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Hereditary uniform property Γ

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Abstract We study the uniform property Γ for separable simple C^*-algebras which have quasitraces and may not be exact. We show that a stably finite separable simple C^*-algebra A with strict comparison and uniform property Γ has tracial approximate oscillation zero and stable rank one. Moreover in this case, its hereditary C^*-subalgebras also have a version of uniform property Γ. If a separable non-elementary simple amenable C^*-algebra A with strict comparison has this hereditary uniform property Γ, then A is $\mathcal{Z}$-stable.

Keywords simple C^*-algebras, uniform property Γ

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1 Introduction

Uniform property Γ was recently introduced in [3] in the study of regularity properties for simple nuclear C^*-algebras, specifically, properties of finite nuclear dimension and $\mathcal{Z}$-stability. More recently, it is shown in [6] that, for a unital separable nuclear simple C^*-algebra A, A has strict comparison and uniform property Γ if and only if A is $\mathcal{Z}$-stable, and if and only if A has finite nuclear dimension, which is a significant recent advance towards the the resolution of Toms-Winter conjecture.

Uniform property Γ is originally only defined for unital C^*-algebras, or those C^*-algebras whose tracial state space is compact. In [5], a stabilized uniform property Γ was introduced and it is shown that, if A is a (non-unital) separable simple nuclear C^*-algebra with strict comparison which has stable rank one and stabilized uniform property Γ, then A is $\mathcal{Z}$-stable.

In this note, we study the uniform property Γ for separable simple C^*-algebras using quasitraces instead of traces. Simple C^*-algebras with strict comparison and uniform property Γ have a very nice matricial structure (see Theorem 3.3). We also find that, if A has strict comparison and uniform property Γ, then A has tracial approximate oscillation zero, and the canonical map $\Gamma : \mathrm{Cu}(A) \to \mathrm{LAff}_+(\widetilde{QT}(A))$ is surjective and has stable rank one, without assuming that A is amenable. In particular, $\mathrm{Cu}(A) \cong \mathrm{Cu}(A \otimes \mathcal{Z})$.

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Moreover, in this case, a version of uniform property Γ holds for hereditary C^*-subalgebras. This property is called hereditary uniform property Γ (see Definition 4.1) which is defined for C^*-algebras whose sets of normalized 2-quasitraces may not be compact, or even empty (but for C^*-algebras having densely defined non-zero traces). Therefore uniform property Γ is a strong condition even in the absence of amenability. However, there are separable simple C^*-algebras which have strict comparison and hereditary uniform property Γ but not $\mathcal{Z}$-stable (see Remark 4.7).

Regarding Toms-Winter conjecture, we also obtain a similar conclusion as in [6] (for non-unital simple C^*-algebras). To be more specific, let A be a (non-unital) stably finite separable non-elementary simple nuclear C^*-algebra with strict comparison. Following [6], we show that A has hereditary uniform property Γ if and only if A is $\mathcal{Z}$-stable. This result is similar to the statement in [5] for non-unital case but we do not assume, as a priori, that A has stable rank one, or $\mathrm{Cu}(A) \cong \mathrm{Cu}(A \otimes \mathcal{K})$ (see Remark 4.5). This is possible because we show that if A has strict comparison and hereditary uniform property Γ, then A has tracial approximate oscillation zero. We also observe that if A is tracially approximately divisible, then A has hereditary uniform property Γ. If A is a separable simple non-elementary amenable C^*-algebra with strict comparison, the converse also holds as, under the assumption that A is amenable, tracial approximate divisibility is equivalent to $\mathcal{Z}$-stability (which is essentially a restatement of Matui-Sato, see also [7]).

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2 Preliminary

Definition 2.1. Let A be a C^*-algebra and $F \subset A$ a subset of A. Denote by $\mathrm{Her}(F)$ the hereditary C^*-subalgebra of A generated by F. Denote by $A^{\mathbf{1}}$ the unit ball of A, and by A_+ the set of all positive elements in A. Put $A_+^{\mathbf{1}} := A_+ \cap A^{\mathbf{1}}$. Denote by $\widetilde{A}$ the minimal unitization of A. Let $\mathrm{Ped}(A)$ denote the Pedersen ideal of A, $\mathrm{Ped}(A)_+ := \mathrm{Ped}(A) \cap A_+$ and $\mathrm{Ped}(A)_+^{\mathbf{1}} := \mathrm{Ped}(A) \cap A_+^{\mathbf{1}}$. Denote by $T(A)$ the tracial state space of A.

Definition 2.2. Let A and B be C^*-algebras and $\varphi : A \to B$ be a linear map. The map φ is said to be positive if $\varphi(A_+) \subset B_+$. The map φ is said to be completely positive contractive, abbreviated to c.p.c., if $\|\varphi\| \leqslant 1$ and $\varphi \otimes \mathrm{id} : A \otimes M_n \to B \otimes M_n$ is positive for all $n \in \mathbb{N}$. A c.p.c. map $\varphi : A \to B$ is called order zero, if for any $x, y \in A_+$, $xy = 0$ implies $\varphi(x)\varphi(y) = 0$ (see Definition 2.3 of [36]). If $ab = ba = 0$, we also write $a \perp b$.

In what follows, $\{e_{i,j}\}_{i,j=1}^n$ (or just $\{e_{i,j}\}$, if there is no confusion) stands for a system of matrix units for M_n and $\iota \in C_0((0,1])$ denotes the identity function on $(0,1]$, i.e., $\iota(t) = t$ for all $t \in (0,1]$.

Notation 2.3. Let $\epsilon > 0$. Define a continuous function $f_\epsilon : [0, +\infty) \to [0,1]$ by

$$f_\epsilon(t) = \begin{cases} 0 & t \in [0, \epsilon/2], \\ 1 & t \in [\epsilon, \infty), \\ \text{linear} & t \in [\epsilon/2, \epsilon]. \end{cases}$$

Definition 2.4. Let A be a C^*-algebra and $a, b \in (A \otimes \mathcal{K})_+$. We write $a \lesssim b$ if there is $x_n \in A \otimes \mathcal{K}$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \|a - x_n^* b x_n\| = 0$. We write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$ both hold. The Cuntz relation $\sim$ is an equivalence relation. Set $\mathrm{Cu}(A) = (A \otimes \mathcal{K})_+/\sim$. Let $\langle a \rangle$ denote the equivalence class of a. We write $[a] \leqslant [b]$ if $a \lesssim b$.

Definition 2.5. Let A be a σ-unital C^*-algebra. A densely defined 2-quasitrace is a 2-quasitrace defined on $\mathrm{Ped}(A)$ (see Definition II.1.1 of [1]). Denote by $\widetilde{QT}(A)$ the set of densely defined quasitraces on $A\otimes\mathcal{K}$. In what follows we will identify A with $A\otimes e_{1,1}$, whenever it is convenient. Let $\tau\in\widetilde{QT}(A)$. Then $\tau(a)\neq\infty$ for any $a\in\mathrm{Ped}(A)_+\setminus\{0\}$.

We endow $\widetilde{QT}(A)$ with the topology in which a net $\{\tau_i\}$ converges to τ if $\{\tau_i(a)\}$ converges to $\tau(a)$ for all $a\in\mathrm{Ped}(A)$ (see also (4.1) on page 985 of [11]).

Denote by $QT(A)$ the set of those $\tau\in\widetilde{QT}(A)$ such that $\|\tau\|=1$.

Note that, for each $a\in(A\otimes\mathcal{K})_+$ and $\varepsilon>0$, $f_\varepsilon(a)\in\mathrm{Ped}(A\otimes\mathcal{K})_+$. Define

$$\widehat{[a]}(\tau):=d_\tau(a)=\lim_{\varepsilon\to 0}\tau(f_\varepsilon(a))\ \text{ for all }\ \tau\in\widetilde{QT}(A).\tag{2.1}$$

Definition 2.6. Let A be a simple C^*-algebra Then A is said to have (Blackadar's) strict comparison, if, given any $a,b\in(A\otimes\mathcal{K})_+$, one has that $a\lesssim b$, whenever

$$d_\tau(a)<d_\tau(b)\ \text{ for all }\ \tau\in\widetilde{QT}(A)\setminus\{0\}.\tag{2.2}$$

Definition 2.7. Let A be a C^*-algebra with $\widetilde{QT}(A)\setminus\{0\}\neq\emptyset$. Let $S\subset\widetilde{QT}(A)$ be a convex subset. Set (if $0\notin S$, we ignore the condition $f(0)=0$)

$$\mathrm{Aff}_+(S)=\{f:C(S,\mathbb{R})_+ : f\text{ affine}, f(s)>0\text{ for }s\neq 0,\ f(0)=0\}\cup\{0\},\tag{2.3}$$
$$\mathrm{LAff}_+(S)=\{f:S\to[0,\infty]:\exists\{f_n\},f_n\nearrow f,\ f_n\in\mathrm{Aff}_+(S)\}.\tag{2.4}$$

For a simple C^*-algebra A and each $a\in(A\otimes\mathcal{K})_+$, the function $\hat{a}(\tau)=\tau(a)$ $(\tau\in S)$ is in general in $\mathrm{LAff}_+(S)$. If $a\in\mathrm{Ped}(A\otimes\mathcal{K})_+$, then $\hat{a}\in\mathrm{Aff}_+(S)$. For $\widehat{[a]}(\tau)=d_\tau(a)$ defined above, we have $\widehat{[a]}\in\mathrm{LAff}_+(\widetilde{QT}(A))$.

We write $\Gamma:\mathrm{Cu}(A)\to\mathrm{LAff}_+(\widetilde{QT}(A))$ for the canonical map defined by $\Gamma([a])(\tau)=\widehat{[a]}=d_\tau(a)$ for all $\tau\in\widetilde{QT}(A)$.

In the case that A is algebraically simple (i.e., A is a simple C^*-algebra and $A=\mathrm{Ped}(A)$), Γ also induces a canonical map $\Gamma_1:\mathrm{Cu}(A)\to\mathrm{LAff}_+(\overline{QT(A)}^w)$, where $\overline{QT(A)}^w$ is the weak*-closure of $QT(A)$. Since, in this case, $\mathbb{R}_+\cdot\overline{QT(A)}^w=\widetilde{QT}(A)$, the map Γ is surjective if and only if Γ_1 is surjective. We would like to point out that, in this case, $0\notin\overline{QT(A)}^w$ and $\overline{QT(A)}^w$ is compact (see Proposition 2.9 of [14]).

The following is known to experts:

Proposition 2.8 (II.4.4 of [1])**.** Let A be a separable C^*-algebra. If $QT(A)\neq\emptyset$ and is compact, then $QT(A)$ is a Choquet simplex.

Proof. If A is unital, by II. 4.4 of [1], $QT(A)$ is a Choquet simplex. If A is not unital, by II. 2.5 of [1], every 2-quasitrace extends to a 2-quasitrace on A with $\tau(1_{\widetilde{A}})=\|\tau\|$. We then view $QT(A)$ as a closed convex subset of Choquet simplex $QT(\widetilde{A})$. On the other hand, any $\tau\in QT(\widetilde{A})$ has the form $\tau=\alpha\tau_0+(1-\alpha)\tau_A$, where $0\leqslant\alpha\leqslant 1$, $\tau_A\in QT(A)$ and τ_0 is the unique tracial state which vanishes on A.

By the Choquet theorem, α and τ_A are uniquely determined by τ. In particular, $QT(A)$ is a face of $QT(\widetilde{A})$. Now suppose that $\tau\in QT(\widetilde{A})$. Then there exists a unique (probability) boundary measure μ on $\partial_e(QT(\widetilde{A}))$ such that

$$f(\tau)=\int_{\partial_e(QT(\widetilde{A}))}f(s)d\mu\ \text{ for all }\ f\in\mathrm{Aff}(QT(\widetilde{A})).\tag{2.5}$$

If $\mu(\{\tau_0\})=\alpha>0$, then $\tau=\alpha\tau_0+(1-\alpha)\tau_A$ for some $\tau_A\in QT(A)$. If $\tau\in QT(A)$, then $\alpha=0$. In other words, μ is concentrated on $\partial_e(QT(A))$. We have just shown that every $\tau\in QT(A)$ is the barycenter of a unique normalized extremal boundary measure. So $QT(A)$ is a Choquet simplex. $\square$

Definition 2.9. Let $l^\infty(A)$ be the C^*-algebra of bounded sequences of A. Recall that $c_0(A):=\{\{a_n\}\in l^\infty(A):\lim_{n\to\infty}\|a_n\|=0\}$ is a (closed two-sided) ideal of $l^\infty(A)$. Let $A_\infty:=l^\infty(A)/c_0(A)$ and $\pi^\infty: l^\infty(A)\to A_\infty$ be the quotient map. We view A as a subalgebra of $l^\infty(A)$ via the canonical map $\iota:a\mapsto\{a,a,,...\}$ for all $a\in A$. In what follows, we may identify a with the constant sequence $\{a,a,...,\}$ in $l^\infty(A)$ whenever it is convenient without further warning.

Put $A'=\{x=\{x_n\}\in l^\infty(A):\lim_{n\to\infty}\|x_na-ax_n\|=0\}$.

Definition 2.10. Let A be a C^*-algebra $QT(A) \neq \{0\}$. Let $\tau \in \widetilde{QT}(A) \setminus \{0\}$. Define, for each $x \in A$,

$$\|x\|_{2,\tau} = \tau(x^*x)^{1/2}. \tag{2.6}$$

Let $S \subset \widetilde{QT}(A) \setminus \{0\}$ be a compact subset. Define

$$\|x\|_{2,S} = \sup\{\tau(x^*x)^{1/2} : \tau \in S\}. \tag{2.7}$$

Put $I_{S,\mathbb{N}} = \{\{x_n\} \in l^\infty(A) : \lim_{n\to\infty} \|x\|_{2,S} = 0\}$.

We would quote the following proposition which follows from II. 2.2 and Theorem I.17 of [1].

Proposition 2.11 (Proposition 3.2 of [18])**.** Let A be a C^*-algebra and $\tau \in QT(A)$, $I = \{x \in A : \tau(x^*x) = 0\}$. Then I is a (closed two-sided) ideal and there is a unique 2-quasitrace $\bar\tau$ on A/I such that

$$\tau(x) = \bar\tau(\rho(x)) \ \text{ for all } \ x \in A, \tag{2.8}$$

where $\rho : A \to A/I$ is the quotient map.

Definition 2.12. Let $\varpi \in \beta(\mathbb{N}) \setminus \mathbb{N}$ be a free ultrafilter. Set

$$c_{0,\varpi} = \{\{x_n\} \in l^\infty(A) : \lim_{n\to\omega} \|x_n\| = 0\}. \tag{2.9}$$

Denote by $\pi_\infty : l^\infty(A) \to l^\infty(A)/c_{0,\varpi}$ the quotient map. Let $S \subset \widetilde{QT}(A)$ be a compact subset. Define

$$I_{S,\varpi} = \{\{x_n\} \in l^\infty(A) : \lim_{n\to\varpi} \|x_n\|_{2,S} = 0\}. \tag{2.10}$$

It is a (closed two-sided) ideal. In the case that $A = \mathrm{Ped}(A)$, we usually consider $I_{\overline{QT(A)}^w,\varpi}$. If A has continuous scale, we consider $I_{QT(A),\varpi}$.

Denote by $\Pi_\varpi : l^\infty(A) \to l^\infty(A)/I_{\overline{QT(A)}^w,\varpi}$ the quotient map. We also write $\Pi : l^\infty(A) \to l^\infty(A)/I_{QT(A),\mathbb{N}}$ for the quotient map.

For convenience, abusing the notation, we may also write A' for $\Pi(A')$ as well as $\Pi_\varpi(A')$.

If $\tau_n \in QT(A)$, for $x = \{x_n\} \in l^\infty(A)$, define

$$\tau_\varpi(x) = \lim_{n\to\varpi} \tau_n(x_n). \tag{2.11}$$

It is a 2-quasitrace on $l^\infty(A)$.

Fix $\{\tau_n\} \subset QT(A)$. Let $J = \{\{x_n\} \in l^\infty(A) : \tau_\varpi(\{x_n^*x_n\}) = 0\}$. Then J is a (closed two sided) ideal of $l^\infty(A)$ and $\tau_\varpi|_J = 0$. If $x = \{x_n\} \in (I_{\overline{QT(A)}^w,\varpi})_{s.a.}$, then,

$$\lim_{n\to\varpi} |\tau_n(x_n)|^2 \leqslant \lim_{n\to\varpi} \tau_n(x_n^*x_n) \leqslant \lim_{n\to\varpi} \|x_n^*x_n\|^2_{2,\overline{QT(A)}^w} = 0. \tag{2.12}$$

In other words, $\tau_\varpi(x) = 0$ and $x \in I_{\overline{QT(A)}^w,\varpi}$.

Since τ_ϖ is a 2-quasitrace on $l^\infty(A)$, by Proposition 4.2 of [18] (see also Proposition 2.11), $\tau_\varpi = \tau_\varpi \circ \pi_J$, where $\pi_J : l^\infty(A) \to l^\infty(A)/J$ is the quotient map. In particular, $\tau_\varpi(x + j) = \tau_\varpi(x)$ for all $x \in l^\infty(A)$ and $j \in J$. Since we have shown $I_{\overline{QT(A)}^w,\varpi} \subset J$, we may also view τ_ϖ as a normalized 2-quasitrace on $l^\infty(A)/I_{\overline{T(A)}^w,\varpi}$. Similarly, we may view τ_ϖ as a normalized 2-quasitrace of $l^\infty(A)/c_{0,\varpi}$.

If $\tau_n = \tau$ for all $n \in \mathbb{N}$, we may write τ instead of τ_ϖ.

Denote by $QT_\varpi(A)$ the set $\{\tau_\varpi : \{\tau_n\} \subset QT(A)\}$.

The following is a variation of II. 2.5 of [1]. Note that δ below depends on ε but not τ.

Lemma 2.13 (cf. II. 2.5 of [1])**.** Let A be a separable C^*-algebra with $QT(A) \neq \emptyset$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ satisfying the following: For any normal elements $a, b \in A^{\mathbf{1}}$ such that $\|ab - ba\|_{2,\overline{QT(A)}^w} < \delta$, then, for any $\tau \in QT(A)$,

$$|\tau(a + b) - \tau(a) + \tau(b)| < \varepsilon. \tag{2.13}$$

Proof. Suppose not, then for some $\varepsilon_0 > 0$, there exists a sequence of pairs of normal elements $a_n, b_n \in A^{\mathbf{1}}$ and a sequence $\{\tau_n\} \subset QT(A)$ such that $\|a_nb_n - b_na_n\|_{2,\overline{QT(A)}^w} < 1/n$ but

$$|\tau_n(a_nb_n) - \tau_n(a_n) + \tau_n(b_n)| \geqslant \varepsilon_0, \ \ n \in \mathbb{N}. \tag{2.14}$$

Put $a = \Pi_\varpi(\{a_n\})$ and $b = \Pi_\varpi(\{b_n\})$. Then a and b are normal and $ab = ba$. Define $\tau_\varpi(\{x_n\}) = \lim_{n\to\varpi} \tau_n(x_n)$ for $\{x_n\} \in l^\infty(A)$. Viewing $\tau_\varpi \in QT(l^\infty(A)/I_{\overline{T(A)}^w,\varpi})$. Then $\tau_\varpi(a+b) = \tau_\varpi(a) + \tau_\varpi(b)$. This contradicts (2.14). □

Proposition 2.14 (cf. Proposition 3.1 of [6], Lemma 4.2 (ii) of [28] and Proposition 4.3.6 of [12])**.** Let A be a separable C^*-algebra with $QT(A) \neq \emptyset$ and $K \subset \partial_e(QT(A))$ a compact subset. Then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$ and finite subset $\mathcal{G} \subset A$ satisfying the following: Suppose that $b \in A^{\mathbf{1}}$ such that

$$\|cb - bc\|_{2,K} < \delta \ \text{ for all } \ \tau \in K \ \text{ and } \ c \in \mathcal{G}. \tag{2.15}$$

Then, for all $a \in \mathcal{F}$,

$$\sup\{|\tau(ab) - \tau(a)\tau(b)| : \tau \in K\} < \varepsilon. \tag{2.16}$$

Proof. One notes that the proof of Proposition 3.1 of [6] works for $QT(A)$. Then the proposition follows from that. □

Definition 2.15 (Definition 4.1, 4.7 and 5.1 of [14])**.** Let A be a C^*-algebra with $\widetilde{QT}(A) \setminus \{0\} \neq \emptyset$. Let $S \subset \widetilde{QT}(A) \setminus \{0\}$ be a compact subset. Define, for each $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+$,

$$\omega(a)|_S = \inf\{\sup\{d_\tau(a) - \tau(c) : \tau \in S\} : c \in \overline{a(A \otimes \mathcal{K})a}, \, 0 \leqslant c \leqslant 1\} \tag{2.17}$$

(see A1 of [10]). The number $\omega(a)|_S$ is called the (tracial) oscillation of a on S.

We are only interested in the case that $\mathbb{R}_+ \cdot S = \widetilde{QT}(A)$. Let $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+$. We write $\Omega^T(a) = 0$ if there exists a sequence $c_n \in \mathrm{Her}(a)_+^{\mathbf{1}}$ with $\lim_{n\to\infty} \omega(c_n)|_S = 0$ such that $\lim_{n\to\infty} \|a - c_n\|_{2,S} = 0$. Note that $\Omega^T(a) = 0$ does not depend on the choice of S (as long as $\mathbb{R}_+ \cdot S = \widetilde{QT}(A)$, see Definition 4.7 of [14]).

A separable simple C^*-algebra A is said to have T-tracial approximate oscillation zero, if for any $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+$, $\Omega^T(a) = 0$. We say that A has tracial approximate oscillation zero if A has T-tracial approximate oscillation zero and has strict comparison.

If A is a separable simple C^*-algebra and $b \in \mathrm{Ped}(A)_+$, then by Brown's stable isomorphism theorem, $\mathrm{Her}(b) \otimes \mathcal{K} \cong A \otimes \mathcal{K}$. So we may view $a \in \mathrm{Ped}(\mathrm{Her}(b) \otimes \mathcal{K})_+$. Note that $\mathrm{Her}(b)$ is algebraically simple. We often assume that A is algebraically simple and choose S to be $\overline{QT(A)}^w$. In that case we will omit S.

3 Uniform property Γ

Let us recall the definition of uniform property Γ. We fix a free ultrafiler $\varpi \in \beta(\mathbb{N}) \setminus \mathbb{N}$.

Definition 3.1 (Definition 2.1 of [3], Definition 2.1 of [16])**.** Let A be a separable C^*-algebra with nonempty and compact $QT(A)$. We say that A has uniform property Γ if, for any $n \in \mathbb{N}$, there exist pairwise orthogonal projections $p_1, p_2, ..., p_n \in (l^\infty(A) \cap A')/I_{QT(A),\varpi}$ (see 2.9) such that, for $1 \leqslant i \leqslant n$,

$$\tau(p_ia) = \frac{1}{n}\tau(a) \ \text{ for all } \ a \in A \ \text{ and } \ \tau \in QT_\varpi(A). \tag{3.1}$$

It should be noted that we do not assume all 2-quasitraces are traces. Let $p = \sum_{i=1}^n p_i$. Then p is a projection and $\tau(pa) = \tau(a)$ for all $\tau \in QT_\varpi(A)$ and $a \in A$. Suppose that $c_k \in (l^\infty(A) \cap A')_+^{\mathbf{1}}$ such that $\Pi_\varpi(\{c_k\}) = p$. Then, for all $a \in A_+$,

$$\|ac_k - a\|_{2,QT(A)}^2 \leqslant \sup\{\tau(a - a^{1/2}c_ka^{1/2}) : \tau \in QT(A)\} \to 0 \ \text{ as } k \to \varpi. \tag{3.2}$$

It follows that $\Pi_\varpi(\iota(a))p = \Pi_\varpi(\iota(a))$ for all $a \in A$. Let $e \in A_+^{\mathbf{1}}$ be a strictly positive element of A. Then $d_\tau(e_A) = 1$ for all $\tau \in QT(A)$. By the Dini theorem, $\tau(e^{1/k}) \nearrow d_\tau(e)$ uniformly on $QT(A)$. By II.2.5 of [1], we extend each $\tau \in QT(A)$ to a 2-quasitrace in $QT(\widetilde{A})$ which we still write τ (so $\tau(1_{\widetilde{A}}) = 1$, see II.2.5 of [1]), if A is not unital. Therefore, for any $\{a_k\} \in l^\infty(A)^{\mathbf{1}}$,

$$\lim_{k\to\infty} \|a_k(1_{\widetilde{A}} - e^{1/k})\|_{2,QT(A)} \leqslant \lim_{k\to\infty} \|1_{\widetilde{A}} - e^{1/k}\|_{2,QT(A)} = 0 \tag{3.3}$$

(see Lemma 3.5 of [18] and also Definition 2.16 of [14]). It follows that $l^\infty(A)/I_{QT(A),\varpi}$ has a unit $E := \Pi_\varpi(\{e^{1/k}\})$. Suppose that $E - p \neq 0$. Then there would be a nonzero element $b = \{b_n\} \in l^\infty(A)_+^{\mathbf{1}}$ such that $p\Pi_\varpi(b) = 0$. Then, for all $k \in \mathbb{N}$,

$$\Pi_\varpi(\iota(e^{1/k}))\Pi_\varpi(b) = \Pi_\varpi(\iota(e^{1/k}))p\Pi_\varpi(b) = 0. \tag{3.4}$$

Or,

$$\Pi_\varpi(E - \iota(e^{1/k}))\Pi_\varpi(b) = \Pi_\varpi(b). \tag{3.5}$$

However, since $\tau(e^{1/k}) \nearrow 1$ uniformly on $QT(A)$, for any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\|E - \iota(e^{1/k})\|_{QT(A),\varpi} < \varepsilon$, whence

$$\|\Pi_\varpi(b)\| < \varepsilon. \tag{3.6}$$

It follows that $p = E = 1_{l^\infty(A)/I_{QT(A),\varpi}}$.

Note that we follow the same sprit in [3], so uniform property Γ, as in Definition 2.1 of [3] (see also [6]), is only defined for separable C^*-algebras with compact $QT(A)$. It is worth mentioning that if A is a σ-unital simple C^*-algebra with nonempty compact $QT(A)$ and with strict comparison, then (by the Dini theorem), A has continuous scale. It follows that A is algebraically simple (see Theorem 3.3 of [20]).

Proposition 3.2 (cf. Corollary 3.2 of [6])**.** Let A be a separable simple C^*-algebra with nonempty compact $QT(A)$. If A has uniform property Γ, then, for any $n \in \mathbb{N}$, there are mutually orthogonal projections $p_1, p_2, ..., p_n \in (l^\infty(A) \cap A')/I_{QT(A),\varpi}$ such that, for $1 \leqslant i \leqslant n$,

$$\tau(p_i) = \frac{1}{n} \text{ for all } \tau \in QT_\varpi(A). \tag{3.7}$$

Conversely, suppose that $\partial_e(T(A))$ is σ-compact and that there are mutually orthogonal projections

$$p_1, p_2, ..., p_n \in (l^\infty(A) \cap A')/I_{QT(A),\varpi}$$

such that, for $1 \leqslant i \leqslant n$, equation (3.7) holds. Then, for any $a \in A$, and $1 \leqslant i \leqslant n$,

$$\tau(p_i a) = \frac{1}{n}\tau(a) \text{ for all } a \in A \text{ and } \tau \in QT(A). \tag{3.8}$$

(Note that, in (3.8), $\tau \in QT(A)$ not in $QT_\varpi(A)$.)

Proof. Suppose that A has uniform property Γ. Then, for any $n \in \mathbb{N}$, there exist mutually orthogonal projections $p_1, p_2, ..., p_n \in (l^\infty(A) \cap A')/I_{QT(A),\varpi}$ such that, for $1 \leqslant i \leqslant n$,

$$\tau(p_i a) = \frac{1}{n}\tau(a) \text{ for all } \tau \in QT_\varpi(A). \tag{3.9}$$

Let $\{p_i^{(m)}\} \in (l^\infty(A) \cap A')_+^{\mathbf{1}}$ be such that $\Pi_\varpi(\{p_i^{(m)}\}) = p_i$, $1 \leqslant i \leqslant n$. Choose a strictly positive element $e \in A_+^{\mathbf{1}}$. Let $\varepsilon \in (0, 1/2)$. Since $QT(A)$ is compact, by the Dini Theorem, there exists $k \in \mathbb{N}$ such that

$$\sup\{1 - \tau(e^{1/k}) : \tau \in QT(A)\} < \varepsilon. \tag{3.10}$$

It follows that, for all $1 \leqslant i \leqslant n$,

$$\tau(p_i) \geqslant \tau(e^{1/k}p_i) = \frac{1}{n}\tau(e^{1/k}) > \frac{1}{n} - \frac{\varepsilon}{n} \text{ for all } \tau \in QT_\varpi(A). \tag{3.11}$$

Let $\varepsilon \to 0$, we obtain that, for all $1 \leqslant i \leqslant n$,

$$\tau(p_i) \geqslant \frac{1}{n} \text{ for all } \tau \in QT_{\varpi}(A). \tag{3.12}$$

Since $\sum_{i=1}^n p_i = 1$, it follows that $\tau(p_i) = \frac{1}{n}$ for all $\tau \in QT_{\varpi}(A)$.

For the second part of the proposition, suppose that there are mutually orthogonal projections $p_1, p_2, ..., p_n \in (l^\infty(A) \cap A')/I_{QT(A),\varpi}$ such that, for $1 \leqslant i \leqslant n$, (3.7) holds. Let $a \in A$. We will show that, for any $\tau \in QT(A)$, (3.8) holds. It suffices to show this for the case that $a \in A_+^{\mathbf{1}}$.

Suppose otherwise. Then there is $a \in A_+^{\mathbf{1}}$ and $\tau \in QT(A)$ such that

$$|\frac{1}{n}\tau(a) - \tau(p_i a)| > \sigma \tag{3.13}$$

for some $1 > \sigma > 0$.

Choose $\varepsilon \in (0, \sigma/16)$. By the Choquet theorem, there exists a probability Borel measure μ_τ on $QT(A)$ concentrated on $\partial_e(QT(A))$ such that, for any $f \in \mathrm{Aff}(QT(A))$,

$$f(\tau) = \int_{\partial_e(QT(A))} f d\mu_\tau. \tag{3.14}$$

Since $\partial_e(QT(A))$ is σ-compact, there exists a compact subset $K \subset \partial_e(QT(A))$ such that

$$\mu(\partial_e(QT(A)) \setminus K) < \varepsilon. \tag{3.15}$$

It follows from Proposition 2.14 (see also Proposition 3.1 of [6]) that there is $\delta > 0$ and finite subset $\mathcal{G} \subset A$ such that if $b \in A_+^{\mathbf{1}}$ such that $\|[x, b]\| < \delta$ for all $x \in \mathcal{G}$, then

$$\sup\{|t(ab) - t(a)t(b)| : \tau \in K\} < \varepsilon. \tag{3.16}$$

Let $\{p_i^{(m)}\} \in (l^\infty(A) \cap A')_+^{\mathbf{1}}$ be such that $\Pi_\varpi(\{p_i^{(m)}\}) = p_i$ $(1 \leqslant i \leqslant n)$. For any $\mathcal{P} \in \varpi$, there is $m \in \mathcal{P}$ such that

$$|\frac{1}{n}\tau(a) - \tau(p_i^{(m)} a)| > \sigma/2, \ \|[x, p_i^{(m)}]\| < \delta \text{ for all } x \in \mathcal{G} \text{ and} \tag{3.17}$$

$$\sup\{|t(p_i^{(m)}) - 1/n| : t \in T(A)\} < \varepsilon. \tag{3.18}$$

Then, by the choice of δ, we estimate that

$$\begin{aligned} |\frac{1}{n}\tau(a) - \tau(ap_i^{(m)})| &= |\int_{\partial_e(QT(A))} \frac{1}{n}\widehat{a} - \widehat{ap_i^{(m)}} d\mu_\tau| \\ &\leqslant \int_{\partial_e(QT(A))} |\frac{1}{n}\widehat{a} - \widehat{ap_i^{(m)}}| d\mu_\tau \\ &< \int_K |\frac{1}{n}\widehat{a} - \widehat{ap_i^{(m)}}| d\mu_\tau + 2\varepsilon \qquad \text{(by (3.15))} \\ &< \int_K |\frac{1}{n}\widehat{a} - \frac{1}{n}\widehat{a}| d\mu_\tau + 4\varepsilon = 4\varepsilon < \sigma/2. \quad \text{(by (3.16) and (3.18))}. \end{aligned}$$

This contradicts (3.17) and the proof is complete. □

If A has strict comparison, then uniform property Γ provides a unital homomorphism $\varphi : M_n \to l^\infty(A)/I_{QT(A),\varpi}$ as follows.

Theorem 3.3. Let A be a non-elementary separable simple C^*-algebra with strict comparison and nonempty compact $QT(A)$. If A has uniform property Γ, then, for any $n \in \mathbb{N}$, there is a unital homomorphism $\varphi : M_n \to l^\infty(A)/I_{QT(A),\varpi}$ such that $\varphi(e_{i,i}) \in (l^\infty(A) \cap A')/I_{QT(A),\varpi}$ and, for all $1 \leqslant i \leqslant n$,

$$\tau(a\varphi(e_{i,i})) = \frac{1}{n}\tau(a) \text{ for all } a \in A \text{ and } \tau \in QT_\varpi(A). \tag{3.19}$$

Proof. By II.2.5 of [1], we extend each $\tau \in QT(A)$ to a 2-quasitrace in $QT(\widetilde{A})$ with $\tau(1_{\widetilde{A}}) = 1$ (if A is not unital).

Fix an integer $n \in \mathbb{N}$ with $n \geqslant 2$. Let $l \in \mathbb{N}$. Choose an integer $m(l) \in \mathbb{N}$ such that

$$|\frac{n}{m(l)}| < \frac{1}{2(n+l)^2}, \ l = 1, 2, ... \tag{3.20}$$

Let $K = nm(l) + n(n+1)/2$.

Since A has uniform property Γ, there exist projections $p_{1,l}, p_{2,l}, ..., p_{K,l} \in (l^\infty(A) \cap A')/I_{QT(A),\varpi}$ such that, for $1 \leqslant i \leqslant n$,

$$\sum_{i=1}^{K} p_{i,l} = 1_{(l^\infty(A)\cap A')/I_{QT(A),\varpi}}, \tag{3.21}$$

$$\tau(p_{i,l}a) = \frac{1}{K}\tau(a) \ \text{ and } \ \tau(p_{i,l}) = \frac{1}{K} \ \text{ for all } \ a \in A \ \text{ and } \ \tau \in QT_\varpi(A). \tag{3.22}$$

We write $P_{i,l} = \{p_{i,l}^{(k)}\}$, where $\{p_{i,l}^{(k)}\} \in (l^\infty(A) \cap A')_+^{\mathbf{1}}$, such that $\Pi_\varpi(P_i) = p_{i,l}$, $1 \leqslant i \leqslant K$. Moreover, $p_{i,l}^{(k)} \perp p_{j,l}^{(k)}$, if $i \neq j$ and $1 \leqslant i, j \leqslant K$. By replacing $p_{i,l}^{(k)}$ by $f_{1/4}(p_{i,l}^{(k)})$ if necessary, we may assume that $\{p_{i,l}^{(k)}\}$ is a permanent projection lifting of $p_{i,l}$ $(1 \leqslant i \leqslant n)$ (see Proposition 6.2 of [14] and Proposition 2.21 of [23]). Therefore, by (1) and (2) of Proposition 6.2 of [14] (see also Proposition 2.21 of [23]), we may assume that

$$\lim_{k\to\varpi} \sup\{\tau(p_{i,l}^{(k)}) - \tau(f_{1/4}(p_{i,l}^{(k)})p_{i,l}^{(k)}) : \tau \in QT(A)\} = 0 \ \text{ and} \tag{3.23}$$

$$\lim_{k\to\varpi} \sup\{d_\tau(p_{i,l}^{(k)}) - \tau((p_{i,l}^{(k)})^2) : \tau \in QT(A)\} = 0. \tag{3.24}$$

Since $\tau((p_{i,l}^{(k)})^2) \leqslant \tau((p_{i,l}^{(k)}))$ for all $\tau \in QT(A)$, we obtain

$$\lim_{k\to\varpi} \sup\{d_\tau(p_{i,l}^{(k)}) - \tau((p_{i,l}^{(k)})) : \tau \in QT(A)\} = 0. \tag{3.25}$$

Since $p_{i,l}$ is a projection, $f_{1/4}(p_{i,l}) = p_{i,l}$ $(1 \leqslant i \leqslant n)$. Consequently,

$$\lim_{k\to\varpi} \|p_{i,l}^{(k)} - f_{1/4}((p_{i,l}^{(k)})\|_{2,QT(A)} = 0. \tag{3.26}$$

Note that (recall that $p_{i,l}^{(k)}$ commutes with $f_{1/4}((p_{i,l}^{(k)}))$)

$$|\tau(p_{i,l}^{(k)}) - \tau(f_{1/4}((p_{i,l}^{(k)})))| \leqslant \tau(1_{\widetilde{A}})^{1/2}\tau((p_{i,l}^{(k)} - f_{1/4}((p_{i,l}^{(k)})))^2)^{1/2} \ \text{ for all } \ \tau \in QT(A). \tag{3.27}$$

By (3.26), we have

$$\lim_{k\to\varpi} \sup\{|\tau(p_{i,l}^{(k)}) - \tau(f_{1/4}((p_{i,l}^{(k)})))| : \tau \in QT(A)\} = 0. \tag{3.28}$$

Let $q_{1,l}$ be $m(l)+1$ copies of $p_{i,l}$'s, $q_{2,l}$ be $m(l)+2$ copies of $p_{i,l}$'s, ..., and $q_{n,l}$ be $m(l)+n$ copies of p_i's. Then

$$\sum_{i=1}^{n} q_{i,l} = \sum_{i=1}^{K} p_{i,l} \ \text{ and } \ \tau(\sum_{i=1}^{n} q_{i,l}) = \frac{nm(l) + n(n+1)/2}{K} = 1 \ \text{ for all } \ \tau \in QT_\varpi(A). \tag{3.29}$$

Write $q_{i,l} = \Pi(\{c_{i,l}^{(k)}\})$, where $c_{i,l}^{(k)}$ is the sum of $m(l)+i$ copies of $p_{i,l}^{(k)}$. Then (see (3.22)),

$$\lim_{k\to\varpi} \sup\{|\tau(ac_{i,l}^{(k)}) - \frac{m(l)+i}{K}\tau(a)| : \tau \in QT(A)\} = 0 \ \text{ and} \tag{3.30}$$

$$\lim_{k\to\varpi} \sup\{|\tau(c_{i,l}^{(k)}) - \frac{m(l)+i}{K}| : \tau \in QT(A)\} = 0 \tag{3.31}$$

for all $a \in A$, Note that, for each fixed n and $1 \leqslant i \leqslant n$,

$$\lim_{l\to\infty} \frac{m(l)+i}{K} = \frac{1}{n}. \tag{3.32}$$

Let $\{\mathcal{F}_k\}$ be an increasing sequence of finite subsets of A such that $\cup_{k=1}^{\infty}\mathcal{F}_k$ is dense in A. Then, for each $l \in \mathbb{N}$, by (3.31), (3.25), and (3.28), as well as (3.30) (recall also $p_{i,l} \in A'$), we find an integer $k(l) \in \mathbb{N}$ such that $k(l) < k(l+1)$,

$$d_\tau(c_{1,l}^{(k(l))}) < d_\tau(c_{2,l}^{(k(l))}) < \cdots < d_\tau(c_{n,l}^{(k(l))}) \ \text{ for all } \ \tau \in QT(A), \tag{3.33}$$

$$\tau(f_{1/4}(c_{i,l}^{(k(l))})) > 1/n - 1/(2(n+l))^2 \ \text{ for all } \ \tau \in QT(A), \tag{3.34}$$

$$\sup\{|\tau(ac_{i,l}^{(k(l))}) - \frac{m(l)+i}{K}\tau(a)| : \tau \in QT(A)\} < 1/l, \tag{3.35}$$

$$\sup\{|\tau(p_{i,l}^{(k)}) - \tau(f_{1/4}((p_{i,l}^{(k)})))| : \tau \in QT(A)\} < 1/l \ \text{ and} \tag{3.36}$$

$$\|[c_{i,l}^{k(l)}, b]\| < 1/l \ \text{ for all } \ b \in \mathcal{F}_k \ \text{ and } \ 1 \leqslant i \leqslant n. \tag{3.37}$$

Since A has strict comparison, by (3.33), we obtain $x_{i,l} \in A$ such that

$$x_{i,l}^*x_{i,l} = f_{1/4}(c_{1,l}^{(k(l))}) \ \text{ and } \ x_{i,l}x_{i,l}^* \in \mathrm{Her}(c_{i,l}^{(k(l))}), \ i = 2,3,...,n. \tag{3.38}$$

Recall that $c_{i,l}^{(k)} \perp c_{j,l}^{(k(l))}$, if $i \neq j$ and $1 \leqslant i,j \leqslant n$. Write $x_{i,l} = u_{i,l}f_{1/4}(c_{1,l}^{(k(l))})^{1/2}$, $1 \leqslant i \leqslant n$.

This provides a homomorphism $\varphi^{(l)} : C_0((0,1]) \otimes M_n \to A$ such that

$$\varphi^{(l)}(\jmath \otimes e_{1,1}) = (x_{2,l}^*x_{2,l})^{1/2} = (f_{1/4}(c_{1,l}^{(k(l))}))^{1/2}, \tag{3.39}$$

$$\varphi^{(l)}(\jmath \otimes e_{1,j}) = x_{j,l}, \ \varphi^{(l)}(\iota \otimes e_{j,1}) = x_{j,l}^*, \ 2 \leqslant j \leqslant n, \tag{3.40}$$

$$\varphi^{(l)}(\jmath \otimes e_{i,j}) = u_{i,l}f_{1/4}(c_{1,l}^{(k(l))})u_{j,l}^*, \ 2 \leqslant i,j \leqslant n, \tag{3.41}$$

$$\varphi^{(l)}(\jmath \otimes e_{i,i}) = (x_{i,l}x_{i,l}^*)^{1/2}, \ (i > 1), \ \text{ and} \tag{3.42}$$

$$\varphi^{(l)}(\jmath \otimes 1_n) = f_{1/4}(c_{1,l}^{k(l)})^{1/2} + \sum_{i=2}^{n}(x_{i,l}x_{i,l}^*)^{1/2}, \tag{3.43}$$

where $\jmath$ is the identify function on $[0,1]$. Define $\psi^{(l)} : M_n \to A$ by $\psi^{(l)}(e_{i,j}) = \varphi^{(l)}(\jmath \otimes e_{i,j})$ $(1 \leqslant i,j \leqslant n)$. Then $\psi^{(l)}$ is an order zero c.p.c. map. We also have (as $l \to \infty$)

$$\|\psi^{(l)}(e_{i,i}) - c_{i,l}^{(k(l))}\|_{_{2,QT(A)}} \to 0 \ \text{ and} \tag{3.44}$$

$$\|\psi^{(l)}(1_n) - \sum_{i=1}^{n} c_{i,l}^{(k(l))}\|_{_{2,QT(A)}} \to 0. \tag{3.45}$$

Define $\Psi = \{\psi^{(l)}\} : M_n \to l^\infty(A)$ and $\varphi = \Pi_\varpi \circ \Psi : M_n \to l^\infty(A)/I_{_{QT(A),\varpi}}$. Then φ is an order zero c.p.c. map. By (3.45), it is unital. Hence φ is a unital homomorphism. Combining (3.35) with (3.32), we obtain that

$$\tau(a\varphi(1_n)) = \frac{1}{n}\tau(a) \ \text{ for all } \ a \in A \ \text{ and } \ \tau \in QT_\varpi(A). \tag{3.46}$$

Note that, by (3.37), $\{c_{i,l}^{(k(l))}\} \in A'$. Thus, by (3.44), we have that $\varphi(e_{i,i}) \in (l^\infty(A) \cap A')/I_{_{QT(A),\varpi}}$. □

Proposition 3.4. Let A be a separable C^*-algebra with nonempty compact $QT(A)$. Suppose that A has uniform property Γ. Then, for any $k \in \mathbb{N}$, $M_k(A)$ also has uniform property Γ.

Proof. Fix $k \in \mathbb{N}$. Let $n \in \mathbb{N}$. Since A has uniform property Γ, there are mutually orthogonal projections $p_1, p_2, ..., p_n \in (l^\infty(A) \cap A')/I_{_{QT(A),\varpi}}$ such that $\sum_{i=1}^n p_i = 1$ and

$$\tau(ap_i) = \frac{1}{n}\tau(a) \ \text{ for all } \ a \in A \ \text{ and } \ \tau \in QT_\varpi(A). \tag{3.47}$$

Put $q_i = p_i \otimes 1_{M_k}$, $i = 1, 2, ..., n$. Then, q_i are projections and $\sum_{i=1}^n q_i = 1_{M_k(C)}$, where $C = l^\infty(A)/I_{QT(A),\varpi}$, and, for any $b = (a_{i,j})_{k\times k} \in M_k(A)$,

$$q_i b = b q_i \ \text{ and } \ \tau(bq_i) = \frac{1}{n}\tau(b) \ \text{ for all } \ \tau \in QT_\varpi(M_k(A)).$$

□

Theorem 3.5. Let A be a non-elementary separable simple C^*-algebra with strict comparison and with nonempty compact $QT(A)$. Suppose that A has uniform property Γ. Then Γ is surjective (see Definition 2.7).

Proof.

Fix $a \in A_+^{\mathbf{1}} \setminus \{0\}$ and $n \in \mathbb{N}$. There is $r \in (0, 1/2)$ such that $f_r(a) > 0$. Set

$$\sigma_0 = \inf\{\tau(f_r(a)) : \tau \in QT(A)\} > 0. \tag{3.48}$$

Choose $m \in \mathbb{N}$ such that $1/m < \sigma_0/8(n+1)$. Since A has uniform property Γ, there is a projection $p \in (l^\infty(A) \cap A')/I_{QT(A),\varpi}$ such that

$$\tau(bp) = \frac{1}{nm}\tau(b) \ \text{ for all } \ \tau \in QT_\varpi(A) \ \text{ and } \ b \in A. \tag{3.49}$$

Fix $\varepsilon \in (0, r/2)$. Then, for $\eta \in \{\varepsilon, \varepsilon/2, \varepsilon/4, \varepsilon/8\}$,

$$\tau(f_\eta(a)p) = \frac{1}{nm}\tau(f_\eta(a)) \ \text{ for all } \ \tau \in QT_\varpi(A). \tag{3.50}$$

Choose $\delta \in (0, 1/(8(n+1)m)^2)$. Recall that $p \in (l^\infty(A) \cap A')/I_{QT(A),\varpi}$. Therefore (by lifting p to a sequence in $l^\infty(A) \cap A'$), we obtain an element $e \in A_+^{\mathbf{1}}$ such that, for any $\eta \in \{\varepsilon, \varepsilon/2, \varepsilon/4, \varepsilon/8\}$ and all $\tau \in QT(A)$,

$$\frac{1}{nm}\tau(f_\eta(a)) + \frac{1}{2(n+1)m^3} > \tau(ef_\eta(a)e) > \frac{1}{nm}\tau(f_\eta(a)) - \frac{1}{2(n+1)m^2}. \tag{3.51}$$

Put $c := ef_{\varepsilon/4}(a)e$. Then, by (3.51),

$$d_\tau(c) \geqslant \tau(ef_{\varepsilon/4}(a)e) > \frac{1}{nm}\tau(f_{\varepsilon/4}(a)) - 1/2(n+1)m^2 \ \text{ for all } \ \tau \in QT(A). \tag{3.52}$$

Choose $b \in (A \otimes \mathcal{K})_+^{\mathbf{1}}$ such that $[b] = (m-1)[c]$. Then, for all $\tau \in QT(A)$,

$$\begin{aligned}
(n+1)\widehat{[b]} &= (n+1)(m-1)\widehat{[c]} > \frac{(n+1)(m-1)}{nm}(\tau(f_{\varepsilon/4}(a))) - 1/2m \\
&> \tau(f_{\varepsilon/4}(a)) + \frac{1}{n}\tau(f_{\varepsilon/4}(a)) - \frac{1}{m} - \frac{1}{nm} - \frac{1}{2m} &\quad (3.53)\\
&\geqslant \tau(f_{\varepsilon/4}(a)) + \frac{\sigma_0}{n} - \frac{1}{m} - \frac{1}{nm} - \frac{1}{2m} &\quad (3.54)\\
&> \tau(f_{\varepsilon/4}(a)) \geqslant d_\tau(f_\varepsilon(a)). &\quad (3.55)
\end{aligned}$$

Since A has strict comparison,

$$(n+1)[b] \geqslant [f_\varepsilon(a)]. \tag{3.56}$$

By (3.51), we also have, for all $\tau \in QT(A)$,

$$\begin{aligned}
n\widehat{[b]} &= n(m-1)\widehat{[c]} \leqslant \frac{m-1}{m}\tau(f_{\varepsilon/4}(a)) + \frac{1}{2m^2} &\quad (3.57)\\
&\leqslant \tau(f_{\varepsilon/4}(a)) - (\frac{\sigma_0}{m} - \frac{1}{2m^2}) \leqslant \tau(f_{\varepsilon/4}(a)) \leqslant d_\tau(a). &\quad (3.58)
\end{aligned}$$

It follows that

$$n[b] \leqslant [a]. \tag{3.59}$$

By Proposition 3.4, (3.56) and (3.59) also hold for any $a \in M_n(A)_+$. It follows that (3.56) and (3.59) hold for any $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+$. We will use an argument of L. Robert to finish the proof.

Let $x' \ll x \in \mathrm{Cu}(A)$. Choose $a \in (A \otimes \mathcal{K})_+^{\mathbf{1}}$ such that $x = [a]$. Then, for some $\varepsilon \in (0, 1/2)$, $x' \leqslant [f_\varepsilon(a)]$. Now $f_{\varepsilon/2}(a) \in \mathrm{Ped}(A \otimes \mathcal{K})_+$. By what has been proved, there is $b \in \mathrm{Ped}(A \otimes K)_+$ such that

$$x' \leqslant [f_\varepsilon(a)] \leqslant [f_\varepsilon(f_{\varepsilon/2}(a))] \leqslant (n+1)[b] \ \text{ and } \ n[b] \leqslant [f_{\varepsilon/2}(a)] \leqslant [a]. \tag{3.60}$$

It follows that A satisfies the property (D) (see Definition 5.5 of [13]). Then, by an argument of L. Robert (see the proof of Proposition 6.2.1 of [26]), Γ is surjective (see Lemma 5.6 of [13]). □

Lemma 3.6. Let A be a separable algebraically simple C^*-algebra with $QT(A) \neq \emptyset$ which has strict comparison and for which the canonical map Γ is surjective. Suppose that there are n mutually orthogonal elements $a_1, a_2, ..., a_n, a_{n+1} \in A_+^{\mathbf{1}}$ such that, for some

$$0 < \eta_1 < \bar{\eta}_1 < \eta_2 < \bar{\eta}_2 < \cdots < \eta_n < \bar{\eta}_n < \eta_{n+1} < \delta/2 \tag{3.61}$$

and $\delta \in (0, 1/2)$,

$$d_\tau(f_{\eta_2}(a_2)) < d_\tau(a_1) \ \text{ and } \tag{3.62}$$

$$d_\tau(f_{\eta_{i+1}}(a_{i+1})) < d_\tau(f_{\bar{\eta}_i}(a_i)) \ \text{ for all } \ \tau \in \overline{QT(A)}^w, \ 2 \leqslant i \leqslant n. \tag{3.63}$$

Then, for any $\sigma \in (0, 1/2)$, there is $d \in \mathrm{Her}(\sum_{i=1}^{n+1} a_i)_+^{\mathbf{1}}$ such that

$$\sum_{i=2}^{n+1} f_\delta(a_i) \leqslant d \ \text{ and } \ \omega(d) < \sigma. \tag{3.64}$$

Proof. We will prove this by induction on n (for any $\sigma \in (0, 1/2)$). For $n = 1$, since A has strict comparison, there is $x \in \mathrm{Her}(a)$, where $a = \sum_{i=1}^{n+1} a_i$ such that

$$x^*x = f_{\delta_1}(a_2) \ \text{ and } \ xx^* \in \mathrm{Her}(a_1), \tag{3.65}$$

where $\eta_2 < \delta_1 < \bar{\eta}_2 < \delta/2$. Put $C_1 := \mathrm{Her}(x^*x + xx^*)$. Define $\psi : C_0((0,1]) \otimes M_2 \to C_1$ by $\psi(\iota \otimes e_{1,1}) = (xx^*)^{1/2}$, $\psi(\iota \otimes e_{2,2}) = (x^*x)^{1/2}$, $\psi(\iota \otimes e_{1,2}) = x$, $\psi(\iota \otimes e_{2,1}) = x^*$. Thus (see Proposition 8.3 of [14], for example) we may write $C_1 = M_2(\mathrm{Her}(x^*x))$. Then, for any $0 < \varepsilon'' < \varepsilon' < \eta_1/2$, by Lemma 8.9 of [14], there exists $c_1 \in \mathrm{Her}(f_{\varepsilon''}(x^*x))_+^{\mathbf{1}}$ and a unitary $U_1 \in \widetilde{C_1}$ such that, with $b_1 = U_1^* \mathrm{diag}(0, c) U_1$,

(1) $f_{\varepsilon'}(x^*x) \leqslant b_1$;

(2) $d_\tau(f_{\varepsilon'}(x^*x)) \leqslant d_\tau(b_1) \leqslant d_\tau(f_{\varepsilon''}(x^*x))$ for all $\tau \in \overline{QT(A)}^w$,

(3) for some $\delta_1' \in (0, 1/2)$,

$$d_\tau(b_1) - \tau(f_{\delta_1'}(b_1)) < \sigma/2(n+1) \ \text{ for all } \ \tau \in \overline{QT(A)}^w, \ \text{ and } \tag{3.66}$$

(4) $U_1^*(g_{\varepsilon''/2}(x^*x) + xx^*)U \in B_1$, where $B_1 := (\mathrm{Her}(b_1)^\perp) \cap C_1$. Note that $b_1 \in C_1 \subset \mathrm{Her}(a_1 + a_2)$ and, by (1) above, $f_\delta(a_2) \leqslant b_1$.

Let a_2'' be a strictly positive element of B_1. Then $a_2'' \in \mathrm{Her}(a)_+^{\mathbf{1}}$ and

$$d_\tau(a_2'') > d_\tau(g_{\varepsilon''/2}(x^*x) + xx^*) > d_\tau(f_{\bar{\eta}_2}(a_2)) \ \text{ for all } \ \tau \in \overline{QT(A)}^w. \tag{3.67}$$

Therefore the lemma holds for $n = 1$.

We assume that lemma holds for $n-1$ (for any $\sigma \in (0, 1/2)$). We will keep the notation just introduced. Then $a_2'' \perp a_i$, $i = 3, 4, ..., n+1$. Moreover, by (3.67),

$$d_\tau(f_{\bar{\eta}_3}(a_3)) < d_\tau(a_2'') \ \text{ for all } \ \tau \in \overline{QT(A)}^w. \tag{3.68}$$

Put $a' := a_2'' + a_3 + a_4 + \cdots + a_{n+1}$. Then, by the inductive assumption (choose $\sigma/2(n+1)$ instead of σ). we obtain $b_2 \in \mathrm{Her}(a')_+^{\mathbf{1}}$ such that

$$f_\delta(\sum_{i=3}^{n+1} a_i) \leqslant b_2 \text{ and } \omega(b_2) < \sigma/2(n+1) \text{ for all } \tau \in \overline{QT(A)}^w. \tag{3.69}$$

Note that $b_1 \perp b_2$ and, by Proposition 4.4 of [14], $\omega(b_1 + b_2) < \sigma$. Moreover,

$$\sum_{i=2}^{n+1} f_\delta(a_i) \leqslant b_1 + b_2. \tag{3.70}$$

This completes the induction and the lemma follows. □

Theorem 3.7. Let A be a separable simple C^*-algebra with strict comparison and with nonempty compact $QT(A)$. Suppose that A also has uniform property Γ. Then

(i) the map Γ is surjective,

(ii) A has tracial approximate oscillation zero,

(iii) A has stable rank one, and

(iv) A has property (TM).

Proof. We have shown that (i) holds (Theorem 3.5). It follows from Theorem 1.1 of [14] that (ii), (iii) and (iv) are equivalent. We will show that (ii) holds.

We need to show that, for any $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+^{\mathbf{1}}$, $\Omega^T(a) = 0$.

Let $\varepsilon > 0$. There is $m \in \mathbb{N}$ such that $\|a - a^{1/2}E_m a^{1/2}\| < \varepsilon/2$, where $E_m = \sum_{i=1}^m e_{i,i}$ and $\{e_{i,j}\}$ is a system of matrix units for $\mathcal{K}$. Note that $a^{1/2}E_m a^{1/2} \in \mathrm{Her}(a)$. Therefore, to show that $\Omega^T(a) = 0$, it suffices to show that $\Omega^T(a^{1/2}E_m a^{1/2}) = 0$. Put $z = E_m a^{1/2}$. Then $z^*z = a^{1/2}E_m a^{1/2}$ and $zz^* = E_m a E_m$.

Therefore, it suffices to show that $\Omega^T(E_m a E_m) = 0$. Consequently, it suffices to show that $\Omega^T(a) = 0$ for any $a \in M_m(A)_+^{\mathbf{1}}$. Since, by Proposition 3.4, $M_m(A)$ also has uniform property Γ, without loss of generality, we may assume that $a \in A_+^{\mathbf{1}}$.

Therefore it suffices to show that, for any $a \in A_+^{\mathbf{1}}$, $\Omega^T(a) = 0$. If $0 \in \overline{\mathbb{R}_+ \setminus \mathrm{sp}(a)}$, then $\Omega^T(a) = 0$. Hence, we may assume that there is $\varepsilon_0 \in (0, 1/2)$ such that $[0, \varepsilon_0] \subset \mathrm{sp}(a)$.

Let $\varepsilon, \sigma \in (0, \varepsilon_0/2)$. By Proposition 5.7 of [14], it suffices to show that there is $d \in \mathrm{Her}(a)_+^{\mathbf{1}}$ such that

$$\|a - ad\|_{_{2,QT(A)}} < \varepsilon \text{ and } \omega(d) < \sigma. \tag{3.71}$$

Fix any $\eta \in (0, (\varepsilon/8)^3)$. Choose $n \in \mathbb{N}$ such $1/n < (\eta/8)^3$.

By Theorem 3.3, there is a unital homomorphism $\varphi : M_{n+1} \to l^\infty(A)/I_{_{QT(A),\varpi}}$ such that $\varphi(e_{i,i}) \in (l^\infty(A) \cap A')/I_{_{QT(A),\varpi}}$, $1 \leqslant i \leqslant n+1$. There exists an order zero c.p.c. map $\Phi = \{\varphi_k\} : M_{n+1} \to l^\infty(A)$ such that $\Pi_\varpi \circ \Phi = \varphi$ and, for all $1 \leqslant i \leqslant n+1$,

$$\tau(b\varphi(e_{i,i})) = \frac{1}{n+1}\tau(b) \text{ for all } b \in A \text{ and } \tau \in QT_\varpi(A). \tag{3.72}$$

Choose

$$0 < r_1 < r_2/2 < r_2 < \cdots < r_{3n+2} < r_{3(n+1)}/2 < r_{3(n+1)} < \eta/2. \tag{3.73}$$

It follows that (recall that $\varphi(e_{i,i}) \in (l^\infty(A) \cap A')/I_{_{QT(A),\varpi}}$), for all $1 \leqslant j \leqslant 3(n+1)$ and $1 \leqslant i \leqslant n+1$,

$$\lim_{k\to\varpi} (\sup_{\tau \in QT(A)} |\tau(f_{r_j}(a)\varphi_k(e_{i,i})) - \frac{1}{n+1}\tau(f_{r_j}(a))|) = 0, \tag{3.74}$$

$$\lim_{k\to\varpi} \|f_{r_j}(a^{1/2}\varphi_k(e_{i,i})a^{1/2}) - f_{r_j}(a)\varphi_k(e_{i,i})\|_{_{2,QT(A)}} = 0 \text{ and} \tag{3.75}$$

$$\lim_{k\to\varpi} \|f_{r_j}(a^{1/2}\varphi_k(e_{i,i})a^{1/2}) - f_{r_j}(\varphi_k(e_{i,i})a\varphi_k(e_{i,i}))\|_{_{2,QT(A)}} = 0. \tag{3.76}$$

Since $\Pi_\varpi(\iota(a^{1/2}))\varphi(e_{i,i})\Pi_\varpi(\iota(a^{1/2})) = \varphi(e_{i,i})\Pi_\varpi(\iota(a))\varphi(e_{i,i})$ for $1 \leqslant i \leqslant n+1$, there are, for each $k \in \mathbb{N}$, mutually orthogonal elements $a_{i,k} \in \mathrm{Her}(a)_+^{\mathbf{1}}$ $(1 \leqslant i \leqslant n+1)$ such that

$$\Pi_\varpi(\{a_{i,k}\}) = \Pi_\varpi(\iota(a^{1/2}))\varphi(e_{i,i})\Pi_\varpi(\iota(a^{1/2})) \text{ and } \tag{3.77}$$

$$\Pi_\varpi(f_{r_j}(\{a_{i,k}\})) = \Pi_\varpi(f_{r_j}(\iota(a^{1/2})))\varphi(e_{i,i})\Pi_\varpi(\iota(a^{1/2})). \tag{3.78}$$

Therefore, for $1 \leqslant j \leqslant 3(n+1)$,

$$\lim_{k\to\varpi}(\sup_{\tau\in QT(A)} |\tau(f_{r_j}(a_{i,k})) - \frac{1}{n+1}\tau(f_{r_j}(a))|) = 0. \tag{3.79}$$

Since A is simple, $QT(A)$ is compact and $[0,\varepsilon_0] \subset \mathrm{sp}(a)$, we have, for any $g \in C_0((0,1])_+^{\mathbf{1}}$ with $g|_{[0,\varepsilon_0]} \neq 0$, that

$$\inf\{\tau(g(a)) : \tau \in QT(A)\} > 0. \tag{3.80}$$

Then, by (3.79), there exists $\mathcal{P} \in \varpi$ such that, for any $k \in \mathcal{P}$,

$$\tau(f_{r_{3j+1}}(a_{i+1,k})) < \tau(f_{r_{3j}}(a_{i,k})) < \frac{1}{n} \text{ for all } \tau \in QT(A), \tag{3.81}$$

$1 \leqslant i \leqslant n$. It follows that

$$d_\tau(f_{r_{3j+2}}(a_{i+1,k})) < d_\tau(f_{r_{3j}}(a_{i,k})) \text{ for all } \tau \in QT(A). \tag{3.82}$$

Keep in mind that (3.73) holds. We also have $a_{i,k} \perp a_{i+1,k}$ $(1 \leqslant i \leqslant n)$. Put $a' := \sum_{i=1}^{n+1} a_{i,k}$ and $c = \sum_{i=2}^{n+1} a_{i,k}$. Then, by Lemma 3.6, we obtain $d \in \mathrm{Her}(a')_+^{\mathbf{1}}$ such that

$$f_\eta(c) \leqslant d \text{ and } \omega(d) < \sigma. \tag{3.83}$$

Note that $a_{i,k} \in \mathrm{Her}(a)$. Therefore $c \in \mathrm{Her}(a)$. We also have that $d \in \mathrm{Her}(a)$. By (3.77) and the fact that φ is unital, we may assume that

$$\|a - a'\|_{2,QT(A)} < (\varepsilon/8)^3. \tag{3.84}$$

Then (see Lemma 3.5 of [18] and also Definition 2.16 of [14])

$$\|a-c\|_{2,QT(A)}^{2/3} \leqslant \|a-a'\|_{2,QT(A)}^{2/3} + \|a'-c\|_{2,QT(A)}^{2/3} < (\varepsilon/8)^2 + (\frac{1}{n+1})^{2/3}. \tag{3.85}$$

It follows that

$$\|a-ad\|_{2,QT(A)}^{2/3} \leqslant \|a-c\|_{2,QT(A)}^{2/3} + \|d\|\|a-c\|_{2,QT(A)}^{2/3} + \|c-cd\|_{2,QT(A)}^{2/3} < (\varepsilon)^2. \tag{3.86}$$

Thus (3.71) holds and the theorem follows.

□

We will now consider simple C^*-algebras A for which $QT(A)$ may not be compact.

4 Hereditary uniform property Γ

Definition 4.1 (Definition 2.1 of [6]). Let A be a separable simple C^*-algebra with $\widetilde{QT}(A) \setminus \{0\} \neq \emptyset$. C^*-algebra A is said to have hereditary uniform property Γ, if for any $e \in \mathrm{Ped}(A \otimes \mathcal{K})_+ \setminus \{0\}$ and any $n \in \mathbb{N}$, there exist pairwise orthogonal projections $p_1, p_2, \ldots, p_n \in (l^\infty(A_e) \cap (A_e)')/I_{\overline{QT(A_e)}^w,\varpi}$, where $A_e = \overline{e(A \otimes \mathcal{K})e}$, such that, for $1 \leqslant i \leqslant n$,

$$\tau(p_i a) = \frac{1}{n}\tau(a) \text{ for all } a \in A_e \text{ and } \tau \in QT_\varpi^w(A_e), \tag{4.1}$$

where $QT_\varpi^w(A_e) = \{\tau_\varpi : \{\tau_n\} \subset \overline{QT(A_e)}^w\}$.

Proposition 4.2 (Proposition 2.2 of [34])**.** Let A be a separable simple C^*-algebra with $\widetilde{QT}(A)\setminus\{0\} \neq \emptyset$. Then the following are equivalent:

(i) A has hereditary uniform property Γ.

(ii) For any $e \in \mathrm{Per}(A\otimes K)_+ \setminus \{0\}$, any finite subset $\mathcal{F} \subset A_e = \overline{e(A\otimes\mathcal{K})e}$, any $\varepsilon > 0$, and any $n \in \mathbb{N}$, there exist pairwise orthogonal elements $e_1, e_2, ..., e_n \in (A_e)_+^{\mathbf{1}}$ such that, for $1 \leqslant i \leqslant n$, and $a \in A_e$, we have

$$\|[x,\, e_i]\|_{2,\overline{QT(A_e)}^w} < \varepsilon,\ \sup_{\overline{QT(A_e)}^w} |\tau(ae_i) - \frac{1}{n}\tau(a)| < \varepsilon \ \text{ and} \tag{4.2}$$

$$\|e_i - e_i^2\|_{2,\overline{QT(A_e)}^w} < \varepsilon. \tag{4.3}$$

(iii) For any $e \in \mathrm{Per}(A\otimes K)_+$, any finite subset $\mathcal{F} \subset A_e = \overline{e(A\otimes\mathcal{K})e}$, any $\varepsilon > 0$, and any $n \in \mathbb{N}$, there exist pairwise orthogonal elements $e_1, e_2, ..., e_n \in (A_e)+^{\mathbf{1}}$ such that, for $1 \leqslant i \leqslant n$, and $a \in A_e$, we have

$$\|[x,\, e_i]\| < \varepsilon,\ \sup_{\overline{QT(A_e)}^w} |\tau(ae_i) - \frac{1}{n}\tau(a)| < \varepsilon \ \text{ and} \tag{4.4}$$

$$\|e_i - e_i^2\|_{2,\overline{QT(A_e)}^w} < \varepsilon. \tag{4.5}$$

Proof. The proof is just a repetition of that of Proposition 2.1 of [34]. □

Theorem 4.3. Let A be a separable non-elementary simple C^*-algebra with strict comparison and nonempty compact $QT(A)$. Suppose that A has uniform property Γ. Then A has hereditary uniform property Γ.

Proof. Let $e_A \in A_+$ be a strictly positive element of A and let $e \in \mathrm{Ped}(A\otimes\mathcal{K})_+^{\mathbf{1}} \setminus \{0\}$. We view A as a hereditary C^*-subalgebra of $A\otimes\mathcal{K}$. Put $A_1 = \overline{e(A\otimes\mathcal{K})e}$. There is $\varepsilon \in (0, 1/2)$ such that $f_\varepsilon(e_A) \neq 0$. Note that $f_\varepsilon(e_A) \in \mathrm{Ped}(A\otimes\mathcal{K})$. Since $e \in \mathrm{Ped}(A\otimes\mathcal{K})$, there is $K \in \mathbb{N}$ such that $[e] \leqslant K[f_\varepsilon(e_A)] \leqslant K[e_A]$. By Theorem 3.7, A has stable rank one. So does $A\otimes\mathcal{K}$. It follows from Proposition 2.1.2 of [26] that there is $x \in A\otimes\mathcal{K}$ such that

$$x^*x = e \ \text{ and } \ xx^* \in M_K(A). \tag{4.6}$$

Thus there is an isomorphism ψ from A_1 to a hereditary C^*-subalgebra of $M_K(A)$ with $\psi(e) \sim e$ (see 1.4 of [8]). Therefore, without loss of generality, we may assume that $e \in M_K(A)_+^{\mathbf{1}}$. Since $M_K(A)$ also has uniform property Γ (see Proposition 3.4), to simplify notation, we may further assume that $e \in A_+^{\mathbf{1}}$.

Fix $n \in \mathbb{N}$. Let $p_1, p_2, ..., p_n \in (l^\infty(A)\cap A')/I_{QT(A),\varpi}$ be mutually orthogonal projections such that, for all $a \in A$,

$$\tau(p_i a) = \frac{1}{n}\tau(a) \ \text{ for all } \ \tau \in QT_\varpi(A),\ 1 \leqslant i \leqslant n. \tag{4.7}$$

Let $p_i^{(k)} \in A_+^{\mathbf{1}}$ be such that $p_i^{(k)} \perp p_j^{(k)}$ if $i \neq j$ $(1 \leqslant i, j \leqslant n)$, $\{p_i^{(k)}\}_{k\in\mathbb{N}} \subset A'$ and $\Pi_\varpi(\{p_i^{(k)}\}) = p_i$, $1 \leqslant i \leqslant n$.

Since, by Theorem 3.7, A has tracial approximate oscillation zero, there is a sequence $\{a_k\}$ in A_1 with $0 \leqslant a_k \leqslant 1$ such that, for any $b \in A_1$,

$$\lim_{k\to\infty} \|b - ba_k\|_{2,QT(A)} = 0 \ \text{ and } \ \lim_{k\to\infty} \omega(a_k) = 0. \tag{4.8}$$

It follows from Proposition 6.2 of [14] that there exists $\{j(k)\} \subset \mathbb{N}$ such that $\Pi(\{a_k^{1/j(k)}\}) = q$ is a projection (recall that $\Pi : l^\infty(A) \to l^\infty(A)/I_{\overline{QT(A)}^w,\mathbb{N}}$ is the quotient map). Put $c_k = a_k^{1/j(k)}$, $k \in \mathbb{N}$. Note that, for any $b \in A_+^{\mathbf{1}}$,

$$\Pi(\iota(b)) = \Pi(\iota(b^{1/2})\{a_k\}\iota(b^{1/2})) \leqslant \Pi(\iota(b^{1/2})\{c_k\}\iota(b^{1/2})) \leqslant \Pi(\iota(b)). \tag{4.9}$$

It follows that, for any $b \in A_1$,

$$\lim_{k\to\infty} \|b - bc_k\|_{2,QT(A)} = 0 = \lim_{k\to\infty} \|b - b^{1/2}c_kb^{1/2}\|_{2,QT(A)}. \tag{4.10}$$

In particular, $\{c_k\} \in (A_1)'$. Let $\{\mathcal{F}_k\}$ be an increasing sequence of finite subsets of A_1 such that its union is dense in A_1. Without loss of generality, we may assume that, for all $k \in \mathbb{N}$, that

$$\|bc_k - b\|_{_{2,QT(A)}} < 1/k \ \text{ and } \ \|c_k b - b\|_{_{2,QT(A)}} < 1/k \ \text{ for all } \ b \in \mathcal{F}_k. \tag{4.11}$$

Put $\mathcal{G}_k = \mathcal{F}_k \cup \{c_1, c_2, ..., c_k\}$. For each $k \in \mathbb{N}$, there exists $\mathcal{P}_k \in \varpi$ such that, for all $m \in \mathcal{P}_k$,

$$\|p_i^{(m)} - (p_i^{(m)})^2\|_{_{2,QT(A)}} < 1/k \ \text{ and,} \tag{4.12}$$

$$\sup\{|\tau(p_i^{(m)}b) - \frac{1}{n}\tau(b)| : \tau \in QT(A)\} < 1/k \ \text{ and } \ \|[p_i^{(m)}, b]\| < 1/k \tag{4.13}$$

$$\text{for all } \ b \in \mathcal{G}_k \ \text{ and } \ 1 \leqslant i \leqslant n. \tag{4.14}$$

We may assume that $\mathcal{P}_k \subset \mathcal{P}_{k+1}$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, choose $m(k) \in \mathcal{P}_k$ such that $m(k) < m(k+1)$ for all $k \in \mathbb{N}$. Define $d_i^{(k)} = p_i^{(m(k))}$, $k \in \mathbb{N}$ and $1 \leqslant i \leqslant n$. Then $d_i = \Pi(\{d_i^{(k)}\})$ is a projection, $d_i d_j = 0$ if $i \neq j$ $(1 \leqslant i, j \leqslant n)$. Moreover,

$$\|d_i^{(k)} - (d_i^{(k)})^2\|_{_{2,QT(A)}} < 1/k, \tag{4.15}$$

$$\sup\{|\tau(d_i^{(k)}b) - \frac{1}{n}\tau(b)| : \tau \in QT(A)\} < 1/k, \tag{4.16}$$

$$\|[d_i^{(k)}, b]\| < 1/k, \ \ b \in \mathcal{F}_k \ \text{ and } \ 1 \leqslant i \leqslant n \ \text{ and} \tag{4.17}$$

$$\|[d_i^{(k)}, c_k]\| < 1/k, \ 1 \leqslant i \leqslant n. \tag{4.18}$$

It follows (by (4.18)) that

$$d_i q = q d_i, \ 1 \leqslant i \leqslant n. \tag{4.19}$$

Put $q_i = d_i q$, $i \in \mathbb{N}$. Then (also by (4.15)), $\{q_i : 1 \leqslant i \leqslant n\}$ are mutually orthogonal projections in $l^\infty(A)/I_{_{QT(A),\mathbb{N}}}$. For any $b \in A_1$, by (4.11), $q\Pi(\iota(b)) = \Pi(\iota(b))q = \Pi(\iota(b))$ in $l^\infty(A)/I_{_{QT(A),\mathbb{N}}}$. Then, for any $\tau \in QT_\varpi(A)$,

$$|\tau(d_i q b) - \tau(d_i b)| = 0. \tag{4.20}$$

It follows that $(1 \leqslant i \leqslant n)$

$$\lim_{k \to \varpi} \sup\{|\tau((d_i^{(k)} c_k) b) - \tau(d_i^{(k)} b)| : \tau \in QT(A)\} = 0. \tag{4.21}$$

Then, by (4.16),

$$\lim_{k \to \varpi} \sup\{|\tau((d_i^{(k)} c_k) b) - \frac{1}{n}\tau(b)| : \tau \in QT(A)\} = 0. \tag{4.22}$$

This also implies that, for $1 \leqslant i \leqslant n$,

$$\tau(q_i b) = \frac{1}{n}\tau(b) \ \text{ for all } \ \tau \in QT_\varpi(A_1) \ \text{ and } \ b \in A_1. \tag{4.23}$$

Put

$$J = \{\{b_k\} \in l^\infty(A_1) : \lim_{k \to \infty} \|b_k\|_{_{2,\overline{QT(A_1)}^w}} = 0\}.$$

Note that $\widetilde{QT}(A_1) = \mathbb{R}_+ \cdot \overline{QT(A_1)}^w$. Since $QT(A)$ is a basis for $\widetilde{QT}(A)$, we then have that (see also Proposition 2.18 of [14])

$$l^\infty(A_1) \cap I_{_{QT(A),\mathbb{N}}} = J. \tag{4.24}$$

By (4.10), (4.17) and (4.19),

$$q_i \Pi(\iota(b)) = \Pi(\iota(b)) q_i, \ \ 1 \leqslant i \leqslant n. \tag{4.25}$$

It remains to show that $q_i \in (l^\infty(A_1) \cap (A_1)')/J$.

By Central Surjectivity of Sato (since we do not assume A is even exact, we apply Proposition 3.10 of [13], see also Proposition 3.8 of [13] and Proposition 2.18 of [14]), we may assume that $q_i \in (l^\infty(A) \cap A')/I_{_{QT(A),\mathbb{N}}}$. The new lifting may be written as $\Pi(\{e_i^{(k)}\}) = q_i$, where $e_i^{(k)} \perp e_j^{(k)}$ for $i \neq j$ $(1 \leqslant i \leqslant n)$ and $\{e_i^{(k)}\} \in (A')_+^{\mathbf{1}}$ and $e_i^{(k)} = d_i^{(k)} c_k + h_k$ for some $\{h_k\} \in I_{_{QT(A),\mathbb{N}}}$. Put $f_i^{(k)} = c_k e_i^{(k)} c_k$, $1 \leqslant i \leqslant n$, $k \in \mathbb{N}$. Then $f_i^{(k)} \in (A_1)'$, since $\{c_k\} \in (A_1)'$. We still have $\Pi(\{f_i^{(k)}\}) = q_i$, $1 \leqslant i \leqslant n$. In other words, $q_i \in (l^\infty(A_1) \cap (A_1)')/J$, $1 \leqslant i \leqslant n$. This completes the proof. $\square$

Proposition 4.4. Let A be a separable simple C^*-algebra with nonempty $QT(A)$ which is compact. Suppose that A has hereditary uniform property Γ. Then A has uniform property Γ.

Proof. Choose any strictly positive element $e \in \mathrm{Ped}(A)_+ \setminus \{0\}$. Then $A_e = A$. Then (3.1) is the same as (4.1). $\square$

Remark 4.5. Theorem 4.3 states that, if a separable simple C^*-algebra A with strict comparison has uniform property Γ, then (4.1) holds for each $e \in \mathrm{Ped}(A \otimes \mathcal{K})_+^{\mathbf{1}}$. This fact may be regarded as the statement that, in this case, the uniform property Γ carries to hereditary C^*-subalgebras as well as $A \otimes \mathcal{K}$, if we restrict ourselves to hereditary C^*-subalgebras of $A \otimes \mathcal{K}$ which are algebraically simple, or rather, to those hereditary C^*-subalgebras of $A \otimes \mathcal{K}$ such that whose quasitraces are bounded. Recall that uniform property Γ is originally only defined on C^*-algebras with compact $T(A)$ (see Definition 2.1 of [3]). It seems to us that Definition 4.1 is an appropriate generalization of the uniform property Γ to separable simple C^*-algebras which do not have continuous scale. A more general version of uniform property Γ (where p_i is not required to be projection) which is called stabilized uniform property Γ was introduced in [5]. However, we prefer to keep the condition that each p_i is a projection intact. The proof of Theorem 4.3 uses the notion of tracial approximate oscillation zero. Theorem 4.9 below shows that if A has strict comparison and hereditary uniform property Γ, then this is also automatic. In particular, A has stable rank one.

Let A be a separable simple C^*-algebra with $T(A) = QT(A) \neq \emptyset$ which has strict comparison. Suppose that A has stabilized uniform property Γ in the sense of Definition 2.5 of [5]. Suppose that $K_0(A)_+ \neq \{0\}$. Then there is a projection $e \in A \otimes \mathcal{K} \setminus \{0\}$. Put $A_1 = e(A \otimes \mathcal{K})e$. Then A_1 is unital. Since A_1 also has stabilized uniform property Γ, A_1 has uniform property Γ (see Proposition 2.6 of [5]). By Theorem 4.3, A has hereditary uniform property Γ. More generally, if there is $e \in \mathrm{Ped}(A \otimes \mathcal{K})_+ \setminus \{0\}$ such that $d_\tau(e)$ is continuous. Set $A_1 = e(A \otimes \mathcal{K})e$. Then $T(A_1)$ is compact. Thus the same argument also implies that A_1 has hereditary uniform property Γ. This is the case if $\mathrm{Cu}(A) \cong \mathrm{Cu}(A \otimes \mathcal{Z})$. So under the assumption that $\mathrm{Cu}(A) \cong \mathrm{Cu}(A \otimes \mathcal{Z})$, the stabilized uniform property Γ is the same as hereditary uniform property Γ.

Theorem 4.6. Let A be a finite separable non-elementary simple C^*-algebra which are tracially approximately divisible (see Definition 5.2 of [15], for example). Then A has hereditary uniform property Γ.

Proof. It follows from Corollary 6.5 of [13] and the proof of Theorem 5.2 of [13] that $W(A)$ is almost unperforated and by Corollary 5.1 of [27] (see also Proposition 4.9 of [15]) that A has a non-zero 2-quasitrace. By Theorem 5.7 of [13], the map Γ is surjective. Choose $e \in \mathrm{Ped}(A \otimes \mathcal{K})_+^{\mathbf{1}} \setminus \{0\}$ such that $d_\tau(e)$ is continuous on $\overline{QT(A)}^w$ and $d_\tau(e) < r$ for all $\tau \in \overline{QT(A)}^w$ and $r \in (0, 1/2)$. By Theorem 6.7 of [13], A has stable rank one. So we may assume that $e \in \mathrm{Ped}(A)_+$.

Put $A_1 = \mathrm{Her}(e)$. Then A_1 has continuous scale (see, for example, Theorem 5.3 of [9]). By Theorem 5.5 of [15], A_1 is tracially approximate divisible. Now $QT(A_1)$ is compact and A_1 has strict comparison (see Theorem 5.2 of [13]).

Now fix $n \in \mathbb{N}$. By Theorem 4.11 of [13], there is a unital homomorphism $\psi : M_n \to (l^\infty(A_1) \cap (A_1)')/I_{_{QT(A_1),\varpi}}$ (note that $I_{_{QT(A_1),\mathbb{N}}} \subset I_{_{QT(A_1),\varpi}}$). Let $p_i = \psi(e_{i,i})$, $1 \leqslant i \leqslant n$. Then $p_i \in (l^\infty(A_1) \cap (A_1)')/I_{_{QT(A_1),\varpi}}$, $1 \leqslant i \leqslant n$, and, for any $a \in A$,

$$\tau(p_i a) = \tau(\varphi(e_{i,i})a) = \frac{1}{n}\tau(a) \ \text{ for all } \ \tau \in QT_\varpi(A_1), \ 1 \leqslant i \leqslant n. \tag{4.26}$$

In other words, A_1 has uniform property Γ. By Theorem 4.3, A_1 has hereditary uniform property Γ. By Brown's stable isomorphism theorem [2], A has hereditary uniform property Γ. $\square$

Remark 4.7. It is known that separable simple C^*-algebras with tracial rank zero are tracially approximately divisible (see Lemma 6.10 of [21]). In fact, any separable simple C^*-algebra A with tracial rank at most one are tracially approximately divisible (see the proof of Theorem 5.4 of [22]). Therefore, by Theorem 4.6, these C^*-algebras have hereditary uniform property Γ (and has strict comparison) but they may not be $\mathcal{Z}$-stable ([25], see also Example 6.10 of [13]).

Theorem 4.8. Let A be a separable simple C^*-algebra with strict comparison and $\widetilde{QT}(A) \setminus \{0\} \neq \emptyset$. Suppose that A has hereditary uniform property Γ. Then the map $\Gamma : \mathrm{Cu}(A) \to \mathrm{LAff}_+(\widetilde{QT}(A))$ is surjective.

Proof. The proof is almost exactly the same as that of Theorem 3.5. But $QT(A)$ will be replaced by $\overline{QT(A)}^w$. The formula (3.48) holds with $QT(A)$ being replaced by $\overline{QT(A)}^w$. The formula in (3.49) holds with $QT_\varpi(A)$ being replaced by $QT_\varpi^w(A)$. Inequalities (3.51) also holds with $QT(A)$ being replaced by $\overline{QT(A)}^w$. Moreover, we also have (3.52) holds with $QT(A)$ being replaced by $\overline{QT(A)}^w$. We then have

$$n[b] \leqslant [a] \ \text{ and } \ [f_\varepsilon(a)] \leqslant (n+1)[b] \tag{4.27}$$

as in the proof of Theorem 3.5. Note that this holds for any $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+^{\mathbf{1}}$ since we assume that A has hereditary uniform property Γ and we may begin with an element $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+^{\mathbf{1}}$. Then the same argument of L. Robert at in the proof of Theorem 3.5 implies that the map Γ is surjective. $\square$

Theorem 4.9. Let A be a separable simple C^*-algebra with strict comparison and $\widetilde{QT}(A) \setminus \{0\} \neq \emptyset$. Suppose that A has hereditary uniform property Γ. Then A has tracial approximate oscillation zero and stable rank one.

Proof. It follows from Theorem 4.8 that the map Γ is surjective. Choose $e \in \mathrm{Ped}(A)_+^{\mathbf{1}} \setminus \{0\}$ such that $d_\tau(e)$ is continuous on $\widetilde{QT}(A)$. Then $\mathrm{Her}(e)$ has continuous scale (see Theorem 5.3 of [9], for example). Since A has hereditary uniform property Γ, $\mathrm{Her}(e)$ has uniform property Γ. It follows from Theorem 3.7 that $\mathrm{Her}(e)$ has tracial approximate oscillation zero and stable rank one. By Brown's stable isomorphism theorem, A has tracial approximate oscillation zero and stable rank one. $\square$

Towards the Toms-Winter conjecture, as in [6] and [5], we have the following.

Theorem 4.10. Let A be a stably finite separable non-elementary amenable simple C^*-algebra. Then the following are equivalent:

(1) A has strict comparison and hereditary uniform property Γ,

(2) $A \cong A \otimes \mathcal{Z}$, and

(3) A has finite nuclear dimension.

Proof. The equivalence of (2) and (3) has been proved (see [4], [3], [35], [30] and [24]).

To see (2) $\Rightarrow$ (1), let A be $\mathcal{Z}$-stable. It is proved in [27] that A has strict comparison. By Theorem 5.9 of [15], A is tracially approximately divisible (see also Theorem 5.2 of [13]). Then, by Theorem 4.6, A has hereditary uniform property Γ.

For (1) $\Rightarrow$ (2), we note that, by Theorem 3.7, the map Γ is surjective. Choose $e \in \mathrm{Ped}(A)_+ \setminus \{0\}$ such that $A_1 = \mathrm{Her}(e)$ has continuous scale. Thus, by Proposition 4.4, A_1 has uniform property Γ. It follows from Theorem 4.6 of [6] that A_1 is uniformly McDuff. By Theorem 5.3 of [9], $T(A_1)$ is compact and A_1 has strict comparison. Then, by a version of Matui-Sato's result, for example, Proposition 4.4 of [7], A_1 is $\mathcal{Z}$-stable and hence A is $\mathcal{Z}$-stable. $\square$

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