Single-Snapshot Nested Virtual Array Completion: Necessary and Sufficient Conditions

Pulak Sarangi, *Member, IEEE*, Mehmet Can Hücümenoğlu, and Piya Pal

***Abstract*—We study the problem of completing the virtual array of a nested array with a single snapshot. This involves synthesizing a virtual uniform linear array (ULA) with the same aperture as the nested array by estimating (or interpolating) the missing measurements. A popular approach for virtual array synthesis involves completing a certain Hankel/Toeplitz matrix from partial observations, by seeking low-rank solutions. However, existing theoretical guarantees for such structured rank minimization (which mostly provide sufficient conditions) do not readily extend to nested arrays. We provide the first necessary and sufficient conditions under which it is possible to exactly complete the virtual array of a nested array by minimizing the rank of a certain Toeplitz matrix constructed using a single temporal snapshot. Our results exploit the geometry of nested arrays and do not depend on the source configuration or on the separation between sources.**

***Index Terms*—Sparse Arrays, Nested Sampling, Array interpolation, Matrix Completion, Direction-of-arrival estimation.**

I. Introduction

SPARSE linear arrays such as nested and coprime are well-known for their enhanced spatial resolution and large degrees of freedom (DoF), which are attributed to their large aperture (spanned using far fewer sensors than a Uniform Linear Array) and the structure and cardinality of their difference coarrays [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12]. The large contiguous difference coarray is typically "synthesized" in the correlation domain, where the unobserved correlation values corresponding to missing sensors get *implicitly interpolated* by computing cross correlations between all pairs of sensor measurements. However, these techniques rely on a large number of temporal snapshots to estimate the spatial correlation matrix [1], [5], [13], [14], [15], [16], [17], which may pose challenges in applications such as automotive radar and joint communication and radar sensing, where the sources/multi paths may be coherent and the environment is dynamic due to the high mobility of the sources. This can significantly limit the number of temporal snapshots available for source localization [18], [19].

In order to exploit the enhanced resolution of sparse arrays in sample-starved regimes, several algorithms have been developed for DOA estimation with a single (or limited) snapshot(s), using both off-grid and grid-based approaches [4], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27]. Another body of work aims at "completing a virtual ULA" with the same aperture as the sparse array, by estimating/interpolating the missing measurements with a single snapshot [23]. The virtual measurements can then be used for diverse tasks such as beamforming and source localization, aided by the enhanced resolution of the filled aperture of the virtual ULA [13], [16], [18], [20], [23]. A popular approach is to synthesize the virtual ULA measurements by using low-rank Toeplitz or Hankel matrix completion [13], [20], [25]. Indeed, the virtual measurements can be arranged in the form of a *low-rank* Hankel/Toeplitz matrix, and the measurements acquired by the sparse array only reveal certain entries of this matrix. In practice, for computational tractability, the rank constraint is often relaxed to a suitable convex surrogate, such as the nuclear norm or atomic norm [13], [18], [20], [25]. Although the aforementioned algorithms can also be applied for nested virtual array completion with only one snapshot, there is currently a disconnect between theory and practice. Existing guarantees for deterministic sparse array completion using nuclear norm minimization involve certain coherence conditions on the virtual Toeplitz/Hankel matrix and utilize specific graph-based array designs [18], [20]. On the other hand, theoretical guarantees for atomic norm minimization typically assume *randomized* sparse arrays, and require the source locations to satisfy a certain minimum separation even in the absence of noise [23], [28], [29]. These results therefore do not apply to deterministic spatial samplers such as nested arrays. Moreover, tight necessary and sufficient conditions remain an open question for single-snapshot virtual array completion via rank minimization.

Our contributions: We address these open questions by providing the first necessary and sufficient conditions for rank-minimization to succeed in synthesizing the virtual array of a nested array with a single snapshot (Theorem 1). Since we consider the original rank-minimization framework, our results also reveal fundamental performance limits of any subsequent relaxation/approximation of the rank function. We guarantee exact interpolation (in absence of noise) regardless of the separation between sources, or coherence of the virtual Toeplitz matrix. Our converse results (necessary conditions) utilize the geometry of nested arrays in order to establish the existence of "ambiguous" source configurations (which we explicitly construct) for which rank-minimization will provably fail.

Notations: Given a vector $\mathbf{z} \in \mathbb{C}^L$, the operator $\mathcal{T}_L(\mathbf{z})$ returns a $L \times L$ Hermitian Toeplitz matrix whose first column is given by $\mathbf{z}$. $\mathcal{R}(\mathbf{A})$ represents the range space of a given matrix $\mathbf{A}$. We denote $\mathbf{A}_{\mathbb{S}}(\boldsymbol{\omega}) = [\mathbf{a}_{\mathbb{S}}(\omega_1), \mathbf{a}_{\mathbb{S}}(\omega_2), \ldots, \mathbf{a}_{\mathbb{S}}(\omega_K)] \in$

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The authors are with the Electrical and Computer Engineering, University of California San Diego, La Jolla, CA 92092 USA (e-mail: psarangi@eng.ucsd.edu; mhucumen@eng.ucsd.edu; pipal@eng.ucsd.edu).

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$\mathbb{C}^{P\times K}$ as the array manifold matrix of an array with sensors located at $n\lambda/2, n\in\mathbb{S}=\{d_1,d_2,\ldots,d_P\}$ (with source wavelength λ), and source frequencies are given by the set $\boldsymbol{\omega}=\{\omega_1,\omega_2,\ldots,\omega_K\}$, with $[\mathbf{a}_{\mathbb{S}}(\omega_k)]_m=e^{j\omega_k d_m}$. We use $[\mathbf{v}]_{i_1:i_2}$ to denote a vector whose entries are given by those at indices $i_1,i_1+1,..,i_2$ of the vector $\mathbf{v}$.

II. Problem Formulation

Consider K far-field narrowband sources impinging from directions $\{\theta_i\}_{i=1}^K$ on a one-dimensional nested array with $P=2M$ sensors whose locations are given by $\mathbb{S}_{\text{nest}}$, where $\mathbb{S}_{\text{nest}}:=\mathbb{S}_1\cup\mathbb{S}_2$ is the union of integer sets $\mathbb{S}_1=\{m-1\}_{m=1}^{M+1}$ and $\mathbb{S}_2=\{m(M+1)-1\}_{m=2}^{M}$. The signal received at the nested array is given by:

$$\mathbf{y}_{\text{nest}}=\mathbf{A}_{\mathbb{S}_{\text{nest}}}(\boldsymbol{\omega})\mathbf{x}+\mathbf{n}, \tag{1}$$

where $\mathbf{x}\in\mathbb{R}^K$ denotes real-valued [1] (deterministic) source signals and $\mathbf{n}$ is an additive noise term. The normalized spatial frequencies are denoted by $\boldsymbol{\omega}=\{\omega_k\}_{k=1}^K$, with $\omega_i=\pi\sin(\theta_i)$.

The difference set $\mathbb{D}_{\mathbb{S}_{\text{nest}}}$ of $\mathbb{S}_{\text{nest}}$ is defined as $\mathbb{D}_{\mathbb{S}_{\text{nest}}}:=\{m-n|m,n\in\mathbb{S}_{\text{nest}}\}$. It is well known that the set of non-negative elements in $\mathbb{D}_{\mathbb{S}_{\text{nest}}}$ are given by $\mathbb{U}:=\{0,1,\ldots,N-1\}$ where $N{=}M(M+1)$[1]. In the absence of noise, we can rewrite (1) as

$$\mathbf{y}_{\text{nest}}=\mathbf{S}_{\text{nest}}\mathbf{y}_{\text{full}},\quad \mathbf{y}_{\text{full}}:=\mathbf{A}_{\mathbb{U}}(\boldsymbol{\omega})\mathbf{x}, \tag{2}$$

where $\mathbf{S}_{\text{nest}}\in\mathbb{R}^{P\times N}$ is a row-selection matrix given by:

$$[\mathbf{S}_{\text{nest}}]_{i,j}=\begin{cases}1, & \text{if } d_i+1=j, d_i\in\mathbb{S}_{\text{nest}}\\ 0, & \text{otherwise}\end{cases}$$

The vector $\mathbf{y}_{\text{full}}$ is a "*virtual measurement*," received by the virtual array $\mathbb{U}$, *with identical source configurations* (same DOAs $\boldsymbol{\omega}$ and source signal $\mathbf{x}$).

Key Question: We are interested in the problem of "sparse array interpolation" with only a *single snapshot*, where the goal is to estimate $\mathbf{y}_{\text{full}}$ from $\mathbf{y}_{\text{nest}}$. As discussed earlier, theoretical guarantees for matrix-completion or atomic norm mimimization based virtual array synthesis do not readily extend to nested arrays. This raises the open question: *What are the necessary and sufficient conditions under which rank-minimization with nested arrays leads to exact virtual array completion?*

III. Guaranteed Single Snapshot Interpolation With Nested Matrix Completion

Consider the noiseless measurement model (1) with $\mathbf{n}=\mathbf{0}$. From (2), it can be seen that when $\mathbf{x}$ is real, the matrix $\mathcal{T}_N(\mathbf{y}_{\text{full}})$ admits the following Vandermonde decomposition:

$$\mathcal{T}_N(\mathbf{y}_{\text{full}})=\mathbf{A}_{\mathbb{U}}(\boldsymbol{\omega})\text{diag}(\mathbf{x})\mathbf{A}_{\mathbb{U}}^H(\boldsymbol{\omega}). \tag{3}$$

Consider the rank-minimization problem

$$\min_{\mathbf{u}\in\mathbb{C}^N}\text{rank}[\mathcal{T}_N(\mathbf{u})]\quad\text{subject to } \mathbf{S}_{\text{nest}}\mathbf{u}=\mathbf{y}_{\text{nest}}. \tag{P1}$$

The following theorem provides necessary and sufficient conditions under which perfect interpolation is possible (in absence of noise) by solving (P1).

[1] A similar setting with real source signals has been considered in [24]. In future, we will extend our theoretical results for the complex case.

Theorem 1: Consider the measurement model (1) with $\mathbf{n}=\mathbf{0}$. If $K\le M$, then (P1) has a unique solution $\mathbf{u}^\star$ satisfying $\mathbf{u}^\star=\mathbf{y}_{\text{full}}=\mathbf{A}_{\mathbb{U}}(\boldsymbol{\omega})\mathbf{x}$, for every $\boldsymbol{\omega}$ and $\mathbf{x}$. Conversely if $K>M$, there exist source configurations with K source angles $\boldsymbol{\omega}_0\in[-\pi,\pi)^K$, and amplitudes $\mathbf{x}_0\in\mathbb{R}^K$, such that one can find a vector $\widehat{\mathbf{y}}$, with $\widehat{\mathbf{y}}\neq\mathbf{y}_{\text{full}}$ (where $\mathbf{y}_{\text{full}}=\mathbf{A}_{\mathbb{U}}(\boldsymbol{\omega}_0)\mathbf{x}_0$), satisfying

$$\mathbf{S}_{\text{nest}}\widehat{\mathbf{y}}=\mathbf{S}_{\text{nest}}\mathbf{y}_{\text{full}},\quad\text{rank}\left(\mathcal{T}_N(\widehat{\mathbf{y}})\right)\le K \tag{4}$$

Proof: We first show that there exists no feasible point $\widetilde{\mathbf{y}}\in\mathbb{C}^N$ of (P1) such that $\text{rank}(\mathcal{T}_N(\widetilde{\mathbf{y}}))<K$. Consider a feasible point $\widetilde{\mathbf{y}}\in\mathbb{C}^N$ and the following block partitioning of the matrix $\mathcal{T}_N(\widetilde{\mathbf{y}})$:

$$\mathcal{T}_N(\widetilde{\mathbf{y}})=\begin{bmatrix}\mathbf{T}_1 & \mathbf{T}_2\\ \text{X} & \text{Z}\end{bmatrix}, \tag{5}$$

where $\mathbf{T}_1\in\mathbb{C}^{(M+1)\times(M+1)}, \mathbf{T}_2\in\mathbb{C}^{M+1\times(N-M-1)}$. We also define a partitioning of the inner ULA manifold $\mathbf{A}_{\mathbb{S}_1}(\boldsymbol{\omega})$ as:

$$\mathbf{A}_{\mathbb{S}_1}(\boldsymbol{\omega})=\begin{bmatrix}\mathbf{1}^\top\\ \mathbf{B}(\boldsymbol{\omega})\end{bmatrix}, \tag{6}$$

where $\mathbf{B}(\boldsymbol{\omega})\in\mathbb{C}^{M\times K}$ is also a Vandermonde matrix due to the structure of the nested array. Since $\widetilde{\mathbf{y}}$ is feasible, we have $\mathbf{S}_{\text{nest}}\widetilde{\mathbf{y}}=\mathbf{y}_{\text{nest}}$, which implies

$$\mathbf{T}_1=\mathcal{T}_{M+1}([\widetilde{\mathbf{y}}]_{1:M+1})=\mathcal{T}_{M+1}([\mathbf{y}_{\text{nest}}]_{1:M+1})=\mathcal{T}_{M+1}(\mathbf{y}_{\mathbb{S}_1}).$$

where $\mathbf{y}_{\mathbb{S}_1}=\mathbf{A}_{\mathbb{S}_1}(\boldsymbol{\omega})\mathbf{x}$. Since $\mathbb{S}_1$ is a ULA, from (3), we have

$$\mathcal{T}_{M+1}(\mathbf{y}_{\mathbb{S}_1})=\mathbf{A}_{\mathbb{S}_1}(\boldsymbol{\omega})\text{diag}(\mathbf{x})\mathbf{A}_{\mathbb{S}_1}^H(\boldsymbol{\omega}). \tag{7}$$

Since $K\le M$, $\text{rank}(\mathcal{T}_{M+1}(\mathbf{y}_{\mathbb{S}_1}))$=$K$. Hence, $\text{rank}(\mathcal{T}_N(\widetilde{\mathbf{y}}))\ge K$, i.e., there exists no feasible point with rank strictly smaller than K.

Suppose $\text{rank}(\mathcal{T}_N(\widetilde{\mathbf{y}}))=K$. We show that $\widetilde{\mathbf{y}}=\mathbf{y}_{\text{full}}$ is the only feasible solution satisfying this property and this will prove that $\mathbf{y}_{\text{full}}$ is the unique solution to (P1). We need to show that $[\widetilde{\mathbf{y}}]_i=[\mathbf{y}_{\text{full}}]_i$ for all $1\le i\le N$. In other words, for every $j'=M+1,M+2,\ldots,N$, we will show that

$$[\widetilde{\mathbf{y}}]_i=[\mathbf{y}_{\text{full}}]_i,\ \forall\, i\le j'. \tag{8}$$

We establish this by induction on j'. The base case $j'=M+1$ follows because $\widetilde{\mathbf{y}}$ is feasible and due to the structure of nested array, we also have $[\widetilde{\mathbf{y}}]_i=[\mathbf{y}_{\mathbb{S}_1}]_i=[\mathbf{y}_{\text{full}}]_i, 1\le i\le M+1$. Next, suppose (8) holds for $j'=j_0$ ($j_0\ge M+1$), and we will show that (8) also holds for j_0+1. Due to the induction hypothesis, showing (8) holds for $j'=j_0+1$ is equivalent to showing $[\widetilde{\mathbf{y}}]_{j_0+1}=[\mathbf{y}_{\text{full}}]_{j_0+1}$. Denote $\bar{\mathbf{T}}:=[\mathbf{T}_1\quad\mathbf{T}_2]\in\mathbb{C}^{M+1\times N}$. Due to the Toeplitz structure, the $(j_0+1)^{\text{th}}$ column of $\bar{\mathbf{T}}$ is given by:

$$\bar{\mathbf{t}}_{j_0+1}=\left[[\widetilde{\mathbf{y}}]_{j_0+1}^*,[\widetilde{\mathbf{y}}]_{j_0}^*,\ldots,[\widetilde{\mathbf{y}}]_{j_0-M+1}^*\right]^\top\overset{(a)}{=}\left[[\widetilde{\mathbf{y}}]_{j_0+1}^*,\bar{\mathbf{v}}^\top\right]^\top, \tag{9}$$

where $\bar{\mathbf{v}}=\left[[\mathbf{y}_{\text{full}}]_{j_0}^*,\ldots,[\mathbf{y}_{\text{full}}]_{j_0-M+1}^*\right]^\top$ and (a) follows from the induction hypothesis. From (2), for $i=1,2,\ldots,M$:

$$[\bar{\mathbf{v}}]_i=\sum_{k=1}^K e^{-j\omega_k(j_0-i)}x_k=\sum_{k=1}^K e^{j\omega_k i}e^{-j\omega_k j_0}x_k, \tag{10}$$

Define $\widetilde{\mathbf{x}}\in\mathbb{C}^K$ as $[\widetilde{\mathbf{x}}]_k=e^{-j\omega_k j_0}x_k$. From (6), we obtain

$$\bar{\mathbf{v}}=\mathbf{B}(\omega)\widetilde{\mathbf{x}}. \tag{11}$$

Now, we use the fact that $\text{rank}(\mathcal{T}_N(\widetilde{\mathbf{y}}))=K=\text{rank}(\mathbf{T}_1)$ which implies that $\text{rank}(\bar{\mathbf{T}})=K$. Therefore, the $(j_0+1)^{\text{th}}$ column of $\bar{\mathbf{T}}$ ($\bar{\mathbf{t}}_{j_0+1}$) satisfies $\bar{\mathbf{t}}_{j_0+1}\in\mathcal{R}(\mathbf{T}_1)$. From the Vandermonde decomposition (7), it can be seen that $\mathbf{A}_{\mathbb{S}_1}(\boldsymbol{\omega})$ is a basis for $\mathcal{R}(\mathbf{T}_1)$, and hence there exists $\mathbf{c} \in \mathbb{C}^K$ such that

$$\bar{\mathbf{t}}_{j_0+1} = \mathbf{A}_{\mathbb{S}_1}(\boldsymbol{\omega})\mathbf{c} = \begin{bmatrix} \mathbf{1}^\top \\ \mathbf{B}(\boldsymbol{\omega}) \end{bmatrix} \mathbf{c}. \tag{12}$$

By combining (9), (11), and (12) we have the following:

$$\bar{\mathbf{t}}_{j_0+1} = \begin{bmatrix} [\widetilde{\mathbf{y}}]^*_{j_0+1} \\ \bar{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{1}^\top \mathbf{c} \\ \mathbf{B}(\boldsymbol{\omega})\mathbf{c} \end{bmatrix} \overset{(a)}{=} \begin{bmatrix} \mathbf{1}^\top \mathbf{c} \\ \mathbf{B}(\boldsymbol{\omega})\widetilde{\mathbf{x}} \end{bmatrix}. \tag{13}$$

From the equality (a), we have $\mathbf{B}(\boldsymbol{\omega})\mathbf{c} = \mathbf{B}(\boldsymbol{\omega})\widetilde{\mathbf{x}}$. Since $K \le M$, $\mathbf{B}(\boldsymbol{\omega})$ is a Vandermonde matrix with full column rank and thus $\mathbf{c} = \widetilde{\mathbf{x}}$. The proof is complete by plugging $\mathbf{c} = \widetilde{\mathbf{x}}$ in (13) $[\widetilde{\mathbf{y}}]^*_{j_0+1} = \sum_{k=1}^K [\widetilde{\mathbf{x}}]_k = \sum_{k=1}^K e^{-j\omega_k j_0} x_k = [\mathbf{y}_{\text{full}}]^*_{j_0+1}$.

For the converse results, we will show the existence of $\boldsymbol{\omega}_0, \mathbf{x}_0$ and $\widehat{\mathbf{y}}$ with the desired properties by considering two cases (1) $2M+1 \le K \le N/2$ and (2) $M < K \le 2M$:

1) ($2M+1 \le K \le N/2$): Consider any $2K$ distinct source angles denoted by the set $\Omega_{2K} := \{\omega_1, \omega_2, \ldots, \omega_{2K}\}$. We define a concatenated matrix $\mathbf{M}(\Omega_{2K}) \in \mathbb{R}^{4M \times 2K}$:

$$\mathbf{M}(\Omega_{2K}) = \left[\mathrm{Re}(\mathbf{A}_{\mathbb{S}_{\text{nest}}}(\Omega_{2K}))^\top \quad \mathrm{Im}(\mathbf{A}_{\mathbb{S}_{\text{nest}}}(\Omega_{2K}))^\top\right]^\top.$$

Since $K \ge 2M+1$, $\mathbf{M}(\Omega_{2K})$ has a non-trivial null space, i.e., there exists $\mathbf{v} \in \mathbb{R}^{2K}, \mathbf{v} \neq \mathbf{0}$ such that

$$\mathbf{M}(\Omega_{2K})\mathbf{v} = \mathbf{0}. \tag{14}$$

Suppose $\mathbf{v}$ has $L \le 2K$ non-zero entries, and without loss of generality, let the indices of the non-zero elements be $\{1, 2, \ldots, L\}$ [2]. We select $\boldsymbol{\omega}_0$ as $\boldsymbol{\omega}_0 = \{\omega_1, \omega_2, \ldots, \omega_K\}$. Now, there can be two possibilities: either $L > K$, or $L \le K$. Suppose $L > K$. In this case, let $\mathbf{x}_0 = -[\mathbf{v}]_{1:K} \in \mathbb{R}^K$ and construct $\widehat{\mathbf{y}}$ as follows. Define $\bar{\boldsymbol{\omega}} := \{\omega_{K+1}, \ldots, \omega_L\}$ and $\bar{\mathbf{x}} := [\mathbf{v}]_{(K+1):L}$. Let $\widehat{\mathbf{y}}$ be given by $\widehat{\mathbf{y}} = \mathbf{A}_{\mathbb{U}}(\bar{\boldsymbol{\omega}})\bar{\mathbf{x}}$. In this case, since $[\mathbf{A}_{\mathbb{U}}(\bar{\boldsymbol{\omega}}), \mathbf{A}_{\mathbb{U}}(\boldsymbol{\omega}_0)]$ is a Vandermonde matrix with L distinct columns, it has full column-rank, since $L \le 2K \le N$. This implies that $\mathbf{A}_{\mathbb{U}}(\bar{\boldsymbol{\omega}})\bar{\mathbf{x}} \neq \mathbf{A}_{\mathbb{U}}(\boldsymbol{\omega}_0)\mathbf{x}_0$ for non-zero $\mathbf{x}_0, \bar{\mathbf{x}}$, and therefore $\widehat{\mathbf{y}} \neq \mathbf{y}_{\text{full}}$. Next consider the case $L \le K$. In this case, let $\mathbf{x}_0$ be given by $\mathbf{x}_0 = [[\mathbf{v}]^\top_{1:L}, \mathbf{1}^\top_{K-L}]^\top$ (where $\mathbf{1}_{K-L} \in \mathbb{R}^{K-L}$ is a vector of all 1's), $\bar{\boldsymbol{\omega}} := [\omega_{L+1}, \ldots, \omega_K]$, $\bar{\mathbf{x}} = \mathbf{1}_{K-L}$, and again construct $\widehat{\mathbf{y}}$ as $\widehat{\mathbf{y}} = \mathbf{A}_{\mathbb{U}}(\bar{\boldsymbol{\omega}})\bar{\mathbf{x}}$. Once again, it can be verified that $\mathbf{y}_{\text{full}} \neq \widehat{\mathbf{y}}$, otherwise it would imply (from the constructions of $\mathbf{x}_0$, $\bar{\mathbf{x}}$ and $\bar{\boldsymbol{\omega}}$) that $\sum_{i=1}^L \mathbf{a}_{\mathbb{U}}(\omega_i)[\mathbf{x}_0]_i = 0$. This cannot happen since $\{\mathbf{a}_{\mathbb{U}}(\omega_i)\}_{i=1}^L$ are L distinct columns of a $N \times L$ Vandermonde matrix (with $L \le N$), and are therefore linearly independent. Therefore, for each construction of $\widehat{\mathbf{y}}$, we have $\mathbf{y}_{\text{full}} \neq \widehat{\mathbf{y}}$, and (14) also implies that $\mathbf{S}_{\text{nest}}\mathbf{y}_{\text{full}} = \mathbf{A}_{\mathbb{S}_{\text{nest}}}(\boldsymbol{\omega}_0)\mathbf{x}_0 = \mathbf{A}_{\mathbb{S}_{\text{nest}}}(\bar{\boldsymbol{\omega}})\bar{\mathbf{x}} = \mathbf{S}_{\text{nest}}\mathbf{A}_{\mathbb{U}}(\bar{\boldsymbol{\omega}})\bar{\mathbf{x}} = \mathbf{S}_{\text{nest}}\widehat{\mathbf{y}}$. Since $\widehat{\mathbf{y}} = \mathbf{A}_{\mathbb{U}}(\bar{\boldsymbol{\omega}})\bar{\mathbf{x}}$, it also holds that $\text{rank}(\mathcal{T}_N(\widehat{\mathbf{y}})) = \text{rank}(\mathbf{A}_{\mathbb{U}}(\bar{\boldsymbol{\omega}})\text{diag}(\bar{\mathbf{x}})\mathbf{A}^H_{\mathbb{U}}(\bar{\boldsymbol{\omega}})) = |K-L| \le K$, as $L \le 2K$.

2) ($M < K \le 2M$): We begin by proving the following fact about the nested array. For every K in the range $M < K \le 2M$, there is at most one $i \in \{2, \ldots, 2M\}$ (i.e. excluding the sensor at 0) such that d_i satisfies $\mod(d_i, K) = 0$. Suppose there exist two sensor locations d_l and d_m for which $\mod(d_l, K) = \mod(d_m, K) = 0$. Since $M < K < 2M+1$, and $d_i = (i-1), d_i \in \mathbb{S}_1$, this would imply that $d_l, d_m \in \mathbb{S}_2$. Therefore, there exist integers z_1, z_2 and $k_1, k_2 \in \{2, \ldots, M\}$, such that $k_1(M+1) - 1 = z_1 K$ and $k_2(M+1) - 1 = z_2 K$, which implies that $(k_2 - k_1)/K = z_2 k_1 - z_1 k_2$. Since $2 \le k_1, k_2 \le M$, we have $-(M-2) \le k_2 - k_1 \le M-2$. But we also have $M < K \le 2M$. Hence, $(k_2 - k_1)/K \in \mathbb{Z}$ can be satisfied only if $k_1 = k_2$. This implies that $d_l = d_m$, and the statement is proved. We now construct $\boldsymbol{\omega}_0$ as $\boldsymbol{\omega}_0 = \{2\pi\frac{k}{K}\}_{k=0}^{K-1}$ [3] and let $\bar{\boldsymbol{\omega}} = \boldsymbol{\omega}_0 + 2\pi\frac{\alpha}{K}$, where α is chosen as follows. If there exists an integer $i_0 \in \{2, 3, \ldots, 2M\}$ such that the sensor location $d_{i_0} = zK$ for some positive integer $z > 0$, then we choose $\alpha = \frac{1}{z}$. Else, α is chosen as an arbitrary real number satisfying $0 < \alpha < 1$. Redefine Ω_{2K} as $\Omega_{2K} := \boldsymbol{\omega}_0 \cup \bar{\boldsymbol{\omega}}$. We construct $\mathbf{w} \in \mathbb{R}^{2K}$ as follows: $[\mathbf{w}]_i = 1, [\mathbf{w}]_{K+i} = -1, \quad 1 \le i \le K$. Clearly, $[\mathbf{w}]_i \neq 0$ for all i. We will show that $\mathbf{w}$ satisfies $\mathbf{M}(\Omega_{2K})\mathbf{w} = \mathbf{0}$. Since the first sensor in nested array is assumed to be at the origin (i.e. $d_1 = 0$), we have $[\mathbf{M}(\Omega_{2K})\mathbf{w}]_1 = \mathbf{1}^\top \mathbf{w} = 0$, and $[\mathbf{M}(\Omega_{2K})\mathbf{w}]_{2M+1} = \mathbf{0}^\top \mathbf{w} = 0$. Consider any i in the range $2 \le i \le 2M$. First assume that the sensor location d_i satisfies $\mod(d_i, K) \neq 0$, implying that $\sin(\frac{\pi d_i}{K}) \neq 0$. Then,

$$[\mathbf{M}(\Omega_{2K})\mathbf{w}]_i = \sum_{k=0}^{K-1} \cos\left(d_i \frac{2\pi k}{K}\right) - \sum_{k=0}^{K-1} \cos\left(d_i \frac{2\pi(k+\alpha)}{K}\right)$$

$$= \frac{\sin(\pi d_i)}{\sin\left(\frac{\pi d_i}{K}\right)}\left[\cos\left(\frac{\pi}{K} d_i (K-1)\right) - \cos\left(\frac{\pi}{K} d_i (K-1+2\alpha)\right)\right] = 0$$

since $\sin(\pi d_i) = 0$ for integer d_i. Similarly,

$$[\mathbf{M}(\Omega_{2K})\mathbf{w}]_{2M+i} = \sum_{k=0}^{K-1} \sin\left(d_i \frac{2\pi k}{K}\right) - \sum_{k=0}^{K-1} \sin\left(d_i \frac{2\pi(k+\alpha)}{K}\right)$$

$$= \frac{\sin(\pi d_i)}{\sin\left(\frac{\pi}{K} d_i\right)}\left[\sin\left(\frac{\pi}{K} d_i (K-1)\right) - \sin\left(\frac{\pi}{K} d_i (K-1+2\alpha)\right)\right] = 0.$$

Finally, suppose there exists i_0 such that $d_{i_0} = zK$. [4] Then, with the aforementioned choice of $\alpha = \frac{1}{z}$, we have $\cos(d_{i_0}\frac{2\pi k}{K}) = \cos(2\pi kz) = 1$ and $\cos(d_{i_0}\frac{2\pi(k+\alpha)}{K}) = \cos(2\pi kz + 2\pi) = 1$. This implies that $[\mathbf{M}(\Omega_{2K})\mathbf{w}]_{i_0} = 0$, as $\sum_i [\mathbf{w}]_i = 0$. Similarly, we have $\sin(d_{i_0}\frac{2\pi k}{K}) = \sin(2\pi kz) = 0$ and $\sin(d_{i_0}\frac{2\pi(k+\alpha)}{K}) = \sin(2\pi kz + 2\pi) = 0$, which implies that $[\mathbf{M}(\Omega_{2K})\mathbf{w}]_{i_0+2M} = 0$ as well. Combining the above results, we showed that $\mathbf{M}(\Omega_{2K})\mathbf{w} = \mathbf{0}$. Let $\mathbf{x}_0, \bar{\mathbf{x}} \in \mathbb{R}^K$ be defined as $[\mathbf{x}_0]_i = -[\mathbf{w}]_i, [\bar{\mathbf{x}}]_i = [\mathbf{w}]_{K+i}, 1 \le i \le K$. As before, construct $\widehat{\mathbf{y}} = \mathbf{A}_{\mathbb{U}}(\bar{\boldsymbol{\omega}})\bar{\mathbf{x}}$. Since $\mathbf{M}(\Omega_{2K})\mathbf{w} = \mathbf{0}$, we again have $\mathbf{S}_{\text{nest}}\mathbf{y}_{\text{full}} = \mathbf{A}_{\mathbb{S}_{\text{nest}}}(\boldsymbol{\omega}_0)\mathbf{x}_0 = \mathbf{A}_{\mathbb{S}_{\text{nest}}}(\bar{\boldsymbol{\omega}})\bar{\mathbf{x}} = \mathbf{S}_{\text{nest}}\mathbf{A}_{\mathbb{U}}(\bar{\boldsymbol{\omega}})\bar{\mathbf{x}} = \mathbf{S}_{\text{nest}}\widehat{\mathbf{y}}$. Since $K \le N/2$, using a similar argument as the previous case, it can again be shown that $\widehat{\mathbf{y}} \neq \mathbf{y}_{\text{full}}$. Furthermore, $\text{rank}(\mathcal{T}_N(\widehat{\mathbf{y}})) = \text{rank}(\mathbf{A}_{\mathbb{U}}(\bar{\boldsymbol{\omega}})\text{diag}(\bar{\mathbf{x}})\mathbf{A}^H_{\mathbb{U}}(\bar{\boldsymbol{\omega}})) = K$. This concludes the proof. $\square$

Theorem 1 guarantees that when $K \le M$, it is possible to perfectly interpolate the missing sensors in a nested array, by solving the rank-minimization problem (P1) *regardless of the separation* between the sources. In fact, it can be shown that the sufficient condition broadly applies to any sparse array with a ULA segment of length at least K. It also shows that, even with a single snapshot, a nested array can identify $O(M)$ sources (by applying any subspace based technique on the output of (P1)). While an ULA can also identify $K \le M$ sources using single-snapshot MUSIC (SS-MUSIC) [30], an interpolated nested array can resolve sources with much smaller separation,

[2] The elements of the set Ω_{2K} can always be permuted to ensure this.

[3] Note that each $\omega_i = \pi \sin\theta_i$ maps to a unique angle in $[-\pi, \pi)$, which can again be uniquely mapped to $\omega_i \in [0, 2\pi)$.

[4] From our previous argument, there can be at most one such sensor.

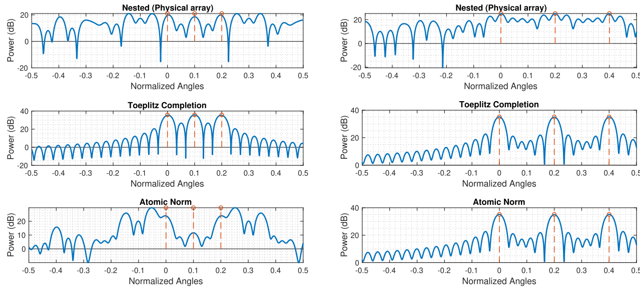

Fig. 1. Comparison of beamforming on the nested array and interpolated virtual array with $K = 3$ sources located at (left) $\boldsymbol{\omega} = \{0.0, 0.1, 0.2\}$ and (right) $\boldsymbol{\omega} = \{0.0, 0.2, 0.4\}$. The total number of sensors is $P = 14$ and the interpolation was performed up to $N = 56$ sensors corresponding to the aperture of the nested array.

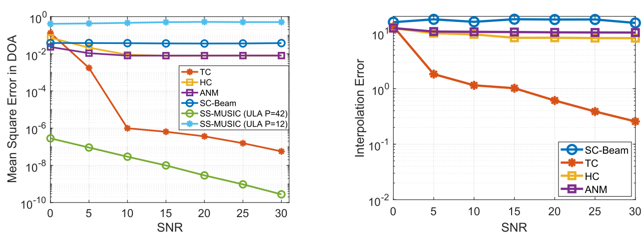

Fig. 2. (Left) MSE in DOA vs. SNR for $K = 5$ sources with a nested array with $P = 12$ sensors and ULA (with $P = 12$ and $P = 42$). (Right) Normalized interpolation error vs. SNR for interpolating up to $N = 42$ sensors corresponding to the aperture of the $P = 12$ sensor nested array.

especially in presence of noise. This is demonstrated in Fig. 2. In the future, we will analyze how the outer ULA $\mathbb{S}_2$ controls this noisy interpolation error.

Theoretical conditions under which nested arrays will provably fail to identify sources with one snapshot were also unavailable. [5] An important contribution of Theorem 1 is to settle this question by showing that the sufficient condition is also necessary. This is done by constructing ambiguous source configurations that exploit the nested geometry.

IV. Simulations

We solve a relaxed version of (P1) by replacing the rank with nuclear norm[6] and the equality constraint by a norm constraint $\|\mathbf{S}_{\text{nest}}\mathbf{u} - \mathbf{y}_{\text{nest}}\|_2 \leq \epsilon$, assuming that the noise is bounded as $\|\mathbf{n}\|_2 \leq \epsilon$. We call this approach Toeplitz Completion (TC). We first show the benefits of interpolation in beamforming with nested arrays using a single snapshot. We consider noiseless measurements acquired by a nested array with $P = 14$ sensors, comprised of three sources with amplitudes $\mathbf{x} = [1, -1, 1]$. In Fig. 1, we plot the beam pattern obtained by interpolating the nested array measurements using TC, and then beamforming with the interpolated measurements for two different DOA configurations. For comparison, we plot the beam pattern obtained from beamforming with the physical nested array (without interpolation). We also perform interpolation with Atomic Norm Minimization (ANM) [23] and plot the resulting beam pattern in Fig. 1 (last row). Due to large side lobes of the nested array, the source locations are not distinguishable when using the physical measurements. On the other hand, using the interpolated signal produced by TC, we can identify three closely spaced sources. Beamforming with the interpolated measurements using ANM fails to resolve sources with small separation (left), and succeeds only when the separation is large enough (right).

We next study the DOA estimation error and the interpolation error in presence of noise. We consider a nested array with $P = 12$ sensors and $K = 5$ sources with spatial frequencies $\omega = \{\pi/20 + 0.1k\}_{k=0}^{4}$, and fixed amplitudes $\mathbf{x} = [1, -1, 1, 1, -1]^\top$. The additive noise is generated as $\mathbf{n} \sim \mathcal{U}(-\sigma/2, \sigma/2)$ and the SNR$=10\log(1/\sigma^2)$ is controlled by varying σ. In Fig. 2 (left) we plot the MSE of DOA estimates (computed over 200 trials) as a function of SNR, by performing Root-MUSIC [32] on the output of TC. We also compare against Successive cancellation beamforming (SC-Beam) [21], ANM [33], and Hankel Completion (HC) [20], all of which permit single-snapshot DOA estimation with (arbitrary) sparse arrays, although their performances vary. We also compare the performance of SS-MUSIC on ULA (with 12 and 42 sensors). The 12-element nested array outperforms the ULA with 12 sensors and comes close to the performance of the 42-element ULA.

In Fig. 2 (right) we also plot the interpolation error $\|\widehat{\mathbf{y}}_{\text{full}} - \mathbf{y}_{\text{full}}\|_2/N$ versus SNR where $\widehat{\mathbf{y}}_{\text{full}}$ is the estimated virtual measurement. For algorithms such as SC-Beam that does not perform explicit interpolation, we generate the interpolated signal using the DOA and source amplitude estimates as $\widehat{\mathbf{y}}_{\text{full}} = \mathbf{A}_{\mathbb{U}}(\widehat{\boldsymbol{\omega}})\widehat{\mathbf{x}}$. In both plots, we observe that the MSE of TC decays sharply with SNR while the other algorithms exhibit saturation. It is to be noted that the theoretical guarantees for these algorithms (if available) do not necessarily apply to deterministic sampling patterns such as nested arrays. Therefore, these techniques may fail to correctly identify all K sources (especially with small separation) with nested arrays, leading to saturation. The steady decay in the MSE of TC with increasing SNR is consistent with Theorem 1, which guarantees that exact interpolation is possible with nested arrays with $K \leq M$ sources by seeking low-rank solutions.

V. Conclusion

We provided necessary and sufficient conditions for rank-minimization based techniques to successfully complete the virtual array of nested arrays in the absence of noise. We showed that if $K \leq M$, one can exactly recover the missing measurements (and synthesize the virtual array) for any source configuration, by solving a Toeplitz matrix completion problem via rank-minimization. In contrast, when $K > M$, there exist source configurations (which depend on the nested geometry) where the recovery will provably fail. Our results indicate that the unique geometry of the nested array allows it to leverage the enhanced resolution of the virtual coarray (via non-linear interpolation), even with a single snapshot. In numerical simulations, the Toeplitz completion approach is observed to be robust to noise and outperforms other single snapshot source localization methods with nested arrays.

[5] Most existing works focus on the multi-snapshot setting [1], [27], [31]

[6] Nuclear norm is just one among many approaches to replace "rank" by a suitable surrogate in order to attain computational tractability.

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