



Sticky particle Cucker–Smale dynamics and the entropic selection principle for the 1D Euler-alignment system

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ABSTRACT

We develop a global wellposedness theory for weak solutions to the 1D Euler-alignment system with measure-valued density, bounded velocity, and locally integrable communication protocol. A satisfactory understanding of the low-regularity theory is an issue of pressing interest, as smooth solutions may lose regularity in finite time. However, no such theory currently exists except for a very special class of alignment interactions. We show that the dynamics of the 1D Euler-alignment system can be effectively described by a nonlocal scalar balance law, the entropy conditions of which serves as an *entropic selection principle* that determines a unique weak solution of the Euler-alignment system. Moreover, the distinguished weak solution of the system can be approximated by the *sticky particle Cucker–Smale dynamics*. Our approach is inspired by the work of Brenier and Grenier on the pressureless Euler equations.

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1. Introduction

We are interested in the following Euler-alignment system

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho \mathbf{u}) = 0, & (x, t) \in \mathbb{R}^d \times \mathbb{R}_+, \\ \partial_t (\rho \mathbf{u}) + \nabla_x \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = \int_{\mathbb{R}^d} \rho(x, t) \rho(y, t) \phi(x - y) (\mathbf{u}(y, t) - \mathbf{u}(x, t)) dy, \end{cases} \quad (1)$$

subject to the initial data

$$\rho(x, 0) = \rho^0(x), \quad \mathbf{u}(x, 0) = \mathbf{u}^0(x).$$

Here $\rho \geq 0$ and $\mathbf{u} \in \mathbb{R}^d$ represent density and velocity, respectively. We shall make the global assumption that ρ is normalized to have total mass 1. The term on the right-hand side of the equation for the momentum $\rho \mathbf{u}$ is the *alignment* force. The function ϕ is called the *communication protocol*, and it governs the strength of the interactions between the ‘agents’ that comprise the density profile. Throughout the paper, we will assume ϕ is non-negative, locally integrable, and radially decreasing.

The Euler-alignment system comes from the theory of collective behavior. Its salient feature is the nonlocal alignment interaction, which for appropriate ϕ leads to a

remarkable long-time behavior referred to as *flocking* (a term intentionally reminiscent of a group of birds). We will discuss flocking in due course, but we do not attempt a comprehensive overview here. The most complete reference is [1], which also contains references to many other excellent reviews.

The nonlocality of (1) is a notorious difficulty in the study of the wellposedness and long-time behavior for this equation. However, there are two important cases where the nonlocality drops out. If $\phi \equiv 0$, then (1) reduces to the well-studied *pressureless Euler equations*. This case does not exhibit the alignment features associated with non-degenerate ϕ , but its more developed wellposedness theory showcases an arsenal of tools that one can try out on the Euler-alignment system. The other situation where nonlocality is not truly present is that of *all-to-all coupling*, where ϕ is a positive constant. Solutions of the all-to-all coupled system exhibit most features of the long-time behavior that one expects for more general ϕ . However, the analysis of this case is simpler: the alignment force reduces to a linear and local damping.

In this paper, we develop a global wellposedness theory for weak solutions of the 1D Euler-alignment system (1) with measure-valued density, and bounded velocity. Our analysis covers the classical setup when the communication protocol ϕ is bounded and Lipschitz. More interestingly, it also works for the case when ϕ is *weakly singular*, namely it has an integrable singularity at the origin. We show an asymptotic flocking behavior for the solutions we construct. Our approach adapts the *sticky particle* approximation, originally developed by Brenier and Grenier [2] to treat the 1D pressureless Euler equations. We require a detailed understanding of the relationship between the discrete and hydrodynamic settings; let us therefore review the derivation of (1) from the Cucker–Smale system.

1.1. A brief derivation of the Euler-alignment system

The Euler-alignment system can be derived as a hydrodynamic version of the celebrated Cucker–Smale system of ODE’s [3, 4]:

$$\begin{cases} \frac{dx_i}{dt} = v_i \\ \frac{dv_i}{dt} = \sum_{\substack{j=1 \\ x_j \neq x_i}}^N m_j \phi(x_j - x_i)(v_j - v_i), \end{cases} \quad (x_i, v_i) \in \mathbb{R}^d \times \mathbb{R}^d, \quad i = 1, \dots, N. \quad (2)$$

This system governs the motion of N agents with masses $m_i \geq 0$, positions x_i and velocities v_i . As the number of agents goes to infinity, one can derive a kinetic formulation, using BBGKY hierarchies [5] or mean-field limits [6–8]. The kinetic distribution function $f(x, v, t)$ solves the Vlasov-type equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (f F(f)) = 0, \\ F(f)(x, v, t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(y, w, t) \phi(x - y)(w - v) dw dy, \end{cases} \quad (x, v, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+. \quad (3)$$

Define the macroscopic density and momentum by

$$\rho(x, t) = \int_{\mathbb{R}^d} f(x, v, t) \, dv, \quad \mathbf{P}(x, t) = \rho(x, t) \mathbf{u}(x, t) = \int_{\mathbb{R}^d} v f(x, v, t) \, dv.$$

Here $\mathbf{u}(\cdot, t)$ is the macroscopic velocity, well-defined on $\Omega(t) := \{x : \rho(x, t) > 0\}$. Taking zeroth and first moments on v of the kinetic system then yields

$$\left\{ \begin{array}{l} \partial_t \rho + \nabla_x \cdot (\rho \mathbf{u}) = 0, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}_+, \\ \partial_t (\rho \mathbf{u}) + \nabla_x \cdot (\rho \mathbf{u} \otimes \mathbf{u} + \mathcal{R}) = \int_{\mathbb{R}^d} \rho(x, t) \rho(y, t) \phi(x - y) (\mathbf{u}(y, t) - \mathbf{u}(x, t)) \, dy, \\ \mathcal{R}(x, t) = \int_{\mathbb{R}^d} (v - \mathbf{u}(x, t)) \otimes (v - \mathbf{u}(x, t)) f(x, v, t) \, dv, \end{array} \right. \quad (4)$$

where $\mathcal{R}$ denotes the Reynolds stress tensor. Finally, one obtains the pressureless Euler-alignment system (1) as a hydrodynamic limit by taking a *monokinetic ansatz* $f(x, v, t) = \rho(x, t) \delta(v - \mathbf{u}(x, t))$, which eliminates $\mathcal{R}$ from (4). Rigorous justification (for bounded ϕ) can be found in [1, 9, 10].

An alternative hydrodynamic limit can be obtained [11–13] by taking an *isothermal ansatz* $f(x, v, t) = (2\pi)^{-d/2} \rho(x, t) \exp(-|v - \mathbf{u}(x, t)|^2/2)$, in which case $\nabla_x \cdot \mathcal{R}$ becomes a pressure term $\nabla_x \rho$. The resulting *isothermal Euler-alignment system* was investigated in [14, 15]. A more general class of *isentropic Euler-alignment system* with a pressure term $\nabla_x(\rho^\gamma)$, $\gamma \geq 1$ was considered in [16, 17]. The regularity theories of these cases are less well-developed than that of their pressureless cousin. We do not treat the pressured system further in the present work, as the assumption of monokineticity is strongly embedded in our framework.

Finally, many authors prefer for technical reasons to replace (1)₂ with the velocity equation

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \int \rho(y, t) \phi(x - y) (\mathbf{u}(y, t) - \mathbf{u}(x, t)) \, dy, \quad (5)$$

and to define $\mathbf{u}(\cdot, t)$ on all of $\mathbb{R}^d$, or on some domain containing $\text{supp} \rho(t)$. We will not use the Eq. (5) directly; the divergence form of (1) is more amenable to the framework we want to build. We mostly do not distinguish between the two formulations in our discussion of the literature.

1.2. Existing wellposedness theory on the pressureless Euler-alignment system

The global regularity theory for smooth solutions of the Euler-alignment system (1) is fairly well-established in one space dimension, with different types of alignment interactions.

The first scenario is when the interaction is *regular*, namely ϕ is bounded and Lipschitz. At the heart of many wellposedness results is the following quantity, introduced in [14]:

$$e(x, t) = \partial_x u(x, t) + \phi * \rho(x, t). \quad (6)$$

Here $*$ denotes convolution in the spatial variable. Remarkably, e satisfies the simple evolution equation

$$\partial_t e + \partial_x(ue) = 0. \quad (7)$$

This structure yields a precise description of global wellposedness via a *critical threshold* condition [14]: if the initial condition is *subcritical*, satisfying $e^0 \geq 0$ on all of $\mathbb{R}$, then the solution stays globally regular; otherwise, *supercritical* initial data lead to shock formations in finite time.

The next scenario is when the interaction is *weakly singular*, that is, when ϕ has an integrable singularity at the origin, e.g. $\phi(r) \sim r^{-s}$ with $s \in (0, 1)$. Under this setup, a similar but distinct critical threshold condition holds [18]; the main additional subtlety occurs when e^0 takes the value 0.

In the presence of vacuum $\rho^0(x) = 0$, the critical threshold conditions are inaccessible from physical initial data (ρ^0, P^0) . The first author [19] has worked with an antiderivative ψ of e , defined by

$$\psi(x, t) = u(x, t) + \Phi * \rho(x, t), \quad (8)$$

where Φ is the unique odd antiderivative of ϕ . The critical threshold condition can then be expressed in terms of ψ^0 , which must be nondecreasing *on the support* of ρ^0 to propagate regularity. This condition is also sufficient when the protocol ϕ is regular; for weakly singular interactions, additional assumptions are needed to propagate regularity. Sharp conditions are not known. In the present work, we make extensive use of the quantity ψ and its discrete analog for the Cucker–Smale system.

Another interesting scenario is when the interaction is *strongly singular*, namely ϕ has a *non-integrable* singularity at the origin. In this case, the alignment produces nonlocal dissipation that regularizes the solution. It has been shown that solutions are globally regular for *all* smooth initial data away from vacuum [20–24]. (However, the non-vacuum stipulation is important here, c.f. [25, 26].) Moreover, for rough initial data, the solutions are instantaneously regularized thanks to the strongly singular interaction, see e.g. [27–30].

The Euler-alignment system (1) is less well understood in higher dimensions, largely because of the lack of a scalar quantity e that solves a simple continuity equation like (7). A natural candidate for a multi-dimensional replacement is $e = \nabla_x \cdot \mathbf{u} + \phi * \rho$, which satisfies the equation $\partial_t e + \nabla_x \cdot (\mathbf{u}e) = (\nabla_x \cdot \mathbf{u})^2 - \text{trace}((\nabla_x \mathbf{u})^2)$. The right-hand side vanishes if the velocity is *unidirectional*, i.e., $\mathbf{u}(x, t) = u(x, t)\mathbf{h}$ with a fixed direction $\mathbf{h} \in \mathbb{R}^d$; in this case, the same threshold as in the 1D setting holds [31]. However, for general $\mathbf{u}$, the term $(\nabla_x \cdot \mathbf{u})^2 - \text{trace}((\nabla_x \mathbf{u})^2)$ does not vanish and is difficult to control (but c.f. [31] for the ‘almost unidirectional’ case). Partial results are available in 2D [32, 33], for radial solutions [34], and recently in higher dimensions [35].

As for the asymptotic behavior, Ha and Liu [7] have shown that the Cucker–Smale dynamics (2) enjoy the *flocking* property: when the communication weight ϕ has a *fat tail*, namely

$$\int_1^\infty \phi(r) \, dr = \infty, \quad (9)$$

then the diameter of the positions of all agents remains uniformly bounded in time, and moreover, the velocities of all agents tend to a common value as time goes to infinity. An analog of this property is inherited by the Euler-alignment system (1), at least

for smooth solutions: it was proved in [33] that if ϕ satisfies (9), then *strong solutions must flock*.

1.3. Weak solutions and the non-uniqueness issue

Though the theory of strong solutions to the Euler-alignment system has been well-developed over the last decade, little is known about weak solutions, even in one dimension. This is a serious gap in the theory, as solutions can lose regularity even if they are initially smooth. Several natural questions arise when considering the possibility of weak solutions:

- How does a solution evolve after the formation of a shock?
- Do weak solutions flock?
- How are weak solutions connected to the Cucker–Smale dynamics?

The global wellposedness theory we develop in this paper will address these points.

Let us remark that when the interaction is strongly singular and the initial data non-vacuous, due to instant regularization, solutions are smooth and the existing theory of strong solutions applies. Therefore, we shall focus on the case when the communication protocol ϕ is *locally integrable*, so the interaction can be either regular or weakly singular. A theory of weak solutions is needed, especially for supercritical initial data.

It is not difficult to formulate a satisfactory definition of a distributional weak solution of the Euler-alignment system (1). However, it is well-known that such solutions are not unique. In fact, Carrillo et al. [36] studied weak solutions to Euler systems with a general class of nonlocal interactions and showed that there exist infinitely many weak solutions that dissipate the kinetic energy. What we need, therefore, is an additional *selection principle* that will single out a unique weak solution.

The non-uniqueness issue is better understood for the 1D pressureless Euler system

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) = 0. \end{cases} \quad (10)$$

A well-known class of entropy inequalities for (10), introduced by Lax [37], requires that for any positive convex entropy η ,

$$\partial_t(\rho \eta(u)) + \partial_x(\rho u \eta(u)) \leq 0. \quad (11)$$

These entropy inequalities can be adapted to the 1D Euler-alignment system, by replacing $\eta(u)$ with $\eta(\psi)$, where ψ is defined in (8). However, the entropy inequalities (11) do not guarantee uniqueness. Bouchut [38] showed there are infinitely many entropic solutions to the 1D pressureless Euler equations that satisfy (11). He also presented an instructive example with atomic initial data: Consider a configuration where two particles move toward each other with velocities $v_1 > v_2$. One can impose different rules when they collide. As long as the collision preserves momentum, and the post-collision velocities $v'_1, v'_2 \in [v_2, v_1]$, this setup generates an entropic solution for the 1D pressureless Euler equations. Two particular solutions are: (i) no collision: $v'_i = v_i, i = 1, 2$;

(ii) completely inelastic collision: $v'_1 = v'_2$. Hence, a stronger selection principle is required to obtain a unique solution.

1.4. Sticky particle dynamics and selection principles

Among all the collision rules, the completely inelastic collision dissipates the most energy. Since the post-collision velocities are the same, the two particles stick to each other and travel together after the collision. This *sticky particle* model was originally proposed by Zeldovich [39]; it generates atomic weak solutions to (10). Grenier [40] also used the sticky particle dynamics to prove existence (but not uniqueness) of solutions to (10). The sticky particle dynamics underlie at some level all of the successful selection principles that we discuss below for the theory of (10) and related systems.

One line of results on the theory of (10) is related to a so-called *generalized variational principle* due to E, Rykov, and Sinai [41], which is compatible with the sticky particle dynamics and can serve as a selection principle. Huang and Wang [42] proved existence and uniqueness of weak solutions satisfying the *one-sided Lipschitz condition*

$$\frac{u(x_2, t) - u(x_1, t)}{x_2 - x_1} \leq \frac{1}{t}, \quad t > 0. \quad (12)$$

Their construction is based on a *generalized potential* and has strong ties to the variational principle of [41] (c.f. also the earlier works [43, 44]). The framework of [42] has been adapted to the 1D Euler-alignment system with all-to-all coupling [45, 46], where the alignment interaction reduces to a linear, local damping. It is not clear whether this approach can be extended to the truly nonlocal case of a general communication protocol ϕ ; to our knowledge, this question has not been treated in the literature.

Another type of selection principle for (10) is based on the entropy conditions for a related scalar equation. This approach was pioneered by Brenier and Grenier [2], who study (10) by connecting it to the *scalar conservation law*

$$\partial_t M + \partial_x(A(M)) = 0, \quad (13)$$

where M is the cumulative distribution function of the density ρ , and the flux A only depends on the initial data ρ^0 and u^0 . The entropy conditions for (13) select a unique solution M , which determines a distinguished weak solution of the pressureless Euler equations through $\rho = \partial_x M$, $\rho u = \partial_x(A(M))$. Hence, the entropy inequalities for the scalar conservation law (13) serve as a satisfactory selection principle for (10). Moreover, the entropy solution of (13) has an elegant connection to the sticky particle dynamics: with a discretized initial condition M_N^0 and flux A_N , the sticky particle dynamics generate the entropy solution of the scalar conservation law by tracking the locations of all the shocks of M_N .

Bouchut and James have developed a related but alternative theory of *duality solutions* [47, 48] for solutions to (10). Their theory relies on properties of monotone solutions to (13), and they prove uniqueness under an assumption similar to (but stronger than) (12). The framework of [2] has been successfully applied to the 1D Euler–Poisson equations [49–51]. To handle the additional nonlocal Poisson force, a nonlocal scalar conservation law (13) is generated, with a time dependent flux $A = A(M, t)$. A similar argument also works for the 1D Euler-alignment system with all-to-all coupling [52].

The sticky particle approach to the pressureless Euler equations and related systems has continued to garner attention, through the lens of optimal transport [49–51, 53, 54], and also from a probabilistic perspective [55–60]. An exhaustive review of the literature on the pressureless Euler system is far beyond the scope of the present work. However, the approaches described above provide sufficient context for the ideas developed in our paper.

1.5. Scalar balance laws for the Euler-alignment system

We are interested in the 1D Euler-alignment system with general alignment interactions.

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) = \rho(\phi * (\rho u)) - \rho u(\phi * \rho). \end{cases} \quad (14)$$

The major challenge is to appropriately treat the nonlocality from the alignment interactions. Unfortunately, the system cannot be connected to a scalar conservation law of the type (13), except for special cases, e.g. constant ϕ .

We introduce a new scalar *balance* law connected to the 1D Euler-alignment system:

$$\partial_t M + \partial_x(A(M)) = (\phi * M)\partial_x M. \quad (15)$$

The contribution from the alignment interaction is split between the flux term and the nonlinear, nonlocal right-hand side of (15). Following the framework of Brenier and Grenier, we establish a global wellposedness theory for (15) and show that there is a unique entropy solution. We then construct a unique weak solution to the 1D Euler-alignment system (14), using the entropy conditions for (15) as our selection principle. The intrinsic nonlocality embedded in (15) requires a significant advancement of the analytical techniques used for (13), in terms of both the global wellposedness theory and (more significantly) the precise connection with (14). For example, we prove that solutions M of (15) are stable with respect to perturbations of the initial data M^0 and the flux A . Our bound is the same as the one for scalar conservation laws (c.f. [Theorem 3, 61]); however, we need to use monotonicity of $M(\cdot, t)$ (built into our definition of entropy solution) in an essential way to treat the nonlocal term, whereas this assumption is not needed to treat (13). Another distinctive feature of (15) is that $\partial_x(A(M))$ does *not* represent the momentum ρu ; rather, it is equal to $\rho\psi$. We use the relation (8) to recover the momentum. Finally, when ϕ is merely locally integrable, the fact that $\phi * M$ may not be differentiable presents challenges in both the existence and stability proofs: The term corresponding to $(\phi * M)\partial_x M$ in our definition of entropy solution is the most subtle with respect to convergence of the discretization. Furthermore, it necessitates an additional application of the BV chain rule and the use of a delicate cancelation in our uniqueness and stability argument.

We also explore the connection between (15) and the *sticky particle Cucker–Smale* dynamics, introduced in Section 3. We show that the entropy solution of (15) can be constructed through an approximation by a sequence of solutions to the sticky particle Cucker–Smale dynamics. Ultimately, our approach yields a uniquely determined solution to (14), through a discrete approximation by sticky particle Cucker–Smale

dynamics, for measure-valued data. Under additional assumptions on the initial conditions and protocol ϕ , we obtain an error estimate for the approximation, with an explicit convergence rate of up to $O(N^{-1})$. The rate echoes the work of Lucier [61] on scalar conservation laws. This paves a clean path for the numerical implementation of our solution through the sticky particle approximation.

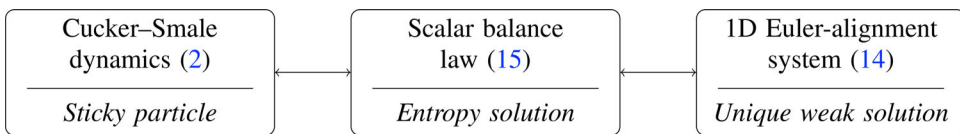
Finally, we study the asymptotic behavior of our constructed solution to (14). Applying uniform estimates on the sticky particle Cucker–Smale dynamics and the convergence result of the approximation, we establish the same flocking property that is enjoyed by strong solutions: If ϕ satisfies (9), our *weak solutions must flock*.

Remark 1.1 (Interpretation of the solution). Our method of construction and the ‘sticky particle’ framework are naturally compatible with the monokinetic ansatz mentioned in Section 1.1. Different closure assumptions (e.g., a Maxwellian) will require different techniques. A recent work of Amadori and Christoforou [62] treats the global existence and asymptotic behavior of weak solutions to the Euler-alignment system with all-to-all coupling ($\phi \equiv 1$) and an additional pressure term resulting from a Maxwellian ansatz. Their solutions are generated by a front-tracking scheme, but their analysis otherwise has little overlap with our work or with the sticky particle-based literature described previously.

In this work, our primary interest in the sticky particle Cucker–Smale dynamics relates to their role in selecting a unique solution of the Euler-alignment system. However, one can interpret them physically as an ‘extreme’ case of strong local alignment. We conjecture that this heuristic can be rigorously justified, and we plan to address it in future work.

1.6. Main results and structure of the paper

We study the following three systems and their connections. The sticky particle collision rule in the Cucker–Smale dynamics corresponds to the entropy conditions for the scalar balance law, which in turn serves as the selection principle to the unique weak solution of the 1D Euler-alignment system.



Our results are summarized in the following points. The explicit statements appear in the text.

- **Existence, Uniqueness, and Stability.**

- For monotone initial data M^0 and Lipschitz flux A , the scalar balance law (15) has a unique entropy solution M , which is BV in space and time (Theorem 5.1(a)). The solution is stable under perturbations of both M^0 and A (Theorem 5.1(c)).
- Given $\rho^0 \in \mathcal{P}_c(\mathbb{R})$ (a compactly supported probability measure) and $u^0 \in L^\infty(d\rho^0)$, a unique solution of (14) can be generated from the unique entropy solution of (15) with corresponding initial data M^0 and flux A (Theorem 6.3, the *entropic selection principle*).

- **Discretization and Approximability.**
 - The sticky particle Cucker–Smale dynamics determine the entropy solution of the scalar balance law (15) for discretized initial data M_N^0 and flux A_N (Theorem 4.1).
 - The sticky particle Cucker–Smale dynamics approximate the unique solution of (15) and (14) for general initial data (Theorems 5.1(b) and 6.5), with an explicit convergence rate depending on ϕ and u^0 (Theorem 6.7).
- **Long-Time Behavior and Flocking.**
 - Our constructed weak solution to the 1D Euler-alignment system (14) exhibits the flocking phenomenon (Theorem 7.2): If ϕ which decays slowly enough, the velocity u converges to a constant, and ρ converges to a traveling wave ρ_∞ .

Outline of the paper

In Section 2, we give a formal derivation of the scalar balance law (15) from the Euler-alignment system and derive the associated Rankine–Hugoniot condition and Oleinik entropy condition. In Section 3, we discuss the properties of the sticky particle Cucker–Smale dynamics that are needed for our wellposedness theory. In Section 4, we connect the sticky particle Cucker–Smale dynamics to the entropy solution of the discretized scalar balance law (15). In Section 5, we present the wellposedness theory for entropy solutions of (15) and prove the convergence of the sticky particle approximation. In Section 6, we establish rigorously the connection between solutions to the scalar balance law (15) and the 1D Euler-alignment system (14), construct a unique weak solution, and study the approximation by the sticky particle Cucker–Smale dynamics. Finally, in Section 7, we study the asymptotic flocking behavior.

2. Derivation of the scalar balance law and entropy conditions

2.1. The scalar balance law

We give a formal derivation of the scalar balance law (15) from the 1D Euler-alignment system (14), assuming all functions involved are as regular as necessary. Rigorous justification of the equivalence between the two systems will be made in Section 6.

We begin by reformulating the 1D Euler-alignment system as follows. Integrating (7) yields

$$\partial_t \psi + u \partial_x \psi = 0, \quad (16)$$

where $\psi = u + \Phi * \rho$ as in (8), and

$$\Phi(x) = \int_0^x \phi(y) \, dy. \quad (17)$$

The 1D Euler-alignment system can then be expressed in terms of the pair $(\rho, \rho\psi)$:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho\psi) + \partial_x(\rho\psi u) = 0. \end{cases} \quad (18)$$

The velocity can be recovered from ψ and ρ via the relation (8):

$$u = \psi - \Phi * \rho. \quad (19)$$

Since ρ and $\rho\psi$ satisfy the same continuity Eq. (18), their primitives

$$M(x, t) = -\frac{1}{2} + \int_{(-\infty, x]} \rho(y, t) dy, \quad Q(x, t) = \int_{(-\infty, x]} \rho\psi(y, t) dy \quad (20)$$

will satisfy the transport equations

$$\partial_t M + u\partial_x M = 0, \quad \partial_t Q + u\partial_x Q = 0.$$

Let A be a map (depending only on the initial data (M^0, Q^0)) such that $A \circ M^0 = Q^0$. If (x, t) lies on a characteristic path originating at $(x^0, 0)$ and governed by the velocity field u , then

$$Q(x, t) = Q^0(x^0) = A(M^0(x^0)) = A(M(x, t)).$$

Using this identity $Q = A(M)$, as well as the definitions of ρ and ψ , we can write

$$u\partial_x M = \rho\psi - \rho\Phi * \rho = \partial_x Q - \partial_x M \cdot (\phi * M) = \partial_x(A(M)) - \partial_x M \cdot (\phi * M),$$

which leads to the scalar balance law (15).

Remark 2.1. We include a shift of $-\frac{1}{2}$ in the definition of M in (20) to make sense of the convolution $\phi * M$. We will always work with ρ such that $\rho(t)$ is supported in a compact interval $[-R(T), R(T)]$ for any $t \in [0, T]$, so that $M(\pm x, t) = \pm \frac{1}{2}$ for any $x \geq R(T)$. For $R \geq R(T)$, we define

$$\phi * M(x, t) = \int_{-2R}^{2R} \phi(z)M(x - z, t) dz, \quad \text{for all } x \in [-R, R]. \quad (21)$$

Since ϕ is even, the choice of R is inconsequential to the value of $\phi * M$; consequently, the definition (21) defines $\phi * M(x, t)$ for all $x \in \mathbb{R}$, $t \in [0, T]$. Moreover,

$$\phi * M(x, t) = \Phi(y)M(x - y) \Big|_{y=-2R}^{y=2R} + \int_{-2R}^{2R} \Phi(y)\rho(x - y, t) dy = \Phi * \rho(x, t),$$

for all $x \in [-R, R]$. Both boundary terms take the value $\frac{1}{2}\Phi(2R)$ and hence cancel with each other.

Finally, it is easy to see that $\phi * M = \Phi * \rho$ is bounded (by $\Phi(2R)$) and continuous (since Φ is).

2.2. Entropy conditions to the scalar balance law

The scalar balance law (15) admits a natural admissibility criterion via entropy inequalities. Let $\eta : [-\frac{1}{2}, \frac{1}{2}] \rightarrow \mathbb{R}$ be a convex and Lipschitz function and suppose $q : [-\frac{1}{2}, \frac{1}{2}] \rightarrow \mathbb{R}$ satisfies $q' = \eta'A'$. The pair (η, q) is known as an entropy/entropy-flux pair, and the entropy inequality for (15) associated to each such pair reads

$$\partial_t(\eta(M)) + \partial_x(q(M)) \leq (\phi * M) \partial_x(\eta(M)), \quad (22)$$

in the sense of distributions. That is, for any nonnegative test function $\zeta \in C_c^\infty(\mathbb{R} \times (0, T))$, we require

$$\int_0^T \int_{\mathbb{R}} [\eta(M)\partial_t \zeta + q(M)\partial_x \zeta + \zeta(\phi * M)\partial_x(\eta(M))] dx dt \geq 0. \quad (23)$$

Let us comment on the last term in (23). When ϕ is bounded and Lipschitz, it is easy to check that $\phi * M$ is Lipschitz. In this case, the integral can be realized as

$$\int_0^T \int_{\mathbb{R}} \zeta(\phi * M) \partial_x(\eta(M)) \, dx \, dt = - \int_0^T \int_{\mathbb{R}} [\partial_x \zeta(\phi * M) + \zeta \partial_x(\phi * M)] \eta(M) \, dx \, dt.$$

For general locally integrable communication protocol ϕ , note that $\eta(M) \in BV_{\text{loc}}(\mathbb{R})$ (e.g. by Lemma 5.2) and is constant outside of $[-R, R]$. Hence, $\partial_x(\eta(M))$ is a Radon measure with support in $[-R, R]$. Since $\phi * M \in C_b([-R, R])$, it follows that the integral is well-defined.

Definition 2.1. We say $M : \mathbb{R} \times [0, T] \rightarrow [-\frac{1}{2}, \frac{1}{2}]$ is an *entropy solution* to the scalar balance law (15) if

- The entropy inequality (22) is satisfied for every entropy/entropy flux pair (η, q) , in the sense outlined in the discussion above.
- $M(\cdot, t)$ is nondecreasing, for any $t \in [0, T]$.
- There exists an $R(T) > 0$, such that $M(\pm x, t) = \pm \frac{1}{2}$, for any $x \geq R(T)$ and $t \in [0, T]$.

We say $M : \mathbb{R} \times [0, +\infty) \rightarrow [-\frac{1}{2}, \frac{1}{2}]$ is an entropy solution if its restriction to any compact time interval $[0, T]$ is an entropy solution in the sense above.

By a standard approximation argument, one deduces that the collection of all entropy/entropy flux pairs in Definition 2.1 may be replaced a smaller class. The entropy solution can be equivalently defined if the entropy inequality (22) is satisfied for every *Kruzkov entropy pair*

$$\eta(m) = |m - \alpha|, \quad q(m) = \text{sgn}(m - \alpha)(A(m) - A(\alpha)), \quad \alpha \in \left[-\frac{1}{2}, \frac{1}{2}\right]. \quad (24)$$

We will use the definition (24) in the proofs of existence and uniqueness of entropy solutions to (15).

Next, we state the Rankine–Hugoniot condition and the Oleinik entropy condition for (15). Since the nonlocal term on the right-hand side of (22) plays a role in the conditions, we shall outline the derivation.

Suppose M takes the values M_ℓ and M_r on two sides of a shock along a curve $C = \{(x, t) : x = s(t)\}$. We denote by $\sigma(t) = \dot{s}(t)$ the speed of the shock. The entropy condition (23) becomes

$$\int_C ([[\eta(M)]] \nu_t(s) + ([[q(M)]] - (\phi * M)[[\eta(M)]]) \nu_x(s)) \zeta \, ds \geq 0,$$

where $\nu = (\nu_x, \nu_t) = (1 + \sigma^2)^{-\frac{1}{2}}(1, -\sigma)$ is the unit normal vector along C , and we have used the notation $[[\eta(M)]] = \eta(M_\ell) - \eta(M_r)$ and $[[q(M)]] = q(M_\ell) - q(M_r)$. This implies

$$(\sigma + (\phi * M))[[\eta(M)]] \leq [[q(M)]], \quad \text{along } C.$$

Taking $\eta = \text{id}$ and $q = A$, and noting that equality should hold in (23), we get the Rankine–Hugoniot condition

$$\sigma + \phi * M = \frac{[[A(M)]]}{[[M]]}. \quad (25)$$

For $M_\ell < M_r$, we take $\eta(m) = (m - \theta)H(m - \theta)$ and $q(m) = (A(m) - A(\theta))H(m - \theta)$ for $\theta \in (M_\ell, M_r)$. Here, H denotes the Heaviside function. This leads to the *Oleinik entropy condition*

$$\sigma + \phi * M \leq \frac{A(\theta) - A(M_\ell)}{\theta - M_\ell}, \quad \theta \in (M_\ell, M_r). \quad (26)$$

3. The sticky particle Cucker–Smale dynamics

3.1. Definitions and notation

Consider a system of particles of positive masses $(m_i)_{i=1}^N$ and configurations $(x_i(t), v_i(t))_{i=1}^N$ following the Cucker–Smale dynamics (2) in one dimension

$$\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = \sum_{\substack{j=1 \\ x_j \neq x_i}}^N m_j \phi(x_j - x_i)(v_j - v_i). \quad (27)$$

We always assume the total mass is 1:

$$\sum_{i=1}^N m_i = 1. \quad (28)$$

The system (27)–(28) allows particles to pass through each other. We propose a modified system: the *sticky particle Cucker–Smale dynamics*, which follows (27) except at times when two or more particles *collide*, i.e., occupy the same position (for the first time). We insist that colliding particles remain stuck together for all future times. We refer to the unmodified system (27)–(28) as the *Cucker–Smale dynamics without collisions*, in order to emphasize the distinction between the two sets of dynamics.

Let us now define precisely the collision rules for the sticky particle Cucker–Smale dynamics. We fix the notation $J_i(t)$ to represent the set of indices j such that particle j is stuck to particle i at time t :

$$J_i(t) := \{j \in 1, \dots, N : x_j(t) = x_i(t)\}.$$

The collision rules have two ingredients:

- Each collision is *completely inelastic*, and particles stick to each other after collisions:

$$J_i(t) \supseteq J_i(s), \quad \text{whenever } t \geq s \geq 0; \quad (29)$$

- The collision conserves momentum:

$$v_i(t+) = \frac{\sum_{j \in J_i(t)} m_j v_j(t-)}{\sum_{j \in J_i(t)} m_j}. \quad (30)$$

We will always index the particles in increasing order from left to right:

$$x_1(t) \leq x_2(t) \leq \dots \leq x_N(t).$$

Since the collision rules do not allow particles to cross each other, the order stays unchanged for all time. It will be convenient to set notation for lowest and highest indices in a given $J_i(t)$:

$$i_*(t) = \min J_i(t), \quad i^*(t) = \max J_i(t).$$

A time t is called a *collision time* if the cardinality of one or more of the J_i 's increases at time t . At collision times, we make the convention that the v_i 's are right continuous, i.e., $v_i(t) = v_i(t+)$. We set $v_j(0-) = v_j^0$ by convention, to account for the possibility that some of the $x_i^0 = x_i(0)$'s might coincide and create a 'collision' at time zero. It is clear that at most $N - 1$ collisions can occur. Thus global-in-time existence and uniqueness of the sticky particle Cucker–Smale dynamics is a triviality.

3.2. Basic properties for the sticky particle Cucker–Smale dynamics

We begin by stating the maximum principle on $(v_i)_{i=1}^N$, which is well-known for the Cucker–Smale dynamics without collisions.

Proposition 3.1. *Suppose $(x_i(t), v_i(t))_{i=1}^N$ follows the sticky particle Cucker–Smale dynamics associated to the data $(x_i^0, v_i^0, m_i)_{i=1}^N$. Then*

$$\min_{1 \leq j \leq N} v_j(s) \leq v_i(t) \leq \max_{1 \leq j \leq N} v_j(s), \quad \text{for any } i = 1, \dots, N, \quad t \geq s \geq 0. \quad (31)$$

Consequently, if $\{x_i^0\}_{i=1}^N \subset [-R^0, R^0]$, then

$$\{x_i(t)\}_{i=1}^N \subset [-R(t), R(t)], \quad R(t) := R^0 + t \max_{1 \leq j \leq N} |v_j^0|, \quad \text{for all } t \geq 0. \quad (32)$$

Proof. On collisionless time intervals, the bounds (31) follow immediately from the dynamics of v_i in (27). It is also clear that the collision rule (30) respects the maximum principle (31). The assertion (32) follows immediately from (31). $\square$

Remark 3.1. A nondegenerate alignment forcing in the velocity equation will often decrease the best possible $R(t)$ in (32). In fact, under appropriate assumptions on the size and support of ϕ , the radius $R(t)$ may be chosen independent of t , and each $v_i(t)$ will converge to the average $\bar{v} = \sum_i m_i v_i^0$ as $t \rightarrow \infty$. These phenomena are known as *flocking* and *velocity alignment* and are well-known for the Cucker–Smale dynamics without collisions, c.f. [7]. We will discuss these properties in the context of the sticky particle Cucker–Smale dynamics and the Euler-alignment system in Section 7.

We now consider some properties of the sticky particle Cucker–Smale dynamics that are unique to the 1D setting. These properties are connected to the quantity

$$\psi_i(t) = v_i(t) + \sum_{j=1}^N m_j \Phi(x_i(t) - x_j(t)), \quad (33)$$

where Φ is the odd antiderivative of ϕ , defined in (17). The quantity ψ_i is a discrete analog of the macroscopic ψ from (8). It plays the same role in many respects that the

particle velocities v_i do in the absence of alignment force. Indeed, $\psi_i = v_i$ in the degenerate case where $\phi \equiv 0$. The quantities ψ_i have been used to study the 1D Cucker–Smale dynamics without collisions [63, 64]. In the 1D sticky particle setup, they play a crucial role in the analysis of collisions. We list two basic properties below.

Proposition 3.2. *The quantities ψ_i have the following properties.*

- (a) *Each $\psi_i(t)$ is constant in time in the absence of collisions; for any t that is not a collision time,*

$$\frac{d}{dt}\psi_i(t) = 0. \quad (34)$$

- (b) *If t is a collision time, we have*

$$\psi_i(t+) = \frac{\sum_{j \in J_i(t)} m_j \psi_j(t-)}{\sum_{j \in J_i(t)} m_j}. \quad (35)$$

Eq. (34) is verified by differentiating the formula for $\psi_i(t)$ and using (27). It is a discrete analog of (16). To prove (35), we apply the collision rule (30) and use the continuity of the trajectories $x_i(t)$:

$$\begin{aligned} \psi_i(t+) &= v_i(t+) + \sum_{k=1}^N m_k \Phi(x_i(t) - x_k(t)) \\ &= \frac{\sum_{j \in J_i(t)} m_j v_j(t-)}{\sum_{j \in J_i(t)} m_j} + \frac{\sum_{j \in J_i(t)} m_j \left(\sum_{k=1}^N m_k \Phi(x_j(t) - x_k(t)) \right)}{\sum_{j \in J_i(t)} m_j} = \frac{\sum_{j \in J_i(t)} m_j \psi_j(t-)}{\sum_{j \in J_i(t)} m_j}. \end{aligned}$$

3.3. The barycentric lemma

We are now in a position to state the following *barycentric* lemma, which is the key to connecting the sticky particle dynamics with the entropy solutions of (15). A similar argument has been implemented in [2, Lemma 2.2] for the pressureless Euler equation, when $\phi \equiv 0$.

Lemma 3.3. *Fix an $i \in \{1, \dots, N\}$ and a time $t > 0$. For any $k \in J_i(t)$, we have*

$$\frac{\sum_{j=i_*(t)}^k m_j \psi_j(t-)}{\sum_{j=i_*(t)}^k m_j} \geq \frac{\sum_{j \in J_i(t)} m_j \psi_j(t-)}{\sum_{j \in J_i(t)} m_j} = \psi_i(t+) \geq \frac{\sum_{j=k}^{i^*(t)} m_j \psi_j(t-)}{\sum_{j=k}^{i^*(t)} m_j}. \quad (36)$$

Proof. It suffices to establish the following monotonicity property:

$$\psi_{i_*(t)}(t-) \geq \psi_{i_*(t)+1}(t-) \geq \dots \geq \psi_{i^*(t)}(t-). \quad (37)$$

However, (37) follows directly from the corresponding obvious monotonicity property for the velocities

$$v_{i_*(t)}(t-) \geq v_{i_*(t)+1}(t-) \geq \dots \geq v_{i^*(t)}(t-),$$

after taking into account that $\psi_j(t-) - v_j(t-)$ is independent of $j \in J_i(t)$. $\square$

4. Entropy solutions to the discretized balance law

In this section, we study the following discretized scalar balance law

$$\partial_t M_N + \partial_x (A_N(M_N)) = (\phi * M_N) \partial_x M_N, \quad M_N(x, 0) = M_N^0(x). \quad (38)$$

as a first step toward understanding (15). Eqs. (38) and (15) are identical except for notation; we write (38) separately to highlight the special discretized initial data M_N^0 and flux A_N under consideration.

Let us describe our hypotheses. We assume M_N^0 is piecewise constant, of the form

$$M_N^0(x) = -\frac{1}{2} + \sum_{j=1}^N m_j H(x - x_j^0), \quad (39)$$

where the m_j 's are all strictly positive and sum to unity, as in (28). We also assume $x_1^0 \leq x_2^0 \leq \dots \leq x_N^0$, and we use H to denote the right-continuous Heaviside function, with $H(0) = 1$. Note that the range of M_N is discrete, consisting of the values $(\theta_i)_{i=0}^N$ defined by

$$\theta_i := -\frac{1}{2} + \sum_{j=1}^i m_j. \quad (40)$$

We define A_N as a continuous and piecewise linear function, with breakpoints only at $(\theta_i)_{i=1}^{N-1}$:

$$A_N : \left[-\frac{1}{2}, \frac{1}{2}\right] \rightarrow \mathbb{R}, \quad A_N \text{ is linear in each interval } [\theta_{i-1}, \theta_i], \text{ for any } i = 1, \dots, N. \quad (41)$$

Our main purpose in this section is to demonstrate that one can generate an entropy solution to (38) using the sticky particle Cucker–Smale dynamics. This builds a connection between the collision rules (29)–(30) and the entropy conditions for the scalar balance law (15). This connection will be further developed later into a selection principle for a unique weak solution of the 1D Euler-alignment system.

Theorem 4.1. *Consider the scalar balance law (38) with discrete initial data M_N^0 and flux A_N , satisfying the hypotheses (39) (for some m_i 's and x_i^0 's as described above) and (41), respectively. For each $i = 1, \dots, N$, define ψ_i^0 via*

$$m_i \psi_i^0 = A_N(\theta_i) - A_N(\theta_{i-1}), \quad (42)$$

and put

$$v_i^0 = \psi_i^0 - \sum_{j=1}^N m_j \Phi(x_i^0 - x_j^0). \quad (43)$$

Let $(x_i(t), v_i(t))_{i=1}^N$ follow the sticky particle Cucker–Smale dynamics associated to the masses $(m_i)_{i=1}^N$ and the initial conditions $(x_i^0, v_i^0)_{i=1}^N$. Then

$$M_N(x, t) = -\frac{1}{2} + \sum_{i=1}^N m_i H(x - x_i(t)) \quad (44)$$

is an entropy solution of the discretized balance law (38). Moreover, we have

$$A_N \circ M_N(x, t) = A_N\left(-\frac{1}{2}\right) + \sum_{i=1}^N m_i \psi_i(t) H(x - x_i(t)). \quad (45)$$

Proof. Since $M_N(\cdot, t)$ in (44) is piecewise constant, it suffices to check that the shock discontinuities along the curves $C_i = \{(x_i(t), t) : t \geq 0\}$ satisfy the Rankine–Hugoniot condition (25) and the Oleinik entropy condition (26), with the shock speed $\sigma_i(t) = v_i(t)$.

Fix a point $(x_i(t), t)$ on C_i . By definition (44), we get

$$M_N(x_i(t)-, t) = \theta_{i_*(t)-1}, \quad M_N(x_i(t)+, t) = M_N(x_i(t), t) = \theta_{i^*(t)}.$$

We denote the jump of a function f across C_i by $[[f]] = f(x_i(t)+) - f(x_i(t)-)$. Thus

$$[[M_N(\cdot, t)]] = \theta_{i^*(t)} - \theta_{i_*(t)-1} = \sum_{j \in J_i(t)} m_j,$$

and from (42)

$$[[A_N \circ M_N(\cdot, t)]] = A_N(\theta_{i^*(t)}) - A_N(\theta_{i_*(t)-1}) = \sum_{j \in J_i(t)} m_j \psi_j^0.$$

Applying (35), we verify the Rankine–Hugoniot condition (25)

$$\frac{[[A_N \circ M_N(\cdot, t)]]}{[[M_N(\cdot, t)]]} = \frac{\sum_{j \in J_i(t)} m_j \psi_j^0}{\sum_{j \in J_i(t)} m_j} = \psi_i(t) = v_i(t) + \phi * M_N(t), \quad \text{along } C_i.$$

Next, we check the Oleinik entropy condition (26), that is,

$$v_i(t) + \phi * M_N(x_i(t), t) \leq \frac{A_N(\theta) - A_N(\theta_{i_*(t)-1})}{\theta - \theta_{i_*(t)-1}}, \quad \theta \in (\theta_{i_*(t)-1}, \theta_{i^*(t)}).$$

Since A_N is piecewise linear, it suffices to check the inequality for $\theta = \theta_k$, $k \in \{i_*(t) - 1, \dots, i^*(t)\}$. Applying Lemma 3.3, we obtain

$$\frac{A_N(\theta_k) - A_N(\theta_{i_*(t)-1})}{\theta_k - \theta_{i_*(t)-1}} = \frac{\sum_{j=i_*(t)}^k m_j \psi_j^0}{\sum_{j=i_*(t)}^k m_j} \geq \frac{\sum_{j \in J_i(t)} m_j \psi_j^0}{\sum_{j \in J_i(t)} m_j} = \psi_i(t) = v_i(t) + \phi * M_N(x_i(t), t).$$

Finally, we check (45). The equality is trivial when $x < x_1(t)$. For $x \geq x_1(t)$, let k be the smallest index such that $x \geq x_k(t)$. Then we have $M_N(x, t) = \theta_k$, so that, recalling (42), we have

$$A_N(M_N(x, t)) = A_N(\theta_0) + \sum_{i=1}^k (A_N(\theta_i) - A_N(\theta_{i-1})) = A_N\left(-\frac{1}{2}\right) + \sum_{i=1}^k m_i \psi_i^0.$$

The conservation of momentum (35) implies that for our choice of x , we have

$$\sum_{i=1}^k m_i \psi_i^0 = \sum_{i=1}^k m_i \psi_i(t) = \sum_{i=1}^N m_i \psi_i(t) H(x - x_i(t)),$$

which ends the proof. □

Remark 4.1. The flux A_N need not be exactly piecewise linear in order for the analysis above to work, but it does need to satisfy certain requirements. More specifically, let $\tilde{A}_N$ be a Lipschitz function such that $\tilde{A}_N(\theta_i) = A_N(\theta_i)$ for each i . One can repeat the analysis of this section, with $\tilde{A}_N$ replacing A_N , and obtain the exact same weak solution $M_N(t)$ of the discretized system (38). This weak solution will be an *entropy* solution of the discretized system (with flux $\tilde{A}_N$) if $\tilde{A}_N \geq A_N$, as one can readily check. On the other hand, if the x_i^0 's are all distinct and if $\tilde{A}_N(m) < A_N(m)$ for some m , then it is immediate that $M_N(t)$ is *not* an entropy solution.

5. The scalar balance law

In this section, we focus on developing global wellposedness theory for the scalar balance law (15), which we recall for the reader's convenience:

$$\partial_t M + \partial_x(A(M)) = (\phi * M)\partial_x M, \quad M(x, 0) = M^0(x). \quad (46)$$

The existence and uniqueness theory for entropy solutions of scalar conservation laws has been well-established. The additional feature of (46) is the right-hand side of the equation, which is both nonlinear and nonlocal, requires extra treatment. We show that the entropy solution of (46), in the sense of Definition 2.1, exists and is unique. Furthermore, it can be approximated by the sticky particle Cucker–Smale dynamics. We also obtain stability bounds with respect to the initial condition M^0 , as well as the flux A . Our main theorem is stated as follows.

Theorem 5.1. *Consider the scalar balance law (46). Assume the initial condition M^0 is a nondecreasing function and that there exists an $R^0 > 0$ such that $M^0(\pm x) = \pm \frac{1}{2}$ for any $x \geq R^0$. Let the flux $A : [-\frac{1}{2}, \frac{1}{2}] \rightarrow \mathbb{R}$ be a Lipschitz function.*

- (a) (Existence and Uniqueness) *Given any $T > 0$, the Cauchy problem (46) has a unique entropy solution*

$$M \in BV(\mathbb{R} \times [0, T]).$$

- (b) (Approximability) *For any $T > 0$, the entropy solution M of (46) on $[0, T]$ can be approximated by the discretized balance law (38), and hence by the sticky particle Cucker–Smale dynamics, in the following sense. There exists a sequence of (explicit) discrete initial data M_N^0 and fluxes A_N , satisfying the hypotheses (39) and (41), respectively, such that the associated entropy solutions M_N of (46) satisfy*

$$M_N - M \rightarrow 0 \quad \text{in } C([0, T]; L^1(\mathbb{R})), \quad (47)$$

and

$$\partial_t M_N(\cdot, t) \xrightarrow{*} \partial_t M(\cdot, t) \quad \text{in } \mathcal{M}(\mathbb{R}). \quad (48)$$

for any $t \in [0, T]$. Here, $\mathcal{M}$ is the space of measures.

- (c) (Stability) *Let $\tilde{M}$ be the entropy solution of the scalar balance law*

$$\partial_t \tilde{M} + \partial_x(\tilde{A}(\tilde{M})) = (\phi * \tilde{M})\partial_x \tilde{M}, \quad \tilde{M}(x, 0) = \tilde{M}^0(x)$$

with initial data $\tilde{M}^0$ and flux $\tilde{A}$ that satisfy the same assumptions as M^0 and A respectively. Then for any $t \geq 0$, we have the following stability bound:

$$\|M(\cdot, t) - \tilde{M}(\cdot, t)\|_{L^1(\mathbb{R})} \leq \|M^0 - \tilde{M}^0\|_{L^1(\mathbb{R})} + t\|A - \tilde{A}\|_{\text{Lip}}. \quad (49)$$

The rest of the section is devoted to the proof of this theorem. Before beginning in earnest, however, we note the following. In our argument, we will need to differentiate the composition of a Lipschitz function and a BV function. To make sense of such an operation, one can use Vol'pert's theory of the BV calculus [65]. The precise version of the BV chain rule that we need is stated in [66, Lemma A2.1].

Lemma 5.2. *Suppose $W \in BV_{\text{loc}}(\mathbb{R})$ and f is Lipschitz. Then $f \circ W$ belongs to $BV_{\text{loc}}(\mathbb{R})$, and in the sense of measures,*

$$\left| \frac{d}{dx} (f \circ W) \right| \leq |f|_{\text{Lip}} \left| \frac{d}{dx} W \right|. \quad (50)$$

5.1. Existence and approximability

We start with the existence part of Theorem 5.1(a). The plan is to construct an entropy solution of (46) using the front-tracking scheme, c.f. [Chapter 14, 67]. A front-tracking approximation of (46) follows precisely the dynamics of the discretized balance law (38); therefore, we will construct a sequence of approximated solutions M_N , extract a limit M , and show that M is an entropy solution of (46).

Step 1: Constructing an approximation sequence

For a given N , we construct an initial condition M_N^0 and flux A_N for the discretized balance law (38). As long as the hypotheses (39) and (41) are satisfied, we can apply Theorem 4.1 to get a solution M_N of the form (38). We give slightly more detail in this step than what is strictly necessary for the proof of Theorem 5.1(a); we do this to allow for the reader to easily compare the approximation scheme we use here with the one we use later in Theorem 6.7.

We begin with an N -tuple of positive masses $(m_{i,N})_{i=1}^N$ that sum to unity (28). We also assume

$$\lim_{N \rightarrow \infty} \max_{1 \leq i \leq N} m_{i,N} = 0. \quad (51)$$

A typical choice is $m_{i,N} = \frac{1}{N}$, so all particles have the same mass. Next, we define $x_{i,N}^0$ by

$$x_{i,N}^0 = \inf \{x : M^0(x) \geq \theta_{i,N}\}, \quad i = 1, \dots, N. \quad (52)$$

where $\theta_{i,N}$ is defined in (40). It is easy to check that $\{x_{i,N}^0\}_{i=1}^N \subset [-R^0, R^0]$. Then, M_N^0 can be constructed from (39). Finally, we define A_N as the piecewise linear approximation of A such that

$$A_N(\theta_{i,N}) = A(\theta_{i,N}), \quad i = 0, \dots, N. \quad (53)$$

The M_N^0 and A_N constructed through the procedure above clearly satisfy the hypotheses (39) and (41). Moreover, they approximate M^0 and A in the following sense.

Lemma 5.3. *The following inequalities hold:*

$$\|M_N^0 - M^0\|_{L^1(\mathbb{R})} \leq 2R^0 \max_{1 \leq i \leq N} m_{i,N}, \quad \sup_{m \in [-\frac{1}{2}, \frac{1}{2}]} |A_N(m) - A(m)| \leq |A|_{\text{Lip}} \max_{1 \leq i \leq N} m_{i,N}. \quad (54)$$

In particular, $M_N^0 - M^0 \rightarrow 0$ in $L^1(\mathbb{R})$ and $A_N \rightarrow A$ uniformly, as $N \rightarrow \infty$.

Proof. Denote $x_{0,N}^0 := -R^0$. We have

$$\begin{aligned} \|M_N^0 - M^0\|_{L^1(\mathbb{R})} &= \sum_{j=1}^N \int_{x_{j-1,N}^0}^{x_{j,N}^0} (M^0(x) - M_N^0(x)) \, dx \leq \sum_{j=1}^N m_{j,N} (x_{j,N}^0 - x_{j-1,N}^0) \\ &\leq 2R^0 \max_{1 \leq j \leq N} m_{j,N}, \end{aligned}$$

which proves the first inequality in (54). Note that we may allow $x_{j-1,N}^0 = x_{j,N}^0$ for some j 's, and the estimate above still holds. As for the second inequality in (54), fix $m \in [-\frac{1}{2}, \frac{1}{2}]$ and choose i such that $m \in [\theta_{i-1,N}, \theta_{i,N}]$. Then (41) and (53) imply

$$A_N(m) - A(m) = \frac{m - \theta_{i-1,N}}{m_i} (A(\theta_{i-1,N}) - A(m)) + \frac{\theta_{i,N} - m}{m_i} (A(\theta_{i,N}) - A(m)),$$

which easily implies the second inequality. $\square$

Step 2: Extracting a limit M

Fix a time $T > 0$. For any $t \in [0, T]$, since $M_N(t)$ is uniformly bounded and nondecreasing, we may apply Helly's selection theorem and find a convergent subsequence $M_{N_k}(t)$ in $L_{\text{loc}}^1(\mathbb{R})$. Using a diagonal argument, we can get a further subsequence, still denoted by M_{N_k} , that is convergent for all rational $t \in [0, T]$ in $L_{\text{loc}}^1(\mathbb{R})$. We provisionally denote the limit by $M(t)$. We want to upgrade the convergence $M_{N_k}(t) - M(t) \rightarrow 0$ from $L_{\text{loc}}^1(\mathbb{R})$ to $L^1(\mathbb{R})$ and also extend our conclusion to irrational times. The following observation will help us achieve this.

Note that by (42), (43), and the monotonicity of Φ , we have the following N -independent bound on the initial velocities $v_{i,N}^0$:

$$|v_{i,N}^0| = \left| \psi_{i,N}^0 - \sum_{j=1}^N m_{j,N} \Phi(x_{i,N}^0 - x_{j,N}^0) \right| \leq |A_N|_{\text{Lip}} + \Phi(2R^0) \leq |A|_{\text{Lip}} + \Phi(2R^0). \quad (55)$$

Then, (32) implies that for $t \in [0, T]$, we have $\{x_{i,N}(t)\}_{i=1}^N \subset [-R(T), R(T)]$, where

$$R(T) = R^0 + T(|A|_{\text{Lip}} + \Phi(2R^0)). \quad (56)$$

It follows that $M_N(\pm x, t) = \pm \frac{1}{2}$ for all $x > R(T)$ and $t \in [0, T]$, and thus we have $M_{N_k}(t) - M(t) \rightarrow 0$ in $L^1(\mathbb{R})$ for all rational times $t \in \mathbb{Q}_+$. The extension of this convergence to irrational times is an easy consequence of the time regularity estimate

$$\begin{aligned} \int_{\mathbb{R}} |M_N(x, t) - M_N(x, s)| \, dx &\leq \sum_{i=1}^N m_i |x_{i,N}(t) - x_{i,N}(s)| \leq \max_i |v_{i,N}^0| \cdot (t - s) \\ &\leq (|A|_{\text{Lip}} + \Phi(2R^0))(t - s). \end{aligned} \quad (57)$$

We used (44) to get the first inequality, then the maximum principle (31) to get the second, and finally the bound (55) to finish.

Combining (57) with the established convergence $M_{N_k}(t) - M(t) \rightarrow 0$ at rational times, we conclude

$$M_{N_k} - M \rightarrow 0 \quad \text{in } C([0, T]; L^1(\mathbb{R})).$$

The limit M has the desired properties: For each $t \in [0, T]$, the function $M(\cdot, t)$ is non-decreasing, with $M(\pm x, t) = \pm \frac{1}{2}$ for all $x \geq R(T)$. Moreover, the time regularity estimate (57) implies a uniform bound

$$\|\partial_t M_{N_k}(x, t)\|_{\mathcal{M}} \leq |A|_{\text{Lip}} + \Phi(2R^0), \quad \text{for all } t \in [0, T].$$

Then, extracting a further subsequence, still denoted by M_{N_k} , we obtain the weak-* convergence

$$\partial_t M_{N_k}(\cdot, t) \xrightarrow{*} \partial_t M(\cdot, t) \quad \text{in } \mathcal{M}(\mathbb{R}).$$

This also allows us to conclude that $M \in BV(\mathbb{R} \times [0, T])$.

Remark 5.1. Once we show M is the *unique* entropy solution of (46) (through an argument independent of the existence proof), we can conclude that the whole sequence M_N converges to M , finishing the proof of Theorem 5.1(b).

Step 3: Verifying the entropy conditions

Finally, we show that the function M we have constructed above is indeed an entropy solution of (46). We do this by verifying the entropy inequality (23) for all Kruzkov entropy pairs (η, q) in (24).

We know from Theorem 4.1 that M_N is an entropy solution of (38). Thus the entropy inequality (23) is satisfied for (η, q_N) , where $\eta(m) = |m - \alpha|$ and $q_N(m) = \text{sgn}(m - \alpha)(A_N(m) - A_N(\alpha))$. It reads

$$\int_0^T \int_{\mathbb{R}} [\eta(M_N) \partial_t \zeta + q_N(M_N) \partial_x \zeta + (\phi * M_N) \zeta \partial_x (\eta(M_N))] \, dx \, dt \geq 0.$$

Now we pass to the limit. To simplify the notation, we write M_N instead of M_{N_k} in what follows. Define $q(m) = \text{sgn}(m - \alpha)(A(m) - A(\alpha))$, as in (24). For a fixed $t \in [0, T]$ (for which we suppress the notation), we use Lemma 5.3 and get

$$\begin{aligned} \|\eta(M_N) - \eta(M)\|_{L^1} &\leq |\eta|_{\text{Lip}} \|M_N - M\|_{L^1} = \|M_N - M\|_{L^1} \rightarrow 0, \\ \|q_N(M_N) - q(M)\|_{L^1} &\leq \|q_N(M_N) - q(M_N)\|_{L^1} + \|q(M_N) - q(M)\|_{L^1} \\ &\leq 2R(T) \|A_N - A\|_{L^\infty([- \frac{1}{2}, \frac{1}{2}])} + |A|_{\text{Lip}} \|M_N - M\|_{L^1} \rightarrow 0. \end{aligned}$$

This establishes convergence for the first two terms. The last term is more subtle; we argue as follows.

First, we claim that $\phi * M_N$ converges uniformly to $\phi * M$ with respect to x , and that this convergence is furthermore uniform with respect to t on $[0, T]$. This is immediate if ϕ is bounded; otherwise we can consider a mollification ϕ_δ of ϕ and estimate as follows:

$$\|\phi * M_N - \phi * M\|_{L^\infty} \leq \|\phi - \phi_\delta\|_{L^1} \|M_N - M\|_{L^\infty} + \|\phi_\delta\|_{L^\infty} \|M_N - M\|_{L^1}.$$

We can first choose δ so that the first term on the right is small, then choose N large enough to finish.

Next, we note that (by Lemma 5.2) $\partial_x(\eta(M_N))$ is a bounded sequence in $\mathcal{M}(\mathbb{R})$:

$$\|\partial_x(\eta(M_N))\|_{\mathcal{M}} \leq |\eta|_{\text{Lip}}.$$

The same bound holds for $\partial_x(\eta(M))$. Now we can write

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{R}} (\phi * M_N) \zeta \partial_x(\eta(M_N)) \, dx \, dt - \int_0^T \int_{\mathbb{R}} (\phi * M) \zeta \partial_x(\eta(M)) \, dx \, dt \right| \\ & \leq \int_0^T \|\zeta(t)\|_{L^\infty} \|\phi * M_N - \phi * M\|_{L^\infty} \|\partial_x(\eta(M_N))\|_{\mathcal{M}} \, dt \\ & \quad + \left| \int_0^T \int_{\mathbb{R}} (\phi * M) \zeta [\partial_x(\eta(M_N)) - \partial_x(\eta(M))] \, dx \, dt \right|. \end{aligned}$$

The first term on the right side of the above inequality goes to zero in light of the above arguments. We can establish the vanishing of the second term similarly: Mollifying $\phi * M$ if necessary, we write

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{R}} (\phi * M) \zeta [\partial_x(\eta(M_N)) - \partial_x(\eta(M))] \, dx \, dt \right| \\ & \leq \int_0^T \|\zeta(\phi * M - (\phi * M)_\delta)\|_{L^\infty} \|\partial_x(\eta(M_N)) - \partial_x(\eta(M))\|_{\mathcal{M}} \, dt \\ & \quad + \int_0^T \|\partial_x(\zeta(\phi * M)_\delta)\|_{L^\infty} \|\eta(M_N) - \eta(M)\|_{L^1} \, dt. \end{aligned}$$

Note that mollification is unnecessary if ϕ is bounded. In any case, the continuity of $\phi * M$ and the compact support of ζ guarantee that $\|\zeta((\phi * M) - (\phi * M)_\delta)\|_{L^\infty}$ can be made as small as desired by choosing δ appropriately, after which we can choose N large enough to make $\|\eta(M_N) - \eta(M)\|_{L^1}$ small. We thus obtain the entropy inequality (23) and conclude that M is an entropy solution of (46).

5.2. Uniqueness and L^1 stability

We now prove the stability estimate (49). Note that uniqueness is a direct consequence if we set $\tilde{M}^0 = M^0$ and $\tilde{A} = A$. We use Kruzkov's *doubling of the variables* strategy, with additional treatment of the nonlocal term on the right-hand side of (46).

For fixed (y, s) , consider the Kruzkov entropy pair (24) with $\alpha = \tilde{M}(y, s)$, and a test function $\zeta(x, t) = w(x, t, y, s)$ to be specified later. Entropy inequality (23) reads

$$\begin{aligned} 0 & \leq \iint |M(x, t) - \tilde{M}(y, s)| \partial_t w(x, t, y, s) \, dx \, dt \\ & \quad + \iint \text{sgn}(M(x, t) - \tilde{M}(y, s)) (A(M(x, t)) - A(\tilde{M}(y, s))) \partial_x w(x, t, y, s) \, dx \, dt \\ & \quad + \iint (\phi * M)(x, t) (\partial_x |M(x, t) - \tilde{M}(y, s)|) w(x, t, y, s) \, dx \, dt. \end{aligned}$$

We omit the bounds of integration in most of the computation below. Unless otherwise specified, the spatial variables x and y are integrated over $\mathbb{R}$, while the time variables t, s are integrated over $[0, T]$.

We perform the analogous manipulations, with $\tilde{A}$ replacing A and the roles of $M(x, t)$ and $\tilde{M}(y, s)$ interchanged. Integrating over the remaining free variables in both cases and adding the results yields

$$\begin{aligned}
 0 \leq & \int \int \int \int |M(x, t) - \tilde{M}(y, s)| (\partial_t w + \partial_s w)(x, t, y, s) \, dx \, dt \, dy \, ds \\
 & + \int \int \int \int \operatorname{sgn}(M(x, t) - \tilde{M}(y, s)) \left[(A(M(x, t)) - A(\tilde{M}(y, s))) \partial_x w(x, t, y, s) \right. \\
 & \quad \left. + (\tilde{A}(M(x, t)) - \tilde{A}(\tilde{M}(y, s))) \partial_y w(x, t, y, s) \right] \, dx \, dt \, dy \, ds \\
 & + \int \int \int \int w(x, t, y, s) \left[(\phi * M)(x, t) \partial_x |M(x, t) - \tilde{M}(y, s)| \right. \\
 & \quad \left. + (\phi * \tilde{M})(y, s) \partial_y |M(x, t) - \tilde{M}(y, s)| \right] \, dx \, dt \, dy \, ds.
 \end{aligned} \tag{58}$$

We introduce the auxiliary variables

$$\bar{x} = \frac{x+y}{2}, \quad \bar{y} = \frac{x-y}{2}, \quad \bar{t} = \frac{t+s}{2}, \quad \text{and} \quad \bar{s} = \frac{t-s}{2},$$

and we take a test function of the form

$$w(x, t, y, s) = b_\varepsilon\left(\frac{x-y}{2}\right) b_\varepsilon\left(\frac{t-s}{2}\right) g\left(\frac{x+y}{2}\right) h_\delta\left(\frac{t+s}{2}\right) = b_\varepsilon(\bar{y}) b_\varepsilon(\bar{s}) g(\bar{x}) h_\delta(\bar{t}),$$

where $b_\varepsilon, g, h_\delta$ are smooth, nonnegative functions satisfying the following properties.

- The functions $(b_\varepsilon)_{\varepsilon>0}$ approximate the Dirac delta distribution as $\varepsilon \rightarrow 0+$. We take b_ε to be a standard mollifier, supported in $(-\varepsilon, \varepsilon)$ and having integral 1.
- The function g is identically 1 on $[-R(T), R(T)]$ and is compactly supported.
- The functions $(h_\delta)_{\delta>0}$ approximate the indicator function of $[s, t]$ as $\delta \rightarrow 0+$. We take h_δ to be identically 1 on $[s, t]$, identically zero outside $[s - \delta, t + \delta]$, and linear on $[s - \delta, s]$ and $[t, t + \delta]$.

To proceed, we shall substitute our test function into (58), using the auxiliary variables. Observe that

$$\partial_t + \partial_s = \partial_{\bar{t}}, \quad \partial_x + \partial_y = \partial_{\bar{x}}, \quad \partial_x - \partial_y = \partial_{\bar{y}}.$$

We now rewrite our inequality in terms of the new variables. In particular, the bracketed part of the second term on the right side of (58) can be rewritten as

$$(A_+(M) - A_+(\tilde{M})) \partial_{\bar{x}} w + (A_-(M) - A_-(\tilde{M})) \partial_{\bar{y}} w,$$

where we have used the shorthand notation

$$A_\pm(m) := \frac{A(m) \pm \tilde{A}(m)}{2}$$

and suppressed the arguments of M and $\tilde{M}$. The latter will be equal to $(x, t) = (\bar{x} + \bar{y}, \bar{t} + \bar{s})$ and $(y, s) = (\bar{x} - \bar{y}, \bar{t} - \bar{s})$, respectively, for the rest of the computation below.

Note that if the fluxes A and $\tilde{A}$ are identical, then A_+ reduces to their common value, while A_- vanishes. Hence, A_- encodes information about stability with respect to the flux.

Next, we rewrite the bracketed part in the last term of (58) (again suppressing arguments) as

$$\frac{\phi * M + \phi * \tilde{M}}{2} \cdot \partial_{\bar{x}} |M - \tilde{M}| + \frac{\phi * M - \phi * \tilde{M}}{2} \cdot \partial_{\bar{y}} |M - \tilde{M}|.$$

Substituting the above into (58) then yields

$$\begin{aligned} 0 \leq & \int \int \int \int \left[|M - \tilde{M}| \partial_{\bar{t}} w + \operatorname{sgn}(M - \tilde{M})(A_+(M) - A_+(\tilde{M})) \partial_{\bar{x}} w \right] d\bar{x} d\bar{t} d\bar{y} d\bar{s} \\ & + \int \int \int \int \operatorname{sgn}(M - \tilde{M})(A_-(M) - A_-(\tilde{M})) \partial_{\bar{y}} w d\bar{x} d\bar{t} d\bar{y} d\bar{s} \\ & + \frac{1}{2} \int \int \int \int (\phi * M + \phi * \tilde{M}) w \cdot \partial_{\bar{x}} |M - \tilde{M}| d\bar{x} d\bar{t} d\bar{y} d\bar{s} \\ & + \frac{1}{2} \int \int \int \int (\phi * M - \phi * \tilde{M}) w \cdot \partial_{\bar{y}} |M - \tilde{M}| d\bar{x} d\bar{t} d\bar{y} d\bar{s} \end{aligned} \quad (59)$$

We want to take $\varepsilon \rightarrow 0$, which will effectively set $\bar{y}$ and $\bar{s}$ equal to zero. Before we can do this, however, we need to deal with the $\bar{y}$ derivatives. We treat the second integrable above first, making use of the following lemma to justify the necessary integration by parts.

Lemma 5.4. *The function*

$$\gamma(M, \tilde{M}) := \operatorname{sgn}(M - \tilde{M})(A_-(M) - A_-(\tilde{M}))$$

is Lipschitz in both variables M and $\tilde{M}$, with

$$|\gamma(\cdot, \tilde{M})|_{\text{Lip}} \leq |A_-|_{\text{Lip}}, \quad |\gamma(M, \cdot)|_{\text{Lip}} \leq |A_-|_{\text{Lip}}.$$

Proof. Fix an $\tilde{M}$ and pick $M_1 < M_2$. We consider two cases. First, if M_1 and M_2 are both greater or both less than $\tilde{M}$, then

$$|\gamma(M_1, \tilde{M}) - \gamma(M_2, \tilde{M})| = |A_-(M_1) - A_-(M_2)| \leq |A_-|_{\text{Lip}} |M_1 - M_2|.$$

If on the other hand we have $M_1 \leq \tilde{M} \leq M_2$, then

$$|\gamma(M_1, \tilde{M}) - \gamma(M_2, \tilde{M})| \leq |A_-|_{\text{Lip}} (\tilde{M} - M_1) + |A_-|_{\text{Lip}} (M_2 - \tilde{M}) = |A_-|_{\text{Lip}} |M_1 - M_2|.$$

The estimate $|\gamma(\cdot, \tilde{M})|_{\text{Lip}} \leq |A_-|_{\text{Lip}}$ follows. The other bound can be obtained in the same way. $\square$

Now, we apply (50) with $f(z) = \gamma(z, \tilde{M})$, $W = M$ and $f(z) = \gamma(M, z)$, $W = \tilde{M}$. Lemma 5.4 yields

$$\left| \frac{d}{d\bar{y}} \gamma(M, \tilde{M}) \right| \leq |A_-|_{\text{Lip}} |\partial_1 M(\bar{x} + \bar{y}, \bar{t} + \bar{s})| + |A_-|_{\text{Lip}} |\partial_1 \tilde{M}(\bar{x} - \bar{y}, \bar{t} - \bar{s})|.$$

Here ∂_1 denotes differentiation with respect to the first (spatial) argument.

We also use the following estimate in the fourth integral of (59):

$$\left| \frac{d}{d\bar{y}} |M(\bar{x} + \bar{y}, \bar{t} + \bar{s}) - \tilde{M}(\bar{x} - \bar{y}, \bar{t} - \bar{s})| \right| \leq |\partial_1 M(\bar{x} + \bar{y}, \bar{t} + \bar{s}) + \partial_1 \tilde{M}(\bar{x} - \bar{y}, \bar{t} - \bar{s})|.$$

We now collect all the estimates above and take $\varepsilon \rightarrow 0$ in (59). For simplicity, we revert to the notation (x, t) in rather than $(\bar{x}, \bar{t})$ in this inequality and the following ones. We obtain

$$\begin{aligned} 0 \leq & \iint [|M(x, t) - \tilde{M}(x, t)| g(x) h'_\delta(t) + \operatorname{sgn}(M - \tilde{M})(A_+(M) - A_+(\tilde{M})) g'(x) h_\delta(t)] dx dt \\ & + \iint |A_-|_{\text{Lip}} (|\partial_x M| + |\partial_x \tilde{M}|) g(x) h_\delta(t) dx dt \\ & + \frac{1}{2} \iint [\Phi * (M + \tilde{M}) \cdot \partial_x |M - \tilde{M}| + (\Phi * |M - \tilde{M}|) \cdot |\partial_x M + \partial_x \tilde{M}|] g(x) h_\delta(t) dx dt. \end{aligned}$$

Next, we recall our choices of g and h_δ . We can drop the second term in the first integral above, since $g' \equiv 0$ in $[-R(T), R(T)]$; we also replace g by 1 for the rest of the terms. Taking $\delta \rightarrow 0$, we get

$$\begin{aligned} & \int |M(x, t) - \tilde{M}(x, t)| dx \\ & \leq \int |M(x, s) - \tilde{M}(x, s)| dx + \frac{1}{2} |A - \tilde{A}|_{\text{Lip}} \int_s^t \int |\partial_x M| + |\partial_x \tilde{M}| dx d\tau \\ & \quad + \frac{1}{2} \int_s^t \int [\Phi * (\partial_x M + \partial_x \tilde{M}) \cdot \partial_x |M - \tilde{M}| + |\partial_x M + \partial_x \tilde{M}| (\Phi * \partial_x |M - \tilde{M}|)] dx d\tau. \end{aligned} \tag{60}$$

Up to this point, we have not used the fact that M and $\tilde{M}$ are nondecreasing. We take advantage of it in this final step by replacing $|\partial_x M|$ with $\partial_x M$, etc. Under these replacements, the second term in (60) becomes $|A - \tilde{A}|_{\text{Lip}}(t - s)$, while the last term in (60) vanishes identically due to the oddness of Φ . The stability bound (49) follows immediately, upon taking $s = 0$.

6. The entropic selection principle for the Euler-alignment system

In this section, we come back to our main 1D Euler-alignment system (14). Recall

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) = \rho(\phi * (\rho u)) - \rho u(\phi * \rho), \end{cases} \quad \begin{cases} \rho(x, 0) = \rho^0(x), \\ u(x, 0) = u^0(x). \end{cases} \tag{61}$$

We construct a uniquely determined weak solution of (61), using the entropy conditions (22) for the scalar balance law (15) in our selection principle. Theorem 6.3 details the process by which our solution is constructed; we prove that the resulting object meets the requirements of Definition 6.2 below. Finally, we explicitly connect our solution to the sticky particle dynamics (27)–(30). We demonstrate in Theorem 6.5 that the sticky particle Cucker–Smale dynamics can always be used to approximate the solution. Moreover, Theorem 6.7 gives a much stronger conclusion under additional hypotheses, by fashioning an explicit rate of convergence of the sticky particle approximation for the density profile.

Let us denote by $\mathcal{P}_c(\mathbb{R})$ the space of probability measures with compact support. We will use the Wasserstein-1 metric to quantify the distance between elements of $\mathcal{P}_c(\mathbb{R})$.

Definition 6.1 (Wasserstein-1 metric). Let $\rho, \tilde{\rho} \in \mathcal{P}_c(\mathbb{R})$. The Wasserstein-1 distance between them is

$$\mathcal{W}_1(\rho, \tilde{\rho}) = \sup_{\text{Lip}(f) \leq 1} \left| \int_{\mathbb{R}} f(x) d\rho(x) - \int_{\mathbb{R}} f(x) d\tilde{\rho}(x) \right|.$$

It is well-known that if M and $\tilde{M}$ are cumulative distribution functions for ρ and $\tilde{\rho}$ defined in (20), respectively, then $\mathcal{W}_1(\rho, \tilde{\rho}) = \|M - \tilde{M}\|_{L^1}$. We will consider $\mathcal{P}_c(\mathbb{R})$ equipped with the Wasserstein-1 metric. In this setting, $\mathcal{W}_1$ convergence is equivalent to weak-* convergence in the sense of measures.

Next, we make precise what we mean by a weak solution of (61).

Definition 6.2 (Weak solution). Let $\rho^0 \in \mathcal{P}_c(\mathbb{R})$ and $u^0 \in L^\infty(d\rho^0)$. Define $P^0 = \rho^0 u^0$, which lies in the space of signed measures $\mathcal{M}(\mathbb{R})$. We say that $(\rho, P) = (\rho, \rho u)$ is a *weak solution* to the Euler-alignment system (61) if for any $T > 0$,

- $\rho \in C([0, T]; \mathcal{P}_c(\mathbb{R}))$.
- $P(\cdot, t) \in \mathcal{M}(\mathbb{R})$ for any $t \in [0, T]$. Moreover, $P(\cdot, t)$ is absolutely continuous with respect to $\rho(\cdot, t)$, with the Radon-Nikodym derivative $u(\cdot, t) \in L^\infty(d\rho(t))$, where $u(\cdot, t)d\rho(\cdot, t) = dP(\cdot, t)$, for any $t \in [0, T]$.
- (ρ, u) satisfies (61) in the sense of distributions.
- The initial condition (ρ^0, P^0) is attained in the following weak sense for every $f \in C_c^\infty(\mathbb{R})$:

$$\lim_{t \rightarrow 0+} \int_{\mathbb{R}} f(x) d\rho(x, t) = \int_{\mathbb{R}} f(x) d\rho^0(x); \quad \lim_{t \rightarrow 0+} \int_{\mathbb{R}} f(x) dP(x, t) = \int_{\mathbb{R}} f(x) dP^0(x). \quad (62)$$

6.1. Construction of the solution

Let us start by introducing the generalized inverse of a nondecreasing function M , defined as

$$M^{-1}(m) = \inf \{x \in \mathbb{R} : M(x) \geq m\}, \quad m \in \left(-\frac{1}{2}, \frac{1}{2}\right].$$

It is a left-continuous function.

Now, we construct our solution through the procedure in the following theorem, which aligns with the formal derivation in Section 2.1.

Theorem 6.3 (The entropic selection principle). Let $\rho^0 \in \mathcal{P}_c(\mathbb{R})$, $u^0 \in L^\infty(d\rho^0)$ and $P^0 = \rho^0 u^0$. We construct a unique pair (ρ, P) from the following procedure:

- (i) Let $M^0(x) = \rho^0((-\infty, x]) - \frac{1}{2}$ and $\psi^0 = u^0 + \Phi * \rho^0$. Define a Lipschitz flux $A : [-\frac{1}{2}, \frac{1}{2}] \rightarrow \mathbb{R}$ by

$$A(m) = \int_{-\frac{1}{2}}^m a(m') dm', \quad \text{where} \quad a(m) = \psi^0 \circ (M^0)^{-1}(m). \quad (63)$$

- (ii) Let M be the unique entropy solution of (15) associated to the initial data M^0 and the flux A .

(iii) Define (ρ, P) from M via the formulas

$$\rho = \partial_x M, \quad P = -\partial_t M = \partial_x(A \circ M) - (\phi * M)\partial_x M. \quad (64)$$

Then (ρ, P) is a weak solution of the 1D Euler-alignment system (61) in the sense of Definition 6.2. Moreover, we can define $u(\cdot, t) = \frac{dP(t)}{d\rho(t)}$ to be the Radon-Nikodym derivative of $P(t)$ with respect to $\rho(t)$. $u(t)$ is uniquely defined $\rho(t)$ -a.e.

Remark 6.1. The definition (63) of the flux A is made to guarantee that $A(M^0) = Q^0 = \int_{(-\infty, \cdot]} \psi^0 d\rho^0$ (c.f. Section 2.1). However, the identity $A(M^0) = Q^0$ determines A uniquely only on $\text{Im}M^0$, which is a proper subset of $[-\frac{1}{2}, \frac{1}{2}]$ if ρ^0 has a singular part. What if one defines A differently away from $\text{Im}M^0$? It turns out that if $\tilde{A}$ is a Lipschitz function that agrees with A on $\text{Im}M^0$, and such that $\tilde{A} \geq A$, then the procedure above produces the same M (and thus the same (ρ, u)) regardless of whether A or $\tilde{A}$ is used. Making sure $\tilde{A}$ is chosen according to these conditions can be thought of as enforcing the ‘stickiness’ at time zero. We have already seen the discrete version of this observation in Remark 4.1. We leave the details to the interested reader.

Proof. Our first step is to check that M^0 and A satisfy the assumptions of Theorem 5.1. The required properties of M^0 follow directly from the fact that ρ^0 is a nonnegative, compactly supported probability measure. Indeed, the number R^0 can be chosen so that $\text{supp}\rho^0 \subset [-R^0, R^0]$. As for A , we note that

$$\|\psi^0\|_{L^\infty(d\rho^0)} \leq \|u^0\|_{L^\infty(d\rho^0)} + \Phi(2R^0),$$

which is bounded. It follows that A as defined in (63) is Lipschitz. Since M^0 and A are of the desired form, we can apply Theorem 5.1 and obtain a unique entropy solution $M \in BV(\mathbb{R} \times [0, T])$ of (46), for any fixed time $T > 0$.

Now, we verify (ρ, P) is a weak solution of (61). First, $\rho \in C([0, T]; \mathcal{P}_c(\mathbb{R}))$ follows from (47), and $P(t) = -\partial_t M(t) \in \mathcal{M}(\mathbb{R})$ is a direct consequence of (48). Let us turn our attention to u . Since $M \in BV(\mathbb{R} \times [0, T])$, we can perform BV calculus, e.g. [Lemma 4.2, 48], and deduce that there exists a measurable function $\psi = \psi(x, t)$, bounded by $|A|_{\text{Lip}}$ and uniquely defined $\rho(t)$ -a.e., such that

$$\partial_x(A(M)) = \psi \partial_x M, \quad \partial_t(A(M)) = \psi \partial_t M, \quad (65)$$

in the sense of measures. Then $P(t)$ defined by (64) is given by

$$P(t) = (\psi - (\phi * M))\partial_x M(t) = (\psi - \Phi * \rho)\rho(t),$$

so that $u = \psi - \Phi * \rho$ inherits the required boundedness and uniqueness properties from ψ and ρ .

Next, we show that (ρ, u) satisfies (61) in a distributional sense. From (46), (64) and (65) we get

$$\begin{aligned} \partial_t \rho &= \partial_t(\partial_x M) = \partial_x(\partial_t M) = -\partial_x P = -\partial_x(\rho u), \\ \partial_t(\rho \psi) &= \partial_{tx}^2(A(M)) = \partial_x(\psi \partial_t M) = -\partial_x(\psi P) = -\partial_x(\rho u \psi). \end{aligned}$$

We then recover the momentum equation in exactly the form (61)₂ as follows:

$$\begin{aligned}\partial_t(\rho u) + \partial_x(\rho u^2) &= (\partial_t(\rho\psi) + \partial_x(\rho u\psi)) - \partial_t(\rho(\Phi * \rho)) - \partial_x(\rho u(\Phi * \rho)) \\ &= -(\partial_t\rho + \partial_x(\rho u))(\Phi * \rho) + \rho(\Phi * \partial_x(\rho u)) - \rho u((\partial_x\Phi) * \rho) \\ &= \rho(\phi * (\rho u)) - \rho u(\phi * \rho).\end{aligned}$$

Finally, we check the initial conditions (62). From continuity of ρ in time, we have $\mathcal{W}_1(\rho(t), \rho^0) \rightarrow 0$, which implies the first equation in (62). For the second equation, we apply (64) to obtain

$$\int_{\mathbb{R}} f(x) dP(x, t) = - \int_{\mathbb{R}} f'(x) A(M(x, t)) dx - \int_{\mathbb{R}} f(x) (\phi * M)(x, t) d\rho(x, t).$$

We can pass to the limit as $t \rightarrow 0+$ for the two terms separately. For the first one, we have

$$\left| \int_{\mathbb{R}} f'(x) A(M(x, t)) dx - \int_{\mathbb{R}} f'(x) A(M^0(x)) dx \right| \leq |A|_{\text{Lip}} \|f'\|_{L^\infty} \|M(\cdot, t) - M^0\|_{L^1} \rightarrow 0.$$

As for the second one, we write

$$\begin{aligned}& \left| \int_{\mathbb{R}} f(x) (\phi * M)(x, t) d\rho(x, t) - \int_{\mathbb{R}} f(x) (\phi * M^0)(x) d\rho^0(x) \right| \\ & \leq \left| \int_{\mathbb{R}} f(x) ((\Phi * \rho)(x, t) - (\Phi * \rho^0)(x)) d\rho(x, t) \right| + \left| \int_{\mathbb{R}} f(x) (\Phi * \rho^0)(x) [d\rho(x, t) - d\rho^0(x)] \right|.\end{aligned}$$

We note that the weak-* convergence $\rho(t) \xrightarrow{*} \rho^0$ implies that the second term above vanishes as $t \rightarrow 0+$, and also that $\Phi * \rho(t) \rightarrow \Phi * \rho^0$ pointwise; the latter allows us to conclude that the first term also vanishes as $t \rightarrow 0+$, after an application of the dominated convergence theorem.

From our construction, we have $P(0) = (\psi^0 - \Phi * \rho^0)\rho^0 = \rho^0 u^0 = P^0$, so the above calculations finish the proof. $\square$

6.2. Approximation by sticky particle Cucker–Smale dynamics

One of the most important features of our entropic selection principle is that it associates to atomic initial data a solution of the Euler-alignment system that is described by the sticky particle Cucker–Smale dynamics. The proposition below gives the precise statement.

Proposition 6.4. *Consider the 1D Euler-alignment system (61) with atomic initial data*

$$\rho_N^0(x) = \sum_{i=1}^N m_{i,N} \delta(x - x_{i,N}^0), \quad P_N^0(x) = \sum_{i=1}^N m_{i,N} v_{i,N}^0 \delta(x - x_{i,N}^0), \quad (66)$$

where the $x_{i,N}^0$'s all belong to a fixed compact set and the $m_{i,N}$'s satisfy (28). Let $(x_{i,N}(t), v_{i,N}(t))$ be the solution of the sticky particle Cucker–Smale dynamics with initial data $(m_{i,N}, x_{i,N}^0, v_{i,N}^0)_{i=1}^N$. The solution of the 1D Euler-alignment system selected by the procedure in Theorem 6.3 takes the form

$$\rho_N(x, t) = \sum_{i=1}^N m_{i,N} \delta(x - x_{i,N}(t)), \quad P_N(x, t) = \sum_{i=1}^N m_{i,N} v_{i,N}(t) \delta(x - x_{i,N}(t)). \quad (67)$$

Proof. Let M_N^0 , ψ_N^0 , A_N , and a_N be defined as in the step (i) of the procedure in [Theorem 6.3](#). Let $\theta_{i,N} = \sum_{j=1}^i m_{j,N}$ for $i = 0, \dots, N$. Clearly M_N^0 can be expressed as

$$M_N^0(x) = -\frac{1}{2} + \sum_{i=1}^N m_{i,N} H(x - x_{i,N}^0), \quad (68)$$

and we have

$$(M_N^0)^{-1}(m) = x_{i,N}^0, \quad m \in (\theta_{i-1}, \theta_i], \quad i = 1, \dots, N.$$

It follows that a_N is piecewise constant and A_N is piecewise linear, with breakpoints at the θ_i 's. Thus the hypotheses of [Theorem 4.1](#) are satisfied. Furthermore, the quantities $\psi_{i,N}^0$ defined as in (42) satisfy

$$\psi_{i,N}^0 := A_N'(m) = a_N(m) = \psi(x_{i,N}^0), \quad m \in (\theta_{i-1}, \theta_i).$$

We define $\tilde{v}_{i,N}^0$ by (43) and verify that it coincides with $v_{i,N}^0$ from (66):

$$\tilde{v}_{i,N}^0 := \psi_{i,N}^0 - \sum_{j=1}^N m_{j,N} \Phi(x_{i,N}^0 - x_{j,N}^0) = (\psi_N^0 - \Phi * \rho_N^0)(x_{i,N}^0) = v_{i,N}^0.$$

Next, we apply [Theorem 4.1](#) to obtain the entropy solution M_N of (38) associated to M_N^0 and A_N . According to (44) and (45), we have

$$\begin{aligned} M_N(x, t) &= -\frac{1}{2} + \sum_{i=1}^N m_{i,N} H(x - x_{i,N}(t)), \\ A_N \circ M_N(x, t) &= \sum_{i=1}^N m_{i,N} \psi_{i,N}(t) H(x - x_{i,N}(t)), \end{aligned} \quad (69)$$

where

$$\psi_{i,N}(t) = v_{i,N}(t) + \sum_{j=1}^N m_{j,N} \Phi(x_{i,N}(t) - x_{j,N}(t)), \quad i = 1, \dots, N.$$

It immediately follows that $\rho_N = \partial_x M_N$ is given by (67).

As for P_N , we write

$$\begin{aligned} P_N(x, t) &= \partial_x (A_N \circ M_N)(x, t) - (\phi * M_N) \partial_x M_N(x, t) \\ &= \sum_{i=1}^N m_{i,N} \psi_{i,N}(t) \delta(x - x_i(t)) - (\Phi * \rho_N)(x, t) \sum_{i=1}^N m_{i,N} \delta(x - x_{i,N}(t)) \\ &= \sum_{i=1}^N m_{i,N} \left[\psi_{i,N}(t) - \sum_{j=1}^N m_{j,N} \Phi(x_{i,N}(t) - x_{j,N}(t)) \right] \delta(x - x_{i,N}(t)) \\ &= \sum_{i=1}^N m_{i,N} v_{i,N}(t) \delta(x - x_{i,N}(t)), \end{aligned}$$

which finishes the proof. □

A direct application of [Theorem 5.1\(b\)](#) gives a convergence result for the sticky particle approximation.

Theorem 6.5. *Let (ρ^0, P^0) satisfy the hypotheses of [Theorem 6.3](#), and let (ρ, P) be the unique weak solution to [\(61\)](#) that it generates. There exists a sequence of (explicitly constructed) atomic initial data $(\rho_N^0, P_N^0)_{N=1}^\infty$ such that if (ρ_N, P_N) denotes the solution associated to (ρ_N^0, P_N^0) by the entropic selection principle, then for any time $t > 0$, as $N \rightarrow \infty$, we have*

$$\mathcal{W}_1(\rho_N(t), \rho(t)) \rightarrow 0, \quad (70)$$

and

$$P_N(t) \xrightarrow{*} P(t), \quad \text{in } \mathcal{M}(\mathbb{R}). \quad (71)$$

Proof. Let A , a , M , and (ρ, u) be defined from (ρ^0, u^0) through [Theorem 6.3](#). We take (ρ_N^0, P_N^0) as in [\(66\)](#), with $(m_{i,N}, x_{i,N}^0)_{i=1}^N$ chosen according to [\(51\)](#) and [\(52\)](#), and $(v_{i,N}^0)_{i=1}^N$ defined by

$$v_{i,N}^0 = \frac{1}{m_{i,N}} \int_{\theta_{i-1,N}}^{\theta_{i,N}} a(m) \, dm - \sum_{j=1}^N m_{j,N} \Phi(x_{i,N}^0 - x_{j,N}^0). \quad (72)$$

By the proof of the above proposition, the solution (ρ_N, P_N) generated by the entropic selection principle is given by [\(67\)](#). Moreover, defining discretized initial data M_N^0 and flux A_N from (ρ_N^0, P_N^0) , we have that M_N defined by [\(69\)](#) is the associated entropy solution of [\(38\)](#). Now, we can apply [Theorem 5.1\(b\)](#). In particular, [\(70\)](#) is equivalent to [\(47\)](#). Since $\partial_t M_N = -P_N$ and $\partial_t M = -P$ in the sense of measures, we get [\(71\)](#) directly from [\(48\)](#). $\square$

Next, we provide a refined estimate of [\(70\)](#), with an explicit convergence rate on the sticky particle approximation to our solution.

For $\rho^0 \in \mathcal{P}_c(\mathbb{R})$, let us denote by $[x_\ell^0, x_r^0]$ the smallest interval such that $\text{supp} \rho^0 \subseteq [x_\ell^0, x_r^0]$. The diameter D^0 is defined as

$$D^0 = \text{diam supp} \rho^0 := x_r^0 - x_\ell^0.$$

We start by constructing a well-prepared atomic approximation of ρ^0 , described in the next proposition.

Proposition 6.6. *Let $\rho^0 \in \mathcal{P}_c(\mathbb{R})$. For any fixed $N \in \mathbb{N}$, there exists $(m_{i,N}, x_{i,N}^0)_{i=1}^N$ such that*

$$\|M^0 - M_N^0\|_{L^1(\mathbb{R})} \leq \frac{D^0}{N}, \quad \|(M^0)^{-1} - (M_N^0)^{-1}\|_{L^\infty(-\frac{1}{2}, \frac{1}{2})} \leq \frac{D^0}{N}, \quad (73)$$

where M_N^0 is defined in [\(68\)](#).

Proof. We first locate all the large internal vacuum intervals of ρ^0 , namely $I_k = (a_k, b_k) \subset [x_\ell^0, x_r^0]$ such that $b_k - a_k > \frac{D^0}{N}$ and $\rho^0(I_k) = 0$. There are clearly fewer than N such intervals. Let $S_N^{(1)} = \{a_k\}_{k=1}^K$ be the collection of all left endpoints of the I_k 's. We

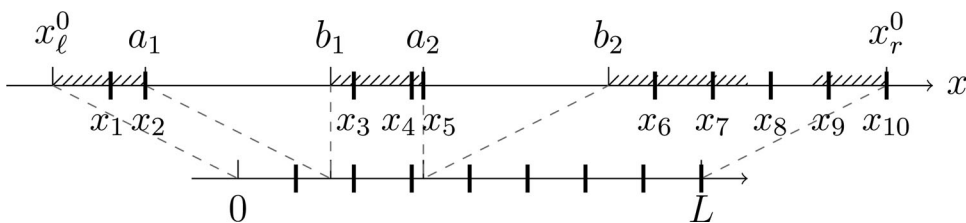


Figure 1. A sample construction of the N -tuple $(x_{i,N}^0)_{i=1}^N$. The shaded areas represent the support of ρ^0 . In this example, $N = 10$, and there are two large vacuum intervals (a_1, b_1) and (a_2, b_2) . Thus $S_N^{(1)} = \{a_1, a_2\}$, and the set $S_N^{(2)}$ is constructed by taking 8 equally distributed points in $[0, L]$ and mapping back to $[x_\ell^0, x_r^0]$.

take $N - K$ additional points, equally distributed, to partition $[x_\ell^0, x_r^0]$ into a total of N intervals. More precisely, we map $[x_\ell^0, x_r^0] \setminus \cup_{k=1}^K I_k$ to a single interval $[0, L]$, with $L < \frac{(N-K)D^0}{N}$, and we take equally distributed nodes $(\frac{iL}{N-K})_{i=1}^{N-K}$. Note that the distance between adjacent nodes is $\frac{L}{N-K} < \frac{D^0}{N}$. If the location of some node coincides with the image of a point in $S_N^{(1)}$, we can perturb the node slightly, in such a way that the distance in $[0, L]$ between any two adjacent nodes is still less than $\frac{D^0}{N}$. Now, we can take the inverse map of the selected nodes to $[x_\ell^0, x_r^0] \setminus \cup_{k=1}^K I_k$ and form a set $S_N^{(2)}$. We then order the set $S_N = S_N^{(1)} \cup S_N^{(2)}$ to obtain an N -tuple $(x_{i,N}^0)_{i=1}^N$. Our construction is illustrated in Figure 1.

We define an important quantity $x_{i,N}^*$ as follows. If $i - 1 \in S_N^{(1)}$, i.e., $x_{i-1,N}^0 = a_k$, we set $x_{i,N}^* = b_k$. If $i - 1 \in S_N^{(2)}$, we set $x_{i,N}^* = x_{i-1,N}^0$. For $i = 1$, we set $x_{1,N}^* = x_\ell^0$. Our construction clearly guarantees

$$0 < x_{i,N}^0 - x_{i,N}^* \leq \frac{D^0}{N}, \quad i = 1, \dots, N. \quad (74)$$

Next, we define $m_{i,N} = \rho^0((x_{i-1,N}^0, x_{i,N}^0])$ for each $i = 1, \dots, N$. (We take $x_{0,N}^0 = -\infty$ for convenience.) By construction, each $m_{i,N}$ is strictly positive. As usual, define $\theta_{i,N} = -\frac{1}{2} + \sum_{j=1}^i m_{j,N}$, $i = 0, 1, \dots, N$. It is easy to check that the $x_{i,N}^0$'s as defined above satisfy

$$x_{i,N}^0 = \inf\{x : M^0(x) \geq \theta_{i,N}\}, \quad i = 1, \dots, N.$$

Moreover, we have

$$M_N^0(x) = -\frac{1}{2} + \sum_{i=1}^N m_{i,N} H(x - x_{i,N}^0) \leq M^0(x),$$

with equality attained at $x = x_{i,N}^0$, as well as when $x \in I_k$. This implies

$$\int_{\mathbb{R}} |M^0(x) - M_N^0(x)| dx \leq \sum_{i=1}^N m_{i,N} (x_{i,N}^0 - x_{i,N}^*) \leq \frac{D^0}{N}.$$

where we have used (74) and $\sum_{i=1}^N m_{i,N} = 1$.

For the second inequality in (73), take any $i = 1, \dots, N$ and $m \in (\theta_{i-1,N}, \theta_{i,N}]$. By definition, $(M_N^0)^{-1}(m) = x_{i,N}^0$. We claim that $(M^0)^{-1}(m) \in [x_{i,N}^*, x_{i,N}^0]$. Indeed, in the case $x_{i-1,N}^0 \in S_N^{(2)}$, we have $M^0(x_{i,N}^*) = \theta_{i-1,N} < m$ and therefore $(M^0)^{-1}(m) \geq x_{i,N}^*$; on the other hand, if $x_{i-1,N}^0 \in S_N^{(2)}$ or if $i = 1$, then since $\rho((x_{i-1,N}^0, x_{i,N}^*)) = 0$, we have $M^0(x_{i,N}^*) = \theta_{i-1,N} < m$, which also implies $(M^0)^{-1}(m) \geq x_{i,N}^*$. Finally, we apply (74) and conclude by writing

$$0 \leq (M_N^0)^{-1}(m) - (M^0)^{-1}(m) \leq x_{i,N}^0 - x_{i,N}^* \leq \frac{D^0}{N}.$$

□

Under an additional regularity assumption on ϕ and on u^0 , we can now apply the stability estimate (49) in Theorem 5.1(c) to obtain an explicit error estimate for a sticky particle approximation to our solution, with the approximate initial density chosen according to the previous proposition.

Theorem 6.7. *Let $\beta, s \in (0, 1]$. Assume there exists a neighbourhood of 0 inside which $\phi(x) \leq c_s |x|^{s-1}$ for some $c_s > 0$. Suppose $\rho^0 \in \mathcal{P}_c(\mathbb{R})$ and $u^0 \in C^\beta([x_\ell^0, x_r^0])$. There exists a sequence of (explicitly constructed) atomic initial data $(\rho_N^0, P_N^0)_{N=1}^\infty$ such that the corresponding solution of (61) satisfies*

$$\mathcal{W}_1(\rho(t), \rho_N(t)) \leq C(1+t)N^{-\gamma}, \quad \gamma = \min\{s, \beta\}, \quad (75)$$

for any $t > 0$, where the constant C depends on D^0 , ϕ , c_s , s , β , and $|u^0|_{C^\beta}$.

Proof. First, we construct the approximated initial data (ρ_N^0, P_N^0) via (66), with $(m_{i,N}, x_{i,N}^0)_{i=1}^N$ chosen according to Proposition 6.6, and $(v_{i,N}^0)_{i=1}^N$ defined as follows. Set $\psi^0 = u^0 + \Phi * \rho^0$ as in Theorem 6.3. Note that our assumption on ϕ guarantees that Φ is locally C^s ; the same is therefore true of $\Phi * \rho^0$. Thus $\psi^0 \in C^\gamma([x_\ell^0, x_r^0])$, with

$$|\psi^0|_{C^\gamma([x_\ell^0, x_r^0])} \leq |u^0|_{C^\gamma([x_\ell^0, x_r^0])} + |\Phi * \rho^0|_{C^\gamma([x_\ell^0, x_r^0])}. \quad (76)$$

We can then simply take $\psi_{i,N}^0 = \psi^0(x_{i,N}^0)$ and define $v_{i,N}^0$ using (43).

Next, we apply the stability estimate (49) with $\tilde{M} = M_N$, to get

$$\mathcal{W}_1(\rho(t), \rho_N(t)) = \|M(\cdot, t) - M_N(\cdot, t)\|_{L^1(\mathbb{R})} \leq \|M^0 - M_N^0\|_{L^1(\mathbb{R})} + t\|a - a_N\|_{L^\infty(-\frac{1}{2}, \frac{1}{2})}.$$

The first term can be estimated directly from (73). We calculate the second term as follows:

$$\begin{aligned} \|a - a_N\|_{L^\infty(-\frac{1}{2}, \frac{1}{2})} &= \|\psi^0 \circ (M^0)^{-1} - \psi_N^0 \circ (M_N^0)^{-1}\|_{L^\infty(-\frac{1}{2}, \frac{1}{2})} \\ &\leq |\psi^0|_{C^\gamma([x_\ell^0, x_r^0])} \|(M^0)^{-1} - (M_N^0)^{-1}\|_{L^\infty(-\frac{1}{2}, \frac{1}{2})}^\gamma + \|(\psi^0 - \psi_N^0) \circ (M_N^0)^{-1}\|_{L^\infty(-\frac{1}{2}, \frac{1}{2})} \\ &\leq |\psi^0|_{C^\gamma([x_\ell^0, x_r^0])} N^{-\gamma}. \end{aligned}$$

Here, since $(M_N^0)^{-1}$ maps $(-\frac{1}{2}, \frac{1}{2})$ to $(x_{i,N}^0)_{i=1}^N$, we only need to make sense of the function ψ_N^0 on $(x_{i,N}^0)_{i=1}^N$, where we have $\psi_N^0(x_{i,N}^0) = \psi^0(x_{i,N}^0)$ by construction. This

eliminates the last term in the penultimate line. The last inequality is then obtained using (73), and (75) follows immediately. $\square$

7. Asymptotic behavior of the solution

In this section, we discuss the asymptotic behaviors of our weak solutions (ρ, P) to the 1D Euler-alignment system (61). The expected *flocking* phenomenon has two main ingredients.

First, we denote by $D(t)$ the diameter of the support of $\rho(\cdot, t)$,

$$D(t) := \text{diam supp } \rho(t) = x_r(t) - x_\ell(t), \quad (77)$$

where $[x_\ell(t), x_r(t)]$ is the smallest interval that contains $\text{supp } \rho(t)$. We say that the solution experiences *flocking* if $D(t)$ remains uniformly bounded for all time, i.e., there exists a constant $\bar{D} > 0$ such that

$$D(t) \leq \bar{D}, \quad \text{for all } t \geq 0. \quad (78)$$

Second, we say that the solution experiences *velocity alignment* if the variation of the velocity $u(\cdot, t)$ decays to zero as time approaches infinity. Since our weak solutions (ρ, P) only determine the velocity $u(\cdot, t)$ uniquely $\rho(t)$ -a.e., we shall make sense of the maximum and minimum velocities as follows:

$$u_+(t) = \sup_{f \in \mathcal{F}} \frac{\int_{\mathbb{R}} f(x) dP(x, t)}{\int_{\mathbb{R}} f(x) d\rho(x, t)}, \quad u_-(t) = \inf_{f \in \mathcal{F}} \frac{\int_{\mathbb{R}} f(x) dP(x, t)}{\int_{\mathbb{R}} f(x) d\rho(x, t)}, \quad (79)$$

where

$$\mathcal{F} = \left\{ f \in C_c^\infty(\mathbb{R}) : \int_{\mathbb{R}} f(x) d\rho(x, t) > 0 \right\}.$$

One can check that the definition of u_+ in (79) is equivalent to the *essential supremum* of u :

$$u_+(t) = \inf \{ c : \rho(t)(\{x : u(x, t) > c\}) = 0 \}.$$

Now, we are ready to define $V(t)$, the variation of $u(\cdot, t)$, by

$$V(t) = u_+(t) - u_-(t). \quad (80)$$

Velocity alignment happens when $V(t) \rightarrow 0$ as time approaches infinity. In particular, if $V(t)$ decays to zero exponentially in time, we say the solution has the *fast alignment* property.

The fact that both mass and momentum are conserved for the Euler-alignment system implies that whenever velocity alignment occurs, the limiting state of u must be given by the ratio $\bar{u} = \frac{\int \rho^0 u^0 dx}{\int \rho^0 dx}$. We say that *strong flocking* occurs (c.f. [1]) if ρ also converges to a limiting state ρ_∞ , in the following sense:

$$\mathcal{W}_1(\rho(\cdot + \bar{u}t, t), \rho_\infty) \rightarrow 0, \quad \text{as } t \rightarrow +\infty. \quad (81)$$

For regular solutions, it has been shown in [33] that if ϕ has a fat tail (9), then *strong solutions must flock*: any smooth solution (ρ, u) of the Euler-alignment system (1)

experiences flocking and fast alignment. (See also [68] for strong flocking in the sense above.) We will show that the same flocking phenomenon occurs for our weak solutions. Our strategy is to obtain flocking estimates for solutions to the sticky particle Cucker–Smale dynamics which are uniform in the number of particles N . Then we will use the convergence results for the approximations from Section 6.2 to pass the properties to (ρ, P) .

7.1. Uniform flocking estimates on the sticky particle approximations

The flocking phenomenon for the Cucker–Smale dynamics (2) has been first studied for general protocols ϕ in [7]. The idea can be easily adapted to the sticky particle dynamics.

Consider a sequence of sticky particle Cucker–Smale dynamics $(m_{i,N}, x_{i,N}(t), v_{i,N}(t))$ that approximates the Euler-alignment system (ρ, P) . The discrete analog of the diameter is

$$D_N(t) := \max_{1 \leq i, j \leq N} |x_{i,N}(t) - x_{j,N}(t)| = x_{N,N}(t) - x_{1,N}(t), \quad (82)$$

and the variation of velocity becomes

$$V_N(t) := \max_{1 \leq i, j \leq N} |v_{i,N}(t) - v_{j,N}(t)| = \max_{1 \leq i \leq N} v_{i,N}(t) - \min_{1 \leq i \leq N} v_{i,N}(t). \quad (83)$$

Note that using the approximations of the initial data $(m_{i,N}, x_{i,N}^0, v_{i,N}^0)$ constructed in Theorem 6.5, it is easy to verify that

$$D_N(0) \leq D^0, \quad V_N(0) \leq V^0. \quad (84)$$

We are ready to establish uniform flocking estimates for the sticky particle Cucker–Smale dynamics.

Theorem 7.1. *Let $(x_{i,N}(t), v_{i,N}(t))_{i=1}^N$ be a sequence of sticky particle Cucker–Smale dynamics associated to the initial data $(m_{i,N}, x_{i,N}^0, v_{i,N}^0)_{i=1}^N$. Define $D_N(t)$ and $V_N(t)$ as in (82) and (83). Assume (84) holds, and that*

$$\sup_{R>0} \Phi(R) > \mathcal{E}^0 := \Phi(D^0) + V^0, \quad (85)$$

where we recall the definition (17) of Φ . Then for all $t \geq 0$, the sticky particle dynamics satisfy the following estimates, uniformly in N .

$$\text{Flocking :} \quad \sup_{t \geq 0} D_N(t) \leq \bar{D} := \Phi^{-1}(\mathcal{E}^0) < +\infty; \quad (86)$$

$$\text{Fast alignment :} \quad V_N(t) \leq V^0 \exp(-\phi(\bar{D})t); \quad (87)$$

Proof. It is well-known that on time intervals where $(x_i(t), v_i(t))_{i=1}^N$ follow the (collision-less) Cucker–Smale dynamics, the quantities $D_N(t)$ and $V_N(t)$ are Lipschitz continuous and satisfy the following differential inequalities at every time t where they are differentiable:

$$\dot{D}_N(t) \leq V_N(t), \quad (88)$$

$$\dot{V}_N(t) \leq -\phi(D_N(t))V_N(t). \quad (89)$$

Define a Lyapunov functional $\mathcal{E}_N(t) = \Phi(D_N(t)) + V_N(t)$. It clearly follows from (88) and (89) that $\mathcal{E}_N(t)$ is nonincreasing along intervals during which no collisions occur. On the other hand, if t is a collision time, then $V_N(t) \leq V_N(t-)$ by the maximum principle, while $\Phi(D_N(t))$ is continuous at time t . Therefore $\mathcal{E}_N(t)$ is nonincreasing on all of $[0, \infty)$.

Let us now assume that (85) holds. Since $\mathcal{E}_N(t)$ is nonincreasing, we have

$$\Phi(D_N(t)) \leq \mathcal{E}_N(t) \leq \mathcal{E}_N(0) \leq \mathcal{E}^0, \quad \text{for all } t \geq 0.$$

This implies (86). Note that Φ is a nondecreasing function in $[0, \infty)$, and the assumption (85) guarantees that $\mathcal{E}^0$ lies in its range. Therefore, $\bar{D} = \Phi^{-1}(\mathcal{E}^0) = \inf\{R : \Phi(R) \geq \mathcal{E}^0\}$ is well-defined and takes a nonnegative finite value.

Finally, combining (86) and (89) and using the fact that ϕ is radially decreasing, we obtain (87). $\square$

7.2. Flocking for the 1D Euler-alignment system

Theorem 7.2. *Let (ρ^0, P^0) satisfy the hypotheses of Theorem 6.3, and (ρ, P) be the associated solution. Assume (85) holds. Then the solution (ρ, P) experiences*

$$\text{Flocking :} \quad \sup_{t \geq 0} D(t) \leq \bar{D} := \Phi^{-1}(\mathcal{E}^0) < +\infty; \quad (90)$$

$$\text{Fast alignment :} \quad V(t) \leq V^0 \exp(-\phi(\bar{D})t); \quad (91)$$

$$\text{Strong Flocking :} \quad \mathcal{W}_1(\rho(\cdot + \bar{u}t, t), \rho_\infty) \rightarrow 0, \quad \text{as } t \rightarrow +\infty, \text{ for some } \rho_\infty \in \mathcal{P}_c(\mathbb{R}). \quad (92)$$

When the communication protocol ϕ has a fat tail (9), we get $\lim_{R \rightarrow \infty} \Phi(R) = \infty$. Therefore, (85) holds for any finite D^0 and V^0 . Hence, Theorem 7.2 implies that our weak solutions must flock in the presence of a fat-tailed ϕ .

Before giving the proof, we state the following Lemma.

Lemma 7.3. *Let $\rho, \tilde{\rho} \in \mathcal{P}_c(\mathbb{R})$. Denote D and $\tilde{D}$ the diameter of the support of ρ and $\tilde{\rho}$, respectively. Suppose $\tilde{D} < D$. Then, there exists a constant $c > 0$, depending only on ρ and $D - \tilde{D}$, such that $\mathcal{W}_1(\rho, \tilde{\rho}) \geq c$.*

Proof. Let $[x_\ell, x_r]$ be the smallest interval that contains $\text{supp} \rho$, so that $D = x_r - x_\ell$. Define $\varepsilon = D - \tilde{D} > 0$. Since $\text{supp} \tilde{\rho}$ has the smaller diameter, we must have $\tilde{\rho}(I) = 0$ for at least one of the intervals $I = [x_\ell, x_\ell + \frac{\varepsilon}{2}]$ or $(x_r - \frac{\varepsilon}{2}, x_r]$. Consider the first case. Let f_N be a Lipschitz function which is supported in $[x_\ell - \varepsilon, x_\ell + \frac{\varepsilon}{2}]$, takes the value 1 in $[x_\ell(t), x_\ell(t) + \frac{\varepsilon}{4}]$, and satisfies $|f_N|_{\text{Lip}} \leq 4\varepsilon^{-1}$. Then

$$\mathcal{W}_1(\rho, \tilde{\rho}) \geq \frac{4}{\varepsilon} \int_{x_\ell}^{x_\ell + \frac{\varepsilon}{4}} d\rho(x) = c_l(\rho, \varepsilon) > 0.$$

A similar argument works for the other case and yields $\mathcal{W}_1(\rho, \tilde{\rho}) \geq c_r(\rho, \varepsilon) > 0$. Thus, we get a positive lower bound $c = \min\{c_l(\rho, \varepsilon), c_r(\rho, \varepsilon)\} > 0$, which only depends on ρ and ε . $\square$

Proof of Theorem 7.2. Let $(\rho_N, P_N)_{N=1}^\infty$ be a sequence of sticky particle approximations. We will establish (90) and (91) using the uniform flocking estimates on (ρ_N, P_N) furnished by Theorem 7.1, as well as the convergence results for the sticky particle approximation in Theorem 6.5. Note that (72) guarantees that $\sum_{i=1}^N m_{i,N} v_{i,N}^0 = \int \rho^0 u^0 dx$ for each N ; therefore we may assume without loss of generality that both are zero, by Galilean invariance.

We first show (90) by contradiction. Assume there exists a time t such that $D(t) = \bar{D} + \varepsilon$, with some $\varepsilon > 0$. We apply Lemma 7.3 below with $\rho = \rho(t)$ and $\tilde{\rho} = \rho_N(t)$ and get $\mathcal{W}_1(\rho(t), \rho_N(t)) \geq c > 0$, where $c = c(\rho, \varepsilon)$ is independent of N . This uniform positive lower bound contradicts the convergence (70).

Next, we turn to (91). Fix a time $t \geq 0$. In view of (79), we can find a sequence of test functions $f_k \in C_c^\infty(\mathbb{R})$, normalized by $\int_{\mathbb{R}} f_k(x) d\rho(x, t) = 1$, such that $u_+(t) = \lim_{k \rightarrow \infty} \int_{\mathbb{R}} f_k(x) dP(x, t)$. For each k , we apply the convergence results (70) and (71) to get

$$\lim_{N \rightarrow \infty} \int_{\mathbb{R}} f_k(x) d\rho_N(x, t) = \int_{\mathbb{R}} f_k(x) d\rho(x, t) = 1, \quad \lim_{N \rightarrow \infty} \int_{\mathbb{R}} f_k(x) dP_N(x, t) = \int_{\mathbb{R}} f_k dP(x, t).$$

We therefore obtain

$$\int_{\mathbb{R}} f_k(x) dP(x, t) = \lim_{N \rightarrow \infty} \frac{\int_{\mathbb{R}} f_k(x) dP_N(x, t)}{\int_{\mathbb{R}} f_k(x) d\rho_N(x, t)}.$$

Similarly, we find a sequence of normalized test functions g_k such that $u_-(t) = \lim_{k \rightarrow \infty} \int_{\mathbb{R}} g_k(x) dP(x, t)$. We may thus write

$$\begin{aligned} \int_{\mathbb{R}} f_k(x) dP(x, t) - \int_{\mathbb{R}} g_k(x) dP(x, t) &= \lim_{N \rightarrow \infty} \left(\frac{\int_{\mathbb{R}} f_k(x) dP_N(x, t)}{\int_{\mathbb{R}} f_k(x) d\rho_N(x, t)} - \frac{\int_{\mathbb{R}} g_k(x) dP_N(x, t)}{\int_{\mathbb{R}} g_k(x) d\rho_N(x, t)} \right) \\ &\leq \limsup_{N \rightarrow \infty} \left(\max_{1 \leq i \leq N} v_{i,N}(t) - \min_{1 \leq i \leq N} v_{i,N}(t) \right) = \limsup_{N \rightarrow \infty} V_N(t) \leq V^0 \exp(-\phi(\bar{D})t), \end{aligned}$$

where we have used the uniform fast alignment estimate (87) in the last inequality. Finally, we take $k \rightarrow \infty$ and use the definition (80) to conclude (91).

We finish by showing that the sequence $(\rho(t))_{t \geq 0}$ is Cauchy in $(\mathcal{P}_c(\mathbb{R}), \mathcal{W}_1)$, which allows us to conclude the existence of a ρ_∞ such that (81) holds. We compute as follows, leveraging the estimates (90) and (91) that we have just established. For $0 \leq t_1 \leq t_2$, we have

$$\begin{aligned}
\mathcal{W}_1(\rho_N(t_1), \rho_N(t_2)) &\leq \sum_{i=1}^N m_{i,N} |x_{i,N}(t_1) - x_{i,N}(t_2)| \leq \sum_{i=1}^N m_{i,N} \int_{t_1}^{t_2} |v_{i,N}(t)| \, dt \\
&\leq \sum_{i=1}^N m_{i,N} \int_{t_1}^{t_2} 2V^0 \exp(-\phi(\bar{D}t)) \, dt \\
&\leq \frac{2V^0}{\phi(\bar{D})} \exp(-\phi(\bar{D})t_1),
\end{aligned}$$

Pick t_1 large enough so that the right side of the above is small. Choosing any $t_2 \geq t_1$, then $N = N(t_1, t_2)$ large enough, we may conclude that

$$\mathcal{W}_1(\rho(t_1), \rho(t_2)) \leq \mathcal{W}_1(\rho(t_1), \rho_N(t_1)) + \mathcal{W}_1(\rho_N(t_1), \rho_N(t_2)) + \mathcal{W}_1(\rho_N(t_2), \rho(t_2))$$

can be made as small as desired. This completes the proof. $\square$

Remark 7.1. The structure of the measure ρ_∞ has been analyzed extensively in [68] in the context of classical solutions (c.f. also [69]). In [68], the authors studied *mass concentration* in the limiting profile ρ_∞ by demonstrating a correspondence between the set where e^0 is zero and the singular support of ρ_∞ . It would be interesting to study the structure of ρ_∞ for the weak solutions considered in this paper—it is far from straightforward to predict the outcome from the initial data. We leave further consideration of this question for future work.

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