Quantitative rigidity of differential inclusions in two dimensions

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Abstract

For any compact connected one-dimensional submanifold $K \subset \mathbb{R}^{2\times 2}$ without boundary which has no rank-one connection and is elliptic, we prove the quantitative rigidity estimate

$$\inf_{M\in K}\int_{B_{1/2}} |Du-M|^2\,dx \le C\int_{B_1} \operatorname{dist}^2(Du,K)\,dx, \qquad \forall u\in H^1(B_1;\mathbb{R}^2).$$

This is an optimal generalization, for compact connected submanifolds of $\mathbb{R}^{2\times 2}$ without boundary, of the celebrated quantitative rigidity estimate of Friesecke, James and Müller for the approximate differential inclusion into $SO(n)$. The proof relies on the special properties of elliptic subsets $K\subset\mathbb{R}^{2\times 2}$ with respect to conformal-anticonformal decomposition, which provide a quasilinear elliptic PDE satisfied by solutions of the exact differential inclusion $Du\in K$. We also give an example showing that no analogous result can hold true in $\mathbb{R}^{n\times n}$ for $n\ge 3$.

1 Introduction

In 1850, Liouville [22] proved that, given a domain $\Omega\subset\mathbb{R}^3$, any smooth map $u\colon\Omega\to\mathbb{R}^3$ satisfying the differential inclusion $Du(x)\in\mathbb{R}_+O(n)$ for all $x\in\Omega$ must be either affine or a Möbius transform. A corollary to Liouville's Theorem is that a C^3 function whose gradient belongs everywhere to $SO(n)$ is an affine mapping. This phenomenon of being able to globally control a map satisfying a certain differential inclusion $Du\in K$ is known as "rigidity". Questions about the stability of differential inclusions under weak convergence and approximate rigidity statements, raised by Tartar in [30, 31], are intimately linked with phenomena of compensated compactness and have been extremely influential in the development of weak convergence methods in PDE.

Here we are interested in quantitative versions of approximate rigidity. In [14] Friesecke, James and Müller solved a long standing open problem by proving an optimal quantitative rigidity estimate for $K=SO(n)$. Specifically, they showed that for every bounded Lipschitz domain $\Omega\subset\mathbb{R}^n$, $n\ge 2$, there exists a constant $C(\Omega)$ such that, for $K=SO(n)$,

$$\inf_{R\in K}\|Dv-R\|_{L^2(\Omega)} \le C(\Omega)\|\operatorname{dist}(Dv,K)\|_{L^2(\Omega)}, \qquad \forall v\in H^1(\Omega;\mathbb{R}^n). \tag{1}$$

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Here and below, $\operatorname{dist}(M, K)$ denotes the distance from a matrix $M \in \mathbb{R}^{n\times n}$ to a subset $K \subset \mathbb{R}^{n\times n}$ measured in the Euclidean norm. This result strengthened earlier work of a series of authors, including John [17], Rešetnjak [26], and Kohn [19], and it has had a number of important applications, in particular to thin film limits of elastic structures [14, 15].

A number of works have extended the above result (1) to cover various larger classes of matrices than $K = SO(n)$. Chaudhuri and Müller [8] and later De Lellis and Székelyhidi [9] considered a set of the form $K = SO(n)A \cup SO(n)B$ where A and B are strongly incompatible in the sense of Matos [24]. Faraco and Zhong [13] proved an analogous quantitative rigidity result with $K = \mathfrak{m}\cdot SO(n)$ where $\mathfrak{m} \subset (0, +\infty)$ is compact. There the infimum in the left-hand side of (1) also needs to include the gradients of Möbius transforms, and the integral is over a smaller subset $\Omega' \subset\subset \Omega$. Similar results for maps defined on the sphere have been obtained recently by Luckhaus and Zemas [23]. Best constants for (1) are investigated by Lewicka and Müller in [21].

Our main result is an optimal generalization of the quantitative rigidity estimate of [14] in the context of compact connected submanifolds $K \subset \mathbb{R}^{2\times 2}$ without boundary.

Theorem 1.1. *Let $K \subset \mathbb{R}^{2\times 2}$ be a smooth, compact and connected 1-manifold without boundary. Assume that K has no rank-one connections, and satisfies the stronger property of being elliptic in the sense that there exists $C_* > 0$ such that*

$$|M - M'|^2 \le C_* \det(M - M') \qquad \forall M, M' \in K. \tag{2}$$

Then for any $u \in H^1(B_1; \mathbb{R}^2)$ we have

$$\inf_{M\in K} \int_{B_{1/2}} |Du - M|^2 \, dx \le C \int_{B_1} \operatorname{dist}^2(Du, K) \, dx, \tag{3}$$

for some constant $C = C(K) > 0$.

Remark 1.2. A covering argument as in [13] shows that the estimate (3) in the balls $B_{1/2} \subset B_1$ automatically improves to

$$\inf_{M\in K} \int_{\Omega'} |Du - M|^2 \, dx \le C \int_{\Omega} \operatorname{dist}^2(Du, K) \, dx, \qquad \forall u \in H^1(\Omega; \mathbb{R}^2),$$

for any bounded domain $\Omega \subset \mathbb{R}^2$ and open subset $\Omega' \subset\subset \Omega \subset \mathbb{R}^2$, and a constant $C = C(K, \Omega', \Omega) > 0$.

This result is optimal among compact connected submanifolds $K \subset \mathbb{R}^{2\times 2}$ without boundary for the following reasons:

- First, it is classical that the no-rank-one-connections assumption is necessary for the rigidity of the exact differential inclusion (see e.g. [25, 18]).
- Second, ellipticity is necessary for the validity of the linearized version of (3) because non-ellipticity would imply that the tangent space $T_M K$ has a rank-one connection for some $M \in K$, and by Remark 2.3 the linearized version of (3) is implied by (3).
- Third, the two previous conditions (no rank-one connections and ellipticity) imply that the connected submanifold without boundary $K \subset \mathbb{R}^{2\times 2}$ must be of dimension 1 [33, Corollary 3.5 & 3.6].

Moreover, we provide in Section 5 an example showing that the two-dimensional setting is also optimal: there exists an elliptic 1-submanifold $K \subset \mathbb{R}^{3\times 3}$ without rank-one connection but which contains a so-called $\mathcal{T}_4$ configuration, a well-known obstruction to compactness of sequences $\{u_k\} \subset H^1$ satisfying $\mathrm{dist}(Du_k, K) \to 0$ in L^2 [7], and therefore to any type of quantitative rigidity estimate.

One of our motivations for studying differential inclusion into general submanifolds $K \subset \mathbb{R}^{2\times 2}$ is our previous work [20] where we obtained a rigidity result for a non-elliptic differential inclusion related to the so-called Aviles-Giga functional, and pointed out the nice consequences that a corresponding quantitative rigidity estimate would have. Theorem 1.1 is not valid for non-elliptic differential inclusions, but the ideas in the present work should be relevant to attain that goal.

While the statements of the quantitative rigidity results of [14, 8, 13] are elementary, their proofs are not. Their starting point, in addition to rigidity of the exact differential inclusion, is a linearized version of (1) for the differential inclusion $Du \in T_{M_0}K$ into a tangent space $T_{M_0}K$. For $K = SO(n)$ and $M_0 = I$, this is Korn's inequality. A natural linearization procedure then provides a quantitative rigidity estimate, but in terms of the L^∞ norm of $\mathrm{dist}(Du, K)$, rather than L^2. Strengthening the L^2 bound on $\mathrm{dist}(Du, K)$ into an L^∞ bound constitutes therefore the main difficulty. A key idea, introduced in [14], is to use the regularity of an elliptic PDE satisfied by solutions of the exact differential inclusion: for $K = SO(n)$ the exact differential inclusion $Du \in SO(n)$ implies that the coordinate functions u_k are harmonic. For $K \subset \mathbb{R}_+ SO(n)$ the coordinate functions satisfy the $(n-2)$-Laplace equation $\mathrm{div}(|\nabla u_k|^{n-2}\nabla u_k) = 0$. Such PDE follows from the universal identity $\mathrm{div}\,\mathrm{cof}(Du) = 0$ (where cof denotes the cofactor matrix), together with identities satisfied by matrices in the specific set K. It is satisfied by solutions of the exact differential inclusion, and for a general map u the error from solving that PDE can be controlled in terms of the right-hand side of (1). This allows to reduce the proof of (1) to maps solving that PDE. Elliptic regularity then provides, via a compactness argument, a uniform bound on $\mathrm{dist}(Du, K)$ and the linearization can be performed.

Following this scheme, the main ingredient to prove Theorem 1.1 is to embed K into the graph of a uniformly monotone vector field: this will be enough to turn the identity $\mathrm{div}\,\mathrm{cof}(Du) = 0$ into a quasilinear elliptic equation for the exact differential inclusion $Du \in K$.

Proposition 1.3. *Let K be as in Theorem 1.1. There exist $G_1, G_2 \colon \mathbb{R}^2 \to \mathbb{R}^2$ smooth, globally Lipschitz vector fields such that*

$$K \subset \left\{ \begin{pmatrix} A \\ iG_1(A) \end{pmatrix} : A \in \mathbb{R}^2 \right\} \cap \left\{ \begin{pmatrix} -iG_2(B) \\ B \end{pmatrix} : B \in \mathbb{R}^2 \right\},$$

and G_1, G_2 are uniformly monotone, that is

$$(G_j(X) - G_j(X')) \cdot (X - X') \geq \lambda |X - X'|^2 \qquad \forall X, X' \in \mathbb{R}^2,$$

for some constant $\lambda > 0$ depending only on K.

Proposition 1.3 relies on remarkable properties of elliptic subsets of $\mathbb{R}^{2\times 2}$ with respect to the decomposition into conformal and anticonformal parts, discovered in [33] and exploited in a striking manner in [5, 11, 12] (see also [18]). It is also related to the classical link between two-dimensional elliptic PDEs of second order and complex Beltrami equations, see e.g. the introduction of [6], and [4, 2, 1, 3] for recent developments on nonlinear Beltrami equations. The proof of Proposition 1.3 is in fact extremely close to [3, Theorem 5], and the main point of Proposition 1.3 is to emphasize that the second order elliptic equations satisfied by the real and imaginary part of solutions to a nonlinear Beltrami equation (as in [3, Theorem 5]) are, in our setting, associated to embeddings of K into graphs: this fact is crucially used in Step 2 of Theorem 1.1's proof, and we find it convenient to state it independently (although it could also be retrieved from the proof of [3, Theorem 5] applied to the nonlinear Beltrami equation associated to K). Then the proof of Theorem 1.1 follows the scheme outlined above.

The article is organized as follows. In Section 2 we establish the two basic prerequisites to Theorem 1, rigidity for the exact differential inclusion and the linearized estimate. In Section 3 we give the proof of Proposition 1.3. In Section 4 we gather these ingredients to prove Theorem 1.1. In Section 5 we describe the counterexample in $\mathbb{R}^{3\times 3}$.

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2 Basic ingredients: rigidity of the exact inclusion and linearized estimate

In this section, let $K \subset \mathbb{R}^{2\times 2}$ be as in Theorem 1.1. We prove the following two Lemmas.

Lemma 2.1. *If $u \in H^1(B_1;\mathbb{R}^2)$ is such that $Du \in K$ a.e., then $Du \equiv M$ for some $M \in K$.*

Under the additional assumption that u coincides with an affine map at the boundary, Lemma 2.1 would follow directly from [32, Theorems 2 & 3]. Without restrictions on boundary values, it is a simple consequence of the smoothness result in [32, § 5] and the fact that our set K is a closed one-dimensional curve, as will be clear from the short proof below.

Lemma 2.2. *For all $M \in K$ and $u \in H^1(B_1;\mathbb{R}^2)$ we have*

$$\inf_{X\in T_M K} \int_{B_1} |Du - X|^2 \, dx \le C \int_{B_1} \mathrm{dist}^2(Du, T_M K)\, dx \tag{4}$$

for some constant $C = C(K) > 0$, where $T_M K$ denotes the linear tangent space to K at M.

Remark 2.3. The linearized estimate (4), or rather its weaker interior version

$$\inf_{X\in T_M K} \int_{B_{1/2}} |Du - X|^2 \, dx \le C \int_{B_1} \mathrm{dist}^2(Du, T_M K)\, dx, \tag{5}$$

is a necessary condition for (3) to be valid. Assume indeed that (3) is verified, fix $u \in C^1(\overline{B}_1;\mathbb{R}^2)$, and apply (3) to $v_\epsilon(x) = Mx + \epsilon u(x)$ for $\epsilon \ll 1$. There exists $M_\epsilon \in K$ such that

$$\begin{aligned}\int_{B_{1/2}} |M - M_\epsilon - \epsilon Du|^2 \, dx &\le C \int_{B_1} \mathrm{dist}^2(M + \epsilon Du, K)\, dx \\ &\le C\epsilon^2 \int_{B_1} \mathrm{dist}^2(Du, T_M K)\, dx + o(\epsilon^2).\end{aligned}$$

Hence, letting $X_\epsilon = \epsilon^{-1}(M - M_\epsilon)$, we have

$$\int_{B_{1/2}} |X_\epsilon - Du|^2\, dx \le C \int_{B_1} \mathrm{dist}^2(Du, T_M K)\, dx + o(1).$$

In particular X_ϵ is bounded, and extracting a converging subsequence we obtain $X \in T_M K$ showing the validity of (5) for $u \in C^1(\overline{B}_1; \mathbb{R}^2)$, and then by density for $u \in H^1(B_1; \mathbb{R}^2)$.

Proof of Lemma 2.1. Let $\ell = |K|$ and $\gamma\colon \mathbb{R}/\ell Z \to K$ be an arc-length parametrization of K. The ellipticity assumption (2) ensures that $\gamma'(t)$ is invertible for all $t \in \mathbb{R}$. Let $u \in H^1(B_1; \mathbb{R}^2)$ such that $Du \in K$ a.e., then u is smooth by [32], and since B_1 is simply connected there exists a smooth lifting $\theta\colon B_1 \to \mathbb{R}$ such that $Du = \gamma(\theta)$. Using that $\mathrm{div}\,\mathrm{cof}(Du) = 0$, where cof denotes the cofactor matrix, we find $\mathrm{cof}(\gamma'(\theta))\nabla\theta = 0$, hence $\nabla\theta = 0$ since $\mathrm{cof}(\gamma'(\theta))$ is invertible. Therefore Du is constant. □

Proof of Lemma 2.2. For $M \in K$ we denote by $P_M\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ the orthogonal projection onto $(T_M K)^\perp$, so that

$$\mathrm{dist}^2(X, T_M K) = |P_M X|^2.$$

We denote by $P_M^{\alpha\beta jk} \in \mathbb{R}$ the coefficients of P_M, that is,

$$(P_M X)_{\alpha\beta} = \sum_{jk} P_M^{\alpha\beta jk} X_{jk} \qquad \forall X = (X_{jk}) \in \mathbb{R}^{2\times 2}.$$

Define $\mathbb{P}_M(i\xi) \in \mathcal{L}(\mathbb{C}^2; \mathbb{C}^{2\times 2})$ by

$$(\mathbb{P}_M(i\xi)v)_{\alpha\beta} = \sum_{jk} P_M^{\alpha\beta jk} v_j \xi_k \qquad \forall \xi, v \in \mathbb{C}^2,$$

so the differential operator $u \mapsto P_M Du$ has symbol $\mathbb{P}_M(i\xi)$, i.e.,

$$(P_M(Df))_{\alpha\beta} = \frac{1}{2\pi} \int \Big(\mathbb{P}_M(i\xi)\hat{f}(\xi)\Big)_{\alpha\beta} e^{ix\cdot\xi} d\xi.$$

We claim that $\mathbb{P}_M(\xi)$ has trivial kernel for all non-zero $\xi \in \mathbb{C}^2$. Let indeed $v \in \mathbb{C}^2$ such that $\mathbb{P}_M(\xi)v = 0$. This implies that $P_M \,\mathrm{Re}(v \otimes i\xi) = P_M \,\mathrm{Im}(v \otimes i\xi) = 0$, because the coefficients $P_M^{\alpha\beta jk}$ are real-valued. In other words, the real and imaginary parts of $v \otimes i\xi$ both belong to $\ker P_M = T_M K$. Since $T_M K$ is a one-dimensional subspace of $\mathbb{R}^{2\times 2}$ which doesn't contain any rank-one matrix, we have $T_M K = \mathbb{R} X_0$ for some invertible matrix X_0. Hence we deduce that $v \otimes i\xi = \lambda X_0$ for some $\lambda \in \mathbb{C}$ and an invertible matrix $X_0 \in T_M K$. But $v \otimes i\xi$ has zero determinant, so $\lambda = 0$ and we must have $v = 0$. This proves that $\mathbb{P}_M(\xi)$ has trivial kernel. Therefore we have the representation formula [28, Theorem 4.1] and the coercive inequality [28, Theorem 8.15] that follows from it,

$$\int_{B_1} |Du|^2\, dx \le C \int_{B_1} |P_M Du|^2\, dx + C \int_{B_1} |u|^2\, dx, \tag{6}$$

for all $u \in H^1(B_1; \mathbb{R}^2)$. (In the notation of [28], $N = 4$, $M = 2$, and the index set $\{1, 2, 3, 4\}$ for j is in our case given by $\{1, 2\}^2$, and we can take $m_j = 1$ for $j \in \{1, 2\}^2$ and $l_i = 0$ for

$i \in \{1,2\}$.) The constant $C > 0$ in (6) depends a priori on the fixed matrix $M \in K$. Denote by $C(M)$ the best possible constant in (6). Then for any $M, M' \in K$ we have

$$\int_{B_1} |Du|^2\,dx \le 2C(M)\int_{B_1} |P_{M'}Du|^2\,dx + C(M)\int_{B_1}|u|^2\,dx \\ + 2C(M)\|P_M - P_{M'}\|^2 \int_{B_1} |Du|^2\,dx.$$

For all $M \in K$, there exists $\delta(M) > 0$ sufficiently small such that for all $M' \in K \cap B_{\delta(M)}(M)$, we have $2C(M)\|P_M - P_{M'}\|^2 < 1/2$. It follows that

$$C(M') \le \frac{2C(M)}{1 - 2C(M)\|P_M - P_{M'}\|^2} < 4C(M) \quad \forall M' \in K \cap B_{\delta(M)}(M).$$

By compactness, we can cover K with a finite collection of balls $\{B_{\delta(M^j)}(M^j) : M^j \in K\}$, hence $C(M) \le 4\max\{C(M^j)\}$ for all $M \in K$, and we can take the constant C in (6) to depend only on K.

Moreover, if $u \in H^1(B_1;\mathbb{R}^2)$ satisfies $P_M Du = 0$ a.e., then $Du = \lambda X_0$ for some $\lambda \in L^2(B_1;\mathbb{R})$, and the distributional identity $0 = \operatorname{div}\operatorname{cof}(Du) = \operatorname{cof}(X_0)\nabla\lambda$ implies that λ is constant, hence $Du \equiv X$ for some $X \in T_M K$.

Therefore (4) follows from (6) via a compactness argument: assume by contradiction the existence of sequences $M^k \in K$, and $u^k \in H^1(B_1;\mathbb{R}^2)$ such that

$$\inf_{X \in T_{M^k}K} \int_{B_1} |Du^k - X|^2\,dx = 1, \quad \int_{B_1} |P_{M^k}Du^k|^2\,dx \longrightarrow 0.$$

For any given k, the function $X \mapsto \int_{B_1} |Du^k - X|^2\,dx$ is a strictly convex quadratic polynomial on the finite-dimensional space $T_{M^k}K$, so the infimum in the left-hand side is attained at a unique $X^k \in T_{M^k}K$. Subtracting from u^k its average and $X^k x$, we may in fact assume

$$\int_{B_1} u^k\,dx = \int_{B_1} Du^k \cdot X\,dx = 0 \qquad \forall X \in T_{M^k}K,$$

and

$$\int_{B_1} |Du^k|^2\,dx = 1, \quad \int_{B_1} |P_{M^k}Du^k|^2\,dx \longrightarrow 0.$$

Thus we may extract subsequences (not relabeled) $u^k \to u$ weakly in $H^1(B_1;\mathbb{R}^2)$ and strongly in $L^2(B_1;\mathbb{R}^2)$, and $M^k \to M \in K$. It follows that $P_{M^k}Du^k \rightharpoonup P_M Du$ in $L^2(B_1;\mathbb{R}^{2\times 2})$, and thus by lower semicontinuity of the L^2 norm under weak convergence, we have $P_M Du = 0$ a.e., which implies $Du \equiv X$ for some $X \in T_M K$. Approximating X by a sequence $X^k \in T_{M^k}K$ and using $Du^k \rightharpoonup Du$ in $L^2(B_1;\mathbb{R}^{2\times 2})$, we deduce that $0 = \int_{B_1} Du \cdot X\,dx = |B_1||X|^2$, and thus $Du \equiv X = 0$. Further u satisfies $\int_{B_1} u\,dx = 0$, which implies $u \equiv 0$. Plugging u^k into (6) gives

$$1 = \int_{B_1} |Du^k|^2\,dx \le C\int_{B_1} |P_{M^k}Du^k|^2\,dx + C\int_{B_1} |u^k|^2\,dx.$$

Passing to the limit as $k \to \infty$ and using the strong L^2 convergence, we have $1 \le C\int_{B_1} |u|^2\,dx = 0$, which gives a contradiction. □

Remark 2.4. The fact that (4) follows from (6) is general and already mentioned in [28, p.74], here we provided details for the reader's convenience. More precisely, if $P_M D$ is replaced by a more general differential operator, the validity of (4) is equivalent to the null space of that differential operator being finite-dimensional [28, Theorem 8.15 & Remark 4], and a compactness argument similar to the one given here shows that the last term in the right-hand side of (6) can be dropped if u is orthogonal to that finite-dimensional null space.

3 Proof of Proposition 1.3

We only prove the existence of G_1, the existence of G_2 is obtained by the same arguments. The proof relies on the properties of the conformal and anticonformal projections of K uncovered in [33, 12]. For any $z_+, z_- \in \mathbb{C}$, we denote by

$$[z_+, z_-] = \begin{pmatrix} \operatorname{Re} z_+ & -\operatorname{Im} z_+ \\ \operatorname{Im} z_+ & \operatorname{Re} z_+ \end{pmatrix} + \begin{pmatrix} \operatorname{Re} z_- & \operatorname{Im} z_- \\ \operatorname{Im} z_- & -\operatorname{Re} z_- \end{pmatrix} \in \mathbb{R}^{2\times 2},$$

the 2×2 matrix whose conformal, respectively anticonformal, part is represented by z_+, respectively z_-. For any $A \in \mathbb{R}^{2\times 2}$, the decomposition $A = [z_+, z_-]$ is unique, and we have the identities

$$\det A = |z_+|^2 - |z_-|^2, \quad |A|^2 = 2|z_+|^2 + 2|z_-|^2, \quad \|A\| = |z_+| + |z_-|,$$

where $|A|$ and $\|A\|$ denote the Hilbert-Schmidt and the operator norms of A, respectively. We denote by $p_+ \colon [z_+, z_-] \mapsto z_+$ the projection onto the conformal part.

Using these notations, the ellipticity assumption (2) is equivalent to

$$2|z_+ - z_+'|^2 + 2|z_- - z_-'|^2 \leq C_* \left(|z_+ - z_+'|^2 - |z_- - z_-'|^2\right), \tag{7}$$

for all $[z_+, z_-], [z_+', z_-'] \in K$, and corresponds exactly to the statement that the curve K is C_*-elliptic in the sense of [12, Def. 1]. This condition, as observed in [33, Theorem 3.2], see also [12, Lemma 1], implies that

$$K = \{[z, H(z)] \colon z \in p_+(K)\},$$

for some k-Lipschitz function $H \colon p_+(K) \to \mathbb{C}$ with $0 \leq k = (C_* - 1)/(C_* + 1) < 1$. Further, the proof of [12, Lemma 2] includes the fact that $p_+(K)$ is a Jordan curve (because if there exist $[z_+, z_-], [z_+', z_-'] \in K$ with $z_+ = z_+'$, then by (7) we have that $[z_+, z_-] = [z_+', z_-']$). Yet further the explicit formula $p_+(A) = (a_{11} + a_{22})/2 - i(a_{12} - a_{21})/2$ for $A = (a_{ij})$ and the smoothness of K imply that $p_+(K) \subset \mathbb{C}$ is smooth.

Lemma 3.1. *The function H admits a smooth extension $H \colon \mathbb{C} \to \mathbb{C}$ which is k-Lipschitz for some (possibly larger) $0 \leq k < 1$.*

Proof of Lemma 3.1. We first fix, thanks to Kirszbraun's theorem, a k-Lipschitz extension $\widehat{H} \colon \mathbb{C} \to \mathbb{C}$. In the rest of the proof we modify $\widehat{H}$ to make it smooth while still agreeing with H on $p_+(K)$, at the cost of slightly increasing its Lipschitz constant.

Let $[z_+(t), z_-(t)]$, $t \in \mathbb{R}/\ell\mathbb{Z}$, denote a smooth arc-length parametrization of K, so that $2|\dot{z}_+|^2 + 2|\dot{z}_-|^2 = 1$, where $\dot{z}_\pm = \frac{d}{dt} z_\pm$. As $z_- = H(z_+)$ and H is k-Lipschitz, it follows that $|\dot{z}_-| \leq k|\dot{z}_+|$, so $|\dot{z}_+|^2 \geq \frac{1}{2(1+k^2)} > 0$. Therefore we may reparametrize and consider

$$K = \{[z(s), H(z(s))] \colon s \in \mathbb{R}/\ell_+\mathbb{Z}\},$$

with $z(s)$ an arc-length parametrization of $p_+(K)$ and ℓ_+ its length, and the map $s \mapsto H(z(s))$ is smooth by smoothness of K.

For small enough $\delta > 0$, the map

$$\begin{aligned}\varphi\colon \mathbb{R}/\ell_+\mathbb{Z} \times (-2\delta, 2\delta) &\to \mathcal{U}_{2\delta} = \{z \in \mathbb{C}\colon \operatorname{dist}(z, p_+(K)) < 2\delta\}\\ (s,r) &\mapsto z(s) + ri\dot{z}(s),\end{aligned}$$

is a smooth diffeomorphism. We first modify $\widehat{H}$ by setting

$$\widetilde{H} = \widehat{H} \circ \Phi, \qquad \Phi(Z) = \begin{cases} z(s) + \lambda(r)i\dot{z}(s) & \text{if } Z = \varphi(s,r) \in \mathcal{U}_{2\delta},\\ Z & \text{otherwise,}\end{cases}$$

where λ is the odd $(1-\delta)^{-1}$-Lipschitz function given for $r > 0$ by

$$\lambda(r) = \begin{cases} 0 & \text{for } 0 \le r \le \delta^2,\\ -\frac{\delta^2}{1-\delta} + \frac{r}{1-\delta} & \text{for } \delta^2 < r \le \delta,\\ r & \text{for } r > \delta.\end{cases}$$

In particular we have

$$\widetilde{H}(Z) = H(z(s)) \qquad \forall Z = \varphi(s,r) \in \mathcal{U}_{\delta^2},$$

so $\widetilde{H}$ is smooth in $\mathcal{U}_{\delta^2}$ (by smoothness of $s \mapsto H(z(s))$ and φ^{-1}) and agrees with H on $p_+(K)$. Note that by definition of φ and λ we have $\Phi(Z) = Z$ in $\mathbb{C} \setminus \mathcal{U}_\delta$ and therefore Φ is Lipschitz in $\mathbb{C}$. Since $D\varphi(s,0) \in SO(2)$ for all $s \in \mathbb{R}/\ell_+\mathbb{Z}$, we have $\|D\varphi\| \le 1 + C\delta$ on $\mathbb{R}/\ell_+\mathbb{Z} \times (-2\delta, 2\delta)$. Further, we have $\|D(\varphi^{-1})\| = 1$ on $p_+(K)$, and $\|D(\varphi^{-1})\| \le 1 + C\delta$ on $\mathcal{U}_{2\delta}$. Denoting by ψ the $(1-\delta)^{-1}$-Lipschitz map $(s,r) \mapsto (s, \lambda(r))$, we write $\Phi(Z) = \varphi(\psi(\varphi^{-1}(Z)))$ and deduce that $\|D\Phi\| \le 1 + C\delta$ a.e. in $\mathcal{U}_{2\delta}$. This inequality is also true in the rest of $\mathbb{C}$ by definition of Φ, so we conclude that $\widetilde{H}$ is $\tilde{k}$-Lipschitz in $\mathbb{C}$, with $\tilde{k} = (1 + C\delta)k < 1$ for small enough $\delta > 0$. Now δ is fixed and we define, for $\epsilon \in (0, \delta^2/4)$,

$$H_\epsilon(z) = \int_{\mathbb{C}} \widetilde{H}(z + \epsilon\chi(z)y)\rho(y)\, dy,$$

for a smooth kernel $\rho \ge 0$ with support in B_1 and $\int \rho(y)\, dy = 1$, and some smooth cut-off function χ with $\mathbf{1}_{\mathcal{U}_{\delta^2/4}} \le 1 - \chi \le \mathbf{1}_{\mathcal{U}_{\delta^2/2}}$. In $\mathcal{U}_{3\delta^2/4}$, the map H_ϵ is smooth thanks to the smoothness of $\widetilde{H}$ in $\mathcal{U}_{\delta^2}$. In $\mathbb{C} \setminus \overline{\mathcal{U}}_{\delta^2/2}$, we have $H_\epsilon(z) = \int_{\mathbb{C}} \widetilde{H}(z + \epsilon y)\rho(y)\, dy$ is also smooth. Therefore the map H_ϵ is smooth in $\mathbb{C}$. Further, $H_\epsilon(z) = \widetilde{H}(z)$ for $z \in \mathcal{U}_{\delta^2/4}$, and thus agrees with H on $p_+(K)$. Finally, denoting by L the Lipschitz constant of χ, we have

$$\begin{aligned}|H_\epsilon(z) - H_\epsilon(z')| &\le \int_{B_1} |\widetilde{H}(z + \epsilon\chi(z)y) - \widetilde{H}(z' + \epsilon\chi(z')y)|\rho(y)\, dy\\ &\le \int_{B_1} \tilde{k}(1 + \epsilon L|y|)|z - z'|\rho(y)\, dy \le \tilde{k}(1 + \epsilon L)|z - z'|.\end{aligned}$$

So H_ϵ is k_ϵ-Lipschitz with $k_\epsilon \le \tilde{k}(1 + \epsilon L) < 1$ for small enough ϵ. □

Lemma 3.2. *The map $F\colon z \mapsto \bar{z}+H(z)$ is a smooth diffeomorphism from $\mathbb{C}$ onto $\mathbb{C}$. Moreover F is $(1+k)$-Lipschitz and F^{-1} is $(1-k)^{-1}$-Lipschitz.*

Proof of Lemma 3.2. For any $w \in \mathbb{C}$ the equation

$$w = \bar{z} + H(z) \quad \Leftrightarrow \quad z = \bar{w} - \bar{H}(z),$$

admits a unique solution $z \in \mathbb{C}$ thanks to the fixed point theorem, since $z \mapsto \bar{w} - \bar{H}(z)$ is k-Lipschitz and $0 \le k < 1$. This shows that F is bijective. The inequalities

$$(1-k)|z-z'| \le |F(z) - F(z')| \le (1+k)|z-z'|,$$

follow directly from the fact that H is k-Lipschitz and imply the announced Lipschitz constants of F and F^{-1}. The inverse F^{-1} is smooth thanks to the Inverse Function Theorem, since $DF(z)\colon h \mapsto \bar{h} + DH(z)h$ is invertible for all $z \in \mathbb{C}$ because $\|DH\| < 1$. □

Proof of Proposition 1.3 completed. Then, identifying $\mathbb{C}$ with $\mathbb{R}^2$, we define

$$G_1(A) = \overline{F^{-1}(A)} - H(F^{-1}(A)),$$

so that a short calculation shows that

$$[z, H(z)] = \begin{pmatrix} A \\ iG_1(A) \end{pmatrix} \qquad \text{for } A = F(z).$$

The map G_1 is smooth and globally Lipschitz with Lipschitz constant $\Lambda = (1+k)/(1-k)$. Moreover, for all $A = F(z)$, $A' = F(z')$, we have

$$\begin{aligned}(G_1(A) - G_1(A')) \cdot (A - A') &= \det \begin{pmatrix} A - A' \\ i(G_1(A) - G_1(A')) \end{pmatrix} \\ &= \det([z - z', H(z) - H(z')]) \\ &= |z-z'|^2 - |H(z) - H(z')|^2 \\ &\ge (1-k^2)|z-z'|^2.\end{aligned}$$

Since F is $(1+k)$-Lipschitz, this implies

$$(G_1(A) - G_1(A')) \cdot (A - A') \ge \lambda |A - A'|^2,$$

for $\lambda = (1-k^2)/(1+k)^2 = (1-k)/(1+k) > 0$. This concludes the proof of Proposition 1.3. □

4 Proof of Theorem 1.1

Step 1. We may assume that u is Lipschitz, thanks to the truncation result [14, Proposition A.1]. Part of the statement of [14, Proposition A.1] is that for any Lipschitz domain $\Omega \subset \mathbb{R}^n$ and any function $\tilde{u} \in W^{1,p}(\Omega; \mathbb{R}^m)$ (for $p \ge 1$), there exists some constant $C(\Omega, m, p) > 0$ such that for any $\lambda > 0$ there exists $\tilde{v} : \Omega \to \mathbb{R}^m$ satisfying $\|D\tilde{v}\|_{L^\infty(\Omega)} \le C(\Omega, m, p)\lambda$ and $\|D\tilde{u} - D\tilde{v}\|^p_{L^p(\Omega)} \le C(\Omega, m, p) \int_{\{x \in \Omega : |D\tilde{u}(x)| > \lambda\}} |D\tilde{u}|^p \, dx$. Let $K \subset B_R \subset \mathbb{R}^{2\times 2}$ for some $R > 0$.

Then for all $X \in \mathbb{R}^{2\times 2}$ with $|X| > 2R$, we have $|X| \leq 2\,\mathrm{dist}(X,K)$. Applying [14, Proposition A.1] with $\lambda = 2R$ gives $v : B_1 \to \mathbb{R}^2$ satisfying

$$\begin{aligned}
\|Dv\|_{L^\infty(B_1)} &\leq 2C_0 R,\\
\int_{B_1} |Du - Dv|^2\,dx &\leq C_0 \int_{\{x\in B_1 : |Du|>2R\}} |Du|^2\,dx\\
&\leq 4C_0 \int_{B_1} \mathrm{dist}^2(Du,K)\,dx,
\end{aligned}$$

for some constant C_0 (depending only on B_1). If there exists $M \in K$ such that

$$\int_{B_{1/2}} |Dv - M|^2\,dx \leq \widetilde{C}(K) \int_{B_1} \mathrm{dist}^2(Dv,K)\,dx,$$

then repeated applications of the triangle inequality give

$$\int_{B_{1/2}} |Du - M|^2\,dx \leq \Big(16\widetilde{C}(K)C_0 + 4\widetilde{C}(K) + 8C_0\Big) \int_{B_1} \mathrm{dist}^2(Du,K)\,dx.$$

Thus if Theorem 1.1 holds for all Lipschitz mappings v for some constant $\widetilde{C}(K)$, then it is also valid for all H^1 mappings with $C(K) = 16\widetilde{C}(K)C_0 + 4\widetilde{C}(K) + 8C_0$.

Step 2. We may assume in addition that $u \in C^2(B_1;\mathbb{R}^2)$ solves

$$\mathrm{div}\,G_1(Du_1) = \mathrm{div}\,G_2(Du_2) = 0 \qquad \text{in } B_1. \tag{8}$$

Consider indeed $w \in C^2(B_1)$ such that $w = u$ on ∂B_1 and

$$\mathrm{div}\,G_1(Dw_1) = \mathrm{div}\,G_2(Dw_2) = 0 \qquad \text{in } B_1.$$

The existence of such w is guaranteed by the ellipticity of the equation $0 = \mathrm{div}\,G_j(Dw_j) = \mathrm{tr}(DG_j(Dw_j)D^2w_j) = 0$, invoking e.g. [16, Theorem 12.5]: the inequality $\lambda|\xi|^2 \leq DG_j(A)\xi\cdot\xi \leq \Lambda|\xi|^2$, valid for all $A,\xi \in \mathbb{R}^2$ thanks to Proposition 1.3, ensures that the eigenvalues of the symmetric part $[DG_j(A)]_s = (DG_j(A) + DG_j(A)^T)/2$ of $DG_j(A)$ are bounded above and below (since $DG_j(A)\xi\cdot\xi = [DG_j(A)]_s\xi\cdot\xi$ for all $\xi \in \mathbb{R}^2$) and in particular condition (ii) in [16, Theorem 12.5] is satisfied. Letting $v = u - w$ and using the uniform monotonicity of G_1 we find

$$\lambda \int_{B_1} |Dv_1|^2\,dx \leq \int_{B_1} (G_1(Du_1) - G_1(Dw_1))\cdot Dv_1\,dx.$$

Since $\mathrm{div}\,G_1(Dw_1) = 0$ and $\mathrm{div}(iDu_2) = 0$ we rewrite this as

$$\begin{aligned}
\lambda \int_{B_1} |Dv_1|^2\,dx &\leq \int_{B_1} (G_1(Du_1) + iDu_2)\cdot Dv_1\,dx\\
&\leq \frac{1}{2\lambda}\int_{B_1} |G_1(Du_1) + iDu_2|^2\,dx + \frac{\lambda}{2}\int_{B_1} |Dv_1|^2\,dx,
\end{aligned}$$

and infer

$$\int_{B_1} |Dv_1|^2\,dx \leq \frac{1}{\lambda^2}\int_{B_1} |G_1(Du_1) + iDu_2|^2\,dx.$$

According to Proposition 1.3 the function $M \mapsto G_1(A) + iB$, where A, B denote the first and second row of the matrix M, vanishes on K. Since that function is Lipschitz we deduce that $|G_1(Du_1) + iDu_2| \leq C \operatorname{dist}(Du, K)$, and therefore

$$\int_{B_1} |Dv_1|^2 \, dx \leq C \int_{B_1} \operatorname{dist}^2(Du, K) \, dx.$$

Applying a similar argument to v_2 we obtain

$$\int_{B_1} |Dv|^2 \, dx \leq C \int_{B_1} \operatorname{dist}^2(Du, K) \, dx.$$

Recalling that $v = u - w$ and using the triangle inequality we deduce

$$\int_{B_1} \operatorname{dist}^2(Dw, K) \, dx \leq C \int_{B_1} \operatorname{dist}^2(Du, K) \, dx,$$
$$\int_{B_{1/2}} |Du - M|^2 \, dx \leq 2 \int_{B_{1/2}} |Dw - M|^2 \, dx + C \int_{B_1} \operatorname{dist}^2(Du, K) \, dx.$$

As a consequence, if Theorem 1.1 is valid for w then we obtain it for u. This proves Step 2.

Step 3. As $u_j \in C^2(B_1)$ satisfies (8), it is a weak solution of

$$\operatorname{div}(\partial_i(G_j(Du_j))) = \operatorname{div}(DG_j(Du_j) D(\partial_i u_j)) = 0 \qquad \text{for } i = 1, 2.$$

Invoking e.g. Theorem 1 in [10, § 6.3], we have that $\|\partial_i u_j\|_{W^{2,2}(B_{1/2})} \lesssim \|\partial_i u_j\|_{L^2(B_1)}$ for any $i, j \in \{1, 2\}$. This combined with the Sobolev embedding theorem implies that

$$\|Du\|_{C^\alpha(\overline{B}_{1/2})} \leq C \|Du\|_{L^2(B_1)},$$

for any $\alpha > 0$ and some constant $C = C(K)$. Thanks to this estimate and the exact rigidity obtained in Lemma 2.1, we may argue exactly as in [13, Lemma 4.5] to deduce that

$$\inf_{M \in K} \|Du - M\|_{L^\infty(B_{1/2})} \leq \rho \left(\int_{B_1} \operatorname{dist}^2(Du, K) \, dx \right), \tag{9}$$

for some function ρ depending only on K and satisfying $\rho(\epsilon) \to 0$ as $\epsilon \to 0$.

Step 4. We finally combine Step 3 with the linearized estimate of Lemma 2.2, to obtain our main estimate (3). The basic idea, as in [14, 13], is to linearize $\operatorname{dist}^2(\cdot, K)$ around $M_0 \in K$ such that $|Du - M_0|$ is uniformly small. When doing so, (3) formally turns into the linearized estimate (4) of Lemma 2.2, and it remains to control the error terms. Due to the modification of u arising from the translation $X \in T_M K$ in the left-hand side of the linearized estimate (4), it is not directly obvious that the error terms are negligible. In [14] this problem is absent because their equivalent of (9) comes with an explicit $\rho(\epsilon) = C\epsilon^{\frac{1}{4}}$. In [13] it is taken care of via a topological degree argument [13, Proposition 4.7] (see also [27]) which allows to avoid the translation. While a similar degree argument could be used to iteratively improve the estimate here, we present a simpler alternative method relying on elementary estimates.

We assume without loss of generality that

$$\int_{B_1} \operatorname{dist}^2(Du, K) \, dx = \epsilon \leq \epsilon_0, \tag{10}$$

where $\epsilon_0 = \epsilon_0(K)$ is to be chosen in the course of the proof. If (10) is not valid then (3) is automatically satisfied for a large enough constant C because the left-hand side of(3) is bounded thanks to Step 1.

We fix $\delta_0 > 0$ depending only on K, such that the nearest-point projection Π_K onto K is uniquely defined and smooth in the neighborhood $\mathcal{N}_{2\delta_0}(K)$. We first choose ϵ_0 small enough that $\rho(\epsilon_0) \leq \delta_0$, so thanks to (9) the projection $\Pi_K(Du)$ is well-defined.

We claim that, for every $M \in K$, there exists $Y_M \in K$ such that

$$\int_{B_{1/2}} |Du - Y_M|^2 \, dx \leq C \int_{B_{1/2}} \mathrm{dist}^2(Du, K) \, dx + C \int_{B_{1/2}} |Du - M|^4 \, dx. \tag{11}$$

Here and in the rest of this proof we denote by $C > 0$ a generic constant depending only on K.

To prove (11), we first invoke Lemma 2.2, according to which we have

$$\inf_{X \in T_M K} \int_{B_{1/2}} |Du - M - X|^2 \, dx \leq C \int_{B_{1/2}} \mathrm{dist}^2(Du - M, T_M K) \, dx.$$

Choosing $X = X_M \in T_M K$ attaining the infimum in the left-hand side, we obtain

$$\int_{B_{1/2}} |Du - M - X_M|^2 \, dx \leq C \int_{B_{1/2}} \mathrm{dist}^2(Du - M, T_M K) \, dx. \tag{12}$$

Moreover the minimizing property of X_M implies that the function $t \mapsto \int_{B_{1/2}} |Du - M - tX_M|^2 \, dx$ has zero derivative at $t = 1$ and hence $\int_{B_{1/2}} (Du - M - X_M) \, dx$ is orthogonal to X_M. We deduce

$$|X_M|^2 = \fint_{B_{1/2}} X_M \cdot (Du - M) \, dx \leq \frac{1}{2} |X_M|^2 + C \int_{B_{1/2}} |Du - M|^2 \, dx,$$

and therefore

$$|X_M|^2 \leq C \int_{B_{1/2}} |Du - M|^2 \, dx. \tag{13}$$

Recalling from the proof of Lemma 2.2 that P_M denotes the orthogonal projection onto $(T_M K)^\perp$, we estimate the integrand in the right-hand side of (12) using

$$\begin{aligned} \mathrm{dist}(Du - M, T_M K) &= |P_M(Du - M)| \\ &\leq |Du - \Pi_K(Du)| + |\Pi_K(Du) - M - (I - P_M)(Du - M)| \\ &\leq |Du - \Pi_K(Du)| + C|Du - M|^2. \end{aligned}$$

The last inequality follows from the fact that $I - P_M = D\Pi_K(M)$. Since $|Du - \Pi_K(Du)| = \mathrm{dist}(Du, K)$, plugging this into (12) we find

$$\int_{B_{1/2}} |Du - M - X_M|^2 \, dx \leq C \int_{B_{1/2}} \mathrm{dist}^2(Du, K) \, dx + C \int_{B_{1/2}} |Du - M|^4 \, dx \tag{14}$$

Now we may choose $Y_M \in K$ such that

$$|M + X_M - Y_M| \leq C|X_M|^2. \tag{15}$$

Indeed, if $|X_M| \leq \delta_0$ then one can simply take $Y_M = \Pi_K(M + X_M)$ and use the fact that $D\Pi_K(M)X_M = (I - P_M)X_M = X_M$, and if $|X_M| \geq \delta_0$ one may take $Y_M = M$. From (14) and (15) we infer

$$\int_{B_{1/2}} |Du - Y_M|^2\,dx \leq C\int_{B_{1/2}} \mathrm{dist}^2(Du, K)\,dx + C\int_{B_{1/2}} |Du - M|^4\,dx + C|X_M|^4.$$

Using (13) and Cauchy-Schwarz to estimate the last term, we deduce (11).

Next we want to discard the last term in the right-hand side of (11). To that end we first fix, thanks to (9)-(10), an $M_0 \in K$ such that $|Du - M_0| \leq 2\rho(\epsilon_0)$ in $B_{1/2}$, and note that

$$\inf_{Y\in K}\int_{B_{1/2}} |Du - Y|^2\,dx = \inf_{Y\in K\cap B_{4\rho(\epsilon_0)}(M_0)}\int_{B_{1/2}} |Du - Y|^2\,dx, \tag{16}$$

because if $Y \in K$ is such that $|Y - M_0| \geq 4\rho(\epsilon_0)$, then in $B_{1/2}$ we have

$$|Du - Y| \geq |Y - M_0| - |Du - M_0| \geq 2\rho(\epsilon_0) \geq |Du - M_0|,$$

and $\widetilde{Y} = M_0$ provides therefore a better choice to optimize the infimum in the left-hand side of (16). Moreover, applying (11) to any $M \in K \cap B_{4\rho(\epsilon_0)}(M_0)$, we have

$$\inf_{Y\in K}\int_{B_{1/2}} |Du - Y|^2\,dx \leq C\int_{B_{1/2}} \mathrm{dist}^2(Du, K)\,dx + C\rho(\epsilon_0)^2\int_{B_{1/2}} |Du - M|^2\,dx,$$

because $|Du - M| \leq |Du - M_0| + |M_0 - M| \leq 6\rho(\epsilon_0)$ in $B_{1/2}$, and we updated the value of the generic constant C. Since this is valid for all $M \in K \cap B_{4\rho(\epsilon_0)}(M_0)$, we deduce

$$\begin{aligned}\inf_{Y\in K}\int_{B_{1/2}} |Du - Y|^2\,dx &\leq C\int_{B_{1/2}} \mathrm{dist}^2(Du, K)\,dx \\ &\quad + C\rho(\epsilon_0)^2 \inf_{Y\in K\cap B_{4\rho(\epsilon_0)}(M_0)}\int_{B_{1/2}} |Du - Y|^2\,dx.\end{aligned}$$

Recalling (16), this implies

$$\begin{aligned}\inf_{Y\in K}\int_{B_{1/2}} |Du - Y|^2\,dx &\leq C\int_{B_{1/2}} \mathrm{dist}^2(Du, K)\,dx \\ &\quad + C\rho(\epsilon_0)^2 \inf_{Y\in K}\int_{B_{1/2}} |Du - Y|^2\,dx.\end{aligned}$$

Since $\rho(\epsilon) \to 0$ as $\epsilon \to 0$, we may choose ϵ_0 such that $C\rho(\epsilon_0)^2 \leq 1/2$, and absorb the last term in the left-hand side, thus proving (3). □

5 A 3×3 counter-example

In this section we prove that the two-dimensional setting of Theorem 1.1 is optimal in the following sense: a connected 1-submanifold of $\mathbb{R}^{3\times 3}$ which has no rank-one connection and is elliptic may not satisfy Šverák's compactness result [32], and even less a quantitative rigidity estimate.

We recall (see e.g. [29, § 2]) that an ordered set of $N \geq 4$ matrices $\{T_i\}_{i=1}^N \subset \mathbb{R}^{m\times n}$ without rank-one connections is said to form a $\mathcal{T}_N$ configuration if there exist matrices $P_i, C_i \in \mathbb{R}^{m\times n}$ and numbers $\kappa_i > 1$ such that

$$\begin{aligned} T_1 &= P + \kappa_1 C_1, \\ T_2 &= P + C_1 + \kappa_2 C_2, \\ &\dots \\ T_N &= P + C_1 + C_2 + \dots C_{N-1} + \kappa_N C_N, \end{aligned}$$

where C_i is rank-one for all i and $\sum_{i=1}^N C_i = 0$.

Proposition 5.1. *There exists a smooth, compact and connected* 1*-submanifold* $K \subset \mathbb{R}^{3\times 3}$ *without boundary which is elliptic and has no rank-one connection, but contains a* $\mathcal{T}_4$ *configuration.*

By a known construction, see e.g. [7, Theorem 3.1], Proposition 5.1 implies the existence of a sequence of maps $u_k\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\int_{B_1} \operatorname{dist}^2(Du_k, K)\, dx \to 0 \qquad \text{as } k \to 0,$$

but (Du_k) is not precompact in $L^2(B_{1/2})$. In particular, one certainly cannot hope for a quantitative estimate

$$\inf_{M\in K} \int_{B_{1/2}} |Du_k - M|^2\, dx \leq \rho\left(\int_{B_1} \operatorname{dist}^2(Du_k, K)\, dx \right),$$

for any function $\rho(\epsilon) \to 0$ as $\epsilon \to 0$.

Proposition 5.1 is a consequence of the construction below. Let $a > 0$ and define matrices T_1, T_2, T_3, T_4 by

$$T_1 = -T_3 = \begin{pmatrix} 1+a & 0 & 0 \\ 0 & 1 & 0 \\ 1+a & 0 & 0 \end{pmatrix}, \qquad T_2 = -T_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1+a & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

and C_1, C_2, C_3, C_4 by

$$C_1 = -C_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad C_2 = -C_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

We have that $T_k - C_k$ is rank-one for $k = 1, 2, 3, 4$, and (with the convention that $C_5 = C_1$)

$$T_k - \frac{2+a}{a}(T_k - C_k) = C_{k+1} \qquad \text{for } k = 1, 2, 3, 4,$$

so $\{T_1, T_2, T_3, T_4\}$ forms a $\mathcal{T}_4$ configuration. Next we construct a curve $K = K_a$ as in Proposition 5.1, which contains this $\mathcal{T}_4$ configuration. Let $\theta_a = \arctan(1/(1+a))$, so

$$\cos\theta_a = \frac{1+a}{r_a}, \qquad \sin\theta_a = \frac{1}{r_a}, \qquad r_a = \sqrt{1+(1+a)^2}.$$

Let $\rho\colon \mathbb{R} \to \mathbb{R}$ be a smooth monotonically increasing function to be determined later that satisfies

- $\rho(\theta + 2\pi) = \rho(\theta) + 2\pi$ for $\theta \in \mathbb{R}$,
- $\rho\left(\hat{\theta}_k\right) = \hat{\theta}_k$ for $\hat{\theta}_k = \theta_a + (k-1)\frac{\pi}{2}$, $k = 1, 2, 3, 4$.

Define $K_a = \Gamma_a(\mathbb{R}/2\pi\mathbb{Z})$, with

$$\Gamma_a(\theta) := \begin{pmatrix} r_a\cos(\theta) & -\sin(8\theta - 8\theta_a) & \sin(6\rho(\theta) - 6\theta_a) \\ \sin(6\theta - 6\theta_a) & r_a\sin(\theta) & \sin(8\rho(\theta) - 8\theta_a) \\ r_a\cos(\theta) & \sin(8\theta - 8\theta_a) & \sin(6\rho(\theta) - 6\theta_a) \end{pmatrix} \tag{17}$$

Then we have

$$T_k = \Gamma_a(\hat{\theta}_k) \text{ for } k = 1, 2, 3, 4,$$

so K_a contains the $\mathcal{T}_4$ configuration $\{T_1, T_2, T_3, T_4\}$. Next we adjust the parameter $a > 0$ and the function ρ in order to ensure that K_a has no rank-one connection and is elliptic.

Notation. With the $M_{i_1 i_2, j_1 j_2}$ minor we mean the determinant of the 2×2 submatrix corresponding to the rows i_1, i_2 and columns j_1, j_2.

Lemma 5.2. *If $a > 0$ is such that $\theta_a \notin \frac{\pi}{48}\mathbb{Z}$, the curve K_a is elliptic, i.e.* $\operatorname{Rank}\Gamma_a'(\theta) > 1$ *for all $\theta \in \mathbb{R}$.*

Proof. The derivative Γ_a' is given by

$$\Gamma_a'(\theta) := \begin{pmatrix} -r_a\sin(\theta) & -8\cos(8\theta - 8\theta_a) & 6\rho'(\theta)\cos(6\rho(\theta) - 6\theta_a) \\ 6\cos(6\theta - 6\theta_a) & r_a\cos(\theta) & 8\rho'(\theta)\cos(8\rho(\theta) - 8\theta_a) \\ -r_a\sin(\theta) & 8\cos(8\theta - 8\theta_a) & 6\rho'(\theta)\cos(6\rho(\theta) - 6\theta_a) \end{pmatrix}$$

Assume $\operatorname{Rank}\Gamma_a'(\theta) \leq 1$ for some $\theta \in \mathbb{R}$. Then calculating the $M_{12,12}$ minor we have

$$-r_a^2\sin(\theta)\cos(\theta) + 48\cos(8\theta - 8\theta_a)\cos(6\theta - 6\theta_a) = 0$$

and calculating the $M_{23,12}$ minor we have

$$48\cos(8\theta - 8\theta_a)\cos(6\theta - 6\theta_a) + r_a^2\sin(\theta)\cos(\theta) = 0.$$

Adding and subtracting these two equations we obtain that

$$\sin(\theta)\cos(\theta) = 0 \text{ and } \cos(8\theta - 8\theta_a)\cos(6\theta - 6\theta_a) = 0.$$

The first equality implies $\theta \in \frac{\pi}{2}\mathbb{Z}$, and then the second equality becomes

$$\cos(8\theta_a)\cos(6\theta_a) = 0,$$

which is impossible for $\theta_a \notin \frac{\pi}{48}\mathbb{Z}$. □

Lemma 5.3. *For any $a > 0$ such that $\theta_a \notin \frac{\pi}{24}\mathbb{Z}$, for any $\epsilon > 0$, there exists a smooth monotonic function $\rho\colon \mathbb{R} \to \mathbb{R}$ with the following properties*

- $\rho(\theta + 2\pi) = \rho(\theta) + 2\pi$ *for* $\theta \in \mathbb{R}$,
- $\rho\left(\hat{\theta}_k\right) = \hat{\theta}_k$ *for* $\theta_k = \hat{\theta}_a + (k-1)\frac{\pi}{2}$, $k = 1, 2, 3, 4$,
- *For any* $\theta, \theta' \in [0, 2\pi) \cap \frac{\pi}{24}\mathbb{Z}$, *we have* $\rho(\theta) - \rho(\theta') \notin \frac{\pi}{12}\mathbb{Z}$,
- $\sup_{\theta \in \mathbb{R}} |\rho(\theta) - \theta| < \epsilon$.

Proof. Let $\delta = \frac{1}{2}\operatorname{dist}(\theta_a, \frac{\pi}{24}\mathbb{Z}) = \frac{1}{2}\operatorname{dist}(\{\hat{\theta}_k\}, \frac{\pi}{24}\mathbb{Z}) > 0$ and fix a smooth function φ such that $\operatorname{supp}\varphi \subset (-\delta, \delta)$, $0 \leq \varphi \leq \varphi(0) = 1$ and $|\varphi'| \leq 2/\delta$. Define ρ on $[0, 2\pi)$ by setting

$$\rho(\theta) = \theta + \sum_{j=1}^{47} t_j\, \varphi\left(\theta - j\frac{\pi}{24}\right) \qquad \text{for } \theta \in [0, 2\pi),$$

where $t_1, \ldots, t_{47} \in (-\eta, \eta)$ are to be fixed later and $\eta = \min(\epsilon, \delta/2)$. The choice of $\delta > 0$ ensures that $\rho(\hat{\theta}_k) = \hat{\theta}_k$, also since $|t_j| < \epsilon$ we have $|\rho - id| < \epsilon$, and finally since $|t_j| < \delta/2$ we have $\rho' > 0$ on $[0, 2\pi)$. Moreover the function $\rho - id$ is identically zero near 0 and 2π, so it can be extended to a smooth 2π periodic function, thus yielding a smooth monotonic extension $\rho\colon \mathbb{R} \to \mathbb{R}$ satisfying $\rho(\theta + 2\pi) = \rho(\theta) + 2\pi$ for $\theta \in \mathbb{R}$. It remains to argue that we can pick $t_1, \ldots, t_{47} \in [0, \eta)$ to ensure that the third condition in Lemma 5.3 is satisfied.

Denote $t_0 = 0$. By induction, we may for each $j = 1, \ldots, 47$ choose $t_j \in [0, \eta)$ to ensure that

$$\rho(j\pi/24) - \rho(\ell\pi/24) = t_j - t_\ell + (j - \ell)\pi/24 \notin \frac{\pi}{12}\mathbb{Z}$$

for all $\ell \in \{0, \ldots, j-1\}$. This is possible because at each step there is only a discrete set of values of t_j to avoid. □

Lemma 5.4. *If $a > 0$ is such that $\theta_a \notin \frac{\pi}{48}\mathbb{Z}$, $\epsilon > 0$ is small enough and ρ is as in Lemma 5.3, then the curve $K_a \subset \mathbb{R}^{3\times 3}$ does not contain rank-one connections.*

Proof. Note as in Lemma 5.2 that the assumption $\theta_a \notin \frac{\pi}{48}\mathbb{Z}$ implies

$$\cos(8\theta_a) \neq 0 \quad \text{and} \quad \cos(6\theta_a) \neq 0.$$

We assume that there exist $\theta \neq \theta' \in \mathbb{R}/2\pi\mathbb{Z}$ such that

$$\operatorname{Rank}\left(\Gamma_a(\theta) - \Gamma_a(\theta')\right) = 1, \tag{18}$$

and we obtain a contradiction. We do this in several steps.

Step 1. We have

$$\theta + \theta' \in \pi\mathbb{Z} \quad \text{and} \quad \theta - \theta' \in \frac{\pi}{3}\mathbb{Z} \cup \frac{\pi}{4}\mathbb{Z}. \tag{19}$$

Proof of Step 1. From (17) calculating the $M_{12,12}$ minor we have

$$\begin{aligned}
0 &= r_a^2 \left(\cos(\theta) - \cos(\theta')\right)\left(\sin(\theta) - \sin(\theta')\right) \\
&\quad + \left(\sin(8\theta - 8\theta_a) - \sin(8\theta' - 8\theta_a)\right)\left(\sin(6\theta - 6\theta_a) - \sin(6\theta' - 6\theta_a)\right) \\
&= -4r_a^2 \sin\left(\frac{\theta + \theta'}{2}\right)\cos\left(\frac{\theta + \theta'}{2}\right)\sin^2\left(\frac{\theta - \theta'}{2}\right) \\
&\quad + 4\sin\left(3\left(\theta - \theta'\right)\right)\sin\left(4\left(\theta - \theta'\right)\right)\cos\left(3\left(\theta + \theta'\right) - 6\theta_a\right)\cos\left(4\left(\theta + \theta'\right) - 8\theta_a\right).
\end{aligned} \tag{20}$$

And calculating the $M_{23,12}$ minor we have

$$\begin{aligned}
0 &= -r_a^2 \left(\cos(\theta) - \cos(\theta')\right)\left(\sin(\theta) - \sin(\theta')\right) \\
&\quad + \left(\sin(8\theta - 8\theta_a) - \sin(8\theta' - 8\theta_a)\right)\left(\sin(6\theta - 6\theta_a) - \sin(6\theta' - 6\theta_a)\right) \\
&= 4r_a^2 \sin\left(\frac{\theta + \theta'}{2}\right)\cos\left(\frac{\theta + \theta'}{2}\right)\sin^2\left(\frac{\theta - \theta'}{2}\right) \\
&\quad + 4\sin\left(3\left(\theta - \theta'\right)\right)\sin\left(4\left(\theta - \theta'\right)\right)\cos\left(3\left(\theta + \theta'\right) - 6\theta_a\right)\cos\left(4\left(\theta + \theta'\right) - 8\theta_a\right).
\end{aligned} \tag{21}$$

Adding and subtracting (20) and (21) we obtain the equations

$$0 = \sin\left(3\left(\theta - \theta'\right)\right)\sin\left(4\left(\theta - \theta'\right)\right)\cos\left(3\left(\theta + \theta'\right) - 6\theta_a\right)\cos\left(4\left(\theta + \theta'\right) - 8\theta_a\right) \tag{22}$$

and

$$0 = \sin\left(\frac{\theta + \theta'}{2}\right)\cos\left(\frac{\theta + \theta'}{2}\right)\sin^2\left(\frac{\theta - \theta'}{2}\right). \tag{23}$$

Since $\theta \neq \theta'$ in $\mathbb{R}/2\pi\mathbb{Z}$, the last factor of (23) is nonzero, so either the first or the second must be zero. This implies $\theta + \theta' \in \pi\mathbb{Z}$. As a consequence, the last two factors in (22) are equal to $\pm\cos(6\theta_a)$ and $\cos(8\theta_a)$ and are nonzero by our choice of a. So one of the first two factors of (22) must vanish, that is, $\theta - \theta' \in \frac{\pi}{3}\mathbb{Z} \cup \frac{\pi}{4}\mathbb{Z}$.

Step 2. We have $\theta - \theta' \in \frac{\pi}{4}\mathbb{Z}$.

Proof of Step 2. Considering the $M_{13,23}$ minor of $\Gamma_a(\theta) - \Gamma_a(\theta')$ we obtain the equation

$$\begin{aligned}
0 &= \left(\sin(8\theta - 8\theta_a) - \sin(8\theta' - 8\theta_a)\right) \\
&\qquad \times \left(\sin(6\rho(\theta) - 6\theta_a) - \sin(6\rho(\theta') - 6\theta_a)\right) \\
&= 4\sin\left(4\left(\theta - \theta'\right)\right)\cos\left(4\left(\theta + \theta'\right) - 8\theta_a\right) \\
&\qquad \times \sin\left(3\left(\rho(\theta) - \rho(\theta')\right)\right)\cos\left(3\left(\rho(\theta) + \rho(\theta')\right) - 6\theta_a\right).
\end{aligned}$$

From Step 1 we have $\theta + \theta' \in \pi\mathbb{Z}$ so the second factor is $\cos(8\theta_a) \neq 0$. The last factor is arbitrarily close to $\pm\cos(6\theta_a) \neq 0$ since $|\rho - id| \leq \epsilon$. The third factor is nonzero by construction of ρ (recall Lemma 5.3) because $\theta, \theta' \in \frac{\pi}{24}\mathbb{Z}$ by (19). So we must have $\sin(4(\theta - \theta')) = 0$ hence $\theta - \theta' \in \frac{\pi}{4}\mathbb{Z}$.

Step 3. We have $\theta - \theta' \in \pi\mathbb{Z}$.

Proof of Step 3. Considering the $M_{12,13}$ minor of $\Gamma_a(\theta) - \Gamma_a(\theta')$ we obtain the equation

$$\begin{aligned} 0 = r_a &\left(\cos(\theta) - \cos(\theta')\right)\left(\sin(8\rho(\theta) - 8\theta_a) - \sin\left(8\rho(\theta') - 8\theta_a\right)\right) \\ &- \left(\sin(6\theta - 6\theta_a) - \sin(6\theta' - 6\theta_a)\right) \\ &\quad \times \left(\sin(6\rho(\theta) - 6\theta_a) - \sin\left(6\rho(\theta') - 6\theta_a\right)\right), \end{aligned}$$

which we rewrite as

$$\begin{aligned} &-4r_a \sin\left(\frac{\theta + \theta'}{2}\right)\sin\left(\frac{\theta - \theta'}{2}\right) \\ &\qquad \times \sin\left(4\left(\rho(\theta) - \rho(\theta')\right)\right)\cos\left(4\left(\rho(\theta) + \rho(\theta')\right) - 8\theta_a\right) \\ &= 4\sin\left(3\left(\theta - \theta'\right)\right)\cos\left(3\left(\theta + \theta'\right) - 6\theta_a\right) \\ &\qquad \times \sin\left(3\left(\rho(\theta) - \rho(\theta')\right)\right)\cos\left(3\left(\rho(\theta) + \rho(\theta')\right) - 6\theta_a\right). \end{aligned} \tag{24}$$

Since $\theta - \theta' \in \frac{\pi}{4}\mathbb{Z}$ and $|\rho - id| \leq \epsilon$, the third factor in the left-hand side of (24) has absolute value $\leq 4\epsilon$. Since $\theta + \theta' \in \pi\mathbb{Z}$, the second factor in the right-hand side of (24) has absolute value equal to $|\cos(6\theta_a)| > 0$, and for small enough ϵ the absolute value of the last factor is $\geq |\cos(6\theta_a)|/2 > 0$. Taking also into account that the first and third factors in the right-hand side of (24) differ from each other by an error $\leq 3\epsilon$, we must have

$$\sin^2\left(3\left(\theta - \theta'\right)\right) \leq c_a\epsilon,$$

for some $c_a > 0$ depending only on a. Because $\theta - \theta' \in \frac{\pi}{4}\mathbb{Z}$ by Step 2, provided ϵ is chosen small enough, this implies $\theta - \theta' \in \frac{\pi}{3}\mathbb{Z} \cap \frac{\pi}{4}\mathbb{Z} = \pi\mathbb{Z}$.

Step 4: Conclusion. From Step 1 and Step 3 we have $\theta + \theta', \theta - \theta' \in \pi\mathbb{Z}$, so $\theta, \theta' \in \frac{\pi}{2}\mathbb{Z}$. Since me may without loss of generality exchange the roles of θ and θ' and choose arbitrary representants in $\mathbb{R}/2\pi\mathbb{Z}$, this amounts to $\theta = 0$ and $\theta' = \pi$, or $\theta = \pi/2$ and $\theta' = 3\pi/2$. In the former case, considering the $M_{12,13}$ minor of $\Gamma_a(0) - \Gamma_a(\pi)$ as in (24) we deduce

$$\sin(4(\rho(\pi) - \rho(0)))\cos(4(\rho(\pi) + \rho(0)) - 8\theta_a) = 0.$$

Using again that $|\rho - id| \leq \epsilon$, the second factor has absolute value $\geq |\cos(8\theta_a)|/2$ provided ϵ is small enough, and the first factor is nonzero by construction of ρ, so we conclude that (18) is not possible for $\theta \neq \theta' \in \mathbb{R}/2\pi\mathbb{Z}$. In the latter case, considering the $M_{12,23}$ minor of $\Gamma_a(\pi/2) - \Gamma_a(3\pi/2)$ and following similar calculations as in (24) leads to

$$\sin(3(\rho(\pi/2) - \rho(3\pi/2)))\cos(3(\rho(\pi/2) + \rho(3\pi/2)) - 6\theta_a) = 0.$$

Finally, we follow exactly the same lines as above to get a contradiction. ∎

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