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Generically Computable Abelian Groups

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Abstract. Generically computable sets, as introduced by Jockusch and Schupp, have been of great interest in recent years. This idea of approximate computability was motivated by asymptotic density problems studied by Gromov in combinatorial group theory. More recently, we have defined notions of generically computable structures, and studied in particular equivalence structures and injection structures. A structure is said to be generically computable if there is a computable substructure defined on an asymptotically dense set, where the functions are computable and the relations are computably enumerable. It turned out that every equivalence structure has a generically computable copy, whereas there is a non-trivial characterization of the injection structures with generically computable copies.

In this paper, we return to group theory, as we explore the generic computablity of Abelian groups. We show that any Abelian p-group has a generically computable copy and that such a group has a Σ_2-generically computably enumerable copy if and only it has a computable copy. We also give a partial characterization of the Σ_1-generically computably enumerable Abelian p-groups. We also give a non-trivial characterization of the generically computable Abelian groups that are not p-groups. AQ1

Keywords: computability · generically computable · Abelian p-group · Σ_n elementary substructure

1 Introduction

Experts in mathematical logic and computability theory show that many interesting problems are undecidable, that is, there is no algorithm for computing a

This research was partially supported by the National Science Foundation SEALS grant DMS-1362273. The work was done partially while the latter two authors were visiting the Institute for Mathematical Sciences, National University of Singapore, in 2017. The visits were supported by the Institute. This material is partially based upon work supported by the National Science Foundation under grant DMS-1928930 while all three authors participated in a program hosted by the Mathematical Sciences Research Institute in Berkeley, California during the Fall 2020 semester. Harizanov was partially supported by the Simons Foundation grant 853762 and NSF grant DMS-2152095.

D. Genova and J. Kari (Eds.): UCNC 2023, LNCS 14003, pp. 1–14, 2023.
https://doi.org/10.1007/978-3-031-34034-5_3

solution to a given problem. Thus it is very important to find unconventional ways in which a solution to the problem may be approximated. The notion of dense computability for sets of natural numbers is that there is an algorithm which computes the solution on an asymptotically dense set. The study of densely computable, generically computable, and coarsely computable sets is now well-established.

The classic motivating example which comes from structure theory is the word problem for finitely generated groups. For many groups with undecidable word problems, including a standard example from [11], the particular words on which it is difficult to decide equality to the identity are very special words (and are even called by this term in some expositions). Thus the problem can be solved on a dense set.

In two recent papers [1,2], the authors have developed the notions of *densely computable* structures and isomorphisms. This builds on the concepts of generically and coarsely computable sets, as studied by Jockusch and Schupp [5,6] and many others, which have been a focus of research in computability. For structures, the question is whether some "large" substructure is computable.

There are, roughly, two extremal possibilities (say, in the case of generic computability):

1. Every countable structure has a generically computable copy, or
2. Any countable structure with a generically computable copy has a computable copy.

It was shown in [1] that each of these can be achieved in certain classes, and that they do not exhaust all possibilities.

The authors also explored these conditions under the added hypothesis that the "large" substructures in question be, in some weak sense, elementary (that is, elements of the substructure satisfy certain formulas which they satisfy in the full structure). Again, we find that there are natural extremal possibilities, and that both they and non-extremal cases are achieved.

Finally, we found that as the elementarity hypotheses are strengthened, all known cases eventually (for Σ_n elementarity at sufficiently large n) trivialize. This demonstrates that these notions of dense computability are structural they depend fundamentally on the semantics of the structure and not only on the density or algorithmic features of the presentation.

1.1 The Model of Computation

It would be worthwhile to distinguish which results in computable structure theory depend on a "special" (and potentially extremely rare) input, and which are less sensitive. To achieve this goal in the context of word problems on groups, Kapovich, Myasnikov, Schupp, and Shpilrain [8] proposed using notions of asymptotic density to state whether a partial recursive function could solve "almost all" instances of a problem.

Jockusch and Schupp [5] generalized this approach to the broader context of computability theory in the following way. For a subset S of $\mathbb{N}$.

1. The density of S up to n, denoted by $\rho_n(S)$, is given by

$$\frac{|S \cap \{0, 1, 2, \ldots, n-1\}|}{n}.$$

2. The asymptotic density of S, denoted by $\rho(S)$, is given by $\lim_{n\to\infty} \rho_n(S)$.

A set A is said to be *generically computable* if and only if there is a partial computable function ϕ such that ϕ agrees with the characteristic function χ_A throughout the domain of ϕ, and such that the domain of ϕ has asymptotic density 1. A set A is said to be *coarsely computable* if and only if there is a *total* computable function ϕ that agrees with χ_A on a set of asymptotic density 1.

The study of generically and coarsely computable sets and some related notions has led to an interesting program of research in recent years; see [6] for a partial survey.

1.2 Densely Computable Structures

A structure $\mathcal{A}$ consists of a set A (the universe or domain of $\mathcal{A}$), together with finitely many functions $\{f_i : i \in I\}$, each f_i of arity p_i, and relations $\{R_j : j \in J\}$, each R_i of arity r_j. The structure $\mathcal{A}$ is said to be computable if the set A and the functions and relations are all computable. A structure $\mathcal{B}$ which is isomorphic to $\mathcal{A}$ is said to be a *copy* of $\mathcal{A}$. Given a structure $\mathcal{A}$, we want to consider what it means to say that $\mathcal{A}$ is generically computable, or "nearly computable" in some other notion related to density. We now present informal versions of the definitions, which will be made precise in Sect. 2. The idea is that $\mathcal{A}$ is generically computable if there is a substructure $\mathcal{D}$ with universe a computably enumerable set D of asymptotic density one which is computable in the following sense: There exist partial computable functions $\{\phi_i : i \in I\}$ and $\{\psi_j : j \in J\}$ such that ϕ_i agrees with f_i on the Cartesian product D^{p_i} and ψ_j agrees with the characteristic function of R_j on D^{r_j}. Similarly $\mathcal{A}$ is coarsely computable if there is a computable structure $\mathcal{E}$ and a dense set D such that the structure $\mathcal{D}$ with universe D is a substructure of both $\mathcal{A}$ and of $\mathcal{E}$ and all relations and functions agree on D. A more interesting variation requires that $\mathcal{D}$ is a Σ_1 elementary submodel of $\mathcal{A}$, more generally a Σ_n elementary submodel. That is, if we are saying that $\mathcal{A}$ is "nearly computable" when it has a dense substructure $\mathcal{D}$ which is computable (computably enumerable), then the substructure should be similar to $\mathcal{A}$ by some standard.

To be precise, recall that $\mathcal{D}$ is an Σ_n elementary substructure of $\mathcal{A}$ provided that, for any Σ_1 formula $\varphi(x_1, \ldots, x_n)$ and any elements $a_1, \ldots, a_n \in D$,

$$\mathcal{A} \models \varphi(a_1, \ldots, a_n) \iff \mathcal{D} \models \varphi(a_1, \ldots, a_n).$$

We will say that the structure $\mathcal{A}$ is *Σ_n-generically computably enumerable* if there is an asymptotically dense set D such that

(a) $\mathcal{D}$ is a Σ_n-elementary substructure of $\mathcal{A}$;

(b) there exist partial computable functions $\{\phi_i : i \in I\}$ such that ϕ_i agrees with f_i on D^{p_i};
(c) each R_j restricted to D^{r_j} is a computably enumerable relation.

We remark that generically computable is the same as generically Σ_0, since $\mathcal{B}$ is a submodel of $\mathcal{A}$ if and only if it preserves all quantifier-free formulas.

The outline of this paper is as follows. Section 2 contains background on asymptotic density, and gives the generalization of generic computability to structures and isomorphisms. Section 3 presents results on generically computable and Σ_n-generically computably enumerable Abelian groups. We show that every countable Abelian p-group has a generically computable copy. We characterize the class of countable Abelian groups which have generically computable copies. We also characterize Abelian p-groups which have Σ_1-generically computably enumerable copies and those which have generically Σ_2-generically computably enumerable copies.

2 Background

In this section, we provide some background on the notions of asymptotic density and generically computable sets. We define the more general notions of Σ_n-generically computably enumerable structures.

The asymptotic density of a set $A \subseteq \omega$ is defined as follows.

Definition 1. *The* asymptotic density *of A is* $\lim_n \frac{|A\cap\{0,1\ldots,n-1\}|}{n}$, *if this exists.*

In [5], Jockusch and Schupp give the following definition, along with the notion of coarsely computable sets, which we will not discuss here.

Definition 2. *Let $S \subseteq \omega$. We say that S is* generically computable *if there is a partial computable function $\Phi : \omega \to 2$ such that $\Phi = \chi_S$ on the domain of Φ, and such that the domain of Φ has asymptotic density 1.*

The most natural notion for a structure seems to be require that the substructure with domain D resembles the given structure $\mathcal{A}$ by agreeing on certain first-order formulas, existential formulas in particular. Throughout this paper, Σ_n represents the n'th level of the arithmetical hierarchy, as described in Soare [12]. Other background on computability may also be found in [12].

We recall the notion of an elementary substructure.

Definition 3. *A substructure $\mathcal{B}$ of the structure $\mathcal{A}$ is said to be a (fully)* elementary *substructure ($\mathcal{B} \prec \mathcal{A}$) if for any $b_1, \ldots, b_k \in \mathcal{B}$, and any formula $\phi(x_1, \ldots, x_k)$,*

$$\mathcal{A} \models \phi(b_1, \ldots, b_k) \iff \mathcal{B} \models \phi(b_1, \ldots, b_k).$$

The substructure $\mathcal{B}$ is said to be a Σ_n elementary substructure ($\mathcal{B} \prec_n \mathcal{A}$) if for any $b_1, \ldots, b_k \in \mathcal{B}$, and any Σ_n formula $\phi(x_1, \ldots, x_k)$,

$$\mathcal{A} \models \phi(b_1, \ldots, b_k) \iff \mathcal{B} \models \phi(b_1, \ldots, b_k).$$

Definition 4. *For any structure $\mathcal{A}$:*

1. *A substructure $\mathcal{B}$ of $\mathcal{A}$, with universe B, is a* computable substructure *if the set B is c.e and each function and relation is computable on B, that is, for any k-ary function f and any k-ary relation R, both $f \upharpoonright B^k$ and $\chi_R \upharpoonright B^k$ are the restrictions to B^k of partial computable functions.*
2. *A substructure $\mathcal{B}$ of $\mathcal{A}$, with universe B, is a* computably enumerable (computably enumerable) structure *if the set B is computably enumerable, each relation is computably enumerable and the graph of each function is computably enumerable (so that the function is partial computable but also total on B).*
3. *$\mathcal{A}$ is* generically computable *if there is a substructure $\mathcal{D}$ with universe a computably enumerable set D of asymptotic density one such that the substructure $\mathcal{D}$ with universe D is a computable substructure.*
4. *$\mathcal{A}$ is* Σ_n-generically computably enumerable *if there is a dense computably enumerable set D such that the substructure $\mathcal{D}$ with universe D is a computably enumerable substructure and also a Σ_n-elementary substructure of $\mathcal{A}$.*

For $n > 0$, any Σ_{n+1}-generically computably enumerable structure Σ_n-generically computably enumerable. For structures with functions but no relations, this also holds for $n = 0$. However, a computably enumerable substructure might not be computable, so a structure $\mathcal{A}$ with relations which is Σ_1-generically computably enumerable is not necessarily generically computable.

Countable Abelian groups have been thoroughly studied by Kaplansky [7], Fuchs [4] and many others. Here is some background from Fuchs [4].

Definition 5. *Let $\mathcal{A}$ be an Abelian group and let p be a prime number.*

1. *$\mathcal{A}$ is a p-group if every element has order a power of p.*
2. *$\mathcal{A}[p]$ is the subgroup of elements with order a power of p.*
3. *The* p-height *$ht_p^{\mathcal{A}}(x)$ of an element $x \in \mathcal{A}$ is the largest n such that $p^n|x$, that is, there exists y such that $p^n y = x$.*
4. *A subgroup $\mathcal{B}$ of $\mathcal{A}$ is* pure *if, for every prime q and every $b \in B$, $ht_q^{\mathcal{B}}(b) = ht_q^{\mathcal{A}}(b)$. The subscript q will be omitted if it is clear from the context.*
5. *$\mathcal{A}$ is* divisible *if every element of $\mathcal{A}$ has infinite height, that is, for every $x \in \mathcal{A}$ and every $n \in \mathbb{N}$, there exists $y \in \mathcal{A}$ such that $x = n \cdot y$.*
6. *A group is* reduced *if it has no divisible subgroup.*

For any prime p, the group $\mathbb{Z}(p^\infty)$ may be realized as the rational numbers with denominators a power of p, with addition modulo one. These groups are said to be *quasicyclic.*

We need the following results from [4].

Theorem 1 (Baer). *Every Abelian group is a direct sum of a divisible group and a reduced group.*

Theorem 2 (Prüfer). *A countable Abelian p-group is a direct sum of cyclic groups if and only if it contains no elements of infinite height.*

Theorem 3 (Szele). *Let $\mathcal{B}$ be a subgroup of the Abelian p-group $\mathcal{A}$ such that $\mathcal{B}$ is the direct sum of cyclic subgroups of the same order p^k, for some finite k. Then $\mathcal{B}$ is a direct summand of $\mathcal{A}$ if and only if $\mathcal{B}$ is a pure subgroup of $\mathcal{A}$.*

The following standard result is Theorem 1 of Kaplansky [7].

Theorem 4. *Any torsion group $\mathcal{A}$ is the direct sum of p-groups $\mathcal{A}[p]$.*

Definition 6. *For an Abelian group $\mathcal{A}$, the* Ulm subgroup *$U(\mathcal{A})$ is the set of elements of infinite height. This operation may be iterated to obtain the Ulm sequence $\mathcal{A}_0 = \mathcal{A}, \mathcal{A}_1 = U(\mathcal{A}), \mathcal{A}_2 = U(\mathcal{A}_1), \dots$ and extended to the transfinite ordinals by $\mathcal{A}^\lambda = \cap_{\alpha<\lambda}\mathcal{A}_\alpha$ and $\mathcal{A}_{\alpha+1} = U(\mathcal{A}_\alpha)$. The* length *of a group is the least α such that $\mathcal{A}_{\alpha+1} = \mathcal{A}_\alpha$.*

Corollary 1. *Let $\mathcal{A}$ be a countable Abelian p-group and let $\mathcal{A} = \mathcal{C} \oplus \mathcal{D}$, where $\mathcal{C}$ has no elements of infinite height and $\mathcal{D}$ is divisible. Then $\mathcal{A}$ has the form $\bigoplus_{i<\omega} \mathbb{Z}(p^{n_i}) \oplus \bigoplus_{i\leq k} \mathbb{Z}(p^\infty)$, where $k \leq \omega$.*

In computability theory, the *character* $\chi(\mathcal{A})$ of an Abelian p-group $\mathcal{A}$ is defined to be the set

$$\{(n,k) \in (\omega \setminus \{0\})^2 : \mathcal{A} \text{ has at least } n \text{ factors of the form } \mathbb{Z}(p^k)\}.$$

We say that $K \subseteq (\omega \setminus \{0\})^2$ is a *character* if whenever $(n+1,k) \in K$, then $(n,k) \in K$. As for injection structures and equivalence structures, it is easy to see that K is a character if and only if $K = \chi(\mathcal{A})$ for some Abelian p-group $\mathcal{A}$.

Computable Abelian p-groups were studied by A. Morozov and the authors in [3]. See Khisamiev [9] for more background.

Proposition 1 (Kulikov). *For any countable Abelian p-group $\mathcal{A}$and any $n,k \geq 1$, $(n,k) \in \chi(\mathcal{A})$ if and only if $\mathcal{A}$ has a pure subgroup isomorphic to $\bigoplus_{i<n} \mathbb{Z}(p^k)$.*

Proposition 2. *Let $\mathcal{A}$ be an Abelian p-group and let n and k be positive integers. Then*

1. *There is a quantifier-free formula $\phi_{n,k}$ such that, for any Abelian group $\mathcal{A}$ and any $a_1, \dots, a_n \in \mathcal{A}$, $\phi_{n,k}(a_1, \dots, a_n)$ if and only if $a_1, \dots, a_n$ are independent elements each of order p^k, that is, if and only if $\langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \cdots \oplus \langle a_n \rangle$ is isomorphic to $\bigoplus_{i<n} \mathbb{Z}(p^k)$.*
2. *There is a Σ_1 formula $\theta_{n,k}$ such that $\mathcal{A} \models \theta_{n,k}$ if and only if $\mathcal{A}$ has a subgroup of the form $\bigoplus_{i<n} \mathbb{Z}(p^{k_i})$, with each $k_i \geq k$.*
3. *There is a Σ_2 formula $\psi_{n,k}$ such that $\mathcal{A} \models \psi_{n,k}$ if and only if $(n,k) \in \chi(\mathcal{A})$, that is, if and only if $\mathcal{A}$ has a pure subgroup of the form $\bigoplus_{i<n} \mathbb{Z}(p^k)$.*

This was used by Khisamiev [9] to obtain the following.

Theorem 5 (Khisamiev). *For any computable p-group $\mathcal{A}$, $\chi(\mathcal{A})$ is a Σ^0_2 set.*

The following was shown in [3].

Proposition 3. *Let K be a Σ_2^0 character and let p be a prime number. Then there is a computable Abelian p-group $\mathcal{A}$ with character K and with infinitely many divisible components.*

Definition 7. *A function $f : \omega^2 \to \omega$ is said to be an s_1-*function *if the following hold:*

1. *For every i and s, $f(i,s) \leq f(i,s+1)$.*
2. *For every i, the limit $m_i = \lim_{s\to\infty} f(i,s)$ exists.*
3. *For every i, $m_i < m_{i+1}$.*

The character K is said to *possess* the s_1-function f if $(1, m_i) \in K$ for each i. Here are some useful results about the characters of Abelian p-groups.

The next lemma is based on Corollary 2.11 and Corollary 2.14 of [3].

Lemma 1. *For any Σ_2^0 character K which is either bounded or possesses a computable s_1-function, there is a computable Abelian p-group $\mathcal{A}$ with character K and no divisible factors.*

3 Σ_n-Generically Computably Enumerable Abelian Groups

This section contains the new results about generically computably enumerable Abelian groups. The following result is immediate from the definition of generically computable structures, and begins to suggest the ubiquity of generically computable copies.

Lemma 2. *Let $\mathcal{A}$ be an Abelian group, and $\mathcal{B}$ an infinite subgroup of $\mathcal{A}$. If $\mathcal{B}$ has a generically computable copy, then $\mathcal{A}$ has a generically computable copy.*

The following phenomenon was unexpected when we first observed the analogous result for equivalence structures.

Proposition 4. *Every countable Abelian p-group $\mathcal{A}$ has a generically computable copy.*

Proof. If the group $\mathcal{A}$ is finite, then of course it is computable. The proof for countably infinite structures is in two steps. First, we show that $\mathcal{A} = (\omega, +_A)$ always has a subgroup $\mathcal{B}$ which is isomorphic to a computable group. Second, we obtain a computable group $\mathcal{D} = (D, +_D)$ isomorphic to $\mathcal{B}$ with universe D a dense co-infinite set, and then extend $\mathcal{D}$ to generically computable $\mathcal{C} = (\omega, +_C)$ isomorphic to $\mathcal{A}$.

The first step is in three cases.

Case 1: $\mathcal{A}$ has a divisible subgroup $\mathcal{B}$. Then it is known that $\mathcal{B}$ has a computable copy.

Case 2: Every element of $\mathcal{A}$ has finite height. Then, by Theorem 2, $\mathcal{A}$ has the form $\bigoplus_{i<\omega} \mathbb{Z}(p^{n_i})$. Let $\{a_i : i < \omega\}$ be a set of generators for $\mathcal{A}$, so that

$\mathcal{A} = \oplus_i \langle a_i \rangle$ and a_i has order p^{n_i} For each i, $p^{n_i - 1} a_i$ has order p. Let $\mathcal{B} = \oplus_i \langle p^{n_i - 1} a_i \rangle$. Then $\mathcal{B}$ is a subgroup of $\mathcal{A}$ isomorphic to $\oplus_{i<\omega} \mathbb{Z}(p)$, which is known to have a computable copy.

Case 3: $\mathcal{A}$ has an element a of infinite height, but no divisible subgroup. Without loss of generaility, we may assume that a has order p. Let $a = a_0$, and for each $n > 0$, choose a_n so that $p^n a_n = a$. For any $m \in \omega$, let $A_m = \{p^{n-m} a_n : n < \omega\}$. In particular, $A_1 = \{p^{n-1} a_n : n > 0\}$, so that every element of A_1 has order p^2. Every element of A_m has order p^{m+1}.

Claim: $\mathcal{A}$ has an element b such that $\{x : px = b\}$ is infinite.

Proof of Claim: Suppose not. Then in particular A_1 is finite. We will construct a divisible subgroup of $\mathcal{A}$, contradicting our assumption. This will be done by finding a sequence $(b_i)_{i<\omega} \subseteq \{a_n : n < \omega\}$ of elements of infinite height, beginning with $b_0 = a$, such that $pb_{n+1} = b_n$ for each n. It will then follow that $\{b_0, b_1, \dots\}$ generates a divisible group. For each element b of A_1, there is some n so that $b = p^{n-1} a_n$. Given that A_1 is finite, there must be some b_1 such that $b_1 = p^{n-1} a_n$ for infinitely many n. It follows that b has infinite height. Let $b_1 = b$ and consider $B_2 = \{p^{n-2} a_n : p^{n-1} a_n = b_1\}$. If B_2 is infinite, then the claim is established. If B_2 is finite, then, as above, there is some $b_2 \in B_2$ such that $b_2 = p^{n-2} a_n$ for infinitely many n. Continuing in this way we reach one of two outcomes.

(1) There will be some n such that $\{x : px = b_n\}$ is infinite.

or

(2) For each n, $pb_{n+1} = b_n$. In this case, $\{b_n : n = 1, 2, \dots\}$ will generate a divisible subgroup.

This completes the proof of the Claim.

Thus we have found b such that $C = \{x : px = b\}$ is infinite.

Let b have order p^r. Then each element of C has order p^{r+1}. It follows that C generates an infinite subgroup $\mathcal{B}$ of $\mathcal{A}$ with all elements of order $\leq p^r$. The group $\mathcal{B}$ is therefore isomorphic to a computable group, as desired.

Now let $\mathcal{D} = (D, +_D)$ be computable and isomorphic to $\mathcal{B}$, where D is asymptotically dense and co-infinite. Let H be a permutation of ω which maps D to B.

Define the extension $\mathcal{C} = (\omega, +_C)$ of $\mathcal{D}$ by

$$x +_C y = H^{-1}(H(x) +_A H(y)).$$

Then H is an isomorphism from $\mathcal{C}$ to $\mathcal{A}$ since $H(x +_C y) = H(x) +_A H(y)$.

In particular, for $x, y \in D$,

$$x +_C y = H^{-1}(H(x) +_A H(y)) = H^{-1}(H(x) +_B H(y)) = x +_D y,$$

since H is a group isomorphism from $\mathcal{D}$ to $\mathcal{B}$.

It follows that $\mathcal{D}$ is a computable subgroup of $\mathcal{C}$. Since D is a dense set, $\mathcal{C}$ is generically computable. So $\mathcal{A}$ is isomorphic to a generically computable group, as desired. □

Next, we consider countable Abelian groups in general. For each such group $\mathcal{A}$, let $\mathcal{A}[p] = \{x \in \mathcal{A} : p^n x = 0 \text{ for some } n\}$.

Theorem 6. *A countable Abelian group has a generically computable copy if and only if either*

1. *$\mathcal{A}[p]$ is infinite for some prime p, or*
2. *$\{p : \mathcal{A}[p] \neq 0\}$ has an infinite computably enumerable subset.*

Proof. Suppose first that $\mathcal{A}$ has a generically computable copy $\mathcal{C}$ and let $\mathcal{D} = (D, +_D)$ be a subgroup of $\mathcal{C}$, where D is a computably enumerable dense set and $+_D$ is computable on D. Suppose that $\mathcal{D}[p]$ is finite for all primes p. Then $\mathcal{D}[p]$ must be nonempty for infinitely many p. Now $\{p : \mathcal{D}[p] \neq 0\}$ is an infinite computably enumerable subset of $\{p : \mathcal{C}[p] \neq 0\} = \{p : \mathcal{A}[p] \neq 0\}$.

Next let p be a prime such that $\mathcal{A}[p]$ is infinite. Then $\mathcal{A}[p]$ has a generically computable copy $\mathcal{B}$. Let $\mathcal{C} = \mathcal{A}[p] \oplus \bigoplus_{q \neq p} \mathcal{A}[q]$. Then $\mathcal{C}$ is isomorphic to $\mathcal{A}$ and $\mathcal{C}[p]$ is generically computable since $\mathcal{C}[p] = \mathcal{B}$ is generically computable.

Finally, suppose that there is an infinite computably enumerable set P of primes p such that $\mathcal{A}[p] \neq 0$. Then $\mathcal{A}$ will have a subgroup isomorphic to $\bigoplus_{p \in P} \mathbb{Z}(p)$, and we proceed as usual. □

Note that Theorem 6 implies that there are countable Abelian groups with no generically computable copy, in contrast to Proposition 4 on primary groups.

We now turn to the topic of Σ_n elementary substructures and Σ_n-generically computably enumerable structures.

Proposition 5. *Let $\mathcal{A}$ be an Abelian group and let $\mathcal{B}$ be a subgroup of $\mathcal{A}$. $\mathcal{B}$ is a Σ_1 elementary subgroup of $\mathcal{A}$ if and only if it satisfies condition*

(*): For any finite subgroup $\mathcal{C}$ of $\mathcal{A}$, there is a subgroup $\mathcal{D}$ of $\mathcal{B}$ isomorphic to $\mathcal{C}$, such that $B \cap C = D \cap C$ and the isomorphism is the identity on $B \cap C$.

Proof. Suppose first that $\mathcal{B}$ is a Σ_1 elementary subgroup of $\mathcal{A}$.

Let $C = \{a_1, \ldots, a_m, b_1, \ldots, b_n\}$ be the domain of a finite subgroup of $\mathcal{A}$ with $B \cap C = \{b_1, \ldots, b_n\}$ and let $\phi(a_1, \ldots, a_m, b_1, \ldots, b_n)$ be a sentence which captures the atomic diagram of $\mathcal{C}$. Then

$$\mathcal{A} \models (\exists x_1)(\exists x_2) \ldots (\exists x_m)\phi(x_1, \ldots, x_m, b_1, \ldots, b_n).$$

Since $\mathcal{B}$ is a Σ_1 elementary submodel, it follows that there are $c_1, \ldots, c_m \in B$ such that

$$\phi(c_1, \ldots, c_m, b_1, \ldots, b_n).$$

Then the subgroup $\mathcal{D}$ with domain $D = \{c_1, \ldots, c_m, b_1, \ldots, b_n)$ is isomorphic to C under the isomorphism mapping each c_i to a_i and mapping each b_j to itself. Furthermore, $B \cap C = D \cap C = \{b_1, \ldots, b_n\}$.

For the other direction, suppose that $\mathcal{B}$ satisfies condition (*). Let $b_1, \ldots, b_n \in B$ and consider an arbitrary Σ_1 formula

$$\varphi(b_1, \ldots, b_n) : (\exists x_1, \ldots, \exists x_m)\theta(x_1, \ldots, x_m, b_1, \ldots, b_n),$$

where θ is quantifier-free. By distributing disjunctions in the usual way, we may assume without loss of generality that θ gives a full description of the subgroup generated by $x_1, \ldots, x_m, b_1, \ldots, b_n$. Suppose now that $\mathcal{A} \models \theta(a_1, \ldots, a_m, b_1, \ldots, b_n)$ and consider the subgroup C generated by $\{a_1, \ldots, a_m, b_1, \ldots, b_n\}$. Then, by assumption, there is a subgroup $\mathcal{D}$ of $\mathcal{B}$ with $B \cap C = D \cap C$ and an isomorphism $F : C \to D$ with $F(b) = b$ for all $b \in B$. It follows that $\mathcal{B} \models \theta(F(a_1), \ldots, F(a_m), b_1, \ldots, b_n)$ and therefore $\mathcal{B} \models \varphi(b_1, \ldots, b_n)$. □

Proposition 6. *Let $\mathcal{A}$ be an Abelian p-group such that $\mathcal{A} = \mathcal{B} \oplus \mathcal{E}$ for some subgroups $\mathcal{B}$ and $\mathcal{E}$, where $\mathcal{B}$ has unbounded character. Then $\mathcal{B}$ is a Σ_1 elementary subgroup of $\mathcal{A}$.*

Proof. We prove this assertion using Proposition 5. Let $\mathcal{C}$ be any finite subgroup of $\mathcal{A}$. Let $\mathcal{B}_0$ be the projection of $\mathcal{C}$ onto $\mathcal{B}$ and let $\mathcal{E}_0$ be the projection onto $\mathcal{E}$. Since $\mathcal{B}$ has unbounded character, there is a subgroup $\mathcal{B}_1$ of $\mathcal{B}$ independent of $\mathcal{B}_0$ and isomorphism ψ from $\mathcal{E}_0$ to $\mathcal{B}_1$. Now let $\mathcal{D} = \{x + y : x \in B_0, y \in B_1\}$ and define the isomorphism from $\mathcal{C}$ to $\mathcal{D}$ by $\phi(b + c) = b + \psi(c)$. Then ϕ is an isomorphism from $\mathcal{C}$ to $\mathcal{D}$ which preserves elements of $\mathcal{B}$. We note that $\mathcal{B} \cap \mathcal{C} = \mathcal{D} \cap \mathcal{C} = \mathcal{B}_0$. Thus, condition (*) is satisfied, and the result follows. □

Proposition 7. *Suppose that $\mathcal{A}$ is a countable Abelian p-group which is a product of cyclic subgroups and let K be a subcharacter of $\chi(\mathcal{A})$. That is, K is a subset of $\chi(\mathcal{A})$ such that, for any n and k, $(n+1, k) \in K$ implies $(n, k) \in K$. Then $\mathcal{A}$ has a pure subgroup $\mathcal{B}$ which is a factor of $\mathcal{A}$.*

Proof. We have $\mathcal{A} = \bigoplus_{i \in \omega} \langle a_i \rangle$, where each $\langle a_i \rangle$ is a pure cyclic subgroup of order p^{n_i}. We can select a subset I of ω so that $\mathcal{B} = \bigoplus_{i \in I} \langle a_i \rangle$ has character K and then $\mathcal{C} = \bigoplus_{i \notin I} \langle a_i \rangle$ is a factor of $\mathcal{A}$, that is, $\mathcal{A} = \mathcal{B} \oplus \mathcal{C}$. □

Proposition 8. *Let $\mathcal{A}$ be a countable Abelian p-group and let $\mathcal{B}$ be a Σ_1 elementary subgroup of $\mathcal{A}$. Then the following conditions hold:*

1. *$\mathcal{B}$ is a pure subgroup of $\mathcal{A}$.*
2. *$\chi(\mathcal{B}) \subseteq \chi(\mathcal{A})$.*
3. *$\mathcal{B} \models \theta_{n,k}$ for any $(n, k) \in \chi(\mathcal{A})$, that is, whenever $\mathcal{A}$ has a pure subgroup of the form $\bigoplus_{i<n} \mathbb{Z}(p^k)$, then $\mathcal{B}$ has a subgroup of the form $\bigoplus_{i<n} \mathbb{Z}(p^{k_i})$, with each $k_i \geq k$.*
4. *If $\mathcal{A}$ has a divisible component, then either $\mathcal{B}$ has a divisible component or $\chi(\mathcal{B})$ is unbounded.*

Proof. Suppose first that $\mathcal{B}$ is a Σ_1 elementary subgroup of the Abelian p-group $\mathcal{A}$.

(1) Let $b \in \mathcal{B}$ and suppose $ht^{\mathcal{A}}(b) \geq n$. Then $b = p^n a$ for some $a \in A$. Thus $\mathcal{A} \models (\exists x) p^n x = b$. Since $\mathcal{B}$ is a Σ_1 elementary subgroup of $\mathcal{B}$, $\mathcal{B} \models (\exists x) p^n x = b$, so that $ht^{\mathcal{B}}(b) \geq n$ as well. It follows that $\mathcal{B}$ is a pure subgroup of $\mathcal{A}$.

(2) Suppose that $(n, k) \in \chi(\mathcal{B})$. Then $\mathcal{B}$ has a pure subgroup $\mathcal{C}$ isomorphic to $\bigoplus_{i<n} \mathbb{Z}(p^k)$. Since $\mathcal{B}$ is pure in $\mathcal{A}$, it follows that $\mathcal{C}$ is a pure subgroup of $\mathcal{A}$. Thus $(n, k) \in \chi(\mathcal{A})$.

For part (3), suppose that $(n, k) \in \chi(\mathcal{A})$. Then $\mathcal{A} \models \theta_{n,k}$. Since $\mathcal{B}$ is a Σ_1 elementary submodel of $\mathcal{A}$ and $\theta_{n,k}$ is a Σ_1 sentence, it follows that $\mathcal{B} \models \theta_{n,k}$, and therefore $\mathcal{B}$ has a subgroup of the form $\bigoplus_{i<n} \mathbb{Z}(p^{k_i})$, with each $k_i \geq k$.

For part (4), suppose that $\mathcal{A}$ has a divisible component. Then $\mathcal{A} \models \theta_{1,k}$ for each k. It follows as above that $\mathcal{B} \models \theta_{1,k}$ for all k and therefore either $\mathcal{B}$ has a divisible component or $\chi(\mathcal{B})$ is unbounded. □

We conjecture that the converse of Proposition 8 also holds.

Proposition 9. *Let $\mathcal{A}$ be a countable Abelian p-group and let $\mathcal{B}$ be a Σ_2 elementary subgroup of $\mathcal{A}$. Then*

1. *$\mathcal{B}$ is a pure subgroup of $\mathcal{A}$.*
2. *$\chi(\mathcal{A}) = \chi(\mathcal{B})$*
3. *If $\mathcal{A}$ has a divisible component, then either $\mathcal{B}$ has a divisible component or $\chi(\mathcal{B})$ is unbounded.*

Proof. First suppose that $\mathcal{B}$ is a Σ_2 elementary subgroup of $\mathcal{A}$.

Parts (1) and (3) follow as in the proof of Proposition 8.

(2) Suppose that $(n, k) \in \mathcal{A}$. Then by Proposition 1, $\mathcal{A}$ has a pure subgroup $\mathcal{C}$ isomorphic to $\oplus_{i<n} \mathbb{Z}(p^k)$. Thus $\mathcal{A} \models \psi_{n,k}$. Since $\mathcal{B}$ is a Σ_2 elementary submodel of $\mathcal{A}$ and $\psi_{n,k}$ is a Σ_2 sentence, it follows that $\mathcal{B} \models \psi_{n,k}$, and therefore $(n, k) \in \chi(\mathcal{B})$. □

We conjecture that the converse of Proposition 9 also holds.

Theorem 7. *Let $\mathcal{A}$ be an Abelian p-group with no elements of infinite height in the reduced part. That is, $\mathcal{A}$ is a product of cyclic and quasi-cyclic components. Then $\mathcal{A}$ has a Σ_1-generically computably enumerable copy if and only if at least one of the following holds:*

(a) $\chi(\mathcal{A})$ is bounded;
(b) $\chi(\mathcal{A})$ has a Σ^0_2 subset K with a computable s_1-function.
(c) $\mathcal{A}$ has a divisible component.

Proof. First suppose that $\mathcal{A}$ has a Σ_1-generically computably enumerable copy. Then $\mathcal{A}$ has a Σ_1 elementary substructure $\mathcal{B}$ which is isomorphic to a computably enumerable structure $\mathcal{C}$. If $\mathcal{A}$ has no divisible component, then $\mathcal{C}$ has no divisible component. If $\chi(\mathcal{A})$ is unbounded, then $\chi(\mathcal{C})$ is unbounded, by Proposition 8. Thus $\mathcal{C}$ has a Σ^0_2 character K with a computable s_1-function, and it follows from Proposition 8 that $\chi(\mathcal{C}) \subseteq \chi(\mathcal{A})$.

The other direction is in three cases.

(a) If $\chi(\mathcal{A})$ is bounded, then $\mathcal{A}$ has a computable copy.

In cases (b) and (c), we will assume that $\chi(\mathcal{A})$ is unbounded and show that there is a structure $\mathcal{B} \subseteq \mathcal{A}$ which is isomorphic to a computable p-group $\mathcal{D}$. Then we will build a copy $\mathcal{C}$ of $\mathcal{A}$ with a dense computable subgroup $\mathcal{D}$ and fill out the rest of $\mathcal{C}$ to make it isomorphic to $\mathcal{A}$, as explained in (b).

(b) In this case, $\mathcal{A}$ has no divisible component, and is a product of cyclic subgroups. Thus by Proposition 7, $\mathcal{A}$ has a pure subgroup $\mathcal{B}$ with character K and $\mathcal{B}$ is a factor of $\mathcal{A}$. It follows from Proposition 6 that $\mathcal{B}$ is a Σ_1 elementary subgroup of $\mathcal{A}$.

By Lemma 1, there is a computable p-group $\mathcal{D}$ with character K isomorphic to $\mathcal{B}$. We may assume that the universe D of $\mathcal{D}$ is a computable asymptotically dense set. Let ϕ be an isomorphism from $\mathcal{D}$ to $\mathcal{B}$ and extend this to a bijection from ω to ω. Then we extend $\mathcal{D}$ to a group $\mathcal{C}$ with universe ω by letting $x +^{\mathcal{C}} y = \phi^{-1}(\phi(x) +^{\mathcal{A}} \phi(y))$. For $x, y \in \mathcal{D}$, we have

$$x +^{\mathcal{C}} y = \phi^{-1}(\phi(x) +^{\mathcal{A}} \phi(y)) = \phi^{-1}(\phi(x +^{\mathcal{D}} y)) = x +^{\mathcal{D}} y,$$

since ϕ is an isomorphism from $\mathcal{D}$ to $\mathcal{B} \subseteq \mathcal{A}$. For arbitrary $x, y \in \omega$,

$$\phi(x +^{\mathcal{C}} y) = \phi(\phi^{-1}(\phi(x) +^{\mathcal{A}} \phi(y))) = \phi(x) +^{\mathcal{A}} \phi(y),$$

so ϕ is an isomorphism from $\mathcal{C}$ to $\mathcal{A}$. Since $\mathcal{B}$ is a Σ_1 elementary subgroup of $\mathcal{A}$, and ϕ is an isomorphism mapping $\mathcal{B}$ to $\mathcal{D}$, it follows that $\mathcal{D}$ is a Σ_1 elementary subgroup of $\mathcal{C}$. Thus $\mathcal{C}$ is Σ_1-generically computably enumerable.

(c) In this case, the divisible component $\mathcal{B}$ will be a Σ_1 elementary substructure and we proceed as in (b) to define a computable group $\mathcal{D}$ with infinitely many divisible components, and extend this to a Σ_1-generically computably enumerable structure which is isomorphic to $\mathcal{A}$. □

We observe that the argument above also proves that $\mathcal{A}$ is Σ_1-generically computably enumerable if and only if it has a subgroup $\mathcal{B}$ which is isomorphic to a computable group.

Theorem 8. *The group $\mathcal{A}$ is Σ_2-generically computably enumerable if and only if it has a computable copy.*

Proof. Suppose that $\mathcal{A} = (\omega, +^{\mathcal{A}})$ is generically Σ_2 and let $\mathcal{D}$ be a dense computably enumerable set such that $\mathcal{D} = (D, +^{\mathcal{A}})$ is a computably enumerable group and also a Σ_2 elementary subgroup of $\mathcal{A}$. Then $\chi(\mathcal{D})$ is a Σ^0_2 set since D is computably enumerable and $\chi(\mathcal{D}) = \chi(\mathcal{A})$ since $\mathcal{D}$ is a Σ_2 elementary submodel of $\mathcal{A}$. If $\chi(\mathcal{A})$ is bounded, then $\mathcal{A}$ has a computable copy. So suppose that $\chi(\mathcal{A})$ is unbounded. If $\mathcal{D}$ has no divisible component, then $\chi(\mathcal{D})$ has a computable s_1 function, so that $\mathcal{A}$ has a computable copy. If $\mathcal{D}$ has a divisible component, then $\mathcal{A}$ also has a divisible component and therefore has a computable copy. □

4 Conclusion and Future Research

We have shown that any Abelian p-group has a generically computable copy and that such a group has a Σ_2-generically computably enumerable copy if and only it has a computable copy. We also gave a partial characterization of the Σ_1-generically computably enumerable Abelian p-groups, and a non-trivial characterization of the generically computable Abelian groups. It remains to consider more general Abelian p-groups with transfinite length.

We obtained necessary conditions for a subgroup of a countable Abelian p-group to be a Σ_1 or a Σ_2 elementary substructure. The conjecture is that these conditions are also necessary. We conjecture that a subgroup of an Abelian p-group is Σ_3 elementary if and only if it is (fully) elementary. This might even hold for Σ_2 elementary substructures.

It is interesting to consider whether any appropriate class of structures (perhaps with bounded Scott rank or some similar condition) would trivialize at some level, and we propose that a general result may be possible. Perhaps a general connection can be made in terms of the level at which Σ_n elementarity implies full elementarity. To our thinking, this recalls the feature of computable categoricity by which every structure with a $\Pi_{\alpha+1}$ Scott sentence is Δ^0_α-categorical [10]. So there might be results in the general hyperarithmetic hierarchy.

Previous papers also examined coarsely computable structures, so future work should examine Σ_n-coarsely computably enumerable Abelian groups.

Generically computable and coarsely computable isomorphisms were also studied in [2]. Future plans involve the study of densely computable isomorphisms for Abelian groups. We have the following preliminary result.

Theorem 9. *Let $\mathcal{A}$ and $\mathcal{B}$ be computable Abelian p-groups each isomorphic to $\bigoplus_{i<\omega} \mathbb{Z}(p) \oplus \bigoplus_{i<\omega} \mathbb{Z}(p^2)$ such that the elements of order p^2 are asymptotically dense.*

Then $\mathcal{A}$ and $\mathcal{B}$ are generically computably isomorphic.

References

1. Calvert, W., Cenzer, D., Harizanov, V.: Densely computable structures. J. Logic Comput. **32**, 581–607 (2022)
2. Calvert, W., Cenzer, D., Harizanov, V.: Generically and coarsely computable isomorphisms. Computability (2023). https://doi.org/10.3233/COM-210382
3. Calvert, W., Cenzer, D., Harizanov, V., Morozov, A.: Effective categoricity of Abelian p-groups. Annals of Pure Applied Logic **159**, 187–197 (2009)
4. Fuchs, L.: Infinite Abelian groups, volume I, Academic Press (1970)
5. Jockusch, C.G., Schupp, P.E.: Generic computability, turing degrees, and asymptotic density. J. London Math. Soc. **85**, 472–490 (2012)
6. Jockusch, C.G. Schupp, P.E.: Asymptotic density and the theory of computability: a partial survey. In: Computability and Complexity, Lecture Notes in Computer Science, vol. 10010, pp. 501–520 (2017)
7. Kaplansky, I.: Infinite Abelian groups, University of Michigan Press (1954)

8. Kapovich, I., Myasnikov, A., Schupp, P., Shpilrain, V.: Generic-case complexity, decision problems in group theory, and random walks. J. Algebra **264**, 665–694 (2003)
9. Khisamiev, N.G.: Constructive Abelian groups, Handbook of Recursive Mathematics, Vol. 2, Stud. Logic Found. Math., vol. 139, Elsevier, pp. 1177–1231 (1998)
10. Montalbán, A.: A robuster Scott rank. Proc. Am. Math. Soc. **143**, 5427–5436 (2015)
11. Rotman, J.J.: The theory of groups, Allyn and Bacon (1965)
12. Soare, R.I.: Recursively enumerable sets and degrees. Springer-Verlag (1987)

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