



Induced Stinespring Factorization and the Wittstock Support Theorem

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Abstract. Given a pair of self-adjoint-preserving completely bounded maps on the same C^* -algebra, say that $\varphi \leq \psi$ if the kernel of φ is a subset of the kernel of ψ and $\psi \circ \varphi^{-1}$ is completely positive. The *Agler class* of a map φ is the class of $\psi \geq \varphi$. Such maps admit colligation formulae, and, in Lyapunov type situations, transfer function type realizations on the Stinespring coefficients of their Wittstock decompositions. As an application, we prove that the support of an extremal Wittstock decomposition is unique.

Mathematics Subject Classification. Primary 46L07; Secondary 46L52, 47A57, 47A48.

Keywords. Stinespring factorization theorem, wittstock decomposition, transfer function realizations.

1. Introduction

First, we recall the following theorem.

Theorem 1.1 (Nevanlinna–Pick interpolation theorem) *Let $z_1, \dots, z_n \in \mathbb{D}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. There is an analytic function $f : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ such that $f(z_i) = \lambda_i$ if and only if the matrix $\left[\frac{1 - \overline{\lambda_i} \lambda_j}{1 - \overline{z_i} z_j} \right]_{i,j}$ is positive semidefinite.*

J. E. Pascoe: Partially supported by National Science Foundation DMS Analysis Grant 1953963. R. Tully-Doyle: Partially supported by National Science Foundation DMS Analysis Grant 2055098.

See [2] for a comprehensive reference on Pick interpolation. Importantly, the elementary so-called “lurking isometry argument” gives that such an f is of the form

$$f(z) = a + b^*z(1 - Dz)^{-1}c$$

where

$$\begin{bmatrix} a & b^* \\ c & D \end{bmatrix}$$

is a unitary operator. Such a formula is often called a transfer function realization.

Let \mathcal{M}, \mathcal{N} be C^* -algebras. Given map $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ we define the n -th induced map $\varphi^{(n)} : M_n(\mathcal{M}) \rightarrow M_n(\mathcal{N})$ by entrywise evaluation as

$$\varphi^{(n)}([m_{ij}]_{ij}) = [\varphi(m_{ij})]_{ij}.$$

We say that φ is **completely bounded** if $\varphi^{(n)}$ are uniformly bounded linear maps. We call φ **real** if $\varphi(H)^* = \varphi(H^*)$. We say that φ is **completely positive** if each $\varphi^{(n)}$ takes positive semidefinite elements to positive semidefinite elements.

Let $X \in M_n(\mathbb{C})$. Define the Lyapunov map $L_X(H) = H - X^*HX$.

Theorem 1.2 (Nevanlinna–Pick interpolation theorem: Lyapunov formulation). *Let $Z \in M_n(\mathbb{C})$ be a strict contraction and $\Lambda \in M_n(\mathbb{C})$. There is an analytic function $f : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ such that $f(Z) = \Lambda$ if and only if $L_\Lambda \circ L_Z^{-1}$ is a completely positive map.*

We give a proof of the above theorem which demonstrates our technique in Sect. 5.1. For a comprehensive generalized reference to this well-known approach, see [5, 6].

Some work shows that if Z is a matrix with z_i on the diagonal and Λ with λ_i on the diagonal, the corresponding Choi matrix is exactly the matrix arising in the classical Nevanlinna–Pick interpolation theorem. The formulation of the problem in terms of complete positivity of some induced map is a powerful idea which has led to broad generalizations in noncommuting variables, especially in terms of the work of Ball–Groenwald–Malakorn [5, 6] and subsequent developments [7–9, 11].

We seek to understand the lurking isometry argument and consequent transfer function realization type objects as fundamental rather than coincidental. That is, we analyze the relationship between two real completely bounded maps φ and ψ such that $\psi \circ \varphi^{-1}$ is completely positive. This framework captures much of operator theoretic interpolation theory arising from generalizations of Nevanlinna–Pick interpolation. The Stinespring theorem on factorization of completely positive maps envelops some of the core ideas of operator theory as special cases, including the GNS construction, Choi’s theorem, and the Sz.-Nagy dilation theorem [15, 16]. The induced Stinespring type theorem likewise envelops and extends Nevanlinna–Pick type interpolation theorems and related realization machinery.

2. Stinespring Factorization, Wittstock Decomposition, and Support

The **Stinespring factorization theorem** [17] states that any completely positive map $\varphi : \mathcal{M} \rightarrow \mathcal{B}(\mathcal{H})$ is of the form

$$\varphi(H) = V^* \pi(H) V$$

where $\pi : \mathcal{M} \rightarrow \mathcal{B}(\mathcal{L})$ is a representation of a C^* -algebra \mathcal{M} on a Hilbert space \mathcal{L} and $V : \mathcal{H} \rightarrow \mathcal{L}$ is a bounded linear operator. We say two representations are **equivalent** if they are unitarily similar. We say π_1 is a **subrepresentation** of π_2 if π_2 is equivalent to representation of the form $\pi_1 \oplus \pi_3$.

Given π_1 and π_2 representations, we say π_1 and π_2 are **totally orthogonal**, denoted $\pi_1 \perp \pi_2$, if there does not exist a subrepresentation of π_1 which is equivalent to a subrepresentation of π_2 . The notion of total orthogonality gives a generalization of Schur's lemma which holds even in the absence of a notion of irreducibility.

Observation 2.1. *Let π_1 and π_2 be representations of some C^* -algebra \mathcal{M} . If $\pi_1 \perp \pi_2$ and A is an operator such that $\pi_1(m)A = A\pi_2(m)$ for all $m \in \mathcal{M}$ then $A = 0$.*

We say $\text{supp } \pi_1 \subseteq \text{supp } \pi_2$ if there does not exist a subrepresentation of π_1 which is totally orthogonal to π_2 . We say $\text{supp } \pi_1 = \text{supp } \pi_2$ if $\text{supp } \pi_1 \subseteq \text{supp } \pi_2$ and $\text{supp } \pi_2 \subseteq \text{supp } \pi_1$. We call the symbol $\text{supp } \pi$ the **support** of π . If φ is a completely positive map, we define the $\text{supp } \varphi$ to be the support of the corresponding representation in its minimal Stinespring factorization.

The **Wittstock decomposition theorem** [14, 18] states that any completely bounded map is in the span of the completely positive maps.

Thus, any real completely bounded map φ can be decomposed as

$$\varphi = \varphi^+ - \varphi^-.$$

We call such a Wittstock decomposition **extremal** if there does not exist a completely positive δ such that $\varphi^+ - \delta$ and $\varphi^- - \delta$ are completely positive. An extremal Wittstock decomposition is similar to the Hahn-Jordan decomposition of measures, though an extremal Wittstock decomposition is not always unique.

However, we prove that the support of a Wittstock decomposition is unique in the following sense.

Theorem 2.2 (Wittstock support theorem). *Let \mathcal{M} be a C^* -algebra. Suppose $\varphi : \mathcal{M} \rightarrow \mathcal{B}(\mathcal{H})$ is a real completely bounded map. Given two extremal Wittstock decompositions*

$$\varphi = \varphi^+ - \varphi^- = \phi^+ - \phi^-,$$

we have that

$$\text{supp } \varphi^+ = \text{supp } \phi^+, \text{supp } \varphi^- = \text{supp } \phi^-.$$

We prove Theorem 2.2 at the end of Sect. 3. Related uniqueness statements about generalized Stinespring representations have been recently obtained by Christensen in [10, Theorem 3.1], who shows that maps of the form $\varphi(H) = W^*\pi(H)V$ have support uniqueness with respect to π under controllability-observability minimality type assumptions. Haagerup's theme of decomposable maps is at least superficially related (see [16]), although there does not appear to be a direct connection. Our results show that in an extremal Wittstock decomposition, one cannot introduce extraneous representations.

3. The Agler Order and Colligations

Let $\mathcal{M}, \mathcal{N}, \tilde{\mathcal{N}}$ be C^* -algebras. Let $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ and $\psi : \mathcal{M} \rightarrow \tilde{\mathcal{N}}$ be real completely bounded maps. We say that $\varphi \leq \psi$ in the **Agler order** if:

- (1) $\ker \varphi \subseteq \ker \psi$,
- (2) the induced map $\gamma = \psi \circ \varphi^{-1}$ is completely positive.

We say φ is **Archimedian** if its range contains a strictly positive element.

The Agler order captures the various Agler models used for Nevanlinna–Pick interpolation in the Schur–Agler, Herglotz–Agler, Pick–Agler classes and so on, codifying the Lyapunov formulation taken in [5–9, 11]. For example, taking $\varphi(H) = H - X^*HX$, $\psi(H) = Y^*H + HY$, we have that $\varphi \leq \psi$ corresponds to there being an analytic function from the disk to the right half plane (a Herglotz function) taking X to Y . (Similarly for noncommutative and commutative multivariable analogues, cf. [1, 12].) The case $\varphi(H) = (X^*H - HX)/2i$, $\psi(H) = Y^*H - HY/2i$ similarly corresponds to the existence of a Nevanlinna model for a Pick function as in [13]. The Agler order abstracts away the domain and range conditions, and interpolation interpretation for the more basal underlying condition of induced complete positivity.

Let $\mathcal{M}, \mathcal{N}, \tilde{\mathcal{N}}$ be C^* -algebras where $\mathcal{N}, \tilde{\mathcal{N}}$ are concrete. Let $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ and $\psi : \mathcal{M} \rightarrow \tilde{\mathcal{N}}$ be real completely bounded maps. We say that $\varphi \preceq \psi$ in the **concrete Agler order** if there exists an operator Γ such that $\Gamma^*\varphi(H)\Gamma = \psi(H)$.

We note, if φ is Archimedian, that $\varphi \leq \psi$ (in the Agler order) if and only if for every representation π of $\tilde{\mathcal{N}}$ there exists a representation $\hat{\pi}$ of \mathcal{N} such that $\hat{\pi} \circ \varphi \preceq \pi \circ \psi$.

Suppose φ and ψ are real completely bounded maps into concrete C^* -algebras. We say ψ is **φ -colligatory** if for all Wittstock decompositions

$$\varphi = \varphi^+ - \varphi^-,$$

$$\psi = \psi^+ - \psi^-,$$

given Stinespring factorizations

$$\begin{aligned}\psi^+ &= \Psi_+^* \pi_1 \Psi_+, \\ \psi^- &= \Psi_-^* \pi_2 \Psi_-, \\ \varphi^+ &= \Phi_+^* \pi_3 \Phi_+, \\ \varphi^- &= \Phi_-^* \pi_4 \Phi_-, \end{aligned}$$

there is a partial isometry

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and operator Γ such that

$$\begin{bmatrix} \Psi_- \\ \Phi_+ \Gamma \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Psi_+ \\ \Phi_- \Gamma \end{bmatrix},$$

where

$$\text{ran } U = \text{span} \bigcup_{H \in \mathcal{M}} \text{ran} \begin{bmatrix} \pi_2(H) \Psi_- \\ \pi_3(H) \Phi_+ \Gamma \end{bmatrix}$$

and

$$\text{ran } U^* = \text{span} \bigcup_{H \in \mathcal{M}} \text{ran} \begin{bmatrix} \pi_1(H) \Psi_+ \\ \pi_4(H) \Phi_- \Gamma \end{bmatrix},$$

and

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \pi_1 & \\ & \pi_4 \end{bmatrix} = \begin{bmatrix} \pi_2 & \\ & \pi_3 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Note that, if $\Phi_+ - D\Phi_-$ is invertible, then

$$\Psi_- = [A + B\Phi_- (\Phi_+ - D\Phi_-)^{-1} C] \Psi_+.$$

We call such an expression a **φ -transfer function realization**. We call U the **colligation operator**.

We prove the following concrete result.

Theorem 3.1. *Let φ and ψ be real completely bounded maps on some C^* -algebra \mathcal{M} mapping into concrete C^* -algebras.*

The following are equivalent:

- (1) $\varphi \preceq \psi$ in the concrete Agler order,
- (2) ψ is φ -colligatory.

Proof. Take Wittstock decompositions

$$\begin{aligned}\varphi &= \varphi^+ - \varphi^-, \\ \psi &= \psi^+ - \psi^-, \end{aligned}$$

and Stinespring factorizations

$$\begin{aligned}
\psi^+ &= \Psi_+^* \pi_1 \Psi_+, \\
\psi^- &= \Psi_-^* \pi_2 \Psi_-, \\
\varphi^+ &= \Phi_+^* \pi_3 \Phi_+, \\
\varphi^- &= \Phi_-^* \pi_4 \Phi_-.
\end{aligned}$$

Since $\psi = \Gamma^* \varphi \Gamma$, we see that

$$\Psi_+^* \pi_1 \Psi_+ - \Psi_-^* \pi_2 \Psi_- = \Gamma^* [\Phi_+^* \pi_3 \Phi_+ - \Phi_-^* \pi_4 \Phi_-] \Gamma$$

Rearranging, we get

$$\Psi_+^* \pi_1 \Psi_+ - \Gamma^* \Phi_-^* \pi_4 \Phi_- \Gamma = \Gamma^* \Phi_+^* \pi_3 \Phi_+ \Gamma + \Psi_-^* \pi_2 \Psi_-$$

Evaluating at W^*V gives

$$\Psi_+^* \pi_1 (W^*V) \Psi_+ + \Gamma^* \Phi_-^* \pi_4 (W^*V) \Phi_- \Gamma = \Psi_-^* \pi_2 (W^*V) \Psi_- + \Gamma^* \Phi_+^* \pi_3 (W^*V) \Phi_+ \Gamma.$$

So,

$$\Psi_+^* \pi_1 (W)^* \pi_1 (V) \Psi_+ + \Gamma^* \Phi_-^* \pi_4 (W)^* \pi_4 (V) \Phi_- \Gamma$$

is equal to

$$\Psi_-^* \pi_2 (W)^* \pi_2 (V) \Psi_- + \Gamma^* \Phi_+^* \pi_3 (W)^* \pi_3 (V) \Phi_+ \Gamma.$$

Factoring, we get that for vectors v, w ,

$$\left\langle \begin{bmatrix} \pi_1(V) \Psi_+ \\ \pi_4(V) \Phi_- \Gamma \end{bmatrix} v, \begin{bmatrix} \pi_1(W) \Psi_+ \\ \pi_4(W) \Phi_- \Gamma \end{bmatrix} w \right\rangle = \left\langle \begin{bmatrix} \pi_2(V) \Psi_- \\ \pi_3(V) \Phi_+ \Gamma \end{bmatrix} v, \begin{bmatrix} \pi_2(W) \Psi_- \\ \pi_3(W) \Phi_+ \Gamma \end{bmatrix} w \right\rangle.$$

So, there is a partial isometry

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and Γ such that

$$\begin{bmatrix} \Psi_- \\ \Phi_+ \Gamma \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Psi_+ \\ \Phi_- \Gamma \end{bmatrix},$$

where

$$\text{ran } U = \text{span} \bigcup_{H \in \mathcal{M}} \text{ran} \begin{bmatrix} \pi_2(H) \Psi_- \\ \pi_3(H) \Phi_+ \Gamma \end{bmatrix}$$

and

$$\text{ran } U^* = \text{span} \bigcup_{H \in \mathcal{M}} \text{ran} \begin{bmatrix} \pi_1(H) \Psi_+ \\ \pi_4(H) \Phi_- \Gamma \end{bmatrix},$$

and

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \pi_1 & \\ & \pi_4 \end{bmatrix} = \begin{bmatrix} \pi_2 & \\ & \pi_3 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

□

We see the immediate corollary for the Agler order.

Corollary 3.2. *Let φ and ψ be real completely bounded maps on some C^* -algebra \mathcal{M} mapping into C^* -algebras $\mathcal{N}, \tilde{\mathcal{N}}$. Suppose φ is Archimedean. The following are equivalent:*

- (1) $\varphi \leq \psi$ in the Agler order,
- (2) For every representation π of $\tilde{\mathcal{N}}$ there exists a representation $\hat{\pi}$ of \mathcal{N} such that $\pi \circ \psi$ is $\hat{\pi} \circ \varphi$ -colligatory.

Say a real completely bounded map φ is of **Lyapunov type** if it admits a Wittstock decomposition $\varphi = \pi - \varphi^-$ where π is a representation and φ^- is strictly completely contractive.

Note that any φ of Lyapunov type is a *fortiori* Archimedean.

Corollary 3.3. *Let φ and ψ be real completely bounded maps on some C^* -algebra \mathcal{M} mapping into concrete C^* -algebras. Suppose φ is of Lyapunov type. The following are equivalent:*

- (1) $\varphi \preceq \psi$ in the concrete Agler order,
- (2) ψ is φ -colligatory,
- (3) ψ has a φ -transfer function realization.

Observation 3.4. *We also note that if $\pi_i = \begin{bmatrix} \hat{\pi}_i \\ \tilde{\pi}_i \end{bmatrix}$ such that $\text{supp } \hat{\pi}_i \perp \text{supp } \tilde{\pi}_j$, then the colligation operator factors as*

$$\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \\ \tilde{C} & \tilde{D} \end{bmatrix}.$$

Thus,

$$\pi \circ \psi^\pm = \hat{\psi}^\pm + \tilde{\psi}^\pm$$

where $\text{supp } \hat{\psi}^\pm \perp \text{supp } \tilde{\psi}^\pm$, and

$$\hat{\pi} \circ \varphi^\pm = \hat{\varphi}^\pm + \tilde{\varphi}^\pm$$

where $\text{supp } \hat{\varphi}^\pm \perp \text{supp } \tilde{\varphi}^\pm$ and, letting $\hat{\varphi} = \hat{\varphi}^+ - \hat{\varphi}^-$ and $\hat{\psi} = \hat{\psi}^+ - \hat{\psi}^-$,

$$\hat{\varphi} \preceq \hat{\psi}, \tilde{\varphi} \preceq \tilde{\psi}.$$

Proof of Theorem 2.2. We consider the case of the positive supports. The negative case is similar.

Let φ have two Wittstock decompositions

$$\varphi = \varphi^+ - \varphi^- = \Psi_+ - \Psi_-,$$

and Stinespring factorizations

$$\begin{aligned}
\psi^+ &= \Psi_+^* \pi_1 \Psi_+, \\
\psi^- &= \Psi_-^* \pi_2 \Psi_-, \\
\varphi^+ &= \Phi_+^* \pi_3 \Phi_+, \\
\varphi^- &= \Phi_-^* \pi_4 \Phi_-.
\end{aligned}$$

By Theorem 3.1, there is a colligation operator such that

$$\begin{bmatrix} \pi_2 \Psi_- \\ \pi_3 \Phi_+ \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \pi_1 \Psi_+ \\ \pi_4 \Phi_- \end{bmatrix}$$

Write

$$\pi_i = \begin{bmatrix} \hat{\pi}_i \\ \tilde{\pi}_i \end{bmatrix}$$

where

$$\text{supp } \pi_1 \perp \text{supp } \tilde{\pi}_i.$$

Note that there is no $\tilde{\pi}_1$; that is, $\hat{\pi}_1 = \pi_1$. By Observation 3.4, factor the colligation operator as

$$\begin{bmatrix} \hat{\pi}_2 \hat{\Psi}_- \\ \tilde{\pi}_2 \tilde{\Psi}_- \\ \hat{\pi}_3 \hat{\Phi}_+ \\ \tilde{\pi}_3 \tilde{\Phi}_+ \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} \pi_1 \Psi_+ \\ \hat{\pi}_4 \hat{\Phi}_- \\ \tilde{\pi}_4 \tilde{\Phi}_- \end{bmatrix}.$$

So, we have that

$$\begin{bmatrix} \tilde{\pi}_2 \tilde{\Psi}_- \\ \tilde{\pi}_3 \tilde{\Phi}_+ \end{bmatrix} = \begin{bmatrix} 0 & \tilde{B} \\ 0 & \tilde{D} \end{bmatrix} \begin{bmatrix} 0 \\ \tilde{\pi}_4 \tilde{\Phi}_- \end{bmatrix}.$$

Hence, by Theorem 3.1, we see that

$$-\tilde{\Psi}_-^* \tilde{\pi}_2 \tilde{\Psi}_- = \tilde{\Phi}_+^* \tilde{\pi}_3 \tilde{\Phi}_+ - \tilde{\Phi}_-^* \tilde{\pi}_4 \tilde{\Phi}_-.$$

If $\tilde{\Phi}_+^* \tilde{\pi}_3 \tilde{\Phi}_+ \neq 0$, taking $\varphi^+ - \tilde{\Phi}_+^* \tilde{\pi}_3 \tilde{\Phi}_+$ and $\varphi^- - \tilde{\Phi}_+^* \tilde{\pi}_3 \tilde{\Phi}_+$ witnesses the nonextremality of the Wittstock decomposition.

4. Truncating Irrelevant Representations and the Commutant Coefficient Theorem

The following proposition shows that one can choose a natural tensored representation in the Stinespring representation of a (homomorphic) noncommutative conditional expectation. Given $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ we use \mathcal{N}' to denote the **commutant** of \mathcal{N} , the set of elements in $\mathcal{B}(\mathcal{H})$ which commute with every element of \mathcal{N} . We say a C^* -algebra is **unital** if it has a multiplicative identity. Given a sub- C^* -algebra \mathcal{N} of a larger C^* -algebra \mathcal{M} , we say \mathcal{N} is unitaly included in \mathcal{M} if the multiplicative identity in \mathcal{N} is the same as for \mathcal{M} .

Proposition 4.1. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{L})$ be a unital C^* -algebra. Let \mathcal{N} be a sub- C^* -algebra unittally included in \mathcal{M} such that \mathcal{M} is generated by \mathcal{N} and \mathcal{N}' . Let $\pi : \mathcal{N} \rightarrow \mathcal{B}(\mathcal{H})$. Let $E : \mathcal{H} \rightarrow \mathcal{L}$ such that $E^*nE = \pi(n)$. There is a representation $\hat{\pi} : \mathcal{M} \rightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$, a unit vector $e_0 \in \mathcal{K}$ and a partial isometry $P : \mathcal{L} \rightarrow \mathcal{H} \otimes \mathcal{K}$ with range containing all vectors of the form $v \otimes e_0$ such that*

$$E^*mE = (I \otimes e_0^*)\hat{\pi}(m)(I \otimes e_0),$$

$$Pm = \hat{\pi}(m)P,$$

and

$$\hat{\pi}|_{\mathcal{N}} = \pi \otimes I.$$

Proof. Consider the minimal Stinespring representation of the map $\varphi(m) = E^*mE = F^*\tilde{\pi}(m)F$ for some isometry F . We will show that the restriction of $\tilde{\pi}$ to \mathcal{N} is unitarily equivalent to a representation of the form

$$\bigoplus_{\alpha \in \mathfrak{A}} P_{\alpha}^* \pi(n) P_{\alpha} \quad (4.1)$$

for some isometries P_{α} which reduce π , where at least one P_{α} is unitary and \mathfrak{A} is an index set. Then, by a Hilbert hotel argument we would be done by pairing up incomplete representations with their orthogonal complements. Specifically, let \mathcal{G} be an infinite dimensional Hilbert space with dimension greater than or equal to that of the representation $\tilde{\pi}$. Taking $\tilde{\pi} \otimes I_{\mathcal{G}}$, we see at least dimension \mathcal{G} complete copies of π and at most the same cardinality of incomplete copies, which is unitarily equivalent to a direct sum of complete copies. (One could do it inductively by taking the missing part of a given representation from the next complete copy, using part of it to complete the representation. The point is that in this process there is always a next complete copy to take from.)

We will show that given $m \in \mathcal{N}'$ and \mathcal{L}_0 containing \mathcal{H} such that \mathcal{N} has the desired form and $\mathcal{L}_0 \neq m\mathcal{H} + \mathcal{L}_0$ then there is a larger \mathcal{L}_1 such that \mathcal{N} has the desired form. Let $\hat{m} = P_{\mathcal{L}_0^{\perp}} m P_{\mathcal{H}}$. Note $nm = mn$. So, since \mathcal{L}_0 reduces n ,

$$n\hat{m} = \hat{m}n.$$

Thus, $\hat{m}^*n\hat{m} = \hat{m}^*\hat{m}n = n\hat{m}^*\hat{m}$. Letting $P = \hat{m}(\hat{m}^*\hat{m})^{\dagger 1/2}$. So $P^*nP = P_{\text{ran } \hat{m}^*}^* n P_{\text{ran } \hat{m}^*}$. Note $P_{\text{ran } \hat{m}^*}^* n P_{\text{ran } \hat{m}^*}$ is a subrepresentation of π .

So there is a minimal \mathcal{L} such that $\mathcal{L} = m\mathcal{H} + \mathcal{L}$ for all m and \mathcal{L} reduces \mathcal{N} . In fact, \mathcal{L} is exactly the minimal reducing subspace for \mathcal{M} containing \mathcal{H} . □

Let π be a representation of a C^* -algebra \mathcal{M} . Call a complete bounded map ψ **π -pure** if

$$\psi = \Psi_+^*(I \otimes \pi)\Psi_+ - \Psi_-^*(I \otimes \pi)\Psi_-.$$

We see the following immediate corollary of the above representation theorem.

Corollary 4.2. *Let φ and ψ be real completely bounded on some C^* -algebra \mathcal{M} . Assume ψ is π -pure. The following are equivalent:*

- (1) $\varphi \leq \psi$ in the Agler order,
- (2) *There exists a representation $\hat{\pi}$ such that $\hat{\pi} \circ \varphi$ is π -pure and ψ is $\hat{\pi} \circ \varphi$ -colligatory.*

The compatibility of representations is important in infinite dimensional noncommutative interpolation problems, where there is some work to show the abstract technique here gives a *bona fide* solution (see, for example, the last section of [11]).

5. Examples from Interpolation Theory

We now sketch several examples of applications of our theory, most of which essentially constitute semidefinite reformulations of classical techniques. However, the final subsection gives an intriguing class of interpolation problems that don't fit into either complex analysis nor into free noncommutative function theory, but are certainly worthy of comprehensive further study.

5.1. Nevalinna–Pick Interpolation in a Lyapunov Formulation

The following example demonstrates how our method works in the classical case.

Given $\|X\| \leq 1$, the Lyapunov map L_X is invertible, and

$$L_X^{-1}(H) = \sum_{n=0}^{\infty} X^{*n} H X^n.$$

Define the operator

$$\Lambda_{XY} = L_Y \circ L_X^{-1}.$$

Theorem 1.2 says that Λ_{XY} must be completely positive for the corresponding interpolation problem to be solvable.

As an example, consider the choice of matrices

$$X = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix}, \quad Y = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Computing the explicit form of L_X^{-1} , we get

$$\begin{aligned} L_X^{-1}(H) &= \sum_{n=0}^{\infty} \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix}^{*n} (h_{ij})_{i,j} \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix}^n \\ &= \left(\frac{h_{ij}}{1 - \bar{z}_i z_j} \right)_{i,j} \end{aligned}$$

Plugging in Y to L_Y gives

$$L_Y(H) = H - \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}^* H \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix},$$

and so

$$\Lambda_{XY}(H) = L_Y \circ L_X^{-1}(H) = \left(h_{ij} \frac{1 - \bar{\lambda}_i \lambda_j}{1 - \bar{z}_i z_j} \right)_{i,j}. \quad (5.1)$$

In this case, the positivity condition becomes

$$H = (h_{ij}) \geq 0 \Rightarrow \left(h_{ij} \frac{1 - \bar{\lambda}_i \lambda_j}{1 - \bar{z}_i z_j} \right)_{i,j} \geq 0,$$

which recovers the Pick condition

$$\left(\frac{1 - \bar{\lambda}_i \lambda_j}{1 - \bar{z}_i z_j} \right)_{i,j} \geq 0.$$

That is, we have recast classical Nevanlinna–Pick interpolation as a question about completely positive maps.

5.1.1. Lurking Isometries. When Λ_{XY} is a completely positive map, the Stinespring theorem allows us to write Λ_{XY} by

$$\Lambda_{XY}(H) = \Gamma^* \pi(H) \Gamma$$

where $\pi : M_n(\mathbb{C}) \rightarrow B(\mathcal{H})$ is a representation.

Note 5.1. In this special case where $H \in M_n(\mathbb{C})$, we know the homomorphisms. There are no closed ideals. All representations of $M_n(\mathbb{C})$ are the same. So write

$$\pi(H) = I \otimes H$$

Since

$$L_Y \circ L_X^{-1}(H) = \Lambda_{XY}(H),$$

we can calculate

$$\begin{aligned} L_Y(H) &= \Lambda_{XY} \circ L_X(H) \\ H - Y^* H Y &= \Gamma^* \pi(H - X^* H X) \Gamma \\ H + \Gamma^* \pi(H) \Gamma &= Y^* H Y + \Gamma^* \pi(X^* H X) \Gamma. \end{aligned}$$

After setting $H = W^* V$ (and conjugation by α, β), we get

$$W^* V + \Gamma^* \pi(W^*) \pi(V) \Gamma = Y^* W^* V Y + \Gamma^* \pi(X)^* \pi(W^*) \pi(V) \pi(X) \Gamma.$$

This is the setup for the so-called lurking isometry argument. That is, there exists $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ so that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V \\ \pi(V)\Gamma \end{bmatrix} = \begin{bmatrix} VY \\ \pi(V)\pi(X)\Gamma \end{bmatrix}, \quad (5.2)$$

where U is a partial isometry (that is, $(U^*U)^2 = U^*U$).

Furthermore,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V \\ \pi(V) \end{bmatrix} = \begin{bmatrix} V \\ \pi(V) \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Note that A, B, C, D factor as $A = \hat{A} \otimes I$, $B = \hat{B} \otimes I$, and so on.

Now set $V = I$. Then (5.2) becomes

$$\begin{bmatrix} Y \\ \Gamma \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I \\ \pi(X)\Gamma \end{bmatrix}$$

leading to the equations

$$\begin{aligned} Y &= A + B\pi(X)\Gamma \\ \Gamma &= C + D\pi(X)\Gamma. \end{aligned}$$

Eliminating Γ gives the (Schur-Agler) transfer function realization

$$Y = A + B\pi(X)(1 - D\pi(X))^{-1}C, \quad (5.3)$$

which points to the existence of an interpolating function in terms of complete positivity. (c.f. [3, 4, 11])

5.2. Two Variable Commutative Nevanlinna–Pick Interpolation

Let X_1, X_2 be commuting matrices. Define

$$\varphi = L_X(H) = \begin{bmatrix} H \\ H \end{bmatrix} - \begin{bmatrix} X_1^* H X_1 & \\ & X_2^* H X_2 \end{bmatrix}$$

and

$$\psi = L_Y(H) = H - Y^* H Y.$$

This setup gives the 2-variable commutative Nevanlinna–Pick interpolation theorem. Similarly, as in [11], our method works on more general semi-algebraic sets in the general noncommutative case.

5.3. Partial Nevanlinna–Pick Interpolation

Let

$$\varphi = H - X^* H X$$

and

$$\psi = w^* H w - v^* H v.$$

Notice that ψ is scalar-valued, and so positivity of $\psi \circ \varphi^{-1}$ implies complete positivity in this case. In the notation of this section, we have $\Psi_+ = w$ and $\Psi_- = v$, as well as $\Phi_+ = I$ and $\Phi_- = X$. By Theorem 3.3 we get the transfer function formulation

$$\begin{aligned} v &= [A + BX(I - DX)^{-1}C]w \\ &= f(X)w \end{aligned}$$

using the usual Schur-Agler representation $f(X) = A + BX(I - DX)^{-1}C$.

If we make the definitions

$$X = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix}, w = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, v = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix},$$

the problem becomes to look at the existence of a function f so that $f(z_i) = \lambda_i$. The technique here generalizes to other settings of solving $f(X)v = w$, including noncommutative Nevanlinna–Pick interpolation.

5.4. Commutant Coefficient Interpolation

Let \mathcal{M} be a C^* -algebra. Consider $L_X(H) = H - X^*HX$ as a map from \mathcal{M} to itself for some $X \in \mathcal{M}$. We see by Corollary 4.2 that if $L_X \leq L_Y$ and X is strictly contractive, then

$$Y = A + B(I \otimes X)(1 - DX)^{-1}C = A + \sum BD^n CX^{n+1}.$$

Author contributions All authors contributed equally to the research and writing of this manuscript.

Funding This work was supported by National Science Foundation DMS Analysis Grant 1953963 and by National Science Foundation DMS Analysis Grant 2055098.

Data Availability Statement Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of Interests The authors have no relevant financial or non-financial interests to disclose.

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Received: September 16, 2022.

Accepted: March 24, 2023.

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