

TO REGULARIZE OR NOT TO REGULARIZE: THE ROLE OF POSITIVITY IN SPARSE ARRAY INTERPOLATION WITH A SINGLE SNAPSHOT

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ABSTRACT

We study single-snapshot nested array interpolation with positive sources. The problem of sparse array interpolation is traditionally cast as a low-rank Toeplitz/Hankel matrix completion problem from partial observations. In recent work, we provided the first necessary and sufficient guarantees for nested array interpolation with real measurements in the rank minimization framework. In this work, we strengthen the sufficiency results by proving that in case of positive sources it is possible to interpolate the nested array by performing a simple convex feasibility search instead of solving a rank minimization problem. Simulations demonstrate that this framework is also effective for noisy measurements, and that noisy nested array interpolation outperforms ULA extrapolation.¹

Index Terms— Sparse Array Interpolation, Nested arrays, Positivity, Matrix Completion.

1. INTRODUCTION

Sparse arrays, such as nested and coprime arrays [1, 2], have recently gained significant recognition due to their superior parameter identifiability and resolution capabilities compared to the conventional Uniform Linear Array (ULA) with the same number of sensors [3–11]. A key advantage of properly designed sparse arrays with P sensors is that their physical aperture and the number of virtual sensors in their so-called difference coarray is comparable to that of a ULA with $\Theta(P^2)$ sensors. The virtual measurements of the difference coarray are typically synthesized by computing the cross-correlations of the physical sensor measurements. However, this may require uncorrelated sources and a large number of temporal snapshots [1, 7, 12–15], which can be problematic in applications with multipath and fast-moving sources, such as automotive radar or autonomous sensing [16, 17].

An alternative approach involves estimating the missing virtual measurements of a ULA with the same aperture as the sparse array [14, 17–19]. The completed measurements can then be utilized to enhance spatial resolution in presence of noise and even outperform direct direction-of-arrival estimation methods, with very few (even a single) snapshot(s)

[20]. Interpolation of the missing measurements is typically cast as a low-rank Toeplitz/Hankel matrix completion problem [17, 19, 21]. Such matrix completion based techniques nevertheless suffer from two critical drawbacks. Firstly, the theoretical recovery guarantees require a certain minimum separation between sources to succeed, even in the absence of noise [18, 22, 23]. Secondly, most of these guarantees are probabilistic in the sense that they are only valid for randomly generated sparse arrays and may not hold for deterministic geometries such as the nested array [18]. Our recent work [20] addressed these shortcomings, and provided guarantees (necessary and sufficient conditions) for single-snapshot rank minimization to successfully interpolate the missing measurements of a nested array for *arbitrary source separations*.

In a variety of applications, such as super-resolution microscopy, the source signals are often assumed to be *positive*. As a result, positivity as a prior has gained a lot of attention for solving inverse problems such as compressed sensing [24–27], super-resolution [7, 28–33], and phase retrieval [34–36]. Incorporating positivity has actually been shown to lead to stronger theoretical guarantees and improved performance. For instance, in the context of super-resolution, the separation requirements for techniques such as TV norm/atomic norm can be relaxed [29, 30]. Recently, [31] also utilized positivity to reduce the computational complexity of the semi-definite program associated with atomic norm minimization.

Contributions: Motivated by the success of positivity as a prior, we investigate its role in sparse array interpolation. Our main contribution is to prove that rank minimization is redundant in the absence of noise. This is made possible by positivity constraints as well as the nested array geometry, which together act as an *implicit regularizer* for finding the low-rank solution without the need for any *explicit regularization*. In fact, it is possible to provably interpolate the virtual array of the nested array by performing a simple feasibility search under mild conditions on the number of sources. The result holds for arbitrary source configurations and separations. We also extend this framework, which is free from any regularization parameters and agnostic of the model order, to the case of noisy measurements.

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2. PROBLEM FORMULATION

Consider K far-field narrowband sources (with wavelength λ) impinging from directions $\theta_1, \theta_2, \dots, \theta_K$ on a one-dimensional nested array with $P = 2M$ sensors located at $n\lambda/2$, $n \in \mathbb{S}_{\text{nest}}$, where $\mathbb{S}_{\text{nest}}$ is an integer set given by

$$\mathbb{S}_{\text{nest}} = \{m-1\}_{m=1}^M \cup \{m(M+1)-1\}_{m=1}^M.$$

Furthermore, consider a “virtual” uniform linear array (ULA) $\mathbb{U} := \{0, 1, \dots, N-1\}$, $N=M(M+1)$. The set $\mathbb{U}$ also coincides with the non-negative half of the difference set of $\mathbb{S}_{\text{nest}}$ [1]. Throughout the paper, we use the notation $\mathbf{A}_{\mathbb{S}}(\omega) = [\mathbf{a}_{\mathbb{S}}(\omega_1), \dots, \mathbf{a}_{\mathbb{S}}(\omega_K)] \in \mathbb{C}^{P \times K}$ to denote the array manifold matrix for any linear array with P sensors, whose locations (normalized w.r.t. $\lambda/2$) are given by the integer set $\mathbb{S}$. The steering vector corresponding to the normalized frequency $\omega_k := \pi \sin(\theta_k)$ is given by $[\mathbf{a}_{\mathbb{S}}(\omega_k)]_m := e^{jd_m \omega_k}$, where $d_m \in \mathbb{S}$. The measurement of the nested array is given by:

$$\mathbf{y}_{\text{nest}} = \mathbf{A}_{\mathbb{S}_{\text{nest}}}(\omega)\mathbf{x} + \mathbf{n} \quad (1)$$

where $\mathbf{x} \in \mathbb{R}^K$ denotes a real-valued (deterministic) source signal, and $\mathbf{n}$ is an additive noise term. In the absence of noise, we can rewrite (1) as

$$\mathbf{y}_{\text{nest}} = \mathbf{S}_{\text{nest}}\mathbf{y}_{\text{full}}, \quad \mathbf{y}_{\text{full}} := \mathbf{A}_{\mathbb{U}}(\omega)\mathbf{x}, \quad (2)$$

where $\mathbf{A}_{\mathbb{U}}(\omega)$ represents the array manifold of the aforementioned virtual ULA with elements at $\mathbb{U}$, and $\mathbf{S}_{\text{nest}} \in \mathbb{R}^{P \times N}$ is a row-selection matrix given by:

$$[\mathbf{S}_{\text{nest}}]_{i,j} = \begin{cases} 1, & \text{if } d_i + 1 = j, d_i \in \mathbb{S}_{\text{nest}} \\ 0, & \text{otherwise.} \end{cases}$$

The vector $\mathbf{y}_{\text{full}}$ can be viewed as a “virtual measurement” received at the virtual array $\mathbb{U}$, with identical source configurations (same ω and $\mathbf{x}$).

2.1. Review of Rank-Constrained Single Snapshot Interpolation with Nested Arrays

The objective of single-snapshot virtual array interpolation is to recover $\mathbf{y}_{\text{full}}$ by estimating the unobserved measurements corresponding to the (missing) sensor locations at $\mathbb{S}_{\text{nest}}^c := \mathbb{U} \setminus \mathbb{S}_{\text{nest}}$ from a single temporal snapshot $\mathbf{y}_{\text{nest}}$. Let $\mathcal{T}_N : \mathbb{C}^N \rightarrow \mathbb{C}^{N \times N}$ be an operator that returns a $N \times N$ Hermitian Toeplitz matrix with $\mathbf{u}$ as its first column. Consider the following set:

$$\mathcal{T}_{\mathbf{y}_{\text{nest}}} := \{\mathbf{T} = \mathcal{T}_N(\mathbf{u}) \mid \mathbf{u} \in \mathbb{C}^N, \mathbf{S}_{\text{nest}}\mathbf{u} = \mathbf{y}_{\text{nest}}\}. \quad (3)$$

In the noiseless setting ($\mathbf{n} = \mathbf{0}$), $\mathcal{T}_{\mathbf{y}_{\text{nest}}}$ denotes the set of all possible Toeplitz Hermitian matrices in $\mathbb{C}^{N \times N}$ whose first column is consistent with the measurement $\mathbf{y}_{\text{nest}}$ at the indices given by $\mathbb{S}_{\text{nest}}$. It is easy to verify that $\mathcal{T}_N(\mathbf{y}_{\text{full}}) \in \mathcal{T}_{\mathbf{y}_{\text{nest}}}$. Notice that the desired virtual measurement $\mathbf{y}_{\text{full}}$ satisfies:

$$\mathcal{T}_N(\mathbf{y}_{\text{full}}) = \mathbf{A}_{\mathbb{U}}(\omega)\text{diag}(\mathbf{x})\mathbf{A}_{\mathbb{U}}(\omega)^H. \quad (4)$$

The above rank-revealing decomposition shows that matrix $\mathcal{T}_N(\mathbf{y}_{\text{full}})$ is of rank- K , whenever $K < P$. This has enabled the use of “low-rank” as a guiding factor for sparse array interpolation, by utilizing ideas from matrix completion [17, 19, 21]. However, the theoretical guarantees for (noiseless) matrix completion (that relies on certain notions of incoherence and random sampling strategies) do not directly apply to deterministic sparse arrays [17–19]. We recently bridged this theoretical gap by developing the first guarantees for exact single-snapshot sparse array interpolation with nested arrays [20]. In particular, we considered the following rank-minimization problem:

$$\min_{\mathbf{T} \in \mathbb{C}^{N \times N}} \text{rank}[\mathbf{T}] \quad \text{subject to } \mathbf{T} \in \mathcal{T}_{\mathbf{y}_{\text{nest}}}. \quad (\text{P1})$$

In [20], we showed that in absence of noise, the solution to (P1) can exactly interpolate the virtual array of the nested array irrespective of the source configuration, provided $K \leq M$, as summarized in the following theorem:

Theorem 1. *Consider the measurement model (1) with $\mathbf{n} = \mathbf{0}$. For every ω and $\mathbf{x} \in \mathbb{R}^K$, (P1) has a unique solution $\mathbf{u}^*$ satisfying $\mathbf{u}^* = \mathbf{y}_{\text{full}} = \mathbf{A}_{\mathbb{U}}(\omega)\mathbf{x}$, if and only if $K \leq M$.*

In recent times, the role of non-negativity in inverse problems such as super-resolution [7, 28–32] and phase retrieval [34–36] has been investigated, since non-negative constraints can themselves act as implicit regularizers, and it may not be necessary to enforce any other priors (such as sparsity). Motivated by these findings, we ask the following question:

(Q): “When $\mathbf{x}$ is non-negative (or positive), is it possible to avoid rank minimization and yet perform exact interpolation from measurements obtained using a nested array?”

Addressing this question will enable us to specialize Theorem 1 to the case of positive sources, and offer additional insights into sparse array interpolation with nested arrays.

3. POWER OF POSITIVITY: OPTIMIZATION-LESS SINGLE SNAPSHOT INTERPOLATION

If $\mathbf{x}$ is not positive, it is not possible to remove the rank constraint and yet hope to exactly recover $\mathbf{y}_{\text{full}}$ from $\mathbf{y}_{\text{nest}}$. Even in the absence of noise, the task of exact interpolation is ill-posed, as there are infinitely many choices to “fill-in” the missing elements in $\mathbb{S}_{\text{nest}}^c$. For instance, a trivial interpolating solution is obtained using zero-padded signal $\mathbf{y}_{\text{zero}} \in \mathbb{C}^N$ defined as:

$$[\mathbf{y}_{\text{zero}}]_i = \begin{cases} [\mathbf{y}_{\text{nest}}]_i, & \text{if } i-1 \in \mathbb{S}_{\text{nest}} \\ 0, & \text{otherwise.} \end{cases}$$

Matrix $\mathcal{T}(\mathbf{y}_{\text{zero}})$ belongs to the set $\mathcal{T}_{\mathbf{y}_{\text{nest}}}$, but its rank can exceed K . Therefore, it is not possible to interpolate the virtual array by arbitrarily picking elements from the set $\mathcal{T}_{\mathbf{y}_{\text{nest}}}$, and we must specifically look for a low-rank solution.

We will now show that when $\mathbf{x}$ is positive, rank constraints indeed become redundant. Consider the noiseless measurement model (1) with a positive source signal $\mathbf{x} \in \mathbb{R}_+^K$. By (4),

the interpolated Toeplitz matrix $\mathcal{T}_N(\mathbf{y}_{\text{full}})$ is positive semidefinite (PSD), i.e., $\mathcal{T}_N(\mathbf{y}_{\text{full}}) \succeq \mathbf{0}$. This additional information can be incorporated into our search for the desired interpolated Toeplitz matrix. To this end, define a new set:

$$\mathcal{T}_{\mathbf{y}_{\text{nest}}}^+ := \mathcal{T}_{\mathbf{y}_{\text{nest}}} \bigcap \mathbf{S}_+^N, \quad (5)$$

where $\mathbf{S}_+^N$ denote the set of (Hermitian) PSD matrices in $\mathbb{C}^{N \times N}$. The set $\mathcal{T}_{\mathbf{y}_{\text{nest}}}^+$ represents a collection of all PSD Toeplitz matrices $\mathcal{T}_N(\mathbf{u})$ such that $\mathbf{u}$ is an interpolating solution. Without this additional positive semidefinite constraint, the interpolating solution is not unique (as argued above) and the set $\mathcal{T}_{\mathbf{y}_{\text{nest}}}$ contains many ambiguous solutions. Remarkably however, the set $\mathcal{T}_{\mathbf{y}_{\text{nest}}}^+$ is a singleton, which makes rank-minimization in (P1) redundant when $\mathbf{x}$ is positive.

Theorem 2. *Consider the noiseless measurement model (1) with $\mathbf{x} > \mathbf{0}$ and $\mathbf{n} = \mathbf{0}$. If $K \leq M$, then the set $\mathcal{T}_{\mathbf{y}_{\text{nest}}}^+ = \{\mathcal{T}_N(\mathbf{y}_{\text{full}})\}$ is singleton.*

Proof. For positive sources, the interpolated Toeplitz matrix $\mathcal{T}_N(\mathbf{y}_{\text{full}}) \in \mathcal{T}_{\mathbf{y}_{\text{nest}}}^+$. Suppose there exists another matrix $\mathbf{T}' (\neq \mathcal{T}_N(\mathbf{y}_{\text{full}})) \in \mathcal{T}_{\mathbf{y}_{\text{nest}}}^+$. As $\mathbf{T}'$ is a PSD Toeplitz matrix, it has the following decomposition due to Caratheodory's Theorem, [37]

$$\mathbf{T}' = \mathbf{A}_{\mathbb{U}}(\boldsymbol{\omega}') \boldsymbol{\Lambda} \mathbf{A}_{\mathbb{U}}^H(\boldsymbol{\omega}'),$$

where $\mathbf{A}_{\mathbb{U}}(\boldsymbol{\omega}') \in \mathbb{C}^{N \times K'}$ is a Vandermonde matrix and $\boldsymbol{\Lambda}$ is a diagonal matrix with positive entries. Let $\mathbf{T}'_{M+1}$ denote the top-left $(M+1) \times (M+1)$ block of $\mathbf{T}'$:

$$\mathbf{T}'_{M+1} = [\mathbf{I}_{M+1} \ \mathbf{0}_{M+1 \times N-M-1}] \mathbf{T}' [\mathbf{I}_{M+1} \ \mathbf{0}_{M+1 \times N-M-1}]^T.$$

Let $\mathbf{A}_{\mathbb{S}_1}(\boldsymbol{\omega}') \in \mathbb{C}^{(M+1) \times K'}$ denote the inner ULA manifold matrix. We can write $\mathbf{T}'_{M+1}$ as:

$$\mathbf{T}'_{M+1} = \mathbf{A}_{\mathbb{S}_1}(\boldsymbol{\omega}') \boldsymbol{\Lambda} \mathbf{A}_{\mathbb{S}_1}^H(\boldsymbol{\omega}'). \quad (6)$$

Since $\mathbf{A}_{\mathbb{S}_1}(\boldsymbol{\omega}')$ is a Vandermonde matrix with distinct elements in $\boldsymbol{\omega}'$, $\text{rank}(\mathbf{T}'_{M+1}) = \min(K', M+1)$. From the feasibility constraint $\mathbf{T}' \in \mathcal{T}_{\mathbf{y}_{\text{nest}}}^+$, we have $\mathbf{T}' = \mathcal{T}_N(\mathbf{u})$ where $\mathbf{S}_{\text{nest}} \mathbf{u} = \mathbf{y}_{\text{nest}}$. Hence, by (4):

$$\mathbf{T}'_{M+1} = \mathcal{T}_{M+1}(\mathbf{y}_{\mathbb{S}_1}) = \mathbf{A}_{\mathbb{S}_1}(\boldsymbol{\omega}) \text{diag}(\mathbf{x}) \mathbf{A}_{\mathbb{S}_1}^H(\boldsymbol{\omega}), \quad (7)$$

where $\mathbf{y}_{\mathbb{S}_1} = \mathbf{A}_{\mathbb{S}_1}(\boldsymbol{\omega}) \mathbf{x}$. Since $K \leq M$, $\text{rank}(\mathcal{T}_{M+1}(\mathbf{y}_{\mathbb{S}_1})) = \text{rank}(\mathbf{A}_{\mathbb{S}_1}(\boldsymbol{\omega})) = \min(K, M+1) = K$. This implies $\text{rank}(\mathbf{T}'_{M+1}) = \min(K', M+1) = K$, and hence, $K' = K$.

Next, we show that $\boldsymbol{\omega} = \boldsymbol{\omega}'$. Suppose there exists i such that $\omega'_i \notin \{\omega_1, \dots, \omega_K\}$. From (6) and (7), we have $\mathbf{a}_{\mathbb{S}_1}(\omega'_i) \in \mathcal{R}(\mathbf{T}'_{M+1}) = \mathcal{R}(\mathbf{A}_{\mathbb{S}_1}(\boldsymbol{\omega}))$. This implies that $\bar{\mathbf{A}} = [\mathbf{A}_{\mathbb{S}_1}(\boldsymbol{\omega}), \mathbf{a}_{\mathbb{S}_1}(\omega'_i)] \in \mathbb{C}^{M+1 \times K+1}$ must be rank-deficient. However, since $\bar{\mathbf{A}}$ is a Vandermonde matrix with all $\omega_1, \dots, \omega_K, \omega'_i$ being distinct, it will be full column-rank. Therefore, we must have $\omega'_i \in \{\omega_1, \dots, \omega_K\}$ for every i which implies $\boldsymbol{\omega} = \boldsymbol{\omega}'$, since $K' = K$. Therefore, when $K \leq M$, $\mathcal{T}_{\mathbf{y}_{\text{nest}}}^+ = \{\mathcal{T}_N(\mathbf{y}_{\text{full}})\}$. $\square$

Theorem 2 shows that thanks to the positivity constraint, it is possible to uniquely identify the subspace corresponding to the full virtual ULA by merely solving a feasibility search problem *without any explicit rank minimization*:

$$\text{find } \mathbf{T} \in \mathbb{C}^{N \times N} \text{ subject to } \mathbf{T} \in \mathcal{T}_{\mathbf{y}_{\text{nest}}}^+. \quad (\text{FEAS+})$$

The feasibility search (FEAS+) is a convex problem and unlike the non-convex rank-minimization, it can be solved in a computationally efficient manner.

Positivity as Implicit Regularization: In a recent work [31], it is shown that under a similar positive source assumptions it is possible to relax the “minimum separation” requirement for the atomic norm minimization regularization scheme. In contrast, our Theorem 2 indicates that owing to the structured nested array geometry no explicit atomic norm minimization is needed. The positive semi-definite constraint alone suffices as an implicit regularizer and is capable of automatically finding the low-rank solution in the noiseless setting, regardless of the minimum separation between sources.

3.1. Extension to Noisy Measurement Model

In the previous section, Theorem 2 showed that a simple feasibility search guarantees perfect interpolation of the nested array measurement in the absence of noise. In the presence of noise, this feasibility search can be modified by expanding the feasible set $\mathcal{T}_{\mathbf{y}_{\text{nest}}}^+$ based on the noise level, ϵ , to

$$\mathcal{T}_{\mathbf{y}_{\text{nest}}, \epsilon}^+ := \mathcal{T}_{\mathbf{y}_{\text{nest}}, \epsilon} \bigcap \mathbf{S}_+^N, \quad (8)$$

where $\mathcal{T}_{\mathbf{y}_{\text{nest}}, \epsilon} := \{\mathbf{T} = \mathcal{T}_N(\mathbf{u}), \text{ s.t. } \|\mathbf{S}_{\text{nest}} \mathbf{u} - \mathbf{y}_{\text{nest}}\|_2 \leq \epsilon\}$. The set $\mathcal{T}_{\mathbf{y}_{\text{nest}}, \epsilon}^+$ comprises all Toeplitz PSD matrices whose first columns are within a ball of radius ϵ over the indices determined by the nested array geometry. Specifically, when $\epsilon = 0$ the set reduces to $\mathcal{T}_{\mathbf{y}_{\text{nest}}}^+$. Given an upper bound on the noise level $\|\mathbf{n}\|_2 \leq \epsilon$, the virtual array can be interpolated by simply picking a point from the convex set $\mathcal{T}_{\mathbf{y}_{\text{nest}}, \epsilon}^+$:

$$\text{find } \mathbf{T} \in \mathbb{C}^{N \times N} \text{ subject to } \mathbf{T} \in \mathcal{T}_{\mathbf{y}_{\text{nest}}, \epsilon}^+. \quad (\text{N-FEAS+})$$

Another approach (which is also regularization-free) would be to solve the following least squares problem:

$$\hat{\mathbf{y}}_{\text{full}, \ell_2} := \underset{\mathbf{y} \in \mathbb{C}^N}{\text{argmin}} \|\mathbf{S}_{\text{nest}} \mathbf{y} - \mathbf{y}_{\text{nest}}\|_2^2 \text{ s.t. } \mathcal{T}_N(\mathbf{y}) \succeq \mathbf{0}. \quad (9)$$

The interpolation scheme in (9) offers the distinct advantage of being agnostic to the noise level ϵ and model order K .

Fig. 1 visualizes the geometry of the feasible set $\mathcal{T}_{\mathbf{y}_{\text{nest}}}^+$ and $\mathcal{T}_{\mathbf{y}_{\text{nest}}, \epsilon}^+$ for an array with two elements $\mathbb{S} = \{0, 1\}$ and a single source with amplitude $x_1 = 2$ and angle $\theta_1 = 0$. We plot the set of points $\mathbf{u} = [u_0, u_1, u_2]^T$ for which $\mathcal{T}_3(\mathbf{u}) \succeq \mathbf{0}$ (in red) and $\|[u_0, u_1]^T - \mathbf{y}_{\text{nest}}\|_2 \leq \epsilon$ (in blue) for two specific noise levels ($\epsilon = 0$ and $1/\sqrt{2}$). Set $\mathcal{T}_{\mathbf{y}_{\text{nest}}, \epsilon}^+$ remains bounded due to the geometry of the PSD cone, and reduces to a singleton when $\epsilon = 0$. This suggests that the interpolation error can be controlled by performing a feasibility search without requiring any rank/sparsity minimization scheme, as will be shown in the numerical simulations of the next section.

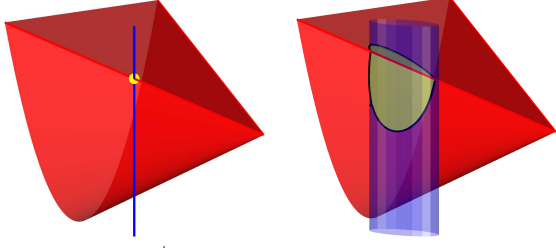


Fig. 1: Feasible set $\mathcal{T}_{\mathbf{y}_{\text{nest}}, \epsilon}^+$ is the intersection of the PSD Toeplitz cone (red) and all vectors $\mathbf{u} \in \mathbb{R}^3$ satisfying $\|\mathbf{S}_{\text{nest}} \mathbf{u} - \mathbf{y}_{\text{nest}}\|_2 \leq \epsilon$ (blue) with noise level $\epsilon = 0$ (left) and $\epsilon = 1/\sqrt{2}$ (right). The intersection, which is shaded in yellow, becomes a single point for $\epsilon = 0$.

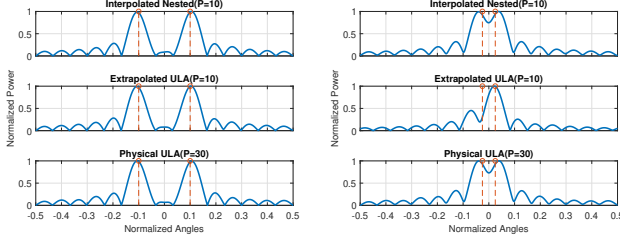


Fig. 2: Beamforming using synthesized virtual array of nested array (top) and ULA (middle), and ULA with same number of physical sensors as the virtual array (bottom). The $K = 2$ sources are separated by $\Delta = 0.2$ (left) and $\Delta = 0.05$ (right). The noise level is $\varsigma = 0.1$.

4. NUMERICAL RESULTS

We consider two equipower sources with angles $\omega = \{-\frac{\Delta}{2}, \frac{\Delta}{2}\}$ and amplitudes $\mathbf{x} = [1, 1]^T$. The number of physical sensors of the nested array and ULA is $P = 10$. The noise is assumed bounded and zero-mean. In particular, the real and imaginary parts of the entries of noise vector $\mathbf{n} \in \mathbb{C}^P$ are independently and uniformly distributed, such that $|n_i| \leq \varsigma \forall i$ and $\|\mathbf{n}\|_2 \leq \varsigma\sqrt{P}$. We set $\varsigma = 0.1$ and consider the minimum ℓ_2 estimator of $\mathbf{y}_{\text{full}} \in \mathbb{C}^N$ defined in (9).

Fig. 2 shows the beam pattern of the interpolated nested array and extrapolated ULA for two different source separations. For comparison, we also plot the beam pattern of a ULA with $P = 30$ sensors (same as the aperture of the nested array). The interpolated nested array detects the two closely placed sources (right), whereas the extrapolated ULA only succeeds when the separation is sufficiently large (left).

Fig. 3 shows the per-sensor interpolation error, defined as $\text{RMSE}[n] = (10^{-3} \sum_{i=1}^{10^3} |\mathbf{y}_{\text{full}} - \hat{\mathbf{y}}_{\text{full}, \ell_2}^{(i)}|_n^2)^{1/2}$, where $n = 1, 2, \dots, N$ is the index of the n th virtual sensor and $\hat{\mathbf{y}}_{\text{full}, \ell_2}^{(i)}$ is the estimate of $\mathbf{y}_{\text{full}}$ in the i th Monte Carlo trial. The nested array achieves a lower error for the interpolated virtual sensors compared to the extrapolated virtual sensors of the ULA for different noise levels ς (left) and source separations Δ (right). Interestingly, the interpolation error may decrease with decreasing source separation despite deteriorating angle estimation performance—cf. Fig. 2 (middle) and Fig. 3 (right).

For the final experiment, we solve the feasibility search (N-FEAS+) by designing an estimator $\hat{\mathbf{y}}_{\text{full}, \epsilon}$ that randomly picks points from $\mathcal{T}_{\mathbf{y}_{\text{nest}}, \epsilon}^+$, with $\epsilon = \varsigma\sqrt{P}$. In particular, the random sampling strategy is implemented by solving

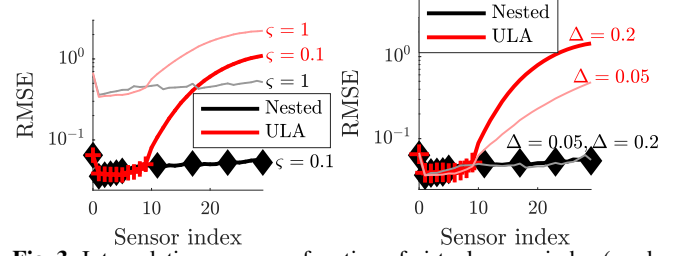


Fig. 3: Interpolation error as a function of virtual sensor index (markers denote physical sensors) for different noise levels ς (left) and source separations Δ (right). The interpolation of the nested array is stable compared to the extrapolation of the ULA.

$\hat{\mathbf{y}}_{\text{full}, \epsilon} := \arg\min_{\mathbf{u}} \|\mathbf{G}\mathbf{u}\|_2$ subject to $\mathcal{T}_N(\mathbf{u}) \in \mathcal{T}_{\mathbf{y}_{\text{nest}}, \epsilon}^+$, where $\mathbf{G} \in \mathbb{C}^{N \times N}$ is a random matrix with i.i.d. complex circularly symmetric Gaussian entries. Given $\mathbf{y}_{\text{nest}}$, estimator $\hat{\mathbf{y}}_{\text{full}, \epsilon}$ is thus a random vector (whereas $\hat{\mathbf{y}}_{\text{full}, \ell_2}$ is deterministic). Fig. 4 shows histograms of the interpolation error of $\hat{\mathbf{y}}_{\text{full}, \epsilon}$ for 10^4 random draws of $\mathbf{G}$, which correspond to random points in $\mathcal{T}_{\mathbf{y}_{\text{nest}}, \epsilon}^+$. We consider the nested array and a single noise realization where $\varsigma \in \{0, 0.3, 1\}$. For comparison, we also plot the histogram of the estimation error *without* the PSD constraint (error of a random point in $\mathcal{T}_{\mathbf{y}_{\text{nest}}, \epsilon}$). When the PSD constraint is omitted, the empirical error distribution has a significantly larger mean and variance than when the PSD constraint is imposed. As the noise level ς decreases, the error distribution in the PSD case concentrates around zero. This is consistent with Theorem 1, which states that in the absence of noise ($\varsigma = 0$), the feasible set (5) is a singleton. Note that without the PSD constraint, the interpolation error remains large even when there is no noise.

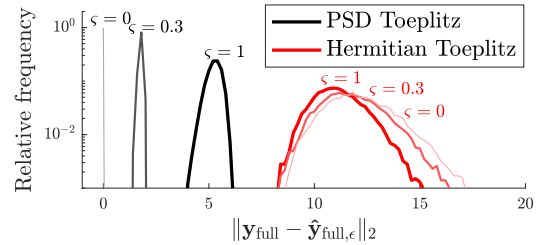


Fig. 4: Interpolation error histograms of randomly chosen points in the set $\mathcal{T}_{\mathbf{y}_{\text{nest}}, \epsilon}^+$ (black) and $\mathcal{T}_{\mathbf{y}_{\text{nest}}, \epsilon}$ (red). Enforcing the PSD constraint results in significantly lower mean error and variance, which decrease as a function of noise level ς .

5. CONCLUSION

This paper showed that it is possible to successfully interpolate the missing measurements of the nested array by a convex feasibility search with a single temporal snapshot, when the source signals are positive. In particular, the set of Toeplitz PSD matrices whose first column is consistent with the measurements at the indices of the nested array sensors is a singleton. Our simulations demonstrate that the feasible set remains bounded in case of noisy measurements, and that noisy nested array interpolation outperforms ULA extrapolation. In future, we plan to theoretically analyze positive virtual array interpolation in the presence of noise.

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