in \mathcal{O} \setminus \mathcal{O}'_1} \mathbb{E} [\exp(tZ_{\mathcal{O}})].$$

It turns out that the MGF of $Z_{\mathcal{O}}$ can be explicitly computed in both Erdős-Rényi random graph and Gaussian model. In particular, applying Lemma 8 yields that $\mathbb{E}[e^{tZ_{\mathcal{O}}}] = L_{|\mathcal{O}|}$ and $L_{\ell} \leq L_2^{\ell/2}$ for $\ell \geq 2$. It follows that

$$\mathbb{E}[\exp(tZ_{\pi'})] = \prod_{\ell=2}^{\binom{n}{2}} L_{\ell}^{N_{\ell}} \leq \prod_{\ell=2}^{\binom{n}{2}} L_2^{\ell N_{\ell}/2} \leq L_2^{\frac{m}{2}},$$

where $m \triangleq \binom{n}{2} - \binom{n-k}{2} - n_2$ and the last inequality follows from $\sum_{\ell=1}^{\binom{n}{2}} \ell N_{\ell} = \binom{n}{2}$ and $N_1 = \binom{n_1}{2} + n_2 = \binom{n-k}{2} + n_2$. Hence,

$$\mathbb{P}\{d(\hat{\pi}_{\text{ML}}, \pi) = k\}$$

$$\begin{aligned} &\stackrel{(a)}{\leq} \exp \left(k \log n + \frac{1}{2} kn \left(1 - \frac{k+2}{2n} \right) \log L_2 \right) \\ &\stackrel{(b)}{\leq} \exp \left(- \left(\frac{\epsilon}{4} - \frac{k+2}{2n} \left(1 + \frac{\epsilon}{4} \right) \right) k \log n \right) \\ &\stackrel{(c)}{\leq} \exp \left(- \frac{\epsilon}{8} k \log n \right), \end{aligned}$$

where (a) holds because $n_2 \leq k/2$ and then $m \geq kn(1 - \frac{k+2}{2n})$; (b) holds by the claim that $n \log L_2 \leq -2(1 + \epsilon/4) \log n$ for appropriately chosen t ; (c) holds for all sufficiently large n given $k/n \leq \frac{\epsilon}{16}$ and $0 < \epsilon < 1$. It remains to check $n \log L_2 \leq -2(1 + \epsilon/4) \log n$ by choosing appropriately t for Erdős-Rényi random graphs and Gaussian model separately.

- For Erdős-Rényi random graphs, in view of (66) in Lemma 8,

$$L_2 = 1 + 2(p_{01}p_{10}(e^t - 1) + p_{00}p_{11}(e^{-t} - 1)).$$

Since $p_{00}p_{11} \geq p_{01}p_{10}$, by choosing the optimal $t \geq 0$ such that $e^t = \sqrt{\frac{p_{00}p_{11}}{p_{01}p_{10}}}$, $L_2 = 1 - 2(\sqrt{p_{00}p_{11}} - \sqrt{p_{01}p_{10}})^2$, and hence

$$\begin{aligned} n \log L_2 &\leq -2n(\sqrt{p_{00}p_{11}} - \sqrt{p_{01}p_{10}})^2 \\ &\leq -2(1 + \epsilon) \log n, \end{aligned}$$

where the last inequality holds by the assumption that $n(\sqrt{p_{00}p_{11}} - \sqrt{p_{01}p_{10}})^2 \geq (1 + \epsilon) \log n$;

- For Gaussian model, in view of (67) in Lemma 8,

$$L_2 = \frac{1}{\sqrt{1 + 4t\rho - 4t^2(1 - \rho^2)}}.$$

By choosing the optimal $t = \frac{\rho}{2(1 - \rho^2)} \geq 0$, $L_2 = \sqrt{1 - \rho^2}$. and hence

$$n \log L_2 \leq -\frac{1}{2} n \rho^2 \leq -2(1 + \epsilon/4) \log n,$$

where the last inequality holds by the assumption that $n\rho^2 \geq (4 + \epsilon) \log n$.

B. Impossibility of Exact Recovery

In this subsection we prove the negative result in Theorem 4. As in Section IV-A, we consider a general correlated Erdős-Rényi graph model, where $\mathbb{P}\{A_{\pi(i)\pi(j)} = a, B_{ij} = b\} = p_{ab}$. We aim to show that if

$$n(\sqrt{p_{00}p_{11}} - \sqrt{p_{01}p_{10}})^2 \leq (1 - \epsilon) \log n, \quad (59)$$

then the exact recovery of the latent permutation π is impossible. Particularizing to the subsampling model parameterized by (11) the condition (59) reduces to

$$nps^2 \left(\sqrt{1 - 2ps + ps^2} - \sqrt{p(1 - s)} \right)^2 \leq (1 - \epsilon) \log n,$$

which, under the assumption that $p = 1 - \Omega(1)$ in Theorem 4, is further equivalent to the desired (13).

Since the true permutation π is uniformly distributed, the MLE $\hat{\pi}_{\text{ML}}$ minimizes the error probability among all estimators. In the sequel, we focus on the case of positive correlation as the other case is entirely analogous. Without

loss of generality, the latent permutation π is assumed to be the identity permutation id .

To prove the failure of the MLE, it suffices to show the existence of a permutation π' that achieves a higher likelihood than the true permutation $\pi = \text{id}$. To this end, we consider the set $\mathcal{T}_2$ of permutation π' such that $d(\pi', \pi) = 2$; in other words, π' coincides with π except for swapping two vertices. Then the cycle decomposition of π' consists of $n - 2$ fixed points and a 2-node orbit (i, j) for some pair of vertices (i, j) . It follows that

$$\begin{aligned} \langle A, B \rangle - \langle A^{\pi'}, B \rangle &= \sum_{k \in [n] \setminus \{i, j\}} (A_{ik} - A_{jk})(B_{ik} - B_{jk}) \\ &\triangleq \sum_{k \in [n] \setminus \{i, j\}} X_{ij,k}, \end{aligned}$$

where $X_{ij,k} \stackrel{\text{i.i.d.}}{\sim} a\delta_{+1} + b\delta_{-1} + (1 - a - b)\delta_0$ for $k \in [n] \setminus \{i, j\}$ with $a \triangleq 2p_{00}p_{11}$, $b \triangleq 2p_{01}p_{10}$ and $a \geq b$ by assumption. We aim to show the existence of $\pi' \in \mathcal{T}_2$ such that $\langle A, B \rangle \leq \langle A^{\pi'}, B \rangle$, which further implies that π' has a higher likelihood than π . We divide the remaining analysis into two cases depending on whether $na \geq (2 - \epsilon) \log n$.

First we consider the case where $na \leq (2 - \epsilon) \log n$. We show that with probability at least $1 - n^{-\Omega(\epsilon)}$, there are at least $n^{\Omega(\epsilon)}$ distinct $\pi' \in \mathcal{T}_2$ such that $\langle A, B \rangle \leq \langle A^{\pi'}, B \rangle$. This implies that the MLE $\hat{\pi}_{\text{ML}}$ coincides with π with probability at most $n^{-\Omega(\epsilon)}$. Specifically, define χ_{ij} as the indicator random variable which equals to 1 if $X_{ij,k} \leq 0$ for all $k \neq i, j$, and 0 otherwise. Then

$$\begin{aligned} \mathbb{P}\{\chi_{ij} = 1\} &= \prod_{k \neq i, j} \mathbb{P}\{X_{ij,k} \leq 0\} \\ &= (1 - a)^{n-2} \geq n^{-2+\epsilon-o(1)}, \end{aligned}$$

where the last inequality holds because $an \leq (2 - \epsilon) \log n$. Let $I = \sum_{1 \leq i \leq n/2} \sum_{n/2 < j \leq n} \chi_{ij}$. Then $\mathbb{E}[I] \geq n^{\epsilon-o(1)}$. Next, we show $\text{var}(I) = o(\mathbb{E}[I]^2)$, which, by Chebyshev's inequality, implies that $I \geq \frac{1}{2}\mathbb{E}[I]$ with high probability. Note that

$$\begin{aligned} \text{var}(I) &= \sum_{1 \leq i, i' \leq n/2} \sum_{n/2 < j, j' \leq n} \left(\mathbb{P}\{\chi_{ij} = 1, \chi_{i'j'} = 1\} \right. \\ &\quad \left. - \mathbb{P}\{\chi_{ij} = 1\} \mathbb{P}\{\chi_{i'j'} = 1\} \right). \end{aligned}$$

We break the analysis into the following four cases depending on the choice of (i, j) and (i', j') .

Case 1: $i = i'$, and $j = j'$. In this case,

$$\begin{aligned} \mathbb{P}\{\chi_{ij} = 1, \chi_{i'j'} = 1\} - \mathbb{P}\{\chi_{ij} = 1\} \mathbb{P}\{\chi_{i'j'} = 1\} \\ \leq \mathbb{P}\{\chi_{ij} = 1\}. \end{aligned}$$

Case 2: $i = i'$, and $j \neq j'$. In this case, note that

$$\begin{aligned} \mathbb{P}\{\chi_{ij'} = 1 \mid \chi_{ij} = 1\} \\ &= \prod_{k \neq i, j'} \mathbb{P}\{X_{ij',k} \leq 0 \mid \chi_{ij} = 1\} \\ &= \mathbb{P}\{X_{ij',j} \leq 0 \mid X_{ij,j} \leq 0\} \end{aligned}$$

$$\begin{aligned} &\times \prod_{k \neq i, j, j'} \mathbb{P}\{X_{ij',k} \leq 0 \mid X_{ij,k} \leq 0\} \\ &\leq (1 - p_{00}p_{11})^{n-2}, \end{aligned}$$

where the last inequality holds because

$$\begin{aligned} \mathbb{P}\{X_{ij',k} \leq 0 \mid X_{ij,k} \leq 0\} \\ &= \frac{p_{11}(1 - p_{00})^2 + p_{01} + p_{10} + p_{00}(1 - p_{11})^2}{1 - 2p_{00}p_{11}} \\ &= \frac{1 - 4p_{00}p_{11} + p_{00}p_{11}(p_{00} + p_{11})}{1 - 2p_{00}p_{11}} \\ &\leq \frac{1 - 3p_{00}p_{11}}{1 - 2p_{00}p_{11}} \leq 1 - p_{00}p_{11}, \end{aligned}$$

and similarly for $\mathbb{P}\{X_{ij',j} \leq 0 \mid X_{ij,j} \leq 0\}$.

Case 3: $i \neq i'$, and $j = j'$. Analogous to Case 2, we have

$$\mathbb{P}\{\chi_{i'j} = 1 \mid \chi_{ij} = 1\} \leq (1 - p_{00}p_{11})^{n-2}$$

Case 4: $i \neq i'$, and $j \neq j'$. In this case,

$$\begin{aligned} \mathbb{P}\{\chi_{i'j'} = 1 \mid \chi_{ij} = 1\} &= \prod_{k \neq i', j'} \mathbb{P}\{X_{i'j',k} \leq 0 \mid \chi_{ij} = 1\} \\ &\leq \prod_{k \neq i, i', j, j'} \mathbb{P}\{X_{i'j',k} \leq 0\} \\ &= (1 - 2p_{00}p_{11})^{n-4}, \end{aligned}$$

where the first inequality holds because for all $k \neq i, i', j, j'$, $X_{i'j',k}$ are independent from $\{X_{ij,k}\}_{k \neq i, j}$. Therefore,

$$\begin{aligned} \mathbb{P}\{\chi_{ij} = 1, \chi_{i'j'} = 1\} - \mathbb{P}\{\chi_{ij} = 1\} \mathbb{P}\{\chi_{i'j'} = 1\} \\ &= (1 - 2p_{00}p_{11})^{2n-6} \left(1 - (1 - 2p_{00}p_{11})^2 \right) \\ &\leq 4p_{00}p_{11}(1 - 2p_{00}p_{11})^{2n-6}. \end{aligned}$$

Combining all four cases above, we get that

$$\begin{aligned} \text{var}(I) &\leq \mathbb{E}[I] \left(1 + n(1 - p_{00}p_{11})^{n-2} \right. \\ &\quad \left. + n^2 p_{00}p_{11}(1 - 2p_{00}p_{11})^{n-4} \right) \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\text{var}(I)}{\mathbb{E}[I]^2} \\ &\leq \frac{1 + n(1 - p_{00}p_{11})^{n-2} + n^2 p_{00}p_{11}(1 - 2p_{00}p_{11})^{n-4}}{n^2(1 - 2p_{00}p_{11})^{n-2}} \\ &= O(n^{-\Omega(\epsilon)}), \end{aligned}$$

where the last inequality holds because $np_{00}p_{11} \leq (1 - \epsilon/2) \log n$. In conclusion, we have shown that $I \geq n^{\Omega(\epsilon)}$ with probability at least $1 - n^{-\Omega(\epsilon)}$. This implies that the MLE exactly recovers the true permutation with probability at most $n^{-\Omega(\epsilon)}$.

Next, we shift to the case where $na \geq (2 - \epsilon) \log n$. The assumption (59) translates to $n(\sqrt{a} - \sqrt{b})^2 \leq 2(1 - \epsilon) \log n$, we have

$$(2 - \epsilon) \log n \left(1 - \sqrt{\frac{b}{a}} \right)^2 \leq na \left(1 - \sqrt{\frac{b}{a}} \right)^2$$

$$\leq 2(1 - \epsilon) \log n.$$

It follows that

$$\begin{aligned} \sqrt{\frac{b}{a}} &\geq 1 - \sqrt{\frac{2(1 - \epsilon)}{2 - \epsilon}} = 1 - \sqrt{1 - \frac{\epsilon}{2 - \epsilon}} \\ &\geq \frac{\epsilon}{2(2 - \epsilon)} \geq \frac{\epsilon}{4} \end{aligned}$$

and hence $\frac{\epsilon^2}{16} \leq \frac{b}{a} \leq 1$. Let T be a set of $2m$ vertices where $m = \lfloor n / \log^2 n \rfloor$. We further partition T into $T_1 \cup T_2$ where $|T_1| = |T_2| = m$. Let $\mathcal{S}$ denote the set of permutations π' that consists of $n - 2$ fixed points and a 2-node orbit (i, j) for some $i \in T_1$ and $j \in T_2$. Then $|\mathcal{S}| = m^2$. Next we show that

$$\begin{aligned} \mathbb{P} \left\{ \exists \pi' \in \mathcal{S} \text{ s.t. } \langle A, B \rangle - \langle A^{\pi'}, B \rangle < 0 \right\} \\ = 1 - o(1). \end{aligned} \quad (60)$$

Fix a $\pi' \in \mathcal{S}$ with (i, j) as its 2-node orbit, i.e., $\pi'(i) = j$, $\pi'(j) = i$, and $\pi'(k) = k$ for any $k \in [n] \setminus \{i, j\}$. Then $\langle A, B \rangle - \langle A^{\pi'}, B \rangle = X_{ij} + Y_{ij}$, where

$$X_{ij} \triangleq \sum_{k \in T^c} X_{ij,k}, \quad \text{and} \quad Y_{ij} \triangleq \sum_{k \in T \setminus \{i, j\}} X_{ij,k}.$$

Note that $\mathbb{E}[Y_{ij}] = (2m - 2)(a - b)$ and $\text{Var}[Y_{ij}] \leq (2m - 2)(a + b)$. Letting

$$\tau \triangleq (2m - 2)(a - b) + \sqrt{(2m - 2)(a + b) \log n},$$

by Chebyshev's inequality,

$$\mathbb{P}\{Y_{ij} \geq \tau\} \leq \frac{1}{\log n}. \quad (61)$$

Define

$$T' = \{(i, j) \in T_1 \times T_2 : Y_{ij} < \tau\}.$$

Then $\mathbb{E}[|(T_1 \times T_2) \setminus T'|] \leq m^2 / \log n$. Hence, by Markov's inequality, $|(T_1 \times T_2) \setminus T'| \geq \frac{1}{2}m^2$ with probability at most $\frac{2}{\log n}$. Hence, we have $|T'| \geq \frac{1}{2}m^2$ with probability at least $1 - \frac{2}{\log n}$.

Note that crucially T' is independent of $\{X_{ij}\}_{i \in T_1, j \in T_2}$. Thus, we condition on the event that $|T'| \geq \frac{1}{2}m^2$ in the sequel. Define $I_n = \sum_{(i,j) \in T'} \mathbf{1}_{\{X_{ij} \leq -\tau\}}$. To bound the $\mathbb{P}\{X_{ij} \leq -\tau\}$ from below, we need the following reverse large-deviation estimate (proved at the end of this subsection):

Lemma 6: Suppose $\{X_k\}_{k \in [n]} \stackrel{\text{i.i.d.}}{\sim} a\delta_{+1} + b\delta_{-1} + (1 - a - b)\delta_0$ with some $a, b \in [0, 1]$ such that $0 \leq a + b \leq 1$, $1 \leq \frac{a}{b} = \Theta(1)$, $an = \omega(1)$, and $(\sqrt{a} - \sqrt{b})^2 \leq 2\frac{\log n}{n}$. For all τ such that $\tau = o(\sqrt{an \log n})$ and $\tau = o(an)$, and any constant $0 < \delta < 1$, there exists n_0 such that for all $n \geq n_0$,

$$\mathbb{P} \left\{ \sum_{k=1}^n X_k \leq -\tau \right\} \geq \exp \left(-n \left(\sqrt{a} - \sqrt{b} \right)^2 - \frac{\delta}{2} \log n \right).$$

To apply this lemma, note that $\tau = O\left(\frac{n(a-b)}{\log^2 n} + \sqrt{\frac{na}{\log n}}\right)$, so

$$\begin{aligned} \frac{\tau}{\sqrt{an \log n}} &= O \left(\left(\sqrt{a} - \sqrt{b} \right) \sqrt{\frac{n}{\log^5 n}} + \frac{1}{\log n} \right) \\ &= O \left(\frac{1}{\log n} \right) \end{aligned}$$

$$\frac{\tau}{an} = O \left(\frac{1}{\log^2 n} + \frac{1}{\sqrt{an \log n}} \right) = O \left(\frac{1}{\sqrt{\log n}} \right),$$

where we used the fact that $m \leq n / \log^2 n$, $b \leq a$, $\sqrt{a} - \sqrt{b} = O(\sqrt{\log(n)/n})$, and $an = \omega(1)$. Then, applying Lemma 6 with $\delta = \frac{\epsilon}{4}$ yields

$$\begin{aligned} \mathbb{E}[I_n] &\geq \frac{1}{2}m^2 \exp \left(-|T^c| \left(\sqrt{a} - \sqrt{b} \right)^2 - \frac{\epsilon}{8} \log n \right) \\ &\geq n^{\epsilon - o(1)}, \end{aligned} \quad (62)$$

where the last inequality holds by assumption that $(\sqrt{a} - \sqrt{b})^2 \leq 2(1 - \epsilon) \frac{\log n}{n}$. Next, we show that $\text{Var}[I_n] / \mathbb{E}[I_n]^2 = o(1)$, which, by Chebyshev's inequality, implies that $I_n \geq \frac{1}{2}\mathbb{E}[I_n]$ with high probability.

Write

$$\begin{aligned} \text{Var}[I_n] &= \sum_{(i,j), (i',j') \in T'} \left(\mathbb{P}\{X_{ij} \leq -\tau, X_{i'j'} \leq -\tau\} \right. \\ &\quad \left. - \mathbb{P}\{X_{ij} \leq -\tau\} \mathbb{P}\{X_{i'j'} \leq -\tau\} \right) \\ &\leq \text{(I)} + \text{(II)}, \end{aligned}$$

where

$$\begin{aligned} \text{(I)} &= \sum_{(i,j) \in T'} \mathbb{P}\{X_{ij} \leq -\tau\} = \mathbb{E}[I_n], \\ \text{(II)} &= \sum_{(i,j), (i',j') \in T', j \neq j'} \mathbb{P}\{X_{ij} \leq 0, X_{i'j'} \leq 0\} \\ &\quad + \sum_{(i,j), (i',j') \in T', i \neq i'} \mathbb{P}\{X_{ij} \leq 0, X_{i'j'} \leq 0\}. \end{aligned}$$

To bound (II), fix $(i, j), (i', j') \in T'$ such that $j \neq j'$. Note that $\{X_{ij,k}\}_{j \in T_2} \stackrel{\text{i.i.d.}}{\sim} \text{Bern}(p_{00})$ conditional on $A_{ik} = 1, B_{ik} = 1$; $\{-X_{ij,k}\}_{j \in T_2} \stackrel{\text{i.i.d.}}{\sim} \text{Bern}(p_{10})$ conditional on $A_{ik} = 1, B_{ik} = 0$; $\{-X_{ij,k}\}_{j \in T_2} \stackrel{\text{i.i.d.}}{\sim} \text{Bern}(p_{01})$ conditional on $A_{ik} = 0, B_{ik} = 1$; $\{X_{ij,k}\}_{j \in T_2} \stackrel{\text{i.i.d.}}{\sim} \text{Bern}(p_{11})$ conditional on $A_{ik} = 0, B_{ik} = 0$. Then, for $\ell, \ell' \in \{0, 1\}$, we define

$$M_{\ell\ell'} = |\{k \in T^c | A_{ik} = \ell, B_{ik} = \ell'\}|,$$

and get that for any $\lambda \geq 0$,

$$\begin{aligned} &\mathbb{P} \left\{ \sum_{k \in T^c} X_{ij,k} \leq 0, \sum_{k \in T^c} X_{i'j',k} \leq 0 \right\} \\ &\stackrel{(a)}{=} \mathbb{E} \left[\mathbb{P} \left\{ \sum_{k \in T^c} X_{ij,k} \leq 0 \middle| M_{11}, M_{10}, M_{10}, M_{00} \right\} \times \right. \\ &\quad \left. \mathbb{P} \left\{ \sum_{k \in T^c} X_{i'j',k} \leq 0 \middle| M_{11}, M_{10}, M_{10}, M_{00} \right\} \right] \\ &\stackrel{(b)}{=} \mathbb{E} \left[\gamma_{00}^{2M_{11}} \gamma_{01}^{2M_{10}} \gamma_{10}^{2M_{10}} \gamma_{11}^{2M_{00}} \right] \\ &\stackrel{(c)}{=} (\gamma_{00}^2 p_{11} + \gamma_{01}^2 p_{10} + \gamma_{10}^2 p_{01} + \gamma_{11}^2 p_{00})^{|T^c|}, \end{aligned}$$

where $\gamma_{\ell, \ell'} = 1 - p_{\ell\ell'} + p_{\ell\ell'} e^{(2|\ell - \ell'| - 1)\lambda}$; (a) holds because $\sum_{k \in T'} X_{ij,k}$ and $\sum_{k \in T'} X_{i'j',k}$ are independent

conditional on $M_{11}, M_{10}, M_{10}, M_{00}$; (b) holds by applying the Chernoff bound; (c) holds by applying the MGF of the multinomial distribution, since $M_{11}, M_{10}, M_{10}, M_{00} \sim \text{Multi}(|T^c|, p_{11}, p_{10}, p_{01}, p_{00})$. Choosing $e^\lambda = \sqrt{\frac{p_{00}p_{11}}{p_{01}p_{10}}} = \sqrt{\frac{a}{b}}$ where $a = 2 p_{00}p_{11}$ and $b = 2 p_{01}p_{10}$, we have

$$\begin{aligned} & \mathbb{P} \left\{ \sum_{k \in T^c} X_{ij,k} \leq 0, \sum_{k \in T^c} X_{ij',k} \leq 0 \right\} \\ & \leq \left(1 - \frac{3}{2} (\sqrt{a} - \sqrt{b})^2 \right)^{|T^c|}. \end{aligned}$$

The same upper bound applies to any $(i, j), (i', j) \in T'$ such that $i \neq i'$.

Therefore, we get that

$$\begin{aligned} \text{(II)} & \leq 2m^3 \left(1 - \frac{3}{2} (\sqrt{a} - \sqrt{b})^2 \right)^{|T^c|} \\ & \leq 2 m^3 \exp \left(-\frac{3}{2} (\sqrt{a} - \sqrt{b})^2 |T^c| \right). \end{aligned}$$

Hence, by Chebyshev's inequality,

$$\begin{aligned} & \mathbb{P} \left\{ I_n \leq \frac{1}{2} \mathbb{E}[I_n] \right\} \\ & \leq \frac{\text{Var}[I_n]}{\frac{1}{4} (\mathbb{E}[I_n])^2} \\ & \leq \frac{4}{\mathbb{E}[I_n]} + \frac{8 \times \text{(II)}}{(\mathbb{E}[I_n])^2} \\ & \stackrel{(a)}{\leq} n^{-\epsilon+o(1)} \\ & \quad + \frac{32}{m} \exp \left(\frac{1}{2} (\sqrt{a} - \sqrt{b})^2 |T^c| + \frac{\epsilon}{8} \log n \right) \\ & \stackrel{(b)}{\leq} n^{-7\epsilon/8+o(1)} \end{aligned}$$

where (a) is due to (62); (b) holds by the assumption that $(\sqrt{a} - \sqrt{b})^2 \leq 2(1 - \epsilon) \frac{\log n}{n}$.

Therefore, with probability $1 - n^{-\Omega(\epsilon)}$ there exists some $(i, j) \in T'$ such that $X_{ij} \leq -\tau$. By definition of T' , we have $Y_{ij} < \tau$ and hence $X_{ij} + Y_{ij} < 0$. Thus, we arrive at the desired claim (60).

Proof of Lemma 6: Let $E_n = \{\sum_{k=1}^n X_k \leq -\tau\}$. Let Q denote the distribution of X_k . The following large-deviation lower estimate based on the data processing inequality is well-known (see [3, Eq. (5.21), p. 167] and [18, Lemma 3]): For any Q' ,

$$Q(E_n) \geq \exp \left(\frac{-nD(Q' \| Q) - \log 2}{Q'(E_n)} \right). \quad (63)$$

Choose

$$Q' = \frac{\alpha - \beta}{2} \delta_{+1} + \frac{\alpha + \beta}{2} \delta_{-1} + (1 - \alpha) \delta_0,$$

where

$$\alpha \triangleq \frac{2\sqrt{ab}}{1 - (\sqrt{a} - \sqrt{b})^2}, \quad \text{and} \quad \beta \triangleq \min \left\{ \frac{\alpha}{2}, \frac{\delta}{8} \sqrt{\frac{b \log n}{n}} \right\}.$$

Note that under the assumption that $1 \leq a/b = \Theta(1)$ and $(\sqrt{a} - \sqrt{b})^2 \leq \frac{\log n}{n}$, we have that $2b \leq \alpha = \Theta(a)$. Moreover, since $\tau = o(\sqrt{an \log n})$ and $\tau = o(an)$, it follows that $\tau = o(\beta n)$. Then,

$$\begin{aligned} Q'(E_n^c) &= Q' \left(\sum_{k=1}^n Y_k \geq -\tau \right) \\ &\stackrel{(a)}{\leq} Q' \left(\sum_{k=1}^n (Y_k - \mathbb{E}[Y_k]) \geq \beta n - \tau \right) \\ &\stackrel{(b)}{\leq} \frac{\sum_{k=1}^n \text{Var}[Y_k]}{(\beta n - \tau)^2} \\ &\stackrel{(c)}{\leq} \frac{\alpha n}{(\beta n - \tau)^2} \stackrel{(d)}{=} o(1), \end{aligned}$$

where (a) holds because $\mathbb{E}[Y_k] = -\beta$; (b) follows from by Chebyshev's inequality as $\{Y_k\}_{i \in [n]}$ are independent and $\tau = o(\beta n)$; (c) holds because $\sum_{k=1}^n \text{Var}[Y_k] \leq \sum_{k=1}^n \mathbb{E}[Y_k^2] \leq \alpha n$; (d) holds because $\tau = o(\beta n)$, and $\frac{\alpha}{\beta^2 n} = \max \left\{ \frac{64\alpha}{\delta^2 b \log n}, \frac{4}{\alpha n} \right\} = o(1)$, in view of that $\delta = \Theta(1)$, $\alpha = \Theta(a) = \Theta(b)$ and $\alpha n = \Theta(an) = \omega(1)$.

Next, we upper bound $D(Q' \| Q)$. We get

$$\begin{aligned} D(Q' \| Q) &= \frac{\alpha - \beta}{2} \log \frac{\alpha - \beta}{2a} + \frac{\alpha + \beta}{2} \log \frac{\alpha + \beta}{2b} \\ &\quad + (1 - \alpha) \log \frac{1 - \alpha}{1 - a - b} \\ &\stackrel{(a)}{=} -\log \left(1 - (\sqrt{a} - \sqrt{b})^2 \right) + \frac{\alpha}{2} \log \frac{(\alpha^2 - \beta^2)}{\alpha^2} \\ &\quad + \frac{\beta}{2} \log \frac{a(\alpha + \beta)}{b(\alpha - \beta)} \\ &\stackrel{(b)}{\leq} -\log \left(1 - (\sqrt{a} - \sqrt{b})^2 \right) + \frac{\beta^2}{\alpha - \beta} + \beta \frac{\sqrt{a} - \sqrt{b}}{\sqrt{b}} \\ &\stackrel{(c)}{\leq} (\sqrt{a} - \sqrt{b})^2 + \frac{\delta \log n}{4n}, \end{aligned}$$

where (a) holds because $\alpha = \frac{2\sqrt{ab}}{1 - (\sqrt{a} - \sqrt{b})^2}$; (b) holds because $\log \frac{\alpha + \beta}{\alpha - \beta} \leq \frac{2\beta}{\alpha - \beta}$ and $\frac{1}{2} \log \frac{a}{b} \leq \frac{\sqrt{a} - \sqrt{b}}{\sqrt{b}}$, in view of $\log(1 + x) \leq x$; (c) holds for all sufficiently large n because $(\sqrt{a} - \sqrt{b})^2 \leq 2 \frac{\log n}{n}$ so that $\log(1 - (\sqrt{a} - \sqrt{b})^2) = -(\sqrt{a} - \sqrt{b})^2 + O(\log^2(n)/n^2)$, $\beta \leq \frac{1}{2}\alpha$ and $\beta \leq \frac{\delta}{8} \sqrt{\frac{b \log n}{n}}$ so that $\frac{\beta^2}{\alpha - \beta} \leq \frac{2\beta^2}{\alpha} \leq \frac{\delta^2 b \log n}{32\alpha n} \leq \frac{\delta^2 \log n}{32n}$, and $\beta \frac{\sqrt{a} - \sqrt{b}}{\sqrt{b}} \leq \frac{\delta}{8} \sqrt{\frac{\log n}{n}} (\sqrt{a} - \sqrt{b}) \leq \frac{\sqrt{2\delta} \log n}{8n}$.

In conclusion, it follows from (63) that

$$\begin{aligned} Q(E_n) &\geq \exp \left(-(1 + o(1)) \left(n (\sqrt{a} - \sqrt{b})^2 + \frac{\delta}{4} \log n + \log 2 \right) \right) \\ &\geq \exp \left(-n (\sqrt{a} - \sqrt{b})^2 - \frac{\delta}{2} \log n \right), \end{aligned}$$

where the last inequality holds for all sufficiently large n in view of $n (\sqrt{a} - \sqrt{b})^2 \leq 2 \log n$. $\square$

APPENDIX

Lemma 7: Fixing π and π' , where π is the latent permutation under $\mathcal{P}$. For any edge orbit O of $\sigma = \pi^{-1} \circ \pi'$ with $|O| = k$ and $t \geq 0$, let $M_k \triangleq \mathbb{E} \left[\exp \left(t \sum_{(i,j) \in O} \bar{A}_{\pi'(i)\pi'(j)} \bar{B}_{ij} \right) \right]$, where $\bar{A}_{ij} = \bar{A}_{ij} - \mathbb{E} A_{ij}$ and $\bar{B}_{ij} = \bar{B}_{ij} - \mathbb{E} B_{ij}$.

- For Erdős-Rényi random graphs,

$$M_k = \left(\frac{T - \sqrt{T^2 - 4D}}{2} \right)^k + \left(\frac{T + \sqrt{T^2 - 4D}}{2} \right)^k, \quad (64)$$

where

$$T = e^{t(1-q)^2} qs + 2 e^{-tq(1-q)} q(1-s) + e^{tq^2} (1 - 2q + qs)$$

$$D = \left(e^{t(1-q)^2 + tq^2} - e^{-2tq(1-q)} \right) q(s - q)$$

and $q \triangleq ps$.

- For Gaussian model, for $t \leq \frac{1}{1+\rho}$,

$$M_k = \left[\left(\frac{\lambda_1 + \lambda_2}{2} \right)^k - \left(\frac{\lambda_1 - \lambda_2}{2} \right)^k \right]^{-1} \quad (65)$$

$$\text{where } \lambda_1 = \sqrt{(1 + \rho t)^2 - t^2} \text{ and } \lambda_2 = \sqrt{(1 - \rho t)^2 - t^2}.$$

Moreover, $M_k \leq M_2^{k/2}$ for all $k \geq 2$.

Proof: For ease of notation, let $\{(a_i, b_i)\}_{i=1}^k$ be independently and identically distributed pairs of random variables with joint distribution P . Let $a_{k+1} = a_1$ and $b_{k+1} = b_1$. Since O is an edge orbit of σ^E , we have $\{A_{\pi(i)\pi(j)}\}_{(i,j) \in O} = \{A_{\pi'(i)\pi'(j)}\}_{(i,j) \in O}$ and $(\pi'(i), \pi'(j)) = (\pi(\sigma(i)), \pi(\sigma(j)))$. Then, we have that

$$\begin{aligned} M_k &= \mathbb{E} \left[\exp \left(\sum_{i=1}^k t(a_{i+1} - \mathbb{E}[a_{i+1}])(b_i - \mathbb{E}[b_{i+1}]) \right) \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\exp \left(\sum_{i=1}^k t(a_{i+1} - \mathbb{E}[a_{i+1}])(b_i - \mathbb{E}[b_{i+1}]) \right) \middle| a_1, a_2, \dots, a_k \right] \right] \\ &= \mathbb{E} \left[\prod_{i=1}^k \mathbb{E} \left[\exp \left(t(a_{i+1} - \mathbb{E}[a_{i+1}])(b_i - \mathbb{E}[b_{i+1}]) \right) \middle| a_i, a_{i+1} \right] \right]. \end{aligned}$$

- For Erdős-Rényi random graphs, $\mathbb{E}[a_i] = \mathbb{E}[b_i] = q$, and $M_k = \text{tr}(M^k)$, where M is a 2×2 row-stochastic matrix with rows and columns indexed by $\{0, 1\}$ and

$$\begin{aligned} M(\ell, m) &= \mathbb{E} \left[\exp \left(t(a_{i+1} - q)(b_i - q) \right) \middle| a_i = \ell, a_{i+1} = m \right] \times \mathbb{P}(a_{i+1} = m). \\ &= \mathbb{E} \left[\exp \left(t(m - q)(b_i - q) \right) \middle| a_i = \ell \right] \mathbb{P}(a_{i+1} = m). \end{aligned}$$

Explicitly, we have

$$\begin{aligned} M_{11} &= e^{tq^2} (1 - \eta) (1 - q) + e^{-tq(1-q)} \eta (1 - q) \\ M_{12} &= e^{-tq(1-q)} (1 - \eta) q + e^{t(1-q)^2} \eta q \\ M_{21} &= e^{tq^2} s (1 - q) + e^{-tq(1-q)} (1 - s) (1 - q) \\ M_{22} &= e^{-tq(1-q)} (1 - s) q + e^{t(1-q)^2} sq. \end{aligned}$$

where $\eta = \frac{q(1-s)}{1-q}$. The eigenvalues of M are $\frac{T - \sqrt{T^2 - 4D}}{2}$ and $\frac{T + \sqrt{T^2 - 4D}}{2}$, where

$$\begin{aligned} T &\triangleq \text{Tr}(M) = e^{t(1-q)^2} qs + 2 e^{-tq(1-q)} q(1-s) \\ &\quad + e^{tq^2} (1 - 2q + qs) \\ D &\triangleq \det(M) = \left(e^{t(1-q)^2 + tq^2} - e^{-2tq(1-q)} \right) q(s - q). \end{aligned}$$

- For Gaussian model, $\mathbb{E}[a_i] = \mathbb{E}[b_i] = 0$, and

$$M_k = \mathbb{E} \left[\prod_{i=1}^k \exp \left(t \rho a_i a_{i+1} + \frac{1}{2} t^2 (1 - \rho^2) a_{i+1}^2 \right) \right],$$

where the equality follows from $b_i \sim \mathcal{N}(\rho a_i, 1 - \rho^2)$ conditional on a_i and $\mathbb{E}[\exp(tZ)] = \exp(t\mu + t^2\nu^2/2)$ for $Z \sim \mathcal{N}(\mu, \nu^2)$.

Let $\lambda_1 = \sqrt{(1 + \rho t)^2 - t^2}$ and $\lambda_2 = \sqrt{(1 - \rho t)^2 - t^2}$, where $t \leq \frac{1}{1+\rho}$. By change of variables,

$$\begin{aligned} M_k &= \int \cdots \int \prod_{i=1}^k \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{(\frac{\lambda_1 + \lambda_2}{2} a_i + \frac{\lambda_1 - \lambda_2}{2} a_{i+1})^2}{2} \right) \\ &\quad da_1 \cdots da_k \\ &\stackrel{(a)}{=} \int \cdots \int \prod_{i=1}^k \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{X_i^2}{2} \right) \\ &\quad dX_1 \cdots dX_k \det(J^{-1}) \\ &\stackrel{(b)}{=} \frac{1}{\left(\frac{\lambda_1 + \lambda_2}{2} \right)^k - \left(\frac{\lambda_1 - \lambda_2}{2} \right)^k}, \end{aligned}$$

where (a) holds by letting $X_i = \frac{\lambda_1 + \lambda_2}{2} a_i + \frac{\lambda_1 - \lambda_2}{2} a_{i+1}$, and denoting J as the Jacobian matrix with $J_{ij} = \frac{\partial X_i}{\partial a_j}$ whose inverse matrix is

$$J = \begin{pmatrix} \frac{\lambda_1 + \lambda_2}{2} & \frac{\lambda_1 - \lambda_2}{2} & 0 & \cdots & 0 \\ 0 & \frac{\lambda_1 + \lambda_2}{2} & \frac{\lambda_1 - \lambda_2}{2} & \cdots & 0 \\ 0 & 0 & \frac{\lambda_1 + \lambda_2}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \frac{\lambda_1 - \lambda_2}{2} \\ \frac{\lambda_1 - \lambda_2}{2} & 0 & \cdots & 0 & \frac{\lambda_1 + \lambda_2}{2} \end{pmatrix};$$

(b) holds because $\det(J^{-1}) = \det(J)^{-1}$, where $\det(J) = \left(\frac{\lambda_1 + \lambda_2}{2} \right)^k - \left(\frac{\lambda_1 - \lambda_2}{2} \right)^k$.

Finally, we prove that $M_k \leq (M_2)^{k/2}$ for $k \geq 2$. Indeed, for the Erdős-Rényi graphs, this simply follows from $x^k + y^k \leq (x^2 + y^2)^{k/2}$ for $x, y \geq 0$ and $k \in \mathbb{N}$. Analogously, for the Gaussian model, this follows from $(a + b)^k - (a - b)^k \geq (4ab)^{k/2}$, which holds by rewriting $(a + b)^2 = (a - b)^2 + 4ab$ and letting $x = (a - b)^2$ and $y = 4ab$. $\square$

Lemma 8: Fixing π and $\tilde{\pi}$, where π is the latent permutation under $\mathcal{P}$. For any edge orbit O of

$\sigma = \pi^{-1} \circ \pi'$ with $|O| = k$ and $t > 0$, let $L_k \triangleq \mathbb{E} \left[\exp \left(t \sum_{(i,j) \in O} A_{\pi'(i)\pi'(j)} B_{ij} - \sum_{(i,j) \in O} A_{\pi(i)\pi(j)} B_{ij} \right) \right]$.

- For general Erdős-Rényi random graphs model,

$$L_k = \left(\frac{T - \sqrt{T^2 - 4D}}{2} \right)^k + \left(\frac{T + \sqrt{T^2 - 4D}}{2} \right)^k, \quad (66)$$

where $T = 1$ and $D = -(p_{01}p_{10}(e^t - 1) + p_{00}p_{11}(e^{-t} - 1))$.

- For Gaussian model, for $t \leq \frac{1}{2(1-\rho)}$,

$$L_k = \left[\left(\frac{\lambda_1 + \lambda_2}{2} \right)^k - \left(\frac{\lambda_1 - \lambda_2}{2} \right)^k \right]^{-1}, \quad (67)$$

where $\lambda_1 = 1$ and $\lambda_2 = \sqrt{1 + 4t\rho - 4t^2(1-\rho^2)}$.

Moreover, $L_k \leq L_2^{k/2}$ for all $k \geq 2$.

Proof: For ease of notation, let $\{(a_i, b_i)\}_{i=1}^k$ be independently and identically distributed pairs of random variables with joint distribution P . Let $a_{k+1} = a_1$ and $b_{k+1} = b_1$. Since O is an edge orbit of σ^E , we have $\{A_{\pi(i)\pi(j)}\}_{(i,j) \in O} = \{A_{\pi'(i)\pi'(j)}\}_{(i,j) \in O}$ and $(\pi'(i), \pi'(j)) = (\pi(\sigma(i)), \sigma(j))$. Then, we have that

$$\begin{aligned} L_k &= \mathbb{E} \left[\exp \left(t \sum_{(i,j) \in O} (A_{\pi'(i)\pi'(j)} - A_{\pi(i)\pi(j)}) B_{ij} \right) \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\exp \left(\sum_{i=1}^k t(a_{i+1} - a_i) b_i \right) \middle| a_1, a_2, \dots, a_k \right] \right] \\ &= \mathbb{E} \left[\prod_{i=1}^k \mathbb{E} \left[\exp(t(a_{i+1} - a_i) b_i) \middle| a_i, a_{i+1} \right] \right]. \end{aligned}$$

- For Erdős-Rényi random graphs, $L_k = \text{tr}(L^k)$, where L is a 2×2 row-stochastic matrix with rows and columns indexed by $\{0, 1\}$ and

$$\begin{aligned} L(\ell, m) &= \mathbb{E}[\exp(t(a_{i+1} - a_i) b_i) | a_i = \ell, a_{i+1} = m] \\ &\quad \mathbb{P}(a_{i+1} = m). \\ &= \mathbb{E}[\exp(t(m - \ell) b_i) | a_i = \ell] \mathbb{P}(a_{i+1} = m). \end{aligned}$$

Explicitly, we have

$$\begin{aligned} L_{11} &= 1 - p_{10} - p_{11} \\ L_{12} &= \left(\frac{p_{01}}{1 - p_{10} - p_{11}} e^t + \frac{p_{00}}{1 - p_{10} - p_{11}} \right) (p_{10} + p_{11}) \\ L_{21} &= \left(\frac{p_{11}}{p_{10} + p_{11}} (e^t - 1) + 1 \right) (p_{10} + p_{11}) \\ L_{22} &= p_{10} + p_{11}. \end{aligned}$$

The eigenvalues of L are $\frac{T - \sqrt{T^2 - 4D}}{2}$ and $\frac{T + \sqrt{T^2 - 4D}}{2}$, where $T = 1$ and

$$D = -(p_{01}p_{10}(e^t - 1) + p_{00}p_{11}(e^{-t} - 1)).$$

- For Gaussian model,

$$L_k = \mathbb{E} \left[\prod_{i=1}^k \exp(t(a_{i+1} - a_i) \rho a_i) \right]$$

$$\begin{aligned} &+ t^2 (a_{i+1} - a_i)^2 (1 - \rho^2) / 2 \Big) \Big] \\ &= \mathbb{E} \left[\exp \left((t^2 (1 - \rho^2) - t\rho) \sum_{i=1}^k (a_i^2 - a_i a_{i+1}) \right) \right] \\ &= \frac{1}{\left(\frac{\lambda_1 + \lambda_2}{2} \right)^k - \left(\frac{\lambda_1 - \lambda_2}{2} \right)^k}. \end{aligned}$$

where the first equality follows from $b_i \sim \mathcal{N}(\rho a_i, 1 - \rho^2)$ conditional on a_i and $\mathbb{E}[\exp(tZ)] = \exp(t\mu + t^2\nu^2/2)$ for $Z \sim \mathcal{N}(\mu, \nu^2)$; the last equality holds by change of variables and Gaussian integral analogous to the proof of Lemma 7.

Finally, $L_k \leq (L_2)^{k/2}$ for $k \geq 2$ follows from the same reasoning as in Lemma 7. $\square$

In view of (64) in Lemma 7, $M_1 = T$ and

$$M_2 = T^2 - 2D, \quad (68)$$

where by letting $a = e^{t(1-q)^2}$, $b = e^{-tq(1-q)}$, and $c = e^{tq^2}$, we get

$$T = aqs + 2bq(1-s) + c(1-2q+qs), \quad D = (ac - b^2)q(s-q).$$

- Suppose $t \leq \log \frac{1}{p}$. Since $1 + x \leq e^x \leq 1 + x + x^2$ for $0 \leq x \leq 1$ and $1 + x + x^2/2 + x^3 \leq e^x \leq 1 + x + x^2/2$ for $-1 \leq x < 0$, we get

$$\begin{aligned} 1 - tq(1-q) + \frac{1}{2}t^2q^2(1-q)^2 - t^3q^3 \\ \leq b \leq 1 - tq(1-q) + \frac{1}{2}t^2q^2(1-q)^2, \end{aligned}$$

and

$$1 + tq^2 \leq c \leq 1 + tq^2 + t^2q^4.$$

It follows that

$$\begin{aligned} T &\leq e^{t(1-q)^2} qs \\ &+ 2 \left(1 - tq(1-q) + \frac{1}{2}t^2q^2(1-q)^2 \right) q(1-s) \\ &+ (1 + tq^2 + t^2q^4)(1 - 2q + qs) \\ &= \alpha_0 + \alpha_1 t + \alpha_2 t^2, \end{aligned}$$

where

$$\alpha_0 = 2q(1-s) + (1-2q+qs) + e^{t(1-q)^2} qs, \quad (69)$$

$$\alpha_1 = -2q(1-q)q(1-s) + q^2(1-2q+qs), \quad (70)$$

$$\alpha_2 = q^2(1-q)^2q(1-s) + q^4(1-2q+qs). \quad (71)$$

Moreover,

$$\begin{aligned} D &\geq e^{t(1-q)^2} (1 + tq^2) q(s-q) \\ &- \left(1 - tq(1-q) + \frac{1}{2}t^2q^2(1-q)^2 \right)^2 q(s-q) \\ &= \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \beta_4 t^4, \end{aligned}$$

where

$$\beta_0 = -q(s-q) + e^{t(1-q)^2} q(s-q), \quad (72)$$

$$\beta_1 = 2q^2(1-q)(s-q) + e^{t(1-q)^2} q^3(s-q), \quad (73)$$

$$\beta_2 = -2q^3(1-q)^2(s-q), \quad (74)$$

$$\beta_3 = q^4(1-q)^3(s-q), \quad (75)$$

$$\beta_4 = -\frac{1}{4}q^5(1-q)^4(s-q). \quad (76)$$

By (68), we get

$$\begin{aligned} M_2 &= T^2 - 2D \\ &\leq (\alpha_0 + \alpha_1 t + \alpha_2 t^2)^2 \\ &\quad - 2(\beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \beta_4 t^4) \\ &= \gamma_0 + \gamma_1 t + \gamma_2 t^2 + \gamma_3 t^3 + \gamma_4 t^4 \end{aligned}$$

where

$$\gamma_0 = \alpha_0^2 - 2\beta_0 \quad (77)$$

$$\gamma_1 = 2\alpha_0\alpha_1 - 2\beta_1 \quad (78)$$

$$\gamma_2 = \alpha_1^2 + 2\alpha_0\alpha_2 - 2\beta_2 \quad (79)$$

$$\gamma_3 = 2\alpha_1\alpha_2 - 2\beta_3 \quad (80)$$

$$\gamma_4 = \alpha_2^2 - 2\beta_4 \quad (81)$$

By (69), (72) and (77), we get

$$\begin{aligned} \gamma_0 &= \alpha_0^2 - 2\beta_0 \\ &= \left(1 - qs + e^{t(1-q)^2} qs\right)^2 \\ &\quad - 2\left(-q(s-q) + e^{t(1-q)^2} q(s-q)\right) \\ &= 1 - 2q^2 + q^2 s^2 + 2e^{t(1-q)^2} q^2(1-s^2) \\ &\quad + e^{2t(1-q)^2} q^2 s^2, \\ &\leq 1 - 2q^2 + q^2 s^2 + 2e^t q^2(1-s^2) + e^{2t} q^2 s^2. \end{aligned}$$

Note that the above bound on γ_0 determines the main terms in our final bound on M_2 . For the remaining terms, we will show $\sum_{i=1}^4 \gamma_i t^i = O(q^3 t(1+t))$ so that they can be made negligible later when we apply the bound on M_2 in the proof of Lemma 3 (see (48)).

By (69), (70), (73) and (78), we get

$$\begin{aligned} \gamma_1 &= 2\alpha_0\alpha_1 - 2\beta_1 \\ &= -2q^2 + 4q^3 + 4q^3 s - 4q^3 s^2 - 4q^4 + 2q^4 s^2 \\ &\quad + (-4q^3 s + 4q^3 s^2 + 2q^4 - 2q^4 s^2) e^{t(1-q)^2} \\ &\stackrel{(a)}{\leq} 4q^3 + 2e^{t(1-q)^2} q^4 \stackrel{(b)}{\leq} 6q^3, \end{aligned}$$

where (a) holds because $-2q^2 + 4q^3 \leq 0$ given $q \leq \frac{1}{2}$, $-4q^3 s^2 - 4q^4 + 2q^4 s^2 \leq 0$ and $-4q^3 s + 4q^3 s^2 + 2q^4 - 2q^4 s^2 \leq 2q^4$; (b) holds because $2e^{t(1-q)^2} q^4 \leq 2e^t q^4 \leq 2q^3$, given $t \leq \log \frac{1}{p}$. By (69), (70), (71), (74) and (79), we get

$$\begin{aligned} \gamma_2 &= \alpha_1^2 + 2\alpha_0\alpha_2 - 2\beta_2 \\ &= 2q^3 - 2q^3 s - q^4 - 4q^4 s + 6q^4 s^2 + 10q^5 s - 8q^5 s^2 \\ &\quad - 6q^6 - 4q^6 s + q^6 s^2 + 6q^7 + 2q^7 s - 2q^8 \\ &\quad + e^{t(1-q)^2} (2q^4 s - 2q^4 s^2 - 2q^5 s + 4q^5 s^2 + 2q^6 s) \\ &\stackrel{(a)}{\leq} 2q^3 - 2q^3 s + q^4 s^2 + 10q^5 s + 2e^t q^4 s \stackrel{(b)}{\leq} 6q^3, \end{aligned}$$

where (a) holds because $-q^4 - 4q^4 s + 6q^4 s^2 \leq q^4 s^2$, $-8q^5 s^2 - 6q^6 - 4q^6 s + q^6 s^2 + 6q^7 + 2q^7 s - 2q^8 \leq 0$,

and $-2q^4 s^2 - 2q^5 s + 4q^5 s^2 + 2q^6 s \leq 0$ given $q \leq \frac{1}{2}$; (b) holds because $-2q^3 s + q^4 s^2 + 10q^5 s \leq 2q^3$ given $q \leq \frac{1}{2}$, $2e^t q^4 s \leq 2q^3 s \leq 2q^3$ given $t \leq \log \frac{1}{p}$. By (70), (71), (75) and (80), we get

$$\begin{aligned} \gamma_3 &= 2\alpha_1\alpha_2 - 2\beta_3 \\ &= -2q^4 s + 12q^5 s - 4q^5 s^2 - 4q^6 - 16q^6 s + 10q^6 s^2 \\ &\quad + 8q^7 - 4q^7 s^2 - 2q^8 + 2q^8 s \\ &\stackrel{(a)}{\leq} -2q^4 s + 12q^5 s \stackrel{(b)}{\leq} 4q^4, \end{aligned}$$

where (a) holds because $-4q^5 s^2 - 4q^6 - 16q^6 s + 10q^6 s^2 + 8q^7 \leq 0$ and $-4q^7 s^2 - 2q^8 + 2q^8 s \leq 0$; (b) holds by $q \leq \frac{1}{2}$. By (71), (76) and (81), we get

$$\begin{aligned} \gamma_4 &= \alpha_2^2 - 2\beta_4 \\ &= \frac{1}{2}(q^5 s + q^6 - 8q^6 s + 2q^6 s^2 + 18q^7 s - 8q^7 s^2 \\ &\quad - 8q^8 - 8q^8 s + 8q^8 s^2 + 8q^9 - 7q^9 s + q^{10}) \\ &\stackrel{(a)}{\leq} \frac{1}{2}(q^5 s + q^6 - 8q^6 s + 2q^6 s^2 + 18q^7 s) \stackrel{(b)}{\leq} 2q^5, \end{aligned}$$

where (a) holds because $-8q^7 s^2 - 8q^8 - 8q^8 s + 8q^8 s^2 + 8q^9 \leq 0$, and $-7q^9 s + q^{10} \leq 0$; (b) holds because $-8q^6 s + 2q^6 s^2 + 18q^7 s \leq 4q^6 s$ and $q^5 s + q^6 + 4q^6 s \leq 4q^5$, given $q \leq \frac{1}{2}$.

Combining the above bounds, we have

$$\begin{aligned} M_2 &\leq 1 - 2q^2 + q^2 s^2 + 6q^3 t + 6q^3 t^2 + 4q^4 t^3 + 2q^5 t^4 \\ &\quad + 2e^t q^2(1-s^2) + e^{2t} q^2 s^2 \\ &= 1 - 2q^2 + q^2 s^2 + q^3 t(6 + 6t + 4qt^2 + 2q^2 t^3) \\ &\quad + 2e^t q^2(1-s^2) + e^{2t} q^2 s^2 \\ &\leq 1 - 2q^2 + q^2 s^2 + 10q^3 t(1+t) + 2e^t q^2(1-s^2) \\ &\quad + e^{2t} q^2 s^2, \end{aligned}$$

where the last inequality holds because $qt^2 \leq 1$ and $q^2 t^3 \leq 1$ given $t \leq \log \frac{1}{p}$.

Moreover, given $M_1 = T$ and $M_2 = T^2 - 2D$,

$$\frac{M_1^2}{M_2} = 1 + \frac{2D}{M_2} \leq 1 + 2D \leq e^2,$$

where the first inequality holds because $M_2 \geq 1$ by definition, and the last inequality holds because

$$\begin{aligned} D &= (ac - b^2)q(s-q) \stackrel{(a)}{\leq} acq(s-q) \\ &\leq e^t(1 + tq^2 + t^2 q^4)q(s-q) \\ &\stackrel{(b)}{\leq} 1 + tq^2 + t^2 q^4 \stackrel{(c)}{\leq} 3, \end{aligned}$$

where (a) holds because $b^2 q(s-q) \geq 0$; (b) holds because $e^t q \leq 1$ given $t \leq \log \frac{1}{p}$; (c) holds because $qt \leq 1$, given $t \leq \log \frac{1}{p}$.

- Suppose $t \leq \frac{1}{2^{10}}$. Since $1 + x + x^2/2 \leq e^x \leq 1 + x + x^2/2 + x^3$ for $0 \leq x \leq 1$ and $1 + x + x^2/2 + x^3 \leq e^x \leq 1 + x + x^2/2$ for $-1 \leq x < 0$, we get

$$\begin{aligned} 1 + t(1-q)^2 + \frac{1}{2}t^2(1-q)^4 &\leq a \\ &\leq 1 + t(1-q)^2 + \frac{1}{2}t^2(1-q)^4 + t^3(1-q)^6; \end{aligned}$$

$$\begin{aligned}
1 - tq(1-q) + \frac{1}{2}t^2q^2(1-q)^2 - t^3q^3 &\leq b \\
&\leq 1 - tq(1-q) + \frac{1}{2}t^2q^2(1-q)^2; \\
1 + tq^2 + \frac{1}{2}t^2q^4 &\leq c \\
&\leq 1 + tq^2 + \frac{1}{2}t^2q^4 + t^3q^6.
\end{aligned}$$

It follows that

$$\begin{aligned}
T &\leq \left(1 + t(1-q)^2 + \frac{1}{2}t^2(1-q)^4 + t^3(1-q)^6\right)qs \\
&\quad + 2\left(1 - tq(1-q) + \frac{1}{2}t^2q^2(1-q)^2\right)q(1-s) \\
&\quad + \left(1 + tq^2 + \frac{1}{2}t^2q^4 + t^3q^6\right)(1-2q+qs) \\
&= 1 + \alpha_1t + \alpha_2t^2 + \alpha_3t^3,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_1 &= (1-q)^2qs - 2q(1-q)q(1-s) \\
&\quad + q^2(1-2q+qs) = q(s-q), \quad (82)
\end{aligned}$$

$$\begin{aligned}
\alpha_2 &= \frac{1}{2}(1-q)^4qs + q^2(1-q)^2q(1-s) \\
&\quad + \frac{1}{2}q^4(1-2q+qs), \quad (83)
\end{aligned}$$

$$\alpha_3 = (1-q)^6qs + q^6(1-2q+qs). \quad (84)$$

And

$$\begin{aligned}
D &\geq \left(1 + t(1-q)^2 + \frac{1}{2}t^2(1-q)^4\right) \times \\
&\quad \left(1 + tq^2 + \frac{1}{2}t^2q^4\right)q(s-q) \\
&\quad - \left(1 - tq(1-q) + \frac{1}{2}t^2q^2(1-q)^2\right)^2q(s-q) \\
&= \beta_1t + \beta_2t^2 + \beta_3t^3 + \beta_4t^4,
\end{aligned}$$

where

$$\begin{aligned}
\beta_1 &= (1-q)^2q(s-q) + q^3(s-q) \\
&\quad + 2q(1-q)q(s-q) \\
&= q(s-q), \quad (85)
\end{aligned}$$

$$\begin{aligned}
\beta_2 &= \frac{1}{2}(1-q)^4q(s-q) + \frac{1}{2}q^5(s-q) \\
&\quad + (1-q)^2q^3(s-q) - q^2(1-q)^2q(s-q) \\
&\quad - q^2(1-q)^2q(s-q) \\
&= \frac{1}{2}(1-4q+4q^2)q(s-q), \quad (86)
\end{aligned}$$

$$\begin{aligned}
\beta_3 &= \frac{1}{2}(1-q)^2q^5(s-q) + \frac{1}{2}(1-q)^4q^3(s-q) \\
&\quad + q(1-q)q^2(1-q)^2q(s-q) \\
&= \frac{1}{2}(q^2-2q^3+q^4)q(s-q), \quad (87)
\end{aligned}$$

$$\begin{aligned}
\beta_4 &= \frac{1}{4}(1-q)^4q^5(s-q) - \frac{1}{4}(1-q)^4q^5(s-q) \\
&= 0. \quad (88)
\end{aligned}$$

By (68), we get that

$$\begin{aligned}
M_2 &= T^2 - 2D \\
&\leq (1 + \alpha_1t + \alpha_2t^2 + \alpha_3t^3)^2 \\
&\quad - 2(\beta_1t + \beta_2t^2 + \beta_3t^3 + \beta_4t^4) \\
&\leq 1 + \gamma_1t + \gamma_2t^2 + \gamma_3t^3 + \gamma_4t^4 + \gamma_5t^5 + \gamma_6t^6,
\end{aligned}$$

where

$$\gamma_1 = \alpha_1 - \beta_1, \quad (89)$$

$$\gamma_2 = \alpha_1^2 + 2\alpha_2 - 2\beta_2, \quad (90)$$

$$\gamma_3 = 2\alpha_1\alpha_2 + 2\alpha_3 - 2\beta_3, \quad (91)$$

$$\gamma_4 = \alpha_2^2 + 2\alpha_1\alpha_3 - 2\beta_4, \quad (92)$$

$$\gamma_5 = 2\alpha_2\alpha_3, \quad (93)$$

$$\gamma_6 = \alpha_3^2. \quad (94)$$

Next, we show $\gamma_1 = 0$ and get a tight bound on γ_2 , which governs our final bound to M_2 . For the remaining terms, we will show $\sum_{i=3}^6 \gamma_i t^i = O(t^3 qs)$ so that they can be made negligible later when we apply the bound on M_2 in the proof of Lemma 3 (See the proof of Case 2 of Erdős-Rényi graphs).

By (82), (85) and (89), we get

$$\gamma_1 = \alpha_1 - \beta_1 = 0.$$

Recall that $\rho = \frac{s(1-p)}{1-ps}$ and $\sigma^2 = ps(1-ps)$. By (82), (83), (86) and (90), we get

$$\begin{aligned}
\gamma_2 &= \alpha_1^2 + 2\alpha_2 - 2\beta_2 \\
&= q^2 + q^2s^2 - 2q^3 - 2q^3s + 2q^4 = \sigma^4(1 + \rho^2).
\end{aligned}$$

By (82), (83), (84), (87) and (91), we get

$$\begin{aligned}
\gamma_3 &= 2\alpha_1\alpha_2 + 2\alpha_3 - 2\beta_3 \\
&= 2qs - 12q^2s + q^2s^2 + 28q^3s - 4q^3s^2 + q^4 - 32q^4s \\
&\quad + 4q^4s^2 - 4q^5 + 22q^5s + 6q^6 - 12q^6s - 4q^7 + 4q^7s \\
&\stackrel{(a)}{\leq} 2qs - 12q^2s + q^2s^2 + 28q^3s \stackrel{(b)}{\leq} 4qs,
\end{aligned}$$

where (a) holds because $-4q^3s^2 + q^4 \leq 0$, $-32q^4s + 4q^4s^2 - 4q^5 + 22q^5s + 6q^6 \leq 0$ and $-12q^6s - 4q^7 + 4q^7s \leq 0$; (b) holds because $-12q^2s + q^2s^2 + 28q^3s \leq -12q^2s + q^2s^2 + 14q^2s \leq 4q^2s \leq 2qs$ given $q \leq \frac{1}{2}$. By (82), (83), (84), (88) and (92), we get

$$\begin{aligned}
\gamma_4 &= \alpha_2^2 + 2\alpha_1\alpha_3 - 2\beta_4 \\
&= \frac{1}{4}(9q^2s^2 - 12q^3s - 56q^3s^2 + 4q^4 + 60q^4s \\
&\quad + 144q^4s^2 - 8q^5 - 146q^5s - 192q^5s^2 + 8q^6 \\
&\quad + 200q^6s + 136q^6s^2 - 12q^7 - 136q^7s - 48q^7s^2 \\
&\quad + q^8 + 32q^8s + 16q^8s^2 + 16q^9 - 16q^9s) \\
&\stackrel{(a)}{\leq} \frac{1}{4}(9q^2s^2 - 12q^3s - 56q^3s^2 + 4q^4 + 60q^4s + 144q^4s^2) \\
&\stackrel{(b)}{\leq} 13q^2s^2,
\end{aligned}$$

where (a) holds because

$$-8q^5 - 146q^5s - 192q^5s^2 + 8q^6 + 200q^6s + 136q^6s^2$$

$$\begin{aligned}
&= (-8q^5 + 8q^6) + (-146q^5s - 192q^5s^2) \\
&\quad + (200q^6s + 136q^6s^2) \\
&\leq -338q^5s^2 + 336q^6s \leq 0,
\end{aligned}$$

and

$$\begin{aligned}
&-12q^7 - 136q^7s - 48q^7s^2 + q^8 + 32q^8s + \\
&16q^8s^2 + 16q^9 \\
&= (-12q^7 + q^8) + (-136q^7s - 48q^7s^2) \\
&\quad + (32q^8s + 16q^8s^2 + 16q^9) \\
&\leq -184q^7s^2 + 64q^8s \leq 0;
\end{aligned}$$

(b) holds because $-12q^3s + 4q^4 \leq -8q^3s$, $60q^4s + 144q^4s^2 \leq 30q^3s + 72q^3s^2$, and then $-8q^3s - 56q^3s^2 + 30q^3s + 72q^3s^2 \leq 24q^3s + 16q^3s^2 \leq 32q^2s^2$, given $q \leq \frac{1}{2}$. By (83), (84) and (93), we get

$$\begin{aligned}
\gamma_5 &= 2\alpha_2\alpha_3 \\
&= q^2(s - 4qs + 2q^2 + 4q^2s - 3q^3) \\
&\quad (s - 6qs + 15q^2s - 20q^3s + 15q^4s \\
&\quad - 6q^5s + q^5 - 2q^6 + 2q^6s) \\
&\leq q^2(s + 4qs + 2q^2 + 4q^2s + 3q^3) \\
&\quad (s + 6qs + 15q^2s + 20q^3s + 15q^4s \\
&\quad + 6q^5s + q^5 + 2q^6 + 2q^6s) \\
&= q^2s^2(1 + 4q + 2qp + 4q^2 + 3q^2p) \\
&\quad (1 + 6q + 15q^2 + 20q^3 + 15q^4 + 6q^5 \\
&\quad + q^4p + 2q^5p + 2q^6) \\
&\leq (14 \cdot 68)q^2s^2 \leq 2^{10}q^2s^2.
\end{aligned}$$

By (84) and (94), we get

$$\begin{aligned}
\gamma_6 &= \alpha_3^2 \\
&= q^2(s - 6qs + 15q^2s - 20q^3s + 15q^4s \\
&\quad - 6q^5s + q^5 - 2q^6 + 2q^6s)^2 \\
&\leq q^2(s + 6qs + 15q^2s + 20q^3s + 15q^4s \\
&\quad + 6q^5s + q^5 + 2q^6 + 2q^6s)^2 \\
&\leq q^2s^2(1 + 6q + 15q^2 + 20q^3 + 15q^4 \\
&\quad + 6q^5 + q^4p + 2q^5p + 2q^6)^2 \\
&\leq (68)^2q^2s^2 \leq 2^{13}q^2s^2.
\end{aligned}$$

Then, we have

$$\begin{aligned}
M_2 &\leq 1 + t^2\sigma^4(1 + \rho^2) + 4t^3qs + 13t^4q^2s^2 \\
&\quad + 2^{10}t^5q^2s^2 + 2^{13}t^6q^2s^2 \\
&\leq 1 + t^2\sigma^4(1 + \rho^2) + 8t^3qs,
\end{aligned}$$

where the last inequality holds because $13t^4 + 2^{10}t^5 + 2^{12}t^6 \leq 4t^3$ given $t \leq \frac{1}{2^{10}}$.

Moreover, given $M_1 = T$ and $M_2 = T^2 - 2D$,

$$\frac{M_1^2}{M_2} = 1 + \frac{2D}{T^2} \leq 1 + 2D \leq e^2,$$

where the first inequality holds because $M_2 \geq 1$ by definition, and the second inequality holds because

$$D = (ac - b^2)q(s - q)$$

$$\begin{aligned}
&\stackrel{(a)}{\leq} acq(s - q) \\
&\leq \left(1 + t(1 - q)^2 + \frac{1}{2}t^2(1 - q)^4 + t^3(1 - q)^6\right) \times \\
&\quad \left(1 + tq^2 + \frac{1}{2}t^2q^4 + t^3q^6\right) q(s - q) \stackrel{(b)}{\leq} 3,
\end{aligned}$$

where (a) holds because $b^2q(s - q) \geq 0$; (b) holds because $1 + t + \frac{1}{2}t^2 + t^3 \leq \frac{3}{2}$ given $t \leq \frac{1}{2^{10}}$.

The following lemma stated in [26, Lemma 10] follows from the Hanson-Wright inequality [17], [23].

Lemma 9 (Hanson-Wright Inequality): Let $U, V \in \mathbb{R}^n$ are standard Gaussian vectors such that the pairs $(U_k, V_k) \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$ are independent for $k = 1, \dots, n$. Let $M \in \mathbb{R}^{n \times n}$ be any deterministic matrix. There exists some universal constant $c > 0$ such that with probability at least $1 - \delta$,

$$\begin{aligned}
&|U^\top MV - \rho \text{Tr}(M)| \\
&\leq c \left(\|M\|_F \sqrt{\log(1/\delta)} + \|M\|_2 \log(1/\delta) \right). \quad (95)
\end{aligned}$$

Lemma 10: The function ϕ in (47), namely

$$\phi(x) \triangleq \frac{\log \frac{1}{x} - 1 + x}{x \left(\log \frac{1}{x} \right) (1 + \log \frac{1}{x})}$$

is monotonically decreasing on $x \in (0, 1)$.

Proof: We have

$$\phi'(x) = \frac{\psi(x)}{x^2 (\log(x) - 1)^2 \log^2(x)}, \quad (96)$$

where

$$\psi(x) \triangleq \log^3(x) + \log^2(x) + (1 - 2x) \log(x) + x - 1.$$

Next, write

$$\psi'(x) = \frac{\kappa(x)}{x}, \quad (97)$$

where

$$\kappa(x) \triangleq 3 \log^2(x) + (2 - 2x) \log(x) - x + 1.$$

Note that for $x \in (0, 1)$,

$$\kappa'(x) = -\frac{2x \log(x) - 6 \log(x) + 3x - 2}{x} \leq 0,$$

where the last inequality holds because $2x \log(x) \geq -1$ and $-6 \log(x) + 3x \geq 3$ for $x \in (0, 1)$. Thus, $\kappa(x)$ is monotone decreasing on $x \in (0, 1)$ and then $\kappa(x) > \kappa(1) = 0$ for $x \in (0, 1)$. By (97), $\psi'(x) \geq 0$ for $x \in (0, 1)$, and then $\psi(x)$ is monotone increasing on $x \in (0, 1)$ with $\psi(x) < \psi(1) = 0$. Then, by (96), $\phi'(x) < 0$ on $x \in (0, 1)$, as desired. $\square$

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