Advances in Mathematics 407 (2022) 108548

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

The royal road to automatic noncommutative real analyticity, monotonicity, and convexity

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A R T I C L E   I N F O

Article history:
Received 27 July 2019
Received in revised form 4 August 2020
Accepted 21 June 2022
Available online 18 July 2022
Communicated by Dan Voiculescu

MSC:
46L52
32A70
30H10

Keywords:
Matrix convex functions
Matrix monotone functions
Loewner's theorem
Automatic analyticity
Noncommutative realizations

A B S T R A C T

It was shown classically that matrix monotone and matrix convex functions must be real analytic by Löwner and Kraus respectively. Recently, various analogues have been found in several noncommuting variables. We develop a general framework for lifting automatic analyticity theorems in matrix analysis from one variable to several variables, the so-called "royal road theorem." That is, we establish the principle that the hard part of proving any automatic analyticity theorem lies in proving the one variable theorem. We use our main result to prove the noncommutative Löwner and Kraus theorems over operator systems as examples, including an analogue of the rational "butterfly realization" of Helton-McCullough-Vinnikov for general analytic functions.

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[1] The authors were generously supported by the Fields Institute, Focus Program on Applications of Noncommutative Functions.

https://doi.org/10.1016/j.aim.2022.108548

1. Introduction

There is no royal road to Löwner's theorem in one variable. (Barry Simon counts 11, or perhaps 12, proofs of the one variable theorem, none of which are regarded as trivial [43]. Thorough treatments are given in [12,17].) However, in this manuscript we give a royal road to the multi-variable Löwner theorem in noncommutative function theory: to bootstrap from the one variable Löwner theorem itself. The purpose of the present quest is to give a general regime for turning one variable theorems in the intersection of classical complex analysis and operator theory into theorems in multiple noncommuting variables using a so-called "royal road theorem" built on the absolute and supreme powers of several complex variables and convexity. We use this "royal road" to prove the analogues of the celebrated theorems of Löwner [26] and Kraus [28] in the multivariable setting as mere examples of a very general analytic technique. (The multivariable Löwner theorem has been established in many settings. In commuting variables, see [3,37]. In noncommuting variables, see [34,39], culminating in essentially the most general framework in [36], which we reprove here using the "royal road" as a shortcut. Convexity theorems are somewhat less generally developed [18,19,22,24,23,20,34].)

Matthew Kennedy gave a talk at the Fields Institute in June of 2019 on recent work with Kenneth Davidson on noncommutative Choquet theory [16]. Prominent in the theory was the role of the matrix convex function. The merit of matrix convex functions was appreciated essentially on the level of classically convex functions. However, as there is a great gulf between positive and completely positive maps, so too should there be between convex and matrix convex functions, as was first discovered by Kraus [28]. In light of the recent progress with respect to the related topic of matrix monotonicity, it seemed clear here that automatic analyticity should hold, and for reasons arising more from complex analysis and the one variable theorem than an artisanal approach starting from scratch. This provided additional motivation for the current endeavor.

1.1. *The classical theorems*

Let $f : (a, b) \to \mathbb{R}$ be a function. We say that f is **matrix monotone** if

$$A \leq B \Rightarrow f(A) \leq f(B)$$

for all A, B self-adjoint of the same size with spectrum in (a, b), where $A \leq B$ means that $B - A$ is positive semidefinite. (The function f is evaluated via the matrix functional calculus.) This evidently innocuous condition is in fact very rigid, as is codified in Löwner's theorem.

Theorem 1.1 *(Löwner 1934). Let $f : (a, b) \to \mathbb{R}$. f is matrix monotone if and only if f is real analytic on (a, b) and analytically continues to the upper half plane in $\mathbb{C}$ as a map into the closed upper half plane.*

For example, the functions x, $\log x$, $\sqrt{x}$, $\tan x$, and $-x^{-1}$ are all matrix monotone on intervals in their domains, but e^x, x^3, and $\sec x$ are not. Note that matrix monotonicity is a geometric property; matrix monotonicity on a single interval implies matrix monotonicity on any interval where the function is real-valued in the real domain for analytic functions. Löwner's theorem arises in many contexts, including mathematical physics [46,45]. Other applications are found, for example, in quantum data processing [4], wireless communications [25,14], matrix means [6] and systems theoretic interpolation problems [5,33].

Nevanlinna [32,30] showed that all such functions on the unit interval are of the form

$$f(x) = a + \int_{[-1,1]} \frac{x}{1+tx}\, d\mu(t)$$

for $a \in \mathbb{R}$ and μ a finite measure supported on $[-1, 1]$. The Nevanlinna representation tells us exactly how to analytically continue a function to the upper half plane.

Let $f : (a, b) \to \mathbb{R}$ be a function. We say that f is **matrix convex** if

$$f\left(\frac{A+B}{2}\right) \leq \frac{f(A)+f(B)}{2}$$

for all A, B self-adjoint with spectrum in (a, b). Löwner's student Kraus proved the following theorem, which is ostensibly more technical, but demonstrates the same essential rigidity.

Theorem 1.2 *(Kraus 1937). Let $f : (-1, 1) \to \mathbb{R}$. f is matrix convex if and only if*

$$f(x) = a + bx + \int_{[-1,1]} \frac{x^2}{1+tx}\, d\mu(t)$$

where $a, b \in \mathbb{R}$ and μ is a finite measure supported on $[-1, 1]$. Note that all such functions analytically continue to the upper half plane.

For example, x^2 is matrix convex, but x^4 is not.

1.2. Free noncommutative function theory

Let R be a real topological vector space. Define the **matrix universe over** R, denoted by $\mathcal{M}(R)$, by

$$\mathcal{M}(R) = \bigcup_{n\in\mathbb{N}} M_n(\mathbb{C}) \otimes_{\mathbb{R}} R,$$

where $M_n(\mathbb{C})$ is the space of n by n matrices over $\mathbb{C}$. The space $\mathcal{M}(R)$ is endowed with the disjoint union topology. Given $V \subset \mathcal{M}(R)$, denote by V_n the set $V \bigcap M_n(\mathbb{C}) \otimes R$. Define the **Hermitian matrix universe over** R, denoted by $\mathcal{S}(R)$, to be

$$\mathcal{S}(R) = \bigcup_{n \in \mathbb{N}} S_n(\mathbb{C}) \otimes_{\mathbb{R}} R,$$

where $S_n(\mathbb{C})$ denotes the space of n by n Hermitian matrices.

A set $G \subset \mathcal{M}(R)$ is defined to be a **(free) domain** if it satisfies the following axioms:

(1) If $X \in G_m$ and $Y \in G_m$ then $X \oplus Y \in G_{n+m}$,
(2) $X \in G_n \Rightarrow U^*XU \in G_n$ for all n by n unitaries U over $\mathbb{C}$
(3) G_n is open for all n.

Let $G \subset \mathcal{M}(R_1)$ be a free domain. We say a function $f : G \to \mathcal{M}(R_2)$ is a **free function** if

(1) $f|_{G_n}$ maps into $\mathcal{M}(R_2)_n$,
(2) $f(X \oplus Y) = f(X) \oplus f(Y)$,
(3) $U^*f(X)U = f(U^*XU)$ for all n by n unitary U over $\mathbb{C}$.

(We note that whenever f is complex analytic, it satisfies a stronger version of the unitary relation - namely, it preserves arbitrary similarities, and therefore agrees with the notion of a free function appearing in other contexts [27,39,44].)

If R is a real operator system – that is, a real subspace containing 1 of self-adjoint elements in a C^*-algebra - then for each n there is a natural ordering on $S_n(\mathbb{C}) \otimes R$, since matrices over R are elements of a larger C^*-algebra. (The Choi-Effros Theorem [15] gives that any abstract Archimedean matrix ordering in a very general sense is equivalent to this situation. That is, this is the most general setup.) Given $A, B \in S_n(\mathbb{C}) \otimes R$, we say $A \leq B$ if $B - A$ is positive semidefinite as an element of $S_n(\mathbb{C}) \otimes R$.

Let R be a complex operator system, a complex subspace of a C^*-algebra containing 1 which is closed under involution. Let R_{sa} be the set of self-adjoint elements in R. We overload the notation for $\mathcal{S}(R)$ and $\mathcal{M}(R)$ by defining

$$\mathcal{S}(R) := \mathcal{S}(R_{sa}), \mathcal{M}(R) := \mathcal{M}(R_{sa}).$$

Given R_1 and R_2 real operator systems and a domain $G \subseteq \mathcal{S}(R_1)$, we say that a free function $f : G \to \mathcal{S}(R_2)$ is **matrix monotone** if

$$A \leq B \Rightarrow f(A) \leq f(B)$$

whenever A and B have the same size. We say a domain $G \subseteq \mathcal{S}(R_1)$ is **convex** if each G_n is convex. For a convex domain $G \subseteq \mathcal{S}(R_1)$, say that a free function $f : G \to \mathcal{S}(R_2)$ is **matrix convex** if

$$f\left(\frac{A+B}{2}\right) \leq \frac{f(A)+f(B)}{2}$$

for all pairs $A, B \in G$ of the same size.

Define the **upper half plane** $\Pi(R) = \{X \in \mathcal{M}(R) | \text{Im } X > 0\}$, where $\text{Im } X = (X - X^*)/2i$, and $A > B$ if the difference is strictly positive definite – that is, the difference is self-adjoint and its spectrum is a subset of $(0, \infty)$. For a convex domain $G \subseteq \mathcal{S}(R)$, define the **tube over** G to be the set

$$T(G) = \{X + iY | X \in G \text{ and } Y = Y^*\}.$$

In several commuting variables, generalizations of Löwner's theorem appear in [3,37]. The proofs are technical and involved, and rely heavily on commutative Hilbert space operator theoretic techniques arising from applications of transfer function realization theory from systems engineering to several complex variables as originated in [1]. The difficulty is a symptom of the fact that the variety of commuting tuples of matrices is full of holes – that is, it is not convex and, thus, unnatural for understanding monotonicity. By contrast, the machinery of several complex variables is apparently much more natural in the noncommutative setting. Noncommutative analogues of Löwner's theorem have previously been established in [39,34]. The culmination of this work appears in [36], where the following theorem was proved in perhaps the highest level of generality that one should expect (although that proof relies on the commuting theorem in [3] and is thus "unnatural").

Theorem 1.3 *(Theorem 1.2, Pascoe [36]). Let R_1 and R_2 be closed real operator systems. Let $G \subseteq \mathcal{S}(R_1)$ be a convex free domain. A function $f : G \to \mathcal{S}(R_2)$ is matrix monotone if and only if f is real analytic on G and analytically continues to $\Pi(R_1)$ as a map into $\overline{\Pi(R_2)}$.*

We give a new proof of this result as Theorem 5.1 using the "royal road".

We note two important examples of matrix monotone functions. The Schur complement $X_{11} - X_{12}X_{22}^{-1}X_{21}$ gives a matrix monotone function on the set $D \subset \mathcal{S}(S_2(\mathbb{C}))$, the space of block 2 by 2 self-adjoint matrices, where X_{22}^{-1} is defined [31]. Another class of examples is the matrix geometric means, originating in mathematical physics [41], which in two variables is given by the formula $X_1^{1/2}(X_1^{-1/2}X_2X_1^{-1/2})^{1/2}X_1^{1/2}$ defined on pairs of positive matrices in $\mathcal{S}(\mathbb{R}^2)$ [29,13,7].

Analogues of Kraus's theorem are less general. One example is the so-called "butterfly realization" developed in [21] for noncommutative rational functions, which captures the essence of the classical case.

Theorem 1.4 *(Theorem 3.3, Helton, McCullough, Vinnikov [21]). Let $r : G \subset \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R})$ denote a noncommutative rational function on a domain G containing 0. If r is matrix convex near 0, then r has a realization of the form*

$$r(X) = r_0 + L(X) + \Lambda(X)^*(1 - \Gamma(X))^{-1}\Lambda(X)$$

for a scalar r_0*, a real linear function* L*,* Λ *affine linear, and* $\Gamma(X) = \sum A_i \otimes X_i$ *for self-adjoint matrices* A_i*.*

We prove the butterfly realization holds for general matrix convex functions in Corollary 4.5. Our butterfly type realization considerably extends and simplifies the original total art in [21].

1.3. The royal road theorem

The main result of the paper is contained in Section 3. It establishes that any class of real free noncommutative functions which consist of locally bounded functions which are analytic on one-dimensional slices in a controlled way and closed under some basic algebraic and analytic procedures are automatically analytic. We call such a class of functions a **sovereign class**. The class of matrix monotone functions and the class of matrix convex functions are each sovereign classes. Once we know such functions are real analytic, algebraic and functional analytic techniques allow us to obtain nice formulas for these functions. The content of our main theorem, Theorem 3.4, states the following:

"Any function in a sovereign class is real analytic".

1.4. Structure of the paper

In Section 2, we discuss analytic continuation in the operator system setting. In Section 3, we describe the structure of the domain and function classes under consideration, the so-called sovereign functions, and show that matrix monotone and matrix convex functions are examples. We also prove the "royal road" theorem, the main engine of the machine under construction, which asserts that sovereign functions are automatically real analytic. In Section 4, we prove analogues of the classical Löwner and Kraus realizations. In Section 5, we show that, in analogy with the classical case, we can deduce analytic continuations from the Löwner and Kraus realizations using the machinery of automatic analyticity in classes of sovereign functions established in Section 3.

2. Prelude: the quantitative wedge-of-the-edge theorem

One of the key notions in the classical and several variable generalizations of the Löwner and Kraus theorems is that of analytic continuation - that is, typically we are interested in extending functions from a "real" domain to some subset of a "complex" set. The edge-of-the-wedge theorem (proven by Bogoliubov and treated by Rudin in a series of lectures [42]) is useful in showing that such a continuation exists. Extremely flexible generalizations of this result to several variables, the so-called "wedge-of-the-edge" theorems, have appeared in [38,35]. Rather than considering the analytic continuation of functions defined on a multi-variate upper half plane (the wedge) through the real boundary (the edge) as in the edge-of-the-wedge theorem, the wedge-of-the-edge theorem concerns analytic continuation within the edge. The key lemma from [38] follows, which we will need to generate quantitative bounds. In this section, we prove a version of the wedge-of-the-edge theorem in the operator system setting.

Lemma 2.1 *(Lemma 2.3, Pascoe [38]). Fix n. Fix $p > 0$. There are constants $J, K > 0$ such that for every measurable $S \subseteq [0,1]^n$ of measure greater than p, and homogeneous polynomial h of degree d in n variables which is bounded by 1 on S, $|h(z)| \leq KJ^d\|z\|^d_\infty$.*

Such an assertion seems foolish, but it is essentially the product of Lagrange interpolation, blind faith, and elbow grease.

Let R_1, R_2 be vector spaces. Define a (noncommutative) **generalized homogeneous polynomial of degree** d to be a (free) function on R_1 such that the restriction to any finite dimensional space is an R_2-valued (noncommutative) homogeneous polynomial of degree d.

Lemma 2.2. *There are universal constants $L, K > 0$ satisfying the following. Let R be an operator system. Let W be the set of positive contractions in R ($\mathcal{S}(R)$ in the noncommutative case). Let h be a (noncommutative) generalized homogeneous polynomial of degree d which is norm bounded by 1 on W. Then, for every $Z \in R$, ($\mathcal{M}(R)$ in the noncommutative case), $\|h(Z)\| \leq KL^d\|Z\|^d$.*

Proof. It is enough to prove the claim when $\|Z\| = 1$, as both sides are homogeneous of degree d. Write $Z = A - B + iC - iD$ for positive A, B, C, D, where the norms of A, B, C, D are less than $2\|Z\|$. The function of four variables $f(x_1, x_2, x_3, x_4) = h((x_1A + x_2B + x_3C + x_4D)/8)$ satisfies the preceding lemma when composed with any norm 1 linear functional for $S = [0,1]^4$, so, by the Hahn-Banach theorem, $\|h(Z)\| = \|f(8, -8, i8, -i8)\| \leq KJ^d 8^d$. □

Let $X \in \mathcal{M}(R)_n$. Define the **complex ball around** X **of radius** ε**,** denoted $B_{\mathbb{C}}(X, \varepsilon)$, to be

$$B_{\mathbb{C}}(X,\varepsilon) = \bigcup_m \{Y \in \mathcal{M}(R)_{mn} | \left\|X^{\oplus m} - Y\right\| < \varepsilon\}.$$

Let $X \in \mathcal{S}(R)_n$. Define the **real ball around** X **of radius** ε**,** denoted $B_{\mathbb{R}}(X, \varepsilon)$ to be

$$B_{\mathbb{R}}(X,\varepsilon) = \bigcup_m \{Y \in \mathcal{S}(R)_{mn} | \left\|X^{\oplus m} - Y\right\| < \varepsilon\}.$$

The following corollary follows immediately from the preceding lemma.

Corollary 2.3 *(The quantitative wedge-of-the-edge theorem). There are universal constants $\delta, \varepsilon > 0$ satisfying the following. Let R be an operator system. Let W be the set of* *positive contractions in R ($\mathcal{S}(R)$ in the noncommutative case). Let h_d be a sequence of (noncommutative) generalized homogeneous polynomials such that h_d has degree d and $\sum_{d=0}^{\infty} \|h_d(X)\|$ is bounded by 1 on W. The formula $\sum_{d=0}^{\infty} h_d(Z)$ defines a (noncommutative) analytic function on $B_{\mathbb{C}}(0, \delta)$ which is bounded by ε.*

3. Automatic analyticity in sovereign classes

Let $G \subseteq \mathcal{M}(R)$. We define the **coordinatization** of G, denoted $G^{(n)}$, to be the natural inclusion of $(G_{mn})_{m=1}^{\infty}$ into $\mathcal{M}(R \otimes M_n(\mathbb{C}))$. Given a free function f defined on G, we define the **coordinatization** of f, denoted $f^{(n)}$, to be the natural map on $G^{(n)}$ defined by $f^{(n)}(X) = f(X)$.

Let a **dominion** $\Join$ be a class of domains satisfying:

Domains: For each $G \in \Join$, there is an operator system R such that $G \subseteq \mathcal{S}(R)$. (Note that we want to allow $\Join$ to be a proper class and contain domains over all operator systems so that we can prove theorems for functions defined over all operator systems simultaneously.)

Translation invariance: For all $G \in \Join$ such that $G \subseteq \mathcal{S}(R)$ for some operator system R and $r \in R$, $G + r \in \Join$.

Closure under intersection: For all $G, H \in \Join$ such that $G, H \subseteq \mathcal{S}(R)$ for some operator system R, $G \cap H \in \Join$.

Closure under coordinatization: If $G \in \Join$, then $G^{(n)} \in \Join$.

Locality: Let $G \in \Join$. For any $X \in G_1$, there is an $\epsilon > 0$ such that $B_{\mathbb{R}}(X, \varepsilon) \subseteq G$ and $B_{\mathbb{R}}(X, \varepsilon) \in \Join$.

Scale invariance: If $t > 0$ and $G \in \Join$, $tG \in \Join$.

We say that a domain G is **convex** if each G_n is convex. An example of a dominion is the class of all open convex domains, which we denote Conv.

A **sovereign class** is a class of functions $\wr$ on domains contained in a dominion $\Join$ satisfying:

Functions: For every $f \in \wr$, there is a $G \in \Join$ and an operator system R such that $f : G \to \mathcal{M}(R)$ is a free function. For all $G \in \Join$, $\wr(G) \subseteq F(G)$, where $\wr(G)$ denotes the functions in $\wr$ on the domain G and $F(G)$ denotes the class of free functions on G.

Local boundedness: Each $f \in \wr$ is locally bounded and measurable on finite dimensional affine subspaces on each level. That is, if $G \subset \mathcal{S}(R)$, $f \in \wr(G)$, and V is a finite dimensional vector subspace of $\mathcal{S}(R)_n$, then $f|_{V \cap G_n}$ is locally bounded and measurable.

Closure under localization: If $f \in \wr(G)$ and $H \subseteq G$ and $H \in \Join$ then $f|_H \in \wr(H)$.

Translation invariance: If $f \in \wr(G)$ then $f(Z - r) \in \wr(G + r)$ for every self-adjoint $r \in R$.

Closure under coordinatization: If $f \in \wr(G)$, then $f^{(n)} \in \wr(G^{(n)})$.

Convexity: The set of functions $\wr(G)$ taking values in $\mathcal{S}(R)$ is convex and closed under pointwise weak limits. (Note that this implies closure under convolution when R is finite dimensional in combination with the property of translation invariance.)

One-variable knowledge: Let $f \in \wr(G)$. Let $A, B \in G_1$. If $A \leq B$ then

$$f_{\overline{AB}}(t) := f\left(\frac{1-t}{2}A + \frac{1+t}{2}B\right) \qquad (-1 < t < 1)$$

analytically continues to the open unit disk $\mathbb{D}$ as a function of t.

Control: There is a map γ taking each pair (X, f), where $f \in \wr(G)$ for some $G \in \bowtie$ and $X \in G$, to a non-negative number or $+\infty$ satisfying:

(1) For each $\varepsilon > 0$ there is a universal constant $c(\varepsilon)$ such that $\inf_{X \in B_{\mathbb{R}}(X_0,\varepsilon)_1} \gamma(X, f) \leq c(\varepsilon)\|f\|_{B_{\mathbb{R}}(X_0,\varepsilon)_1}$ whenever f is defined on all of $B_{\mathbb{R}}(X_0, \varepsilon)_1$. (We set $\|f\|_{B_{\mathbb{R}}(X_0,\varepsilon)_1} = \sup_{B_{\mathbb{R}}(X_0,\varepsilon)_1} \|f\|$. The definition here is rigged so that near any point X there is a point with good "control".)

(2) There is a universal positive valued function e on $\mathbb{R}^+$ satisfying the following. Write $f_{\overline{AB}}(t) = \sum a_n t^n$. Then

$$\|a_n\| \leq \gamma\left(\frac{A+B}{2}, f\right) e(\|B - A\|).$$

(Note that, if the class is closed under composition with positive, norm one, linear functionals λ, and $\gamma(X, \lambda(f)) \leq \gamma(X, f)$, it is sufficient to check properties (1) and (2) when $R_2 = \mathbb{R}$ by the Hahn-Banach theorem.)

(3) If $H \subseteq G$ and $X \in H$ then $\gamma(X, f|_H) = \gamma(X, f)$.

(4) $\gamma(X, f) = \gamma(X^{\oplus N}, f)$.

We consider two specific sovereign classes: monotone functions, and convex functions on the dominion Conv.

We define the **positive-orthant norm of the n-th derivative at X**, denoted $\|D^n f(X)\|_+$, to be

$$\|D^n f(X)\|_+ = \sup_{\|H\|=1, H>0, m} \|D^n f(X^{\oplus m})[H]\|,$$

where $D^n f(X)[H] = \frac{d^n}{dt^n} f(X + tH)\Big|_{t=0}$. If the n-th derivative does not exist in some positive direction, we formally set $\|D^n f(X)\|_+ = \infty$.

Proposition 3.1. *The matrix monotone functions on domains in* Conv *are a sovereign class.*

Proof. Monotone functions are functions. To see local boundedness, note that $f(X+1)$ and $f(X-1)$ bound $f(X+H)$ for all $\|H\|<1$. That is, as

$$X-1 \leq X+H \leq X+1,$$

monotonicity implies

$$f(X-1) \leq f(X+H) \leq f(X+1).$$

The restriction of a monotone function to a convex set remains a monotone function. Likewise, coordinatization preserves monotonicity as does translation. That the monotone functions on a domain G taking values in $\mathcal{S}(R)$ are a convex set follows from the fact that the defining inequality for monotonicity is linear.

We now consider one variable knowledge. Let $A \leq B$, $A, B \in G_1$. Consider the function $f_{\overline{AB}}(t) = f(\frac{1-t}{2}A + \frac{1+t}{2}B)$. For each positive norm one linear functional λ, we have that $\lambda \circ f_{\overline{AB}}$ is matrix monotone, and therefore by Löwner's theorem

$$\lambda \circ f_{\overline{AB}}(t) = \sum a_n^\lambda t^n$$

defines a function on a neighborhood of the closed unit disk. Define

$$\Delta^n(t) = \frac{\sum_{k=0}^n (-1)^{n-k}\binom{n}{k} f_{\overline{AB}}(kt)}{t^n}.$$

Note that

$$a_n^\lambda = \frac{1}{n!}\lim_{t\to 0} \lambda \circ \Delta^n(t).$$

Therefore,

$$\sup_{t\in[0,1]\setminus\{0\}} |\lambda \circ \Delta^n(t)| < \infty,$$

and by the principle of uniform boundedness,

$$\sup_{t\in[0,1]\setminus\{0\}} \|\Delta_n(t)\| < \infty.$$

(This is not precisely the typical formulation of the principle of uniform boundedness; see Corollary 6.2 in the Appendix.) Note that because Δ_n is bounded, $f_{\overline{AB}}$ is $(n-1)$-times differentiable, and therefore we merely need to show the convergence of the series $\sum a_n t^n$, where $a_n = f^{(n)}(0)/n!$. Now, it suffices to show that the sequence $(a_n)_{n=0}^\infty$ is bounded. If not, there will be a positive linear functional such that $(\lambda(a_n))_{n=0}^\infty$ is unbounded, again by the principle of uniform boundedness, which contradicts the convergence of the series for $\lambda \circ f_{\overline{AB}}(t)$. Therefore, we obtain one variable knowledge.

Let

$$\gamma(X, f) = \|f(X)\| + \|Df(X)\|_+ .$$

We will show that γ is a control function for the class of matrix monotone functions. Note that if λ is a positive norm one linear functional, then $\lambda(f)$ is again matrix monotone. (Namely, λ is completely positive.) So without loss of generality, it suffices to consider $f : G \to \mathcal{S}(\mathbb{R})$. Fix $\varepsilon > 0$. Suppose that $B_{\mathbb{R}}(X, \varepsilon)$ is contained in the domain of f. Without loss of generality, $0 = X$. Fix $H \geq 0$ in $B_{\mathbb{R}}(0, \varepsilon)$. So $f(zH)$ has a Nevanlinna type representation given by

$$\begin{aligned} f(zH) &= a_0 + \int_{[-1,1]} \frac{z}{tz+1}\, d\mu(t) \\ &= a_0 + z \sum_{i=0}^{\infty} \int t^i z^i \, d\mu(t). \end{aligned}$$

Therefore, $f(zH) = a_0 + \sum_{n=1}^{\infty} a_n z^n$ where $a_n = \int t^{n-1} \mathrm{d}\mu(t)$ when $n \geq 1$. Note that $|a_n| \leq \int |t|^{n-1}\, d\mu(t) \leq \int d\mu(t) = a_1$ for $n \geq 1$. Moreover,

$$f(zH) - f(-zH) = 2z \sum_{i=0}^{\infty} \int z^{2i} t^{2i} \, d\mu.$$

This shows that

$$\|Df(0)[H]\| \leq \|f\|_{B_{\mathbb{R}}(0,\varepsilon)} .$$

Therefore,

$$\|Df(0)\|_+ \leq \frac{1}{\varepsilon} \|f\|_{B_{\mathbb{R}}(0,\varepsilon)} ,$$

and thus $\gamma(X, f)$ is bounded by $(1 + \frac{1}{\varepsilon}) \|f\|_{B_{\mathbb{R}}(0,\varepsilon)}$. That is, in the definition of control function, $c(\varepsilon) = 1 + \frac{1}{\varepsilon}$. For this class, taking the infimum in property (1) is redundant. Property (2) follows from the fact that $|a_n| \leq a_1$ in the above argument after a change of variables, where the scaling factor is captured by e. The last control two properties follow from the definition of γ. □

Proposition 3.2. *The locally bounded matrix convex functions on domains in* Conv *are a sovereign class.*

Proof. Convex functions are functions. Note that we have assumed local boundedness. (This assumption cannot be dropped as all linear functions are convex, including the unbounded ones.) The restriction of a convex function to a subdomain remains convex.

The coordinatization of a convex function is convex, as is a translation. Convexity follows from the fact that the defining inequality for matrix convexity is linear. By the Kraus theorem, these functions satisfy one variable knowledge via an argument similar to Proposition 3.1.

Let

$$\gamma(X,f) = \|f(X)\| + \|Df(X)\|_{+} + \left\|D^2 f(X)\right\|_{+}.$$

Note that if λ is a positive norm one linear functional, then $\lambda(f)$ is again convex. (Namely, λ is completely positive.) So without loss of generality, it suffices to consider $f : G \to \mathcal{S}(\mathbb{R})$. We will show that γ is a control function for the class of matrix convex functions. Fix $\varepsilon > 0$. Suppose that $B_{\mathbb{R}}(X,\varepsilon)$ is contained in the domain of f. Without loss of generality, $0 = X$. Fix H in $B_{\mathbb{R}}(0,\varepsilon)$. The function $f(zH)$ has a Kraus type representation

$$f(zH) = a + bz + \int\limits_{[-1,1]} \frac{z^2}{tz+1}\, d\mu(t).$$

Therefore,

$$f(zH) = a + bz + z^2 \sum_{n=0}^{\infty} \int\limits_{[-1,1]} t^n z^n \, d\mu(t).$$

Writing $f(zH) = \sum a_i z_i$, we have that $a_n = \int_{[-1,1]} t^{n-2} \mathrm{d}\mu(t)$ for $n \geq 2$. Note that $|a_n| \leq \int |t|^{n-2}\, d\mu(t) \leq \int_{[-1,1]} d\mu(t) = a_2$ for $n \geq 2$.

We have

$$f(zH) + f(-zH) - 2f(0) = 2z^2 \sum_{n=0}^{\infty} \int z^{2n} t^{2n}\, d\mu(t).$$

This shows that

$$\left\|D^2 f(0)[H]\right\| \leq 4\, \|f\|_{B_{\mathbb{R}}(0,\varepsilon)}.$$

Therefore,

$$\left\|D^2 f(0)\right\|_{+} \leq \frac{1}{\varepsilon^2}\, \|f\|_{B_{\mathbb{R}}(0,\varepsilon)}.$$

Set $M = \|f\|_{B_{\mathbb{R}}(0,\varepsilon)}$.

Then,

$$|bz| - \left|\sum_{n=2}^{\infty} a_n z^n\right| \leq \left|bz + \sum_{n=2}^{\infty} a_n z^n\right|$$

$$= |f(zH) - f(0)|$$
$$\leq 2M,$$

which gives

$$\begin{aligned}
|b| &\leq \left| z \sum_{n=0}^{\infty} a_{n+2} z^n \right| + \frac{2M}{|z|} \\
&\leq \left| z \sum_{n=0}^{\infty} a_2 z^n \right| + \frac{2M}{|z|} \\
&\leq \left| \frac{z}{1-z} \right| \left| D^2 f(0)[H] \right| + \frac{2M}{|z|} \\
&\leq \left| \frac{z}{1-z} \right| 4M + \frac{2M}{|z|}.
\end{aligned}$$

Pick $z = \frac{1}{2}$. Then

$$\|Df(0)[H]\| = \|b\| \leq 8M.$$

Therefore,

$$\|Df(0)\|_+ \leq \frac{1}{\varepsilon} 16M.$$

Thus, combining these estimates, γ is a control function with the remaining properties following similarly to the previous argument in Proposition 3.1. □

We note that any matrix convex function on a finite dimensional space will be continuous and thus locally bounded. Some sort of topological restriction, such as local boundedness, is necessary, as arbitrary linear maps on any operator system are not necessarily bounded but are definitely convex, as all linear functions are convex.

Lemma 3.3. *Any function in a sovereign class is real analytic at each level on each finite dimensional affine subspace.*

Proof. Without loss of generality, we will assume R_1 is finite dimensional. Fix $X \in G_n$. Without loss of generality, $0 = X \in G_1$ by closure under coordinatization and translation. Also without loss of generality, assume that $B_{\mathbb{R}}(X, 2) \subset G$ by locality of the dominion. Let φ be a compactly supported positive smooth function on R_1 such that $\int_{R_1} \phi(r) \mathrm{d}r = 1$. Define $\varphi_\alpha(x) = \frac{1}{\alpha^m} \varphi\left(\frac{1}{\alpha} x\right)$ where $m = \dim R_1$. Consider

$$f_\alpha(Y) = (\varphi_\alpha * f)(Y) = \int_{R_1} f(Y - r)\varphi_\alpha(r)\mathrm{d}r.$$

As a sovereign class of functions is closed under convolution, for any fixed $\varepsilon > 0$, the function f_α will well defined on the domain $B_{\mathbb{R}}(X, 2-\varepsilon)$ for small enough α and so it will again be in the class on that domain. Let $\delta < 1$ be the constant occurring in the quantitative wedge-of-the-edge theorem. Note that

$$\lim_{\alpha\to 0} \|f_\alpha\|_{B_{\mathbb{R}}(X,\delta/2)_1} \leq \|f\|_{B_{\mathbb{R}}(X,\delta)_1}.$$

Therefore, for α sufficiently small,

$$\|f_\alpha\|_{B_{\mathbb{R}}(X,\delta/2)_1} \leq 2\|f\|_{B_{\mathbb{R}}(X,\delta)_1}.$$

For small enough α, choose $Y \in B_{\mathbb{R}}(X, \delta/2)_1$ such that

$$\gamma(Y, f_\alpha) \leq 2c(\delta/2)\|f_\alpha\|_{B_{\mathbb{R}}(X,\delta/2)_1} \leq 4c(\delta/2)\|f\|_{B_{\mathbb{R}}(X,\delta)_1}.$$

(Such a Y exists by control property (1)). Note that $f_\alpha|_{B_{\mathbb{R}}(X,2-\varepsilon)_1}$ is smooth at Y and by the one variable knowledge $f_\alpha(Y+Z) = \sum_{d=0}^\infty h_d(Z)$ on positive contractions in R_1, where each h_d is a homogenous polynomial of degree d. By the control properties, we see that $\sum_{d=0}^\infty \|h_d(Z)\|$ is bounded by some M on the positive contractions as we have uniform bounds on the Taylor coefficients by control property (2), and therefore by the quantitative wedge-of-the-edge theorem, f_α continues to a function bounded by $M\varepsilon$ on $B_{\mathbb{C}}(Y, \delta)_1$. Therefore, f extends analytically and is bounded by $M\varepsilon$ on $B_{\mathbb{C}}(Y, \delta)_1$ by a normal families argument. As $B_{\mathbb{C}}(X, \delta/2)_1 \subseteq B_{\mathbb{C}}(Y, \delta)_1$, we are done. □

Let $G \subset \mathcal{S}(R_1)$ be a real domain. Let $f : G \to \mathcal{M}(R_2)$. Fix $X \in G_n$. We say that a free function f is **real analytic at** X if there is a $\delta > 0$ such that for any choice of $H_1, \ldots, H_k$, the induced free function has a power series expansion

$$f(X + \sum H_i t_i) = \sum a_\alpha t^\alpha \text{ for all } \left\|\sum H_i t_i\right\| < \delta.$$

That is, the function is real analytic on finite dimensional slices. Equivalently, $f^{(n)}(X + Y) = \sum h_j(Y)$ is convergent on $B_{\mathbb{C}}(X, \delta)$ for some noncommutative generalized homogeneous polynomials h_j.

Theorem 3.4 *(The royal road theorem). Any function in a sovereign class is real analytic.*

Proof. Fix $X \in G_n$. Without loss of generality, $0 = X \in G_1$ by closure under coordinatization and translation. Also without loss of generality, assume that $B_{\mathbb{R}}(X, 1) \subset G$. Therefore, since f is real analytic at each level by Lemma 3.3, $f(X) = \sum h_d(X)$ for some noncommutative homogenous generalized polynomials h_d on the set of positive contractions in $\mathcal{S}(R_1)$. Moreover, the series is bounded on smaller balls by the control properties (2) and (4), as we have uniform bounds on the Taylor coefficients on each positively oriented one dimensional slice. Thus, by the noncommutative quantitative wedge-of-the-edge theorem, the function f must be bounded and analytic on $B_{\mathbb{C}}(X, \delta)$ for some $\delta > 0$. This establishes the claim. $\square$

4. Realizations and the Kraus theorem

We adopt the (by now standard) Helton convention of suppressing tensor notation for products of operators A and noncommutative indeterminants x_i; that is, we write Ax_i for $A \otimes x_i$.

In the following section, we will usually assume that $R_1 = \mathbb{R}^d$ and always that R_2 is contained in some concrete $\mathcal{B}(\mathcal{K})$. We will frequently use free noncommutative power series of the form

$$f(Z) = \sum_{\alpha} c_\alpha Z^\alpha,$$

where α runs over all words in the formal noncommuting letters $x_1, \ldots, x_d$, where the empty word will be denoted by 1. (Words are the natural multi-indices in the noncommutative setting.) Various series representations can be derived via model-realization theory [27,9,10,2,11] with many results for the homogenous expansion.

4.1. Monotonicity

The following lemma is essentially [39, Theorem 4.16] lifted to the multi-dimensional output setting. The lemma establishes the positivity of certain infinite block matrices assembled from the coefficients of our function, which will be later used in a Gelfand-Naimark-Segal type construction to obtain our representation formulae. Define the x_i-**localizing matrices**, denoted C_i, via the following formula:

$$C_i = [c_{\beta^* x_i \alpha}]_{\alpha,\beta}$$

where α, β range over all monomials. Note that in one variable, the monomials can be indexed by the non-negative integers, and the single localizing matrix is actually a Hankel matrix whose positivity was established in Nevanlinna's solution to the Hamburger moment problem [32].

Lemma 4.1. *Suppose that $f(X) = \sum c_\alpha X^\alpha$ is analytic on $B_{\mathbb{C}}(0, 1) \subseteq \mathcal{S}(\mathbb{R}^d)$ and that f is matrix monotone. For each $i = 1, \ldots, d$, the x_i-localizing matrices (with operator entries) satisfy*

$$C_i = [c_{\beta^* x_i \alpha}]_{\alpha,\beta} \geq 0$$

where α, β range over all monomials.

Proof. Note

$$Df(X)[H] = \sum_{\alpha,\beta,i} c_{\beta^* x_i \alpha} X^{\beta^*} H_i X^{\alpha}.$$

We can write

$$Df(X)[H] = \sum_i (I_{\mathcal{K}} \otimes K_X)^* (C_i \otimes H_i)(I_{\mathcal{K}} \otimes K_X)$$

where K_X is the vector-valued free function $(X^\alpha)_\alpha$. Taking $H_i = vv^*$, and the rest zero then defining a vector-valued function $K_X^v(w) = (I_{\mathcal{K}} \otimes (v^* X^\alpha)_\alpha)w$ for v, X and compatible vector w, we see, by monotonicity, that $K_X^v(w)^* C_i K_X^v(w) \geq 0$. So it suffices to show that the range of $K_X^v(w) = (I_{\mathcal{H}} \otimes (v^* X^\alpha)_\alpha)w$ is dense. It is an elementary exercise to show that their span is dense, say by viewing the ambient setting as a kind of reproducing kernel Hilbert space. (See, for example, [39, Proposition 3.9].) Therefore, it is sufficient to show that the range is closed under taking sums. One checks that

$$K_{X_1}^{v_1}(w_1) + K_{X_2}^{v_2}(w_2) = K_{X_1 \oplus X_2}^{v_1 \oplus v_2}(w_1 \oplus w_2).$$

So, we are done. □

Theorem 4.2. *Let f be a matrix monotone function whose power series converges absolutely and uniformly on $B_{\mathbb{C}}(0, 1+\varepsilon) \subseteq \mathcal{S}(\mathbb{R}^d)$. Let $\mathcal{H}_i$ be the Hilbert space equipped with the inner product*

$$\langle \alpha \otimes v, \beta \otimes w \rangle_{\mathcal{H}_i} = w^* c_{\beta^* x_i \alpha} v.$$

Let $\mathcal{H} = \oplus \mathcal{H}_i$ and P_i be the projection onto $\mathcal{H}_i$. Note that

$$\langle \alpha \otimes v, \beta \otimes w \rangle_{\mathcal{H}} = \sum_i w^* c_{\beta^* x_i \alpha} v.$$

Define $A : \mathcal{H} \to \mathcal{H}$ by

$$A(\alpha \otimes v) = \sum_i (x_i \alpha) \otimes v,$$

where A is defined to be zero on the orthocomplement of the span of vectors of the form $\alpha \otimes v$. Let Q be the map taking $k \in \mathcal{K}$ to $1 \otimes k \in \mathcal{H}$. The operator A is a bounded self-adjoint contraction on $\mathcal{H}$, and

$$f(Z) = a_0 - Q^*(A - \sum_i P_i Z_i^{-1})^{-1} Q.$$

Proof. To see that A is self-adjoint, compute

$$\begin{aligned}
\langle A(\alpha \otimes v), \beta \otimes w\rangle &= \left\langle \sum_i x_i\alpha \otimes v, \beta \otimes w \right\rangle \\
&= \sum_i \langle x_i\alpha \otimes v, \beta \otimes w\rangle \\
&= \sum_i \sum_j w^* c_{\beta^* x_j x_i \alpha} v \\
&= \sum_j \sum_i w^* c_{\beta^* x_j x_i \alpha} v \\
&= \sum_j \langle \alpha \otimes v, x_j\beta \otimes w\rangle \\
&= \left\langle \alpha \otimes v, \sum_j x_j\beta \otimes w \right\rangle \\
&= \langle \alpha \otimes v, A\beta \otimes w\rangle .
\end{aligned}$$

To see that A is contractive, we will use the fact that

$$\|A\| = \rho(A) = \sup_{\|v\|=1} \sup_{\alpha} \liminf_{n\to\infty} \|A^n \alpha \otimes v\|^{1/n} .$$

Write

$$\begin{aligned}
\|A^n(\alpha \otimes v)\|^2 &= \langle A^n(\alpha \otimes v), A^n(\alpha \otimes v)\rangle \\
&= \left\langle (\sum x_i)^n \alpha \otimes v, (\sum x_i)^n \alpha \otimes v \right\rangle \\
&= \sum_{|\omega|=2n+1} v^* c_{\alpha^*\omega\alpha} v \\
&\leq \sum_{\omega} |v^* c_\omega v| .
\end{aligned}$$

The power series converges uniformly and absolutely on the ball of radius 1, and thus the coefficients are uniformly bounded. This implies that $\rho(A) \leq 1$.

We will now establish that $AP_j(\alpha \otimes v) = x_i\alpha \otimes v$.

$$\begin{aligned}
\langle AP_j(\alpha \otimes v), \beta \otimes w\rangle &= \langle P_j\alpha \otimes v, A\beta \otimes w\rangle \\
&= \left\langle P_j(\alpha \otimes v), \sum_i x_i\beta \otimes w \right\rangle \\
&= \sum_i \langle P_j(\alpha \otimes v), x_i\beta \otimes w\rangle
\end{aligned}$$

$$= \sum_i w^* c_{\beta^* x_i x_j \alpha} v$$
$$= \langle x_j \alpha \otimes v, \beta \otimes w \rangle .$$

We now compute the realization to see that it agrees with f.

$$\begin{aligned} w^* f(Z) v &= \sum_\alpha w^* c_\alpha v Z^\alpha \\ &= w^* c_1 v + \sum_i \sum_\alpha w^* c_{x_i \alpha} v Z^{x_i \alpha} \\ &= w^* c_1 v + \sum_i \sum_\alpha \langle P_i(\alpha \otimes v), 1 \otimes w \rangle Z^{x_i \alpha} \\ &= w^* c_1 v + \sum_i \sum_\alpha \langle P_i (AP)^\alpha (1 \otimes v), 1 \otimes w \rangle Z^{x_i \alpha} \\ &= w^* c_1 v - \left\langle (A - \sum_i P_i Z_i^{-1})^{-1} (1 \otimes v), (1 \otimes w) \right\rangle_{\mathcal{H}} \\ &= w^* c_1 v - w^* Q^* (A - \sum_i P_i Z_i^{-1})^{-1} Q v. \quad \square \end{aligned}$$

We note that, in general, noncommutative Pick functions have representations of the form $a_0 - E((A - Z^{-1})^{-1})$ whenever they are analytic on a neighborhood of 0 and R_1 is a C^*-algebra, where E is a completely positive map [47,40]. The theory of such "Cauchy transforms" is well understood in the context of free probability [8,48].

4.2. Convexity

Lemma 4.3. *Suppose that f is analytic on $B_{\mathbb{C}}(0,1) \subseteq \mathcal{S}(\mathbb{R}^d)$ and that f is matrix convex. The block matrix (with operator entries),*

$$C = [c_{\beta^* \alpha}]_{\alpha, \beta} \geq 0$$

where α, β range over all monomials of degree greater than or equal to 1.

Proof. Note

$$D^2 f(X)[H] = 2 \sum_{\alpha,\beta,\gamma,i,j} c_{\beta^* x_i \gamma x_j \alpha} X^{\beta^*} H_i X^\gamma H_j X^\alpha \geq 0.$$

Under the substitution

$$X \mapsto \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}, H \mapsto \begin{bmatrix} 0 & Xv \\ (Xv)^* & 0 \end{bmatrix},$$

and taking the $1,1$ entry of the above relation, we see that

$$\sum_{\alpha,\beta,i,j} c_{\beta^* x_i x_j \alpha} X^{\beta^* x_i} v v^* X^{x_j \alpha} \geq 0.$$

Therefore, considering the function $K_X^v(w) = (I_{\mathcal{K}} \otimes (v^* X^\alpha)_\alpha) w$ we see again that the range is dense, so we are done. □

The following theorem is related to the "butterfly realization" for noncommutative rational functions in [21].

Theorem 4.4. *Let f be a matrix convex function whose power series converges absolutely and uniformly on $B_{\mathbb{C}}(0, 1+\varepsilon) \subseteq \mathcal{S}(\mathbb{R}^d)$. Let $\mathcal{H}$ be a Hilbert space equipped with the inner product*

$$\langle \alpha \otimes v, \beta \otimes w \rangle = w^* c_{\beta^* \alpha} v$$

where α, β range over all monomials with degree greater than or equal to 1 and v, w range over $\mathcal{K}$. Define the self-adjoint operators T_i by

$$T_i(\alpha \otimes v) = x_i \alpha \otimes v.$$

Let Q_i be the map taking $v \in \mathcal{K}$ to $x_i \otimes v \in \mathcal{H}$. The operators T_i are contractions and

$$f(Z) = a_0 + L(Z) + \left(\sum Q_i Z_i^*\right)^* \left(I - \sum T_i Z_i\right)^{-1} \left(\sum Q_i Z_i\right)$$

for some choice of a_0 and continuous linear function L.

Proof. That the realization formula is equivalent to the function when the T_i are contractions is a standard algebraic manipulation. The nontrivial part of the proof, then, is to show that the T_i are contractive.

We proceed by a spectral radius argument as before.

$$\begin{aligned} \|T_i^n(\alpha \otimes v)\|^2 &= \langle T_i^n \alpha \otimes v, T_i^n \alpha \otimes v \rangle \\ &= v^* c_{\alpha^* x_i^{2n} \alpha} v. \end{aligned}$$

The coefficients must be uniformly bounded, as the power series converges uniformly and absolutely on the ball of radius 1. This completes the proof. □

We remark that the construction of the realization is essentially canonical, and therefore must have maximal domain, (as opposed to our *a priori* assumption of a ball) as the realization at any point can be used to determine the realization at any other point on connected sets. (That is, a matrix convex function with a realization as above defined on a convex domain G must have $I - \sum T_i Z_i$ positive for all $Z \in G$.) Moreover, by a limiting argument, a matrix convex function on a domain containing 0 over a general operator system should be of the form:

$$f(Z) = a_0 + L(Z) + \Lambda(Z^*)^*(I - \Gamma(Z))^{-1}\Lambda(Z)$$

where $\Lambda : R_1 \to \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $\Gamma : R_1 \to \mathcal{B}(\mathcal{H})$ are linear maps. The boundedness of Λ follows from the continuity of the second derivative, the continuity of Γ follows from the fact that the spectral radius is bounded, essentially the same argument as before. That is, we have the following corollary.

Corollary 4.5 *(A noncommutative Kraus theorem). Let R_1, R_2 be real operator systems. Let $G \subseteq \mathcal{S}(R_1)$ be a convex domain. Let $f : G \to \mathcal{S}(R_2)$ be a locally bounded free function on a convex domain $G \subseteq \mathcal{S}(R_1)$ with $B \in G_1$. The function f is matrix convex if and only if*

$$f(Z + B) = a_0 + L(Z) + \Lambda(Z^*)^*(I - \Gamma(Z))^{-1}\Lambda(Z)$$

where $\mathcal{H}$ is a Hilbert space, $L : R_1 \to \mathcal{B}(\mathcal{K})$, $\Lambda : R_1 \to \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $\Gamma : R_1 \to \mathcal{B}(\mathcal{H})$ are completely bounded linear maps, where L and Γ are self-adjoint valued.

Proof. Without loss of generality $B = 0$, $f(0) = 0$ and $Df(0) = 0$. Moreover, we assume f has a uniformly convergent homogeneous power series on the unit ball, which exists by real analyticity.

Let $\mathcal{R}$ denote the collection of finite dimensional operator system subspaces of R_1.

Fix $R \in \mathcal{R}$. Pick a basis $r_1, \dots, r_n$. Consider the induced function $g(X) = f(\sum r_i X_i)$. We see that

$$g(Z) = (\sum Q_i Z_i^*)^*(I - \sum T_i Z_i)^{-1}(\sum Q_i Z_i).$$

Call the representing Hilbert space $\mathcal{H}_R$. Now, $f|_R(Z) = \Lambda_R(Z^*)(I - \Gamma_R(Z))^{-1}\Lambda_R(Z)$. Taking the second derivative, we get

$$\begin{aligned}
&\Lambda_R(H^*)^*(I - \Gamma_R(Z))^{-1}\Lambda_R(H)+\\
&\Lambda_R(Z^*)^*(I - \Gamma_R(Z))^{-1}\Gamma_R(H)(I - \Gamma_R(Z))^{-1}\Lambda_R(H)+\\
&\Lambda_R(H^*)^*(I - \Gamma_R(Z))^{-1}\Gamma_R(H)(I - \Gamma_R(Z))^{-1}\Lambda_R(Z)+\\
&\Lambda_R(Z^*)^*(I - \Gamma_R(Z))^{-1}\Gamma_R(H)(I - \Gamma_R(Z))^{-1}\Gamma_R(H)(I - \Gamma_R(Z))^{-1}\Lambda_R(Z).
\end{aligned}$$

Under the substitution

$$Z \mapsto \left[\begin{smallmatrix} 0 & 0 \\ 0 & Z \end{smallmatrix}\right], H \mapsto \left[\begin{smallmatrix} 0 & H \\ H & 0 \end{smallmatrix}\right],$$

taking the 1, 1 entry we get

$$\Lambda_R(H)(I - \Gamma_R(Z))^{-1}\Lambda_R(H).$$

The geometric expansion of this formula converges uniformly and absolutely. Therefore for contractions, $\Gamma_R(Z)^n\Lambda_R(H)$ is eventually contractive. Now, taking Z to be a strictly block upper triangular matrix with $Z_1, \ldots, Z_n \in B_{\mathbb{R}}(0,1)$ on the upper diagonal, we see that $\Gamma_R(Z_1)\Gamma_R(Z_2)\ldots\Gamma_R(Z_n)\Lambda_R(H)$ must be contractive for n large enough, and therefore the joint spectral radius of the set $\{\Gamma_R(Z)|Z \in B_{\mathbb{R}}(0,1)_m\}$ is less than or equal to 1 for each m.

By canonicity of the construction, if $R \subseteq S$, $\mathcal{H}_R$ embeds into $\mathcal{H}_S$ (for example we could have extended the basis we chose for R in our original construction to a basis for S.) Moreover $\Lambda_S|_R = \Lambda_R$ under this identification and $\Gamma_S|_R = \Gamma_R \oplus J_{SR}$ for some linear map J_{SR}. So, ordering the sets in $\mathcal{R}$ under inclusion, we can take a direct limit to obtain Γ, Λ as desired. □

5. Löwner and Kraus type continuation theorems

Theorem 5.1. *Let R_1, R_2 be real operator systems. Let $G \subseteq \mathcal{S}(R_1)$ be a convex domain. A free function $f : G \to \mathcal{S}(R_2)$ is matrix monotone if and only if it analytically continues to the upper half plane.*

Proof. We essentially follow [36], except we need not appeal to the perhaps technically daunting Agler, McCarthy, and Young theorem [3]. Note that it is enough to show that f analytically continues at each level to a Pick function – that is an analytic function from $\Pi(R_1)_1$ to $\overline{\Pi(R_2)}_1$ - and therefore, by coordinatization, it is enough to show that this occurs at level 1. Moreover, it suffices to consider the case of finite dimensional R_1. Moreover, we can assume 0 is in G.

The function f will analytically continue to a Pick function if and only if $\lambda \circ f$ analytically continues to a Pick function for all positive linear functionals λ on R_2. Therefore, it is enough to consider the case where R_2 is one dimensional.

Pick $Z \in \Pi(R_1)_1$. Pick $H_1, \ldots, H_n > 0$ such that there is a point $(z_1, \ldots, z_n) \in \Pi(\mathbb{R}^n)_1$ with $Z = \sum H_i z_i$ and the H_i span R_1. Now, $f(\sum H_i x_i)$ is a matrix monotone function of x and therefore analytically continues to the upper half plane $\Pi(\mathbb{R}^n)_1$ by the realization formula in Theorem 4.2, which pulls back to $\Pi(R_1)_1$. (Note, as we choose additional H_i, we exhaust more and more of $\Pi(R_1)_1$.) □

Theorem 5.2. *Let R_1, R_2 be real operator systems. Let $G \subseteq \mathcal{S}(R_1)$ be a convex domain. If a free function $f : G \to \mathcal{S}(R_2)$ is matrix convex and locally bounded then f analytically continues to the tube*

$$T(G) = \{X + iY | X \in G \text{ and } Y = Y^*\}.$$

Proof. Let $Z \in T(G)$. Without loss of generality, $Z \in T(G)_1$. We will show that f is bounded on a noncommutative ball around Z.

First, write $Z = X + iY$. Without loss of generality, $X = 0$ and f is bounded and analytic on $B_{\mathbb{C}}(0, 1+\varepsilon)$. Pick $W \in B_{\mathbb{C}}(0, 1)$. By the realization formula in Corollary 4.5,

$$f(Z) = a_0 + L(Z) + \Lambda(Z^*)^*(I - \Gamma(Z))^{-1}\Lambda(Z).$$

Therefore,

$$\|f(Z+W)\| \leq \|a_0\| + \|L\| \, \|Z+W\| + \frac{1}{\varepsilon}\|\Lambda\|^2 \, \|Z+W\|^2 .$$

This shows that f analytically continues to a neighborhood of Z, which establishes the claim. □

6. Appendix: the principle of uniform boundedness for closed cones

The following is a variant of the principle of uniform boundedness that works for closed cones instead of an entire Banach space.

Theorem 6.1. *Let X be a Banach space. Let $\mathcal{C}$ be a closed cone. Let $\mathcal{T}$ be a collection of bounded linear operators on X. If*

$$\sup_{T \in \mathcal{T}} \|Tc\| < \infty$$

for every $c \in \mathcal{C}$, then, there exists a constant K such that

$$\|Tc\| \leq K\|c\|$$

for every $T \in \mathcal{T}$ and $c \in \mathcal{C}$

Proof. The proof is essentially the same as the classical principle of uniform boundedness.

Let

$$\mathcal{C}_n = \{c \in \mathcal{C} | \sup_{T \in \mathcal{T}} \|Tc\| \leq n\}.$$

Note each $\mathcal{C}_n$ is closed. Note that $\bigcup_{n \in \mathbb{N}} \mathcal{C}_n = \mathcal{C}$. Therefore, by the Baire category theorem, there exists an N such that $\mathcal{C}_N$ has nonempty interior. Thus, there is some $B(c_0, \varepsilon) \cap \mathcal{C} \subseteq \mathcal{C}_N$. If $\|c\| = 1$, we have

$$\begin{aligned} \|Tc\| &= \frac{1}{\varepsilon}\|T(c_0 + \varepsilon c) - Tc_0\| \\ &\leq \frac{1}{\varepsilon}\|T(c_0 + \varepsilon c)\| + \frac{1}{\varepsilon}\|Tc_0\| \end{aligned}$$

$$\leq \frac{2n}{\varepsilon}.$$

Therefore, $\|Tc\| \leq \frac{2n}{\varepsilon}\|c\|$. □

We use the following corollary for an operator system R which arises from applying the previous theorem with $\mathcal{C}$ the set of positive linear functionals and viewing a set $\mathcal{X} \subseteq R$ as linear functionals on linear functionals by double duality.

Corollary 6.2. *Let R be a operator system. Let $\mathcal{X} \subseteq R$. If*

$$\sup_{X \in \mathcal{X}} |\lambda(X)| < \infty$$

for every positive linear functional λ, then the set $\mathcal{X}$ is bounded.

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