Thin Set Versions of Hindman's Theorem

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1 Introduction

This paper is part of a line of research on the computability-theoretic and reverse-mathematical strength of versions of Hindman's Theorem [6] that began with the work of Blass, Hirst, and Simpson [1], and has seen considerable interest recently. We assume basic familiarity with computability theory and reverse mathematics, at the level of the background material in [8], for instance. On the reverse mathematics side, the two major systems with which we will be concerned are RCA_0, the usual weak base system for reverse mathematics, which corresponds roughly to computable mathematics; and ACA_0, which corresponds roughly to arithmetic mathematics. For principles P of the form (8X) [(X) ! (9Y) (X; Y)], we call any X such that (X) holds an instance of P, and any Y such that (X; Y) holds a solution to X.

We begin by introducing some related combinatorial principles. For a set S, let $[S]^n$ be the set of n-element subsets of S. Ramsey's Theorem (RT) is the statement that for every n and every coloring of $[N]^n$ with nitely many colors, there is an innite set H that is homogeneous for c, which means that all elements of $[H]^n$ have the same color. There has been a great deal of work on computability-theoretic and reverse-mathematical aspects of versions of Ramsey's Theorem, such as RT^n_k, which is RT restricted to colorings of $[N]^n$ with k many colors. (See e.g. [8].)

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The Thin Set Theorem is another variant of Ramsey's Theorem that has been studied from this perspective. It follows easily from Ramsey's Theorem itself.

Denition 1.1. Thin Set Theorem (TS): For every n and every coloring c : $[N]^n$! N, there is an innite set T N and an i such that $c(s) = i$ for all s 2 $[T]^n$. We call such a set T a thin set for c. TS^n is the restriction of TS to colorings of $[N]^n$.

Jockusch [9] showed that there is a computable instance of RT^3_2 such that any solution computes the halting problem ;^{0}. As shown by Simpson [18], Jockusch's construction can also be used to prove that RT^3_2 (and hence RT) implies ACA_0 over RCA_0. Wang [19] showed that TS, on the other hand, does not have this much power. Indeed, it has a property known as strong cone avoidance, which implies in particular that for every coloring c : $[N]^n$! N and every noncomputable X, there is an innite thin set for c that does not compute X. It also follows from strong cone avoidance that TS does not imply ACA_0 over RCA_0.

As shown by Seetapun [17], RT^2_k also fails to imply ACA_0. Indeed, Liu [11, 12] showed that it does not imply the weaker system WKL_0, which consists of RCA_0 together with Weak K•nig's Lemma, or the even weaker system $WWKL_0$ consisting of RCA_0 together with Weak Weak K•nig's Lemma. Patey [14] showed that the same is true of TS.

We now turn to Hindman's Theorem. For a set S N, let fs(S) be the set of sums of nonempty nite sets of distinct elements of S.

Denition 1.2. Hindman's Theorem (HT): For every coloring of N with nitely many colors, there is an innite set S N such that all elements of fs(S) have the same color.

Blass, Hirst, and Simpson [1] showed that such an S can always be computed in the (! + 1)st jump of the coloring, and that there is a computable coloring such that every such S computes ;^{0}. By analyzing these proofs they showed that HT is provable in ACA_0^+ (the system consisting of RCA_0 together with the statement that !th jumps exist) and implies ACA_0 over RCA_0. The exact computability-theoretic and reverse-mathematical strength of HT remains open.

There has recently been interest in studying restricted versions of HT such as the following. (See e.g. [2].)

Denition 1.3. $\mathsf{HT}^{\leq n}$ is HT restricted to sums of at most n many elements, and $\mathsf{HT}^{=n}$ is HT restricted to sums of exactly n many elements. $\mathsf{HT}^{\leq n}_k$ and $\mathsf{HT}^{=n}_k$ are the corresponding restrictions to colorings with k many colors.

Dzhafarov, Jockusch, Solomon, and Westrick [5] showed that $\mathsf{HT}^{\leq 3}_3$ implies ACA_0 over RCA_0. Carlucci, Kolodzieczyk, Lepore, and Zdanowski [3] did the same for $\mathsf{HT}^{\leq 2}_4$. These principles are also complex in a more heuris-tic sense: There is no known way to prove even $\mathsf{HT}^{\leq 2}_2$ other than to give a proof of the full HT, which has led Hindman, Leader, and Strauss [7] to ask whether every proof of $\mathsf{HT}^{\leq 2}$ is also a proof of HT. This question can be formalized by asking whether $\mathsf{HT}^{\leq 2}$ (or $\mathsf{HT}^{\leq 2}_2$) implies HT, say over RCA_0. A related open question is whether $\mathsf{HT}^{\leq 2}_2$ is provable in ACA_0.

The principle $\mathsf{HT}^{=2}$ is quite dierent, as $\mathsf{HT}^{=2}_k$ follows easily from RT^2_k. Indeed, it was not clear even whether this principle is computably true until the work of Csima, Dzhafarov, Hirschfeldt, Jockusch, Solomon, and Westrick [4], who showed that it is not, and that indeed there is a computable instance of $\mathsf{HT}^{=2}_2$ with no Σ^0_2 solutions. (The same had been shown for RT^2_2 by Jockusch [9], who also showed that every computable instance of RT^2_2 has a Π^0_2 solution, which implies that the same is true of $\mathsf{HT}^{=2}_2$.) They also showed that there is a computable instance of $\mathsf{HT}^{=2}_2$ such that every solution has DNC degree relative to $\emptyset'$, and adapted this proof to show that $\mathsf{HT}^{=2}_2$ implies the principle RRT^2_2, a version of the Rainbow Ramsey Theorem, over RCA_0. (See Section 3 for denitions.)

In this paper, we study further versions of Hindman's Theorem, obtained by combining HT and its variants with the Thin Set Theorem.

Denition 1.4. thin-HT: For every coloring $c : \mathbb{N} \to \mathbb{N}$, there is an innite set $S \subseteq \mathbb{N}$ such that $fs(S)$ is thin for c. We denite restrictions such as thin-$\mathsf{HT}^{\leq n}$ analogously.

In Section 2, we give similar lower bounds on the complexity of thin-HT as Blass, Hirst, and Simpson [1] gave for HT, which suggests that thin-HT behaves like HT at least to some extent. Indeed, it seems possible that thin-HT is equivalent to HT over RCA_0. The situation for restricted versions is dierent, however. Clearly, thin-$\mathsf{HT}^{=n}$ follows from TS^n, but in fact so does thin-$\mathsf{HT}^{\leq n}$, due to the following fact.

Lemma 1.5. For each n and k, the following holds in $\mathsf{RCA}_0 + \mathsf{TS}^n$: Given $c_i : [\mathbb{N}]^{m_i} \to \mathbb{N}$ for $i \leq k$, with $m_i \leq n$ for all $i \leq k$, there is a single innite set T and a j such that $c_i(s) \neq j$ for each c_i and each $s \in [T]^{m_i}$ with $i \leq k$.

Proof. We use the fact that TS^n implies TS^m for each $m < n$, and proceed by external induction to prove the stronger assertion that for each $j \; 6 \; k$, $RCA_0 + TS^n$ proves that there is an innite set T and an innite set C such that $c_i(s) \not 2 \; C$ for each c_i and each $s \; 2 \; [T]^{m_i}$ with $i \; 6 \; j$.

We do the base and inductive cases simultaneously. For $j + 1 > 0$, assume that that the assertion holds for j and let T and C be as above. For $j + 1 = 0$, let $T = C = N$. Dene $d : [T]^{m_{j+1}} \; ! \; N$ as follows. Partition C into innitely many innite sets $A_0; A_1; : : :$. Let $d(s) = 0$ if either $c_{j+1}(s) \; 2 \; A_0$ or $c_{j+1}(s) \not 2 \; C$, and for $i > 0$, let $d(s) = i$ if $c_{j+1}(s) \; 2 \; A_i$. By $TS^{m_{j+1}}$, there is an innite $U \quad T$ that is thin for d. Let $i \not 2 \; d([U]^{m_{j+1}})$ and let $D = A_i$. Then U and D are innite sets such that $c_i(s) \not 2 \; D$ for each c_i and each $s \; 2 \; [U]^{m_i}$ with $i \; 6 \; j + 1$. □

This lemma allows us to get thin-HT^{6n} from TS^n by taking a coloring $c : N \; ! \; N$ and considering the colorings that map $fa_0; : : : ; a_jg$ to $c(a_0 + \quad + a_j)$ for each $j < n$.

There are also dierences that have nothing to do with computability theory and reverse mathematics between thin-HT^{6n} on the one hand, and thin-HT and HT^{6n} on the other. The former remains true if we allow sums of non-distinct elements, but it is not dicult to show that the latter two do not. Similarly, the former remains true for colorings $S \; ! \; N$, where $S \quad N$ is any innite set, while the latter two again do not.

Nevertheless, even thin-$HT^{=2}$ still has a signicant level of complexity. In Section 3, we show that all of the lower bounds mentioned above obtained in [4] for $HT^{=2}$ still hold for thin-$HT^{=2}$.

In Section 4 we mention some open questions arising from our results, and briey discuss version of HT obtained by combining it with thin set theorems for colorings with nitely many colors.

2 Encoding $;^0$ into thin-HT

In this section, we show how to build on the proof of Theorem 2.2 of Blass, Hirst, and Simpson [1], which shows that there is a computable instance of HT such that every solution computes $;^0$, to show that the same is true of thin-HT. We then derive a reverse-mathematical consequence of our proof.

Theorem 2.1. There is a computable instance of thin-HT such that every solution computes $;^0$.

Proof. As in the proof of Theorem 2.2 of [1], we write each number $x > 0$ as $2^{n_0} + \ + 2^{n_k}$ with $n_0 < \ < n_k$, and dene $(x) = n_0$ and $(x) = n_k$. A set S has 2-apartness if for every $x; y$ 2 S with $x < y$, we have $(x) < (y)$. Lemma 4.1 of [1] shows that from any innite S we can compute an innite set T with 2-apartness such that fs(T) fs(S) (and hence if fs(S) is thin for a coloring, so is fs(T)).

Let $x = 2^{n_0} + \ + 2^{n_k}$ with $n_0 < \ < n_k$. Say that $(n_i; n_{i+1})$ is a short gap in x if there is an $m < n_i$ such that m 2 $;^0[n_{i+1}]$ but m 2 $;^0$. Say that $(n_i; n_{i+1})$ is a very short gap in x if there is an $m < n_i$ such that m 2 $;^0[n_{i+1}]$ but m 2 $;^0[n_k]$. Let sg(x) and vsg(x) be the numbers of short gaps and very short gaps in x, respectively. Note that sg is not a computable function, but vsg is.

Fix a bijection between N and the set of pairs $(p; i)$ where p is prime and 1 6 $i < p$, and identify N with this set via this bijection. Dene the coloring c by letting $c(x) = (p; i)$ where p is the least prime that does not divide vsg(x) and vsg(x) = i mod p. We say that x has color $(p; i)$ if $c(x) = (p; i)$, and we also say that x has color $(p; 0)$ or $(p; p)$ if it has color $(q; i)$ for some $q > p$, i.e., if every prime less than or equal to p divides vsg(x).

Let Y be such that fs(Y) is an innite thin set for c. We can assume that Y has 2-apartness, by Lemma 4.1 of [1], as mentioned above. This condition ensures that if $x; y$ 2 fs(Y) and $(x) < (y)$, and we express x and y as sums of sets F and G of distinct elements of Y, respectively, then F and G are disjoint, and hence $x + y$ 2 fs(Y). Say that S fs(Y) is -bounded if there is a bound on the values of (x) for x 2 S (which includes the case $S = ;$). Note that fs(Y) itself is not -bounded. Note also that the union of nitely many -bounded sets is -bounded. Say that a color j is almost absent from fs(Y) if the set of x 2 fs(Y) that have color j is -bounded. (This denition includes the case $j = (p; 0)$, or equivalently $j = (p; p)$.)

Lemma 2.2. There are p and 0 6 $i < p$ such that $(p; i + 1)$ is almost absent from fs(Y) but $(p; i)$ is not.

Proof. Let p be least such that there is a j for which $(p; j)$ is almost absent from fs(Y), which exists since fs(Y) is thin. If $p = 2$ then $(p; j + 1)$ cannot be almost absent, since every number has color $(p; j)$ or $(p; j + 1)$. Now suppose that $p > 2$ and q is the preceding prime. Since $(q; 0)$ is not almost absent from fs(Y) and every number that has color $(q; 0)$ has color $(p; j)$ for some j, there is some k such that $(p; k)$ is not almost absent. In either case, since having color $(p; 0)$ is the same as having color $(p; p)$, the lemma follows. □

Fix p and i as in the above lemma.

Lemma 2.3. Let 1 6 j < p. Then S = fx 2 fs(Y) : sg(x) = j mod pg is -bounded.

Proof. Suppose S is not -bounded. Let q_0 < < $q_{m\ 1}$ be the primes less than p. Since there are only nitely many sequences (k_0; : : : ; $k_{m\ 1}$) with k_i < q_i, there is such a sequence for which T = fx 2 S : (8' < m) sg(x) = $k_{\prime}$ mod $q_{\prime}$g is not -bounded.

Since j = 0 mod p, and hence $q_0 q_{m\ 1}$j = 0 mod p, there is a multiple n of $q_0 q_{m\ 1}$ such that nj = 1 mod p (where $q_0 q_{m\ 1}$ = 1 if p = 2). Since T is not -bounded, there are x_0 < < $x_{n\ 1}$ 2 T such that each (x_{k+1}) is suciently large relative to (x_k) to ensure that ((x_k); (x_{k+1})) is not a short gap. Then the short gaps in x_0 + + $x_{n\ 1}$ are exactly the short gaps in x_0; : : : ; $x_{n\ 1}$, so sg(x_0 + + $x_{n\ 1}$) = sg(x_0) + + sg($x_{n\ 1}$). The latter is equal to nj mod p = 1 mod p, since each $x_{\prime}$ is in S, and is also equal to n$k_{\prime}$ mod $q_{\prime}$ for each ' < m, and hence equal to 0 mod $q_{\prime}$ for each ' < m, since n = 0 mod $q_{\prime}$.

Since (p; i) is not almost absent from fs(Y), there is a y 2 fs(Y) that has color (p; i) such that (y) > ($x_{n\ 1}$), and every number less than ($x_{n\ 1}$) that is in ;^{0} is already in ;^{0}[(y)]. Note that vsg(y) = 0 mod $q_{\prime}$ for each ' < m, as otherwise c(y) would be of the form ($q_{\prime}$; k) for some 1 6 k < $q_{\prime}$. Now vsg(x_0 + + $x_{n\ 1}$ + y) = vsg(y) + sg(x_0 + + $x_{n\ 1}$), which is equal to i + 1 mod p, and to 0 mod $q_{\prime}$ for all ' < m. So x_0 + + $x_{n\ 1}$ + y has color (p; i + 1). As we can choose x_0 so that (x_0) is arbitrarily large, (p; i + 1) is not almost absent from fs(Y), contradicting the choice of i. □

So by removing nitely many elements from Y if needed, we can assume that p divides sg(x) for all x 2 fs(Y). We can now argue as in the proof of Claim 2 in the proof Theorem 2.2 of [1] to compute ;^{0} from Y : Given n, nd x; y 2 Y such that x < y and n < (x). The short gaps in x + y are the ones in x, the ones in y, and possibly ((x); (y)). But if the latter is a short gap, then sg(x + y) = sg(x) + sg(y) + 1, which is impossible since p divides all three numbers. Thus n 2 ;^{0} iff n 2 ;^{0}[(y)]. □

The above proof can be carried out in relativized form in RCA_0 except for two issues: One is that in RCA_0 we cannot show that the union of nitely many -bounded sets is -bounded, which in general requires the $_1$-bounding principle. Another is that being almost absent is a $_2$ condi-tion, so we cannot conclude in RCA_0 that there is a least p such that there is a j for which (p; j) is almost absent from fs(Y). Since $^0_1$-bounding follows from $_2$-induction over RCA_0, adding the latter to RCA_0 is sucient to get around these issues, so we have the following.

Theorem 2.4. thin-HT implies ACA_0 over $RCA_0 + I_2$.^{0}

We do not know whether the use of I_2 ^{0} in this theorem can be removed.

3 Hard Instances of thin-HT$^{=2}$

In this section, we show that all the lower bounds on the complexity of $HT_2^{=2}$ obtained by Csima, Dzhafarov, Hirschfeldt, Jockusch, Solomon, and Westrick [4] still hold for thin-HT$^{=2}$. (Of course, all upper bounds on the complexity of $HT_2^{=2}$ automatically hold for thin-HT$^{=2}$, as the latter follows easily from the former.) As in that paper, we use the computable version of the Lovasz Local Lemma due to Rumyantsev and Shen [15, 16]. In particular, we use the following consequence of Corollary 7.2 in [16] given in [4], with an addendum on uniformity as noted at the end of Section 4 of [4]. This uniformity, which in [4] is used only to obtain results on Weihrauch reducibility, will be essential in all our results, as their proofs will require applying Theorem 3.1 innitely often.

Theorem 3.1 (essentially Rumyantsev and Shen [16]). For each q 2 (0; 1) there is an M such that the following holds. Let $F_0; F_1; : : :$ be a computable sequence of nite sets, each of size at least M. Suppose that for each m > M and n, there are at most 2^{qm} many j such that jF_jj = m and n 2 F_j, and that there is a computable procedure P for determining the set of all such j given m and n. Then there is a computable c : N ! 2 such that for each j the set F_j is not homogeneous for c. Furthermore, c can be obtained uniformly computably from $F_0; F_1; : : :$ and P (for a xed q).

We will also rely in this section on arguments in [4] when they carry through in this case in an entirely analogous way.

We now introduce a notion of largeness that will be key to our iterated applications of Theorem 3.1. As in [4], we will be diagonalizing against $_2^0$ sets, so this notion will be dened in terms of sets that are c.e. relative to ;^{0}. For a set A and a number s, we write s + A for the set fs + a : a 2 Ag. We write W_e for the eth enumeration operator. Given e and s, for each x 2 $W_e^{;^0}[s]$, let t_x be the least t such that x 2 $W_e^{;^0}[u]$ for all u 2 [t; s]. (I.e.,

t_x measures how long x has been in $W_e^{;0}$.) Order the elements of $W_e^{;0}[s]$ by letting x y if either $t_x < t_y$ or both $t_x = t_y$ and $x < y$. Let $E_e^n[s]$ be the set consisting of the least n many elements of $W_e^{;0}[s]$ under this ordering, or $E_e^n[s] = {}_e[0; n)$ if $W_e^{;0}[s]$ has fewer than n many elements. If there is an s such that $E_e^n[t] = E_e^n[s]$ for all $t > s$ then let $E_e^n = E_e^n[s]$.

Denition 3.2. For a binary function f, say that a set D is f-large if for all e and k such that $E_e^{f(e;k)}$ is dened, we have $|D \setminus (s + E_e^{f(e;k)})| > k$ for all suciently large s.

Note that $\mathbb{N}$ is g-large for the function $g(e; k) = k$, and that f-largeness is preserved under nite dierence. The following lemma captures the key property of this notion of largeness.

Lemma 3.3. From a binary function f and an f-large set D, we can uniformly compute a binary function $\hat{f}$ and a splitting $D = D^0 \sqcup D^1$ such that each D^i is $\hat{f}$-large.

Before proving this lemma, let us derive some of its consequences, beginning with computability-theoretic lower bounds on the complexity of thin-$\mathrm{HT}^{=2}$. A function f is diagonally noncomputable (DNC) relative to an oracle X if $f(e) \neq \Phi_e^X(e)$ for all e such that $\Phi_e^X(e)$ is dened, where Φ_e is the eth Turing functional. A degree is DNC relative to X if it computes a function that is DNC relative to X. An innite set A is eectively immune relative to X if there is an X-computable function f such that if $W_e^X \subseteq A$ then $|W_e^X| < f(e)$.

Theorem 3.4 (Jockusch [10]). A degree is DNC relative to X if and only if it computes a set that is eectively immune relative to X.

The proof of the following theorem shows how to obtain a hard computable instance of thin-$\mathrm{HT}^{=2}$ from Lemma 3.3.

Theorem 3.5. There is a computable instance of thin-$\mathrm{HT}^{=2}$ such that any solution is eectively immune relative to $;^0$, and hence has DNC degree relative to $;^0$.

Proof. Let $D_0 = \mathbb{N}$ and $f_0(e; k) = k$. Given D_n and f_n, let $\hat{f}_n$ and D_n^i be as in Lemma 3.3, let $f_{n+1} = \hat{f}_n$, and let $D_{n+1} = D_n^1$. Note that the D_n are uniformly computable. Let $c(x)$ be the largest $n \leqslant x$ such that $x \in D_n$. Then

c is a computable coloring of $\mathbb{N}$. If $c(x) = n$ and $x > n$ then $x \in D_n$ but $x \notin D_m$ for $m > n$, so $x \in D'_n$. Thus for each n, we have that the difference between $c^{-1}(n)$ and D'_n is finite, and hence $c^{-1}(n)$ is f_n-large.

Let S be a solution to c as an instance of thin-$\mathrm{HT}^{=2}$, and let n be such that $c(x + y) \neq n$ for all distinct $x, y \in S$. For any e, if $|W_e^{\emptyset'}| > f_n(e, 1)$ then $E_e^{f_n(e,1)} \subseteq W_e^{\emptyset'}$ is defined, and hence $c^{-1}(n) \cap (s + E_e^{f_n(e,1)}) \neq \emptyset$ for all sufficiently large s. In other words, if s is sufficiently large then there is an $x \in E_e^{f_n(e,1)}$ such that $c(x + s) = n$. It follows that $E_e^{f_n(e,1)} \nsubseteq S$, and hence $W_e^{\emptyset'} \nsubseteq S$, since $E_e^{f_n(e,1)} \subseteq W_e^{\emptyset'}$. Thus we conclude that if $W_e^{\emptyset'} \subseteq S$ then $|W_e^{\emptyset'}| < f_n(e, 1)$. Since $f_n(e, 1)$ is computable as a function of e, it follows that S is effectively immune relative to $\emptyset'$, and hence has DNC degree relative to $\emptyset'$. □

No infinite Δ^0_2 set can be effectively immune relative to $\emptyset'$, so we have the following.

Corollary 3.6. There is a computable instance of thin-$\mathrm{HT}^{=2}$ with no Δ^0_2 solution.

It follows that thin-HT is not provable in WKL_0, since the latter has ω-models consisting entirely of Δ^0_2 sets. It was noted in [4] that $\mathrm{HT}^{=2}_2$ does not imply WKL_0, and hence neither does thin-$\mathrm{HT}^{=2}$. Thus thin-$\mathrm{HT}^{=2}$ and WKL_0 are incomparable over RCA_0. In fact, as mentioned in the introduction, Patey [14] showed that TS does not imply WKL_0, or even WWKL_0, and we can easily adapt the proof of Theorem 3.5 to thin-$\mathrm{HT}^{=n}$ for any $n > 2$, so we have the following.

Corollary 3.7. For each $n > 1$, both thin-$\mathrm{HT}^{=n}$ and thin-$\mathrm{HT}^{\leq n}$ are incomparable with $(\mathrm{W})\mathrm{WKL}_0$ over RCA_0.

Arguing as in the proof of Corollary 3.6 of [4], we have the following.

Corollary 3.8. There is a computable instance of thin-$\mathrm{HT}^{=2}$ such that all solutions are hyperimmune.

The reverse-mathematical analog of the existence of degrees that are DNC over the jump is the principle 2-DNC, defined e.g. in Section 4 of [4]. Miller [unpublished] showed that 2-DNC is equivalent, both over RCA_0 and in the sense of Weihrauch reducibility, to the following version of the Rainbow Ramsey Theorem, which was shown by Patey [13] to be strictly weaker than TS^2.

Denition 3.9. $RRT^2{:}_2$ Let $c : [N]^2$! N be such that jc ^{1}(i)j 6 2 for all i. Then there is an innite set R such that c is injective on $[R]^2$.

As discussed in [4], the proof of Theorem 3.1 carries through in RCA_0, from which it will follow that so does the proof of Lemma 3.3 that we will give below. Thus the proof of Theorem 3.5 also carries through in RCA_0, except for one issue: Having $jW_e^{;0}j >$ m does not necessarily imply in RCA_0 that E_e^m is dened. (The issue is that RCA_0 does not imply the $^0_1$-bounding principle.) However, we can get around this problem exactly as in Section 4 of [4], by using the principle 2-EI dened there, thus obtaining the following.

Theorem 3.10. thin-$HT^{=2}$ implies RRT^2_2 over RCA_0.

We can also obtain a Weihrauch reduction from RRT^2_2 to a version of thin-$HT^{=2}$ as in the nal paragraph of Section 4 of [4], but we have to be a bit careful because in the proof of Theorem 3.5, the function witnessing that S is eectively immune relative to $;^0$ is obtained uniformly not from S, but from an n such that c(x + y) = n for all distinct x; y 2 S. Let strong thin-$HT^{=2}$ be the version of thin-$HT^{=2}$ where a solution to an instance c consists of both a solution S to c as an instance of thin-$HT^{=2}$ and an n as above. Then we have the following.

Theorem 3.11. RRT^2_2 is Weihrauch-reducible to strong thin-$HT^{=2}$.

We do not know, however, whether this theorem remains true if we replace strong thin-$HT^{=2}$ by thin-$HT^{=2}$.

None of the above results depend on the addition function in particular, and can be adapted as in [4] to any function f : $[N]^2$! N that is addition-like, which means that

1. f is computable,
2. there is a computable function g such that f(fx; yg) > n for all y > g(x; n), and
3. there is a b such that for all x = y, there are at most b many z's for which f(fx; zg) = f(fx; yg).

We nish this section by proving Lemma 3.3.

Proof of Lemma 3.3. Let f be a binary function and D an f-large set. We will apply Theorem 3.1 to obtain a computable $c : \mathbb{N} \to 2$. We then dene $D^i = \{n \in D : c(n) = i\}$. The value of q will not matter here, so let us x $q = \frac{1}{2}$. Let M be as in Theorem 3.1.

Let g be a computable injective binary function with computable image such that $kg(e,k) \leq 2^{\frac{g(e,k)}{2}}$ and $g(e,k) \geq M$ for all e and k.

Say that s is acceptable for e,k if $|D \cap (s + E_e^{f(e,kg(e,k))}[s])| \geq kg(e,k)$ and for every $t < s$ such that $(s + E_e^{f(e,kg(e,k))}[s]) \cap (t + E_e^{f(e,kg(e,k))}[t]) = \emptyset$, we have $E_e^{f(e,kg(e,k))}[s] = E_e^{f(e,kg(e,k))}[t]$. If s is acceptable for e,k then let $F_{e,k,s,0}$ be the rst $g(e,k)$ many elements of $s + E^{f(e,kg(e,k))}[s]$, let $F_{e,k,s,1}$ be the next $g(e,k)$ many elements of $s + E^{f(e,kg(e,k))}[s]$, and so on, until $F_{e,k,s,k-1}$.

Let F consist of all $F_{e,k,s,j}$ for all e,k, all s acceptable for e,k, and all $j < k$. Then we can arrange the elements of F into a computable sequence of nite sets, each of size at least M. Fix x and m. If m is not in the image of g then there are no elements of F of size m. Otherwise, there is a unique pair e,k such that $m = g(e,k)$, and all elements of F of size m that contain x are of the form $F_{e,k,s,j}$ for some $s \leq x$. We can computably determine all such sets from m and x, and the denition of acceptability means that there are at most $kg(e,k) \leq 2^{\frac{m}{2}}$ many such sets.

Thus the hypotheses of Theorem 3.1 hold, and hence there is a c, obtained uniformly computably from f and D, such that none of the sets in F are homogeneous for c. Let $\hat{f}(e,k) = f(e,kg(e,k))$ and let $D^i = \{n \in D : c(n) = i\}$. Fix e and k such that $E_e^{\hat{f}(e,k)}$ is dened. If s is suciently large then s is acceptable for e,k, and $F_{e,k,s,j} \subseteq s + E_e^{\hat{f}(e,k)}$ for all $j < k$. For each $j < k$ and $i < 2$, there is at least one $x \in F_{e,k,s,j}$ such that $c(x) = i$. Since the $F_{e,k,s,j}$ are disjoint, $|D^i \cap (s + E_e^{\hat{f}(e,k)})| \geq k$. Thus D^i is $\hat{f}$-large. □

4 Open Questions

In this section, we collect a few open questions and possible directions for further work arising from the above results.

Question 4.1. Does thin-HT imply ACA_0 over RCA_0 (i.e., without assuming $I\Sigma^0_2$)?

Of course, one way to give a positive answer to this question would be to show that thin-HT implies $I\Sigma^0_2$ over RCA_0. If that is not the case, then it could be interesting to try to determine the rst-order part of thin-HT.

Question 4.2. Is thin-HT provable in ACA_0?

Question 4.3. Does thin-HT imply HT, say over RCA_0?

In the spirit of Hindman, Leader, and Strauss [7], we can also ask the less formal question of whether there is a proof of thin-HT that is not already a proof of HT.

Question 4.4. Is RRT^2_2 Weihrauch-reducible to thin-$HT^{=2}$ (as opposed to strong thin-$HT^{=2}$)?

Question 4.5. What is the exact relationship between thin-$HT^{=2}$ and each of TS^2, RRT^2_2, and $HT^{=2}$?

There are also versions of the Thin Set Theorem for colorings with nitely many colors. For example, an instance of TS^n_k is a coloring c of $[N]^n$ with k many colors, and a solution to this instance is an innite set T such that $jc([T]^n)j < k$. This principle and RT^n_k form the two ends of a spectrum of principles $RT^n_{k;j}$ for $1 6 j < k$, where an instance is a coloring c of $[N]^n$ with k many colors, and a solution to this instance is an innite set T such that $jc([T]^n)j 6 j$. It would be interesting to pursue versions of HT based on these principles. One might hope to show, for instance, that there is a boundary between principles that \behave like HT", e.g. HT^{62}_4, which as mentioned in the introduction was shown to imply ACA_0 in [3]; and those that \behave like versions of TS / RT", e.g. the thin version of HT^{62}_4, which can easily be shown to follow from $RT^2_{4;2}$.

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