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The Kottwitz conjecture for unitary PEL-type Rapoport–Zink spaces

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Abstract. In this paper we study the cohomology of PEL-type Rapoport–Zink spaces associated to unramified unitary similitude groups over $\mathbb{Q}_p$ in an odd number of variables. We extend the results of Kaletha–Minguez–Shin–White and Mok to construct a local Langlands correspondence for these groups and prove an averaging formula relating the cohomology of Rapoport–Zink spaces to this correspondence. We use this formula to prove the Kottwitz conjecture for the groups we consider.

Contents

1. Introduction

Shimura varieties play an important role in the global Langlands program, which predicts a link between automorphic representations of linear algebraic groups and Galois representations. Rapoport and Zink ([45]) introduced p-adic analogues of Shimura varieties defined as moduli spaces of p-divisible groups with additional structures. The ℓ-adic ($\ell \neq p$) cohomology of these spaces should provide local incarnations of the Langlands correspondences and

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this is the subject of the Kottwitz conjecture ([44, Conjecture 7.3]). The goal of this paper is to prove the Kottwitz conjecture in the case of PEL-type Rapoport–Zink spaces associated to unramified unitary similitude groups over $\mathbb{Q}_p$ in an odd number of variables. Prior to our work, the conjecture was proven for Lubin-Tate spaces by [9, 10, 19]. By duality [13, 15, 47], the conjecture is also known in the Drinfeld case. The case of unramified EL-type Rapoport–Zink spaces was proven by [14, 53] and the case of unramified unitary PEL-type spaces of signature $(1, n-1)$ was proven by [42]. Hansen, Kaletha, and Weinstein ([18]) have proven, for all local shtuka spaces, a weakened form of the Kottwitz conjecture where, in particular, they do not consider the action of the Weil group.

We now describe our results in more detail. One considers triples (G, b, μ) such that G is a connected reductive group over $\mathbb{Q}_p$ and μ is a minuscule cocharacter of G and b is an element of the Kottwitz set $B(\mathbb{Q}_p, G, \mu)$. Then Rapoport–Zink attach to triples (G, b, μ) of PEL-type a tower of rigid spaces $\mathcal{M}_{K_p}$ indexed by compact open subgroups $K_p \subset G(\mathbb{Q}_p)$.

Attached to the group G and the element b is a connected reductive group J_b that is an inner form of a Levi subgroup of the quasi-split inner form G^* of G. The element b is said to be basic when J_b is in fact an inner form of G. The tower $(\mathcal{M}_{K_p})_{K_p \subset G(\mathbb{Q}_p)}$ carries an action of $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$. For each $i \geq 0$ one can take the compactly supported ℓ-adic cohomology $H^i_c(\mathcal{M}_{K_p}, \overline{\mathbb{Q}}_\ell)$ of $\mathcal{M}_{K_p}$ and hence consider, for each irreducible admissible representation ρ of $J_b(\mathbb{Q}_p)$, the cohomology space

$$H^{i,j}(G, b, \mu)[\rho] := \varinjlim_{K_p} \operatorname{Ext}^j_{J_b(\mathbb{Q}_p)}(H^i_c(\mathcal{M}_{K_p}, \overline{\mathbb{Q}}_\ell), \rho),$$

as a representation of $G(\mathbb{Q}_p) \times W_E$, where W_E is the Weil group of the reflex field E of μ. We now further assume that we can give G the structure of an extended pure inner twist (G, ϱ, z) of G^*. Then the Kottwitz conjecture describes the homomorphism of Grothendieck groups

$$\operatorname{Mant}_{G,b,\mu} : \operatorname{Groth}(J_b(\mathbb{Q}_p)) \to \operatorname{Groth}(G(\mathbb{Q}_p) \times W_E)$$

given by

$$\operatorname{Mant}_{G,b,\mu}(\rho) := \sum_{i,j} (-1)^{i+j} H^{i,j}(G, b, \mu)[\rho](-\dim \mathcal{M}^{an}),$$

in the case when b is basic and ρ is an irreducible admissible representation of $J_b(\mathbb{Q}_p)$ with supercuspidal L-parameter. This means that under the local Langlands correspondence, the L-parameter $\varphi : W_{\mathbb{Q}_p} \times SL_2(\mathbb{C}) \to {}^L J_b$ is trivial when restricted to the $SL_2(\mathbb{C})$-factor and does not factor through a proper Levi subgroup of ${}^L J_b$.

The Kottwitz conjecture states the following:

Conjecture 1.1 (Kottwitz conjecture, [44, Conjecture 7.3]). *For irreducible admissible representations ρ of $J_b(\mathbb{Q}_p)$ with supercuspidal L-parameter, we have the following equality in $\operatorname{Groth}(G(\mathbb{Q}_p) \times W_E)$:*

$$\operatorname{Mant}_{G,b,\mu}(\rho) = \sum_{\pi \in \Pi_\varphi(G, \varrho)} [\pi][\operatorname{Hom}_{S_\varphi}(\iota_{\mathfrak{w}}(\pi)^\vee \otimes \iota_{\mathfrak{w}}(\rho), r_\mu \circ \varphi) \otimes |\cdot|^{\langle \rho_G, \mu \rangle}],$$

where $\Pi_\varphi(G, \varrho)$ is the L-packet of irreducible admissible representations of $G(\mathbb{Q}_p)$ attached to φ.

We have not defined all the notation appearing in this conjecture, but this is described in detail in Section 5. One can extend the conjecture to general G as in [18, Conjecture 1.0.1] using the theory of rigid inner twists.

The main goal of this paper is to prove Conjecture 1.1 when $G = GU$ is an unramified unitary similitude group over $\mathbb{Q}_p$ in an odd number of variables and the datum (GU, b, μ) is basic and of PEL-type. Of course, to make sense of the Kottwitz conjecture for GU, one needs to establish the local Langlands correspondence for this group and show it satisfies an expected list of desiderata. In particular, one needs to check that the L-packet $\Pi_\varphi(G, \varrho)$ has the expected structure determined by a certain group S_φ related to the centralizer of φ in J_b and satisfies the endoscopic character identities.

Prior to this work, such a local Langlands correspondence was known for unitary groups by the works [40, Theorems 2.5.1 and 3.2.1] and [25, Theorem 1.6.1]. These authors work with the arithmetic normalization of the local Langlands correspondence whereby the Artin map is normalized so that uniformizers correspond to arithmetic Frobenius morphisms. However, it is more convenient for us to work with the opposite normalization. In Theorem 2.8 we use Kaletha's results in [23] on the compatibility of local Langlands correspondence and the contragredient to define a local Langlands correspondence for unitary groups under the geometric normalization whereby the Artin map takes uniformizers to geometric Frobenius morphisms.

We next construct a local Langlands correspondence for our groups GU by lifting the result for unitary groups to the group $U \times Z(GU)$ and then descending it to GU. We can carry out such an analysis because the map $U \times Z(GU) \to GU$ is a surjection on $\mathbb{Q}_p$ points for odd unitary groups. This property fails in the even case and is in fact the main reason we consider odd unitary similitude groups. We get:

Theorem 1.2 (Theorems 2.8 and 2.12, Section 3.2). *There exists a local Langlands correspondence for odd unramified unitary similitude groups that satisfies the properties of [25, Theorem 1.6.1], in particular, the endoscopic character identities. By construction, the correspondence is compatible with that of [25,40] via restriction of irreducible admissible representations to $U(\mathbb{Q}_p) \subset GU(\mathbb{Q}_p)$ and projection of Langlands parameters along ${}^L GU \to {}^L U$.*

With the local Langlands correspondence in hand, we can describe our proof of Conjecture 1.1 for the groups we consider. Our method of proof is similar to that of [53] and crucially uses the endoscopic averaging formulas of [6]. We briefly describe these formulas for a connected reductive group G. Suppose that $\mathfrak{e} = (H, s, {}^L\eta)$ is an elliptic endoscopic datum for G. Then there exists a complicated map

$$\mathrm{Red}^{\mathfrak{e}}_b : \mathrm{Groth}^{st}(H(\mathbb{Q}_p)) \to \mathrm{Groth}(J_b(\mathbb{Q}_p)),$$

whose precise definition is given in Section 5.2. We remark that $\mathrm{Groth}^{st}(H(\mathbb{Q}_p))$ denotes the subgroup of $\mathrm{Groth}(H(\mathbb{Q}_p))$ with stable virtual character. Associated to each A-parameter ψ^H of H, we have a stable character denoted by $S\Theta_{\psi^H}$. Suppose that ψ is an A-parameter of G with parameter ψ^H of H such that $\psi = {}^L\eta \circ \psi^H$. Then the endoscopic averaging formula for G is the following identity in $\mathrm{Groth}(G(\mathbb{Q}_p) \times W_E)$:

$$(1.1)\qquad \sum_{b \in B(\mathbb{Q}_p, G, \mu)} \mathrm{Mant}_{G,b,\mu}(\mathrm{Red}^{\mathfrak{e}}_b(S\Theta_{\psi^H})) = \sum_{\pi_p \in \Pi_\psi(G,\varrho)} \sum_{\rho} \langle \pi_p, s \rangle\, \mathrm{tr}(s \mid V) \frac{\dim \rho}{\dim} \ldots$$

where the first sum on the right-hand side is over irreducible factors of the representation r ı and V is the -isotypic part of r ı . The element s equals the image under of .1; I; I/ 2 W_{Q_p} SL_2.C/ SL_2.C/, where the second SL_2.C/ is the Arthur factor.

Note that equation (1.1) is an averaging formula in the sense that it gives a description of $Mant_{G;b;}$ ı Red^e_b summing over the set B.Q_p; G; /. One expects that this summation results in large cancellations of the individual terms (see [8] for a description of this in the G D GL_n case). The phrase “averaging formula” first appeared in this context in [44, footnote 4] while the formula itself was proven for trivial e (i.e. .H; s; ^{L}/ D .G; 1; id/) in the Lubin–Tate case in [19] and the EL-type case in [53]. Equation (1.1) for non-trivial e was first formulated in [6]. For our application to the Kottwitz conjecture, it is crucial that we establish (1.1) in cases where e is non-trivial. In general, one expects to need these endoscopic cases in applications relating to L-parameters with non-singleton L-packets.

The averaging formula is derived in [6] for PEL-type groups under a substantial list of assumptions. In this paper, we verify these assumptions for discrete parameters and hence prove:

Theorem 1.3. For discrete L-parameters of GU, the endoscopic averaging formulas hold.

For the sake of completeness, we briefly recall the strategy of the proof of this result as well as explain the important assumptions. The proof is via global methods. Thus we consider a global unitary similitude group GU defined over Q and a Shimura variety Sh attached to GU which “globalizes” our Rapoport–Zink space. In particular, we have GU_{Q_p} D GU. We deduce the averaging formula by combining the Mantovan formula ([37, Theorem 22], [34, Theorem 6.26])

(1.2) $$H_c.Sh; L/ \text{ D } \sum_{b \in B.Q_p;GU; /} Mant_{GU;b;}.H_c.Ig_b; L//$$

and the trace formulas for Shimura and Igusa varieties ([29, Theorem 7.2], [49, Theorem 13.1], [50, Theorem 7.2]). We denote respectively by H_c .Sh; L/ and H_c .Ig_b; L/ the alternating sums of the compactly supported cohomology of Shimura and Igusa varieties evaluated at the `-adic sheaf L associated to some irreducible algebraic representation of GU.

To carry out this approach, we need to define global A-parameters of GU without referring to the conjectural global Langlands group. We do so by adapting Arthur’s approach (also used in [25,40]) where global parameters correspond to self-dual formal sums of cuspidal automorphic representations of GL_n. For us, a parameter $_{GU}$ of GU consists of a pair . $_U$;/ such that $_U$ is a global parameter of U in the sense of [40] and is an automorphic character of Z.GU/.A/. We attach global A-packets to these parameters in the generic case and prove they satisfy the global multiplicity formula (Proposition 2.26).

One important step in the proof of the averaging formula is the process of stabilization and destabilization of the trace formula for the cohomology of Shimura and Igusa varieties following [29] and [50]. The goal is to relate both sides of equality (1.2) to the global multiplicity formula. In order to achieve this, we need to prove a technical hypothesis concerning stable orbital integrals. More precisely, let H be an endoscopic group of GU and f^H a test function satisfying some local “cuspidality” conditions. We want to show that $ST^H_{ell}.f^H/$ D $ST^H_{disc}.f^H/$, where $ST^H_{ell}.f^H/$ is a sum of stable orbital integrals of H with respect to f^H and $ST^H_{disc}.f^H/$ is, loosely speaking, the traces of all automorphic representations of $H(\mathbb{A})$ evaluated against f^H. This hypothesis is proven in Section 4.2.

Once we have done the destabilization step, we can put everything into equation (1.2) and derive the averaging formula. However, at this point equality (1.2) is still quite complicated and we need to solve a lifting problem in order to extract the desired information. More precisely, for our choice of connected reductive group $\mathbf{GU}$ over $\mathbb{Q}$ such that $\mathbf{GU}_{\mathbb{Q}_p} = GU$ and a discrete L-parameter ϕ_{GU} of GU, we need to construct global L-parameters $\dot\phi_{GU}$ lifting ϕ_{GU} and satis-fying a number of conditions. For instance, we need to precisely control the centralizer group of $\dot\phi_{GU}$ in $\widehat{GU}_{\mathbb{Q}_p}$. These lifting problems are studied in [5, 25] and we adapt their arguments to the unitary similitude case (Section 4.3).

With the endoscopic averaging formula in hand, we prove the Kottwitz conjecture in Section 6. To do so, we observe that $\mathrm{Red}^{e}_{b}(S_{\psi,H}) = 0$ whenever b is non-basic and ψ is super-cuspidal. Hence, in this case, the only term on the left-hand side of the endoscopic averaging formula is the one for b basic. We then combine the formulas for each elliptic e to deduce the conjecture.

2. Automorphic representations

2.1. The groups. Let F be a field of characteristic 0, E a quadratic extension of F and fix an algebraic closure $\overline{F}$. Let $J \in GL_n(F)$ be the anti-diagonal matrix defined by $J = (J_{i,j})$ such that $J_{i,j} = (-1)^{i+1}\delta_{i,n+1-j}$. We define quasi-split groups $U_{E/F}(n)$ and $GU_{E/F}(n)$ over F as follows. Set

$$U_{E/F}(n)(\overline{F}) = GL_n(\overline{F}) \quad \text{and} \quad GU_{E/F}(n)(\overline{F}) = GL_n(\overline{F}) \times GL_1(\overline{F}). \text{ Then}$$

we give $GU_{E/F}(n)(\overline{F})$ an action of $\Gamma_F := \mathrm{Gal}(\overline{F}/F)$ whereby $\sigma \in \Gamma_F$ acts by

$$\sigma_{GU} : \begin{cases} (g,c) \mapsto (\sigma(g), \sigma(c)), & \sigma \in \Gamma_E, \\ (g,c) \mapsto (\sigma(c) J \sigma(g)^{-t} J^{-1}, \sigma(c)), & \sigma \notin \Gamma_E. \end{cases}$$

We get an action of Γ_F on $U_{E/F}(n)(\overline{F})$ by restriction.

We also need to define slightly more general groups $G(U(n_1) \times \cdots \times U(n_k))$ by

$$G(U(n_1) \times \cdots \times U(n_k)) := \{(g_1, \ldots, g_k) \in GU(n_1) \times \cdots \times GU(n_k) : c(g_1) = \cdots = c(g_k)\}.$$

In this paper, we only consider the case where F is one of $\mathbb{Q}_v$ or $\mathbb{Q}$. We now fix for once and for all a prime p and a quadratic imaginary extension $E/\mathbb{Q}$ that is inert at p. At each place v of $\mathbb{Q}$ we get a rank two étale algebra E_v over $\mathbb{Q}_v$. Since we will not change E, we can unambiguously use the notations $\mathbf{U}(n)$ and $\mathbf{GU}(n)$ (resp. $U(n)$ and $GU(n)$) for the global (resp. local, for v that do not split over E) quasi-split groups we have defined. To simplify notation, we will typically refer to inner twists of $U(n)$ (resp. $GU(n)$) by U (resp. GU).

The global groups we consider in this paper will be inner forms of $\mathbf{GU}(n)$ coming from Hermitian forms. Namely, let V be an n-dimensional E-vector space equipped with a Hermitian form $\langle \cdot, \cdot \rangle$. Let $\mathbf{GU}(V)$ (resp. $\mathbf{U}(V)$) be the algebraic groups defined over $\mathbb{Q}$ by

$$\mathbf{GU}(V)(R) = \{(g, c(g)) \in GL(V \otimes_{\mathbb{Q}} R) \times \mathbb{G}_m(R) : \langle gx, gy \rangle = c(g)\langle x, y \rangle, \; x, y \in V \otimes_{\mathbb{Q}} R\}$$

and

$$U(V)(R) = \{g \in GL(V \otimes_{\mathbb{Q}} R) : \langle gx, gy\rangle = \langle x, y\rangle,\ x, y \in V \otimes_{\mathbb{Q}} R\}$$

for any $\mathbb{Q}$-algebra R. To simplify notation, we will often denote these groups by GU (resp U).

In this paper we will assume that n is an odd number and that the localization GU_v at every finite place v is quasi-split. Such groups exist and the quasi-split condition we impose at the finite places does not constrain the isomorphism class of the group at the Archimedean place. Indeed we can define

$$I_{r,s} := \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix},$$

where I_r is the $r \times r$ identity matrix. Then for V an n-dimensional E-vector space,

$$\langle x, y\rangle := \sigma(x)^t I_{r,s}\, y,$$

for $r + s$ odd and $\sigma \in \Gamma_{E/\mathbb{Q}}$ the non-trivial element, gives a unitary similitude group of type (r,s) at the Archimedean place that is quasi-split at the finite places.

Recall that a reductive group G over a number field F arises as an extended pure inner twist of its quasi-split form G^* if there exists a tuple (G, ϱ, z) such that $\varrho : G^* \to G$ is an isomorphism over some finite extension K/F and $z \in Z^1_{bas}(\mathcal{E}_3(K/F), G^*(K))$ is such that for each $\sigma \in \Gamma_{K/F}$ and each $e \in \mathcal{E}_3(K/F)$ projecting to σ, we have

$$\varrho^{-1} \circ \sigma(\varrho) = \mathrm{Int}(z(e)).$$

The set $Z^1_{bas}(\mathcal{E}_3(K/F), G(K))$ is defined as in [32]. We record the following lemma.

Lemma 2.1. *The groups $GU(V)$ (resp. $U(V)$) defined above arise as extended pure inner twists of $GU(n)$ (resp. $U(n)$).*

Proof. In the case that G has connected center, it is known by [32, Proposition 10.4] that all inner twists of G come from extended pure inner twists. In our case, we have

$$Z(U(n)) \cong U(1) \quad \text{and} \quad Z(GU(n)) = \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m,$$

so this is indeed the case. □

We also consider extended pure inner twists for connected reductive groups over $F = \mathbb{Q}_v$. The definition is the same except for we have $z \in Z^1_{bas}(\mathcal{E}_{iso}(K/F), G(K))$ (where $\mathcal{E}_{iso}(K/F)$ is the local gerb $\mathcal{E}(K/F)$ in [32]). As in [32], we define

$$B(F, G)_{bas} := \varinjlim_K H^1_{bas}(\mathcal{E}_3(K/F), G(K))$$

for F a number field and

$$B(F, G)_{bas} := \varinjlim_K H^1_{bas}(\mathcal{E}_{iso}(K/F), G(K)),$$

for F a finite extension of $\mathbb{Q}_v$.

A maximal torus T defined over $\mathbb{Q}_v$ of $GU(n)$ and with maximal split rank is given by the diagonal subgroup. We have

$$T(\mathbb{Q}_v) = \{(t_1, \dots, t_n) \in (E_v^\times)^n : \exists c \in \mathbb{Q}_v^\times\ \forall i \in \{1, \dots, n\},\ t_i\, \sigma(t_{n+1-i}) = c\}.$$

The maximal split subtorus A of T is isomorphic to $(\mathbb{Q}_v^\times)^{\frac{n-1}{2}} \times \mathbb{Q}_v^\times$. The relative Weyl group is

$$W_{re} = (\mathbb{Z}/2\mathbb{Z})^{\frac{n-1}{2}} \rtimes S_{\frac{n-1}{2}};$$

where $S_{\frac{n-1}{2}}$ is the permutation group of $\{1, \dots, \frac{n-1}{2}\}$. The normalizer of A inside $GU(n)(\mathbb{Q}_v)$ is generated by A and the following elements:

$$s_{i,j} = \begin{pmatrix} I^{i,j}_{\frac{n-1}{2}} & & \\ & 1 & \\ & & I^{\frac{n+1}{2}+i,\frac{n+1}{2}+j}_{\frac{n-1}{2}} \end{pmatrix}; \quad A_k = I_n^{\frac{n+1}{2}-k,\frac{n+1}{2}+k};$$

where $1 \le i, j, k \le \frac{n-1}{2}$ and $I_n^{i,j}$ is the matrix with 1 in the positions $(i,j), (j,i)$ and (k,k) for $k \neq i, j$ and 0 elsewhere.

A minimal parabolic subgroup of $GU(n)$ is

$$P_{min} = \left\{ \begin{pmatrix} t_1 & & & & & & * \\ & \ddots & & & & & \\ & & t_{\frac{n-1}{2}} & & & & \\ & & & x & & & \\ & & & & c\,t^{-1}_{\frac{n-1}{2}} & & \\ & & & & & \ddots & \\ 0 & & & & & & c\,t_1^{-1} \end{pmatrix} \; t_i, x \in E_v^\times, \; x\bar{x} = c \right\} \cap GU(n)(\mathbb{Q}_v).$$

From the description of unitary similitude groups, we see that there is an embedding $E_v^\times \hookrightarrow Z(GU(n))(\mathbb{Q}_v)$ given by

$$t \mapsto \operatorname{diag}(t, \dots, t).$$

The tuple $(P_{min}, T, \{E_{i,i+1}\}_{1 \le i \le n-1})$ gives a $\Gamma_{\mathbb{Q}_v}$-stable splitting of $U(n)$.

Note that we can identify $\widehat{GU(n)}$ with $GL_n(\mathbb{C}) \times \mathbb{C}^\times$ and $\widehat{U(n)}$ with $GL_n(\mathbb{C})$. Fix the standard F-splittings of $GL_n(\mathbb{C}) \times \mathbb{C}^\times$ and $GL_n(\mathbb{C})$ consisting of the $(\hat{T}, \hat{B}, \{E_{i,i+1}\}_{1 \le i \le n-1})$, where $\hat{T}$ and $\hat{B}$ are the diagonal subgroup and upper triangular subgroup respectively. The action of the Weil group $W_{\mathbb{Q}_v}$ on these dual groups factors through $\Gamma_{E_v/\mathbb{Q}_v}$ and the non-trivial element σ of $W_{E_v/\mathbb{Q}_v}$ acts via

$$\sigma((g, c)) = (J g^{-t} J^{-1}, c \det(g))$$

and

$$\sigma(g) = (J g^{-t} J)$$

respectively (see [41, p. 38] for details).

A maximal torus defined over $\mathbb{Q}_v$ of $G(U(n_1) \times \dots \times U(n_k))$ with maximal split rank is given by

$$T = \{((t_{1,1}, \dots, t_{1,n_1}), \dots, (t_{k,1}, \dots, t_{k,n_k})) \in (E_v^\times)^{n_1 + \dots + n_k} \mid \exists c \in \mathbb{Q}_v^\times \; \forall i \in \{1, \dots, k\} \; \forall j \in \{1, \dots, n_i\}, \; t_{i,j}\,\bar{t}_{i,n_i+1-j} = c\}.$$

If we denote I, resp. J the set of indexes i such that n_i is odd, resp. even, then a maximal split sub-torus of T is isomorphic to

$$A = \mathbb{Q}_p^\times \times \prod_{i \in I} (\mathbb{Q}_p^\times)^{\frac{n_i-1}{2}} \times \prod_{j \in J} (\mathbb{Q}_p^\times)^{\frac{n_j}{2}}.$$

The relative Weyl group is

$$W_{\mathrm{rel}} = \prod_{i\in I} (\mathbb{Z}/2\mathbb{Z})^{\frac{n_i-1}{2}} \rtimes S_{\frac{n_i-1}{2}} \times \prod_{j\in J} (\mathbb{Z}/2\mathbb{Z})^{\frac{n_j}{2}} \rtimes S_{\frac{n_j}{2}}.$$

Lemma 2.2. We have the equality

$$G(U(n_1)\times\cdots\times U(n_k))(\mathbb{Q}_v) = (U(n_1)\times\cdots\times U(n_k))(\mathbb{Q}_v)E_v^\times,$$ where $E_v^\times$ embeds into $G(U(n_1)\times\cdots\times U(n_k))(\mathbb{Q}_v)$ via the diagonal embedding.

Proof. For simplicity, we prove the equality when $k = 1$. The general case follows by the same argument.

We just need to show that $c(E_v^\times) = c(GU(n)(\mathbb{Q}_v))$. Because $GU(n)(\mathbb{Q}_v)$ is quasi-split, we have the Bruhat decomposition

$$GU(n)(\mathbb{Q}_v) = \coprod_{w\in W_{\mathrm{rel}}} P_{\min}\, w\, P_{\min}.$$

We see that $c(P_{\min}\, w\, P_{\min}) = c(P_{\min}\, w)$ and $c(w) = 1$ by the above description of the normalizer of A. Hence we have $c(GU(n)(\mathbb{Q}_v)) = c(P_{\min})$ and then $c(GU(n)(\mathbb{Q}_v)) = c(T)$ since $c(U_{P_{\min}}) = 1$, where $U_{P_{\min}}$ is the unipotent radical of $P_{\min}$. By the assumption n is odd and the description of T, we have

$$c(GU(n)(\mathbb{Q}_v)) = \{x\bar{x} : x \in E_v^\times\}.$$ Moreover, by the above injection $E_v^\times \hookrightarrow Z(GU(n))(\mathbb{Q}_v)$, we also see that

$$c(E_v^\times) = \{x\bar{x} : x \in E_v^\times\}.$$

Therefore $c(E_v^\times) = c(GU(n)(\mathbb{Q}_v))$. □

We now recall some facts from the theory of endoscopy.

Definition 2.3 (cf. [6, Definition 2.1]). A refined endoscopic datum for G a connected reductive group over F is a triple (H, s, η) such that

- H is a quasi-split reductive group over F,
- $s \in Z(\hat{H})^{\Gamma_F}$,
- $\eta : \hat{H} \to \hat{G}$ such that the conjugacy class of η is Γ_F-stable and $\eta(\hat{H}) = Z_{\hat{G}}(\eta(s))^{\circ}$. Suppose that $(H, s, \eta), (H', s', \eta')$ are refined endoscopic data. Then we say that an isomorphism $\alpha : H \to H'$ is an isomorphism of endoscopic data if $\hat{\alpha}(s') = s$ and $\eta \circ \hat{\alpha}$ and η' are conjugate in $\hat{G}$. We say that a refined endoscopic datum (H, s, η) is elliptic if $(Z(\hat{H})^{\Gamma_F})^{\circ} \subset Z(\hat{G})$. We denote the set of isomorphism classes of refined endoscopic data of G by $\mathcal{E}^r(G)$.

We record a set of representatives for the isomorphism classes of refined elliptic endoscopic data for $U(n_1)\times\cdots\times U(n_k)$ and $G(U(n_1)\times\cdots\times U(n_k))$. The description for the global case is analogous. Compare with [41, Proposition 2.3.1] but note that we have more isomorphism classes because we consider refined endoscopic data. For each i, choose non-negative natural numbers n_i^+ and n_i^- such that $n_i^+ + n_i^- = n_i$.

In the unitary case, let H be the group $U(n_1^+) \times U(n_1^-) \times \cdots \times U(n_k^+) \times U(n_k^-)$, let η be the block diagonal embedding of dual groups and let

$$s = (I_{n_1^+}, -I_{n_1^-}, \dots, I_{n_k^+}, -I_{n_k^-}).$$

These elliptic endoscopic data are all non-isomorphic and give a representative of each elliptic isomorphism class.

In the unitary similitude case we let H be $G(U(n_1^+) \times U(n_1^-) \times \cdots \times U(n_k^+) \times U(n_k^-))$, let η be the block diagonal embedding of dual groups, and let

$$s = (I_{n_1^+}, -I_{n_1^-}, \dots, I_{n_k^+}, -I_{n_k^-}, 1).$$

We further require that $n_1^- + \cdots + n_k^-$ is even.

In each case, we can extend η to get a map ${}^L\eta$ of L-groups. This is done explicitly in [41, Proposition 2.3.2] (cf. [25, p. 52]).

2.2. The Langlands correspondence for unitary groups.

In this subsection, we will review the Langlands correspondences for unitary groups in the local and global settings, largely following the works of [25, 40].

2.2.1. Local unitary groups.

We start by considering a local field $\mathbb{Q}_v$ for v any place of $\mathbb{Q}$. The local Langlands group is defined by $L_{\mathbb{Q}_v} := W_{\mathbb{R}}$ if $v = \infty$ and by $W_{\mathbb{Q}_p} \times SL_2(\mathbb{C})$ if $v = p$ is a prime. For a connected reductive group G, we also set ${}^LG = \widehat{G} \rtimes W_{\mathbb{Q}_v}$ as a topological group where $\widehat{G}$ is the Langlands dual group of G. In our case we see that

$${}^LU(n) = GL_n(\mathbb{C}) \rtimes W_{\mathbb{Q}_v},$$

and the group W_{E_v} acts trivially on $GL_n(\mathbb{C})$.

Definition 2.4. A local L-parameter for a connected reductive group G defined over $\mathbb{Q}_v$ is a continuous morphism $\phi : L_{\mathbb{Q}_v} \to {}^LG$ which commutes with the canonical projections of $L_{\mathbb{Q}_v}$ and ${}^LG$ to $W_{\mathbb{Q}_v}$ and such that ϕ sends semisimple elements to semisimple elements.

We denote $\Phi(G)$ the set of $\widehat{G}$-conjugacy classes of L-parameters. An L-parameter ϕ is called bounded (resp. discrete) if its image in ${}^LG$ projects to a relatively compact subset of $\widehat{G}$ (resp. if its image is not contained in any proper parabolic subgroup of ${}^LG$). We denote by $\Phi_{bdd}(G)$ (resp. $\Phi_2(G)$) the subset of $\Phi(G)$ consisting of bounded (resp. discrete) L-parameters.

For global classifications, we will also need the notion of a local Arthur parameter.

Definition 2.5. A local A-parameter for a connected reductive group G defined over $\mathbb{Q}_v$ is a continuous morphism $\psi : L_{\mathbb{Q}_v} \times SL_2(\mathbb{C}) \to {}^LG$ such that the projection of $\psi(W_{\mathbb{Q}_v})$ to $\widehat{G}$ is bounded.

We denote by $\Psi(G)$ the set of equivalence classes of A-parameters. We also denote the set $\Psi^+(G)$ of the equivalence classes of continuous morphisms ψ as above but where $\psi|_{L_{\mathbb{Q}_v}}$ is not necessarily bounded. An A-parameter ψ (or $\psi \in \Psi^+(G)$) is said to be generic if $\psi|_{SL_2(\mathbb{C})}$ is trivial. Thus, generic A-parameters correspond to bounded L-parameters. Associated to each

$\psi \in \Psi^{+}(G)$ we have $\phi_\psi \in \Phi(G)$ given by

$$\phi_\psi(w; g) = \psi\left(w; g; \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right):$$

We also have a "standard base change" morphism of L-groups ([40, p. 9]):

$$\eta_B : {}^{L}U(n) \to {}^{L}\mathrm{Res}_{E_v/\mathbb{Q}_v} GL_{n;E_v};$$

which allows us to identify the L-parameters of $U(n)$ with self-dual L-parameters of $GL_{n;E_v}$. More precisely, in the terminology of [40], we set $\kappa = 1$ and choose χ to be trivial. Moreover, there is a bijection

$$\Phi(\mathrm{Res}_{E_v/\mathbb{Q}_v} GL_{n;E_v}) \cong \Phi(GL_{n;E_v});$$

given by projection of ${}^{L}\mathrm{Res}_{E_v/\mathbb{Q}_v} GL_{n;E_v}$ onto the first $GL_n(\mathbb{C})$-factor. If $\phi \in \Phi(U(n))$ is an L-parameter, then $\eta_B \circ \phi$ composed with this bijection is just $\phi|_{L_{E_v}}$. By [40, Lemma 2.2.1], the image of $\Phi(U(n))$ by η_B is the set of self-dual parameters in $\Phi(GL_{n;E_v})$ with parity 1 (as defined in [40]).

For each A-parameter $\psi \in \Psi^{+}(G)$ we define centralizer groups as below, which play an important role in the local and global theory. Completely analogous definitions exist for L-parameters:

$$S_\psi := \mathrm{Cent}(\mathrm{Im}\,\psi, \hat{G}); \quad \bar{S}_\psi := S_\psi / Z(\hat{G})^{\Gamma_{\mathbb{Q}_v}}; \quad \mathcal{S}_\psi := \pi_0(\bar{S}_\psi);$$

$$S_\psi^{\mathrm{rad}} := (S_\psi \cap \hat{G}_{\mathrm{der}})^{\circ}; \quad S_\psi^{\natural} := S_\psi / S_\psi^{\mathrm{rad}}:$$

Remark 2.6. For $G = U(n)$, the group S_ϕ is in general a product of symplectic, orthogonal, and linear groups. Therefore, $\pi_0(S_\phi)$ will always be a finite product of groups isomorphic to $\mathbb{Z}/2\mathbb{Z}$ coming from the component group of the orthogonal factors of S_ϕ. We have

$$S_\phi^{\natural} / Z(\widehat{U(n)})^{\Gamma_{\mathbb{Q}_v}} = \mathcal{S}_\phi$$

(although note that it is possible that $Z(\widehat{U(n)})^{\Gamma_{\mathbb{Q}_v}} \not\subset S_\phi^{\mathrm{rad}}$). For discrete (and hence supercuspidal) L-parameters, $S_\phi = S_\phi^{\natural}$. For $G = GU(n)$, for n odd, the relevant centralizer groups of a parameter ϕ_{GU} are completely determined by the corresponding groups for the parameter ϕ_U equal to the composition of ϕ_{GU} with ${}^{L}GU(n) \to {}^{L}U(n)$ (see Lemma 2.18).

We also need to introduce some notation for representations. We denote the set of isomorphism classes of irreducible admissible representations of a connected reductive group G by $\Pi(G)$. We denote the set of tempered, essentially square integrable, and unitary representations by $\Pi_{\mathrm{temp}}(G)$, $\Pi_2(G)$, and $\Pi_{\mathrm{unit}}(G)$ respectively. Denote $\Pi_{\mathrm{temp}}(G) \cap \Pi_2(G)$ by $\Pi_{2;\mathrm{temp}}(G)$.

The following theorem gives the local Langlands correspondence for extended pure inner twists of $U(n)$ over $\mathbb{Q}_v$. We first fix some more notation. Fix an odd natural number n and let $(U; \varrho; z)$ be an extended pure inner twist of $U(n)$. Fix a $\Gamma_{\mathbb{Q}_v}$-invariant splitting of $\hat{U}$. Then $(U; \varrho; z)$ induces a unique isomorphism

$${}^{L}U \cong {}^{L}U(n);$$

preserving the chosen $\Gamma_{\mathbb{Q}_v}$-splittings and we often identify these groups via this isomorphism. The cocycle z and the map

$$B(\mathbb{Q}_v; U(n)) \to X^{*}(Z(\widehat{U(n)})^{\Gamma_{\mathbb{Q}_v}})$$

defines a character $_z$ 2 X.Z.U.n/1/€$_{Q_v}$/ by z ! $_z$. We now fix a non-trivial character ' WQ ! C. Together with our chosen splitting of U.n/, this gives a Whittaker datum w of U.n/. Attached to each refined endoscopic datum .H; s; / of U we have a canonical local transfer factor •Œw;%;z• normalized as in [6, Section 4.1]. These transfer factors correspond to the •$_D$ factors of [33, Section 5]. Since U has simply connected derived subgroup, we can extend to a map ^{L} of L-groups.

Remark 2.7. We stress that in this paper, we are using the geometric normalization of the Langlands correspondence. This means that our Artin map is normalized to map a geometric Frobenius morphism to a uniformizer and explains why we normalize our transfer factors using the •$_D$ normalization. This normalization is consistent with [19] and [6] but is the inverse of the normalization in [25].

Theorem 2.8 ([25, Theorem 1.6.1], [40, Theorems 2.5.1 and 3.2.1]). Fix a field Q_v over which all groups are defined, an odd natural number n, and an extended pure inner twist .U; %; z/ of U.n/. Fix a non-trivial character ' WQ$_v$! C. Together with our fixed splitting of U.n/, this gives a Whittaker datum w of U.n/. Then:

(1) For each generic 2 ‰.U.n// (or equivalently 2 ˆ$_{bdd}$.U.n//), there exists a finite set … .U; %/ endowed with a morphism to …$_{unit}$.U/. Our choice of w defines a map

$$_w \text{ W.. .U; \%/ ! Irr.S}^{\natural}\text{; }_z\text{/; ! h; i;}$$

where Irr.S$^\natural$; $_z$/ is the set of irreducible representations of S$^\natural$ restricting on Z.U/$^{€Q_v}$ to $_z$.

(2) The morphism … .U; %/ ! …$_{unit}$.U/ is injective and its image is contained in …$_{temp}$.U/. If Q_v is non-Archimedean, then the map … .U; %/ ! Irr.S$^\natural$/ is a bijection.

(3) For each 2 …$_{unit}$.U/ in the image of … .U; %/, the central character ! WZ.U/ ! C has a Langlands parameter given by the composition

$$L_{Q_v} \;!\; {}^LU \overset{\det\rtimes id}{!} C \rtimes W_{Q_v}:$$

(4) Let .H; s; ^{L}/ be a refined endoscopic datum and let ^{H} 2 ‰.H/ be a generic parameter such that ^{L} ı ^{H} D . If f ^{H} 2 H.H/ and f 2 H.U/ are two •Œw; %; z•-matching functions, then we have

$$\sum_{^H 2 …_H .H/} h^H; s_H i\, tr.^H j f^H / D e.U/ \sum_{2 … .U;\%/} h; ^L.s/ s\; i\, tr. j f /;$$

where e. / is the Kottwitz sign.

(5) We have

$$…_{temp}.U/ D \coprod_{2ˆ_{bdd}.U.n//} ….U; \%/$$

and

$$…_{2;temp}.U/ D \coprod_{2ˆ_{2;bdd}.U.n//} ….U; \%/:$$

Proof. The contents of this theorem appear in the works of Mok ([40, Theorem 2.5.1]) and Kaletha–Minguez–Shin–White ([25, Theorem 1.6.1]) except using the arithmetic normalization of the Langlands correspondence. Hence our main task is to explain how we can use these results to define a geometrically normalized correspondence.

For 2 ‰.U.n// a generic parameter, we let …A.U; %/ denote the packet of representations assigned to by [25, Theorem 1.6.1] (the letter A stands for arithmetic normalization) and define … .U; %/ to consist of the contragredients of the representations in …A.U; %/. By the compatibility of the local Langlands correspondence with contragredients (proven in our case in Proposition 2.10, cf. [23, equation (1.2)]), this is the same as saying that the packet … .U; %/ of [25] is assigned to the parameter L C ı , where L C is the extension to L U.n/ of the Chevalley involution, C , of G b as described in [23, pp. 3–4].

We now define w. For convenience, we will denote by A w the maps given by the arithmetic normalization. Then we define for – 2 … .U; %/ that

w.–/ D A w 1.–;

where if w is the Whittaker datum .B; Qv/, then w 1 is the datum .B; Qv 1/. Equivalently by taking the contragredient, we have

w./ D A./w J C 1:b

We now verify the endoscopy character identity which is (4) in the theorem. To this end, fix f 2 H.U/ and f H 2 H.H/ a •Œw; %; z•-matching function. By Lemma 3.5, we have that if iU WU.Qp/ ! U.Qp/ is the inverse map, then f H ı iH and f ı iU are matching for the transfer factors •0Œw 1; %; z•with respect to the endoscopic datum .H; s 1; L/. We use the letter • (D •D) resp. •0 to denote the transfer factors that are compatible with the geometric normalization resp. arithmetic normalization of the local Artin reciprocity map. Then we will show in Proposition 2.9 that

X hH; s Hi tr.H j f H/ D X hH; s Hi tr.H j f H/
H 2 … H.H/ H 2 …LCHı A H.H/
D X hH; s Hi tr.H j f H ı iH/: H 2 …A H.H/

We now apply the endoscopic character identity proven in [25, Theorem 1.6.1] to get that the above equals

e.U/ X tr.A w 1./ j L.s 1/ s /tr. j f ı i/
2 …A.U;%/
D e.U/ X tr.w./– j L.s 1/ s /tr.– j f ı i/: 2 … .U;%/

Now, since tr.– j f / D tr. j f ı i/ (by Lemma 2.11), we get that the above equals

e.U/ X tr.w./ j L.s/ s /tr. j f /;
2 … .U;%/

as desired. □

Proposition 2.9. Continue with the notation as in Theorem 2.8. Let $\psi \in \Psi(U(n))$ be a generic A-parameter. Then we have the following equality for $f \in H(U(n))$:

$$\sum_{\pi \in \Pi^A_\psi(U)} \langle \pi, s\rangle \operatorname{tr}(\pi \mid f) = \sum_{\pi \in \Pi^A_{{}^L C_U \circ \psi}(U)} \langle \pi, s\rangle \operatorname{tr}(\pi \mid f \circ i_U).$$

Proof. Thanks to the results in [25, 40], the arguments in [23, Theorem 4.8] also work in our case. Indeed, the group $U(n)$ can be extended to a (twisted) endoscopic datum

$$e = (U(n), s, \xi),$$

of the triple $(\mathrm{Res}_{E/\mathbb{Q}_v} GL_n, \theta, 1)$ for a suitable outer automorphism θ of $\mathrm{Res}_{E/\mathbb{Q}_v} GL_n$ preserv-ing the standard splitting. Then $\xi \circ \psi$ is a Langlands parameter of $\mathrm{Res}_{E/\mathbb{Q}_v} GL_n$ and denote by Π the representation of $\mathrm{Res}_{E/\mathbb{Q}_v} GL_n(\mathbb{Q}_v) = GL_n(E)$ assigned to $\xi \circ \psi$ by the local Langlands correspondence. The representations Π and $\Pi \circ \theta$ are isomorphic, and there is a unique isomorphism $A : \Pi \to \Pi \circ \theta$ which preserves the w-Whittaker functionals. Then we have the distribution

$$f^n \mapsto T^{w}_{\psi,\theta}(f^n) = \operatorname{tr}\Big(v \mapsto \int_{GL_n(E)} f^n(g)\Pi(g) A v \, dg\Big).$$

Then by [40, Theorem 3.2.1] the linear form

$$f \mapsto S_\psi(f) = \sum_{\pi \in \Pi^A_\psi(U)} \langle \pi, s\rangle \operatorname{tr}(\pi \mid f)$$

is the unique distribution on $H(U(n))$ having the properties that

$$S_\psi(f) = T^{w}_{\psi,\theta}(f^n)$$

for all $f \in H(U(n))$ and $f^n \in H(GL_n(E))$ such that the (θ, ω)-twisted orbital integrals of f^n match the stable integrals of f with respect to $\mathfrak{e}' = [w; e; z_e]$.

Once we have this characterization, the proof of [23, Theorem 4.8] works without any change since [23, Proposition 4.4, Corollary 4.5 and Corollary 4.7] are valid for quasi-split unitary groups. □

Proposition 2.10. Let $\psi \in \Psi(U(n))$ be a generic A-parameter and w a Whittaker datum. Let π be a representation in $\Pi^A_\psi(U)$ and denote $\iota^A_w(\pi) = \eta$. Then:

- the contragredient representation $\pi^\vee$ belongs to the L-packet $\Pi^A_{{}^L C \circ \psi}(U)$,
- $\iota^A_{w^{-1}}(\pi^\vee) = (\eta \circ C^{-1})^\vee$.

Proof. These results are completely analogous to [23, Theorem 4.9]. The same arguments carry over to our case since an analogue of [23, Theorem 4.8] is still valid for unitary groups (Proposition 2.9). □

We also have the following basic fact.

Lemma 2.11. For (π, V) an admissible representation of $G(\mathbb{Q}_p)$ for G a reductive group and $f \in H(G)$, we have

$$\operatorname{tr}(\pi^\vee \mid f) = \operatorname{tr}(\pi \mid f \circ i).$$

Proof. Pick some compact open subgroup $K \subset G(\mathbb{Q}_p)$ such that f is K-bi-invariant, and let $(\check{\pi}, \check{V})$ denote the contragredient of π so that $\check{V} \subset V^*$ is the subspace of smooth vectors in the dual vector space V^* of V. Then we note that $(\check{V})^K = (V^K)^*$ since each vector in $(V^*)^K$ is by definition smooth.

The operator $\check{\pi}(f)$ acts on $(V^K)^*$ as the dual of the operator $\pi(f \circ i)$. Indeed, for $v \in (V^K)^*$ and $w \in V^K$,

$$\begin{aligned}
\check{\pi}(f)(v)(w) &= \int_{G(\mathbb{Q}_p)} f(g)\check{\pi}(g)(v)(w)\,dg \\
&= \int_{G(\mathbb{Q}_p)} (f \circ i)(g^{-1})v(\pi(g^{-1})w)\,dg \\
&= v\Big(\int_{G(\mathbb{Q}_p)} (f \circ i)(g)\pi(g)w\,dg\Big) \\
&= \pi(f \circ i)^*(v)(w),
\end{aligned}$$

where the third equality uses the fact that G is unimodular. This implies the desired equality of traces. □

When we consider global parameters, we will also need a version of Theorem 2.8 for $\psi \in \Phi^{+}(U(n))$. The following theorem is essentially contained within the union of remarks in [40, p. 33] and [25, Section 1.6.4].

Theorem 2.12. *Fix a field $\mathbb{Q}_v$ over which all groups are defined, an odd natural number n and let (U, ϱ, z) be an extended pure inner twist of $U(n)$. Fix a non-trivial character $\psi : \mathbb{Q}_v \to \mathbb{C}^{\times}$. Together with our chosen splitting of $U(n)$, this gives a Whittaker datum $\mathfrak{w}$ of $U(n)$. Then:*

(1) *For each generic $\psi \in \Phi^{+}(U(n))$, there exists a finite set $\Pi_{\psi}(U, \varrho)$ of possibly reducible or non-unitary representations of U. Our choice of $\mathfrak{w}$ defines a map*

$$\iota_{\mathfrak{w}} : \Pi_{\psi}(U, \varrho) \to \mathrm{Irr}(S_{\psi}^{\natural}, \chi_z), \quad \pi \mapsto \langle \pi, \cdot \rangle,$$

where $\mathrm{Irr}(S_{\psi}^{\natural}, \chi_z)$ denotes the set of irreducible representations of $S_{\psi}^{\natural}$ with central character χ_z. Each $\pi \in \Pi_{\psi}(U, \varrho)$ has a central character ω_{π}, these characters are the same for each element of $\Pi_{\psi}(U, \varrho)$.

(2) *Let $(H, s, {}^L\eta)$ be a refined endoscopic datum and let $\psi^H \in \Phi^{+}(H)$ be a generic parameter such that ${}^L\eta \circ \psi^H = \psi$. If $f^H \in \mathcal{H}(H)$ and $f \in \mathcal{H}(U)$ are two $(\mathfrak{w}, \varrho, z)$-matching functions, then we have*

$$\sum_{\pi^H \in \Pi_{\psi^H}(H)} \langle \pi^H, s_{\psi^H} \rangle \operatorname{tr}(\pi^H \mid f^H) = e(U) \sum_{\pi \in \Pi_{\psi}(U, \varrho)} \langle \pi, {}^L\eta(s) s_{\psi} \rangle \operatorname{tr}(\pi \mid f);$$

where $e(\cdot)$ is the Kottwitz sign.

Proof. We sketch the proof following ideas in [25, 40]. The proof of (1) is in [25, Section 1.6.4]. They choose a standard parabolic subgroup $P = MN_P$ of $U(n)$ that transfers to U, a parameter $\psi_M \in \Phi(M)$ and a character $\lambda \in \mathrm{Hom}_{\mathbb{Q}_v}(M, \mathbb{G}_m) \otimes \mathbb{R}$ that induces a central parameter $: W_{\mathbb{Q}_v} \to {}^L M$ satisfying that $_{M}$ agrees with under the L-embedding ${}^L M \hookrightarrow {}^L U(n)$. Choose a representative (U, ϱ, z) in its equivalence class so that the restriction (M, ϱ_M, z_M) to M is also an extended pure inner twist.

They then define $\Pi (U, \varrho)$ by

$$\Pi (U, \varrho) := \{ I_P^U(_M \otimes) : _M \in \Pi_{_M}(M, \varrho_M)\},$$

where I_P^G denotes normalized parabolic induction and is the character of $M(\mathbb{Q}_v)$ corresponding to . Note that by definition of parabolic induction, if $_M$ has central character $\omega_{_M}$, then $I_P^U(_M)$ will have central character $\delta_P^{1/2}\omega_M$. Since each element of $\Pi_{_M}(M, \varrho_M)$ has the same central character, this will also be true of $\Pi (U, \varrho)$.

From the explicit description of S given in [25, p. 62], it follows that $S = S_{_M}$. In [25, Section 1.6.4] they show that

$$S^{rad} Z(\widehat{U(n)})^{\Gamma_{\mathbb{Q}_v}} = S^{rad} Z(\widehat{M})^{\Gamma_{\mathbb{Q}_v}},$$

and that $_z$ and $_{z_M}$ both extend uniquely to give the same character z_z of $S^{rad} Z(\widehat{U(n)})^{\Gamma_{\mathbb{Q}_v}}$ that is trivial on S^{rad}. Now, we have an identification

$$\mathrm{Irr}(S^\natural, _z) = \mathrm{Irr}(S^\natural_{_M}, _{z_M}),$$

as both parametrize irreducible representations of S that restrict to z_z on $S^{rad} Z(\widehat{U(n)})^{\Gamma_{\mathbb{Q}_v}}$. One can now define

$$\langle , s\rangle = \langle _M, s_M\rangle$$

for $s \in S^\natural$.

It remains to verify the endoscopic character identity. Fix a refined endoscopic datum $(H, s, {}^L)$ for $U(n)$ such that $= {}^L \circ ^H$ for some $^H \in \Phi^{\mathbb{C}}(H)$. Then ${}^L(s) \in S \widehat{M}$. In view of [6, Proposition 3.10], there exist a refined endoscopic datum $(H_M, s_M, {}^L_M)$ and a parameter $^{H_M} \in \Phi(H_M)$ corresponding to the pair $(_M, {}^L(s))$. It is clear from construction that under the map $Y : \mathcal{E}^r(M) \to \mathcal{E}^r(U(n))$ of [6, Section 2.5], the image of the class of $(H_M, s_M, {}^L_M)$ equals the class of $(H, s, {}^L)$. Now by [6, Proposition 2.20], we can choose a refined datum equivalent to $(H_M, s_M, {}^L_M)$ fitting into an embedded datum $(H, H_M, s, {}^L)$. We observe that H_M is a Levi subgroup of H.

Now, ${}^L j_M$ induces a map $Z(\widehat{M}) \hookrightarrow Z(\widehat{H_M})$ and hence yields a central parameter $_H$ of H_M. It is easy to see that by definition

$$^{H_M} _H = ^H,$$

under the natural inclusion ${}^L H_M \hookrightarrow {}^L H$. Hence, we can define a packet $\Pi_{_H}(H)$ and pairing

$$\langle , \rangle : \Pi_{_H}(H) \times S^\natural_{_H} \to \mathbb{C},$$

using the above procedure.

We need to verify that the resulting pairing satisfy the endoscopic character identity. Let f, f^H be $[w, \varrho, z]$-matching functions. Let $f_P \in \mathcal{H}(M)$ and $f^H_{P_H} \in \mathcal{H}(H_M)$ be the corresponding constant term functions. By [57, paragraph at the top of p. 237 and the remark on p. 239] it follows that

$$\mathrm{tr}(I_P^U(_M) \mid f) = \mathrm{tr}(_M \mid f_P),$$

and similarly for $f^H_{P_H}$. We can restrict the splitting of $U(n)$ to M and together with the character φ, this gives a Whittaker datum $\mathfrak{w}_M$. By [6, Proposition 5.3], the corresponding canonical transfer factor $\Delta[\mathfrak{w}_M, \varrho_M, z_M]$ satisfies

$$\Delta[\mathfrak{w}_M, \varrho_M, z_M](\gamma_H, \gamma) = |D^{U(n)}_M(\gamma)|^{-\frac{1}{2}} \, |D^H_{H_M}(\gamma_H)|^{\frac{1}{2}} \, \Delta[\mathfrak{w}, \varrho, z](\gamma_H, \gamma)$$

for regular $\gamma \in M(\mathbb{Q}_v)$, $\gamma_H \in H_M(\mathbb{Q}_v)$ and where we recall that $D^G_M(\gamma)$ is defined to equal $\det(1 - \mathrm{ad}(m))|_{\mathrm{Lie}(G)/\mathrm{Lie}(M)}$.

We now claim that f_P and $f^H_{P_H}$ are $\Delta[\mathfrak{w}_M, \varrho_M, z_M]$-matching. If we can show this then we will have

$$\begin{aligned}
\sum_{\pi_H \in \Pi_{\psi_H}(H)} \langle \pi_H, s_{\psi_H} \rangle \operatorname{tr}(\pi_H \mid f^H) &= \sum_{\pi_{H_M} \in \Pi_{\psi_{H_M}}(H_M)} \langle \pi_{H_M}, s_{\psi_{H_M}} \rangle \operatorname{tr}(\pi_{H_M} \mid f^H_{P_H}) \\
&= e(M) \sum_{\pi_M \in \Pi_{\psi_M}(M, \varrho_M)} \langle \pi_M, {}^L\eta(s) s_{\psi_M} \rangle \operatorname{tr}(\pi_M \mid f_P) \\
&= e(U) \sum_{\pi \in \Pi_\psi(U, \varrho)} \langle \pi, {}^L\eta(s) s_\psi \rangle \operatorname{tr}(\pi \mid f),
\end{aligned}$$

as desired. Note that in the above we use that $e(M) = e(U)$ which is part of [26, proposition on p. 292].

Suppose $\gamma_H \in H_M(\mathbb{Q}_v)$ and $\gamma \in M(\mathbb{Q}_v)$ are strongly regular elements that transfer to each other. Then by [57, Lemma 9], we have the following equality of orbital integrals (and analogously for f^H):

$$O^U_\gamma(f) = |D^U_M(\gamma)|^{-\frac{1}{2}} O^M_\gamma(f_P),$$

and hence, since f and f^H are $\Delta[\mathfrak{w}, \varrho, z]$-matching:

$$\begin{aligned}
SO^{H_M}_{\gamma_H}(f^H_{P_H}) &= |D^H_{H_M}(\gamma_H)|^{\frac{1}{2}} SO^H_{\gamma_H}(f^H) \\
&= |D^H_{H_M}(\gamma_H)|^{\frac{1}{2}} \sum_{\gamma' \sim_{st} \gamma} \Delta[\mathfrak{w}, \varrho, z](\gamma_H, \gamma') O^U_{\gamma'}(f) \\
&= |D^H_{H_M}(\gamma_H)|^{\frac{1}{2}} |D^U_M(\gamma')|^{-\frac{1}{2}} \sum_{\gamma' \sim_{st, U} \gamma} \Delta[\mathfrak{w}, \varrho, z](\gamma_H, \gamma') O^M_{\gamma'}(f_P) \\
&= \sum_{\gamma' \sim_{st, M} \gamma} \Delta[\mathfrak{w}_M, \varrho_M, z_M](\gamma_H, \gamma') O^M_{\gamma'}(f_P),
\end{aligned}$$

as desired. Note that we use that the number of conjugacy classes of γ in the stable class is the same for U and M (this follows from the injection $H^1(\mathbb{Q}_v, M) \hookrightarrow H^1(\mathbb{Q}_v, U)$). □

2.2.2. Global unitary groups. We now consider the global situation. Recall that we have fixed a quadratic imaginary extension $E/\mathbb{Q}$ and are considering global unitary groups $U = U(V)$ that are quasi-split at the finite places and with fixed quasi-split inner form $U(n)$. By Lemma 2.1 we give U the structure (U, ϱ, z) of an extended pure inner twist of $U(n)$. We also fix a global Whittaker datum $\mathfrak{w}$ of $U(n)$.

Due to the lack of a global L-group, we rely on the cuspidal automorphic representations of $GL_n(\mathbb{A}_E)$ to define the notion of global parameters as in [5] (cf. [25]). Let $\Psi(n)$ denote the set of all formal sums

$$\psi^n = \ell_1(\mu_1 \boxtimes \nu_1) \boxplus \cdots \boxplus \ell_r(\mu_r \boxtimes \nu_r),$$

where ℓ_i are positive integers, μ_i are cuspidal automorphic representations of $GL_{n_i}(\mathbb{A}_E)$ and ν_i are algebraic representations of $SL_2(\mathbb{C})$ such that $\mu_i \boxtimes \nu_i$ are pairwise disjoint and $\sum_{i=1}^r \ell_i n_i \dim \nu_i = n$.

We denote $(\mu \boxtimes \nu)^\star = \mu^\star \boxtimes \nu$, where $\mu^\star = (\mu^c)^\vee$ the conjugate dual representation of μ. Now for $\psi^n = \ell_1(\mu_1 \boxtimes \nu_1) \boxplus \cdots \boxplus \ell_r(\mu_r \boxtimes \nu_r)$. We say that ψ^n is generic if ν_i is the trivial representation of $SL_2(\mathbb{C})$ for all i. We say that ψ^n is self-dual if there exists an involution $i \mapsto i^*$ of $\{1, \ldots, r\}$ such that $(\mu_i \boxtimes \nu_i)^\star = \mu_{i^*} \boxtimes \nu_{i^*}$ and $\ell_i = \ell_{i^*}$. From a self-dual formal sum ψ^n, we can construct a group $\mathcal{L}_{\psi^n}$ and a map ([40, pp. 22–23, Definition 2.4.3])

$$\tilde{\psi}^n : \mathcal{L}_{\psi^n} \times SL_2(\mathbb{C}) \to {}^L GL_{n,E}.$$

We have a standard base change map $\eta_B : {}^L U(n) \to {}^L GL_{n,E}$ defined analogously to the local case.

Definition 2.13. The set of global L-parameters $\Psi(U(n))$ of $U(n)$ is the set consisting of pairs $\psi = (\psi^n, \tilde{\psi})$, where ψ^n is a self-dual formal sum and

$$\tilde{\psi} : \mathcal{L}_{\psi^n} \times SL_2(\mathbb{C}) \to {}^L U(n)$$

is a map such that $\tilde{\psi}^n = \eta_B \circ \tilde{\psi}$. The parameter ψ is called generic if ψ^n is generic.

We remark that $\tilde{\psi}$ is determined by the base change map η_B and $\tilde{\psi}^n$, and as in the local case, from the map $\tilde{\psi}$, we can define various centralizer groups S_ψ, $\bar{S}_\psi$, $\mathcal{S}_\psi$, $S_\psi^\natural$. For later use, we denote $\Psi_2(U(n))$ to be the set of global parameters $\psi = (\psi^n, \tilde{\psi})$ such that $|\bar{S}_\psi|$ is finite.

There is a localization morphism

$$\Psi(U(n)) \to \Psi^+(U(n)_v), \quad \psi \mapsto \psi_v,$$

see [40, pp. 18–19]. More precisely, if v is a place of $\mathbb{Q}$ that splits in E, then $E_v = E_w \oplus E_{\bar{w}}$ and $U(n)_v \cong GL_{n,E_w}$, where w, $\bar{w}$ are the primes of E above v. Moreover, there is an identification $\mathbb{Q}_v = E_w$ and therefore we can define $\psi_v = \psi^n_w$. If v is a place of $\mathbb{Q}$ that does not split in E, then E_v is a quadratic extension of $\mathbb{Q}_v$. By [40, Theorem 2.4.10] the localization ψ^n_v of ψ^n factors through the base change map η_B so that it defines a parameter ψ_v in $\Psi^+(U(n)_v)$.

According to Theorem 2.8 and Theorem 2.12, for each $\psi_v \in \Psi^+(U(n)_v)$ we have a packet $\Pi_{\psi_v}(U(n)_v, \varrho_v)$ together with a map

$$\Pi_{\psi_v}(U_v, \varrho_v) \to \mathrm{Irr}(S^\natural_{\psi_v}, \chi), \quad \pi_v \mapsto \langle \pi_v, \cdot \rangle.$$

We denote

$$\Pi_\psi(U, \varrho) := \left\{ \bigotimes_v{}' \pi_v : \pi_v \in \Pi_{\psi_v}(U_v, \varrho_v), \langle \pi_v, \cdot \rangle = 1 \text{ for almost all } v \right\}.$$

Since the localization maps $\psi \mapsto \psi_v$ induce the localization maps $S_\psi^\natural \to S_{\psi_v}^\natural$ for centralizer groups ([25, p. 71]), we can associate to each $\bigotimes_v \pi_v \in \Pi_\psi(U, \varrho)$ a character of $S_\psi^\natural$ by the formula

$$\langle \pi, s \rangle := \prod_v \langle \pi_v, s_v \rangle, \quad s \in S_\psi^\natural;$$

where s_v is the image of s by the localization morphism $S_\psi^\natural \to S_{\psi_v}^\natural$. The global pairing $\langle\cdot,\cdot\rangle$ descends to a character of $\overline{S}_\psi$ (see [25, p. 89]).

Definition 2.14. Let $\Pi_\psi(U,\varrho,\omega) := \{\pi \in \Pi_\psi(U,\varrho) : \langle\cdot,\pi\rangle = \varepsilon_\psi\}$, where ε_ψ is the Arthur character of $\overline{S}_\psi$ (see [40, equation (2.5.5)]). Recall that if ψ is a generic parameter, then $\varepsilon_\psi = 1$.

Let $\mathfrak{h}$ be the standard maximal Cartan subalgebra in the Lie algebra of $\mathrm{Res}_{E/\mathbb{Q}}U(\mathbb{C})$ and let $|\,|$ be a fixed Weyl-Hermitian metric on the dual of $\mathfrak{h}$. Let π be an automorphic representation of $U(\mathbb{A})$. Then the local factor π_v is unramified if v does not belong to some finite set of places S. Thus we get a Hecke string $c = (c_v)_{v\notin S}$, where c_v is the semisimple conjugacy class corresponding to π_v via the Satake transform. Moreover, the infinitesimal character of its Archimedean components gives a linear form λ on $\mathfrak{h}$. Denote $\mathrm{im}(\lambda)$ its imaginary part.

Following [25, Section 3.3], for each global parameter ψ we define

$$L^2_{\mathrm{disc},\psi}(U(\mathbb{Q})\backslash U(\mathbb{A})) = \sum_{c \mapsto c(\psi^n)} L^2_{\mathrm{disc},t(\psi^n),c}(U(\mathbb{Q})\backslash U(\mathbb{A})),$$

where the sum runs over the set of Hecke strings c which map to $c(\psi^n)$ via the base change map η_B and where $L^2_{\mathrm{disc},t,c}(U(\mathbb{Q})\backslash U(\mathbb{A}))$ is the direct sum of automorphic representations π such that $|\mathrm{im}(\lambda)| = t$ and c_v corresponds to π_v via the Satake transform away from a sufficiently large finite set.

Theorem 2.15 ([25, Theorem 1.7.1]). There is an isomorphism of $U(\mathbb{A})$-modules

$$L^2_{\mathrm{disc}}(U(\mathbb{Q})\backslash U(\mathbb{A})) \simeq \bigoplus_{\psi\in\Psi(U(n))} L^2_{\mathrm{disc},\psi}(U(\mathbb{Q})\backslash U(\mathbb{A})).$$

If ψ is generic then

$$L^2_{\mathrm{disc},\psi}(U(\mathbb{Q})\backslash U(\mathbb{A})) = 0 \text{ if } \psi \notin \Psi_2(U(n)),$$

$$L^2_{\mathrm{disc},\psi}(U(\mathbb{Q})\backslash U(\mathbb{A})) \simeq \bigoplus_{\pi\in\Pi_\psi(U,\varrho,\omega)} \pi \text{ if } \psi \in \Psi_2(U(n)).$$

In particular, if π is an automorphic representation of the unitary group U belonging to a generic global packet, then the automorphic multiplicity, m, equals 1.

In order to show that $m = 1$ when π is an automorphic representation of the unitary group U belonging to a generic global packet, it is enough to show that if π belongs to $L^2_{\mathrm{disc},\psi_1}(U(\mathbb{Q})\backslash U(\mathbb{A}))$ and $L^2_{\mathrm{disc},\psi_2}(U(\mathbb{Q})\backslash U(\mathbb{A}))$, then $\psi_1 = \psi_2$.

By the definition of $L^2_{\mathrm{disc},\psi_1}(U(\mathbb{Q})\backslash U(\mathbb{A}))$, we can identify the Hecke string of ψ_1^n and the base change of the Hecke string c. Similarly, we can also identify the Hecke string of ψ_2^n and the base change of the Hecke string c. Thus ψ_1^n and ψ_2^n have the same Hecke string. By the strong multiplicity one theorem for isobaric automorphic representations of $GL_n(\mathbb{A}_E)$, we conclude that $\psi_1^n = \psi_2^n$ and hence $\psi_1 = \psi_2$.

Remark 2.16. We remark that the proof of the theorem as we have stated it here is completed in [25] up to assumptions in [40]. For instance, the careful reader will note that [25, Theorem 1.7.1] requires that U arises as a pure inner twist of $U(n)$. Indeed, this will be true since we are assuming U comes from a Hermitian form (Lemma 2.1). However, the work of [40] assumes that the weighted fundamental lemma and analogues of the unpublished papers [A25], [A26], [A27] referenced in [5] hold for U.n/, and these results are not available at the time of writing.

2.3. The Langlands correspondence for unitary similitude groups. In this subsection, we want to transfer the results about automorphic representations from unitary groups to unitary similitude groups (with an odd number of variables).

2.3.1. Local unitary similitude groups. Let v be a finite place of Q that does not split over E, let n be an odd positive integer, and let GU be an inner form of GU.n/, defined over Q_v, and denote the corresponding unitary group by U. Fix a €$_{Q_v}$-invariant splitting of GU and restrict to get a €$_{Q_v}$-invariant splitting of U. Fix also a character ' WQ_v ! C. This data gives us Whittaker data w_{GU} and w_U of GU.n/ and U.n/ respectively.

We give GU the structure of an extended pure inner twist .GU; %$_{GU}$; z_{GU}/ of GU.n/. We also fix an extended pure inner twist .U; %$_U$; z_U/ of U.n/. Note that this induces an extended pure inner twist of GU.n/ that on the level of cocycles is given by composing z_U with U.n/ ! GU.n/ and that this induced twist is trivial since the map

B.Q_v; U.n//$_{bas}$! B.Q_v; GU.n//$_{bas}$

is trivial. In particular, this induced extended pure inner twist need not be isomorphic to .GU; %$_{GU}$; z_{GU}/. In fact, our constructions in this section will not depend on .U; %$_U$; z_U/. Note also that since we are assuming n is odd, GU will automatically be quasi-split. By Lemma 2.2, we have GU.Q_v/ D U.Q_v/E_v and then the following result:

Corollary 2.17. There is a natural bijection between the set ….GU/ and the set of pairs .; /, where 2 ….U/ and is a character of E_v such that j_E ^{T} U.Q_v/ D !j_E ^{T} U.Q_v/, for ! the central character of .

We use this corollary to define A-packets of representations for GU and the associated A-parameters. Fix a character of Z.GU/ corresponding to a morphism

z WL$_{Q_v}$! 2U.n/$_{ab}$ Ì W$_{Q_v}$ D .C C/ Ì W$_{Q_v}$;

and a parameter $_U$ 2 ‰^{C}.U.n// given by

$_U$ WL$_{Q_v}$ SL$_2$.C/ ! ^{L}U D GL$_n$.C/ Ì W$_{Q_v}$;

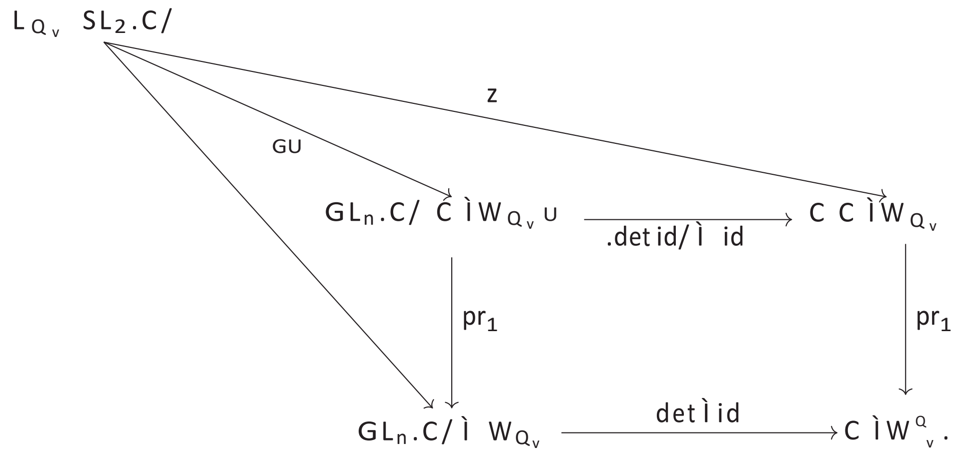
such that ! j_E ^{T} U.Q_v/ D j_E ^{T} U.Q_v/ for one (hence any) 2 … $_U$.U; %$_U$/. We can view

^{L}GU.n/ D GL$_n$.C/ C Ì W$_{Q_v}$

as a product of ^{L}U.n/ D GL$_n$.C/ Ì W$_{Q_v}$ and 2U.n/$_{ab}$ Ì W$_{Q_v}$ D .C C/ Ì W$_{Q_v}$ over C Ì W$_{Q_v}$, where the first projection is given by g Ì ! det g Ì and the second is given by .x; y/ Ì ! x Ì . The above pair . $_U$; / then defines a unique morphism

$_{GU}$ WL$_{Q_v}$ SL$_2$.C/ ! GL$_n$.C/ C Ì W$_{Q_v}$:

Conversely, each $_{GU}$ 2 ‰^{C}.GU.n// gives rise to a pair . $_U$; /. We summarize these relationships in the following commutative diagram:



We now define the A-packet associated to ψ_{GU} and $(GU, \varrho_{GU}, z_{GU})$ assuming it has been defined for ψ_U and (U, ϱ_U, z_U). We have

$$\Pi_{\psi_{GU}}(GU, \varrho_{GU}) := \{\pi(\chi, \pi') : \pi' \in \Pi_{\psi_U}(U, \varrho),\ \omega_{\pi'}|_{E_v^{\times} \cap U(\mathbb{Q}_v)} = \chi|_{E_v^{\times} \cap U(\mathbb{Q}_v)}\}.$$

We now use the internal structure of $\Pi_{\psi_U}(U, \varrho_U)$ to describe that of $\Pi_{\psi_{GU}}(GU, \varrho_{GU})$. Let us first describe the relations between the various centralizer groups for ψ_U and ψ_{GU}.

Lemma 2.18. With ψ_U and ψ_{GU} as above, we have

$$S_{\psi_{GU}} = S^+_{\psi_U} \times \mathbb{C}^\times, \qquad \overline{S}_{\psi_U} = \overline{S}_{\psi_{GU}}, \qquad S^\natural_{\psi_{GU}} = \pi_0(S^+_{\psi_U}) \times \mathbb{C}^\times,$$

where $S^+_{\psi_U} = \{g \in S_{\psi_U} : \det g = 1\}$.

Proof. For (g, c) and (x, t) in $GL_n(\mathbb{C}) \times \mathbb{C}^\times$ and $\sigma \in W_{\mathbb{Q}_v}$ projecting to the non-trivial element of $\Gamma_{E_v/\mathbb{Q}_v}$, we have

$$(g, c)(x, t) \rtimes \sigma (g^{-1}, c^{-1}) = (gx, ct) \rtimes \sigma (g^{-1}, c^{-1})$$
$$= (gxJg^tJ^{-1}, t \det g^{-1}) \rtimes \sigma,$$

where the second equality comes from the action of σ on (g^{-1}, c^{-1}). In particular, we have $(g, c) \in S_{\psi_{GU}}$ if and only if $g \in S_{\psi_U}$ and $\det g = 1$. In other words, $S^+_{\psi_U} \times \mathbb{C}^\times = S_{\psi_{GU}}$.

We now prove that $\overline{S}_{\psi_U} = \overline{S}_{\psi_{GU}}$. By a direct calculation, we see that

$$Z(\hat{U}_n)^{\Gamma_{\mathbb{Q}_v}} = \{\pm \mathrm{id}_n\} \cong \mathbb{Z}/2\mathbb{Z},$$

and $Z(\widehat{GU}_n)^{\Gamma_{\mathbb{Q}_v}} = \mathrm{id}_n \times \mathbb{C}^\times$ (because n odd). Hence $\overline{S}_{\psi_{GU}} = \pi_0(S^+_{\psi_U})$. We also remark that the equality $gxJg^tJ^{-1} = x$ implies $(\det g)^2 = 1$. Therefore, for every $g \in S_{\psi_U}$ we have $(\det g)^2 = 1$. Moreover, since $\det(-\mathrm{id}_n) = -1$, we have

$$S^+_{\psi_U} \cong S_{\psi_U} / Z(\hat{U}_n)^{\Gamma_{\mathbb{Q}_v}}.$$

Thus, $\overline{S}_{\psi_{GU}} = \overline{S}_{\psi_U}$ as desired. Finally, we have $S^{\mathrm{rad}}_{\psi_{GU}} = (S_{\psi_{GU}} \cap (SL_n(\mathbb{C}) \times 1))^\circ = (S^+_{\psi_U})^\circ$ which implies the description of $S^\natural_{\psi_{GU}}$ in the statement of the lemma. □

We now construct a pairing

h ; i$_{GU}$ W.. $_{GU}$.GU; %$_{GU}$/ S$^{\backslash}$ $_{GU}$! C :

Let .; / 2 … $_{GU}$.GU; %$_{GU}$/. Then 2 … $_{U}$.U; %$_{U}$/ and by Theorems 2.8 and 2.12 there is an associated character h; i$_{U}$ WS $^{\backslash}$ $_{U}$! C . Note that since S $^{rad}_{U}$ D .S $^{C}_{U}$/^{I} , we can restrict this character to

S $^{C}_{U}$=S$^{rad}_{U}$ D $_0$.S $^{C}_{U}$/:

We claim this character does not depend on our choice of .U; %$_{U}$; z$_{U}$/. Indeed, all inner twists of U.n/ are trivial so up to equivalence, the only dependence is on z$_{U}$. This dependence is described in [25, Theorem 1.6.1 (2)] where they observe that modifying z$_{U}$ corresponds to taking the tensor product of h; i with a certain character of S$^{\backslash}$ $_{U}$ induced from a map

S $_{U}$! Œ1U.n/=U.n/$_{sc}$1•$^{€_{Q_v}}$:

This map is induced by the determinant map on matrices and hence contains S $^{C}_{U}$ in its kernel. This implies our claim.

Via z$_{GU}$ and the map

WB.Q$_v$; GU.n// ! X.Z.GU.n2/$^{€_{Q_v}}$ / D X.1 C/;

we get a character $_{z_{GU}}$ of 1 C. In Lemma 2.18, we showed that S$^{\backslash}$ $_{GU}$ D $_0$.S $^{C}_{U}$/ C. Hence we define

h.; /; .s; c/i$_{GU}$ D h; si$_{U}$ $_{z_{GU}}$.c/:

Suppose that $_{GU}$ 2 ‰.GU.n// is generic. We show that

.; / ! h.; /; i$_{GU}$

is bijective onto Irr.S$^{\backslash}$ $_{GU}$; $_{z_{GU}}$/ by constructing an inverse. To this end, we pick a character $_{GU}$ of S$^{\backslash}$ $_{GU}$ which restricts on Z.GU.n2//$^{€_{Q_v}}$ to the character $_{z_{GU}}$. As Z.U1.n//$^{€_{Q_v}}$ and $_0$.S $^{C}_{U}$/ generate S$^{\backslash}$ $_{U}$ and have trivial intersection, there is then a unique character $_{U}$ of S$^{\backslash}$ $_{U}$ that restricts to $_{z_{U}}$ on Z.U/$^{€_{Q_v}}$ and $_{GU}$ on $_0$.S $^{C}_{U}$/. By .2/ of Theorem 2.8, there then exists a 2 … $_{U}$.U; %$_{U}$/ that gets mapped to $_{U}$, and by construction, .; / maps to $_{GU}$. Hence $_{GU}$! .; / is our desired inverse.

We have now proven:

Theorem 2.19. Parts (1) and (2) of Theorem 2.8 and part (1) of Theorem 2.12 hold for GU for non-Archimedean v.

In the Archimedean case, these results are known by work of Langlands and Shelstad.

In the next section, we will prove that this pairing also satisfies the endoscopic character identities.

We record the following proposition for later use.

Proposition 2.20 ([39, Section 8.4.4]). Let $_{U}$ WW$_{Q_v}$ SL$_2$.C/ ! ^{L}U.n/ be a dis-crete L-parameter which is trivial over SL$_2$.C/. Then the packet …$_{U}$.U; %$_{U}$/ contains only supercuspidal representations. These L-parameters are called supercuspidal.

Corollary 2.21. From the above description of local L-packets of GU, it follows that the L-packet of a supercuspidal L-parameter of GU will consist entirely of supercuspidal representations.

Remark 2.22. Suppose that is as above and $(H, s, {}^L)$ is an elliptic endoscopic datum and ${}^H : W_{\mathbb{Q}_v} \times SL_2(\mathbb{C}) \to {}^L H$ an L-parameter such that ${}^L \circ {}^H =$. Then ^{H} is also supercuspidal and hence the packet $\Pi_H(H)$ contains only supercuspidal representations.

2.3.2. Global unitary similitude groups. Fix a Hermitian form V and global group $U = U(V)$ and $GU = GU(V)$. As in the local case, we give GU and U the structure of extended pure inner twists $(GU, \%_{GU}, z_{GU})$ and $(U, \%_U, z_U)$. We begin by recalling the following result which relates automorphic representations of $U(\mathbb{A})$ and of $GU(\mathbb{A})$.

Proposition 2.23 ([12, Section CHL.IV.C, Proposition 1.1.4]). Fix $n \in \mathbb{N}$ odd. Let be an irreducible automorphic representation of $GU(\mathbb{A})$ whose restriction to $U(\mathbb{A})$ contains an irreducible automorphic representation . If has multiplicity 1 in the discrete spectrum of $U(\mathbb{A})$, then has multiplicity 1 in the discrete spectrum of $GU(\mathbb{A})$. Moreover, is the unique automorphic representation of $GU(\mathbb{A})$ with central character and containing in its restriction.

Let be an automorphic central character of $GU(\mathbb{A})$ and $_U := |_{Z(U(\mathbb{A}))}$ its restriction to the center of $U(\mathbb{A})$. Consider $_U$ a generic A-parameter for a global unitary group whose automorphic representations have $_U$ as central character. The generic condition ensures the multiplicity one property of these automorphic representations by Theorem 2.15. As in the local case, a pair $(_U,)$ satisfying the above conditions determines a generic A-parameter for GU. In the following, we will denote such an A-parameter by $_{GU}$ if is clear from the context. We define the associated A-packet $\Pi_{_{GU}}(GU, \%_{GU},)$ to consist of the whose central character is and whose restriction to $U(\mathbb{A})$ belongs to $\Pi_{_U}(U, \%_U,)$.

Now, by the proof of [12, Section CHL.IV.C, Proposition 1.1.4], we have

$$L^2(\backslash U(\mathbb{A}), _U) = \mathrm{Res}^{GU(\mathbb{A})}_{U(\mathbb{A})} L^2(\backslash GU(\mathbb{A}),),$$

where $= GU(\mathbb{Q})Z(GU)(\mathbb{A})$ and $= \cap U(\mathbb{A})$. In particular, it follows from Theorem 2.15 that we can lift every representation $\in \Pi_{_U}(U, \%_U, _U)$ to a representation of $GU(\mathbb{A})$ whose central character is . Combining with Proposition 2.23, we see that there is a bijection between $\Pi_{_U}(U, \%_U, _U)$ and $\Pi_{_{GU}}(GU, \%_{GU}, _{GU})$.

We now give a description of $\Pi_{_{GU}}(GU, \%_{GU}, _{GU})$ in the spirit of Definition 2.14. We have defined global generic A-parameters of GU in terms of their counterpart for U. We define the centralizer groups for such parameters of GU using the analogous groups for U and using Lemma 2.18 as our guide.

Remark 2.24. It would perhaps be possible to define these parameters and their centralizer groups in analogy with our definitions for U using cuspidal automorphic representations and the methods of [5, 25, 40]. For simplicity, we choose not to do this in the present paper.

Definition 2.25. Let $_{GU} = (_U,) \in \Phi(GU, n)$ be a generic parameter. We define

$$S_{_{GU}} := S_{_U} \mathbb{C}^{\times}, \quad \overline{S}_{_{GU}} := \overline{S}_{_U}, \quad S^{\natural}_{_{GU}} := \pi_0(S_{_U}/\mathbb{C}^{\times}).$$

We now discuss localization. First, by the localization map for algebraic cocycles (see [32, Section 7]), the extended pure inner twists $(GU, \varrho_{GU}, z_{GU})$ and (U, ϱ_U, z_U) give rise to local extended pure inner twists $(GU_v, \varrho_{GU_v}, z_{GU_v})$ and $(U_v, \varrho_{U_v}, z_U^v)$ for each place v of $\mathbb{Q}$.

Let $(\psi_U, \tilde{\psi})$ be a generic A-parameter. At each place v of $\mathbb{Q}$, we get a local parameter ψ_{U_v} as well as a local character $\tilde{\psi}_v$. We define the localization of $(\psi_U, \tilde{\psi})$ at v to be $(\psi_{U_v}, \tilde{\psi}_v)$. The localization map $S^\natural_{\psi_U} \to S^\natural_{\psi_{U_v}}$ restricts to give a map $S^+_{\psi_U} \to S^+_{\psi_{U_v}}$ and hence we get a localization map

$$S^\natural_{\psi_{GU}} \to S^\natural_{\psi_{GU_v}}.$$

Similarly, we get a localization map $\overline{S}_{\psi_{GU}} \to \overline{S}_{\psi_{GU_v}}$.

We now define

$$\Pi_{\psi_{GU}}(GU, \varrho_{GU}) := \left\{ \pi = \bigotimes_v{}' \pi_v : \pi_v \in \Pi_{\psi_{GU_v}}(GU_v, \varrho_{GU_v}),\ \langle \pi_v, \cdot \rangle_{GU_v} = 1 \text{ for almost all } v \right\}.$$

We associate to each $\pi = \bigotimes_v \pi_v \in \Pi_{\psi_{GU}}(GU, \varrho_{GU})$ a character of $S^\natural_{\psi_{GU}}$. Each π_v corresponds to a pair $(\pi_v^\flat, \chi_v)$, where $\pi_v^\flat \in \Pi(U)$. We then define a global pairing by the formula

$$\langle \pi, (s, c) \rangle_{GU} := \prod_v \langle (\pi_v^\flat, \chi_v), (s_v, c_v) \rangle_{GU_v}, \qquad (s, c) \in S^\natural_{\psi_{GU}},$$

where (s_v, c_v) is the image of (s, c) under the localization map defined above. We claim that $\langle \pi, \cdot \rangle$ descends to a character on $\overline{S}_{\psi_{GU}}$. Indeed, by definition we have

$$\langle \pi, (s, c) \rangle_{GU} = \prod_v \langle \pi_v^\flat, s_v \rangle_{U^v}\, z_{GU_v}(c_v).$$

We showed previously that $\prod_v \langle \pi_v^\flat, s_v \rangle_{U_v}$ descends to $\overline{S}_{\psi_{GU}} = \overline{S}_{\psi_U}$ and $\prod_v z_{GU_v}(c_v)$ is trivial by [32, Proposition 15.6].

Proposition 2.26. For ψ_{GU} a generic A-parameter of GU, we have the following equality of sets:

$$\Pi_{\psi_{GU}}(GU, \varrho_{GU}, \epsilon_{GU}) = \{ \pi \in \Pi_{\psi_{GU}}(GU, \varrho_{GU}) : \langle \pi, \cdot \rangle_{GU} = \epsilon_{GU} \}.$$

We note that since we are assuming ψ_{GU} is generic, we in fact have $\epsilon_{GU} = 1$.

Proof. The left-hand side consists of all pairs $(\pi^\flat, \chi)$ such that $\pi^\flat \in \Pi_{\psi_U}(U, \varrho_U, \epsilon_U)$. By definition, we have

$$\Pi_{\psi_U}(U, \varrho_U, \epsilon_U) = \{ \pi \in \Pi_{\psi_U}(U, \varrho_U) : \langle \pi, \cdot \rangle_U = \epsilon_U \}.$$

Hence we just need to show that $\langle \pi^\flat, \cdot \rangle_U$ is trivial if and only if $\langle (\pi^\flat, \chi), \cdot \rangle_{GU}$ is. But this is clear since these are the same character of $\overline{S}_{\psi_{GU}} = \overline{S}_{\psi_U}$. □

Remark 2.27. For our purposes, we also need to generalize the above description of automorphic representations to the groups $G(U(n_1) \times \cdots \times U(n_k))(\mathbb{A})$ with $n_1 + \cdots + n_k = n$ odd. In this case, Proposition 2.23 still holds true ([12, Section CHL.IV.C, Proposition 1.3.5]) and then the above process can be applied without any major change.

3. Endoscopic character identities

Fix v a finite place of $\mathbb{Q}$ and let $E/\mathbb{Q}_v$ be a quadratic extension and n an odd natural number. Our goal in this section is to prove the endoscopic character identities for elliptic endoscopic groups of $G(U(n_1) \times \cdots \times U(n_r))$ with $n_1 + \cdots + n_r = n$ and $U(n_i)$ an inner form of $U(n_i)$. We prove this using the fact that these identities hold for $U(n_1) \times \cdots \times U(n_r)$ as in [25, 40]. Note that we are not assuming all n_i are odd, though at least one must be since n is. We show that

- if the endoscopic character identities hold for $U(n_1) \times \cdots \times U(n_r)$, then they also hold for $U(n_1) \times \cdots \times U(n_r) \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m$, where $\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m$ embeds diagonally into the center of $G(U(n_1) \times \cdots \times U(n_r))$,
- if the endoscopic character identities hold for $U(n_1) \times \cdots \times U(n_r) \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m$, then they hold for $G(U(n_1) \times \cdots \times U(n_r))$.

We recall the statement of the endoscopic character identity for an extended pure inner twist (G, ϱ, z) of a quasi-split reductive group G^* over $\mathbb{Q}_v$ with refined endoscopic datum $(H, s, {}^L\eta)$. Fix a local Whittaker datum $\mathfrak{w}$ of G giving a Whittaker normalized transfer factor $\Delta[\mathfrak{w}, \varrho, z]$ (as in [24, Section 4.3]) between $(H, s, {}^L\eta)$ and G. Suppose that $f \in \mathcal{H}(G)$ and $f^H \in \mathcal{H}(H)$ are $\Delta[\mathfrak{w}, \varrho, z]$-matching functions.

Let $\psi \in \Psi^+(G)$ and $\psi^H \in \Psi^+(H)$ be such that $\psi = {}^L\eta \circ \psi^H$. Let $\Pi_{\psi^H}(H), \Pi_\psi(G, \varrho)$ denote the respective A-packets for the parameters. Then the endoscopic character identity states that

$$\sum_{\pi^H \in \Pi_{\psi^H}(H)} \langle \pi^H, s_{\psi^H} \rangle \operatorname{tr}(\pi^H \mid f^H) = e(G) \sum_{\pi \in \Pi_\psi(G, \varrho)} \langle \pi, s_\psi s \rangle \operatorname{tr}(\pi \mid f); \tag{3.1}$$

where $\langle \pi, s \rangle$ is as defined in Theorem 2.8 and Theorem 2.12. The elements s_ψ and s_{ψ^H} are defined to be the image of $(1, -1)$ under ψ and ψ^H respectively and $e(G)$ is the Kottwitz sign.

According to a theorem of Harish-Chandra, the trace distribution $f \mapsto \operatorname{tr}(\pi \mid f)$ is given by integrating against the Harish-Chandra character, which is a locally constant function Θ_π of $G(\mathbb{Q}_v)_{sr}$ (where $G(\mathbb{Q}_v)_{sr}$ denotes the strongly regular semisimple elements of $G(\mathbb{Q}_v)$). Then the above identity is equivalent to the equality

$$\int_{H(\mathbb{Q}_v)_{sr}} \sum_{\pi^H \in \Pi_{\psi^H}(H)} \langle \pi^H, s_{\psi^H} \rangle f^H(g) \Theta_{\pi^H}(g)\, dg = e(G) \int_{G(\mathbb{Q}_v)_{sr}} \sum_{\pi \in \Pi_\psi(G, \varrho)} \langle \pi, s_\psi s \rangle f(g) \Theta_\pi(g)\, dg.$$

We remark that a Harish-Chandra character exists for parabolically induced representations $I_P^G(\sigma)$ by [57, Theorem 3] and that this holds even in the case where the induction is not irreducible. Hence, $\pi \in \Pi_\psi(G, \varrho)$ have Harish-Chandra characters even in the case where $\psi \in \Psi^+(G)$.

3.1. Endoscopic identities for $U(n_1) \times \cdots \times U(n_r) \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m$.

In this subsec-tion we use the notation U to denote the group $U(n_1) \times \cdots \times U(n_r)$. Our goal is to prove the endoscopic character identities for $U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m$ using the fact that these identities are known for U by [25] (Theorems 2.8 and 2.12 in the present paper).

In fact, we will prove the following more general result. Fix quasi-split reductive groups G_i^* for $i \in \{1, 2\}$. Let (G_i, ϱ_i, z_i) be extended pure inner twists of G_i^*. Let $(H_i, s_i, {}^L\eta_i)$ be refined endoscopic data for G_i. We denote by $(H_1 \times H_2, s_1 \times s_2, {}^L\eta_1 \times {}^L\eta_2)$ the corresponding endoscopic datum of $G_1 \times G_2$. Fix a character $\psi : \mathbb{Q}_v \to \mathbb{C}^\times$ and $\mathbb{Q}_v$-splittings of G_i^*. This induces a Whittaker datum $\mathfrak{w}_i$ of G_i^* as well as the Whittaker datum $\mathfrak{w}_1 \times \mathfrak{w}_2$ of $G_1^* \times G_2^*$. We will prove that if the endoscopic character identities are satisfied for G_i and $(H_i, s_i, {}^L\eta_i)$, then they are also satisfied for $G_1 \times G_2$ and $(H_1 \times H_2, s_1 \times s_2, {}^L\eta_1 \times {}^L\eta_2)$.

Fix $\psi^{G_1 \times G_2} \in \Psi^+(G_1 \times G_2)$ and suppose $\psi^{H_1 \times H_2} \in \Psi^+(H_1 \times H_2)$ is such that

$$\psi^{G_1 \times G_2} = ({}^L\eta_1 \times {}^L\eta_2) \circ \psi^{H_1 \times H_2}.$$

Then $\psi^{H_1 \times H_2}$ factors as a product of parameters ψ^{H_1} of H_1 and ψ^{H_2} of H_2. As a result, $\psi^{G_1 \times G_2}$ factors as a product of parameters ψ^{G_1} of G_1 and ψ^{G_2} of G_2 such that

$$\psi^{G_i} = {}^L\eta_i \circ \psi^{H_i}.$$

We need to show that for two arbitrary $[\mathfrak{w}_1 \times \mathfrak{w}_2, \varrho_1 \times \varrho_2, z_1 \times z_2]$-matching functions $f \in \mathcal{H}(G_1 \times G_2)$ and $f' \in \mathcal{H}(H_1 \times H_2)$, the following identity holds:

$$\int_{(H_1 \times H_2)(\mathbb{Q}_v)_{sr}} \sum_{\pi' \in \Pi_{\psi^{H_1 \times H_2}}} \langle \pi', s_{\psi^{H_1 \times H_2}} \rangle f'(g) \Theta_{\pi'}(g)\, dg \tag{3.2}$$
$$= e(G_1 \times G_2) \int_{(G_1 \times G_2)(\mathbb{Q}_v)_{sr}} \sum_{\pi \in \Pi_{\psi^{G_1 \times G_2}}} \langle \pi, s\, s_{\psi^{G_1 \times G_2}} \rangle f(g) \Theta_\pi(g)\, dg.$$

The packets $\Pi_{\psi^{H_1 \times H_2}}(H_1 \times H_2)$ resp. $\Pi_{\psi^{G_1 \times G_2}}(G_1 \times G_2, \varrho_1 \times \varrho_2)$ consist of represen-tations of the form $\pi^{H_1} \otimes \pi^{H_2}$ resp. $\pi^{G_1} \otimes \pi^{G_2}$, where π^{H_i} resp. π^{G_i} are representations in $\Pi_{\psi^{H_i}}(H_i)$ resp. $\Pi_{\psi^{G_i}}(G_i, \varrho_i)$. The pairings $\langle \pi^{H_1} \otimes \pi^{H_2}, \cdot \rangle$ resp. $\langle \pi^{G_1} \otimes \pi^{G_2}, \cdot \rangle$ are defined as $\langle \pi^{H_1}, \cdot \rangle \langle \pi^{H_2}, \cdot \rangle$ resp. $\langle \pi^{G_1}, \cdot \rangle \langle \pi^{G_2}, \cdot \rangle$. It is not difficult to see that

$$\Theta_{\pi^{G_1} \otimes \pi^{G_2}} = \Theta_{\pi^{G_1}} \otimes \Theta_{\pi^{G_2}}.$$

It is also a basic property of the Kottwitz sign that $e(G_1 \times G_2) = e(G_1)e(G_2)$.

Moreover, a function $f \in \mathcal{H}(G_1 \times G_2)$ can be written as a sum of functions of the form $f_1 \otimes f_2$, where $f_1 \in \mathcal{H}(G_1)$ and $f_2 \in \mathcal{H}(G_2)$. Hence, for every such $f_1 \otimes f_2$ we have an equality between the quantities

$$e(G_1 \times G_2) \int_{(G_1 \times G_2)(\mathbb{Q}_v)_{sr}} \sum_{\pi \in \Pi_{\psi^{G_1 \times G_2}}} \langle \pi, s\, s_{\psi^{G_1 \times G_2}} \rangle (f_1 \otimes f_2)(x) \Theta_\pi(x)\, dx$$

and

$$e(G_1) \int_{G_1(\mathbb{Q}_v)_{sr}} \sum_{\pi^{G_1} \in \Pi_{\psi^{G_1}}} \langle \pi^{G_1}, s\, s_{\psi^{G_1}} \rangle f_1(x) \Theta_{\pi^{G_1}}(x)\, dx$$
$$\cdot\, e(G_2) \int_{G_2(\mathbb{Q}_v)_{sr}} \sum_{\pi^{G_2} \in \Pi_{\psi^{G_2}}} \langle \pi^{G_2}, s\, s_{\psi^{G_2}} \rangle f_2(y) \Theta_{\pi^{G_2}}(y)\, dy.$$

Similarly, for every $f_1^{H_1} \otimes f_2^{H_2}$ with $f_1^{H_1} \in \mathcal{H}(H_1)$ a matching function of f_1 and $f_2^{H_2} \in \mathcal{H}(H_2)$ a matching function of f_2 we have an equality between

$$\int_{(H_1 \times H_2)(\mathbb{Q}_v)_{sr}} \sum_{\pi' \in \Pi_{\phi_{H_1 \times H_2}}} \langle \pi', s_{H_1 \times H_2} \rangle (f_1^{H_1} \otimes f_2^{H_2})(x) \Theta_{\pi'}(x)\, dx$$

and

$$\int_{H_1(\mathbb{Q}_v)_{sr}} \sum_{\pi_{H_1} \in \Pi_{\phi_{H_1}}} \langle \pi_{H_1}, s_{H_1} \rangle f_1^{H_1}(x) \Theta_{\pi_{H_1}}(x)\, dx \int_{H_2(\mathbb{Q}_v)_{sr}} \sum_{\pi_{H_2} \in \Pi_{\phi_{H_2}}} \langle \pi_{H_2}, s_{H_2} \rangle f_2^{H_2}(y) \Theta_{\pi_{H_2}}(y)\, dy.$$

In order to prove equation (3.2), it suffices to prove that for each $f_1 \otimes f_2 \in \mathcal{H}(G_1 \times G_2)$, we may choose a $[\mathfrak{w}_1 \times \mathfrak{w}_2, \varrho_1 \times \varrho_2, z_1 \times z_1]$-matching function $f_1^{H_1} \otimes f_2^{H_2} \in \mathcal{H}(H_1 \times H_2)$ such that $f_i^{H_i} \in \mathcal{H}(H_i)$ and $f_i \in \mathcal{H}(G_i)$ are $[\mathfrak{w}_i, \varrho_i, z_i]$-matching. This follows from the following lemma.

Lemma 3.1. *If $f_i^{H_i} \in \mathcal{H}(H_i)$ and $f_i \in \mathcal{H}(G_i)$ are $[\mathfrak{w}_i, \varrho_i, z_i]$-matching functions, then*

$$f_1^{H_1} \otimes f_2^{H_2} \in \mathcal{H}(H_1 \times H_2) \quad \text{and} \quad f_1 \otimes f_2 \in \mathcal{H}(G_1 \times G_2)$$

are $[\mathfrak{w}_1 \times \mathfrak{w}_2, \varrho_1 \times \varrho_2, z_1 \times z_2]$-matching functions.

Proof. Pick $\gamma_H = (\gamma_{H_1}, \gamma_{H_2}) \in (H_1 \times H_2)(\mathbb{Q}_v)$ such that γ_H is strongly regular and transfers to a strongly regular $\delta = (\delta_1, \delta_2) \in (G_1 \times G_2)(\mathbb{Q}_v)$. Then we need to show that

$$SO^{H_1 \times H_2}_{\gamma_H}(f_1^{H_1} \otimes f_2^{H_2}) = \sum_{\delta' \sim_{st} \delta} \Delta[\mathfrak{w}_1 \times \mathfrak{w}_2, \varrho_1 \times \varrho_2, z_1 \times z_1](\gamma_H, \delta') O^{G_1 \times G_2}_{\delta'}(f_1 \otimes f_2) \tag{3.3}$$

where the sum is taken over the set of δ' that are stably conjugate to δ. By definition, for $\delta_i \in G_i$ and $f_i \in C_c^\infty(G_i)$ we have

$$O^{G_1 \times G_2}_{\delta}(f_1 \otimes f_2) = O^{G_1}_{\delta_1}(f_1) O^{G_2}_{\delta_2}(f_2).$$

Moreover, an element $(\gamma_1', \gamma_2') \in H_1 \times H_2$ is stable conjugate to $(\gamma_1, \gamma_2) \in H_1 \times H_2$ if and only if γ_1' is stable conjugate to γ_1 in H_1 and γ_2' is stable conjugate in γ_2 in H_2. Therefore we have

$$SO^{H_1 \times H_2}_{\gamma_H}(f_1^{H_1} \otimes f_2^{H_2}) = SO^{H_1}_{\gamma_{H_1}}(f_1^{H_1}) SO^{H_2}_{\gamma_{H_2}}(f_2^{H_2})$$

and similarly

$$\begin{aligned} &\sum_{\delta' \sim_{st} \delta} \Delta[\mathfrak{w}_1 \times \mathfrak{w}_2, \varrho_1 \times \varrho_2, z_1 \times z_1](\gamma_H, \delta') O^{G_1 \times G_2}_{\delta'}(f_1 \otimes f_2) \\ &= \sum_{\delta_1' \sim_{st} \delta_1,\ \delta_2' \sim_{st} \delta_2} \Delta[\mathfrak{w}_1 \times \mathfrak{w}_2, \varrho_1 \times \varrho_2, z_1 \times z_1](\gamma_H, \delta') O^{G_1}_{\delta_1'}(f_1) O^{G_2}_{\delta_2'}(f_2). \end{aligned} \tag{3.4}$$

We will prove in Lemma 3.4 that

$$\Delta[\mathfrak{w}_1 \times \mathfrak{w}_2, \varrho_1 \times \varrho_2, z_1 \times z_1](\gamma_H, \delta') = \Delta[\mathfrak{w}_1, \varrho_1, z_1](\gamma_{H_1}, \delta_1') \Delta[\mathfrak{w}_2, \varrho_2, z_2](\gamma_{H_2}, \delta_2').$$

We can then rewrite the right-hand side of (3.4) as

$$\sum_{\pi_1} \langle \mathfrak{w}_1,\varrho_1,z_1\rangle(\pi^{H_1},\pi_1)\,\Theta^{G_1}_{\pi_1}(f_1) \sum_{\pi_2} \langle \mathfrak{w}_2,\varrho_2,z_2\rangle(\pi^{H_2},\pi_2)\,\Theta^{G_2}_{\pi_2}(f_2)$$

and because $f_i^{H_i}$ and f_i are $\langle \mathfrak{w}_i,\varrho_i,z_i\rangle$ matching functions, this is exactly

$$SO^{H_1}_{H_1}(f_1^{H_1})\,SO^{H_2}_{H_2}(f_2^{H_2}).$$

In other words, equation (3.3) is true. □

3.2. Endoscopic identities for $G(U(n_1)\times\cdots\times U(n_r))$. We now have the endoscopic character identities for $U\times \operatorname{Res}_{E/\mathbb{Q}_v}\mathbb{G}_m$ and need to show they also hold for GU, where we use the letter GU to denote the group $G(U(n_1)\times\cdots\times U(n_r))$ until the end of this section. We have a surjection of algebraic groups

$$P: U\times \operatorname{Res}_{E/\mathbb{Q}_v}\mathbb{G}_m \to GU,$$

with kernel isomorphic to $U(1)$.

We fix quasi-split groups $U\times\operatorname{Res}_{E/\mathbb{Q}_v}\mathbb{G}_m$ and GU as well as an extended pure inner twist (GU,ϱ_{GU},z_{GU}) of GU. The projection P induces a surjection

$$B(\mathbb{Q}_v, U\times\operatorname{Res}_{E/\mathbb{Q}_v}\mathbb{G}_m)\to B(\mathbb{Q}_v, GU),$$

hence (after possibly modifying (GU,ϱ_{GU},z_{GU}) in its isomorphism class) we can choose an extended pure inner twist $(U\times\operatorname{Res}_{E/\mathbb{Q}_v}\mathbb{G}_m,\varrho_U,z_U)$ such that P takes ϱ_U to ϱ_{GU} and z_U to z_{GU}. The extended pure inner twist $(U\times\operatorname{Res}_{E/\mathbb{Q}_v}\mathbb{G}_m,\varrho_U,z_U)$ restricts to give (U,ϱ_U^0,z_U^0) and $(\operatorname{Res}_{E/\mathbb{Q}_v}\mathbb{G}_m,\varrho_{\mathbb{G}_m},z_{\mathbb{G}_m})$. We fix compatible $\Gamma_{\mathbb{Q}_p}$-splittings of these groups as well as a character $\psi:\mathbb{Q}_v\to\mathbb{C}^\times$. Hence we get compatible Whittaker data which we denote by $\mathfrak{w}_U$ and $\mathfrak{w}_{GU}$ respectively.

A crucial input in the case we consider (where $n=n_1+\cdots+n_r$ is odd) is that the projection P is also a surjection on $\mathbb{Q}_v$-points. This follows from Lemma 2.2. Hence we get a map

$$\operatorname{Irr}(GU(\mathbb{Q}_v))\hookrightarrow \operatorname{Irr}((U\times\operatorname{Res}_{E/\mathbb{Q}_v}\mathbb{G}_m)(\mathbb{Q}_v)),$$

given by pullback. The image of this map is the set of irreducible representations π such that $\omega_\pi|_{U(1)(\mathbb{Q}_v)}=1|_{U(1)(\mathbb{Q}_v)}$, where ω_π is the central character of π and the $U(1)$ in question is the kernel of P. If this is satisfied by a single member of an A-packet of $U\times\operatorname{Res}_{E/\mathbb{Q}_v}\mathbb{G}_m$, then it will be satisfied by the entire packet since elements of an A-packet have the same central character ([25, Theorem 1.6.1] and Theorem 2.12). In light of Theorem 2.19, the A-packets of GU are in a natural way a subset of the A-packets of $U\times\operatorname{Res}_{E/\mathbb{Q}_v}\mathbb{G}_m$.

Since the kernel of P is compact, it follows that any $f\in\mathcal{H}(GU)$ lifts to an element $f'\in\mathcal{H}(U\times\operatorname{Res}_{E/\mathbb{Q}_v}\mathbb{G}_m)$. Suppose π is an admissible representation of $GU(\mathbb{Q}_v)$ and π' is the pullback to $\operatorname{Irr}(U\times\operatorname{Res}_{E/\mathbb{Q}_v}\mathbb{G}_m)$. Then to prove the endoscopic character identities for GU it will be necessary to relate $\operatorname{tr}(\pi\mid f)$ and $\operatorname{tr}(\pi'\mid f')$. We have

$$\begin{aligned}\Theta_{\pi'}(f') &= \int_{(U\times\operatorname{Res}_{E/\mathbb{Q}_v}\mathbb{G}_m)(\mathbb{Q}_v)} f'(g)\Theta_{\pi'}(g)\,dg\\ &= \int_{GU(\mathbb{Q}_v)} f(g)\Theta_\pi(g)\,dg\int_{U(1)(\mathbb{Q}_v)}dz = \operatorname{Vol}(U(1)(\mathbb{Q}_v))\,\Theta_\pi(f),\end{aligned}$$

where the middle equality holds by [43, equation (3.21)].

Analogously in the endoscopic case, we have a map

$$P^H : H \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m \to G(H),$$

with kernel $U(1)$, where $H = \prod_{i=1}^{r} U(n_i^+) \times U(n_i^-)$ such that $n_i = n_i^+ + n_i^-$ is an endoscopic group of U and $G(H)$ is the associated similitude group. Suppose $n = n_1 + \cdots + n_r$ is odd. By Lemma 2.2, the map is a surjection on $\mathbb{Q}_v$-points.

We fix a refined endoscopic datum $(G(H), s, {}^L\eta)$ for GU as in Section 2.1. The map $\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m \cong Z(GU) \hookrightarrow GU$ induces a map of L-groups ${}^LGU \to {}^L(\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m)$. We get an analogous map for $G(H)$ and one checks there is an induced map

$${}^L\tilde{\eta} : {}^L(\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m) \to {}^L(\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m),$$

giving a commutative diagram

$$\begin{array}{ccc} {}^LG(H) & \xrightarrow{\ {}^L\eta\ } & {}^LGU \\ \downarrow & & \downarrow \\ {}^L(\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m) & \xrightarrow{\ {}^L\tilde{\eta}\ } & {}^L(\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m). \end{array}$$

We now fix an endoscopic datum of $U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m$ which we denote by

$$(H \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m, s', {}^L\eta')$$

as follows. We set $s' = {}^LP^H(s)$ and we fix ${}^L\eta'$ such that the restriction to $\hat{H}$ induces an elliptic endoscopic datum for U as in Section 2.1 compatible with our fixed datum for GU and such that ${}^L\eta'$ restricted to $\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m$ is just ${}^L\tilde{\eta}$. In particular, we have a commutative diagram:

(3.5)
$$\begin{array}{ccc} {}^L(H \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m) & \xrightarrow{\ {}^L\eta'\ } & {}^L(U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m) \\ {\scriptstyle {}^LP^H}\uparrow & & \uparrow{\scriptstyle {}^LP} \\ {}^LG(H) & \xrightarrow{\ {}^L\eta\ } & {}^LGU. \end{array}$$

We now prove the following lemma.

Lemma 3.2. *Using the above normalizations, if $f \in \mathcal{H}(GU)$ and $f^H \in \mathcal{H}(G(H))$ are $(\mathfrak{w}_{GU}, \varrho_{GU}, z_{GU})$-matching, then the pullbacks*

$$f' \in \mathcal{H}(U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m) \quad \text{and} \quad f'^H \in \mathcal{H}(H \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m)$$

are $(\mathfrak{w}_U, \varrho_U, z_U)$-matching.

We begin by proving an auxiliary lemma.

Lemma 3.3. *For $(\gamma, z) \in U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m(\mathbb{Q}_v)$, the map P gives a bijection between conjugacy classes in $U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m(\mathbb{Q}_v)$ that are stably conjugate to (γ, z) and conjugacy classes in $GU(\mathbb{Q}_v)$ that are stably conjugate to z. The analogous result also holds for the map P^H.*

Proof. If (γ', z') is conjugate or stable conjugate to (γ, z) in $U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m(\mathbb{Q}_v)$, then it is clear that $\gamma' z'$ and γz are conjugate or stably conjugate in $GU(\mathbb{Q}_v)$. Now, suppose that $g, \gamma z \in GU(\mathbb{Q}_v)$ are conjugate or stably conjugate. Then they must have the same similitude factor. In particular, this means that gz^{-1} has trivial similitude factor and so

$$(gz^{-1}, z) \in U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m(\mathbb{Q}_v);$$

and clearly $P(gz^{-1}, z) = g$.

We now aim to show that (gz^{-1}, z) is conjugate or stably conjugate to (γ, z). To simplify the notation, we just show that (gz^{-1}, z) and (γ, z) are conjugate (although the argument to show stable conjugacy is similar).

Let $x \in GU(\mathbb{Q}_v)$ be such that $xgx^{-1} = \gamma z$. We want to show that x can be chosen to be an element of $U(\mathbb{Q}_v)$. Since the map P is surjective on $\mathbb{Q}_v$ points, we can write $x = ur$ such that $u \in U(\mathbb{Q}_v)$ and $r \in \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m(\mathbb{Q}_v)$. Then r lies in the center of $GU(\mathbb{Q}_v)$ and hence we have $ugu^{-1} = \gamma z$ as desired. Finally, we finish the argument by observing that

$$(u, 1)(gz^{-1}, z)(u, 1)^{-1} = (\gamma, z)$$

since the restriction of P to the first component is an injection. □

We now prove Lemma 3.2.

Proof. We choose a strongly regular semisimple $(\gamma_H, z) \in H \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m(\mathbb{Q}_v)$ that transfers to a strongly regular $(\gamma, z) \in U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m(\mathbb{Q}_v)$. Then we need to show that

$$SO_{(\gamma_H, z)}(f'^{H}) = \sum_{(\gamma', z) \sim_{st} (\gamma, z)} \Delta[w_U, \varrho_U, z_U]((\gamma_H, z), (\gamma', z)) O_{(\gamma', z)}(f').$$

Expanding this is equivalent to showing that

$$\sum_{(\gamma'_H, z) \sim_{st} (\gamma_H, z)} \int_{H \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m / T_{(\gamma_H, z)}} f^{H'}(h(\gamma'_H, z)h^{-1})\, dh$$

equals

$$\sum_{(\gamma', z) \sim_{st} (\gamma, z)} \Delta[w_U, \varrho_U, z_U]((\gamma_H, z), (\gamma', z)) \int_{U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m / T_{(\gamma', z)}} f'(g(\gamma, z)g^{-1})\, dg.$$

Note that the kernels of P^H, P are contained within $T_{(\gamma_H, z)}$ and $T_{(\gamma', z)}$ respectively. Hence we have $(U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m)(\mathbb{Q}_v)/T_{(\gamma, z)}(\mathbb{Q}_v) = GU(\mathbb{Q}_v)/T_{\gamma z}(\mathbb{Q}_v)$ and the analogous statement also holds for P^H.

By Lemma 3.3, we can rewrite the equation above as

$$\sum_{\gamma'_H z \sim_{st} \gamma_H z} \int_{G(H)/T_{\gamma'_H z}} f^{H}(h \gamma'_H z h^{-1})\, dh$$

equals

$$\sum_{\gamma' z \sim_{st} \gamma z} \Delta[w_U, \varrho_U, z_U]((\gamma_H, z), (\gamma', z)) \int_{GU/T_{\gamma' z}} f(g \gamma' z g^{-1})\, dg.$$

In Lemma 3.8 we prove that there is an equality of transfer factors

$$\Delta[\mathfrak{w}_U, \varrho_U, z_U](\phi_H, z, (\phi, z)) = \Delta[\mathfrak{w}_{GU}, \varrho_{GU}, z_{GU}](\phi_H z, \phi z).$$

Hence, the above equation reduces to

$$SO_{\phi_H z}(f^H) = \sum_{z} \Delta[\mathfrak{w}_{GU}, \varrho_{GU}, z_{GU}](\phi_H z, \phi z) O_{\phi z}(f), \quad \phi z \sim_{st}$$

which is true by assumption. □

With this lemma in hand, we now prove the endoscopic character identities. Pick a parameter $\phi \in \Phi^+(GU)$ and let $\phi' \in \Phi^+(U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m)$ be the composition of ϕ with the map ${}^L GU \to {}^L(U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m)$. We suppose ϕ factors through ${}^L G(H)$ and pick $\phi_{G(H)}$ so that $\phi = {}^L\eta \circ \phi_{G(H)}$. We can write $\phi' = \phi_U \times \phi_{\mathbb{G}_m}$, where ϕ_U is the image of ϕ under the map ${}^L GU \to {}^L U$. Diagram (3.5) implies that there is a parameter ϕ'_H such that $\phi' = {}^L\eta' \circ \phi'_H$.

Fix matching functions $f \in \mathcal{H}(GU)$, $f^H \in \mathcal{H}(G(H))$. Write

$$s = (\hat{s}, c) \in \hat{H} \times \hat{\mathbb{G}}_m = \widehat{G(H)}.$$

Now, for ϕ in the previous paragraph with packet $\Pi_\phi(GU, \varrho_{GU}, z_{GU})$, we have by the definition of the pairing $\langle\cdot,\cdot\rangle_{GU}$ in Section 2.3.1 that

$$\begin{aligned} e(GU) &\sum_{\pi \in \Pi_\phi(GU, \varrho_{GU}, z_{GU})} \langle \pi, s\rangle_{GU} \operatorname{tr}(\pi \mid f) \\ &= e(GU) \sum_{\pi \in \Pi_\phi} \langle \pi_U, \hat{s}\rangle_U \, z_{GU}(c) \operatorname{tr}(\pi \mid f), \end{aligned}$$

where on the right-hand side, π corresponds to $\pi_U \in \mathrm{Irr}(U \times \mathrm{Res}_{E/\mathbb{Q}_p} \mathbb{G}_m)$. We showed above that there is a natural bijection

$$\Pi_\phi(GU, \varrho_{GU}) \to \Pi_{\phi'}(U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m, \varrho_U),$$

and we related the traces of corresponding representations. The pairing

$$\langle\cdot,\cdot\rangle_{U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m} : \Pi_{\phi'}(U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m) \times S^\natural_{\phi'} \to \mathbb{C}$$

is given as a product of the pairings for U and $\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m$, and we remark that the pairing on $\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m$ is given by z_{GU} from the way we chose $(\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m, \varrho_{\mathbb{G}_m}, z_{\mathbb{G}_m})$. Hence we have the above equals

$$e(GU) \frac{1}{\mathrm{Vol}(U(1)(\mathbb{Q}_v))} \sum_{\pi' \in \Pi_{\phi'}} \langle \pi', s\rangle_{U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m} \operatorname{tr}(\pi' \mid f').$$

Now, using that $e(GU) = e(U) = e(U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m)$ (see [26, p. 292]) we can apply the previously established endoscopic character identity for $U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m$ to get that the above equals

$$\frac{1}{\mathrm{Vol}(U(1)(\mathbb{Q}_v))} \sum_{\pi_H \in \Pi_{\phi'_H}} \langle \pi_H, \tilde{s}\rangle_{H \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m} \operatorname{tr}(\pi_H \mid f'^H).$$

Finally, we relate this to $G(H)$ using that $G(H)$ and $H \cong \mathrm{Res}_{E_v/\mathbb{Q}_v} \mathbb{G}_m$ are both assumed to be trivial extended pure inner forms so that the pairings are especially simple. We get

$$\sum_{\pi_{G(H)} \in \Pi_{\phi_{G(H)}}} \langle \pi_{G(H)}, s_{\phi_{G(H)}} \rangle_{G(H)} \operatorname{tr}(\pi_{G(H)} \mid f^H),$$

which is the desired formula.

3.3. Transfer factor identities. In this subsection, we prove a number of identities relating various transfer factors. These identities are used in the previous subsections. Remark that we use the letter Δ resp. Δ' to denote the transfer factors that are compatible with the geometric normalization resp. arithmetic normalization of the local Artin reciprocity map.

3.3.1. Transfer factors of a product. We temporarily return to the notation of Section 3.1. We denote by G the group $G_1 \times G_2$ and by G^* the group $G_1^* \times G_2^*$.

We prove the following lemma

Lemma 3.4. *Let $(\gamma_1, \gamma_2) \in (H_1 \times H_2)(\mathbb{Q}_v)_{sr}$ and $(\delta_1, \delta_2) \in (G_1 \times G_2)(\mathbb{Q}_v)_{sr}$ be related elements. We have*

$$\begin{aligned}\Delta[\mathfrak{w}_1 \times \mathfrak{w}_2, \varrho_1 \times \varrho_2, z_1 \times z_2]&((\gamma_1, \gamma_2), (\delta_1, \delta_2)) \\ &= \Delta[\mathfrak{w}_1, \varrho_1, z_1](\gamma_1, \delta_1)\Delta[\mathfrak{w}_2, \varrho_2, z_2](\gamma_2, \delta_2).\end{aligned}$$

Proof. Each transfer factor is a product of terms

$$\epsilon_L^{G_i}(V^{G_i}, \psi)\Delta_I^{G_i}\Delta_{II}^{G_i}\Delta_{III_2,D}^{G_i}\Delta_{IV}^{G_i}\langle \mathrm{inv}[z_i](\gamma_i, \delta_i), s_{\gamma_i}\rangle^{-1}.$$

We state everything for G_i but the definitions are analogous for G. We now explain the terms in the above formula. Notably, all the terms except the last only depend on G_i^* and H_i (as opposed to G_i). Fix a $\delta_i' \in G_i^*(\mathbb{Q}_v)$ such that δ_i' is stably conjugate to $\varrho_i^{-1}(\delta_i)$. Recall that we have fixed $\mathbb{Q}_v$-splittings $(T_i, B_i, \{X_{i,\alpha}\})$ for G_i^* as well as the $\mathbb{Q}_v$-splitting $(T = T_1 \times T_2, B = B_1 \times B_2, \{X_\alpha\} = \{X_{1,\alpha}\} \cup \{X_{2,\alpha}\})$ of G^*.

Now, V^{G_i} is the degree 0 virtual Galois representation $X^*(T_i) \otimes \mathbb{C} - X^*(T_{H_i}) \otimes \mathbb{C}$ and ψ is the additive character we fixed in order to define our Whittaker datum. The term $\epsilon_L(V^{G_i}, \psi)$ is the local ϵ-factor of this representation normalized as in [56, Section 3.6]. We also know that $\epsilon_L(V, \psi)$ is additive for degree 0 virtual representations V (see [56, Theorem. 3.4.1]), therefore

$$\epsilon_L^G(V^G, \psi) = \epsilon_L^{G_1}(V^{G_1}, \psi)\epsilon_L^{G_2}(V^{G_2}, \psi).$$

We denote by S_i the centralizer of δ_i' and $S_i^{H_i}$ the centralizer of γ_i so that $S = S_1 \times S_2$ and $S^H = S_1^{H_1} \times S_2^{H_2}$ are the centralizers of (δ_1', δ_2') resp. (γ_1, γ_2).

We put

$$D_G((\delta_1, \delta_2)) = \prod_{\alpha} |\alpha((\delta_1, \delta_2)) - 1|^{\frac{1}{2}},$$

where the product is over all roots of S in G. Similarly

$$D_{G_i}(\delta_i) = \prod_{\alpha} |\alpha(\delta_i) - 1|^{\frac{1}{2}},$$

where the product is over all roots of S_i in G_i. In particular, we have

$$D_G((\gamma_1,\gamma_2)) = D_{G_1}(\gamma_1)D_{G_2}(\gamma_2).$$

We define $D_H(\delta_1,\delta_2)$ and $D_{H_i}(\delta_i)$ analogously and we also have the equality

$$D_H(\delta_1,\delta_2) = D_{H_1}(\delta_1)D_{H_2}(\delta_2).$$

By definition, $\Delta_{IV} = D_G D_H^{-1}$ so that we have

$$\Delta_{IV}^{G}((\delta_1,\delta_2),(\gamma_1,\gamma_2)) = \Delta_{IV}^{G_1}(\delta_1,\gamma_1)\Delta_{IV}^{G_2}(\delta_2,\gamma_2).$$

For the other terms in the definition of the transfer factors, we need to explain the notions of a-data and χ-data. A set of a-data for the set $R(T,G)$ of absolute roots of S in G is a function

$$R(T,G) \to \overline{\mathbb{Q}}_v^{\times}, \quad \alpha \mapsto a_\alpha$$

which satisfies $a_{-\alpha} = -a_\alpha$ and $a_{\sigma\alpha} = \sigma(a_\alpha)$ for $\sigma \in \Gamma_{\mathbb{Q}_v}$. We recall the notion of χ-data. For $\alpha \in R(T,G)$, we set

$$\Gamma_\alpha = \mathrm{Stab}(\alpha,\Gamma) \quad \text{and} \quad \Gamma_{\pm\alpha} = \mathrm{Stab}(\{\alpha,-\alpha\},\Gamma);$$

and denote F_α, $F_{\pm\alpha}$ the fixed fields of Γ_α resp. $\Gamma_{\pm\alpha}$. A set of χ-data is then a set of characters

$$\chi_\alpha : W_{F_\alpha} \to \mathbb{C}^{\times}$$

satisfying the conditions

$$\chi_{-\alpha} = \chi_\alpha^{-1}; \qquad \chi_{\sigma\alpha} = \chi_\alpha \circ \sigma^{-1};$$

and if $[F_\alpha : F_{\pm\alpha}] = 2$, then $\chi_\alpha|_{F_{\pm\alpha}^{\times}}$ is non-trivial but trivial on the subgroup of norms from $F_\alpha^{\times}$.

Since $\Gamma_{\mathbb{Q}_v}$ acts on $G_{\overline{\mathbb{Q}}}$ and preserves $(G_i)_{\mathbb{Q}_v}$, it follows that if $(a_\alpha)_{\alpha\in R(S_i,G_i)}$ and $(\chi_\alpha)_{\alpha\in R(S_i,G_i)}$ are a-data resp χ-data of (S_i,G_i), then $(a_\alpha)_{\alpha\in R(S,G)}$ and $(\chi_\alpha)_{\alpha\in R(S,G)}$ are a-data resp χ-data of (S,G).

Now, we define

$$\Delta_{II}^{G_i} = \prod_\alpha \chi_\alpha\left(\frac{\alpha(\gamma_i)-1}{a_\alpha}\right)$$

where the product is taken over the set $R(S_i,G_i) \setminus \varphi_{\delta,\gamma}^{-1}R(S_i^{H_i},H_i)$.

We have a similar formula for Δ_{II}^{G} in which the product runs over the set

$$R(S,G) \setminus \varphi_{\delta,\gamma}^{-1}R(S^{H},H) = R(S_1,G_1) \setminus \varphi_{\delta_1,\gamma_1}^{-1}R(S_1^{H_1},H_1) \sqcup R(S_2,G_2) \setminus \varphi_{\delta_2,\gamma_2}^{-1}R(S_2^{H_2},H_2).$$

In particular, we have

$$\Delta_{II}^{G} = \Delta_{II}^{G_1}\Delta_{II}^{G_2}.$$

Next, we want to show that

$$\Delta_{I}^{G} = \Delta_{I}^{G_1}\Delta_{I}^{G_2}.$$

To this end, for $i \in \{1,2\}$ one constructs an element $\lambda_i \in H^1(\Gamma_{\mathbb{Q}_v},(S_i)_{sc})$ and then uses the Tate–Nakayama duality for tori in order to get a pairing $\langle\,,\rangle$ between $H^1(\Gamma_{\mathbb{Q}_v},(S_i)_{sc})$ and $\pi_0([\hat{S}_i/Z(\hat{G}_i)]^{\Gamma_{\mathbb{Q}_v}})$. One can view s_i as an element of $[Z(\hat{H}_i)/Z(\hat{G}_i)]^{\Gamma_{\mathbb{Q}_v}}$, embed the latter into $\hat{S}_i^H/Z(\hat{G}_i)$, and transport it to $\hat{S}_i/Z(\hat{G}_i)$ by the admissible isomorphism $'_{;\imath}$. We then define

$$\Omega_i^G = \langle h_i, s_i \rangle .$$

Because $S = S_1 \times S_2$ and $S_{sc} = (S_1)_{sc} \times (S_2)_{sc}$, to show the necessary product relation for this term, it is enough to show that $= {}_1 \cdot {}_2$.

We recall the construction of for G and S. Write $\Omega(T, G)$ for the absolute Weyl group and let $g \in G$ be such that $gTg^{-1} = S$. For each $\in \Gamma_{\mathbb{Q}_v}$ there exists $\omega(\ ) \in \Omega(T, G)$ such that for all $t \in T$,

$$\omega(\ )(t) = g^{-1}(gtg^{-1})g.$$

Let $\omega(\ ) = s_{\alpha_1} \cdots s_{\alpha_k}$ be a reduced expression and let n_i be the image of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ under the homomorphism $SL_2 \to G$ attached to the simple root vector X_{α_i}. Then $n(\ ) = n_1 \cdots n_k$ is independent of the choice of the reduced expression. So $\in H^1(\Gamma_{\mathbb{Q}_v}, S_{sc})$ is defined by the following 1-cocycle:

$$\mapsto g\Big(\prod_{\alpha} \alpha^{\vee}(a_{\alpha})\, n(\ )\Big) g^{-1} \cdot g(\ g^{-1}) ;$$

where the product runs over the subset $\{\alpha > 0;\ ^{-1}\alpha < 0\}$ of $R(S, G)$, where positivity is determined by the Borel subgroup gBg^{-1}. The construction is analogous for G_i.

Now, we have

(1) $B = B_1 \times B_2$,

(2) $T = T_1 \times T_2$, $S = S_1 \times S_2$,

(3) $R(S, G) = R(S_1, G_1) \bigsqcup R(S_2, G_2)$ so that $(X_\alpha)_G = (X_\alpha)_{G_1} \bigsqcup (X_\alpha)_{G_2}$.

We see that

$$\Omega(T, G) = \Omega(T_1, G_1) \times \Omega(T_2, G_2);$$

and we can take $g = g_1 \times g_2$ so that $\omega(\ )_G = \omega(\ )_{G_1} \times \omega(\ )_{G_2}$. Therefore,

$$n(\ )_G = n(\ )_{G_1} \times n(\ )_{G_2}.$$

We conclude that $= {}_1 \cdot {}_2$.

We are now going to show that

$$\Omega^G_{III_{2,D}} = \Omega^{G_1}_{III_{2,D}} \cdot \Omega^{G_2}_{III_{2,D}}.$$

The construction is as follows. First, we associate to the fixed -datum a $\hat{G}$-embedding (see [36, Section (2.6)])

$$_G : {}^L S \to {}^L G.$$

Next via the admissible isomorphism $'_{;\imath}$ the -datum can be transferred to S^H and gives an L-embedding $^H : {}^L S^H \to {}^L H$. The admissible isomorphism $'_{;\imath}$ also provides dually an L-isomorphism $^L'_{;\imath} : {}^L S \to {}^L S^H$. The composition

$$' = {}^L \circ {}^H \circ {}^L'_{;\imath}$$

gives another L-embedding ${}^L S \to {}^L G$. Via conjugation by an element of $\hat{G}$, we can arrange that $_G$ and $'$ coincide on $\hat{S}$ so that $' = a \cdot {}_G$ for some $a \in Z^1(W_{\mathbb{Q}_v}, \hat{S})$.

The term $\Delta^{G}_{\mathrm{III}_2;D}$ is given by $\langle a, \gamma\rangle$, where the paring $\langle\ ,\ \rangle$ is the Langlands correspondence for tori under the geometric normalization. More precisely, the element a of $Z^1(W_F, \widehat{S})$ is an L-parameter of S. By the local Langlands correspondence for tori, a gives rise to a character $\langle a, \cdot\rangle$ of S.

In our case, we have

$$S = S_1 \times S_2 \quad \text{and} \quad \gamma = (\gamma_1, \gamma_2),$$

so it suffices to show that $a = a_1 \times a_2$. In order to verify that, we need to review carefully the formation of the L-embedding $\xi : {}^LS \to {}^LG$ associated to a χ-datum [36, Section (2.6)].

Fix a Borel pair $(\mathcal{B}, \mathcal{T})$ of $\widehat{G}$ as well as a Borel subgroup B_S (possibly not defined over $\mathbb{Q}_v$) of G containing S. The pair (B_S, S) yields a set of positive coroots of S and equivalently a set of elements of $X^*(\widehat{S})$. Then ξ is defined so that the restriction to $\widehat{S}$ maps $\widehat{S}$ to $\widehat{T}$ by the unique isomorphism mapping our chosen subset of $X^*(\widehat{S})$ to the set of positive roots of $\widehat{T}$ determined by $\widehat{T}$.

To specify ξ, we have only to give a homomorphism

$$w \mapsto \xi(w) = \xi_0(w) \times w;$$

where $\xi_0(w) \in \mathrm{Norm}(\widehat{T}, \widehat{G})$. We require that if $w \mapsto \sigma$ under $W_{\mathbb{Q}_v} \to \Gamma_{\mathbb{Q}_v}$, then $\mathrm{Int}(\xi(w))$ acts on $\widehat{T}$ as the transport by ξ of the action of $\sigma \in \Gamma_{\mathbb{Q}_v}$ on $\widehat{S}$.

We then define

$$\xi(w) = r_p(w) n(\omega(\sigma)) \times w$$

for $w \in W_{\mathbb{Q}_v}$ and $w \mapsto \sigma$ under $W_{\mathbb{Q}_v} \to \Gamma_{\mathbb{Q}_v}$. The term $n(\omega(\sigma))$ is defined above, in the definition of Δ_{I} and we have already seen that $n(\omega(\sigma)) = n(\omega(\sigma))_{G_1} \times n(\omega(\sigma))_{G_2}$.

We recall briefly the construction of $r_p(w)$. We denote by R the set $R_{-}(G, S)$ and define Σ to be the group of automorphisms of R generated by $\Gamma_{\mathbb{Q}_v}$ and $\pm$, where ϵ acts on $X^*(S)$ by $\epsilon(t) = -t$ (as in [36, Lemma 2.1A]). The group Σ acts on R and divides it into Σ-orbits $R = R_1 \sqcup \dots \sqcup R_k$. For each Σ-orbit R_i, we define an element $r_p^i(w)$ and then take the product over the orbits to obtain $r_p(w)$. Since $R_G = R_{G_1} \sqcup R_{G_2}$ and the group Σ preserves R_{G_1}, R_{G_1}, we have

$$r_p(w)_G = r_p(w)_{G_1} r_p(w)_{G_2}.$$

This implies the desired product identity for $\Delta^{G}_{\mathrm{III}_2;D}$.

Finally, we show that

$$\langle \mathrm{inv}[z_1 \times z_2]((\gamma_1, \gamma_2), (\gamma_1, \gamma_2)), s_1 \times s_2\rangle = \langle \mathrm{inv}[z_1](\gamma_1, \gamma_1), s_1\rangle \langle \mathrm{inv}[z_2](\gamma_2, \gamma_2), s_2\rangle.$$

We have a natural isomorphism

$$B(\mathbb{Q}_v, S) = B(\mathbb{Q}_v, S_1 \times S_2),$$

that maps the class of $g^{-1}(z_1 \times z_2)\sigma(g)$ to the product of the classes of $g_1^{-1}z_1\sigma(g_1)$ and $g_2^{-1}z_2\sigma(g_2)$. Moreover, this product decomposition respects the Kottwitz maps

$$\kappa_i : B(\mathbb{Q}_v, S_i) \to X_*(S_i)_{\Gamma_{\mathbb{Q}_v}};$$

defining the above pairings. This implies the desired product formula. □

3.3.2. Transfer factors and changing the normalization.

Lemma 3.5. Let $f \in \mathcal{H}(U)$ and $f^H \in \mathcal{H}(H)$ be $\Delta[w^{-1}, \varrho, z]$-matching functions for an endoscopic datum $(H, s, {}^L\eta)$ of U. If $\iota_U : U(\mathbb{Q}_v) \to U(\mathbb{Q}_v)$ and $\iota_H : H(\mathbb{Q}_v) \to H(\mathbb{Q}_v)$ are the inverse functions, then $f^H \circ \iota_H$ and $f \circ \iota_U$ are matching for the transfer factors $\Delta'[w, \varrho, z]$ with respect to the endoscopic datum $(H, s^{-1}, {}^L\eta)$.

Proof. We consider first the ordinary endoscopic case. Suppose $\gamma_H \in H(\mathbb{Q}_v)$ is strongly regular and transfers to a strongly regular element $\gamma \in U(\mathbb{Q}_v)$. By hypothesis, we have

$$SO^H_{\gamma_H}(f^H) = \sum_{\gamma' \sim_{st} \gamma} \Delta[w^{-1}, \varrho, z](\gamma_H, \gamma') O^U_{\gamma'}(f).$$

Then we need to show that

$$SO^H_{\gamma_H}(f^H \circ \iota_H) = \sum_{\gamma' \sim_{st} \gamma} \Delta'[w, \varrho, z](\gamma_H, \gamma') O^U_{\gamma'}(f \circ \iota_U).$$

Since

$$SO^H_{\gamma_H}(f^H \circ \iota_H) = SO^H_{\gamma_H^{-1}}(f^H) \quad \text{and} \quad O^U_{\gamma'}(f \circ \iota_U) = O^U_{(\gamma')^{-1}}(f),$$

it suffices to show that the transfer factor $\Delta[w^{-1}, \varrho, z](\gamma_H^{-1}, (\gamma')^{-1})$ with respect to the endoscopic datum $(H, s, {}^L\eta)$ is the same as the transfer factor $\Delta'[w, \varrho, z](\gamma_H, \gamma')$ with respect to the endoscopic datum $(H, s^{-1}, {}^L\eta)$.

Recall that the transfer factor $\Delta'[w, \varrho, z]$ is a product of terms

$$\epsilon_L(V, \psi)\Delta_I^{-1}\Delta_{II}\Delta_{III_2}\Delta_{IV}\langle \mathrm{inv}[z](\gamma_H, \gamma), s\rangle$$

which we need to use χ-data and a-data in order to define and moreover the transfer factors do not depend on the choices of χ-data and a-data.

By [33, Section 5.1], the transfer factor $\Delta[w, \varrho, z]$ is defined by the same formula, except that one replaces the term Δ_{III_2} by $\Delta_{III_2, D}$, inverts Δ_I and inverts $\langle \mathrm{inv}[z](\gamma, \gamma), s\rangle$. If one keeps track of the dependence on χ-data and a-data, then $\Delta_{III_2, D, \chi^{-1}}(\gamma_H^{-1}, (\gamma')^{-1}) = \Delta_{III_2, \chi}(\gamma_H, \gamma')$. By using the definitions of the terms appearing in the transfer factors which we recalled in Lemma 3.4, we have

$$\epsilon_L(V, \psi)\Delta^{-1}_{I, a}[s^{-1}]\Delta_{IV}(\gamma_H, \gamma') = \epsilon_L(V, \psi)\Delta_{I, a}[s]\Delta_{IV}(\gamma_H^{-1}, (\gamma')^{-1}),$$

since these terms do not depend on χ-data and where the $\Delta_{I,a}[s]$ notation keeps track of whether we plug in s or s^{-1} into the pairing defining Δ_I. Moreover,

$$\Delta_{II, \chi^{-1}, a^{-1}}(\gamma_H^{-1}, (\gamma')^{-1}) = \Delta_{II, \chi, a}(\gamma_H, \gamma').$$

Thus we have

$$\begin{aligned}\Delta'[w, \varrho, z](\gamma_H, \gamma') &= \epsilon_L(V, \psi)\Delta^{-1}_{I, a}[s^{-1}]\Delta_{IV}(\gamma_H, \gamma')\Delta_{II, \chi, a}(\gamma_H, \gamma') \\ &\quad \Delta_{III_2, \chi}(\gamma_H, \gamma')\langle \mathrm{inv}[z](\gamma, \gamma), s^{-1}\rangle \\ &= \epsilon_L(V, \psi)\Delta_{I, a}[s]\Delta_{IV}(\gamma_H^{-1}, (\gamma')^{-1})\Delta_{II, \chi^{-1}, a^{-1}}(\gamma_H^{-1}, (\gamma')^{-1})\Delta_{III_2, D, \chi^{-1}} \\ &\quad (\gamma_H^{-1}, (\gamma')^{-1})\langle \mathrm{inv}[z](\gamma^{-1}, (\gamma^{-1})), s\rangle^{-1}.\end{aligned}$$

Therefore $\Delta[\mathfrak{w}, \varrho, z](\gamma_H^{-1}, (\delta')^{-1})$ with respect to the endoscopic datum $(H, s, {}^L\eta)$ is nearly the same as $\Delta'[\mathfrak{w}, \varrho, z](\gamma_H, \delta')$ with respect to the endoscopic datum $(H, s^{-1}, {}^L\eta)$. The only difference is that in the above second product, the term Δ_I is defined with respect to a-data and the term Δ_{II} is defined with respect to the a^{-1}-data. However, the Δ_I and $\epsilon_L(V, \psi)$ terms also depend on the Whittaker datum. According to [23, p. 16], we have

$$\epsilon_L(V, \psi)\, \Delta_{I,a}(\gamma_H^{-1}, (\delta')^{-1}) = \epsilon_L(V, \psi^{-1})\, \Delta_{I,a^{-1}}(\gamma_H^{-1}, (\delta')^{-1}).$$

Since inverting the character ψ leads to the inverse Whittaker datum $\mathfrak{w}^{-1}$, the second product is actually the transfer factor $\Delta[\mathfrak{w}^{-1}, \varrho, z](\gamma_H^{-1}, (\delta')^{-1})$ with respect to the endoscopic datum $(H, s, {}^L\eta)$.

For the twisted endoscopic case, the same arguments still work. Indeed, in this case $H = G^{\theta}$ and we need to show that

$$SO^H_{\gamma_H}(f^H \langle \cdot \rangle_H) = \sum_{\delta'_{st}} \Delta'[\mathfrak{w}, \varrho, z](\gamma_H, \delta')\, O^U_{\delta'}(f \langle \cdot \rangle_U).$$

Since

$$SO^H_{\gamma_H}(f^H \langle \cdot \rangle_H) = SO^H_{\gamma_H^{-1}}(f^H) \quad \text{and} \quad O^U_{\delta'}(f \langle \cdot \rangle_U) = O^U_{(\delta')^{-1}}(f);$$

it suffices to show that the transfer factor $\Delta[\mathfrak{w}^{-1}, \varrho, z](\gamma_H^{-1}, (\delta')^{-1})$ with respect to the endo-scopic datum $(H, s, {}^L\eta)$ is the same as the transfer factor $\Delta'[\mathfrak{w}, \varrho, z](\gamma_H, \delta')$ with respect to the endoscopic datum $(H, s^{-1}, {}^L\eta)$. By the results in [33, Sections 5.3 and 5.4], we know that the twisted transfer factor $\Delta'[\mathfrak{w}, \varrho, z]$ is a product of terms

$$\epsilon_L(V, \psi)(\Delta_I^{new})^{-1} \Delta_{II} \Delta_{III_2}^{-1} \Delta_{IV} \langle \mathrm{inv}[z](\gamma, \delta), s \rangle$$

and the twisted transfer factor $\Delta_D[\mathfrak{w}, \varrho, z]$ is a product of terms

$$\epsilon_L(V, \psi) \Delta_I^{new} \Delta_{II} \Delta_{III_2}^{new} \Delta_{IV} \langle \mathrm{inv}[z](\gamma, \delta), s \rangle^{-1}.$$

Since $\Delta_{III_2}^{new}$ is the term Δ_{III_2} computed for the inverse set of χ-data, we see that

$$\Delta^{new}_{III_2,\chi^{-1}}(\gamma_H^{-1}, (\delta')^{-1}) = \Delta_{III_2,\chi}(\gamma_H, \delta').$$

Moreover,

$$(\Delta_I^{new})^{-1}(\gamma_H, \delta')[s^{-1}] = \Delta_I^{new}(\gamma_H, \delta')[s].$$

Thus we have

$$\begin{aligned}
\Delta'[\mathfrak{w}, \varrho, z](\gamma_H, \delta') &= \epsilon_L(V, \psi)(\Delta^{new}_{I,a})^{-1}[s^{-1}]\, \Delta_{IV}(\gamma_H, \delta')\, \Delta_{II,\chi,a}(\gamma_H, \delta') \\
&\quad \Delta_{III_2,\chi}(\gamma_H, \delta') \langle \mathrm{inv}[z](\gamma, \delta), s^{-1} \rangle \\
&= \epsilon_L(V, \psi) \Delta^{new}_{I,a}[s]\, \Delta_{IV}(\gamma_H^{-1}, (\delta')^{-1})\, \Delta_{II,\chi^{-1},a^{-1}}(\gamma_H^{-1}, (\delta')^{-1})\, \Delta^{new}_{III_2,\chi^{-1}} \\
&\quad (\gamma_H^{-1}, (\delta')^{-1}) \langle \mathrm{inv}[z](\gamma^{-1}, \delta^{-1}), s \rangle^{-1}.
\end{aligned}$$

As in the standard endoscopy case, the second product is actually the twisted transfer factor $\Delta[\mathfrak{w}^{-1}, \varrho, z](\gamma_H^{-1}, (\delta')^{-1})$ with respect to the endoscopic datum $(H, s, {}^L\eta)$. □

3.3.3. Endoscopy for $\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m$. We now study the endoscopy of $\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m$.

We must have $H = \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m$ and pick $s \in \hat{H}^{\Gamma_{\mathbb{Q}_v}}$. We will be most interested in the case where ${}^L\eta|_{\hat{H}}$ is the identity map and so we assume this is the case. Then ${}^L\eta$ is determined up to conjugacy by an element of $H^1(W_{\mathbb{Q}_v}, \widehat{\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m})$. By the Langlands correspondence for tori, this cocycle corresponds to a character χ of $\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m(\mathbb{Q}_v) = E^{\times}$.

We now study transfer factors for the endoscopic datum (H, s, η) of $\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m$. Recall that we have fixed an extended pure inner twist $(\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m, \varrho_{\mathbb{G}_m}, z_{\mathbb{G}_m})$. Consider $z_H \in H(\mathbb{Q}_v)$ which transfers to $z \in \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m$ and $z \in (\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m)$. Our goal is to compute the transfer factor $\Delta[w_{\mathbb{G}_m}, \varrho_{\mathbb{G}_m}, z_{\mathbb{G}_m}](z_H, z)$.

Lemma 3.6. We have

$$\Delta[w_{\mathbb{G}_m}, \varrho_{\mathbb{G}_m}, z_{\mathbb{G}_m}](z_H, z) = \chi(z)\langle \mathrm{inv}[z_{\mathbb{G}_m}](z, z), s\rangle^{-1}.$$

Proof. We will calculate each term in the definition of transfer factor. The virtual representation V in this case is 0 so that the factor $\epsilon(V, \psi) = 1$. The terms Δ_{IV}, Δ_{II} are trivial since $\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m$ has no absolute roots. The term Δ_I is trivial since the group $S^{\natural} = Z(\widehat{\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m})$ is trivial.

We now compute Δ_{III_2}. The L-maps $\xi_{(\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m)}$, ξ_H and ${}^L\varphi_{z,z}$ are all the identity. Hence, by comparing $\xi' = \xi \circ {}^L\varphi_{z,z}$ with $\xi_{\mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m}$, we see that $\Delta_{III_2} = \chi(z)$.

The final term then contributes the factor $\langle \mathrm{inv}[z_{\mathbb{G}_m}](z, z), s\rangle^{-1}$, completing the argument. □

3.3.4. Transfer factors for GU and $U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m$. We use the notation of Section 3.2. We denote the Whittaker datum and extended pure inner twists of U induced by restriction from $U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m$ by $w_U^{\flat}$ and $(U, \varrho_U^{\flat}, z_U^{\flat})$. We record the following lemma:

Lemma 3.7. Suppose $\gamma_H \in H(\mathbb{Q}_v)$ and $\gamma \in U(\mathbb{Q}_v)$ are strongly regular and related. Then we have the following equality:

$$\Delta[w_U^{\flat}, \varrho_U^{\flat}, z_U^{\flat}](\gamma_H, \gamma) = \Delta[w_{GU}, \varrho_{GU}, z_{GU}](\gamma_H, \gamma) \langle \mathrm{inv}[z_{GU}](\gamma, \gamma), s\rangle \langle \mathrm{inv}[z_U^{\flat}](\gamma, \gamma), s\rangle^{-1}.$$

Proof. This is [58, Lemma 3.6] adapted to the non-quasi-split setting. □

Finally, we prove the following lemma:

Lemma 3.8. Suppose

$$(\gamma, z) \in (U \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m)(\mathbb{Q}_v)_{sr}$$

and

$$(\gamma_H, z_H) \in (H \times \mathrm{Res}_{E/\mathbb{Q}_v} \mathbb{G}_m)(\mathbb{Q}_v)_{sr}$$

are related. Then we have an equality of transfer factors

$$\Delta[w_U, \varrho_U, z_U]((\gamma_H, z_H), (\gamma, z)) = \Delta[w_{GU}, \varrho_{GU}, z_{GU}](\gamma_H z_H, \gamma z).$$

Proof. First of all, by Lemma 3.4 we have

$$\Delta[w_U, \varrho_U, z_U]((\gamma_H, z_H), (\gamma, z)) = \Delta[w_U^{\flat}, \varrho_U^{\flat}, z_U^{\flat}](\gamma_H, \gamma)\, \Delta[w_{\mathbb{G}_m}, \varrho_{\mathbb{G}_m}, z_{\mathbb{G}_m}](z_H, z).$$ By

Lemma 3.6, this equals

$$\Delta[w_U^{\flat}, \varrho_U^{\flat}, z_U^{\flat}](\gamma_H, \gamma)\, \chi(z) \langle \mathrm{inv}[z_{\mathbb{G}_m}](z, z), s\rangle^{-1};$$

and by Lemma 3.7 we have

•Œw⁰U; %⁰U; z⁰U.• H; / D •ŒwGU; %GU; zGU.• H; /
hinvŒzGU.• ; /; sihinvŒz0 • ; U/; si 1:

Since the Kottwitz set and the Kottwitz map respect products, we get

hinvŒz⁰U.• ; /; sihinvŒzGm.• z; z/; si D hinvŒzU.• ; z/; .; z//; si:

By the functoriality of the Kottwitz map,

hinvŒzU.• ; z/; .; z//; si D hinvŒzGU.• z; z/; si:

Hence we get

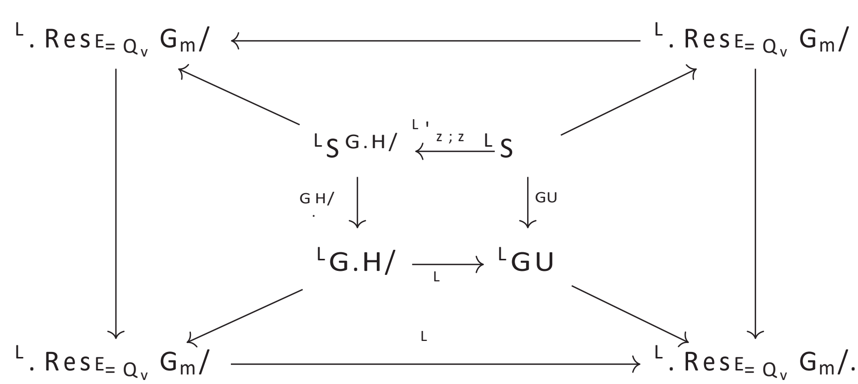
•ŒwU; %U; zU.• H; zH/; .; z// D •ŒwGU; %GU; zGU.• H; /
hinvŒzGU.• ; /; sihinvŒzGU.• z; z/; si 1:

On the other hand, by [36, Lemma 4.4A], there is a character ⁰ on .ResE = Qv Gm/.Qv/ such that

•ŒwGU; %GU; zGU.• HzH; z/ D •ŒwGU; %GU; zGU.• H; /⁰.z/
hinvŒzGU.• ; /; sihinvŒzGU.• z; z/; si 1:

Hence, it remains to show that ⁰.z/ D .z/. We recall that is the character arising from the construction of the •III2 -term of the transfer factor for ResE = Qv Gm. From the description in [36, Lemma 4.4A], ⁰ is the restriction to Z.GU/ D ResE = Qv Gm of the character arising from the •III2 -term of the transfer factor for GU.

The characters and are determined by the failure of the following diagram to commute:



We explain this diagram. The objects SG.H/ and S are maximal tori in their respective groups that are isomorphic by an admissible embedding L' z;z. The maps G.H/ and GU are the L-embeddings constructed in [36, Section (2.6)] from a choice of -data. The lower two diag-onal maps in the diagram are induced by the embeddings ResE = Qv Gm Š Z.GU/ ,! GU and ResE = Qv Gm Š Z.G.H// ,! G.H/. Since the images of these embeddings lie in the image of the embeddings S ,! G and SG.H/ ,! G.H/ respectively, we get induced maps

ResE = Qv Gm ,! SG.H/ and ResE = Qv Gm ,! S:

These induce the upper diagonal maps in the above diagram. The outer vertical arrows are then defined so that the left and right trapezoids commute. Note that by definition of n.w/ and r .w/, the vertical maps ^{L}. $Res_{E=Q}$ G_m/ ! ^{L}. $Res_{E=Q}$ G_m/ are both the identity. The bottom trapezoid commutes by construction. Finally, the top map in the diagram is defined so that the top trapezoid commutes and will agree with ^{L} on $Res_{E=Q}$ G_m and map .1; w/ to .1; w/.

Then the outer square fails to commute by the cocycle 2 v^1.W_Q ; $Res_{E=Q}$ G_m/ and the inner square faivs to commute by ^{0}2 Z^1.W_Q ; T/. Since the trapezoids all commute, these cocycles agree under the natural map Z^1.W_{Q_v} T/ ! Z^1.W_{Q_v}; $Res_{E=Q}$ G_m/. This is the desired result. □

4. Properties of the local and global correspondences

In this section we prove a number of properties and compatibilities of the local and global Langlands correspondences. These properties are needed to derive our main theorem.

4.1. Unramified representations. In this subsection we suppose that v is a finite place of Q and that $E_v = Q_v$ is unramified. We let .GU; id; 1/ and .U; id; 1/ be the trivial extended pure inner twists of GU.n/ and U.n/ respectively. Let GU.Z_v/ be the standard hyperspecial subgroup. Then we say that is GU.Z_v/-spherical if it has non-trivial GU.Z_v/-invariants.

Proposition 4.1. Let $_{GU}$ WL_{Q_v} ! ^{L}GU.n/ 2 ‰^{C}.GU.n// be a generic parameter. Then … $_{GU}$.GU; id/ contains a GU.Z_v/-spherical representation if and only if $_{GU}$ is unramified. In that case, … $_{GU}$.GU; id/ contains a unique GU.Z_v/-spherical representation , which satisfies h; i D 1. The same results hold true for U.

Proof. Suppose z 2 ….GU/ and 2 ….U/ such that z is a lift of . By Corollary 2.17, we see that z is spherical if and only if is. Moreover, by the construction local packets for GU.Q_v/, we have that h; i D 1 if and only if h; i D 1. Therefore it suffices to prove the proposition for unitary groups.

We mimic the proof of Lemma 4:1:1 in [54]. Denote by f the characteristic function of the standard special maximum compact subgroup of U.Q_v/. If $_U$ is unramified, then by proposition [40, Proposition 7.4.3] we have

$$1 D \sum_{2 … _U.U;id/} tr. j f /:$$

In other words, the packet … $_U$.U; id/ contains an unramified representation. The uniqueness comes from Theorem 2:5:1a in [40].

Suppose now that $_U$ is ramified. Then the base change L-parameter $_B$ ı $_U$ is also ramified. By the local Langlands correspondence for GL_n.E_v/, one gets a representation of GL_n.E_v/ corresponding to $_B$ ı $_U$. Then, as in [40, Section 3.2], one lifts to a representa-tion z of GL_n.E_v/ Ì GL$_n$.E_v/ Ì hi, where is the automorphism g ! J_n.g/ ^{t}J ^{1} of $Res_{E_v=Q}$ $GL_{n;E_v}$. Hence the corresponding representation of GL_n.O_{E_v}/Ì is ramified. We want to show that $\sum_{2…}$ h; xi tr. j f / D 0 for every x 2 S$_U$. If we denote f_N the characteristic function of GL_n.O_{E_v}/Ì , then f_N .$_B$ ı $_U$/ D 0. The twisted fundamental lemma implies that f_N is the twisted transfer of f and hence by [40, Theorem 3.2.1 (a)] we have

$$\sum_{\pi\in\Pi_{\psi_U}(U,\mathrm{id})}\mathrm{tr}(\pi\mid f)=\sum_{\pi\in\Pi_{\psi_U}(U,\mathrm{id})}\langle\pi,1\rangle\,\mathrm{tr}(\pi\mid f)=f_N(\widetilde{\pi}_{\psi_U})=0.$$

By the same argument we have

$$\sum_{\pi^H\in\Pi_{\psi_U^H}(H,\mathrm{id})}\langle\pi^H,1\rangle\,\mathrm{tr}(\pi^H\mid f_H)=0,$$

for every refined endoscopic datum $(H,s,{}^L\eta)$ of U, where f_H is the characteristic function of a hyperspecial subgroup $H(\mathbb{Z}_v)$ of H. By the fundamental lemma, f_H is the transfer of f. Then, again by [40, Theorem 3.2.1], we have

$$\sum_{\pi\in\Pi_{\psi_U}(U,\mathrm{id})}\langle\pi,x\rangle\,\mathrm{tr}(\pi\mid f)=\sum_{\pi^H\in\Pi_{\psi_U^H}(H,\mathrm{id})}\langle\pi^H,1\rangle\,\mathrm{tr}(\pi^H\mid f_H)=0,$$

where (ψ_U,x) corresponds to $(H,s,{}^L\eta,\psi_U^H)$ under [6, Proposition 3.10]. Hence we conclude that $\mathrm{tr}(\pi\mid f)=0$ for every $\pi\in\Pi_{\psi_U}(U,\mathrm{id})$. Therefore the packet $\Pi_{\psi_U}(U,\mathrm{id})$ does not con-tain any unramified representations.

We now consider the case of general $\psi_{GU}\in\Psi^{+}(GU(n))$. This follows from the fact that $I_P^{GU}(\pi)$ is $GU(\mathbb{Q}_v)$-spherical if and only if π is $M(\mathbb{Q}_v)$-spherical for M a standard Levi subgroup with parabolic subgroup P. □

4.2. On the hypothesis $ST^H_{\mathrm{ell}}(f^H)=ST^H_{\mathrm{disc}}(f^H)$. In this subsection, we prove that for (H,s,η) a refined elliptic endoscopic datum of $GU=GU(V)$ and $f^H\in C_c^\infty(H(\mathbb{A}))$ that is stable cuspidal at infinity and cuspidal at a finite place v, we have an equality of traces:

$$ST^H_{\mathrm{ell}}(f^H)=ST^H_{\mathrm{disc}}(f^H).$$

We begin with some preparatory notation and lemmas. Let G be a connected reductive group defined over $\mathbb{Q}$ and let χ be a sufficiently regular (in the sense of Lemma 5.11) quasi-character of $A_G(\mathbb{R})^0$ and $C_c^\infty(G(\mathbb{R}),\chi^{-1})$ be the set of functions $f_\infty: G(\mathbb{R})\to\mathbb{C}$ smooth, with compact support modulo $A_G(\mathbb{R})^0$ and such that for every $(z,g)\in A_G(\mathbb{R})^0\times G(\mathbb{R})$,

$$f_\infty(zg)=\chi^{-1}(z)f_\infty(g).$$

Fix K_G a maximal compact subgroup of $G(\mathbb{R})$.

Definition 4.2 (Stable cuspidal function at infinity). We say that $f_\infty\in C_c^\infty(G(\mathbb{R}),\chi^{-1})$ is stable cuspidal if f_∞ is left and right K_G-finite and if the function

$$\Pi_{\mathrm{temp}}(G(\mathbb{R}),\chi)\to\mathbb{C},\quad\pi\mapsto\mathrm{tr}(\pi\mid f_\infty)$$

vanishes outside $\Pi_{\mathrm{disc}}(G(\mathbb{R}))$ and is constant in the L-packets of $\Pi_{\mathrm{disc}}(G(\mathbb{R}),\chi)$.

Definition 4.3 (Cuspidal function). We say that $f_v\in C_c^\infty(G_v(\mathbb{Q}_v),\chi^{-1})$ is cuspidal if for each proper Levi subgroup $M\subsetneq G$ we have that the constant term, $f_{v,M}$, vanishes (as defined in [16, equation (7.13.2)]).

We record the following well-known lemma.

Lemma 4.4. If $f_\infty \in C_c^\infty(G(\mathbb{R}), \chi^{-1})$ is a stable cuspidal function and (H, s, η) is an endoscopic triple of G, there exists a stable cuspidal transfer function $f_\infty^H \in C_c^\infty(H(\mathbb{R}), \chi^{-1})$ of f_∞.

Proof sketch. Due to [48], we can find a function $f_\infty^H \in C_c^\infty(H(\mathbb{R}), \chi^{-1})$ that transfers to f_∞. Define the function F on the set of unitary tempered representations of $H(\mathbb{R})$ by setting

$$F(\pi) = \frac{1}{|\Pi_\phi(H(\mathbb{R}))|} \sum_{\pi' \in \Pi_\phi(H(\mathbb{R}))} \operatorname{tr}(\pi' \mid f_\infty^H),$$

where ϕ is the L-parameter of π. Then F must be supported on finitely many discrete series packets since f_∞ is stable cuspidal and (H, s, η) is elliptic. Hence, by [11, Theorem 1] there exists a function $f_\infty^{\prime H} \in C_c^\infty(H(\mathbb{R}), \chi^{-1})$ that is stable cuspidal and $F(\pi) = \operatorname{tr}(\pi \mid f_\infty^{\prime H})$. Thus, $f_\infty^{\prime H}$ has the same stable orbital integrals as f_∞^H. This implies that $f_\infty^{\prime H}$ is a stable cuspidal transfer of f_∞. □

We recall that $ST_{\mathrm{ell}}^H(f^H)$ is defined by the formula

$$ST_{\mathrm{ell}}^H(f^H) := \sum_{\gamma_H} \tau(H) SO_{\gamma_H}(f^H),$$

where the sum is over a set of representatives of the (G_U, H)-regular, semisimple, $\mathbb{Q}$-elliptic, stable conjugacy classes in $H(\mathbb{Q})$.

Definition 4.5. We define the term $ST_{\mathrm{disc}}^H(f^H)$ to equal

$$\sum_{\psi \in \Psi_2(H)} \frac{1}{|\overline{S}_\psi|} \sum_{\pi \in \Pi_\psi(H, \mathrm{id})} \langle 1, \pi \rangle \operatorname{tr}(\pi \mid f^H),$$

where ψ is such that one (hence any) π in $\Pi_\psi(H, \mathrm{id})$, the restriction of the central character of π to $A_G(\mathbb{R})^0$ is equal to χ.

Note that we have suppressed the term $\epsilon_\psi(s_\psi)$ from this expression because our assumption on χ implies that all ψ are generic by Lemma 5.11.

Separately, we have for every Levi subgroup M of H the term ST_M^H defined in [41, p. 86] as well as the term ST^H defined by

$$ST^H := \sum_M (n_M^H)^{-1} ST_M^H,$$

for certain constants $(n_M^H)^{-1}$.

We prove the following standard result.

Lemma 4.6. Suppose $h = h^\infty h_\infty \in C_c^\infty(H(\mathbb{A}), \chi^{-1})$ is stable cuspidal at infinity and cuspidal at a finite place. Then:

- For any $M \subsetneq H$ we have
$$ST_M^H(h) = 0.$$
- If $M = H$, then
$$ST_H^H(h) = ST_{\mathrm{ell}}^H(h).$$

Proof. To prove the first part, we note that by definition, for M a proper Levi subgroup, the "constant term" h^{∞}_{M} is 0 (for instance see the definition before [1, Theorem 7.1]). This implies that $ST^{H}_{M}(h) = 0$.

We now prove the second part. We first show that $S\Phi_H(\gamma_H; h_\infty) = SO_{\gamma_H}(h_\infty)$. By [2, Theorem 5.1], we have

$$O_{\gamma_H}(h_\infty) = \Phi_H(\gamma_H; h_\infty) = v(I_H)^{-1} \sum \Phi_H(\gamma_H^{-1}; \Pi)\,\mathrm{tr}(\pi \mid h_\infty); \ldots$$

where the sum is over discrete series L-packets of $H(\mathbb{R})$ with central character χ_H (the unique character of $A_H(\mathbb{R})^{\circ}$ such that if a parameter φ_H has central character restricting to χ_H, then ${}^{L}\eta \circ \varphi_H$ has central character χ). The representation π is some representative of Π, and the value of $\mathrm{tr}(\pi \mid h_\infty)$ does not depend on the choice of representative since h_∞ is stable cuspidal. The $^{-1}$ in this formula that is seemingly at odds with the formula of Arthur is explained by [16, Section (7.19)].

Therefore we have

$$SO_{\gamma_H}(h_\infty) = \sum_{\gamma'_H \sim_{st} \gamma_H} e(I_{\gamma'_H})\Phi_H(\gamma'_H; h_\infty).$$

Now, by definition,

$$S\Phi_H(\gamma_H; h_\infty) = v(\bar{I}_H)^{-1} \sum_{\Pi} \Phi_H(\gamma_H^{-1}; \Pi)\,\mathrm{tr}(\Pi \mid h_\infty).$$

Since h_∞ is stable cuspidal, we have $\mathrm{tr}(\Pi \mid h_\infty) = |\Pi|\,\mathrm{tr}(\pi \mid h_\infty)$. Furthermore, it follows from the definitions and basic properties of the Kottwitz sign that

$$e(I_H)/v(\bar{I}_H) = (-1)^{q(I_H)}\mathrm{Vol}(I_H(\mathbb{R})/A_H(\mathbb{R})^{0}) = v(I_H)/d(I_H);$$

where $d(I_H) = |\ker(H^1(\mathbb{R}; T) \to H^1(\mathbb{R}; I_H))|$ for T an elliptic maximal torus of I_H.

Finally, we put everything together to get

$$\begin{aligned}
SO_{\gamma_H}(h_\infty) &= \sum_{\gamma'_H \sim_{st} \gamma_H} e(I_{\gamma'_H})\Phi_H(\gamma'_H; h_\infty) \\
&= \sum_{\gamma'_H \sim_{st} \gamma_H} \frac{d(I_{\gamma'_H})}{v(\bar{I}_{\gamma'_H})} \sum_{\Pi} \Phi_H(\gamma'^{-1}_H; \Pi)\,\mathrm{tr}(\pi \mid h_\infty) \\
&= \sum_{\gamma'_H \sim_{st} \gamma_H} \frac{d(I_{\gamma'_H})}{|\Pi|} S\Phi_H(\gamma'_H; h_\infty) \\
&= S\Phi_H(\gamma_H; h_\infty).
\end{aligned}$$

The last equality follows from the fact that $S\Phi_H(\gamma_H; h_\infty)$ only depends on the stable class of γ_H and

$$\sum_{\gamma'_H \sim_{st} \gamma_H} \frac{d(I_{\gamma'_H})}{|\Pi|} = 1.$$

Indeed, $|\Pi|$ is well known to equal $|\ker(H^1(\mathbb{R}; T) \to H^1(\mathbb{R}; H))|$ for T an elliptic maximal torus of H. Hence, it suffices to show that

$$\sum_{\gamma'_H \sim_{st} \gamma_H} d(I_{\gamma'_H}) = |\ker(H^1(\mathbb{R}; T) \to H^1(\mathbb{R}; H))|.$$

To see this, first note that the set of conjugacy classes that are stably conjugate to γ_H is in natural bijection with $\ker(H^1(\mathbb{R}, I_{\gamma_H}) \to H^1(\mathbb{R}, H))$. For each such conjugacy class, we can choose a representative $\gamma_H^c \in T$. This follows from the fact that since H contains an elliptic maximal torus, any elliptic element of $H(\mathbb{R})$ is contained in an elliptic maximal torus and all elliptic maximal tori are conjugate in $H(\mathbb{R})$. Then the set of classes in $H^1(\mathbb{R}, T)$ mapping to the class of γ'_H in $H^1(\mathbb{R}, I_{\gamma_H})$ is in bijection with $\ker(H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, I_{\gamma'_H}))$.

It then follows that

$$ST^H_H(h) = \iota(H) \sum_{\gamma_H} SO_{\gamma_H}(h),$$

where the sum is over stable conjugacy classes in $H(\mathbb{Q})$ that are semisimple and elliptic in $H(\mathbb{R})$.

Since h_∞ is stable cuspidal, its orbital integrals vanish on γ_H that are not elliptic at $\mathbb{R}$, so we may as well impose this condition. By [41, Proposition 3.3.4, Remark 3.3.5] we may also restrict the sum to γ_H that are (GU, H)-regular. We then see that this is equal to $ST^H_{\mathrm{ell}}(h)$. □

Suppose now that $f \in \mathcal{H}(GU(\mathbb{A}))$ is stable cuspidal at infinity and cuspidal at a finite place. Then by the above Lemma 4.4 and [3, Lemma 3.4], for each elliptic endoscopic datum (H, s, η), we can find a function f^H that is stable cuspidal at infinity, cuspidal at a finite place, and a transfer of f.

Our proof of the main result of this section will be by induction. We now state the key formulas we will need. First, we have the following theorem of Morel:

Theorem 4.7. (See [41, Theorem 5.4.1]) Let G be a connected reductive group over $\mathbb{Q}$. Let $f = f^\infty f_\infty$, where $f_\infty \in C_c^\infty(G(\mathbb{R}), \mathbb{C})$ and $f^\infty \in C_c^\infty(G(\mathbb{A}_f), \mathbb{C})$. Assume that f_∞ is stable cuspidal and that for every $(H, s, \eta) \in \mathcal{E}(G)$, there exists a transfer f^H of f. Then

$$T^G(f) = \sum_{(H,s,\eta)\in\mathcal{E}(G)} \iota(G, H) ST^H(f^H),$$

where $\mathcal{E}(G)$ is the set of isomorphism classes of elliptic endoscopic triples in the sense of Kottwitz and we recall that $T^G(f)$ is defined to be the trace of f on $L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$.

Fix an odd positive integer n. By Proposition 2.23 and Remark 2.27 we have the following formula for each group G' of the form $G(U(n_1) \times \cdots \times U(n_k))$ such that $\sum_{i=1}^k n_i = n$. We note that all such groups are quasi-split.

For a function $f^{G'} \in \mathcal{H}(G'(\mathbb{A}))$,

$$T^{G'}(f^{G'}) = \sum_{\psi\in\Psi_2(G')} \sum_{\pi\in\Pi_\psi(G',\varrho,1)} \mathrm{tr}\, \pi(f^{G'});$$

where $\Pi_\psi(G', \varrho, 1)$ is the subset of $\Pi_\psi(G', \varrho)$ containing those π with trivial character $\langle\cdot, \pi\rangle$. We will now prove by induction that for each group G' that we consider and for each $f^{G'} \in \mathcal{H}(G'(\mathbb{A}))$ stable cuspidal at infinity, we have

$$ST^{G'}(f^{G'}) = ST^{G^c}_{\mathrm{disc}}(f^{G'}).$$

We induct on $\sum_{i=1}^k n_i^2$. Hence, the base case is when each $n_i = 1$. Such a group G' is a torus and hence has no non-trivial elliptic endoscopy. In particular, by Theorem 4.7 we have that

$$T^{G'}(f^{G'}) = ST^{G'}(f^{G'})$$

and hence it suffices to show that $T^{G^0}(f^{G^0}) = ST^{G^0}_{\mathrm{disc}}(f^{G^0})$. By property (v) since there is no non-trivial endoscopy, each $\overline{\mathcal{S}}_\psi = 1$ and hence $\langle \cdot, \pi\rangle$ is the trivial character for all π. The result follows.

We now settle the inductive step. Suppose we have shown $ST^{G^0}(f^{G^0}) = ST^{G^0}_{\mathrm{disc}}(f^{G^0})$ for each G^0 satisfying $\sum_{i=1}^{k} n_i^2 \leq N$, and suppose that G^0 satisfies $\sum_{i=1}^{k} n_i^2 = N+1$. Pick a function $f^{G^+} \in \mathcal{H}(G^+(\mathbb{A}))$ that is stable cuspidal at infinity and for each elliptic endoscopic datum (H, s, η) of G^0 we pick by Lemma 4.4 a transfer $f^H \in \mathcal{H}(H(\mathbb{A}))$ that is stable cuspidal at infinity.

Then we can write Theorem 4.7 in the form

$$T^{G^0}(f^{G^0}) = ST^{G^0}(f^{G^0}) + \sum_{(H,s,\eta)\in\mathcal{E}(G^0)} \iota(G^0, H) ST^H(f^H);$$

where for each non-trivial elliptic endoscopic group H appearing in the sum on right-hand side, we have verified $ST^H(f^H) = ST^H_{\mathrm{disc}}(f^H)$ by inductive assumption.

To conclude, it suffices to show that we have an equality

$$T^{G^0}(f^{G^0}) = ST^{G^+}_{\mathrm{disc}}(f^{G^0}) + \sum_{(H,s,\eta)\in\mathcal{E}(G^0)} \iota(G^0, H) ST^H_{\mathrm{disc}}(f^H):$$

We prove this by arguing as in [55, p. 30] (cf. [27, Section 12]). Indeed, we have

$$\sum_{(H,s,\eta)\in\mathcal{E}(G^0)} \iota(G^0, H) ST^H_{\mathrm{disc}}(f^H) = \sum_{(H,s,\eta)\in\mathcal{E}(G^0)} \iota(G^0, H) \sum_{\psi\in\Psi_2(H)} \frac{1}{|\overline{\mathcal{S}}_\psi|} \sum_{\pi\in\Pi_\psi(H,\mathrm{id})} \langle 1, \pi\rangle \operatorname{tr}(\pi \mid f^H):$$

Now, we apply at each place the endoscopic character identity we proved in Section 3 and argue as for the equation [55, equation (11)] to get that the above equals

$$\sum_{\psi\in\Psi_2(G^0)} \sum_{s\in\overline{\mathcal{S}}_\psi} \frac{1}{|\overline{\mathcal{S}}_\psi|} \sum_{\pi\in\Pi_\psi(G^0,\mathrm{id})} \langle s, \pi\rangle \operatorname{tr}(\pi \mid f^{G^0}):$$

Now we use that

$$\sum_{s\in\overline{\mathcal{S}}_\psi} \frac{1}{|\overline{\mathcal{S}}_\psi|} \langle s, \pi\rangle$$

is 1 if $\pi \in \Pi_\psi(G, \mathrm{id}, 1)$ and 0 otherwise to get that the above equals

$$\sum_{\psi\in\Psi_2(G^0)} \sum_{\pi\in\Pi_\psi(G^0,\mathrm{id},1)} \operatorname{tr}(\pi \mid f^{G^0});$$

which equals $T^{G^0}(f^{G^0})$ as desired.

4.3. Some special global liftings. In this subsection, we work over $\mathbb{Q}_v$ for a fixed finite place v. Now consider $\psi_{GU} : W_{\mathbb{Q}_v} \times SL_2(\mathbb{C}) \to {}^L GU(n) = (GL_n(\mathbb{C}) \times \mathbb{C}^\times) \rtimes W_{\mathbb{Q}_v}$ a dis-crete L-parameter. We denote by ψ_U the L-parameter of $U(n)$ obtained from ψ_{GU} by the projection ${}^L GU(n) \to {}^L U(n)$. There is a (standard) base change morphism

$$\eta_{B,GU} : \Psi(GU(n)) \to \Psi(GL_{E_v}(n) \times \mathbb{G}_m): \tag{4.1}$$

Denote by φ^n_{GU} the image of φ_{GU} by this morphism. Then φ^n_{GU} is just the restriction of φ_{GU} to $W_{E_v} \times SL_2(\mathbb{C})$. Since W_{E_v} acts trivially on $GL_n(\mathbb{C}) \times \mathbb{C}^\times$, if we denote φ^n_U the projection of φ^n_{GU} to $GL_n(\mathbb{C})$, then it is an n dimensional representation of $W_{E_v} \times SL_2(\mathbb{C})$ and moreover φ^n_U is the image of φ_U by the (standard) base-change morphism for U.

Since φ_U is a discrete L-parameter, the group S_{φ_U} is finite (see [27, Lemma 10.3.1]) and we can write $\varphi^n_U = (\varphi_1 \otimes \nu_1) \oplus \cdots \oplus (\varphi_r \otimes \nu_r)$, where φ_i are simple L-parameters of general linear groups and ν_i are irreducible representations of $SL_2(\mathbb{C})$. By the computation in [25, pp. 62–63], all the φ_i are conjugate-orthogonal and we have

$$S_{\varphi_U} \simeq S^{\natural}_{\varphi_U} \simeq \prod_{i=1}^{r} O(1;\mathbb{C}) \simeq \prod_{i=1}^{r} \mathbb{Z}/2\mathbb{Z}.$$

Moreover, the group

$$Z(\hat{U}(n))^{\Gamma_{\mathbb{Q}_v}} = \{\pm \mathrm{id}\}$$

embeds diagonally into S_{φ_U}. Furthermore, $\det(-\mathrm{id}) = -1$ and $S_{\varphi_U} = \{\pm \mathrm{id}\} \times S^{+}_{\varphi_U}$. By Lemma 2.18, we have

$$\bar{S}_{\varphi_{GU}} \simeq \bar{S}_{\varphi_U} = S_{\varphi_U}/\{\pm \mathrm{id}\} = S^{+}_{\varphi_U} \simeq \prod_{i=1}^{r-1} \mathbb{Z}/2\mathbb{Z}$$

and

$$S_{\varphi_{GU}} \simeq S^{\natural}_{\varphi_{GU}} \simeq S^{+}_{\varphi_U} \times \mathbb{C}^\times \simeq \left(\prod_{i=1}^{r-1} \mathbb{Z}/2\mathbb{Z}\right) \times \mathbb{C}^\times.$$

Let $\psi_{GU} = (\psi_U, \chi)$ be a discrete (in particular generic) global A-parameter of $GU(\mathbb{A})$. The corresponding A-packet consists of automorphic representations of $GU(\mathbb{A})$ whose central character is χ and whose restriction to $U(\mathbb{A})$ is an automorphic representation in the A-packet of ψ_U. Again, we denote by $\psi^n_U = \Pi_1 \boxplus \cdots \boxplus \Pi_m$ the isobaric sum of automorphic representations of $GL_n(\mathbb{A}_E)$ corresponding to ψ_U. As in the local case, we see that S_{ψ_U} is finite and by [25, p. 69] we have then

$$S_{\psi_U} \simeq S^{\natural}_{\psi_U} = \prod_{i=1}^{m} O(1;\mathbb{C}) \simeq \prod_{i=1}^{m} \mathbb{Z}/2\mathbb{Z};$$

with the group $Z(\hat{U}(n))^{\Gamma_{\mathbb{Q}}} = \{\pm \mathrm{id}\}$ embedded diagonally into S_{ψ_U} and an isomorphism $S_{\psi_U} \simeq \{\pm \mathrm{id}\} \times S^{+}_{\psi_U}$. Thus

$$\bar{S}_{\psi_{GU}} = \bar{S}_{\psi_U} = S_{\psi_U}/\{\pm \mathrm{id}\} \simeq \prod_{i=1}^{m-1} \mathbb{Z}/2\mathbb{Z}$$

and

$$S_{\psi_{GU}} \simeq S^{\natural}_{\psi_{GU}} \simeq S^{+}_{\psi_U} \times \mathbb{C}^\times \simeq \left(\prod_{i=1}^{m-1} \mathbb{Z}/2\mathbb{Z}\right) \times \mathbb{C}^\times.$$

We say that a global parameter $\psi_{GU} = (\psi_U, \chi)$ is a global lifting of φ_{GU} if we have $(\psi_{U_v}, \chi_v) = \varphi_{GU}$. In this case, there exist morphisms $\iota: S_{\psi_U} \to S_{\varphi_U}$, $\tilde{\iota}: S_{\psi_{GU}} \to S_{\varphi_{GU}}$ and $\bar{\iota}: \bar{S}_{\psi_U} \to \bar{S}_{\varphi_U}$. Since the local and global parameters φ_U and ψ_U are discrete, these maps are injective (see [40, pp. 28–31] for more details). In this subsection, we construct some global liftings $\psi_{GU} = (\psi_U, \chi)$ such that the above maps ι, $\tilde{\iota}$ and $\bar{\iota}$ have some special properties.

4.3.1. First construction. (Cf. [25, Lemma 4.2.1].) We choose auxiliary places u, u' of $\mathbb{Q}$ such that u splits over E as $u = w\bar{w}$ and u' is inert. Therefore $U(\mathbb{Q}_u)$ is isomorphic to $GL_n(E_w)$. By [52, Theorem 5.7], there exists a cuspidal automorphic representation π of $U(\mathbb{A})$ satisfying the following properties;

(i) π_∞ is discrete series corresponding to a regular highest weight and with sufficiently regular infinitesimal character in the sense of [42, Definition 2.2.10],

(ii) π_v belongs to the packet $\Pi_{\psi_U}(U_v, \varrho_{U_v})$,

(iii) π_u is a supercuspidal representation of $GL_n(E_w)$.

(iv) $\pi_{u'}$ is any prescribed supercuspidal representation of $U(\mathbb{Q}_{u'})$.

Note that such a π will be cohomological by the first condition and the remark at the end of [30, Section 2].

By [17, Lemma 4.1.2], we can extend π to an algebraic cuspidal automorphic representation $\bar{\pi}$ of $GU(\mathbb{A})$. Furthermore, we can assume that $\bar{\pi}$ is cohomological since π is.

Consider the exact sequence

$$1 \to U \to GU \xrightarrow{c} \mathbb{G}_m \to 1.$$

Since π_v belongs to the packet $\Pi_{\psi_U}(U_v, \varrho_{U_v})$, the central character $\omega_{\bar{\pi}_v}$ and the central character ω_{GU} of any representation in $\Pi_{\psi_{GU}}(GU_v, \varrho_{GU_v})$ must agree on $(Z(GU) \cap U)(\mathbb{Q}_v)$. The map c restricted to $Z(GU)$ has kernel equal to $Z(GU) \cap U$ so that $\omega_{\bar{\pi}_v} \omega_{GU}^{-1}$ factors to give a character of $\mathrm{im}(c)$ which (since n is odd) is the norm subgroup $N_{E_v/\mathbb{Q}_v} \subset \mathbb{Q}_v^\times$. We can choose a lift of this character to $\mathbb{Q}_v^\times$ and hence we conclude that there is some character $\chi: \mathbb{Q}_v^\times \to \mathbb{C}^\times$ such that $\bar{\pi}_v \otimes (\chi \circ c)$ belongs to the packet $\Pi_{\psi_{GU}}(GU_v, \varrho_{GU_v})$.

There is an isomorphism of topological groups

$$\mathbb{Q}^\times \mathbb{R}_{>0} \prod_p \mathbb{Z}_p^\times \to \mathbb{G}_m(\mathbb{A}); \qquad (r, t, (u_p)) \mapsto (rt, ru_2, ru_3, \dots).$$

Then there is a character $\tilde{\chi}$ of $\mathbb{Q}^\times \mathbb{R}_{>0} \prod_p \mathbb{Z}_p^\times$ such that $\tilde{\chi}$ is trivial on $\mathbb{Q}^\times \mathbb{R}_{>0}$ and satisfies $\tilde{\chi}|_{\mathbb{Z}_v^\times} = \chi|_{\mathbb{Z}_v^\times}$ and $\tilde{\chi}(1, 1, (1)) = 1$. This character descends to a Hecke character χ of $\mathbb{G}_m(\mathbb{Q}) \backslash \mathbb{G}_m(\mathbb{A})$ such that $\chi_v = \chi \otimes \chi'$, where $\chi': \mathbb{Q}_v^\times \to \mathbb{C}^\times$ is an unramified character and χ_∞ is trivial. In particular, if we denote $\tilde{\pi} := \bar{\pi} \otimes (\chi \circ c)$, it is still cohomological (since $\bar{\pi}$ is) and the local representation $\tilde{\pi}_v$ belongs to the packet $\Pi_{\psi_{GU}}(GU_v, \varrho_{GU_v})$ up to an unramified character twist.

Therefore the global parameter $\psi_{GU} = (\psi_U, \eta)$ is a globalization of ψ_{GU}, up to an unramified twist (where ψ_U is the global parameter of π and η corresponds to the central character of $\tilde{\pi}$). Since π_∞ has sufficiently regular infinitesimal character, ψ_U is generic (Lemma 5.11). The third condition implies that ψ_U is a cuspidal automorphic representation of $GL_n(\mathbb{A}_E)$ which is self-dual and conjugate orthogonal. Therefore we have $S_{\psi_U} = \{\pm \mathrm{id}\}$ ([25, p. 69]) so $\bar{S}_{\psi_U} = \{\mathrm{id}\}$. The above second condition implies that ψ_U is a global lift of ψ_U. Since the map is injective, we see that is the diagonal embedding of $\{\pm \mathrm{id}\}$ into S_{ψ_U}.

Moreover, since $\bar{S}_{\psi_{GU}} = \bar{S}_{\psi_U} = \{\mathrm{id}\}$ and $S^\natural_{\psi_{GU}} = \bar{S}_{\psi_U}$, the map is the trivial map. The group $S^\natural_{\psi_U}$ is also trivial and the map is given by

$$S_{\psi_{GU}} \cong \mathbb{C}^\times \to S_{\psi_{GU}} \cong S_{\psi_U}(\mathbb{C}); \qquad t \mapsto (\mathrm{id}, t).$$

Thus, we have proved the following lemma.

Lemma 4.8. Let ϕ_{GU} be a discrete L-parameter of the group $GU(n)$ defined over $\mathbb{Q}_v$. Then there exists a generic global parameter ψ_{GU} such that:

(i) ψ_{GU} is a globalization of ϕ_{GU} up to an unramified twist.

(ii) We have $S_{\psi_{GU}} \cong \mathbb{C}^\times$ and the map is given by
$$S_{\psi_{GU}} \cong \mathbb{C}^\times \to S_{\phi_{GU}} \cong S_{\phi_U} \times \mathbb{C}^\times; \qquad t \mapsto (\mathrm{id}, t).$$

(iii) Any automorphic representation $\tilde{\pi}$ in the global packet $\Pi_{\psi_{GU}}(GU, \varrho_{GU})$ is cuspidal and cohomological.

4.3.2. *Second construction.* (We adapt [25, proof of Lemma 4.4.1].) Consider an element $s = (x_i)_{i=1}^r \in S_{\phi_U} = \prod_{i=1}^r \mathbb{Z}/2\mathbb{Z}$ whose image in $\bar{S}_{\phi_U}$ is denoted by $\bar{s}$. We can suppose that $x_i = 1$ for $i \in X \subset \{1, \dots, r\}$ and $x_i = -1$ for $i \in Y \subset \{1, \dots, r\}$. Denote
$$\phi_X^{n_X} = \bigoplus_{i \in X} \phi^i_{n_i} \quad \text{and} \quad \phi_Y^{n_Y} = \bigoplus_{i \in Y} \phi^i_{n_i}$$
(where $n_X = \sum_{i \in X} n_i$ and $n_Y = \sum_{i \in Y} n_i$). We choose an auxiliary inert place u_0 and a supercuspidal L-parameter ϕ_{u^0} of U_{u^0} such that $\phi^n_{u^0} = \phi^{n_X}_{u^0} \oplus \phi^{n_Y}_{u^0}$ and the parameters $\phi^{n_X}_{u^0}$ (resp. $\phi^{n_Y}_{u^0}$) are simple and of degree n_X (resp. n_Y). In particular, the packet $\Pi_{\phi_{u^0}}(U_{u^0}, \varrho_{U_{u^0}})$ contains only supercuspidal representations.

Since all the $\phi^i_{n_i}$ are conjugate orthogonal, by [40, Lemma 2.2.1], the L-parameters $\phi^{n_X}_X$ resp. $\phi^{n_Y}_Y$ come from L-parameters ϕ_X (resp. ϕ_Y) of unitary groups $U(n_X)_v$ (resp. $U(n_Y)_v$) by the base change map η_B (see (4.1)). Similarly, the L-parameters $\phi^{n_X}_{u^c}$ (resp. $\phi^{n_Y}_{u^c}$) come from L-parameters $\phi_{u^0;X}$ (resp. $\phi_{u^0;Y}$) of unitary groups $U(n_X)_{u^0}$ resp. $U(n_Y)_{u^0}$ by the base change map η_B. Now as in the first construction, for these L-parameters we can construct cuspidal automorphic representations Π_X resp. Π_Y of $U_{n_X}(\mathbb{A})$ resp. $U_{n_Y}(\mathbb{A})$, in particular, $(\Pi_X)_{u^0}$ resp. $(\Pi_Y)_{u^0}$ are the supercuspidal representations whose L-parameters are $\phi_{u^0;X}$, resp. $\phi_{u^0;Y}$. These cuspidal automorphic representations give rise to cuspidal automorphic representations $\Pi^{n_X}_X$ resp. $\Pi^{n_Y}_Y$ of $GL_{n_X}(\mathbb{A}_E)$ resp. $GL_{n_Y}(\mathbb{A}_E)$. Since these automorphic representations are self-dual and conjugate-orthogonal, the isobaric sum $\Pi^{n_X}_X \boxplus \Pi^{n_Y}_Y$ factors through the base change map η_B ([25, Proposition 1.3.1], [40, p. 27]). Denote this global L-parameter of $U(\mathbb{A})$ by ψ_U. Again by [25, p. 69] we know that $S_{\psi_U} \cong \prod_{i \in \{X,Y\}} \mathbb{Z}/2\mathbb{Z}$. As in the first construction, the L-parameter ψ_U is generic (Lemma 5.11) and is a global lift of ϕ_U. Moreover, the localization map is defined as follows:
$$S_{\psi_U} \to S_{\phi_U}; \qquad (x_1, x_2) \mapsto (\underbrace{x_1, \dots, x_1}_{i \in X}, \underbrace{x_2, \dots, x_2}_{i \in Y}).$$

Taking the quotient by $\{\pm \mathrm{id}\}$, we see that $\bar{S}_{\psi_U} = S_{\psi_U}/\{\pm \mathrm{id}\} \cong \mathbb{Z}/2\mathbb{Z}$ and the map is given by
$$\bar{S}_{\psi_U} \to \bar{S}_{\phi_U}; \qquad (-1) \mapsto \bar{s}.$$

Now take an automorphic representation Π of $U(\mathbb{A})$ in the packet $\Pi_{\psi_U}(U, \varrho_U)$. The automorphic representation Π is cuspidal since Π_{u^0} is a representation whose L-parameter is ϕ_{u^0}. By the same argument as in the first construction, we can extend it to an automorphic representation $\tilde{\Pi}$ of $GU(\mathbb{A})$ such that $\tilde{\Pi}_v$ belongs to the packet $\Pi_{\phi_{GU}}(GU_v, \varrho_{GU_v})$ up to an unramified twist. Thus the global parameter ψ_{GU} of $\tilde{\Pi}$ is a globalization of ϕ_{GU}. We have then $S_{\psi_U} \cong \mathbb{Z}/2\mathbb{Z}$ and $S_{\psi_{GU}} = S_{\psi_U} \times \mathbb{C}^\times$.

Furthermore, if the element s belongs to $S^{+}_{\psi_U}$, then $(x_1; x_2)$ belongs to $S^{+}_{\psi_U}$ since the map η is injective and restricts to a map from $S^{+}_{\psi_U}$ to $S^{+}_{\psi_U}$. Therefore, we have the following description of the map $\tilde{\eta}$:

$$S_{\psi_{GU}} \cong \mathbb{Z}/2\mathbb{Z}^{+} \to S_{\psi_{GU}} \cong S^{+,+}_{\psi_U} \qquad 1^t \mapsto 1^t; \qquad -1^t \mapsto s^t.$$

Thus we have proved the following lemma.

Lemma 4.9. Let ϕ_{GU} be a discrete L-parameter of the group $GU(n)$ defined over $\mathbb{Q}_p$ and $s \in S^{+}_{\phi_U}$. Then there exists a generic global parameter $\dot{\psi}_{GU}$ and an inert place u^0 such that:

(i) $\dot{\psi}_{GU}$ is a globalization of ϕ_{GU} up to an unramified twist.

(ii) We have $S_{\dot{\psi}_{GU}} \cong \mathbb{Z}/2\mathbb{Z}^{+}$ and the map $\tilde{\eta}$ is given by

$$S_{\dot{\psi}_{GU}} \cong \mathbb{Z}/2\mathbb{Z}^{+} \to S_{\phi_{GU}} \cong S^{+,+}_{\phi_U} \qquad 1^t \mapsto 1^t; \qquad -1^t \mapsto s^t.$$

(iii) Any automorphic representation $\dot{\pi}$ in the global packet $\Pi_{\dot{\psi}_{GU}}(GU; \varrho_{GU})$ is cohomological. Moreover, $\dot{\pi}_{u^0}$ is supercuspidal.

4.4. Galois representations associated to global cohomological generic parameters. We have fixed a quadratic imaginary extension E of $\mathbb{Q}$. In this subsection, we associate representations of Γ_E to certain global parameters.

Let $(\psi_U; \chi)$ be a global A-parameter of a global unitary similitude group GU. In particular, ψ_U is a global parameter for the corresponding unitary group U. We suppose further that the localization at infinity $(\psi_{U\infty}; \chi_\infty)$ is regular and sufficiently regular so that ψ_U will be generic.

We first associate a Γ_E representation to ψ_U. Associated to ψ_U, we have the quadratic base change, $\tilde{\psi}_U$, which is an automorphic representation of $GL_n(\mathbb{A}_E)$. Since the global parameter is generic, the representation $\tilde{\psi}_U$ is of the form $\pi_1 \boxplus \cdots \boxplus \pi_k$, where π_i are self dual cuspidal generic and cohomological automorphic. Now, fix a place ℓ of $\mathbb{Q}$ and an isomorphism $\iota_\ell: \overline{\mathbb{Q}}_\ell \to \mathbb{C}$. Then by [51, Theorem 1.2], for each representation π_i there is a unique ℓ-adic Γ_E-representation ρ_i such that for each place P of E not dividing ℓ, we have the following isomorphism of Weil–Deligne representations:

$$WD(\rho_i|_{\Gamma_{E_P}})^{F\text{-ss}} \cong \iota_\ell^{-1} L((\pi_i)_P);$$

where $L((\pi_i)_P)$ is the local parameter associated to $(\pi_i)_P$ under the local Langlands correspondence.

Similarly, if we denote $\rho = \rho_1 \oplus \cdots \oplus \rho_k$, then for each place P dividing q and not dividing ℓ, we have

$$WD(\rho|_{\Gamma_{E_P}})^{F\text{-ss}} = \iota_\ell^{-1} L((\pi_1 \boxplus \cdots \boxplus \pi_k)_P):$$

Denote by $\psi_{U P}$ the localization of ψ_U at P. By the definition of localization map of global parameters ([40, pp. 18–19]), we see that the local L-parameter (not necessarily bounded) corresponding to $WD(\rho|_{\Gamma_{E_P}})^{F\text{-ss}}$ is ψ_{UP} if q is split in E. If q is inert in E then $q = P$ and E_P is a quadratic extension of $\mathbb{Q}_q$. In this case $WD(\rho|_{\Gamma_{E_P}})^{F\text{-ss}}$ corresponds to the image of ψ_{UP} via the base change map η_B and equals $\psi_{UP}|_{L_{E_P}}$.

The central character ω gives rise to a character of $GL_1(\mathbb{A}_E)$ and hence an ℓ-adic character ω'. The pair (ρ, ω') then gives us a morphism

$$\tilde{\rho}: W_E \to GL_n(\overline{\mathbb{Q}}_\ell) \times \overline{\mathbb{Q}}_\ell^\times.$$

From the local-global compatibility properties of ρ, we conclude that for every place P dividing a prime $q \neq \ell$, the restriction $\tilde{\rho}|_{W_{E_P}}$ equals $\phi(U;q)|_{W_{E_P}}$, where $\phi(U;q)$ is the localization of the global parameter $\phi(U)$ at the prime q.

5. Rapoport–Zink spaces and an averaging formula

5.1. Rapoport–Zink spaces. We continue with our fixed prime number p as before. Let

$$\breve{\mathbb{Q}}_p := \widehat{\mathbb{Q}_p^{ur}} = \mathrm{Frac}\, W(\overline{\mathbb{F}}_p)$$

be the completion of the maximal unramified extension of $\mathbb{Q}_p$ and σ the geometric Frobenius automorphism of $\breve{\mathbb{Q}}_p / \mathbb{Q}_p$.

We will be interested in the subset $B(\mathbb{Q}_p, G, \mu)$ of $B(\mathbb{Q}_p, G)$ associated with a minuscule cocharacter $\mu: \mathbb{G}_{m/\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ as defined in [31, Section 6.2]. The Bruhat ordering on the image of the Newton map induces a partial order on $B(\mathbb{Q}_p, G, \mu)$.

Definition 5.1. A Rapoport–Zink data of simple unramified unitary PEL type $(E_p, *, V, \langle\cdot,\cdot\rangle, GU, \mu, b)$ consists of the following:

- an unramified extension E_p of degree 2 of $\mathbb{Q}_p$ with a non-trivial involution $*$, a E_p-vector space V of dimension n,
- a symplectic Hermitian form $\langle\cdot,\cdot\rangle: V \times V \to \mathbb{Q}_p$ for which there is a self-dual lat-tice Λ,
- a conjugacy class of minuscule cocharacters $\mu: \mathbb{G}_{m\,\overline{\mathbb{Q}}_p} \to GU_{\overline{\mathbb{Q}}_p}$, where GU is the similitude unitary group defined over $\mathbb{Q}_p$ by
$$GU(R) = \{g \in GL(V \otimes R) : \langle gv, gw\rangle = c(g)\langle v, w\rangle;\ v, w \in V \otimes R\}$$
for all $\mathbb{Q}_p$-algebras R and $c(g) \in R$; we also suppose that $c \circ \mu(z) = z$, where c is the similitude factor of GU,
- a σ-conjugacy class $b \in B(\mathbb{Q}_p, GU, \mu)$.

The cocharacter μ is determined by a pair of integers $(d, n-d)$ such that d (resp. $n-d$) is the dimension of the weight 1 (resp. 0) weight space of μ.

To such a data, we associate the isocrystal $N = (V \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p, b \circ (\mathrm{id} \otimes \sigma))$ with an action $\iota: O_{E_p} \to \mathrm{End}(N)$ and an alternating non-degenerate form $\langle\cdot,\cdot\rangle: N \times N \to \breve{\mathbb{Q}}_p(n)$, where $n = \mathrm{val}(c(b))$. By Dieudonné's theory, the isocrystal N corresponds to a p-divisible group $(\mathbb{X}, \iota, \lambda)$ defined over $\overline{\mathbb{F}}_p$ provided with an action of O_{E_p} and a polarization λ.

Theorem 5.2 ([45, Theorem 3.25]). *Let $\mathcal{M}$ be the functor associating to each $O_{\breve{\mathbb{Q}}_p}$-scheme S on which p is locally nilpotent the set of pairs (X, ρ), where:*

- *X is a p-divisible group over S with a p-principle polarization λ_X and an action ι_X such as the Rosati involution inducing by λ_X induces $*$ on O_{E_p}.*

a O_{E_p}-linear quasi-isogeny

$$\rho : X \times_S \bar{S} \to \mathbb{X} \times_{\mathrm{Spec}(\bar{\mathbb{F}}_p)} \bar{S},$$

such that $\rho^\vee \circ \lambda_{\mathbb{X}} \circ \rho$ is a $\mathbb{Q}_p^\times$-multiple of λ_X in $\mathrm{Hom}_{O_E}(X, X^\vee) \otimes_{\mathbb{Z}} \mathbb{Q}$ (here, $\bar{S}$ is the modulo p reduction of S).

We also require that (X, ι_X) satisfies the Kottwitz determinant condition. More precisely, under the action of E_p, we have a decomposition $\mathrm{Lie}(X) = \bigoplus \mathrm{Lie}(X)$; then $\mathrm{Lie}(X)$ is locally free of rank p. This functor is then represented by a formal scheme defined over $\mathrm{Spf}(O_{\breve{\mathbb{Q}}_p})$.

In order to introduce the usual level structures, we work with the rigid generic fiber $\mathcal{M}^{an}$ of $\mathcal{M}$ over $\breve{\mathbb{Q}}_p$. Set $C_0 = \{g \in GU(\mathbb{Q}_p) : g\Lambda = \Lambda\}$, a maximal compact subgroup of $GU(\mathbb{Q}_p)$.

Definition 5.3. Let $T/\mathcal{M}^{an}$ be the local system defined by the p-adic Tate module of the universal p-divisible group on $\mathcal{M}$. For $K \subset C_0$ we define $\mathcal{M}_K$ as the étale covering of $\mathcal{M}^{an}$ which classifies the O_{E_p}-trivializations modulo K of T by Λ. We also require that the trivialization preserves the alternating form up to $\mathbb{Q}_p^\times$.

We have, in particular, that $\mathcal{M}^{an} = \mathcal{M}_{C_0}$. We then get a tower $(\mathcal{M}_{K_p})_{K_p}$ of analytic spaces on $\breve{\mathbb{Q}}_p$ provided with finite étale transition maps

$$\pi_{K_p', K_p} : \mathcal{M}_{K_p'} \to \mathcal{M}_{K_p}$$

(for $K_p' \subset K_p$) which forget the level structure. The map π_{K_p', K_p} is Galois of Galois group K_p / K_p' if K_p' is normal in K_p.

Let $J_b(\mathbb{Q}_p)$ be the group of O_{E_p}-linear quasi-isogenies g of $\mathbb{X}$ such that $\lambda \circ g$ is a $\mathbb{Q}_p^\times$-multiple of $g \circ \lambda$. The group $J_b(\mathbb{Q}_p)$ acts on the left on $\mathcal{M}$ by the formula

$$(X, \rho) \cdot g = (X, \rho \circ g^{-1}) \qquad \text{for all } g \in J_b(\mathbb{Q}_p) \text{ and all } (X, \rho) \in \mathcal{M}.$$

We say that a simple unramified unitary Rapoport–Zink datum $(E_p, \ast, V, \langle \cdot, \cdot \rangle, GU, \mu, b)$ is basic if the associated group $J_b(\mathbb{Q}_p)$ is an inner form of GU. The above datum is basic if and only if b is the unique minimal element in $B(\mathbb{Q}_p, GU, \mu)$. In this case, we also say that b is basic.

Let $\ell \neq p$ be a prime number. Let $K_p \subset C_0$ be a level. As in [14, Remark 2.6.3] we denote

$$H_c^\bullet(\mathcal{M}_{K_p}, \bar{\mathbb{Q}}_\ell) := \varinjlim_V \varprojlim_n H_c^\bullet(V \hat{\otimes}_{\breve{\mathbb{Q}}_p} \mathbb{C}_p, \mathbb{Z}/\ell^n \mathbb{Z}) \otimes \bar{\mathbb{Q}}_\ell$$

where V runs through the relatively compact open subsets of $\mathcal{M}_{K_p}$.

The group $J_b(\mathbb{Q}_p)$ acts on $\mathcal{M}_{C^0}$ and this action extends to $\mathcal{M}_{K_p}$ so that $J_b(\mathbb{Q}_p)$ acts on $H_c^\bullet(\mathcal{M}_{K_p}, \bar{\mathbb{Q}}_\ell)$. Since n is odd, the reflex field of the conjugacy class of μ is E_p. We can also define an action of the Weil group W_{E_p} on these cohomology groups thanks to the Rapoport–Zink descent data defined as below.

Let $\sigma_{E_p} : \breve{\mathbb{Q}}_p \to \breve{\mathbb{Q}}_p$ the relative Frobenius automorphism with respect to E_p. We denote by $\bar{\sigma}_{E_p}$ the Frobenius morphism induced on $\bar{\mathbb{F}}_p$. For X a p-divisible group defined over $\bar{\mathbb{F}}_p$, we note $F_{E_p} : X \to \bar{\sigma}_{E_p}^* X$ the relative Frobenius morphism. We construct a functor isomorphism $\alpha : \mathcal{M} \to \sigma_{E_p}^* \mathcal{M}$ as follows.

For S a $\mathcal{O}_{\breve{\mathbb{Q}}_p}$-scheme on which p is nilpotent as well as a point $(X,\iota,\lambda,\rho)\in\mathcal{M}(S)$, the point $(X^\sharp,\iota^\sharp,\lambda^\sharp,\rho^\sharp)$ associated in $\mathcal{M}(S)$ is defined as follows: $X^\sharp := X$ with the action of $\iota^\sharp := \iota_X$, with the polarization $\lambda^\sharp := \lambda_X$ and $\rho^\sharp := \rho\circ F^{-1}$. Note that the isomorphism of functors $\alpha:\mathcal{M}\to\mathcal{M}$ is the Rapoport–Zink descent data associated with $\mathcal{M}$. As the descent data commute with the action of $J_b(\mathbb{Q}_p)$, the groups $H^i(\mathcal{M}_{K_p},\mathbb{Q}_\ell)$ has an action of $J_b(\mathbb{Q}_p)\times W_{E_p}$. In addition, when K_p varies, the system $(H^i_c(\mathcal{M}_{K_p},\mathbb{Q}_\ell))_{K_p}$ has an action of $GU(\mathbb{Q}_p)$. Thus, this system has an action of $GU(\mathbb{Q}_p)\times J_b(\mathbb{Q}_p)\times W_{E_p}$. Let ρ be an admissible ℓ-adic repre-sentation of $J_b(\mathbb{Q}_p)$, we define

$$H^{i,j}(GU,b,\mu)[\rho] := \varinjlim_{K_p}\mathrm{Ext}^j_{J_b(\mathbb{Q}_p)}(H^i_c(\mathcal{M}_{K_p},\overline{\mathbb{Q}}_\ell),\rho).$$

By [38, Theorem 8], the $H^{i,j}(GU,b,\mu)[\rho]$ are admissible and are zero for almost all $i,j\geq 0$. Finally, we define the homomorphism of Grothendieck groups

$$\mathrm{Mant}_{GU,b,\mu}:\mathrm{Groth}(J_b(\mathbb{Q}_p))\to\mathrm{Groth}(G(\mathbb{Q}_p)\times W_E)$$

by

$$\mathrm{Mant}_{GU,b,\mu}(\rho) := \sum_{i,j}(-1)^{i+j}H^{i,j}(GU,b,\mu)[\rho](-\dim\mathcal{M}^{an}).$$

5.2. An averaging formula for the cohomology of Rapoport–Zink spaces. In this subsection we deduce an averaging formula for the cohomology of Rapoport–Zink spaces using the results of [6].

We begin with some endoscopic preliminaries. To state the formula, we need the follow-ing notion of endoscopic data for Levi subgroups.

Definition 5.4 (cf. [6, Definition 2.18]). Let $M\subset G$ be a Levi subgroup. We say that

- (H,H_M,s,η) is an embedded endoscopic datum of G relative to M and a fixed split-ting $(\mathcal{T}_H,\mathcal{B}_H,\{\mathcal{X}_{H,\alpha}\})$ if (H,s,η) is a refined endoscopic datum of G and the restriction $(H_M,s,\eta|_{\widehat{H}_M})$ gives a refined endoscopic datum of M.
- two embedded endoscopic data (H,H_M,s,η) and (H',H'_M,s',η') are isomorphic if there exists an isomorphism $\alpha:H\to H'$ of refined endoscopic data (H,s,η) and (H',s',η') whose restriction α_M to H_M gives an isomorphism of (H_M,s,η) and (H'_M,s',η'). We denote the set of isomorphism classes of embedded endoscopic data of G relative to M by $\mathcal{E}^e(M;G)$.

We now fix a refined elliptic endoscopic datum (H,s,η) of GU. Note that for each standard Levi subgroup $M\subset G$, there is a natural forgetful map

$$Y^e:\mathcal{E}^e(M;GU)\to\mathcal{E}^r(GU).$$

We define $\mathcal{E}^i(M;GU;H)$ to be the set of embedded endoscopic data (H',H'_M,s',η') such that $H'=H$ and whose class lies in the fiber $(Y^e)^{-1}((H,s,\eta))$ modulo the relation that two data (H,H_M,s,η) and (H,H'_M,s',η') are equivalent if there exists an inner automorphism α of H inducing an isomorphism of the embedded endoscopic data.

Fix a maximal torus $\widehat{T}_H\subset\widehat{H}$ and define

$$\widehat{\varphi}:=\eta(\widehat{T}_H)\subset\widehat{GU}.$$

By [6, comment before Proposition 2.27], we have that the set $\mathcal{E}^i(M; G^* \mid H)$ is parametrized by the set of double cosets $W(\hat{T}, \hat{M}) \backslash W(M, H) / W(\hat{T}_H, \hat{H})$, where $W(\hat{T}, \hat{M})$ and $W(\hat{T}_H, \hat{H})$ are the Weyl groups of $\hat{M}$ and $\hat{H}$ respectively and $W(M, H)$ is defined in [6, Definition 2.23].

Finally, for an inner form J of M, we define the subset $\mathcal{E}^i_{\mathrm{eff}}(J; G^* \mid H) \subset \mathcal{E}^i(M; G^* \mid H)$ to consist of those equivalence class of endoscopic data (H, H_M, s, η) such that there exists a maximal torus of H that transfers to J.

We now fix $b \in B(\mathbb{Q}_p, G^*)$ and let $b \in G^*(\breve{\mathbb{Q}}_p)$ be a decent lift. We get a standard Levi subgroup M_b of G^* and an extended pure inner twist J_b of M_b. Let $\nu_b : \mathbb{D} \to A_{M_b}$ (where A_{M_b} is the maximal split torus in the center of M_b) denote the image of the Newton map applied to b. Fix $(H, s, {}^L\eta)$ an elliptic endoscopic group of G^* and a set, $X^{\mathfrak{e}}_{J_b}$, of representatives of $\mathcal{E}^i_{\mathrm{eff}}(J_b; G^* \mid H)$. Furthermore, for each $(H, H_{M_b}, s, \eta') \in X^{\mathfrak{e}}_{J_b}$ we may choose an extension

$$ {}^L\eta' : {}^LH \to {}^LG^* $$

of η'. We also get a natural map $A_{M_b} \hookrightarrow A_{H_{M_b}}$ induced by η' and we define ν to be the com-position of ν_b with this map. The cocharacter ν_b defines a parabolic subgroup $P(\nu_b)$ of G^* as follows. Choose $m \in \mathbb{Z}^{+}$ so that $m\nu_b \in X_*(A_{M_b})$. Then $m\nu_b$ gives also a cocharacter of T, and this defines a parabolic subgroup by

$$ P(\nu_b)(R) = \left\{ x \in G^*(R) : \lim_{t \to 0} (m\nu_b)(t)\, x\, (m\nu_b)(t)^{-1} \text{ exists} \right\}. $$

It is clear that $P(\nu_b)$ does not depend on m and also, since ν_b is dominant, that $P(\nu_b)$ is a stan-dard parabolic subgroup. Similarly, ν defines a standard parabolic subgroup $P(\nu)$ of H and we let $P(\nu)^{op}$ denote the opposite parabolic subgroup relative to B_H. The Levi subgroup associated to $P(\nu_b)$ is the centralizer of $m\nu_b$ in G^*, which is M_b. Similarly, H_{M_b} is the centralizer of $m\nu$ in H and hence Levi subgroup of $P(\nu)$ (indeed, $m\nu$ is non-vanishing on the roots of H outside of H_{M_b} since $m\nu_b$ is non-vanishing on these roots thought of as roots of G^* via ${}^L\eta'$.

We then make the following definition.

Definition 5.5. We define

$$ \mathrm{Red}^{\mathfrak{e}}_b : \mathrm{Groth}^{st}(H(\mathbb{Q}_p)) \to \mathrm{Groth}(J_b(\mathbb{Q}_p)) $$

by

$$ \pi \mapsto \sum_{X^{\mathfrak{e}}_{J_b}} \mathrm{Trans}^{H_{M_b}}_{J_b}\left(\mathrm{Jac}^H_{P(\nu)^{op}}(\pi) \otimes \delta^{\frac{1}{2}}_{P(\nu_b)}\right) $$

where $\mathrm{Trans}^{H_{M_b}}_{J_b}$ denotes the endoscopic transfer of distributions from $H_{M_b}(\mathbb{Q}_p)$ to $J_b(\mathbb{Q}_p)$ and $\mathrm{Groth}(J_b(\mathbb{Q}_p))$ denotes the Grothendieck group of admissible representations of $J_b(\mathbb{Q}_p)$ and $\mathrm{Groth}^{st}(H(\mathbb{Q}_p))$ is the subgroup of $\mathrm{Groth}(H(\mathbb{Q}_p))$ consisting of those elements with stable distribution character.

Our aim in this subsection is to establish the theorem below using the results of [6].

Theorem 5.6. Let $(H, s, {}^L\eta)$ be a refined elliptic endoscopic datum of G^*. Let

$$ \phi : W_{\mathbb{Q}_p} \to {}^LG^* $$

be a discrete Langlands parameter such that there exists a Langlands parameter ^{H} of H with $= {}^L \circ {}^H$. Then we have the following equality in $\mathrm{Groth}(GU(\mathbb{Q}_p) \times W_E)$:

$$\sum_{b \in B(\mathbb{Q}_p, GU,)} \mathrm{Mant}_{GU,b,}(\mathrm{Red}_b^{e}(S_{,H})) = \sum \sum_{p \in (GU, \varrho)} \langle_p, (s)\rangle \frac{\mathrm{tr}((s) \mid V)}{\dim} \otimes |\;|^{\langle_{GU},\rangle};$$

where the first sum on the right-hand side is over irreducible factors of the representation $r \circ$ and V is the -isotypic part of $r \circ$.

This theorem is [6, Theorem 6.4]. To verify this theorem, we essentially just need to check a number of hypotheses from [6].

First, we need a global group $\mathbf{GU}$ such that $\mathbf{GU}_{\mathbb{Q}_p} \cong GU$ and such that there exists a Shimura datum $(\mathbf{GU}, X)$ of PEL type such that the global conjugacy class of cocharacters of $\mathbf{GU}$ associated to X localizes to the conjugacy class of . Since is assumed minuscule, its weights are equal to 1 and 0. In particular, is determined by a pair (p, q) such that $p + q = n$ and p denotes the number of 1 weights and q denotes the number of 0 weights.

We fix n an odd positive integer and define $\mathbf{GU}$ to be the group $\mathbf{GU}(p, q)$ coming from the Hermitian form $I_{p,q}$ as in Section 2. Following [41, Section 2.1], we have a PEL Shimura datum $(\mathbf{GU}, X)$ for this group (in Morel's notation, this is the datum $(\mathbf{GU}, X, h)$). As we observed in Section 2, the group $\mathbf{GU}$ can be equipped with the structure of an extended pure inner twist (GU, ϱ, z). As in [7], this twist gives us for each refined endoscopic datum $(H, s,)$ of $\mathbf{GU}$ a normalized transfer factor at each place v.

We observe that, in accordance with [6, Sections 4.1 and 5.1], we have GU_{der} is simply connected and $\mathbf{GU}_{\mathbb{Q}_p}$ is unramified. The center $Z(\mathbf{GU})$ is isomorphic to $\mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m$ which has split rank equal to 1. Since $E/\mathbb{Q}$ is an imaginary quadratic extension, the split rank of $Z(\mathbf{GU})_{\mathbb{R}}$ also equals 1.

We verify that $\mathbf{GU}$ satisfies the Hasse principle. By [27, Lemma 4.3.1] it suffices to show that $\ker^1(\mathbb{Q}, GU/GU_{der}) = \ker^1(\mathbb{Q}, \mathbb{G}_m)$ vanishes but this latter group is trivial.

We now note an important difference between the exposition in [6, Section 4] and our current situation. This is that the group $\mathbf{GU}$ will not in general be anisotropic modulo center. For this reason, the stabilization of the trace formula carried out in that paper does not carry over exactly to our case.

Instead, we use Morel's work on the cohomology of these Shimura varieties to establish the desired stabilization. However, Morel's work is on the intersection cohomology of Shimura varieties whereas we need to study compactly supported cohomology. We introduce some necessary notation.

Let $K \subset \mathbf{GU}(\mathbb{A}_f)$ be a compact open subgroup that factors as $K^p K_p$, where K_p is a hyperspecial subgroup of $\mathbf{GU}(\mathbb{Q}_p)$. Following the notation of [41], we let $\overline{M}^K(\mathbf{GU}, X)$ be the Baily–Borel–Satake compactification of the Shimura variety $M^K(\mathbf{GU}, X)$. Fix primes p and ℓ and an algebraic representation V of $\mathbf{GU}$, where we choose the highest weight of V to be "sufficiently regular" in the sense of [42, Definition 2.2.10]. Let $L \subset \mathbb{C}$ be a number field containing the field of definition of V and let be a place of L over ℓ. Then let $IC^K V$ denote the intersection complex on $\overline{M}^K(\mathbf{GU}, X)$ with coefficients in $\overline{\mathbb{Q}}_\ell$ and where V is the evident -adic realization of the local system associated with V. Then we define an element $W_{,K}$ in the Grothendieck group of $\mathcal{H}_K \times W_E$ representations on $\overline{\mathbb{Q}}_\ell$ vector space by W^{*}

$$W^{*}_{\varrho,K} := \sum_{i\geq 0} (-1)^i [H^i(M^K(G,X)_{\overline{\mathbb{Q}}}, IC^K V)].$$

Similarly, we define the element $W^{c}_{\varrho,K}$ in the Grothendieck group of $\mathcal{H}_K \times W_E$ representations on $\overline{\mathbb{Q}}_\ell$ vector spaces by

$$W^{c}_{\varrho,K} := \sum_{i\geq 0} (-1)^i [H^i_c(M^K(G,X)_{\overline{\mathbb{Q}}}, V)].$$

Fix a place $\mathfrak{p}$ of E above p and let $\Phi_{\mathfrak{p}}$ be a lift of the geometric Frobenius at $\mathfrak{p}$ and fix a positive integer j. We will consider functions $f \in \mathcal{H}_K$ such that $f = f^{\infty,p,v} 1_{K_p} f_v f_\infty$, where f_v is cuspidal and f_∞ is stable cuspidal. For instance, f_v could be a coefficient for a supercuspidal representation. Recall that these terms were defined in Section 4.2.

Lemma 5.7. *Suppose that f is cuspidal at a finite place. Then we have*

$$\mathrm{tr}(W^{*}_{\varrho,K} \mid f^{\infty} \times \Phi_{\mathfrak{p}}^{j}) = \mathrm{tr}(W^{c}_{\varrho,K} \mid f^{\infty} \times \Phi_{\mathfrak{p}}^{j}).$$

Proof. Indeed, this follows from the fact we have a natural W-equivariant morphism for each i,

$$H^i_c(M^K(G,X)_{\overline{\mathbb{Q}}}, V) \to H^i(M^K(G,X)_{\overline{\mathbb{Q}}}, IC^K V);$$

and the cuspidal part of $H^i(M^K(G,X)_{\overline{\mathbb{Q}}}, IC^K V)$ lies in the image of this map (see, for instance, [42, Proposition 3.2]). □

We remark on the definitions of the functions $f^H, f_H^{(j)} \in \mathcal{H}(H(\mathbb{A}))$ defined in [6, Section 4] and [41, Section 6.2] respectively. These functions depend on the chosen normalization of transfer factors at each place of $\mathbb{Q}$. We explain in the following paragraph that a priori these functions differ by a constant, but if one modifies Morel's normalization of transfer factors to agree with that of [6, Section 4], then the resulting function $f_H^{(j)}$ can be chosen to equal f^H.

Morel's normalization of transfer factors away from p and ∞ is arbitrary up to the global constraint given by [28, Conjecture 6.10 (b)]. At $v \neq p, \infty$ the definitions of f^H and $f_H^{(j)}$ coincide up to differences in transfer factor normalization. At p, Morel normalizes her transfer factors as in [29, p. 180]. If one chooses a different normalization at p, then Kottwitz explains ([29, pp. 180–181]) how to modify the function $f^{(j)}_{H,p}$ by a constant such that it satisfies and analogous fundamental lemma formula. At $v = \infty$, Morel uses the normalization given in [29, p. 184]. We can again modify the function $f^{(j)}_{H,\infty}$ by a constant so that it satisfies the same formulas. Hence, so long as one modifies the normalizations of the transfer factors at each place in such a way that the global constraint is still satisfied, one gets an analogous modification of the function $f_H^{(j)}$ satisfying the same transfer formulas. By examining the constructions at each place, it is clear that if $f_H^{(j)}$ is modified to be compatible with our chosen normalization of transfer factors, then the functions $f_H^{(j)}$ and f^H can be chosen to be equal.

Since the transfer of a cuspidal function is cuspidal [3, Lemma 3.4] and f^H_∞ is stable cuspidal by definition, we have that f^H satisfies the hypotheses of Lemma 5.7 and Lemma 4.6. In particular, we have the following proposition.

Proposition 5.8. Suppose f^{∞} is cuspidal at a finite place and factors as $f^{p,\infty} 1_{K_p}$. Then

$$\mathrm{tr}(W^{\mathscr{K}} \mid f^{\infty} \times \Phi_{\mathfrak{p}}^{j}) = \sum_{(H,s,\eta)\in \mathcal{E}(GU)} \iota(GU,H)\, ST^{H}_{\mathrm{ell}}(f^{H}).$$

Proof. By Lemma 5.7 and [41, Theorem 7.1.7] (keeping in mind her remark that the result holds for general p) we have

$$\mathrm{tr}(W^{\mathscr{K}} \mid f^{\infty} \times \Phi_{\mathfrak{p}}^{j}) = \sum_{(H,s,\eta)\in \mathcal{E}(GU)} \iota(GU,H)\, ST^{H}(f^{H}).$$

Now, we apply Lemma 4.6 to the right-hand side to get the desired equality. □

At this point, we have finished using the work of Morel and have arrived at the formula [6, equation (4.17)]. We now need to show that we can perform the destabilization procedure as in [6, Section 4.7]. To do so we need to prove that we have a sufficiently good theory of the Langlands correspondence for GU and its localizations. Globally, we will work with "automorphic parameters" in the style of [5, 25] and as we defined in Section 2.3.2. Since our ultimate goal is to prove a local formula, these parameters are sufficient for our purpose. We list the following properties we need and where these facts have been proven.

- (i) We need a construction of local Arthur packets of generic parameters at all localizations of GU and descriptions of the elements in each local A-packet in terms of representations of the various centralizer groups (Theorem 2.19).
- (ii) The local packets must satisfy the endoscopic character identities (Section 3).
- (iii) A local generic A-packet contains a K-unramified representation if and only if the parameter is unramified. In the case that an A-parameter is unramified, this K-unramified representation is unique (Section 4.1).
- (iv) We need a construction for global Arthur packets for generic "v-cuspidal" parameters. These consist of parameters that are supercuspidal at some fixed local place v. We need a description of the global A-packet in terms of the local packets (Section 2.3.2).
- (v) We need v-cuspidal parameters to satisfy a version of [6, Proposition 3.10]. This proposition gives a bijection up to equivalence between pairs $(\psi, s) \in \Psi(G)_S$ and tuples $((H, s, {}^L\eta), \psi^H)$ for $(H, s, \eta) \in \mathcal{E}^r(G)$ and $\psi^H \in \Psi(H)$. (This is discussed in [5, p. 36].)
- (vi) We need a decomposition of the generic v-cuspidal part of $L^2_{\mathrm{disc}}(GU(\mathbb{Q}) \backslash GU(\mathbb{A}))$ in terms of global Arthur packets and this decomposition should satisfy the global multiplicity formula (Section 2.3.2).
- (vii) We need to attach to a global generic parameter a global Galois representation whose localizations at each place are compatible with the corresponding localization of the global parameter (Section 4.4).

With these properties in hand, we can now apply the results of Section 4.2 (which is analogous to [6, Assumption 4.8]) to get

$$\mathrm{tr}(W^{\mathscr{K}} \mid f^{\infty} \times \Phi_{\mathfrak{p}}^{j}) = \sum_{(H,s,\eta)\in \mathcal{E}(GU)} \iota(GU,H)\, ST^{H}_{\mathrm{disc}}(f^{H}).$$

Following the argument of [6, Section 4.7], we derive the formula

$$\mathrm{tr}(W^*_{\xi,K} \mid f^{\infty} \times \phi_p^j) = \sum_{[\psi]\in\Psi(S)} \sum_{\pi^{\infty}\in\Pi_{\psi^{\infty}}(GU,\varrho^{\infty})} \sum \cdots \; m(\pi^{\infty};\psi)\,\mathrm{tr}(\pi^{\infty} \mid f^{\infty})\,A(\pi_p;\psi),$$

where the first sum is over equivalence classes of v-cuspidal parameters and $A(\pi_p;\psi)$ is the ϕ_p^j-trace of a certain representation determined by ψ.

We now define

$$W^* = \lim_{\to} W^*_{\xi,K}$$

in the Grothendieck group of $GU(\mathbb{A}_f) \times W_E$-representations. Suppose π_f is a representation of $GU(\mathbb{A}_f)$ appearing in W^* whose associated automorphic A-parameter is v-cuspidal. We need to compute the π_f-isotypic part, $W^*[\pi_f]$, of W^*. To do so, we apply the argument at the end of [6, Section 4.7]. This argument requires the existence of a compact open $K \subset GU(\mathbb{A}_f)$ such that $\pi_f^K \neq 0$ and a function $f^{\infty} \in H(GU(\mathbb{A}_f)//K)$ that is non-vanishing on π_f but vanishes on every other admissible $GU(\mathbb{A}_f)$-representation appearing in W^*. Our present situation is complicated by the fact that we also need f to be v-cuspidal. More precisely, for the argument at the end of [6, Section 4.7] to go through, we need the following lemma.

Lemma 5.9. *Let π_f be an admissible representation of $GU(\mathbb{A}_f)$ such that the A-parameter at v is supercuspidal. There exists a compact open $K \subset GU(\mathbb{A}_f)$ such that $\pi_f^K \neq 0$ and K factors as $K^v K_v$ and there exists a v-cuspidal function $f^{\infty} \in H(GU(\mathbb{A}_f); K)$ such that $\mathrm{tr}(\pi_f \mid f^{\infty}) \neq 0$ and for any π'_f with non-trivial K-invariants and appearing in either W^* or*

$$\sum_{\psi} \sum_{\pi^{\infty}\in\Pi_{\psi^{\infty}}(GU,\varrho^{\infty})} \sum \cdots \; m(\pi^{\infty};\psi)(s_{\psi})(-1)^{q(GU)}\cdots V(\psi;\pi),$$

we have

$$\mathrm{tr}(\pi'_f \mid f^{\infty}) = 0.$$

Proof. The set R' of isomorphism classes of π'_f satisfying the above conditions is finite. Hence we can find a function $f^{v,\infty}$ such that $\mathrm{tr}((\pi'_f)^v \mid f^{v,\infty}) = 0$ for all $\pi'_f \in R'$ unless $(\pi'_f)^v \cong \pi_f^v$ in which case the trace is non-zero. Now, at v we have that $(\pi_f)_v$ is supercuspidal and so we choose $f_v \in H(GU(\mathbb{Q}_v); K_v)$ to be a coefficient for $(\pi_f)_v$. Then $f^{v,\infty} f_v$ has the desired properties. Indeed, any π'_f not isomorphic to π_f will differ from π_f either at v or away from it, and hence $\mathrm{tr}(\pi'_f \mid f^{v,\infty} f_v) = 0$. □

Following the argument at the end of [6, Section 4.7], we conclude that

$$W^*[\pi_f] = \sum_{\psi} \sum_{\pi^{\infty}\in\Pi_{\psi^{\infty}}(GU,z^{\mathrm{iso}}_{\infty})} \sum \cdots \; m(\pi^{\infty};\psi)[\pi_f] \cdots (s_{\psi})(-1)^{q(GU)}(\pi^{\infty})\, V(\psi;\cdots)[\pi_f] \tag{5.1}$$

in $\mathrm{Groth}(GU(\mathbb{A}_f) \times W_E)$.

So far we have discussed the computation of the cohomology of Shimura varieties and arrived at equation (5.1). We now want to carry out an analogous computation for the compactly supported cohomology of Igusa varieties as in [6, Section 5] starting from the stable trace formula as in [50, Theorem 1.1]. In this case the stabilization in [6, Section 5] does not require that GU is anisotropic modulo center and so that argument goes through essentially unchanged. The only difference is that in this paper, we only prove the equality of $ST^H_{\mathrm{ell}}(f^H)$ and $ST^H_{\mathrm{disc}}(f^H)$ in the case that f^H is cuspidal at a finite place. In particular, this means that we again need a lemma analogous to Lemma 5.9. In this case, the precise conditions on f^{∞} are slightly different since the trace formula for Igusa varieties is stated for acceptable functions in the sense of [49, Definition 6.2].

Lemma 5.10. Let π_f be an irreducible admissible representation of $GU(\mathbb{A}^p_f) \times J_b(\mathbb{Q}_p)$ such that the corresponding local A-parameter at v is supercuspidal. Let

$$K \subset GU(\mathbb{A}^p_f) \times J_b(\mathbb{Q}_p)$$

be a compact open subgroup such that $\pi_f^K \neq \emptyset$ and K factors as $K^{v,p} K_v K_p$. Let R be a finite set of isomorphism classes of irreducible admissible $GU(\mathbb{A}^p_f) \times J_b(\mathbb{Q}_p)$ representations such that $\pi_f \in R$. Then there exists a v-cuspidal function $f^{\infty} \in \mathcal{H}(GU(\mathbb{A}^p_f) \times J_b(\mathbb{Q}_p); K)$ that is acceptable in the sense of [49, Definition 6.2] such that f^{∞} factors as $f^{p,v,\infty} f_p f^p_v$ and $\mathrm{tr}(\pi'_f \mid f^{\infty}) \neq 0$ for $\pi'_f \in R$ if and only if $\pi'_f \cong \pi_f$.

Proof. Consider the linear map from v-cuspidal functions to $\mathbb{C}^{|R|}$ given by

$$f^{\infty} \mapsto (\mathrm{tr}(\pi_1 \mid f^{\infty}), \dots, \mathrm{tr}(\pi_n \mid f^{\infty})),$$

where $R = \{\pi_1, \dots, \pi_n\}$. It suffices to show this map is surjective. If the map is not surjective, then its image is a proper subspace and hence lies in a hyperplane of $\mathbb{C}^{|R|}$. Hence we can find some element $c_1, \dots, c_n \in \mathbb{C}[R]$ such that for all v-cuspidal f^{∞}, we have

$$c_1 \mathrm{tr}(\pi_1 \mid f^{\infty}) + \cdots + c_n \mathrm{tr}(\pi_n \mid f^{\infty}) = 0.$$

Now, by the argument of [49, Lemma 6.4] and also [49, Lemma 6.3] it follows that every $f^{\infty} = f^{p,v,\infty} f_p f_v$ that is cuspidal at v satisfies

$$\mathrm{tr}(c_1\pi_1 + \cdots + c_n\pi_n \mid f^{\infty}) = 0.$$

By the argument of Lemma 5.9, we can find an f^{∞} that does not vanish at $c_1\pi_1 + \cdots + c_n\pi_n$. This is a contradiction and implies our desired result. □

At this point, we have verified the assumptions of [6, Sections 4–5]. It remains to check those of [6, Section 6]. We first note that the Mantovan formula is known for the PEL-type Shimura varieties we consider. Indeed, this is [34, Theorem 6.32].

It remains to check [6, Assumptions 6.2 and 6.3]. We record some useful lemmas.

Lemma 5.11. Suppose π is a discrete automorphic representation of $GU(\mathbb{A})$ contained in an A-packet Π_{ψ}. Suppose further that the infinitesimal character of π_{∞} is sufficiently regular in the sense of [42, Definition 2.2.10]. Then the A-parameter associated to ψ is generic.

Proof. Standard. For instance see [25, Lemma 4.3.1]. □

Lemma 5.12. Suppose π is a discrete automorphic representation of $GU(\mathbb{A})$ contained in an A-packet and such that π_∞ has sufficiently regular infinitesimal character. Then this is the unique A-packet containing π. Moreover, if $\tilde{\pi}$ is another discrete automorphic represen-tation of $GU(\mathbb{A})$ such that $\tilde{\pi}_\infty$ has sufficiently regular infinitesimal character and such that $\pi^\infty \simeq \tilde{\pi}^\infty$ then π and $\tilde{\pi}$ are in the same A-packet.

Proof. Suppose π belongs to two A-packets with associated A-parameters (ψ_{U1}, χ_1) and (ψ_{U2}, χ_2). Since χ_1 and χ_2 correspond to the central character of π, they are equal. We need to show that ψ_{U1}, ψ_{U2} are also equal. At almost all finite unramified places v where π_v is unramified, the localizations $(\psi_{U1})_v$ and $(\psi_{U2})_v$ are equal. Indeed, our sufficiently regu-lar assumption implies that these parameters are generic. Following [40, p. 189], these local parameters factor through ${}^L M$ where M is the minimal Levi subgroup of U_v and correspond to the same spherical parameter of M (for more details, see [40, p. 189]). This implies that ψ_{U1}^{n} and ψ_{U2}^{n} give rise to the same Hecke string. Then, by [22] and [4, Theorem 4.3], we see that ψ_{U1} and ψ_{U2} are equal. It is clear that the second statement also follows from exactly the same argument. □

Before verifying [6, Assumptions 6.2 and 6.3], we need to understand the effect of an unramified twist on the $\mathrm{Mant}_{GU,b,\mu}$ map. Let $c : GU(\mathbb{Q}_p) \to \mathbb{Q}_p^\times$ be the similitude factor character. For b non-basic, the group $J_b(\mathbb{Q}_p)$ is an inner form of a Levi subgroup $M_b(\mathbb{Q}_p)$ of $GU(\mathbb{Q}_p)$. Then the similitude character c restricted to $M_b(\mathbb{Q}_p)$ can be transferred to $J_b(\mathbb{Q}_p)$. Hence by abuse of language, we also denote c the corresponding character on $J_b(\mathbb{Q}_p)$.

Lemma 5.13. Let $(E_p, \ast, V, \langle\cdot,\cdot\rangle, GU, \mu, b)$ be an unramified unitary Rapoport–Zink PEL datum and suppose that $\omega : \mathbb{Q}_p^\times \to \overline{\mathbb{Q}}_\ell^\times$ is an unramified character. Then the following holds in $\mathrm{Groth}(GU(\mathbb{Q}_p) \times W_{E_p})$:

$$\mathrm{Mant}_{GU,b,\mu}(\rho \otimes (\omega \circ c)) = \mathrm{Mant}_{GU,b,\mu}(\rho) \otimes (\omega \circ c) \otimes (\omega \circ \mathrm{Art}_{E_p})^{-1}.$$

Proof. This lemma is an analogue of [53, Lemma 4.9] and the same proof applies in our situation. Thus we just briefly give an idea of how to proceed.

Define a character θ of $J_b(\mathbb{Q}_p) \times GU(\mathbb{Q}_p) \times W_{E_p}$ such that

$$\theta^{-1} := (\omega \circ c) \otimes (\omega \circ c) \otimes (\omega \circ \mathrm{Art}_{E_p}^{-1}).$$

In $\mathrm{Groth}(GU(\mathbb{Q}_p) \times W_{E_p})$ we have

$$\begin{aligned}&\lim_{\to K_p} \mathrm{Ext}^{\bullet}_{J_b(\mathbb{Q}_p)}(H^j_c(\mathcal{M}_{K_p}, \overline{\mathbb{Q}}_\ell), \rho \otimes (\omega \circ c))\\ &\simeq \lim_{\to K_p} \mathrm{Ext}^{\bullet}_{J_b(\mathbb{Q}_p)}(H^j_c(\mathcal{M}_{K_p}, \overline{\mathbb{Q}}_\ell) \otimes \theta^{-1}, \rho) \otimes (\omega \circ c) \otimes (\omega \circ \mathrm{Art}_{E_p}^{-1}).\end{aligned}$$

Then we prove that for each level K_p, there is an isomorphism of $\overline{\mathbb{Q}}_\ell$-vector spaces

$$H^j_c(\mathcal{M}_{K_p}, \overline{\mathbb{Q}}_\ell) \simeq H^j_c(\mathcal{M}_{K_p}, \overline{\mathbb{Q}}_\ell) \otimes \theta$$

such that the resulting bijection of direct limits

$$\lim_{\to K_p} H^j_c(\mathcal{M}_{K_p}, \overline{\mathbb{Q}}_\ell) \simeq \lim_{\to K_p} H^j_c(\mathcal{M}_{K_p}, \overline{\mathbb{Q}}_\ell) \otimes \theta$$

is compatible with the action of $J_b(\mathbb{Q}_p) \times GU(\mathbb{Q}_p) \times W_{E_p}$.

Note that there is a $J_b.Q_p/$-equivariant map (see [45, Section 3.52])

$$WM_{K_p} \ ! \quad \bullet \ WDHom_Z.X.GU/; Z/;$$

and moreover there is a natural way to define an action of $GU.Q_p/ W_{E_p}$ on $\bullet$. We can then prove the lemma by using the fact that acts trivially on $.J_b.Q_p/ \ GU.Q_p/ \ W_{E_p}/^1$ and that there is an $J_b.Q_p/ \ GU.Q_p/ \ W_{E_p}$ -equivariant bijection ([14, Remark 2.6.11]).

$$\lim_{\substack{! \\ K_p}} H^j_c.M_{K_p}; \overline{Q}_\ell/ \quad ' \quad \bigoplus_{\bar{i} 2 \bullet = J_b.Q_p/GU.Q_p/W_{E_p}} c\text{-}ind^{J_b.Q_p/GU.Q_p/W_{E_p}}_{.J_b.Q_p/GU.Q_p/W_{E_p}/^1} \lim_{\substack{! \\ K_p}} H^j_c.M^{.i/}_{K_p}; \overline{Q}_\ell/ \ ;$$

where $M^{.i/}_{K_p}$ is the inverse image of i by and $.J_b.Q_p/ \ GU.Q_p/ \ W_{E_p}/^1$ is the subgroup of $J_b.Q_p/ \ GU.Q_p/ \ W_{E_p}$ that acts trivially on $\bullet$. □

We can now settle [6, Assumptions 6.2] in the cases we need. Let $_p$ be a representation of $GU.Q_p/$ and $_1$ a discrete automorphic representation of $GU.A/$ such that $._1/ \ \check{S}_{p\,p}$. Suppose further that $^1_1$ appears in either the formula for the cohomology of Igusa varieties or W^C. Then since V has sufficiently regular infinitesimal character, it follows that the same is true of $._1/_1$. Now suppose $_2$ is a discrete automorphic representation of $GU.A/$ appearing in either of the above formulas and such that $^1 \ \check{S} \ ^1_1$. We then have by Lemma 5.12 that $_2$ and $_1$ are in the same packet.

We now tackle [6, Assumptions 6.2]. For a fixed discrete series representation $_p$ of $GU.Q_p/$ with local parameter $_{GU}$, we have the local centralizer group S_{GU}. For any global A-parameter $_{GU}$ such that $_{GU_p} \ D \ _{GU}$, we have a natural embedding $S_{GU} \ ,! \ S_{GU}$. Note that [6, formula immediately before Assumption 6.3] includes a sum indexed over a set of representatives X_{GU} of $\overline{S}_{GU}$. We must show that we can pick different globalizations, $_{GU}$, of $_{GU}$ to derive the formula below [6, Assumption 6.2] for each element of S_{GU}.

Suppose first that $s \ 2 \ S_{GU}$ projects to the identity element of $\overline{S}_{GU}$ such that

$$s \ D \ .1; t/ \ 2 \ S_{GU} \ D \ S_U \ {}^C C:$$

By Lemma 4.8, we can choose $_{GU}$ so that the image of S_{GU} in S_{GU_p} is $^1.1; t^0/ \ Wt^0 \ 2 \ C^º$ and the packet $…_{GU_p}.GU.Q_p/; \%_{GU}/$ differs from the packet $…_{GU}.GU.Q_p/; \%_{GU}/$ by an unramified twist of the form $! \ ı \ c$. Then we simply pick X_{GU} to contain the unique element of S_{GU} mapping to s. This establishes the formula for s projecting to the identity of $\overline{S}_{GU_p}$. By Lemma 5.13, we obtain the formula for s projecting to the identity of $\overline{S}_{GU}$.

Now suppose we pick $s \ 2 \ S_{GU}$ that projects to a non-identity element $\bar{s}$. By Lemma 4.9, we may choose $_{GU}$ such that the image of S_{GU} in S_{GU_p} is precisely the pre-image of $^1 1; \bar{s}^º$ under the map

$$S_{GU_p} \ ! \ \overline{S}_{GU_p};$$

and the packet $…_{GU_p}.GU.Q_p/; \%_{GU}/$ differs from the packet $…_{GU}.GU.Q_p/; \%_{GU}/$ by an unramified twist of the form $! \ ı \ c$. Choose X_{GU} to contain the unique elements mapping to $s; 1$ and denote these x_s and x_1 respectively. Then each side of the formula before [6, Assumptions 6.2] for the parameter $_{GU_p}$ has two terms indexed by x_s and x_1 respectively. Again, by Lemma 5.13, we can derive the same formula for $_{GU}$. The x_1 terms are already known to be equal by the previous paragraph. It therefore follows that the x_s terms are equal as well.

This completes the verification of Theorem 5.6.

6. Proof of the main theorem

To prove the Kottwitz conjecture for the groups we consider, we use Theorem 5.6. We remark that since GU is quasi-split, the sign $e(GU) = 1$, and since we consider only supercuspidal parameters in this section, the elements s_ϕ, s_{ϕ^H} are trivial. We recall that we have fixed an extended pure inner twist (GU, ϱ, z) of $GU(n)$, where all groups are defined over $\mathbb{Q}_p$, and that J_b has the structure of an extended pure inner twist (J_b, ϱ_b, z_b) of GU and hence $(J_b, \varrho_b \circ \varrho, z + z_b)$ of $GU(n)$.

First of all, we show that

$$\mathrm{Red}^e_b\Big(\sum_{\pi^H \in \Pi_{\phi^H}(H)} \langle \pi^H, s_{\phi^H}\rangle \pi^H\Big) = 0$$

for b non-basic, (H, s, η) an elliptic endoscopic datum of GU and ϕ a supercuspidal parameter.

Indeed, the parameter ϕ^H is again a supercuspidal L-parameter. In particular, the representations π^H are supercuspidal. Now by definition we have

$$\mathrm{Red}^e_b = \sum \delta_{P(b)^e}^{\frac{1}{2}}\, \mathrm{Trans}_{J_b}^{H_M} \circ \mathrm{Jac}^H_{P(b)^{op}} \circ \chi_{J_b}.$$

As b is non-basic, the group J_b is an inner form of a proper Levi subgroup of GU. Suppose that $P(b)^{op} = H$. In this case H equals H_M and is isomorphic to an endoscopic group of J_b. This is a contradiction because by the classification of the endoscopic groups of GU and its Levi subgroups, we know that the elliptic endoscopic groups of GU are not endoscopic groups of any proper Levi subgroup of GU. We conclude that $P(b)^{op}$ is a proper parabolic subgroup of H so that

$$\mathrm{Red}^e_b\Big(\sum_{\pi^H \in \Pi_{\phi^H}(H)} \langle \pi^H, s_{\phi^H}\rangle \pi^H\Big) = 0,$$

as desired.

Now, for b basic, the main formula of Theorem 5.6 becomes

$$\mathrm{Mant}_{GU,b,\mu}\Big(\mathrm{Trans}^H_{J_b} \sum_{\pi^H \in \Pi_{\phi^H}} \langle \pi^H, 1\rangle \pi^H\Big) = \sum_{\rho} \sum_{\pi \in \Pi_\phi(GU,\varrho)} \langle \pi, \eta(s)\rangle \frac{\mathrm{tr}(\eta(s) \mid V)}{\dim \rho}\, \rho \boxtimes |\cdot|^{-\langle \rho_{GU}, \mu\rangle}.$$

The endoscopic character identity (Equation (3.1)) and definition of $\mathrm{Trans}^H_{J_b}$ (see [20, p. 1634], for instance) immediately implies that

$$\mathrm{Trans}^H_{J_b} \sum_{\pi^H \in \Pi_{\phi^H}} \langle \pi^H, 1\rangle \pi^H = \sum_{\pi_{J_b} \in \Pi_\phi(J_b, \varrho_b \circ \varrho)} \langle \pi_{J_b}, \eta(s)\rangle \pi_{J_b}.$$

Substituting into the previous equation gives

$$\mathrm{Mant}_{GU,b,\mu}\Big(\sum_{\pi_b \in \Pi_\phi(J_b, \varrho_b \circ \varrho)} \langle \pi_{J_b}, \eta(s)\rangle \pi_{J_b}\Big) = \sum_{\rho} \sum_{\pi \in \Pi_\phi(GU,\varrho)} \langle \pi, \eta(s)\rangle \frac{\mathrm{tr}(\eta(s) \mid V)}{\dim \rho}\, \rho \boxtimes |\cdot|^{-\langle \rho_{GU}, \mu\rangle}.$$

Now, fix $\rho_{J_b} \in \Pi(J_b, \varrho_b \mid \varrho)$ and multiply the above equation (where we renotate the ρ_{J_b} in the summation as ρ'_{J_b}) by $\langle \rho_{J_b}, \iota(s)\rangle^{-1}$. One can check that $\langle \rho_{J_b}, \iota(s)\rangle^{-1}\langle \rho'_{J_b}, \iota(s)\rangle$ and $\langle \rho_{J_b}, \iota(s)\rangle^{-1}\langle \pi, \iota(s)\rangle \operatorname{tr}(\iota(s) \mid V)$ depend only on the image $\iota(s) \in \bar{S}$. Indeed, it suffices to show each expression vanishes on $Z(\widehat{G})^{\Gamma_{\mathbb{Q}_p}}$ since $\bar{S} = S/Z(\widehat{G})^{\Gamma_{\mathbb{Q}_p}}$. The first expres-sion does so since $\langle \rho'_{J_b}, \cdot\rangle$ and $\langle \rho_{J_b}, \cdot\rangle$ have the same central character restricted to $Z(\widehat{G})^{\Gamma_{\mathbb{Q}_p}}$ (see the definition of this pairing in Section 2.3.1). The second does so because $\langle \rho_{J_b}, \cdot\rangle^{-1}\langle \pi, \cdot\rangle$ has central character equal to $\kappa(b)^{-1} = \hat{\mu}$ while the action of $\iota(s)$ on the image of r_{μ} is by $\hat{\mu}$. Therefore, by a slight abuse of notation, we may regard these expressions as functions of $s \in \bar{S}$. We remark also that every element of $\bar{S}$ has a representative of the form $\iota(s)$ for an elliptic endoscopic datum $(H, s, {}^{L}\eta)$. Indeed, this is [6, Corollary 3.13] (see also [6, Remark 3.14]).

We then average over $\bar{S}$. This gives an equality between

$$\operatorname{Mant}_{G,b,\mu}(\pi)\ \frac{1}{|\bar{S}|}\sum_{s \in \bar{S}}\ \sum_{\rho'_{J_b} \in \Pi(J_b, \varrho_b \mid \varrho)} \langle \rho_{J_b}, s\rangle^{-1}\langle \rho'_{J_b}, s\rangle\, \rho'_{J_b}$$

and

$$\frac{1}{|\bar{S}|}\sum_{s \in \bar{S}}\ \sum_{\rho}\ \sum_{\pi \in \Pi(G, \varrho)} \langle \rho_{J_b}, s\rangle^{-1}\langle \pi, s\rangle \frac{\operatorname{tr}(s \mid V)}{\dim \rho}\, |\cdot|^{-\langle 2\rho_G, \nu\rangle}\, \pi.$$

Now, for any irreducible representation τ of $\bar{S}$, we have $\frac{1}{|\bar{S}|}\sum_{s \in \bar{S}} \tau(s)$ is 1 if τ is trivial and 0 otherwise. Hence we get the equality

$$\operatorname{Mant}_{G,b,\mu}(\rho_{J_b}) = \frac{1}{|\bar{S}|}\sum_{s \in \bar{S}}\ \sum_{\rho}\ \sum_{\pi \in \Pi(G, \varrho)} \langle \rho_{J_b}, s\rangle^{-1}\langle \pi, s\rangle \frac{\operatorname{tr}(s \mid V)}{\dim \rho}\, |\cdot|^{-\langle 2\rho_G, \nu\rangle}\, \pi.$$

We now isolate the term for a fixed $\pi_G \in \Pi(G, \varrho)$ and representation ρ. It is

$$\frac{1}{|\bar{S}|}\sum_{s \in \bar{S}} \langle \rho_{J_b}, s\rangle^{-1}\langle \pi_G, s\rangle \frac{\operatorname{tr}(s \mid V)}{\dim \rho}\, |\cdot|^{-\langle 2\rho_G, \nu\rangle}\, \pi_G. \tag{6.1}$$

We would like to relate this to the term

$$\dim \operatorname{Hom}_{S}(\tau_{\rho_{J_b}} \otimes \tau_{\pi_G}^{\vee}, V),$$

which appears in the statement of the Kottwitz conjecture. Note that the dimension does not change if we tensor both $\tau_{\rho_{J_b}} \otimes \tau_{\pi_G}^{\vee}$ and V by the character $\tau_{\rho_{J_b}}^{\vee} \otimes \tau_{\pi_G}$. We observe that the resulting representation $V \otimes \tau_{\rho_{J_b}}^{\vee} \otimes \tau_{\pi_G}$ is trivial on $Z(\widehat{G})^{\Gamma_{\mathbb{Q}_p}}$ and hence factors through $\bar{S}$. Hence, it suffices to compute the dimension as an $\bar{S}$-represen-tation where it is given by the formula

$$\frac{1}{|\bar{S}|}\sum_{s \in \bar{S}} \operatorname{tr}(s \mid V \otimes \tau_{\rho_{J_b}}^{\vee} \otimes \tau_{\pi_G}) = \frac{1}{|\bar{S}|}\sum_{s \in \bar{S}} \langle \rho_{J_b}, s\rangle^{-1}\langle \pi_G, s\rangle \operatorname{tr}(s \mid V), \tag{6.2}$$

where we have the same abuse of notation as before that $\langle h_{J_b}, s_i^{-1}\rangle\langle h_{GU}, s_i\rangle \operatorname{tr}(s \mid V)$ only depends on $s \in S$ but the individual terms in the product require taking a lift to S. Comparing equations (6.1) and (6.2), we see that the expression in equation (6.1) becomes

$$\frac{\dim \operatorname{Hom}_{S}(\, , V)}{\dim}\,[\,][\,] \mid \, \mid^{\langle h_{GU}, i\rangle}.$$

This equals

$$[\,][\operatorname{Hom}_{S}(\, , V)] \mid \, \mid^{\langle h_{GU}, i\rangle};$$

and summing over , we get

$$\begin{aligned}&\operatorname{Mant}_{GU,b,}(J_b)\\ &= \sum [\,][\operatorname{Hom}_{S}(\, , r)] \mid \, \mid^{\langle h_{GU}, i\rangle} \quad (GU \in \Pi(GU, \%))\end{aligned}$$

In conclusion we have proven

Theorem 6.1 (Kottwitz conjecture). *For irreducible admissible representations J_b of $J_b(\mathbb{Q}_p)$ with supercuspidal L-parameter , we have the equality*

$$\begin{aligned}&\operatorname{Mant}_{GU,b,}(J_b)\\ &= \sum [\,][\operatorname{Hom}_{S}(\, , r)] \mid \, \mid^{\langle h_{GU}, i\rangle} \quad (GU \in \Pi(GU, \%))\end{aligned}$$

in $\operatorname{Groth}(GU(\mathbb{Q}_p) \times W_E)$.

A. Some computations with GU(3)

In this appendix we use the averaging formula (Theorem 5.6) to compute $\operatorname{Mant}_{G,b,}()$ in a few cases for $G = GU(3)$ where the parameter of is not supercuspidal.

Let $E/\mathbb{Q}_p$ be the quadratic unramified extension with non-trivial Galois automorphism and corresponding quasi-split unitary group $GU(3)$. Recall that the diagonal torus T gives a maximal torus of $GU(3)$ of maximal split rank and satisfies

$$T(\mathbb{Q}_p) = \left\{ \left(\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix}, c \right) : (t_1)(t_3) = c = (t_2)(t_2),\ t_i \text{ with } c \in E \right\}.$$

The torus T is a Levi subgroup of $GU(3)$.

Let GU be the trivial extended pure inner form of $GU(3)$. We let be the cocharacter of GU given by

$$z \mapsto \left(\begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, z \right).$$

Let P be the parabolic subgroup with Levi factor equal to T and such that is dominant with respect to the positive root system determined by P. Let $_G$ be the half sum of the positive absolute roots of GU. Then we have

$$\langle \rho_G, \mu \rangle = 1.$$

The modulus character δ_P on T is given by $t \mapsto |2\rho_G(t)|$.

We now describe the set $B(GU, \mu)$. To begin, we have that $Z(\widehat{GU})^\Gamma$ consists of pairs $(\lambda I, c) \in GL_3(\mathbb{C}) \times \mathbb{C}^\times$ and hence $X^*(Z(\widehat{GU})^\Gamma) \cong \mathbb{Z}$. Then there is a unique basic element b of $B(GU, \mu)$ whose image under the Kottwitz map is $1 \in X^*(Z(\widehat{GU})^\Gamma)$.

We also have

$$\widehat{T}^\Gamma = \left\{ \left(\begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-1} \end{pmatrix}, c \right) : t, c \in \mathbb{C}^\times \right\}.$$

Hence $X^*(\widehat{T}^\Gamma) \cong \mathbb{Z} \times \mathbb{Z}$ and the non-basic elements of $B(GU, \mu)$ are in bijection with pairs $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that $x > 0$. Moreover, the pair (x, y) corresponds to an element whose slope cocharacter has weights $((\frac{x+y}{2}, \frac{y}{2}, \frac{y-x}{2}); y)$. As $\bar{\mu} = \mu^{+}(\,)$ has weights $((1, \frac{1}{2}, 0); 1)$, one can check that the element $(1, 1) \in \mathbb{Z} \times \mathbb{Z} \cong X^*(\widehat{T}^\Gamma)$ has slope cocharacter equal to that of $\bar{\mu}$ and gives the other element b' of $B(GU, \mu)$. We have $J_b \cong GU$ and $J_{b'} \cong T$.

Let ϕ be an $\mathrm{St}_H(\,)$ parameter of $GU(3)$ in the notation of [46, Chapter 12] and [21]. Then $r \circ \phi = (\,\otimes \mathrm{std}_-) + (\,\otimes \mathrm{triv})$ for certain characters and . The L-packet of ϕ is $\{\pi^s, \pi^2\}$, where π^s is supercuspidal and π^2 is discrete series but not supercuspidal. The representation π^2 shows up as the parabolic induction $I_P(\,)$ for a certain character of T. This induction is reducible and the other Jordan–Hölder factor is a non-tempered representation π^n.

We first consider the averaging formula for the trivial endoscopic group. This gives

$$\begin{aligned} \mathrm{Mant}_b(\pi^2 + \pi^s) + \mathrm{Mant}_{b'}(J_{P^{op}}(\pi^2) \otimes \delta_P^{\frac{1}{2}}) &= [\pi^2] \\ &+ [\pi^s] \boxtimes [\chi(\phi_2) + \chi(\phi_2) \otimes \chi(1)] \boxtimes [\,]^{\frac{3}{}} \end{aligned}$$

On the other hand, by the Harris–Viehmann conjecture (known in our present situation since GU is HN-reducible, cf. [44, Theorem 8.8]), we get that

$$\begin{aligned} \mathrm{Mant}_{b'}(J_{P^{op}}(\pi^2) \otimes \delta_P^{\frac{1}{2}}) = \mathrm{Ind}_P\, \mathrm{Mant}_{b''}(J_{P^{op}}(\pi^2) \otimes \delta_P^{\frac{1}{2}}) \\ = [\theta_P\, J_{P^{op}}(\pi^2) \boxtimes (\,_2)]; \quad \tfrac{1}{} \end{aligned}$$

for a certain $b'' \in B(T)$. Hence, we have

$$\begin{aligned} \mathrm{Mant}_b(\pi^2 + \pi^s) = [\pi^2] + [\pi^s] \boxtimes [\chi(\phi_2) + \chi(\phi_2) + \chi(1)] \boxtimes [\,]^{\frac{3}{}} \\ - [\theta_P\, J_{P^{op}}(\pi^2) \boxtimes (\,_2)]: \quad \tfrac{1}{} \end{aligned}$$

Now, we consider the non-trivial endoscopic group

$$H = G(U(2) \times U(1)),$$

with $s = (\,I_2; I_1)$. We pick η such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}; e; f \mapsto \left(\begin{pmatrix} a & 0 & b \\ 0 & e & 0 \\ c & 0 & d \end{pmatrix}; f \right):$$

We need to compute W.T; H/. These are the w 2 W.T; G/ such that .w ı / ^{1}.T/ is €-invariant up to conjugacy in H. This implies that

w 2 ^{1}./; .13/º:

It follows that W.T; G/ n W.T; H/=W.H; / has a single element which we can take to be the endoscopic datum .T; s; id/. We fix P_H the parabolic of H with Levi factor equal to T determined by . We have ı$_{P_H}$ is given by t ! j^2.t/j.

Let ^{e} be the parameter of H whose composition with ^{L} is . Let ^{e} be the unique element in the packet of ^{e}. Then we have that the left-hand side of the endoscopic averaging formula is:

$$\mathrm{Mant}_b.\mathrm{Trans}_H^{GU}.^e/// C \ \mathrm{Mant}_{b^0}..\mathrm{Trans}^T J_{P_H^{op}}.^e// ˝ \ ı^2/;^{\frac{1}{P}}$$

where the Trans_T^T term denotes endoscopic transfer between .T; s; id/ and T. The representation $J_{P_H^{op}}.^e/$ is a character of T with parameter $_T$. The local Langlands correspondence associates to this representation an irreducible representation of the group $S^{\backslash}_T$ which in this case equals $^{€}_{Q_p}$. In particular, $J_{P_H^{op}}.^e/$ is associated to the character b_T which is the image under the Kottwitz map of b^{00}. Hence, we get

$$\mathrm{Trans}_T^T J_{P_H^{op}}.^e/ D .s/ J_{P^{op}}.^e_H/ ˝ \ ı_P \ D \ ^{\frac{1}{2}} J_{P^{op}}.^e/_H ˝ \ ı_P : \ \ ^{\frac{1}{2}}$$

To figure out the right-hand side of the averaging formula, we need to understand which representations of the centralizer group of correspond to ^{s} and ^{2}. The centralizer group of (according to [35]) corresponds to the matrices

$$8\,0\,0 \quad a \quad 0 \quad 0^1 \quad 1 \quad 9 ; \quad \text{BB} \ @@0 \ 1 \ 0\text{A} ; c\text{A} \ \text{W} a \ D \ \cdot \ 1 \ : \quad 0 \ 0 \ a$$

In the unitary case, [46, Proposition 13.1.3 (d)] indicates that the unitary group representations corresponding to ^{2} corresponds to the trivial character of the centralizer group and ^{s} corresponds to the non-trivial character. By our parametrization of the L-packets in the unitary similitude case, we get that the characters attached to both ^{2} and ^{s} are trivial on the simil-itude factor that the ^{2} character corresponds to the trivial character of the Z = 2 Z factor and ^{s} corresponds to the non-trivial character.

Hence, the endoscopic averaging formula becomes

$$\mathrm{Mant}_b.^2 \quad ^s/ C \ \mathrm{Mant}_{b^0}. \ J_{P_H^{op}}.^e/ ˝ \ ı^2/^{\frac{1}{P}}$$
$$D \ ^2 \ Œ. \ _2/ \ \tfrac{1}{} \ . \ _2/^3 C \ . \ 1/• C \ ^s \ Œ \ _2/ C \ .\tfrac{1}{}\,_2/ \ \tfrac{3}{} \ . \ 1/•$$

Using Harris–Viehmann to compute Mant_{b^0} as above, we get

$$\mathrm{Mant}_{b^0}. \ J_{P_H^{op}}.^e/ ˝ \ ı_P/^{\frac{1}{2}} D \ Œ J_{P_H^{op}}.^e/• \ Œ \ _2/• \ \tfrac{1}{}$$

Hence,

$$\mathrm{Mant}_b.^2 \quad ^s/ D \ ^2 \ Œ. \ _2/ \ \tfrac{1}{} \ . \ _2/^3 C \ . \ 1/•$$
$$C \ ^s \ Œ \ _2/ C \ \tfrac{1}{}\,_2/ \ \tfrac{3}{} \ . \ 1/•$$
$$C \ Œ J_{P_H^{op}}.^e/• \ Œ \ \tfrac{1}{}/• \ _2$$

To finish the computation, we use that I P J P op .2/ D 2 C n D I P J P op . We then get

Mant b .2 C s/ D Œ2C s• Œ 2/ C . 1 2/C . 1/•3
Œ2C n•. 2/; Mant b .2 1
s/ D 2 Œ. 1/ 2 . 3 2/C . 1/•
C s Œ 1/ C 2 3/ 2 . 1/•
C Œ2C n•. 1/: 2

Proposition A.1. We have

Mant b .s/ D s . 1/ C 2 . 2/ 3 n . 2/ 1

and

Mant b .2/ D s Œ 2/ C . 1 2/•C 2 3 . 1/:

Additionally, we can consider the A-parameter 0 whose associated A-packet is ¹n; sº and do the same computations. We remark that we have not proven the averaging formula in this case although we still expect it to hold. We also remark that the element s 0 is non-trivial and so the stable distribution attached to the packet ¹n; sº is actually n s while the distribution n C s is unstable.

In this case, the trivial endoscopic group gives us the formula

Mant b .n s/ C Mant b0 .J P op .n/ ˝ I 2 / 1 P
D n Œ 2/ C 1 2/ C . 3 1/• s Œ 2/ C 1 2/ C . 3 1/•;

and hence

Mant b .n s/ D n Œ 2/ C . 1 2/ C . 3 1/•
s Œ 2/ C 1 2/ C . 3 1/• Œ2C n•. 2/: 3

In the non-trivial endoscopic case, we get

Mant b .n C s/ C Mant b0 ..Trans T J P op T .n// ˝ I 2 / 1 P H
D n Œ. 2/ 1 . 3 2/C . 1/•C s Œ. 2/ 1 . 3 2/C . 1/•

Hence,

Mant b .n C s/ D n Œ. 2/ 1 . 3 2/C . 1/•
C s Œ. 2/ 1 . 3 2/C . 1/•C Œ2C n•. 2/: 3

Using these equations, we deduce

Mant b .n/ D n . 1/ s Œ 2/ C 1 2/• 3

and

Mant b .s/ D n . 2/ C s 1 1/ C 2 . 2/: 3

We briefly explain how these results relate to Ito and Mieda's computation in [21]. Firstly, we note that our definition of Mant has a twist by j j $^{h_G;i}$ which explains why our Galois parts have different twists from theirs. Secondly, we do not restrict to supercuspidal parts and so we have several extra terms that do not appear in their computation. Thirdly, Mant_b is an alternating sum of ext groups of cohomology whereas they compute isotopic components of cohomology. So for instance, their computation of $M^i.^n/$ contains a ^{s} . $\frac{1}{2}/$ piece in the i D 3 degree (middle degree is i D 2). In our computation, this corresponds to the fact that our ^{s} . $\frac{3}{2}/$ term appears with a negative sign. The supercuspidal part of $\mathrm{Mant}_b.^n/$ also contains an extra ^{s} term as compared to $M^i.^n/$ because ^{n} appears in a non-trivial extension with ^{2}.

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