GEOMETRIC DILATIONS AND OPERATOR ANNULI

SCOTT MCCULLOUGH AND JAMES E. PASCOE[1]

ABSTRACT. Fix $0 < r < 1$. The dilation theory for the quantum annulus $\mathbb{QA}_r$, consisting of those invertible Hilbert space operators T satisfying $\|T\|, \|T^{-1}\| \leq r^{-1}$, is determined. The proof technique involves a geometric approach to dilation that applies to other well known dilation theorems. The dilation theory for the quantum annulus is compared, and contrasted, with the dilation theory for other canonical operator annuli.

1. INTRODUCTION

Following M. Mittal in [Mi] (see also the references therein; e.g., [Pau88, Sh74]), given $0 < r < 1$, we call $\mathbb{QA}_r$, the collection of invertible Hilbert space operators T satisfying $\|T\|, \|T^{-1}\| \leq r^{-1}$, the *quantum annulus*. In this article we determine the dilation theory for $\mathbb{QA}_r$. The result is stated in Subsection 1.1 immediately below. The key proof technique – an adaptation of Nelson's trick [N] – is of independent interest. We illustrate how the trick provides a geometric alternate approach to the Sz.-Nagy dilation theorem and related results. See Section 2. Later in this introduction we discuss connections between the quantum annulus and other operator annuli [BY+, CG, Ts22a, Ts22b].

1.1. **The quantum annulus.** For a Hilbert space H, let $B(H)$ denote the bounded operators on H. An invertible operator $T \in B(H)$ has a $\mathbb{C}[x, x^{-1}]$-*dilation* to an invertible operator $J \in B(K)$ if there is an isometry $V : H \to K$ such that

$$T^n = V^* J^n V, \tag{1.1}$$

for all integers n. For expository ease, we will often drop the $\mathbb{C}[x, x^{-1}]$ modifier.

It is evident that if T dilates to J and J is in the quantum annulus, then T is also in the quantum annulus; that is, $\mathbb{QA}_r$ is *closed with respect* $\mathbb{C}[x, x^{-1}]$-*compressions*. Hence the collection $\mathbb{QA}_r$ is, what we call here, a $\mathbb{C}[x, x^{-1}]$-*dilation family* [Ag88] in the sense that it is closed with respect to (a) direct sums, (b) unital $*$-representations and (c) $\mathbb{C}[x, x^{-1}]$-compressions.

2010 *Mathematics Subject Classification.* 47A20 (Primary); 47B20, 47B91 (Secondary).

Key words and phrases. quantum annulus, complete Pick kernel, operator extension, dilation theory.

Research Supported by NSF grant DMS-1953963.

The dilation of equation (1.1) is *trivial* if the range of V reduces T. An element $T \in \mathbb{QA}_r$ is *dilation extremal in* $\mathbb{QA}_r$ if all of its dilations to $\mathbb{QA}_r$ are trivial; that is if $J \in \mathbb{QA}_r$ and equation (1.1) holds, then the range of V reduces J. This notion of dilation extremal has connections with boundary representations in the sense of Arveson. See [Ar, DK, DM, MuSo] and the references therein.

Let $\mathcal{Q}\mathbb{A}_r$ denote those operators J that, up to unitary equivalence, take the form

$$J = U \begin{pmatrix} rI_{K_{+1}} & 0 \\ 0 & r^{-1}I_{K_{-1}} \end{pmatrix},$$

where U is unitary, J acts on the Hilbert space direct sum $K_{+1} \oplus K_{-1}$ and $I_{K_{\pm 1}}$ is the identity on $K_{\pm 1}$. It is immediate that $J \in \mathbb{QA}_r$. As an alternate description, $J \in \mathcal{Q}\mathbb{A}_r$ if and only if there exists projection $P_\pm$ that sum to the identity such that $J^*J = r^2P_+ + r^{-2}P_-$. Theorem 1.1 below, proved in Section 3, describes the dilation theory of $\mathbb{QA}_r$, where $\mathbb{QA}_r[H]$ denotes those elements of $\mathbb{QA}_r$ that act on the Hilbert space H.

Theorem 1.1. *The collection $\mathcal{Q}\mathbb{A}_r$ has the following properties.*

(a) If $J \in \mathcal{Q}\mathbb{A}_r$, then $J \in \mathbb{QA}_r$ is dilation extremal in $\mathbb{QA}_r$;

(b) If $J \in \mathcal{Q}\mathbb{A}_r[K]$ and $\pi : B(K) \to B(G)$ is a unital $*$-representation on a Hilbert space G, then $\pi(J) \in \mathcal{Q}\mathbb{A}_r[G]$.

(c) If $T \in \mathbb{QA}_r$, then T dilates to a $J \in \mathcal{Q}\mathbb{A}_r$.

Thus $\mathcal{Q}\mathbb{A}_r$ is closed with respect to arbitrary direct sums, unital $*$-representations and restrictions to reducing subspaces and each element of $\mathbb{QA}_r$ dilates to an element of $\mathcal{Q}\mathbb{A}_r$. Moreover, item (a) says that $\mathcal{Q}\mathbb{A}_r$ is the smallest subcollection of $\mathbb{QA}_r$ with these properties. Hence $\mathcal{Q}\mathbb{A}_r$ is canonical and deserves the moniker *dilation boundary* of $\mathbb{QA}_r$. (See [Ag88].) The proof of item (c), the dilation, uses Nelson's trick to produce a geometric dilation. See Section 3.

1.2. **The Pick annulus.** Let $\mathbb{A}_r$ denote the annulus,

$$\mathbb{A}_r = \{z \in \mathbb{C} : r < |z| < \frac{1}{r}\}.$$

The recent papers [BY+, Ts22b] consider the family of operators associated to the kernel $k_r : \mathbb{A}_r \times \mathbb{A}_r \to \mathbb{C}$ defined by

$$k_r(z,w) = \frac{r^{-2} - r^2}{r^{-2} + r^2 - zw^* - (zw^*)^{-1}}.$$

The Agler model theory for this family was determined in [BY+]. In [Ts22b] the spectral bound for $\mathbb{PA}_r$ is determined, with the lower bound also obtained in [BY+]. (Tsikalas also

obtains a lower bound for the spectral constant for the quantum annulus in [Ts22b].) See Section 5.1.

For a self-adjoint operator A on Hilbert space, $A \succeq 0$ indicates A is positive semidefinite. Let $\mathbb{PA}_r$ denote those invertible Hilbert space operators T satisfying $\frac{1}{k_r}(T, T^*) \succeq 0$ in the *hereditary* sense that

$$\frac{1}{k_r}(T, T^*) = \frac{1}{r^{-2} - r^2}\left(r^{-2} + r^2 - T^*T - T^{-*}T^{-1}\right) \succeq 0. \tag{1.2}$$

While not obvious, if $T \in \mathbb{PA}_r[H]$, then T is in $\mathbb{QA}_r[H]$. See [BY+, Proposition 2.2] and [Ts22b, Lemma 4.1].

Because the collection $\mathbb{PA}_r$ has an hereditary definition (all the adjoints are on the left in equation (1.2)), it is closed with respect to restrictions to $\mathbb{C}[x, x^{-1}]$-invariant subspaces: if $Y \in \mathbb{PA}_r[K]$ and $H \subseteq K$ is a subspace invariant for both Y and Y^{-1}, then the restriction of Y to H is in $\mathbb{PA}_r[H]$. Thus $\mathbb{PA}_r$ is a *family*, in the sense of Agler [Ag88], with respect to $\mathbb{C}[x, x^{-1}] : \mathbb{PA}_r$ is closed with respect to (a) direct sums, (b) unital $*$-representations and (c) restrictions to $\mathbb{C}[x, x^{-1}]$-invariant subspaces.

An operator $T \in B(H)$ *lifts* to an operator $J \in B(K)$ if there is an isometry $V : H \to K$ such that $VT = JV$. Equivalently, T is, up to unitary equivalence, the restriction of J to an invariant subspace. In the case that both J and T are invertible, it follows that $VT^n = J^nV$ for all integers n. In particular, $T^n = V^*J^nV$ and hence T dilates to J.

Let $\mathscr{P}\mathbb{A}_r$ denote the collection of Hilbert space operators that have, up to unitary equivalence, the form

$$J = \begin{pmatrix} rV & \frac{1}{r}E \\ 0 & \frac{1}{r}W^* \end{pmatrix} = \begin{pmatrix} V & E \\ 0 & W^* \end{pmatrix} \begin{pmatrix} rI_{K_{+1}} & 0 \\ 0 & r^{-1}I_{K_{-1}} \end{pmatrix},$$

where

$$U = \begin{pmatrix} V & E \\ 0 & W^* \end{pmatrix},$$

is a unitary matrix. In particular, V and W are isometries and E is a partial isometry with initial space $\ker W^*$ and final space $\ker V^*$. A calculation shows $\frac{1}{k_r}(J, J^*) = -\frac{1}{k_r}(J^*, J)$ is the projection onto $\ker V^*$. Thus elements in $\mathscr{P}\mathbb{A}_r$ are distinguished members of $\mathbb{PA}_r$.

The following lifting theorem, which essentially identifies the Agler model theory for $\mathbb{PA}_r$ was established in [BY+].

Theorem 1.2 ([BY+])**.** *If $T \in B(H)$ is invertible, then T is in $\mathbb{PA}_r[H]$ if and only if T lifts to a J in $\mathscr{P}\mathbb{A}_r$.*

The *Agler boundary* [Ag88] of $\mathbb{PA}_r$ is the smallest subcollection $\partial\mathbb{PA}_r$ of $\mathbb{PA}_r$ that is (1) closed with respect to (a) direct sums, (b) unital *-representations, and (c) restrictions to reducing subspaces; and (2) each T in $\mathbb{PA}_r$ lifts to a J in $\partial\mathbb{PA}_r$. While it is not evident, families have such a boundary [Ag88]. For the family $\mathbb{PA}_r$, it is straightforward to verify that $\mathscr{P}\mathbb{A}_r$ satisfies the conditions (1). On the other hand, as observed in Section 4, each $J \in \mathscr{P}\mathbb{A}_r$ is *extremal* – if J lifts to $F \in \mathbb{PA}_r$ as $VJ = JV$, then the range of V reduces F – and is therefore in $\partial\mathbb{PA}_r$. Thus Theorem 1.2 has the following corollary.

Corollary 1.3. *The collection $\mathscr{P}\mathbb{A}_r$ is the Agler boundary of $\mathbb{PA}_r$.*

Since the kernel k_r is a complete Pick kernel [AM][Ts22b] (see also [AHMR19, AHMR21, H+] for recent results for complete Pick kernels), the family $\mathbb{PA}_r$ is also a dilation family. Indeed, if $T \in B(H)$ is invertible and there is a $Y \in \mathbb{PA}_r[K]$ and an isometry $V : H \to K$ such that $T^n = V^*Y^nV$ for all integers n, then[1]

$$\begin{aligned}(r^{-2} - r^2)\,\frac{1}{k_r}(T, T^*) &= r^2 + r^{-2} - T^*T - T^{-*}T^{-1} \\ &= r^2 + r^{-2} - V^*J^*VV^*JV - V^*J^{-*}VV^*J^{-1}V \\ &\succeq r^2 + r^{-2} - V^*J^*JV - V^*J^{-*}J^{-1}V \\ &= V^*(r^2 + r^{-2} - J^*J - J^{-*}J^{-1})V \\ &= (r^{-2} - r^2)\,V^*\frac{1}{k_r}(Y, Y^*)V \succeq 0,\end{aligned}$$

and hence $T \in \mathbb{PA}_r[H]$. A consequence, stated precisely in Corollary 1.4 below, is that the dilation theory and the Agler model theory for $\mathbb{PA}_r$ are essentially the same.

Corollary 1.4. *For an invertible operator $T \in B(H)$, the following are equivalent.*

(i) $T \in \mathbb{PA}_r$;

(ii) T *dilates to a* $Y \in \mathbb{PA}_r$;

(iii) T *lifts to a* $J \in \mathscr{P}\mathbb{A}_r$.

Moreover,

(a) if $J \in \mathscr{P}\mathbb{A}_r$, then J is dilation extremal; that is, any dilation of J to an element of $\mathbb{PA}_r$ is trivial;

(b) The collection $\mathscr{P}\mathbb{A}_r$ is closed with respect to arbitrary direct sums, unital $*$-representations and restrictions to reducing subspaces.

[1] It is enough that $T^n = V^*J^nV$ for $n = \pm 1$.

1.3. **The annulus as a spectral set.** Let $R(\mathbb{A}_r)$ denote the algebra of rational functions with poles off $\overline{\mathbb{A}_r}$. An operator T with spectrum in $\overline{\mathbb{A}_r}$ has $\mathbb{A}_r$ as *spectral set* if, for each $\psi \in R(\mathbb{A}_r)$,

$$\|\psi(T)\| \leq \|\psi\|_\infty,$$

where $\|\psi\|_\infty$ is the sup norm of ψ on $\overline{\mathbb{A}_r}$.

Let $\mathbb{SA}_r$ denote the collection of operators with $\mathbb{A}_r$ as a spectral set. Let $\mathscr{S}\mathbb{A}_r$ denote the normal operators with spectrum in the boundary of $\mathbb{A}_r$. In [Ag85] Agler proves the following analog of the Sz.-Nagy dilation theorem for $\mathbb{A}_r$. If $T \in \mathbb{SA}_r[H]$ and $\sigma(T) \subseteq \mathbb{A}_r$, then there exists a Hilbert space K an operator $N \in \mathscr{S}\mathbb{A}_r[K]$ and an isometry $V : H \to K$ such that

$$\psi(T) = V^*\psi(N)V,$$

for all $\psi \in R(\mathbb{A}_r)$.

The collection $\mathbb{SA}_r$ is both a family in the sense of Agler and a dilation family. The statement of Agler's dilation theorem essentially identifies the normal operators with the spectrum in the boundary of $\mathbb{A}_r$ as the dilation boundary of $\mathbb{SA}_r$.

A normal operator with spectrum in the boundary of $\mathbb{PA}_r$ has, up to unitary equivalence, the form

$$\begin{pmatrix} U_{+1} & 0 \\ 0 & U_{-1} \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix},$$

where $U_{\pm 1}$ are unitary operators. As noted in [BY+, Ts22b], there are the proper inclusions

$$\mathbb{SA}_r \subsetneq \mathbb{PA}_r \subsetneq \mathbb{QA}_r.$$

Evidently from the discussions above, but somewhat surprisingly, the same chain of inclusions holds for the dilation boundaries; that is,

$$\mathscr{S}\mathbb{A}_r \subsetneq \mathscr{P}\mathbb{A}_r \subsetneq \mathscr{Q}\mathbb{A}_r.$$

Remark 1.5. The family $\mathbb{SA}_r$ is naturally identified with unital contractive (equivalently unital completely contractive) representations of $R(\mathbb{A}_r)$ on Hilbert space. Namely, an operator $T \in \mathbb{SA}_r[H]$ determines the representation $\pi_T : R(\mathbb{A}_r) \to B(H)$ given by $\pi_T(\psi) = \psi(T)$. Both $\mathbb{QA}_r$ and $\mathbb{PA}_r$ are also naturally identified with collections of representations of $\mathbb{C}[x, x^{-1}]$ (or $R(\mathbb{A}_r)$).

1.4. **Reader's guide.** Section 2 contains geometric arguments in favor of the Sz.-Nagy dilation theorem, its multivariable non-commutative analog for row contractions and the dilation theory for doubly commuting contractions all using Nelson's trick. The trick is used in Section 3 to prove Theorem 1.1.

Section 4 provides some details for the proofs of Corollaries 1.3 and 1.4. We emphasize the key ingredient is Theorem 1.2.

The paper concludes with a few remarks about spectral constants for $\mathbb{QA}_r$ and musing about Nelson's trick and Ando's Theorem in Section 5

2. Automorphic Nelson's Trick

To prove the principal part of Theorem 1.1, item (c), we adapt a method found in [N] that replaces the positive definite (diagonal) matrix in the singular value decomposition of a square matrix with automorphisms of the unit disc. Similar methods were employed by [H17] for multivariable weighted shifts. Hartz notes the trick also appears in [Pa02] as well as in [Pi01]. In this section we illustrate the technique by indicating how it applies to several familiar dilation results. Namely the Sz.-Nagy dilation theorem (Subsection 2.1), the Bunce-Frazho-Popescu multivariable noncommutative theory of *row contractions* [B, Fr82, Fr84, Po89, Po91] (Subsection 2.2) and doubly commuting contractions (Subsection 2.3). In Section 3 we apply the method to the quantum annulus.

2.1. **Unitary dilations of contractions.** For a square matrix T of size n with $\|T\| < 1$, Nelson's construction proceeds as follows. The singular value decomposition of T has the form $T = UDW$, where D is a diagonal matrix whose diagonal entries λ_j are the eigenvalues of $(T^*T)^{\frac{1}{2}}$ and U, W are unitary matrices. In particular $0 \leq \lambda_j < 1$. Let

$$b_j(z) = \frac{\lambda_j - z}{1 - \lambda_j z}.$$

Thus each b_j is an automorphism of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let

$$D(z) = \begin{pmatrix} b_1(z) & 0 & 0 & \cdots & 0 \\ 0 & b_2(z) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & b_n(z) \end{pmatrix} \tag{2.1}$$

and note that $D(0) = D$. Let $F(z) = UD(z)W$. Thus $F : \mathbb{D} \to M_n(\mathbb{C})$ is a bounded analytic function such that $F(0) = T$ and $F(\zeta)^*F(\zeta) = I_n$ for $|\zeta| = 1$. We note that our construct slightly refines Nelson's in that originally he used independent variables on the diagonal, which, as we now turn to dilation theory, is less desirable from a minimality perspective.

We now show how to produce the desired dilation of T. Let $H^2(\mathbb{D})$ denote the usual Hardy-Hilbert space. By construction, the operator M_F of multiplication by F on $H^2(\mathbb{D})\otimes\mathbb{C}^n$ is an isometry. (Note that the cokernel has dimension at most n.) The mapping $V : \mathbb{C}^n \to$

$H^2(\mathbb{D})\otimes\mathbb{C}^n$ defined by $Vx = 1\otimes x$ is an isometry and

$$M_F^* V x = M_F^* 1\otimes x = 1\otimes F(0)^* x = 1\otimes T^* x = VT^* x.$$

Hence T^* *lifts* to the coisometry M_F^*, providing an explicit, geometric, unitary dilation of T.

2.2. **Row contractions.** The Nelson argument extends to the Bunce-Frazho-Popescu multivariable noncommutative theory of *row contractions* [B, Fr82, Fr84, Po89, Po91]. Given a tuple $T = (T_1, \dots, T_{\mathtt{g}})$ of $n\times n$ matrices, view T as the $n\times n\mathtt{g}$ matrix

$$T = \begin{pmatrix} T_1 & \cdots & T_{\mathtt{g}}\end{pmatrix} \in M_{n,n\mathtt{g}}(\mathbb{C}).$$

Suppose T is a strict contraction; that is $\|T\| < 1$.

Once again, consider the singular value decomposition

$$T = UEW,$$

where

$$E = \begin{pmatrix} D & 0 & \dots & 0\end{pmatrix}$$

for a diagonal $n\times n$ matrix D with nonnegative entries that are strictly less than one. In particular, the unitary matrices U and W have sizes n and $n\mathtt{g}$ respectively. Define $D(z)$ is as in equation (2.1). Thus $D(0) = D$ and D is unitary valued on the boundary of the disc.

Write W as a block $\mathtt{g}\times\mathtt{g}$ matrix with entries $W_{j,k}\in M_n(\mathbb{C})$ and define $F_j : \mathbb{D}\to M_n(\mathbb{C})$ by $F_j(z) = UD(z)W_{1,j}$. Let M_j denote the operator of multiplication by F_j on $H^2(\mathbb{D})\otimes\mathbb{C}^n$ and observe

$$M_j^* 1\otimes x = 1\otimes F_j(0)^* x = 1\otimes T_j^* x. \tag{2.2}$$

Let $\mathbb{F}_g^2$ denote the freely noncommutative Fock space on $\mathtt{g}$ letters $x = \{x_1, \cdots, x_{\mathtt{g}}\}$ and let $\varnothing$ denote the empty word (vacuum state). Thus $\mathbb{F}_g^2$ is the Hilbert space with orthonormal basis the words in x. Let $S_j : \mathbb{F}_g^2\to\mathbb{F}_g^2$ denote the shift determined by $S_j f = x_j f$ for f a finite $\mathbb{C}$-linear combination of words and note that $S_j^* S_k = \delta_{j,k} I$; that is, the S_j are isometries with pairwise orthogonal ranges.

Define $\iota : H^2(\mathbb{D})\to\mathbb{F}_g^2\otimes H^2(\mathbb{D})$ by $\iota h = \varnothing\otimes h$. Thus ι is an isometry. On the Hilbert space

$$K = \left[H^2(\mathbb{D})\otimes\mathbb{C}^n\right] \oplus \oplus_2^{\mathtt{g}}\left[\mathbb{F}_g^2\otimes H^2(\mathbb{D})\otimes\mathbb{C}^n\right],$$

let

$$J_j = \begin{pmatrix} M_j & 0 \\ \begin{bmatrix}\iota\otimes UW_{2,j}\\ \vdots \\ \iota\otimes UW_{\mathtt{g},j}\end{bmatrix} & I\otimes S_j\otimes I\end{pmatrix}.$$

For instance, with $\mathsf{g} = 3$,

$$J_j = \begin{pmatrix} M_j & 0 & 0 \\ \iota \otimes UW_{2,j} & S_j \otimes I & 0 \\ \iota \otimes U^* W_{3,j} & 0 & S_j \otimes I \end{pmatrix}.$$

Observe that $J_j^* J_k = \delta_{j,k} I$. Thus $\{J_1, \dots, J_{\mathsf{g}}\}$ is a family of isometries with orthogonal ranges.

Define $V : \mathbb{C}^n \to K$ by $Vx = [1 \otimes x] \oplus 0$. Thus V is an isometry and, in view of equation (2.2),

$$J_j^* V = V T_j^*.$$

2.3. **Doubly commuting contractions via Nelson.** We note that Nelson's trick also works for a tuple of *doubly commuting* contractions. See [NFBL, BNS] and the references therein. Suppose $T_1, \dots, T_d$ are commuting $n \times n$ matrices that are strict contractions. Suppose further they are invertible and *doubly commute*, meaning for each $j \neq k$,

$$T_j^* T_k = T_k T_j^*.$$

Each T_j has its polar decomposition, $T_j = U_j D_j$, where $D_j = (T_j^* T_j)^{\frac{1}{2}}$ and $U_j = T_j D_j^{-1}$. Thus each D_j is a positive definite strict contraction and each U_j is unitary. The doubly commuting hypothesis implies the D_j commute with one another, the U_j doubly commute and $D_j U_k = U_k D_j$ for $j \neq k$. Since the D_j are commuting self-adjoint matrices, they are simultaneously diagonalizable. Hence, for each $1 \leq j \leq d$, there are pairwise orthogonal projections $P_{j,1}, P_{j,2}, \dots, P_{j,m_j}$ that sum to the identity and distinct $0 < \lambda_{j,\alpha} < 1$ such that

$$D_j = \sum_{\alpha=1}^{m_j} \lambda_{j,\alpha} P_{j,\alpha}.$$

Further, for all j, k, α, β, the operators $P_{j,\alpha}$ and $P_{k,\beta}$ commute and, for $j \neq k$, the operators U_k and $P_{j,\beta}$ commute. Let

$$b_{j,\alpha} = \frac{\lambda_{j,\alpha} - z}{1 - \lambda_{j,\alpha} z}$$

and

$$D_j(z) = \sum_{\alpha=1}^{m_j} b_{j,\alpha}(z) P_{j,\alpha}.$$

It follows that $D_j(z)$ and U_k commute for $j \neq k$, and the $D_j(z)$ commute with one another. Hence the resulting matrix functions $F_j(z) = U_j D_j(z)$ pointwise doubly commute and are unitary valued on the boundary of the disc. Thus the operators M_j of multiplication by F_j on $H^2(\mathbb{D}) \otimes \mathbb{C}^n$ are isometries and doubly commute. With the usual isometry $V : \mathbb{C}^n \to H^2(\mathbb{D}) \otimes \mathbb{C}^n$ defined by $Vh = 1 \otimes h$, one finds $V T_j^* = M_j^* V$ for each j.

3. The Quantum Annulus

In this section we prove Theorem 1.1. A preliminary version of the dilation, item (c), is established using Nelson's trick in Subsection 3.1. The remaining items of the theorem are proved in Subsection 3.2 and are then used, in conjunction with the result of Subsection 3.1, to complete the proof of item (c) in Subsection 3.3.

3.1. **Nelson's trick applied to $\mathbb{QA}_r$.** In this subsection we apply Nelson's trick to obtain Lemma 3.1 below, an initial version of item (c) of Theorem 1.1. The proof of Theorem 1.1 concludes in subsection 3.3. Let $\sigma(T)$ denote the spectrum of a bounded operator T on Hilbert space.

Lemma 3.1. *Suppose $T \in \mathbb{QA}_r[H]$. If $\sigma((T^*T)^{\frac{1}{2}}) \subseteq (r, r^{-1})$ is finite, then there exists a Hilbert space K, an operator $J \in \mathcal{Q}\mathbb{A}_r[K]$ and an isometry $V : H \to K$ such that $T^n = V^*J^nV$ for all integers n.*

Proof. The operator T has the polar decomposition

$$T = UP,$$

where P is positive semidefinite and U is unitary. Indeed, since T^*T is invertible, $P = (T^*T)^{\frac{1}{2}}$ and $U = TP^{-1}$. By hypothesis, there exists a finite set $\mathfrak{F} \subseteq (r, r^{-1})$ such that P has spectral decomposition,

$$P = \sum_{\lambda \in \mathfrak{F}} \lambda E_\lambda,$$

where $\{E_\lambda : \lambda \in \mathfrak{F}\}$ are pairwise orthogonal projections that sum to the identity.

Let $\mathbb{T}$ denote the unit circle, viewed as the boundary of $\mathbb{D}$. Let $I^{\pm 1}$ denote the upper and lower half of $\mathbb{T}$ respectively. It is well known that the unit disc $\mathbb{D}$ is the universal cover of $\mathbb{A}_r$. Namely, there exists an onto analytic function $\vartheta : \mathbb{D} \to \mathbb{A}_r$ with $\vartheta(0) = 1$ and mapping $I^{\pm 1}$ to the inner and outer boundary components $\{|z| = r^{\pm 1}\}$ of $\mathbb{A}_r$ respectively that extends across $\mathbb{T}$ except at ± 1. Given $r < \lambda < r^{-1}$, there exists a Möbuis automorphism m_λ of the unit disc so that $\vartheta_\lambda = \vartheta \circ m_\lambda$ maps $\mathbb{D}$ onto $\mathbb{A}_r$ and sends 0 to λ. There exists arcs $I_\lambda^{\pm 1}$ whose disjoint union is the boundary of the unit disc, save for two points, that are mapped, under ϑ_λ, to the inner and outer boundaries of $\mathbb{A}_r$ respectively.

Define $F : \mathbb{D} \to B(H)$ by $F(z) = UP(z)$, where

$$P(z) = \sum_{\lambda \in \mathfrak{F}} \vartheta_\lambda(z) E_\lambda.$$

Observe F is analytic on $\mathbb{D}$, extends to $\mathbb{T}$ except at finitely many point, and $F(0) = T$.

Let $L^2 = L^2(\mathbb{T})$ denote the usual L^2 space of the unit circle $\mathbb{T}$. The characteristic functions $\chi_\lambda^{\pm 1}$ of the intervals $I_\lambda^{\pm 1}$ induce projection operators $Q_\lambda^{\pm 1}$ on $L^2(\mathbb{T})$ by sending

f to $\chi_\lambda^{\pm 1} f$. Moreover, the projections $Q_\lambda^{\pm 1}$ are orthogonal and sum to the identity. Let $K = L^2(\mathbb{T}) \otimes H$. It follows that the $2|\mathfrak{F}|$ projections $\{Q_\lambda^{\pm 1} \otimes E_\lambda : \lambda \in \mathfrak{F}\}$ are pairwise orthogonal and sum to the identity.

Let M_F denote the operator of multiplication by F on $K = L^2(\mathbb{T}) \otimes H$. By construction, given $f \in L^2(\mathbb{T})$, for $h \in H$ and $\zeta \in \partial\mathbb{D}$ and each $\lambda \in \mathfrak{F}$,

$$M_F(\zeta)[Q_\lambda^{\pm 1} \otimes E_\lambda](f \otimes h)(\zeta) = M_F(\zeta)\chi_\lambda^{\pm 1} f(\zeta) \otimes E_\lambda h = \vartheta_\lambda(\zeta)\chi_\lambda^{\pm 1} f(\zeta) \otimes U E_\lambda h.$$

Hence, if also $f' \in L^2(\mathbb{T})$ and $h' \in h$, then

$$\begin{aligned}
&\langle M_F^* M_F [Q_\lambda^{\pm 1} \otimes E_\lambda](f \otimes h), f' \otimes h' \rangle \\
&\qquad = \langle\, \vartheta_\lambda(\zeta)\chi_\lambda^{\pm 1} f(\zeta) \otimes U E_\lambda h, \sum_{\mu \in \mathfrak{F}, j = \pm 1} \vartheta_\mu(\zeta)\chi_\mu^j f'(\zeta) \otimes U E_\mu h' \,\rangle \\
&\qquad = r^{\pm 2} \langle \chi_\lambda^{\pm 1} f(\zeta) \otimes E_\lambda h, \chi_\lambda^{\pm 1} f'(\zeta) \otimes E_\lambda h' \rangle \\
&\qquad = r^{\pm 2} \langle Q_\lambda^{\pm 1} \otimes E_\lambda (f \otimes h), f' \otimes h' \rangle.
\end{aligned}$$

It follows that

$$M_F^* M_F Q_\lambda^{\pm 1} \otimes I = r^{\pm 2} Q_\lambda^{\pm 1} \otimes I.$$

Thus

$$M_F^* M_F = r^2 Q_\lambda^+ \otimes I + r^{-2} Q_\lambda^- \otimes I$$

and therefore $M_F \in \mathscr{Q}\mathbb{A}_r$.

Define $V : H \to K = L^2(\mathbb{T}) \otimes H$ by $Vh = 1 \otimes h$. Thus V is an isometry. Moreover, since F is analytic, if $h, g \in H$, and $n \in \mathbb{Z}$, then

$$\langle M_F^n V h, V g \rangle = \langle M_F^n 1 \otimes h, 1 \otimes g \rangle = \langle F^n(0) h, g \rangle = \langle T^n h, g \rangle$$

and we obtain

$$T^n = V^* M_F^n V$$

for all $n \in \mathbb{Z}$. ∎

3.2. **The dilation boundary of $\mathbb{Q}\mathbb{A}_r$.** In this subsection we prove items (a) and (b) of Theorem 1.1 and provide an alternate characterizations of the dilation boundary of $\mathbb{Q}\mathbb{A}_r$.

3.2.1. *Proof of Theorem 1.1 items (a) and (b).*

Proposition 3.2. *An invertible operator T is in $\mathbb{Q}\mathbb{A}_r$ if and only if*

$$r^{-2} + r^2 - T^*T - T^{-1}T^{-*} \succeq 0$$

and $T \in \mathscr{Q}\mathbb{A}_r$ if and only if

$$r^{-2} + r^2 - T^*T - T^{-1}T^{-*} = 0. \tag{3.1}$$

Proof. Suppose T is invertible and let $A = T^*T$. By definition, $T \in \mathbb{QA}_r$ if and only $r^2 \preceq A \preceq r^{-2}$ if and only if $\sigma(A)$, the spectrum of A, lies in the interval $[r^2, r^{-2}]$.

Let

$$A = \int_{\sigma(A)} \lambda \, dE(\lambda)$$

denote the spectral decomposition of A. Since $T^{-1}T^{-*} = A^{-1}$,

$$r^{-2} + r^2 - T^*T - T^{-1}T^{-*} = \int_{\sigma(A)} \left(r^{-2} + r^2 - (\lambda + \lambda^{-1})\right) \, dE.$$

On the other hand $f : \mathbb{R} \to \mathbb{R}$ defined by $f(t) = r^{-2} + r^2 - (t + t^{-1})$ is positive if and only if $r^2 < t < r^{-2}$ and is 0 if and only if either $t = r^2$ or $t = r^{-2}$. Thus $\sigma(A) \subseteq [r^2, r^{-2}]$ if and only if

$$\int_{\sigma(A)} f(t) \, dE(t) \succeq 0$$

and the first part of the proposition is proved. Moreover,

$$0 = r^2 + r^{-2} - A - A^{-1} = \int_{\sigma(A)} f(t) \, dE(t)$$

if and only if $\sigma(A) \subseteq \{r^2, r^{-2}\}$ if and only if $A = r^2 E(\{r^2\}) + r^{-2} E(\{r^{-2}\})$ if and only if $T \in \mathscr{Q}\mathbb{A}_r$, completing the proof of the proposition. ∎

Proof of Theorem 1.1 item (a). Suppose $J \in \mathscr{Q}\mathbb{A}_r[H]$ and $F \in \mathbb{QA}_r[K]$ and there is an isometry $V : H \to K$ such that $J^n = V^*F^nV$ for all $n \in \mathbb{Z}$. Using both parts of Proposition 3.2,

$$\begin{aligned} 0 &= r^2 + r^{-2} - J^*J - J^{-1}J^{-*} \\ &= r^2 + r^{-2} - V^*F^*(VV^*)FV - V^*F^{-1}(VV^*)F^{-*}V \\ &\succeq V^*(r^2 + r^{-2} - F^*F - F^{-1}F^{-*})V \succeq 0. \end{aligned}$$

It follows that $V^*F^*(VV^*)FV = V^*F^*FV$ and also $V^*F^{-1}VV^*F^{-*}V = V^*F^{-1}F^{-*}V$. Thus the range of V is invariant for both F and F^{-*}. In particular, $F^{-*}V = VV^*F^{-*}V = VJ^{-*}$. Thus $F^*V = VJ^*$ and hence the range of V is invariant for F^*. Hence the range of V reduces F. ∎

Proof of Theorem 1.1 item (b). Simply note that if $J^*J = r^2P_+ + r^{-2}P_-$, where $P_\pm$ are orthogonal projections that sum to the identity and π is a unital $*$-representation, then $\pi(P_\pm)$ are orthogonal projections that sum to the identity and

$$\pi(J)^*\pi(J) = \pi(J^*J) = r^2\pi(P_+) + r^{-2}\pi(P_-).$$

∎

3.2.2. *Boundary representations.* We call, $\mathfrak{H}_r$, the universal unital C^*-algebra with generators $\mathfrak{t}$ and $\mathfrak{d}$ satisfying the relations $\mathfrak{t}\mathfrak{d} = 1 = \mathfrak{d}\mathfrak{t}$ and

$$r^{-2} + r^2 - \mathfrak{t}^*\mathfrak{t} - \mathfrak{d}\mathfrak{d}^* = 0$$

the *donut C^*-algebra*. (Compare Proposition 3.2.) Naturally we write $\mathfrak{d} = \mathfrak{t}^{-1}$. The existence of $\mathfrak{H}_r$ is guaranteed since if $T \in \mathscr{Q}\mathbb{A}_r$ and $D = T^{-1}$, then T and D satisfy the relations. By Proposition 3.2, if $\pi : \mathfrak{H}_r \to B(H)$ is a unital $*$-representation, then $J = \pi(\mathfrak{t}) \in \mathscr{Q}\mathbb{A}_r$ and in particular $\|\mathfrak{t}\| \leq r^{-2}$ and $\|\mathfrak{t}^{-1}\| \leq r^{-2}$. Classically, the von Neumann inequality is equivalent to saying that the map taking continuous functions on the unit circle to algebra generated by a contraction T such that $e^{in\theta}$ is mapped to T^n and $e^{-in\theta}$ to $(T^*)^n$ is a completely positive map. Similarly, the natural map from from the donut algebra induced by an element of the donut algebra is completely positive.

Note that we may also view $\mathfrak{H}_r$ as the completion of the algebra of trigonometric polynomials $\mathfrak{P} = \{\sum_{-N}^{N} p_n z^n\}$ endowed with the family of norms on $M_n(\mathfrak{P})$ given by

$$\|p\|_n = \sup\{\|p(T)\| : T \in \mathbb{Q}\mathbb{A}_r\} = \sup\{\|p(J)\| : J \in \mathscr{Q}\mathbb{A}_r\},$$

using Ruan's characterization of operator algebras.

3.3. **The proof of Theorem 1.1 item (c).** Fix $T \in \mathbb{Q}\mathbb{A}_r[H]$ and let $T = UP$ denote its polar decomposition. Thus, as before $P = (T^*T)^{\frac{1}{2}}$ and $U = TP^{-1}$. Using Proposition 3.2, let

$$P = \int_{[r,r^{-1}]} \lambda \, dE$$

denote the spectral decomposition of P. Given a positive integer m, choose a measurable simple function s_m taking values in (r, r^{-1}) that approximates λ uniformly within $\frac{1}{m}$ on $[r, r^{-1}]$. Let

$$P_m = \int s_m(\lambda) \, dE$$

and let $T_m = UP_m$. It follows that T_m is in $\mathbb{Q}\mathbb{A}_r[H]$ and satisfies the hypotheses of Lemma 3.1. Hence there exists a Hilbert space K_m, an operator $J_m \in \mathscr{Q}\mathbb{A}_r$ and an isometry $V_m : H \to K_m$ such that, for $n \in \mathbb{Z}$,

$$T_m^n = V_m^* J_m^n V_m.$$

Let $J = \oplus J_m$ acting on the Hilbert space $K = \oplus K_m$. Thus $J \in \mathscr{Q}\mathbb{A}_r$ and moreover, if $p(z) = \sum_{j=-N}^{N} p_j z^j$ is a $d \times d$ matrix-valued polynomial and $p(J) \succeq 0$, then $p(T_m) \succeq 0$ for each m. Since (T_m) converges to T in operator norm, we conclude that $p(T) \succeq 0$. By Arveson's extension theorem [Ag82], there exists a Hilbert space L, an isometry $V : H \to L$, and a unital $*$-representation $\pi : B(K) \to B(L)$ such that

$$T^n = V^* \pi(J)^n V$$

for $n \in \mathbb{Z}$. Item (c) of Theorem 1.1 now follows from item (b).

Remark 3.3. Combining Theorem 1.1 with Proposition 3.2 shows $J \in \mathbb{QA}_r$ is dilation extremal (in $\mathbb{QA}_r$) if and only if $J \in \mathscr{Q}\mathbb{A}_r$. On the other hand, it is possible to prove directly that T satisfies equation (3.1) if and only if T is dilation extremal and thus deduce item (c) of Theorem 1.1 as a consequence of results in [Ag88, Ar, DM, DK].

In Agler's operator model theory, a pleasing fact is that the boundary of a family has a C-star characterization. For the dilation family $\mathbb{QA}_r$, the identity of equation (3.1) is such a condition. ∎

4. The Boundaries of the Pick Annulus

This section contains proofs of the parts of Corollaries 1.3 and 1.4 not covered by Theorem 1.2 or the discussion in the introduction. Thus, what remains to be proved are items (a) and (b) of Corollary 1.4. Namely, that the collection $\mathscr{P}\mathbb{A}_r$ (1) is closed under (a) $*$-representations, (b) restrictions to reducing subspaces, and (c) arbitrary direct sums and (2) all dilations (and hence lifts) of a $J \in \mathscr{P}\mathbb{A}_r$ to an $F \in \mathbb{PA}_r$ are trivial.

4.1. The dilation boundary of $\mathbb{PA}_r$.

The following lemma establishes item (b) of Corollary 1.4.

Lemma 4.1. *An invertible operator J is in $\mathscr{P}\mathbb{A}_r$ if and only if*

$$\frac{1}{k_r}(J, J^*) = -\frac{1}{k_r}(J^*, J) \tag{4.1}$$

is a projection. Thus $\mathscr{P}\mathbb{A}_r$ is closed with respect to arbitrary direct sums, restrictions to reducing subspaces and unital $*$-representations.

Proof. Direct computation shows if $J \in \mathscr{P}\mathbb{A}_r$, then the relevant conditions are satisfied.

Now suppose J is an invertible operator satisfying the given conditions. Let $\mu = r^{-2} + r^2$ and $\nu = r^{-2} - r^2$. Multiplying equation (4.1) by ν gives,

$$\mu - J^*J - J^{-*}J^{-1} = -\mu + JJ^* + J^{-1}J^{-*}.$$

Rearranging,

$$\mu - J^*J - (J^*J)^{-1} = -\mu + JJ^* + (JJ^*)^{-1}. \tag{4.2}$$

Since J and J^* are in $\mathbb{QA}_r$, the left hand side of equation (4.2) is positive semidefinite and the right hand side is negative semidefinite. Hence both sides are 0 and therefore,

by Proposition 3.2, $J \in \mathscr{Q}\mathbb{A}_r$. Thus there exists a unitary U such that (up to unitary equivalence),

$$J = U \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}.$$

It follows that

$$\nu P_{2,2} = \nu \frac{1}{k_r}(J, J^*) = \mu - \begin{pmatrix} r^2 & 0 \\ 0 & r^{-2} \end{pmatrix} - U \begin{pmatrix} r^{-2} & 0 \\ 0 & r^2 \end{pmatrix} U^*,$$

where $P = \frac{1}{k_r}(J, J^*)$ is a projection. Simplifying,

$$\begin{pmatrix} r^{-2} & 0 \\ 0 & r^2 \end{pmatrix} - U \begin{pmatrix} r^{-2} & 0 \\ 0 & r^2 \end{pmatrix} U^* = \nu P, \tag{4.3}$$

Writing $U = (U_{j,k})_{j,k=1}^2$ and comparing the $(2,2)$ (block) entries from equation (4.3) and using the fact that U is unitary,

$$\nu P_{2,2} = r^2 - r^{-2} U_{2,1} U_{2,1}^* - r^2 U_{2,2} U_{2,2}^* = -(r^{-2} - r^2) U_{2,1} U_{2,1}^* = -\nu\, U_{2,1} U_{2,1}^*.$$

Since both $U_{2,1}U_{2,1}^*$ and $P_{2,2}$ are positive semidefinite, both $U_{2,1}$ and $P_{2,2}$ are 0. Hence U is block upper triangular and thus $J \in \mathscr{P}\mathbb{A}_r$.

To prove the last statement, note the conditions characterizing membership in $\mathscr{P}\mathbb{A}_r$ are all invariant under arbitrary direct sums, restrictions to reducing subspaces and unital $*$-representations. ∎

To prove item (a) of Corollary 1.4, suppose $J \in \mathscr{P}\mathbb{A}_r$ dilates to $F \in \mathbb{P}\mathbb{A}_r$. Thus J is in $\mathscr{Q}\mathbb{A}_r$ and dilates to $F \in \mathbb{Q}\mathbb{A}_r$. By item (a) of Theorem (1.1), the dilation is trivial.

4.2. **The Agler boundary of $\mathbb{P}\mathbb{A}_r$.** To this point, we have seen that $\mathscr{P}\mathbb{A}_r$ is closed with respect to direct sums, unital $*$-representations and restrictions to reducing subspaces. Moreover, Theorem 1.2 says each element of $\mathbb{P}\mathbb{A}_r$ lifts to an element of $\mathscr{P}\mathbb{A}_r$. To show that $\mathscr{P}\mathbb{A}_r$ is the smallest subcollection of $\mathbb{P}\mathbb{A}_r$ with these properties, and is thus the Agler boundary, it suffices to show that each $J \in \mathscr{P}\mathbb{A}_r$ is extremal in $\mathbb{P}\mathbb{A}_r$; that is, if $J \in \mathscr{P}\mathbb{A}_r[H]$ and $F \in \mathbb{P}\mathbb{A}_r[K]$ and there is an isometry $V : H \to K$ such that

$$VJ = FV, \tag{4.4}$$

then the range of V reduces J. To prove this statement, observe that equation (4.4) immediately implies

$$J^n = V^* F^n V$$

for all integers n. Hence J dilates to F. By item (a), the range of V reduces F.

Remark 4.2. Lemma 4.1 provides a C-star characterization of the boundary of $\mathbb{PA}_r$. (Compare with Remark 3.3.)

Also note, in Lemma 4.1, the condition $\frac{1}{k_r}(J, J^*)$ is a projection can be replaced with $\frac{1}{k_r}(J, J^*)$ is positive semidefinite.

5. Further remarks

The paper concludes with a few remarks about spectral constants for $\mathbb{QA}_r$ and musing about Nelson's trick and Ando's Theorem in Subsections 5.1, and 5.2 respectively.

5.1. **Spectral constants.** To more easily connect with the existing literature, we now work with the annulus,

$$\mathfrak{A}_q = \{z \in \mathbb{C} : q < |z| < 1\}.$$

It is conformally equivalent to the annulus $\mathbb{A}_{q^{\frac{1}{2}}}$. We update the definitions of $\mathbb{PA}_q$ and $\mathbb{QA}_q$ accordingly.

Operators in $\mathbb{PA}_q$ and $\mathbb{QA}_q$ do not necessarily have the annulus as a spectral set, but one can ask, what are the *spectral constants*

$$\kappa_{\mathcal{F}} = \sup\{\|\psi(T)\| : T \in \mathcal{F}, \quad \psi \in R(\mathfrak{A}_q), \quad \|\psi\|_\infty \leq 1\},$$

for $\mathcal{F}$ either $\mathbb{PA}_q$ or $\mathbb{QA}_q$. Note that it suffices to optimize not over all of $\mathbb{PA}_q$ or $\mathbb{QA}_q$, but just over their dilation boundaries.

Tsikalas [Ts22b] shows $\kappa_{\mathbb{PA}_q}$ is $\sqrt{2}$ independent of q (in [BY+] the inequality $\kappa_{\mathbb{PA}_q} \leq \sqrt{2}$ is obtained). For $\mathbb{QA}_q$ there are the estimates

$$2 \leq \kappa_{\mathbb{QA}_q} \leq 1 + \sqrt{2}, \tag{5.1}$$

with the lower bound due to Tsikalas [Ts22a] and the upper bound to Crouzeix and Greenbaum [CG]. The estimate in [Ts22a] is obtained by a clever choice of element of the dilation boundary of $\mathbb{QA}_r$.

Let $\mathscr{A}(\mathfrak{A}_q)$ denote the annulus algebra, consisting of those functions continuous on $\overline{\mathfrak{A}_q}$ and analytic on $\mathfrak{A}_q$. Fisher proves that convex combinations of inner functions are dense in $\mathscr{A}(\mathfrak{A}_q)$ [Fi]. It follows that to determine spectral constants, it suffices to optimize over inner functions in $\mathscr{A}(\mathfrak{A}_q)$. For the annulus it is particularly easy to numerically compute the inner functions.

The zero set $Z(\psi)$ of an inner function $\psi \in \mathscr{A}(\mathfrak{A}_q)$ is finite. Moreover, the modulus of the product (counting with multiplicity) of the zeros of ψ is q^k for some natural number k. Conversely, given a finite subset F of $\mathfrak{A}_q$ such that the modulus of the product of the

elements of F is q^k for some natural number k, then there is an inner function ψ such that $Z(\psi) = F$, determined uniquely up to a rotation.

One way to construct these inner functions is as follows [McSh]. Let $f(\alpha, t)$ denote the Jordan-Kronecker function,

$$f(\alpha, t) = \sum_{n=-\infty}^{\infty} \frac{\alpha^n}{1 - tq^{2n}}.$$

Given $w \in \mathfrak{A}_q$, let

$$B_w(z) = f(z\overline{w}, |w|^2)$$

The function $B_w : \mathfrak{A}_q \to \mathbb{C}$ has constant (but different) modulus on each boundary component of $\mathfrak{A}_q$ and vanishes precisely at w. Given $W = \{w_1, \dots, w_m\} \subseteq \mathfrak{A}_q$ and a positive integer k such that $|\prod w_j| = q^k$, let

$$\tau(z) = \frac{1}{z^k} \prod B_{w_j}(z).$$

The function $\psi_W : \mathfrak{A}_q \to \mathbb{D}$ defined by

$$\psi_W(z) = \frac{\tau(z)}{\tau(1)}$$

vanishes precisely on the set $\{w_1, \dots, w_m\}$ and has modulus 1 on the boundary of $\mathfrak{A}_q$.

To establish the lower bound from equation (5.1), Tsikalas [Ts22a] applies the functions

$$f_n(z) = \frac{z^n + \frac{q^n}{z^n}}{1 + q^n}$$

to a clever choice of operator from $\mathscr{Q}\mathbb{A}_q$ that, in a certain sense, contains arbitrarily large permutations. While not inner, the functions f_n are, loosely, asymptotically inner. Numerical experiments suggest 2 is the optimal spectral constant for the quantum annulus.

5.2. **Ando's inequality.** An optimist hopes Nelson's trick can be applied to prove the following two variable analogue of the von Neumann inequality.

Theorem 5.1 (Ando [An])**.** *Let p be a polynomial. Let T_1, T_2 be bounded operators on some Hilbert space such that $\|T_1\|, \|T_2\| \leq 1$ and T_1 commutes with T_2. Then,*

$$\|p(T_1, T_2)\| \leq \sup_{z \in \mathbb{D}^2} |p(z)|.$$

Ando's inequality fails for commuting triples [Par, V74].

Recall the following result of Gerstenhaber.

Theorem 5.2 (Gerstenhaber [G])**.** *The algebra generated by a commuting pair of n by n matrices is at most n dimensional.*

It is known that for 4-tuples of commuting matrices, the above theorem fails, and it is unknown what happens for triples. The variety of commuting pairs is irreducible with diagonalizable elements being dense, whereas for large dimensions for triples it is known not to be [NS].

We conjecture that there should be a Nelson's trick type argument in 2 variables. Such an argument cannot work in 3 variables because of the failure of irreducibility of the variety. In turn, failure of irreducibility should imply that the distinguished boundary (or the place where functions are forced to take their maximum) contains nonunitary points in 3 or more variables.

References

[Ag85] Agler, Jim Rational dilation on an annulus. Ann. of Math. (2) 121 (1985), no. 3, 537—563. 5

[Ag88] Jim Agler, *An abstract approach to model theory,* Surveys of some recent results in operator theory, Vol. II, 1–23, Pitman Res. Notes Math. Ser., 192, Longman Sci. Tech., Harlow, 1988. 1, 2, 3, 4, 13

[Ag82] Jim Agler, *The Arveson extension theorem and coanalytic models,* Integral Equations Operator Theory 5 (1982), no. 5, 608–631. 12

[AM] Jim Agler and John E. McCarthy, *Complete Nevanlinna-Pick kernels,* J. Funct. Anal. 175 (2000), no. 1, 111–124. 4

[AHMR21] Alexandru Aleman, Michael Hartz, John E. McCarthy and Stefan Richter, *Weak products of complete Pick spaces,* Indiana Univ. Math. J. 70 (2021), no. 1, 325–352. 4

[AHMR19] Alexandru Aleman, Michael Hartz, John E. McCarthy and Stefan Richter, *Interpolating sequences in spaces with the complete Pick property,* Int. Math. Res. Not. IMRN 2019, no. 12, 3832–3854. 4

[An] T. Ando, *On a pair of commutative contractions,* Acta Sci. Math. (Szeged), 24 (1963), 88–90. 16

[Ar] William Arveson, *The noncommutative Choquet boundary,* J. Amer. Math. Soc. 21 (2008), no. 4, 1065–1084. 2, 13

[BNS] Bhattacharyya, T.; Narayanan, E. K.; Sarkar, Jaydeb *Analytic model of doubly commuting contractions,* Oper. Matrices 11 (2017), no. 1, 101–113. 8

[BY+] Gleaner Bello and Dmitry Yakubovich, *An Operator Model in the Annulus*, https://arxiv.org/abs/2106.08757(v3). 1, 2, 3, 5, 15

[B] J. Bunce, *Models for n-tuples of non-commuting operators,* J. Funct. Anal. 57 (1984), 21–30 6, 7

[CG] Michel Crouzeix and Anne Greenbaum, *Spectral sets: numerical range and beyond,* In: SIAM J. Matrix Anal. Appl. 40.3 (2019), pp. 1087–1101. 1, 15

[DK] Kenneth R. Davidson and Matthew Kennedy, *The Choquet boundary of an operator system,* Duke Math. J. 164 (2015), no. 15, 2989–3004. 2, 13

[DM] Michael Dritschel and Scott McCullough, *Boundary representations for families of representations of operator algebras and spaces,* J. Operator Theory 53 (2005), 159–167. 2, 13

[Fi] Stephen Fisher, *Another theorem on convex combinations of unimodular functions,* Bull. Amer. Math. Soc. 75 (1969), 1037–1039. 15

[Fr82] A. Frazho, *Models for non-commuting operators,* J. Funct. Anal. 48 (1982), 1–11. 6, 7

[Fr84] A. Frazho, *Complements to models for noncommuting operators,* J. Funct. Anal. 59 (1984), 445–461. 6, 7

[G] M. Gerstenhaber, *On dominance and varieties of commuting matrices,* Ann. of Math. (1961): 324–348. 16

[H+] Michael Hartz, *Every complete Pick space satisfies the column-row property,* https://arxiv.org/abs/2005.09614 4

[H17] Michael Hartz, *Von Neumann's inequality for commuting weighted shifts,* Indiana Univ. Math. J. 66 (2017), no. 4, 1065–1079. 6

[McSh] Scott McCullough and Li-Chien Shen, *On the Szegö kernel of an annulus* Proc. Amer. Math. Soc. 121 (1994), no. 4, 1111–1121. 16

[Mi] Mittal, Meghna Function theory on the quantum annulus and other domains. Thesis (Ph.D.)–University of Houston. 2010. 141 pp. ISBN: 978-1124-46385-8 1

[MuSo] Muhly, Paul S.; Solel, Baruch *An algebraic characterization of boundary representations,* Nonselfadjoint operator algebras, operator theory, and related topics, 189–196, Oper. Theory Adv. Appl., 104, Birkhäuser, Basel, 1998. 2

[N] Edward Nelson, *The distinguished boundary of the unit operator ball,* Proc. Amer. Math. Soc. 12 (1961), 994–995. 1, 6

[NFBL] Nagy-Foias Béla Sz.-Nagy, Ciprian Foais, Hari Bercovici and László Kérchy, *Harmonic analysis of operators on Hilbert space,* Second edition. Revised and enlarged edition. Universitext. Springer, New York, 2010. xiv+474 pp. ISBN: 978-1-4419-6093-1 8

[NS] Nham Ngo, Klemen Šivic, *On varieties of commuting nilpotent matrices,* Lin. Alg. Appl. 452 (2014): 237-262. 17

[Par] S. Parrott, *Unitary dilations for commuting contractions,* Pacific Journal of Mathematics, 34(2), 481-490. (1970) 16

[Pau88] V.I. Paulsen, *Toward a theory of K-spectral sets,* "Surveys of Some Recent Results in Operator Theory", Vol. I, 221–240, Pitman Res. Notes Math. Series, 171, Longman Sci. Tech., Harlow, 1988. 1

[Pa02] V.I. Paulsen, *Completely Bounded Maps and Operator Algebras,* Cambridge Univ. Press, Cambridge 2002. 6

[Po89] Gelu Popescu, *Isometric dilations for infinite sequences of noncommuting operators,* Trans. Amer. Math. Soc. 316 (1989), 523–536. 6, 7

[Po91] G. Popescu, *von Neumann inequality for* $(B(H)^n)_1$, Math. Scand. 68 (1991), 292–304. 6, 7

[Pi01] Giles Pisier, *Similarity Problems and Completely Bounded Maps,* Second, expanded edition; In- cludes the solution to "The Halmos problem", Lecture Notes in Mathematics, vol. 1618, Springer-Verlag, Berlin, 2001. 6

[Sh74] A. L. Shields, *Weighted shift operators and analytic function theory,* Topics in operator theory, 49–128, Math. Surveys, No. 13, Amer. Math. Soc., Providence, R.I., 1974. 1

[Ts22a] Georgios Tsikalas, *A note on a spectral constant associated with an annulus,* Operators and Matrices, 16 (2022) no. 1, 95–99. 1, 15, 16

[Ts22b] Georgios Tsikalas, *A von Neumann type inequality for an annulus,* J. Math. Anal. Appl. 506 (2022), no. 2, Paper No. 125714, 12 pp. 1, 2, 3, 4, 5, 15

[V74] N. Varopoulos *On an inequality of von Neumann and an application of the metric theory of tensor products to operators theory,* Journal of Functional Analysis 16.1 (1974): 83-100. 16

Department of Mathematics, University of Flordia, Gainesville, FL
Email address: sam@ufl.edu

Department of Mathematics, University of Flordia, Gainesville, FL
Email address: pascoe@ufl.edu