Local noncollapsing for complex Monge–Ampère equations

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Abstract. We prove a local volume noncollapsing estimate for Kähler metrics induced from a family of complex Monge–Ampère equations, assuming a local Ricci curvature lower bound. This local volume estimate can be applied to establish various diameter and gradient estimate.

1. Introduction

The non-local collapsing of Ricci flow on Riemannian manifolds is a fundamental theorem for the compactness of Ricci flows [20], which states that along the flow the volume of the geodesic balls is uniformly bounded below by a positive constant, if the scalar curvature along the flow is bounded. In Kähler geometry, it has generated great interest to study degenerating families of Kähler metrics, satisfying certain complex Monge–Ampère (MA) equations [22]. Analogous to Ricci flow, it is tempting to study the non-local collapsing of Kähler metrics along a degenerating family.

Complex Monge–Ampère equations have been an important tool in the construction of canonical Kähler metrics, ever since Yau's solution to the Calabi conjecture [26]. A priori estimates on the Kähler potentials are the keys to study the geometry of the Kähler metrics. By using pluripotential theory, Kołodziej [15] first proved the sharp C^0 estimate for complex MA equations, assuming the right side belongs to some Orlicz space. Kołodziej's approach was generalized to a family of complex MA equations, allowing the Kähler classes to degenerate in [4–6, 16, 27]. Recently, a PDE-based method [9] was utilized to give a new and uniform proof of C^0 estimates for a class of fully nonlinear partial differential equations, which include in particular the complex MA and Hessian equations. In [7], X. Fu and the authors first revealed the relationship between the C^0 estimate of the Kähler potentials with the global diameter bound of the Kähler metrics, generalizing the distance estimate in [22]. The current paper aims to study how the C^0 estimates of Kähler potentials affect the geometry locally.

Let (X, ω_X) be a compact Kähler manifold of complex dimension n. Suppose that χ is a smooth closed $(1, 1)$-form such that the class $[\chi]$ is *nef*, which means the class $[\chi]$ lies in the

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closure of the Kähler cone of X. For $t \in (0, 1]$, we consider a family of smooth closed forms

$$\hat{\omega}_t = \chi + t\omega_X.$$

Though $\hat{\omega}_t$ may not be Kähler forms, each $[\hat{\omega}_t]$ is a Kähler class. We consider the following complex Monge–Ampère equation:

$$(1.1) \qquad (\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^n = c_t e^F \omega_X^n, \quad \sup_X \varphi_t = 0,$$

where F is a smooth function normalized such that

$$\int_X e^F \omega_X^n = \int_X \omega_X^n = V, \quad c_t = \frac{V_t}{V} = O(t^{n-\nu}), \quad V_t = \int_X \hat{\omega}_t^n$$

and $\nu \in \{0, \ldots, n\}$ is the numerical dimension of the class $[\chi]$.

We denote $\omega_t = \hat{\omega}_t + i\partial\bar{\partial}\varphi_t$ to be the Kähler metric satisfying (1.1) and write g_t for the associated Riemannian metric. The following is our main theorem.

Theorem 1.1. *For any $p > n$ and $R_0 \in (0, 1]$, if ω_t solves (1.1) for $t \in (0, 1]$ and the Ricci curvature* $\mathrm{Ric}(g_{t_0})$ *satisfies*

$$\mathrm{Ric}(g_{t_0}) \geq -\frac{K^2}{R_0^2} \quad \text{on } B_{g_{t_0}}(z_0, 2R_0)$$

for some $K \geq 0$, $t_0 \in (0, 1]$ and $z_0 \in X$, then

$$(1.2) \qquad \frac{\mathrm{Vol}_{g_{t_0}}(B_{g_{t_0}}(z_0, R_0))}{\mathrm{Vol}_{g_{t_0}}(X)} \geq CR_0^{\frac{2np}{p-n}}$$

for some constant $C > 0$ depending only on n, p, K, ω_X, χ and $\|e^F\|_{L^1(\log L)^p}$.

We remark that if $p \to \infty$, then the power exponent of R_0 in (1.2) tends to $2n$, which is natural since the metrics ω_t are close to Euclidean ones when R_0 is small. However, an example on Riemann surfaces (i.e. $n = 1$) shows that this power exponent $\frac{2np}{p-n}$ is sharp in some sense (see Example 3.1). If $e^F \in L^q(X, \omega_X^n)$ for some $q > 1$, the right side of (1.2) can be made as $C_\varepsilon R_0^{2n+\varepsilon}$ for any $\varepsilon > 0$ with the constant $C_\varepsilon > 0$ depending additionally on ε.

Estimate (1.2) can be viewed as an analogue of Perelman's κ-noncollapsing theorem [20] as the following. Let $g(t)$ be the smooth solution of the Ricci flow on a real n-dimensional compact Riemannian M for $t \in [0, T)$ with the initial metric g_0. Let $(x_0, t_0) \in M \times [0, T)$ and $r_0 \in (0, (t_0)^{\frac{1}{2}})$. If the scalar curvature satisfies

$$\mathrm{R} < r_0^{-2} \quad \text{on } B_{g(t_0)}(x_0, r_0),$$

then

$$\mathrm{Vol}_{g(t_0)}(B(x_0, t_0, r_0)) \geq \kappa r_0^n$$

for some constant $\kappa > 0$ that only depends on n and $\nu[g_0, 2T]$, Perelman's μ-functional at the initial time.

We define the *radius of Ricci curvature lower bound* at z for the metric g_t by

$$\bar{r}_{g_t}(z) = \sup\left\{r > 0 : \mathrm{Ric}(g_t) \geq -\frac{1}{r^2} \text{ in } B_{g_t}(z, r)\right\}$$

as an analogue of the curvature radius introduced in [1]. An equivalent way to state Theorem 1.1 is that under the same setup

$$\frac{\text{Vol}_{g_t}(B_{g_t}(z, \bar{r}_{g_t}(z)))}{\text{Vol}_{g_t}(X)} \geq C \bar{r}_{g_t}(z)^{\frac{2np}{p-n}} \quad \text{for any } z \in X$$

for some constant $C > 0$ depending only on n, p, ω_X, χ and $\|e^F\|_{L^1(\log L)^p}$.

The main idea of the proof of Theorem 1.1 is motivated by that in [9, 10], that is, we compare the Kähler metric ω_t to some auxiliary complex Monge–Ampère equation, with the function on the right side being truncated squared distance function of g_t. Additionally we also need the Riemannian geometric tools like the Laplacian comparison and volume comparison theorems.

Theorem 1.1 immediately implies the following diameter bound established in [7], by choosing $R_0 = 1$ in Theorem 1.1.

Corollary 1.1. *For any $p > n$, if ω_t solves* (1.1) *for $t \in (0, 1]$ and the Ricci curvature* $\text{Ric}(g_t)$ *is bounded below by*

$$\text{Ric}(g_t) \geq -K^2$$

for some $K \geq 0$, then there exists $C > 0$ depending only on n, p, K, ω_X, χ and $\|e^F\|_{L^1(\log L)^p}$ such that

$$\text{Diam}(X, g_t) \leq C$$

for all $t \in (0, 1]$.

We also establish a local gradient estimate as an application of Theorem 1.1. We consider the following complex Monge–Ampère equation:

$$(1.3) \qquad (\omega_X + i\partial\bar{\partial}\varphi)^n = e^F \omega_X^n, \quad \sup_X \varphi = 0$$

on a compact Kähler manifold (X, ω_X) of complex dimension n with $F \in C^\infty(X)$ satisfying the normalization condition $\int_X e^F \omega_X^n = \int_X \omega^n$.

Theorem 1.2. *Let φ be the solution of equation* (1.3) *and let g be the Kähler metric corresponding to the Kähler form $\omega = \omega_X + i\partial\bar{\partial}\varphi$. Given $p > n$ and $R_0 \in (0, 1]$, if*

$$\text{Ric}(g) \geq -\frac{K^2}{R_0^2} \quad \text{on } B_g(z_0, 2R_0)$$

for some $K \geq 0$ and $z_0 \in X$, then

$$|\nabla\varphi|_g^2 \leq C \quad \text{on } B_g(z_0, R_0)$$

for some constant $C > 0$ depending on n, p, ω_X, $\|e^F\|_{L^1(\log L)^p}$, K and R_0.

We will prove Theorem 1.1 in Section 2. In Section 3, we construct an example and show that the exponent in (1.2) is sharp. In Section 4, we will prove Theorem 1.2 and discuss some other applications of Theorem 1.1 and Corollary 1.1 on the diameter bound of Kähler metrics satisfying certain complex Monge–Ampère equations.

2. Proof of Theorem 1.1

Since $\widehat{\omega}_t$ may not be Kähler, we define the "envelope" associated to $\widehat{\omega}_t$:

$$\mathcal{V}_t = \sup\{v : v \in PSH(X, \widehat{\omega}_t), \; v \le 0\}.$$

It is known that for each $t \in (0, 1]$, $\mathcal{V}_t$ is a $C^{1,1}$ function. We recall the following uniform L^∞-estimates on φ_t, the solution to (1.1).

Lemma 2.1 ([7, 11]).   *There is a constant $C_0 > 0$ depending only on n, p, ω_X, χ and $\|e^F\|_{L^1(\log L)^p}$ such that*

$$\sup_X |\varphi_t - \mathcal{V}_t| \le C_0 \quad \text{for all } t \in (0, 1].$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1.   We break the proof into four steps.

Step 1.   We fix a family of smooth positive functions $\eta_k : \mathbb{R} \to \mathbb{R}_+$ such that $\eta_k(x)$ converges uniformly and decreasingly to the function $x \cdot \chi_{\mathbb{R}_+}(x)$ as $k \to \infty$. We solve the auxiliary complex Monge–Ampère equations

$$(2.1) \qquad (\widehat{\omega}_t + i\partial\bar{\partial}\psi_{t,k})^n = c_t \frac{\eta_k(R_0^2 - d_t^2)}{A_{k,t}} e^F \omega_X^n, \quad \sup_X \psi_{t,k} = 0,$$

where

$$0 < A_{k,t} = \frac{c_t}{V_t} \int_X \eta_k(R_0^2 - d_t^2) e^F \omega_X^n$$

is the normalizing constant making equation (2.1) solvable by Yau's theorem [26]. Here we write $d_t(x) = d_{g_t}(x, z_0)$ to be the geodesic distance of x to the fixed point z_0 under the *varying* metric g_t. We note that although the positive function on the right-hand side of (2.1) is only Lipschitz but not necessarily smooth, the solution $\psi_{t,k}$ is still in $C^{2,\alpha}(X)$ for some $\alpha > 0$, which follows from the regularity theory of complex Monge–Ampère equations (see, e.g., [2, 19]). We observe that by the dominated convergence theorem

$$A_{k,t} \to A_t := \frac{c_t}{V_t} \int_{B_{g_t}(z_0, R_0)} (R_0^2 - d_t^2) e^F \omega_X^n \quad \text{as } k \to \infty.$$

Step 2.   We aim to compare $\psi_{t,k}$ with the solution φ_t. As in [9, 10], we look at the test function

$$\Phi := -\varepsilon(-\psi_{t,k} + \varphi_t + C_1)^{\frac{n}{n+1}} + (R_0^2 - d_t^2),$$

where $C_1 = C_0 + 1$ and $C_0 > 0$ is the constant in Lemma 2.1, and $\varepsilon > 0$ is chosen as

$$(2.2) \qquad \varepsilon = \left(\frac{n^3 + (n+1)(4n+2K)}{n^2}\right)^{\frac{n}{n+1}} A_{k,t}^{\frac{1}{1+n}} =: C_2 A_{k,t}^{\frac{1}{1+n}},$$

where we fix the constant $C_2 > 0$ which depends on n and K. As an initial observation we note that on X

$$-\psi_{t,k} + \varphi_t + C_1 = -(\psi_{t,k} - \mathcal{V}_t) + (\varphi_t - \mathcal{V}_t) + C_0 + 1 \ge 1,$$

by Lemma 2.1 and the fact $\psi_{t,k} \le \mathcal{V}_t$ for each k. Thus the function $\Phi < 0$ on $X \backslash B_{g_t}(z_0, R_0)$.

We claim that $\Phi \leq 0$ on X. To prove this, let $x_{\max} \in X$ be a maximum point of Φ. If $x_{\max} \notin B_{g_t}(z_0, R_0)$, we are done. So we may assume $x_{\max} \in B_{g_t}(z_0, R_0)$. Applying Calabi's trick if necessary [21], we may assume d_t^2 is smooth at $x_{\max}$. Then at $x_{\max}$ we have (write $\Delta = \Delta_{\omega_t}$)

$$
\begin{aligned}
0 &\geq \Delta_{\omega_t} \Phi(x_{\max}) \\
&= \frac{n\varepsilon}{n+1}(-\psi_{t,k} + \varphi_t + C_1)^{-\frac{1}{1+n}}(\Delta\psi_{t,k} - \Delta\varphi_t) \\
&\quad + \frac{n\varepsilon}{(n+1)^2}(-\psi_{t,k} + \varphi_t + C_1)^{-\frac{n+2}{n+1}}|\nabla(\psi_{t,k} - \varphi_t)|_{\omega_t}^2 - \Delta d_t^2 \\
&\geq \frac{n\varepsilon}{n+1}(-\psi_{t,k} + \varphi_t + C_1)^{-\frac{1}{1+n}}(\mathrm{tr}_{\omega_t}\,\omega_{\psi_{t,k}} - \mathrm{tr}_{\omega_t}\,\omega_t) - 2d_t\Delta d_t - 2 \\
&\geq \frac{n^2\varepsilon}{n+1}(-\psi_{t,k} + \varphi_t + C_1)^{-\frac{1}{1+n}}\left(\frac{\omega_{\psi_{t,k}}^n}{\omega_t^n}\right)^{\frac{1}{n}} - \frac{n^2\varepsilon}{n+1}(-\psi_{t,k} + \varphi_t + C_1)^{-\frac{1}{1+n}} \\
&\quad - 2d_t\left(\frac{2n-1}{d_t} + \frac{K}{R_0}\right) - 2 \\
&\geq \frac{n^2\varepsilon}{n+1}(-\psi_{t,k} + \varphi_t + C_1)^{-\frac{1}{1+n}}\left(\frac{\eta_k(R_0^2 - d_t^2)}{A_{k,t}}\right)^{\frac{1}{n}} - \frac{n^2\varepsilon}{n+1} - 4n - 2K \\
&\geq \frac{n^2\varepsilon}{n+1}(-\psi_{t,k} + \varphi_t + C_1)^{-\frac{1}{1+n}}\left(\frac{R_0^2 - d_t^2}{A_{k,t}}\right)^{\frac{1}{n}} - \frac{n^2\varepsilon}{n+1} - 4n - 2K,
\end{aligned}
$$

where we write $\omega_\phi = \hat\omega_t + i\partial\bar\partial\phi$ for a function $\phi \in \mathrm{PSH}(X, \hat\omega_t)$, in the fifth line we apply the arithmetic-geometric inequality and the fact $\mathrm{tr}_{\omega_t}\,\omega_t = n$, in the sixth line we apply the Laplace comparison theorem [21] of the distance function which holds under our assumption that $\mathrm{Ric}(\omega_t) \geq -K^2 R_0^{-2}$ on the geodesic ball $B_{g_t}(z_0, 2R_0)$, in the seventh line we use the equations satisfied by $\psi_{t,k}$ and φ_t and in the last line we apply the choice of η_k which satisfies $\eta_k(s) \geq s$ for $s \geq 0$. Hence at $x_{\max}$ we have

$$
\begin{aligned}
R_0^2 - d_t^2 &\leq A_{k,t}\left(n + \frac{(n+1)(4n+2K)}{n^2\varepsilon}\right)^n(-\psi_{t,k} + \varphi_t + C_1)^{\frac{n}{n+1}} \\
&< \varepsilon(-\psi_{t,k} + \varphi_t + C_1)^{\frac{n}{n+1}},
\end{aligned}
$$

by the choice the ε in (2.2). This finishes the proof of the claim that $\Phi \leq 0$ on X.

Step 3. From $\Phi \leq 0$ we infer that on $B_{g_t}(z_0, R_0)$

$$
\frac{(R_0^2 - d_t^2)^{\frac{n+1}{n}}}{A_{k,t}^{1/n}} \leq C_2^{\frac{n+1}{n}}(-\psi_{t,k} + \varphi_t + C_1).
$$

We can view $\psi_{t,k}$ as a $C_3\omega_X$-PSH function for some $C_3 > 0$ depending only on χ and ω_X. So the Hörmander–Tian estimate [14, 24] holds for each $\psi_{t,k}$. We thus have a small constant $\alpha = \alpha(\omega_X, \chi, K) > 0$ such that the following Trudinger-type inequality holds:

$$
(2.3) \qquad \int_{B_{g_t}(z_0,R_0)} e^{\alpha\frac{(R_0^2 - d_t^2)^{\frac{n+1}{n}}}{A_{k,t}^{1/n}}}\omega_X^n \leq \int_X e^{\alpha C_2^{\frac{n+1}{n}}(-\psi_{t,k} + \varphi_t + C_1)}\omega_X^n \leq C
$$

for some uniform constant $C > 0$, where we have chosen $\alpha > 0$ small so that $\alpha C_2^{\frac{n+1}{n}}$ is smaller than the α-invariant of $(X, C_3\omega_X)$.

Step 4.   It then follows from a generalized Young's inequality that

$$v^p e^F \le e^F (1 + |F|^p) + C_p e^{2v}$$

which applied to $v = \alpha \frac{(R_0^2 - d_t^2)^{\frac{n+1}{n}}}{2A_{k,t}^{1/n}}$ yields by (2.3) that

$$(2.4) \qquad \int_{B_{g_t}(z_0, R_0)} (R_0^2 - d_t^2)^{\frac{n+1}{n} p} e^F \omega_X^n \le C A_{k,t}^{\frac{p}{n}}$$

for some constant $C > 0$ depending on n, p, ω_X, χ, K and $\|e^F\|_{L^1(\log L)^p}$. Letting $k \to \infty$, inequality (2.4) implies that

$$\int_{B_{g_t}(z_0, R_0)} (R_0^2 - d_t^2)^{\frac{n+1}{n} p} e^F \omega_X^n \le C A_t^{\frac{p}{n}}.$$

On the other hand, by the Hölder inequality we have

$$\begin{aligned}
A_t &= \frac{c_t}{V_t} \int_{B_{g_t}(z_0, R_0)} (R_0^2 - d_t^2) e^F \\
&\le \left(\int_{B_{g_t}(z_0, R_0)} (R_0^2 - d_t^2)^{\frac{n+1}{n} p} e^F \omega_X^n \right)^{\frac{n}{p(n+1)}} \cdot \left(\int_{B_{g_t}(z_0, R_0)} e^F \omega_X^n \right)^{\frac{1}{q}} \\
&\le C A_t^{1/(n+1)} \left(\int_{B_{g_t}(z_0, R_0)} e^F \omega_X^n \right)^{\frac{1}{q}},
\end{aligned}$$

where $q = \frac{p(n+1)}{p(n+1)-n}$ is the conjugate of p. We thus conclude that

$$(2.5) \qquad A_t \le C \left(\int_{B_{g_t}(z_0, R_0)} e^F \omega_X^n \right)^{\frac{n+1}{nq}} = C \left(\int_{B_{g_t}(z_0, R_0)} e^F \omega_X^n \right)^{1 + \frac{p-n}{np}}.$$

Note that $\frac{c_t}{V_t}$ is uniformly bounded, so (2.5) shows that there exists a uniform constant $C > 0$ such that

$$\int_{B_{g_t}(z_0, R_0)} (R_0^2 - d_t^2) e^F \omega_X^n \le C \left(\int_{B_{g_t}(z_0, R_0)} e^F \omega_X^n \right)^{1 + \frac{p-n}{np}}.$$

In particular, we obtain

$$(2.6) \qquad R_0^2 \int_{B_{g_t}(z_0, R_0/2)} e^F \omega_X^n \le C \left(\int_{B_{g_t}(z_0, R_0)} e^F \omega_X^n \right)^{1 + \frac{p-n}{np}}.$$

Multiplying c_t on both sides of (2.6), we get

$$(2.7) \qquad R_0^2 \, \mathrm{Vol}_{g_t}\left(B_{g_t}\left(z_0, \frac{R_0}{2}\right) \right) \le \frac{C}{c_t^{\frac{p-n}{np}}} (\mathrm{Vol}_{g_t}(B_{g_t}(z_0, R_0)))^{1 + \frac{p-n}{np}}.$$

By volume comparison [21], the function $(0, 2R_0) \ni r \mapsto r^{-2n} e^{-\frac{K}{R_0} r} \, \mathrm{Vol}_{g_t}(B_{g_t}(z_0, r))$ is non-increasing, which implies

$$\mathrm{Vol}_{g_t}(B_{g_t}(z_0, R_0)) \le C(n, K) \, \mathrm{Vol}_{g_t}\left(B_{g_t}\left(z_0, \frac{R_0}{2}\right) \right).$$

Combined with (2.7), this implies that

$$R_0^{\frac{2np}{p-n}} V_t \leq C \, \mathrm{Vol}_{g_t}(B_{g_t}(z_0, R_0)),$$

where as usual $V_t = \int_X \omega_t^n$ and $C > 0$ is a uniform constant. This finishes the proof of Theorem 1.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. An example

In this section, we will construct an example and demonstrate that the exponent $\frac{2np}{p-n}$ is sharp in the estimate of Theorem 1.1. In other words, the estimate in Theorem 1.1 may fail with the exponent replaced by $\frac{2np}{p-n} - \varepsilon$ for any $\varepsilon > 0$.

Example 3.1. Let $\mathbf{D} \subset \mathbb{C} \subset \mathbb{CP}^1$ be the disk with radius $\frac{1}{2}$. Consider the function $\varphi(z) = (-\log|z|^2)^{-a}$ for some $a > 0$. We calculate the "metric" defined by $i\partial\bar{\partial}\varphi$:

$$\omega = i\partial\bar{\partial}\varphi = a(a+1)\frac{i\,dz \wedge d\bar{z}}{|z|^2(-\log|z|^2)^{a+2}} = e^F i\,dz \wedge d\bar{z}.$$

Straightforward calculations show that $\|e^F\|_{L^1(\log L)^p(\mathbf{D})}$ is bounded for any $1 < p < a+1$ and is unbounded when $p \geq a+1$. Fix a point $z_0 \in \mathbf{D}\setminus\{0\}$ close to 0, and a point $w \in \mathbf{D}$ with $\arg z_0 = \arg w$. By the rotational symmetry of ω, we see that

$$d_\omega(z_0, w) = \sqrt{a(a+1)} \int_{\min\{|z_0|, |w|\}}^{\max\{|z_0|, |w|\}} \frac{dr}{r(-\log r^2)^{1+\frac{a}{2}}}$$
$$= \sqrt{\frac{a+1}{2^a a}} \left(\frac{1}{(-\log\max\{|z_0|, |w|\})^{\frac{a}{2}}} - \frac{1}{(-\log\min\{|z_0|, |w|\})^{\frac{a}{2}}} \right).$$

In particular, letting $w \to 0$, we see that

$$d_\omega(z_0, 0) = \sqrt{\frac{a+1}{2^a a}} \frac{1}{(-\log|z_0|)^{\frac{a}{2}}} =: 6R_0 > 0.$$

Take $z_0^\pm \in \mathbf{D}$ with the same arguments as z_0 and

$$|z_0^+| = |z_0|^{2^{\frac{2}{a}}} < |z_0| \quad \text{and} \quad |z_0^-| = |z_0|^{(\frac{2}{3})^{\frac{a}{2}}} > |z_0|$$

so that

$$(-\log|z_0^+|)^{a/2} = 2(-\log|z_0|)^{\frac{a}{2}} \quad \text{and} \quad (-\log|z_0^-|)^{\frac{a}{2}} = \frac{2}{3}(-\log|z_0|)^{\frac{a}{2}}.$$

Then it follows that

$$d_\omega(z_0, z_0^\pm) = \frac{1}{2}\sqrt{\frac{a+1}{2^a a}} \frac{1}{(-\log|z_0|)^{\frac{a}{2}}} = 3R_0.$$

On the other hand, the length of the circles around 0 in the annulus $\{|z_0^+| \leq |z| \leq |z_0^-|\}$ is given by (again by the rotational symmetry of ω these circles are ω-geodesics)

$$L_\omega(\text{circle}) = \sqrt{a(a+1)} \int_0^{2\pi} \frac{d\theta}{(-\log|z|^2)^{1+\frac{a}{2}}} = \frac{2\pi\sqrt{a(a+1)}}{(-\log|z|^2)^{1+\frac{a}{2}}} < \frac{R_0}{10}$$

if $|z_0| > 0$ small enough. This implies that

$$B_\omega(z_0, 2R_0) \subset \{|z_0^+| \le |z| \le |z_0^-|\} \subset B_\omega(z_0, 4R_0) \subset \mathbf{D}\backslash\{0\}.$$

By straightforward calculations, we have

$$\begin{aligned}
\mathrm{Ric}(\omega) &= i\partial\bar\partial \log(|z|^2(-\log|z|^2)^{a+2}) \\
&= -\frac{a+2}{|z|^2(-\log|z|^2)^2} i\,dz\wedge d\bar z \\
&= -\frac{a+2}{a(a+1)}(-\log|z|^2)^a \cdot \omega \\
&\ge -\frac{K^2}{R_0^2}\omega \quad \text{in the annulus } \{|z_0^+| \le |z| \le |z_0^-|\} =: \mathcal{A},
\end{aligned}$$

where $K > 0$ is a constant depending only on $a > 0$. Finally, we calculate the ω-volume of the above annulus as

$$\mathrm{Vol}_\omega(\mathcal{A}) = \int_{|z_0^+|}^{|z_0^-|} \frac{2\pi r\,dr}{r^2(-\log r^2)^{a+2}} = \frac{C_a}{(-\log|z_0|)^{a+1}} = C_a' R_0^{\frac{2(a+1)}{a}}.$$

Note that $e^F \in L^1(\log L)^p$ for *any* $1 < p < a+1$ and $\mathrm{Vol}_\omega(\mathcal{A}) \ge \mathrm{Vol}_\omega(B_\omega(z_0, 2R_0))$. We see that the exponent $\frac{2np}{p-n}$ of R_0 in Theorem 1.1 is sharp. Moreover, the Ricci curvature $\mathrm{Ric}(\omega)$ is not bounded below on the whole $\mathbf{D}$ since it decays to $-\infty$ near 0.

Though ω in this example is a "singular" Kähler metric on a local domain, we can regularize it near 0 to make it a genuine Kähler metric, and glue it to $\mathbb{CP}^1$ to get an example on compact Kähler manifolds (cf., e.g., [23]). One can naturally generalize the above example to higher dimensions and indeed the exponent $\frac{2np}{p-n}$ is sharp.

4. Applications of Theorem 1.1

We discuss some geometric applications of the noncollapsing result in Theorem 1.1. We will show a local gradient estimate of the Kähler potential, if the Ricci curvature is bounded below locally. Under certain assumption on the Ricci curvature lower bound, we will prove the diameter bound and a local noncollapsing result for Kähler metrics along the normalized Kähler–Ricci flow on minimal Kähler manifolds.

4.1. MA equations with a fixed background Kähler metric.   As the first application of Theorem 1.1, we prove Theorem 1.2 as generalization and a new proof of the global gradient estimate in [7].

Let (X, ω_X) be a given compact Kähler manifold. We consider the following complex Monge–Ampère equation:

$$(4.1) \qquad\qquad (\omega_X + i\partial\bar\partial\varphi)^n = e^F \omega_X^n, \quad \sup_X \varphi = 0.$$

Under the assumption of locally Ricci curvature lower bound, we prove the following local gradient estimate on φ. We write $\omega = \omega_X + i\partial\bar\partial\varphi$, which satisfies (4.1) and denote g the associated Riemannian metric of ω. We recall the statement of Theorem 1.2 below.

Theorem 4.1. *Given $p > n$ and $R_0 \in (0, 1]$, suppose $\mathrm{Ric}(g) \geq -\frac{K^2}{R_0^2}$ on the geodesic ball $B_g(z_0, 2R_0)$. Then*

$$|\nabla\varphi|_g^2 \leq C \quad on \ B_g(z_0, R_0)$$

for some constant $C > 0$ depending on n, p, ω_X, $\|e^F\|_{L^1(\log L)^p}$, K and R_0.

We first recall the L^∞ estimate on φ in [9, 15]:

(4.2) $$\|\varphi\|_{L^\infty} \leq \overline{C}(n, p, \omega_X, \|e^F\|_{L^1(\log L)^p}).$$

To prove Theorem 4.1, we need the following mean value inequality [21].

Lemma 4.1. *Let (M, g) be a Riemannian manifold such that $B_g(p, 2R_0)$ is relatively compact in M with $R_0 \in (0, 1]$. Suppose that $u \geq 0$ is a nonnegative function on $B_g(p, 2R_0)$ satisfying*

$$\Delta_g u \geq -\frac{A^2}{R_0^2} \quad and \quad \mathrm{Ric}(g) \geq -\frac{K^2}{R_0^2} \quad on \ B_g(p, 2R_0).$$

Then the following mean value inequality holds:

$$\sup_{B_g(p,\tau R_0)} u^2 \leq \frac{C}{\mathrm{Vol}_g(B_g(p, 2R_0))} \int_{B_g(p,2R_0)} (u^2 + 1) \, dV_g$$

for some $C > 0$ depending on $\tau \in [1, \frac{4}{3}]$, K and A.

Proof. We consider $v = u + 1 \geq 1$, and v satisfies

$$\Delta_g v \geq -\frac{A^2}{R_0^2} v.$$

We look at the product manifold $B_g(p, 2R_0) \times \mathbb{R}$ with the metric $\widehat{g} = g + ds^2$, where $s \in \mathbb{R}$ is the natural coordinate. Define

$$\widehat{v} = e^{\frac{As}{R_0}} v$$

to be function on $B_g(p, 2R_0) \times \mathbb{R}$, and it satisfies $\Delta_{\widehat{g}} \widehat{v} \geq 0$. Clearly

$$\mathrm{Ric}(\widehat{g}) \geq -\frac{K^2}{R_0^2}$$

and we can then apply the standard mean value inequality (cf. [21, Chapter 2, Theorem 6.2]) to conclude that

$$\sup_{B_g(p,\tau R_0)\times(-\tau R_0,\tau R_0)} \widehat{v}^2 \leq \frac{C}{R_0 \, \mathrm{Vol}_g(B_g(p, 2R_0))} \int_{-2R_0}^{2R_0} \int_{B_g(p,2R_0)} \widehat{v}^2 \, dV_g \, ds$$

$$\leq \frac{C}{\mathrm{Vol}_g(B_g(p, 2R_0))} \int_{B_g(p,2R_0)} v^2 \, dV_g \, ds$$

from which the lemma follows. $\square$

Lemma 4.2. *Under the same assumptions as in Theorem 4.1, we have*

$$\mathrm{tr}_\omega \omega_X \leq \exp(C R_0^{-\frac{np}{p-n}}) \quad on \ B_g\left(z_0, \frac{3R_0}{2}\right)$$

for some $C > 0$ depending on n, p, ω_X, $\|e^F\|_{L^1(\log L)^p}$ and $K \geq 0$.

Proof.   It follows from the Schwarz-lemma-type inequality that on $B_g(z_0, 2R_0)$

$$(4.3) \qquad \Delta_\omega \operatorname{tr}_\omega \omega_X \geq -\frac{K^2}{R_0^2} \operatorname{tr}_\omega \omega_X - C_0 (\operatorname{tr}_\omega \omega_X)^2 + \frac{|\nabla \operatorname{tr}_\omega \omega_X|_\omega^2}{\operatorname{tr}_\omega \omega_X},$$

where $C_0 > 0$ is an upper bound of the bisectional curvature of ω_X on $B_g(z_0, 2R_0)$. Straightforward calculations show that

$$\Delta_\omega \log \operatorname{tr}_\omega \omega_X \geq -\frac{K^2}{R_0^2} - C_0 \operatorname{tr}_\omega \omega_X.$$

Hence the function $u := (\log \operatorname{tr}_\omega \omega_X)_+ - C_0 \varphi$ satisfies

$$\Delta u \geq -\frac{K^2}{R_0^2} - C_0 n = -\frac{A^2}{R_0^2} \quad \text{with } A^2 = K^2 + C_0 n R_0^2.$$

We apply Lemma 4.1 to conclude that (we denote $B_R = B_g(z_0, R)$ for simplicity)

$$\sup_{B_{3R_0/2}} [(\log \operatorname{tr}_\omega \omega_X)_+]^2 \leq 2 \sup_{B_{3R_0/2}} u^2 + 2(C_0 \overline{C})^2$$

$$\leq \frac{C}{\operatorname{Vol}_g(B_{2R_0})} \int_{B_{2R_0}} [(\log \operatorname{tr}_\omega \omega_X)_+]^2 \, dV_g + C$$

$$\leq \frac{C}{R_0^{2np/(p-m)}} \int_X \operatorname{tr}_\omega \omega_X \, \omega^n + C \leq \frac{C}{R_0^{2np/(p-n)}},$$

where in the third line we use the calculus inequality $[(\log x)_+]^2 \leq x$ for any $x > 0$ and Theorem 1.1. Hence we have

$$\sup_{B_{3R_0/2}} \operatorname{tr}_\omega \omega_X \leq \exp\left(CR_0^{-\frac{np}{p-n}}\right). \qquad \square$$

Proof of Theorem 4.1.   By the Bochner formula, we have on $B_g(z_0, 2R_0)$

$$\Delta_\omega |\nabla \varphi|_\omega^2 = |\nabla \nabla \varphi|_\omega^2 + |\nabla \bar{\nabla} \varphi|_\omega^2 + \operatorname{Ric}_g(\nabla \varphi, \bar{\nabla} \varphi) - 2 \operatorname{Re}\langle \nabla \varphi, \bar{\nabla} \operatorname{tr}_\omega \omega_X \rangle_\omega$$

$$\geq |\nabla \nabla \varphi|_\omega^2 + |\nabla \bar{\nabla} \varphi|_\omega^2 - K^2 R_0^{-2} |\nabla \varphi|_\omega^2 - 2|\nabla \varphi||\nabla \operatorname{tr}_\omega \omega_X|_\omega,$$

which by Kato's inequality[1] implies that

$$\Delta_\omega |\nabla \varphi|_\omega \geq -K^2 R_0^{-2} |\nabla \varphi|_\omega - |\nabla \operatorname{tr}_\omega \omega_X|_\omega.$$

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[1] Kato's inequality states that $2|\nabla|\nabla \varphi||_\omega^2 \leq |\nabla \nabla \varphi|_\omega^2 + |\nabla \bar{\nabla} \varphi|_\omega^2$, which can be verified by the Cauchy–Schwarz inequality. It suffices to verify this inequality when $|\nabla \varphi| > 0$. In normal coordinates of ω, we calculate

$$4|\nabla \varphi|^2 |\nabla|\nabla \varphi||^2 = |\nabla|\nabla \varphi|^2|^2$$

$$= (\varphi_i \varphi_{\bar{i}})_{\bar{j}} (\varphi_k \varphi_{\bar{k}})_j$$

$$= \varphi_{i\bar{j}} \varphi_{\bar{i}} \varphi_{kj} \varphi_{\bar{k}} + \varphi_{i\bar{j}} \varphi_{\bar{i}} \varphi_k \varphi_{\bar{k}j} + \varphi_i \varphi_{\bar{i}\bar{j}} \varphi_{kj} \varphi_{\bar{k}} + \varphi_i \varphi_{\bar{i}\bar{j}} \varphi_k \varphi_{\bar{k}j}$$

$$\leq 2|\nabla \varphi|^2 |\nabla \nabla \varphi||\nabla \bar{\nabla} \varphi| + |\nabla \varphi|^2 |\nabla \bar{\nabla} \varphi|^2 + |\nabla \varphi|^2 |\nabla \nabla \varphi|^2$$

$$= |\nabla \varphi|^2 (|\nabla \nabla \varphi| + |\nabla \bar{\nabla} \varphi|)^2 \leq 2|\nabla \varphi|^2 (|\nabla \nabla \varphi|^2 + |\nabla \bar{\nabla} \varphi|^2).$$

Cancelling $2|\nabla \varphi|^2$ on both sides gives the desired Kato's inequality.

Define $u = |\nabla\varphi|_\omega + \varphi^2 + (\mathrm{tr}_\omega \, \omega_X)^2$. We calculate using (4.3)

$$\Delta_\omega u \geq -K^2 R_0^{-2}|\nabla\varphi|_\omega - |\nabla \, \mathrm{tr}_\omega \, \omega_X| + 2|\nabla\varphi|_\omega^2 + 2n\varphi - 2\varphi \, \mathrm{tr}_\omega \, \omega_X$$

$$+ 2|\nabla \, \mathrm{tr}_\omega \, \omega_X|_\omega^2 - 2\frac{K^2}{R_0^2}(\mathrm{tr}_\omega \, \omega_X)^2 - 2C_0(\mathrm{tr}_\omega \, \omega_X)^3$$

$$\geq -C(R_0) \quad \text{in } B_g\left(z_0, \frac{3R_0}{2}\right),$$

where in the last inequality the Schwarz inequality, (4.2) and Lemma 4.2. It then follows from Lemma 4.1 that

$$\sup_{B_{R_0}} u^2 \leq \frac{C(R_0)}{\mathrm{Vol}_g(B_g(z_0, \frac{3R_0}{2}))} \int_{B_{3R_0/2}} (1 + u^2)\omega^n$$

$$\leq \frac{C(R_0)}{R_0^{2np/(p-n)}} \int_X |\nabla\varphi|_\omega^2 \omega^n + C(R_0).$$

The proof is complete by observing that

$$\int_X |\nabla\varphi|_\omega^2 \omega^n = \int_X (-\varphi)(\omega - \omega_X) \wedge \omega^{n-1} \leq \int_X (-\varphi)\omega^n$$

and the latter integral is bounded due to (4.2). □

We remark that if the Ricci curvature $\mathrm{Ric}(\omega)$ is bounded below on X, then Proposition 4.1 implies a global gradient estimate of φ, which was proved in [7] using maximum principle. Note that in [7], the assumption $i\partial\bar{\partial}F \leq A\omega_X$ was made to ensure the Ricci curvature being bounded below. With this extra assumption, the bound on $\|e^F\|_{L^1(\log L)^p}$ with $p > n$ together with Corollary 1.1 imply the diameter bound of (X, ω), where $\omega = \omega_\varphi$ satisfies (4.1).

Finally, we mention that when the function e^F on the right side of (4.1) belongs to L^q for some $q > 1$, the diameter bound of ω_φ has been proved in [18], using the Hölder continuity of the solution φ established in [3, 17] (see [12]). When $\|e^F\|_{L^1(\log L)^p} \leq A$ for some $A > 0$ and $p > 3n$, the diameter bound of ω_φ has been obtained in [12] after they establish the modulus of continuity of the Kähler potentials.

4.2. Normalized Kähler–Ricci flow. As in the previous subsection, we assume that X is a minimal Kähler manifold (i.e. K_X is *nef*, and we do not assume K_X is semi-ample) with ω_X a given Kähler metric. Let Ω be a smooth volume form such that $\chi = i\partial\bar{\partial}\log\Omega \in |K_X|$ is a representative of the canonical class of X. Multiplying a constant to ω_X if necessary we may assume that $\chi \leq \omega_X$. We consider the following normalized Kähler–Ricci flow:

$$(4.4) \qquad \frac{\partial\omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) - \omega_t, \quad \omega_t|_{t=0} = \omega_X.$$

It is well known [25] that (4.4) exists for all $t \in [0, \infty)$ and (4.4) is equivalent to the following parabolic complex Monge–Ampère equation:

$$(4.5) \qquad \frac{\partial\varphi_t}{\partial t} = \log\frac{(\chi + e^{-t}(\omega_X - \chi) + i\partial\bar{\partial}\varphi_t)^n}{e^{-(n-v)t}\Omega} - \varphi_t, \quad \varphi_t|_{t=0} = 0$$

with $\omega_t = \chi + e^{-t}(\omega_X - \chi) + i\partial\bar{\partial}\varphi_t > 0$.

Lemma 4.3. *There is a uniform constant $C > 0$ such that*

$$\sup_X \dot{\varphi}_t = \sup_X \frac{\partial \varphi_t}{\partial t} \leq C \quad \text{and} \quad \sup_X \varphi_t \leq C.$$

Proof. Taking $\frac{\partial}{\partial t}$ on both sides of (4.5), we get

$$\frac{\partial \dot{\varphi}_t}{\partial t} = \Delta_{\omega_t} \dot{\varphi}_t - e^{-t} \operatorname{tr}_{\omega_t}(\omega_X - \chi) - \dot{\varphi}_t + (n - v), \quad \dot{\varphi}_t|_{t=0} = 0,$$

so by maximum principle and $\omega_X - \chi \geq 0$ it follows that for $\varphi_{t,\max} = \max_X \dot{\varphi}_t$

$$\frac{d}{dt} \dot{\varphi}_{t,\max} \leq -\dot{\varphi}_{t,\max} + (n - v)$$

multiplying both sides by e^t and integrating over t it yields that

$$\dot{\varphi}_{t,\max} \leq (n - v)(1 - e^{-t}) \leq n - v,$$

which gives the upper bound of $\dot{\varphi}_t$. To see the upper bound of φ_t, we calculate (set $V = \int_X \Omega$)

$$\begin{aligned}
\frac{d}{dt} \frac{1}{V} \int_X \varphi_t \Omega &= \frac{1}{V} \int_X \log \frac{\omega_t^n}{e^{-(n-v)t} \Omega} \Omega - \frac{1}{V} \int_X \varphi_t \Omega \\
&= (n - v)t + \frac{1}{V} \int_X \log \frac{\omega_t^n}{\Omega} \Omega - \frac{1}{V} \int_X \varphi_t \Omega \\
&\leq (n - v)t + \log\left(\frac{1}{V} \int_X \omega_t^n\right) - \frac{1}{V} \int_X \varphi_t \Omega \\
&\leq (n - v)t + \log\left(\frac{C}{V} e^{-(n-v)t}\right) - \frac{1}{V} \int_X \varphi_t \Omega \\
&\leq C - \frac{1}{V} \int_X \varphi_t \Omega,
\end{aligned}$$

where in the third line above we applied Jensen's inequality. Integrating both sides of the inequality above it follows that $\frac{1}{V} \int_X \varphi_t \Omega \leq C$ for some uniform $C > 0$. The desired upper bound of φ_t follows from the mean value theorem. $\quad\square$

We re-write equation (4.5) as

$$(4.6) \qquad \left(\chi + e^{-t}(\omega_X - \chi) + i\partial\bar{\partial}\varphi_t\right)^n = e^{-t(n-v)} e^{\varphi_t + \dot{\varphi}_t} \Omega.$$

Integrating both sides of (4.6) and applying Lemma 4.1, we get $\int_X e^{\varphi_t} \Omega \geq c > 0$ for some uniform $c > 0$, which implies that $\sup_X \varphi_t \geq -C$, where $C > 0$ is independent of t. We can now apply the L^∞-estimate of φ_t as the solution to the "elliptic" complex MA equation (4.6) with

$$e^F := e^{\varphi_t + \dot{\varphi}_t} \leq C$$

from Lemma 4.3, and it follows that

$$(4.7) \qquad \sup_X |(\varphi_t - \sup_X \varphi_t) - \mathcal{V}_t| \leq C \implies \sup_X |\varphi_t - \mathcal{V}_t| \leq C,$$

in which we use $|\sup_X \varphi_t| \leq C$ and $\mathcal{V}_t$ is the envelope associated to $\chi + e^{-t}(\omega_X - \chi)$. With (4.7) we are ready to state the local noncollapsing along the normalized Kähler–Ricci flow (4.4).

Proposition 4.1. *Let X be a minimal Kähler manifold and let ω_t be the solution to the Kähler–Ricci flow* (4.4). *Let the assumptions be as above. Then the following holds. For any $\varepsilon > 0$, $t \in [0, \infty)$, $z_0 \in X$, and $R_0 \in (0, 1]$, if $\mathrm{Ric}(\omega_t) \geq -\frac{K^2}{R_0^2}$ on $B_{\omega_t}(z_0, 2R_0)$, then there exists a constant $c_\varepsilon = c_\varepsilon(n, \chi, \omega_X, K, \Omega, \varepsilon) > 0$ such that*

$$\frac{\mathrm{Vol}_{\omega_t}(B_{\omega_t}(z_0, R_0))}{V_t} \geq c_\varepsilon R_0^{2n+\varepsilon}.$$

The proof of this proposition is almost the same as that of Theorem 1.1, given the L^∞-estimate (4.7) of φ_t. In fact, the proof is even simpler since the function $e^F = e^{\varphi_t + \dot{\varphi}_t}$ is bounded in L^∞-norm. We leave the details to interested readers.

If the Ricci curvature ω_t is uniformly bounded below for $t \in [0, \infty)$, the local noncollapsing in Proposition 4.1 will imply the uniform diameter bound of (X, ω_t), thus providing a new proof of one of the results in [8], which studied the Kähler–Ricci flow on minimal manifolds with numerical dimension $\nu = n$. By volume comparison, it will follow that $\mathrm{Vol}_{\omega_t}(B_{\omega_t}(z_0, r)) \geq cV_t r^{2n}$ for any $r \in (0, 1]$. This volume decay property appears in the assumptions of the main theorems in [13]. It suggests that this extra assumption may be superfluous for Kähler–Ricci flow.

References

[1] *J. Cheeger* and *G. Tian*, Curvature and injectivity radius estimates for Einstein 4-manifolds, J. Amer. Math. Soc. **19** (2006), no. 2, 487–525.

[2] *X. Chen* and *W. He*, The complex Monge–Ampère equation on compact Kähler manifolds, Math. Ann. **354** (2012), no. 4, 1583–1600.

[3] *J.-P. Demailly*, *S. Dinew*, *V. Guedj*, *H. H. Pham*, *S. Kołodziej* and *A. Zeriahi*, Hölder continuous solutions to Monge–Ampère equations, J. Eur. Math. Soc. (JEMS) **16** (2014), no. 4, 619–647.

[4] *J.-P. Demailly* and *N. Pali*, Degenerate complex Monge–Ampère equations over compact Kähler manifolds, Internat. J. Math. **21** (2010), no. 3, 357–405.

[5] *S. Dinew* and *Z. Zhang*, On stability and continuity of bounded solutions of degenerate complex Monge–Ampère equations over compact Kähler manifolds, Adv. Math. **225** (2010), no. 1, 367–388.

[6] *P. Eyssidieux*, *V. Guedj* and *A. Zeriahi*, A priori L^∞-estimates for degenerate complex Monge–Ampère equations, Int. Math. Res. Not. IMRN **2008** (2008), Article ID rnn 070.

[7] *X. Fu*, *B. Guo* and *J. Song*, Geometric estimates for complex Monge–Ampère equations, J. reine angew. Math. **765** (2020), 69–99.

[8] *B. Guo*, On the Kähler Ricci flow on projective manifolds of general type, Int. Math. Res. Not. IMRN **2017** (2017), no. 7, 2139–2171.

[9] *B. Guo*, *D. H. Phong* and *F. Tong*, On L^∞ estimates for complex Monge-Ampère equations, preprint 2021, https://arxiv.org/abs/2106.02224.

[10] *B. Guo*, *D. H. Phong* and *F. Tong*, Stability estimates for the complex Monge–Ampère and Hessian equations, preprint 2021, https://arxiv.org/abs/2106.03913.

[11] *B. Guo*, *D. H. Phong*, *F. Tong* and *C. Wang*, On L^∞ estimates for Monge–Ampère and Hessian equations on nef classes, preprint 2021, https://arxiv.org/abs/2111.14186; to appear in Anal. PDE.

[12] *B. Guo*, *D. H. Phong*, *F. Tong* and *C. Wang*, On the modulus of continuity of solutions to complex Monge–Ampère equations, preprint 2021, https://arxiv.org/abs/2112.02354.

[13] *M. Hallgren*, Ricci flow with Ricci curvature and volume bounded below, preprint 2021, `https://arxiv.org/abs/2104.03386`.

[14] *L. Hörmander*, An introduction to complex analysis in several variables, Van Nostrand, Princeton 1973.

[15] *S. Kołodziej*, The complex Monge–Ampère equation, Acta Math. **180** (1998), no. 1, 69–117.

[16] *S. Kołodziej*, The Monge–Ampère equation on compact Kähler manifolds, Indiana Univ. Math. J. **52** (2003), no. 3, 667–686.

[17] *S. Kołodziej*, Hölder continuity of solutions to the complex Monge–Ampère equation with the right-hand side in L^p: The case of compact Kähler manifolds, Math. Ann. **342** (2008), no. 2, 379–386.

[18] *Y. Li*, On collapsing Calabi–Yau fibrations, J. Differential Geom. **117** (2021), no. 3, 451–483.

[19] *B. Guo, D. H. Phong* and *J. Sturm*, Green's functions and complex Monge–Ampère equations, preprint 2022, `https://arxiv.org/abs/2202.04715`.

[20] *G. Perelman*, The entropy formula for the Ricci flow and its geometric applications, preprint 2022, `https://arxiv.org/abs/math/0211159`.

[21] *R. Schoen* and *S.-T. Yau*, Lectures on differential geometry. Vol. 2, International Press, Cambridge 1994.

[22] *J. Song*, Riemannian geometry of Kähler–Einstein currents, preprint 2014, `https://arxiv.org/abs/1404.0445`.

[23] *G. Székelyhidi*, Degenerations of $\mathbf{C}^n$ and Calabi–Yau metrics, Duke Math. J. **168** (2019), no. 14, 2651–2700.

[24] *G. Tian*, On Kähler–Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$, Invent. Math. **89** (1987), no. 2, 225–246.

[25] *G. Tian* and *Z. Zhang*, On the Kähler-Ricci flow on projective manifolds of general type, Chinese Ann. Math. Ser. B **27** (2006), no. 2, 179–192.

[26] *S. T. Yau*, On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I, Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411.

[27] *Z. Zhang*, On degenerate Monge–Ampère equations over closed Kähler manifolds, Int. Math. Res. Not. IMRN **2006** (2006), Article ID 63640.

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