

# Polynomial bounds for chromatic number

## II. Excluding a star-forest

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### **Abstract**

The Gyárfás-Sumner conjecture says that for every forest  $H$ , there is a function  $f_H$  such that if  $G$  is  $H$ -free then  $\chi(G) \leq f_H(\omega(G))$  (where  $\chi, \omega$  are the chromatic number and the clique number of  $G$ ). Louis Esperet conjectured that, whenever such a statement holds,  $f_H$  can be chosen to be a polynomial. The Gyárfás-Sumner conjecture is only known to be true for a modest set of forests  $H$ , and Esperet's conjecture is known to be true for almost no forests. For instance, it is not known when  $H$  is a five-vertex path. Here we prove Esperet's conjecture when each component of  $H$  is a star.

# 1 Introduction

The Gyárfás-Sumner conjecture [6, 20] asserts:

**1.1 Conjecture:** *For every forest  $H$ , there is a function  $f$  such that  $\chi(G) \leq f(\omega(G))$  for every  $H$ -free graph  $G$ .*

(We use  $\chi(G)$  and  $\omega(G)$  to denote the chromatic number and the clique number of a graph  $G$ , and a graph is  $H$ -free if it has no induced subgraph isomorphic to  $H$ .) This remains open in general, though it has been proved for some very restricted families of trees (see, for example, [1, 7, 8, 9, 11, 13, 14]).

A class  $\mathcal{C}$  of graphs is  $\chi$ -bounded if there is a function  $f$  such that  $\chi(G) \leq f(\omega(G))$  for every graph  $G$  that is an induced subgraph of a member of  $\mathcal{C}$  (see [15] for a survey). Thus the Gyárfás-Sumner conjecture asserts that, for every forest  $H$ , the class of all  $H$ -free graphs is  $\chi$ -bounded. Esperet [5] conjectured that every  $\chi$ -bounded class is *polynomially  $\chi$ -bounded*, that is,  $f$  can be chosen to be a polynomial. Neither conjecture has been settled in general. There is a survey by Schiermeyer and Randerath [19] on related material.

In particular, what happens to Esperet's conjecture when we exclude a forest? For which forests  $H$  can we show the following?

**1.2 Esperet's conjecture:** *There is a polynomial  $f_H$  such that  $\chi(G) \leq f_H(\omega(G))$  for every  $H$ -free graph  $G$ .*

Not for very many forests  $H$ , far fewer than the forests that we know satisfy 1.1. For instance, 1.2 is not known when  $H = P_5$ , the five-vertex path. (This case is of great interest, because it would imply the Erdős-Hajnal conjecture [3, 4, 2] for  $P_5$ , and the latter is currently the smallest open case of the Erdős-Hajnal conjecture.)

We remark that, if in 1.2 we replace  $\omega(G)$  by  $\tau(G)$ , defined to be the maximum  $t$  such that  $G$  contains  $K_{t,t}$  as a subgraph, then all forests satisfy the modified 1.2. More exactly, the following is shown in [16]:

**1.3** *For every forest  $H$ , there is a polynomial  $f_H$  such that  $\chi(G) \leq f_H(\tau(G))$  for every  $H$ -free graph  $G$ .*

One difficulty with 1.2 is that we cannot assume that  $H$  is connected, or more exactly, knowing that each component of  $H$  satisfies 1.2 does not seem to imply that  $H$  itself satisfies 1.2. For instance, while  $H = P_4$  satisfies 1.2, we do not know the same when  $H$  is the disjoint union of two copies of  $P_4$ .

As far as we are aware, the only forests that were already known to satisfy 1.2 are those of the following three results, and their induced subgraphs (a *star* is a tree in which one vertex is adjacent to all the others):

**1.4** *The forest  $H$  satisfies 1.2 if either:*

- *$H$  is the disjoint union of copies of  $K_2$  (S. Wagon [21]); or*
- *$H$  is the disjoint union of  $H'$  and a copy of  $K_2$ , and  $H'$  satisfies 1.2 (I. Schiermeyer [18]); or*
- *$H$  is obtained from a star by subdividing one edge once (X. Liu, J. Schroeder, Z. Wang and X. Yu [12]).*

In the next paper of this series [17] we will show a strengthening of the third result of 1.4, that is, 1.2 is true when  $H$  is a “double star”, a tree with two internal vertices, the most general tree that does not contain a five-vertex path. Our main theorem in this paper is a strengthening of the second result of 1.4:

**1.5** *If  $H$  is the disjoint union of  $H'$  and a star, and  $H'$  satisfies 1.2, then  $H$  satisfies 1.2.*

A *star-forest* is a forest in which every component is a star. From 1.5 and the result of [17], we deduce

**1.6** *If  $H'$  is a double star, and  $H$  is the disjoint union of  $H'$  and a star-forest, then  $H$  satisfies 1.2.*

As far as we know (although it seems unlikely), these might be all the forests that satisfy 1.2.

## 2 The proof

We will need the following well-known version of Ramsey’s theorem:

**2.1** *For  $k \geq 1$  an integer, if a graph  $G$  has no stable subset of size  $k$ , then*

$$|V(G)| \leq \omega(G)^{k-1} + \omega(G)^{k-2} + \cdots + \omega(G).$$

*Consequently  $|V(G)| < \omega(G)^k$  if  $\omega(G) > 1$ .*

**Proof.** The claim holds for  $k \leq 2$ , so we assume that  $k \geq 3$  and the result holds for  $k - 1$ . Let  $X$  be a clique of  $G$  of cardinality  $\omega(G)$ , and for each  $x \in X$  let  $W_x$  be the set of vertices nonadjacent to  $x$ . From the inductive hypothesis,  $|W_x| \leq \omega(G)^{k-2} + \cdots + \omega(G)$  for each  $x$ ; but  $V(G)$  is the union of the sets  $W_x \cup \{x\}$  for  $x \in X$ , and the result follows by adding. This proves 2.1. ■

If  $X \subseteq V(G)$ , we denote the subgraph induced on  $X$  by  $G[X]$ . When we are working with a graph  $G$  and its induced subgraphs, it is convenient to write  $\chi(X)$  for  $\chi(G[X])$ . Now we prove 1.5, which we restate:

**2.2** *If  $H'$  satisfies 1.2, and  $H$  is the disjoint union of  $H'$  and a star, then  $H$  satisfies 1.2.*

**Proof.**  $H$  is the disjoint union of  $H'$  and some star  $S$ ; let  $S$  have  $k + 1$  vertices. Since  $H'$  satisfies 1.2, and  $\chi(G) = \omega(G)$  for all graphs with  $\omega(G) \leq 1$ , there exists  $c'$  such that  $\chi(G) \leq \omega(G)^{c'}$  for every  $H'$ -free graph  $G$ . Choose  $c \geq k + 2$  such that

$$x^c - (x - 1)^c \geq 1 + x^{k+2} + x^{k(k+2)+c'}$$

for all  $x \geq 2$  (it is easy to see that this is possible).

Let  $G$  be an  $H$ -free graph, and write  $\omega(G) = \omega$ ; we will show that  $\chi(G) \leq \omega^c$  by induction on  $\omega$ . If  $\omega = 1$  then  $\chi(G) = 1$  as required, so we assume that  $\omega \geq 2$ . Let  $n = \omega^{k+1}$ . If every vertex of  $G$  has degree less than  $\omega^c$ , then the result holds as we can colour greedily, so we assume that some vertex  $v$  has degree at least  $\omega^c$ . Let  $N$  be the set of all neighbours of  $v$  in  $G$ . Let  $X_1$  be the largest clique contained in  $N$ ; let  $X_2$  be the largest clique contained in  $N \setminus X_1$ ; and in general, let  $X_i$  be the largest clique contained in  $N \setminus (X_1 \cup \cdots \cup X_{i-1})$ . Since  $|N| \geq \omega^c \geq n\omega$  (because  $c \geq k + 2$ ), it follows

that  $X_1, \dots, X_n \neq \emptyset$ . Let  $X = X_1 \cup \dots \cup X_n$ , and  $X_0 = N \setminus X$ , and  $t = |X_n|$ . Thus  $1 \leq t \leq \omega - 1$  (because  $\omega(G[N]) < \omega$ ).

$$(1) \chi(N \cup \{v\}) \leq t^c + n\omega.$$

From the choice of  $X_n$ , it follows that the largest clique of  $G[X_0]$  has cardinality at most  $t < \omega$ . From the inductive hypothesis,  $\chi(X_0) \leq t^c$ , and since  $X \cup \{v\}$  has cardinality at most  $n\omega$ , it follows that  $\chi(N \cup \{v\}) \leq t^c + n\omega$ . This proves (1).

For each stable set  $Y \subseteq X$  with  $|Y| = k$ , let  $A_Y$  be the set of vertices in  $V(G) \setminus (N \cup \{v\})$  that have no neighbour in  $Y$ . Let  $A$  be the union of all the sets  $A_Y$ , and  $B = V(G) \setminus (A \cup N \cup \{v\})$ .

$$(2) \chi(A) \leq (n\omega)^k \omega^{c'}.$$

For each choice of  $Y$ ,  $G[A_Y]$  is  $H'$ -free (because  $Y \cup \{v\}$  induces a copy of  $S$  with no edges to  $A_Y$ ), and so  $\chi(A_Y) \leq \omega^{c'}$ . Since there are at most  $|X|^k \leq (n\omega)^k$  choices of  $Y$ , it follows that the union  $A$  of all the sets  $A_Y$  has chromatic number at most  $(n\omega)^k \omega^{c'}$ . This proves (2).

(3) For each  $b \in B$ ,  $b$  has fewer than  $\omega^k$  non-neighbours in  $X$ .

Let  $Z$  be the set of vertices in  $X$  nonadjacent to  $b$ . Since  $b \notin A$ ,  $G[Z]$  has no stable set of cardinality  $k$ ; and since it also has no clique of cardinality  $\omega$ , 2.1 implies that  $|Z| \leq (\omega - 1)^k < \omega^k$ . This proves (3).

$$(4) \chi(B) \leq (\omega - t)^c.$$

Suppose that  $C \subseteq B$  is a clique with  $|C| = \omega - t + 1$ . For each  $c \in C$ , (3) implies that  $c$  has a nonneighbour in fewer than  $\omega^k$  of the cliques  $X_1, \dots, X_n$ ; and so fewer than  $(\omega - t + 1)\omega^k$  of the cliques  $X_1, \dots, X_n$  contain a vertex with a non-neighbour in  $C$ . Since  $(\omega - t + 1)\omega^k \leq \omega^{k+1} = n$ , there exists  $i \in \{1, \dots, n\}$  such that every vertex in  $X_i$  is adjacent to every vertex of  $C$ , and so  $C \cup X_i$  is a clique. Since  $|X_i| \geq |X_n| = t$ , it follows that  $|C \cup X_i| > \omega$ , a contradiction. Thus there is no such clique  $C$ , and so  $\omega(G[B]) \leq \omega - t$ ; and from the inductive hypothesis (since  $t > 0$ ) it follows that  $\chi(B) \leq (\omega - t)^c$ . This proves (4).

From (1), (2), (4) we deduce that

$$\chi(G) \leq \chi(N \cup \{v\}) + \chi(A) + \chi(B) \leq t^c + n\omega + (n\omega)^k \omega^{c'} + (\omega - t)^c.$$

Since  $1 \leq t \leq \omega - 1$  and  $c \geq 1$ , it follows that  $t^c + (\omega - t)^c \leq 1 + (\omega - 1)^c$ , and so

$$\chi(G) \leq 1 + n\omega + (n\omega)^k \omega^{c'} + (\omega - 1)^c \leq \omega^c$$

from the choice of  $c$  and since  $\omega \geq 2$ . This proves 1.5. ■

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