Geom. Funct. Anal. Vol. 32 (2022) 1335–1356
https://doi.org/10.1007/s00039-022-00620-9
Published online October 22, 2022

GAFA Geometric And Functional Analysis

DIAMETER ESTIMATES FOR LONG-TIME SOLUTIONS OF THE KÄHLER–RICCI FLOW

Wangjian Jian and Jian Song

Abstract. It is well known that the Kähler–Ricci flow on a Kähler manifold X admits a long-time solution if and only if X is a minimal model, i.e., the canonical line bundle K_X is nef. The abundance conjecture in algebraic geometry predicts that K_X must be semi-ample when X is a projective minimal model. We prove that if K_X is semi-ample, then the diameter is uniformly bounded for long-time solutions of the normalized Kähler–Ricci flow. Our diameter estimate combined with the scalar curvature estimate in Song and Tian (Am J Math 138(3):683–695, 2016) for long-time solutions of the Kähler–Ricci flow are natural extensions of Perelman's diameter and scalar curvature estimates for short-time solutions on Fano manifolds. As an application, the normalized Kähler–Ricci flow on a minimal threefold X always converges sequentially in Gromov–Hausdorff topology to a compact metric space homeomorphic to its canonical model X_{can}.

1 Introduction

The behavior of long-time solutions of the Kähler–Ricci flow has been extensively studied [Cao85, Tsu88, TZ06, ST12, Zha09, TWY18, Wan18, GSW16, TZ16, TZ21, FL, JS, FZ15, FZ20, STZ] after the fundamental work of Hamilton [Ham82], Perelman [Pere1, Pere2] and the pioneering work of Song-Tian [ST07] in the framework of the analytic minimal model program with Ricci flow [ST17]. In this paper, we consider the normalized Kähler–Ricci flow on an n-dimensional Kähler manifold X defined by

$$\begin{cases} \dfrac{\partial g}{\partial t} = -\mathrm{Ric}(g) - g, \\ \\ g|_{t=0} = g_0. \end{cases} \tag{1.1}$$

with the initial Kähler metric g_0. It is well-known [Tsu88, TZ06] that the Kähler–Ricci flow (1.1) admits a long-time solution if and only if K_X is nef, i.e., for any holomorphic curve C of X,

$$K_X \cdot C = \int_C \eta \geq 0,$$

where η is any smooth closed $(1,1)$-form in the class of $[K_X] = -c_1(X)$. The abundance conjecture in birational geometry predicts that the canonical bundle K_X is nef if and only it is semi-ample, i.e., mK_X is globally generated for some sufficiently large $m \in \mathbb{Z}^+$. A Kähler manifold of nef canonical bundle is also called a minimal model. The abundance conjecture always holds for Kähler manifolds of general type or of complex dimension no greater than three [Kaw85, Miy98, Kaw92, CHP16]. The deep and subtle relationship between these two notations of positivity in algebraic geometry is also reflected in the canonical metric structures of the underlying Kähler manifolds.

We will assume the canonical bundle K_X is semi-ample for most parts of the paper. When K_X is semi-ample, the pluricanonical system induces a unique holomorphic map

$$\Phi : X \to X_{\mathrm{can}}$$

from X to its unique canonical model X_{can} as a normal projective variety. The Kodaira dimension of X, denoted by $\mathrm{kod}(X)$, is defined to be the complex dimension of X_{can}.

When X is of general type, i.e. $\dim X_{\mathrm{can}} = \mathrm{kod}(X) = \dim X = n$, Φ is a birational morphism. When K_X is positive, it is proved in [Cao85] that the Kähler–Ricci flow (1.1) converges smoothly to the unique Kähler-Einstein metric on X, as the alternative proof of the celebrated theorem of Aubin [Aub78] and Yau [Yau78] for the existence of Kähler-Einstein metrics on Kähler manifolds with negative first Chern class. When K_X is not strictly positive, the flow must develop long-time singularities, exactly at $\mathcal{S}_X$, the critical set of Φ. Tsuji [Tsu88] proves that the flow in fact converges smoothly on

$$X^\circ = X \setminus \mathcal{S}_X$$

to the unique smooth Kähler-Einstein metric g_{KE} on X°, which can uniquely extend to a global Kähler-Einstein current on X_{can} with bounded local potentials [TZ06] based on the L^∞-estimate of [Kol98, Zha06]. Furthermore, the scalar curvature [Zha09] and diameter [Wan18] are uniformly bounded along the flow. It is proved in [Song2] that the metric completion of $(X^\circ_{\mathrm{can}}, g_{KE})$ is a compact metric space homeomorphic to X_{can} as a projective variety and so the limiting metric space of the Kähler–Ricci flow must coincide with X_{can} as well [Wan18].

When $1 \le \mathrm{kod}(X) \le n-1$, Φ gives an holomorphic fibration of X over X_{can} and the general fibres are smooth Kähler manifolds with vanishing first Chern class. We define $\mathcal{S}_{X_{\mathrm{can}}}$ to be the critical values of Φ and let

$$X^\circ_{\mathrm{can}} = X_{\mathrm{can}} \setminus \mathcal{S}_{X_{\mathrm{can}}}, \quad X^\circ = \Phi^{-1}(X^\circ_{\mathrm{can}}), \ \mathcal{S}_X = \Phi^{-1}(\mathcal{S}_{X_{\mathrm{can}}}).$$

Then $\mathcal{S}_X$ is the set of all singular fibres of Φ and X°_{can} is the Zariski open set of X_{can} that contains all the smooth points of X_{can} over which the fibres of Φ are nonsingular. Obviously, X° is an open Zariski open set of X containing all the

nonsingular fibres of Φ. The Kähler–Ricci flow (1.1) will naturally collapse all the Calabi-Yau fibres in terms of volume or Kähler classes. It is proved in a series of papers [ST07, ST12, ST16] that the collapsing flow converges with uniformly bounded scalar curvature to a unique smooth twisted Kähler-Einstein metric g_{can} on X°_{can} satisfying

$$\mathrm{Ric}(g_{\mathrm{can}}) = -g_{\mathrm{can}} + g_{WP} \tag{1.2}$$

on the Zariski open set X°_{can}, where g_{WP} is the Weil-Petersson metric for the variation of Calabi-Yau fibres. The canonical metric g_{can} extends to a unique twisted Kähler-Einstein current ω_{KE} with bounded local potentials and g_{WP} also extends to a closed positive current on X_{can} [GS]. Furthermore, $(X^{\circ}_{\mathrm{can}}, g_{\mathrm{can}})$ has bounded diameter [FGS20] and its metric completion is a compact metric space [STZ], which is essential for the diameter estimate in [STZ] as well as in our work for the long time solutions of the flow. In the special case when $\dim X = 2$ or the general fibre of X over X_{can} is a complex torus, it is proved in [STZ] the diameter is indeed uniformly bounded, after applying Tian-Zhang's relative volume comparison of [TZ21]. To apply such a relative volume comparison, one needs to obtain a uniform bound for the Ricci curvature in a neighborhood of a nonsingular fibre of X over X_{can}. In general, it is rather difficult to verify the Ricci curvature bound except for the case when the general fibre is a complex torus because the curvature is uniformly bounded along the flow near each nonsingular torus fibre. In this paper, we will apply recent results of [Bam] to obtain a relative volume comparison for the Ricci flow by removing the assumption on Ricci curvature and to establish a uniform diameter bound after combining the techniques developed in [STZ]. The following theorem is the main result of the paper.

THEOREM 1.1. *Suppose X is an n-dimensional Kähler manifold with semi-ample canonical line bundle K_X. Then for any initial Kähler metric g_0, the normalized Kähler–Ricci flow (1.1) admits a long-time solution $g(t)$ for $t \in [0, \infty)$ and there exists $C = C(g_0) > 0$ such that for all $t \geq 0$,*

$$\sup_X |R(t)| + diam_{g(t)}(X) \leq C, \tag{1.3}$$

where $R(t)$ is the scalar curvature of $g(t)$ and $diam_{g(t)}(X)$ is the diameter of $(X, g(t))$.

We remark that the scalar curvature bound is already proved in [ST16] ([Zha09] for the case of general type). The first named author further proves in [Jia20] that the scalar curvature in fact converges to $-\mathrm{kod}(X)$ on X°. The diameter bound is proved for minimal models of general type in [Wan18]. The uniform scalar curvature and diameter bound in Theorem 1.1 for the long-time solution of the Kähler–Ricci flow is a natural extension and analogue of Perelman's scalar curvature and diameter estimate [Pere2, Ses08] for the Kähler–Ricci flow of finite time extinction, i.e. the Kähler–Ricci flow on Fano manifolds with initial Kähler metric in the first Chern class. Perelman's scalar curvature estimate is essential for the convergence of the

Fano Ricci flow [TZ07, TZZZ13, CW17, Bam18, DS10, HL, GPS] in relation to the Hamilton-Tian conjecture. The remaining case for scalar curvature and distance estimates is the non-extinct finite time solutions of the Kähler–Ricci flow. This corresponds to the analytic minimal model program proposed in [ST17] and is related to geometric surgeries and local uniformization problem in [SW16, SW14, SY12, Song1].

The following volume estimate follows from our relative volume comparison and Theorem 1.1.

COROLLARY 1.1. *Suppose X is an n-dimensional Kähler manifold with semi-ample canonical line bundle K_X. Let $g(t)$ be the long-time solution of the normalized Kähler–Ricci flow (1.1) with any initial Kähler metric g_0 for $t \in [0,\infty)$. Then there exists $c = c(g_0) > 0$ such that for any $p \in X$, $t \geq 0$ and $0 < r < diam_{g(t)}(X)$,*

$$c\ Vol_{g(t)}(X) \leq \frac{Vol_{g(t)}(B_{g(t)}(p,r))}{r^{2n}} \leq c^{-1}\, Vol_{g(t)}(X), \tag{1.4}$$

where $B_{g(t)}(p,r))$ is the metric ball center at p with radius r with respect to $g(t)$.

In fact, the estimate (1.4) can be made more explicit by applying the volume estimate in [ST16]. There exists $C = C(g_0)$ such that for all $t \geq 0$,

$$C^{-1}e^{-(n-\kappa)t} \leq \mathrm{Vol}_{g(t)}(X) \leq Ce^{-(n-\kappa)t},$$

where $\kappa = \mathrm{kod}(X)$.

The following convergence result follows naturally by the uniform diameter bound and Corollary 1.1.

COROLLARY 1.2. *Suppose X is an n-dimensional Kähler manifold with semi-ample canonical line bundle K_X. Let $g(t)$ be the long-time solution of the normalized Kähler–Ricci flow (1.1) with any initial Kähler metric g_0 for $t \in [0,\infty)$. Then $g(t)$ converges to the twisted Kähler-Einstein g_{can} in $C^0(X^\circ)$ as $t \to \infty$, and for any $t_j \to \infty$, after possibly passing to a subsequence, $(X, g(t_j))$ converges in Gromov–Hausdorff topology to a compact metric space (X_∞, d_∞) satisfying the following conditions.*

(1) *Let (Y, d_Y) be the metric completion of (X°_{can}, g_{can}). Then the identity maps from $X^\circ_{can} \subset X_\infty$ and $X^\circ_{can} \subset Y$ to $X^\circ_{can} \subset X_{can}$ can extend uniquely to the following two Lipschitz maps*

$$\Psi : (Y, d_Y) \to (X_\infty, d_\infty),\ \ \Upsilon : (X_\infty, d_\infty) \to (X_{can}, g_{FS}),$$

where g_{FS} is the restriction of the Fubini-Study metric to X_{can} from the pluri-canonical map Φ.

(2) *If $kod(X) = 0, 1, 2, n$, then both Ψ and Υ are homeomorphic.*

Unfortunately, we are unable to identify Y (or X_∞) topologicially with the projective variety $X_{\rm can}$ in general cases. We instead propose the following conjecture.

CONJECTURE 1.1. *Ψ is an isometry and Υ is an homeomorphism.*

Indeed, the conjecture holds when $\mathrm{kod}(X) = 1$ [TZ21, STZ], or more generally when $\dim \mathcal{S}_{X_{\rm can}} = 0$. Finally, we can give a fairly complete description for limiting behavior of the long-time solutions of the Kähler–Ricci flow on minimal threefolds by applying Theorem 1.1 and Corollary 1.2.

COROLLARY 1.3. *Let X be a minimal threefold. Let $g(t)$ be the long-time solution of the Kähler–Ricci flow (1.1) with any initial Kähler metric g_0 for $t \in [0, \infty)$. Then there exists $A = A(g_0) > 0$ such that for all $t \in [0, \infty)$,*

$$\sup_X |R(t)| + \mathit{diam}(X, g(t)) \le A,$$

and for any $t_j \to \infty$, after passing to subsequence, $(X, g(t_j))$ converges in Gromov–Hausdorff topology to a compact metric space (X_∞, d_∞) homeomorphic to the canonical model X_{can} of X.

We give a brief outline of the paper. In §2, we give the background for long-time solutions of the Kähler–Ricci flow and some basic analytic and geometric estimates. In §3, we apply the work of [Bam] to derive a relative volume comparison for the Ricci flow. We establish the uniform diameter bound for long-time solutions of the Kähler–Ricci flow in §4. In §5, we prove the geometric convergence.

2 Basic Estimates

Let X be an n-dimensional Kähler manifold. X is called a minimal model if the canonical bundle K_X is nef. The abundance conjecture in birational geometry predicts that K_X must be semi-ample if K_X is nef.

From now on, we will assume that K_X is semi-ample. The canonical ring $R(X, K_X)$ is therefore finitely generated, and the pluricanonical system $|mK_X|$ induces a holomorphic map

$$\Phi : X \to X_{\rm can} \subset \mathbb{P}^N \tag{2.5}$$

for sufficiently large $m \in \mathbb{Z}^+$, where $X_{\rm can}$ is the canonical model of X. In fact, both Φ and $X_{\rm can}$ are uniquely determined independent of the choice of m. The Kodaira dimension of X is defined to be

$$\mathrm{kod}(X) = \dim X_{\rm can}. \tag{2.6}$$

We always have

$$0 \le \mathrm{kod}(X) \le \dim X = n.$$

In particular,

(1) If $\mathrm{kod}(X) = n$, X is birationally equivalent to its canonical model X_{can}, and X is called a minimal model of general type.
(2) If $0 < \mathrm{kod}(X) < n$, X admits a Calabi-Yau fiberation

$$\pi : X \to X_{\mathrm{can}}$$

over X_{can} and a general fibre is a smooth Calabi-Yau manifold of complex dimension $n - \mathrm{kod}(X)$.
(3) If $\mathrm{kod}(X) = 0$, X_{can} is a point and X is a Calabi-Yau manifold with $c_1(X) = 0$.

Now we will reduce the normalized Kähler–Ricci flow to a parabolic Monge-Ampère equation. Let $\mathcal{O}_{\mathbb{P}^N}(1)$ be the hyperplane bundle over $\mathbb{P}^N$ in (2.5) and $\omega_{FS} \in [\mathcal{O}_{\mathbb{P}^N}(1)]$ be a Fubini-Study metric on $\mathbb{P}^N$. Then there exists $m > 0$ such that

$$mK_X = \pi^* \mathcal{O}_{\mathbb{P}^N}(1).$$

We define

$$\chi = \frac{1}{m} \pi^* \omega_{FS} \in [K_X]$$

and χ is a smooth nonnegative closed $(1,1)$-form on X. There also exists a smooth volume form Ω on X such that

$$\mathrm{Ric}(\Omega) = -\sqrt{-1}\partial\overline{\partial} \log \Omega = -\chi.$$

Let ω_0 be the initial Kähler metric of the normalized Kähler–Ricci flow (1.1) on X. Then Kähler class evolving along the normalized Kähler–Ricci flow is given by

$$[\omega(t)] = (1 - e^{-t})[K_X] + e^{-t}[\omega_0]$$

and so $[\omega(t)]$ is a Kähler class for all $t \in [0, \infty)$. Therefore the normalized Kähler–Ricci flow starting with ω_0 on X has a smooth global solution on $X \times [0, \infty)$. We define the reference metric

$$\omega_t = (1 - e^{-t})\chi + e^{-t}\omega_0.$$

Then the Kähler–Ricci flow is equivalent to the following Monge-Ampere flow.

$$\frac{\partial \varphi}{\partial t} = \log \frac{e^{(n-\kappa)t}(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi)^n}{\Omega} - \varphi, \tag{2.7}$$

where

$$\omega_t = \chi + e^{-t}(\omega_0 - \chi)$$

and

$$\kappa = \mathrm{kod}(X) = \dim X_{\mathrm{can}}.$$

We define $\mathcal{S}_{X_{\rm can}}$ to be the critical values of Φ and let

$$X^\circ_{\rm can} = X_{\rm can} \setminus \mathcal{S}_{X_{\rm can}}, \quad X^\circ = \Phi^{-1}(X^\circ_{\rm can}),\ \mathcal{S}_X = \Phi^{-1}(\mathcal{S}_{X_{\rm can}}). \tag{2.8}$$

When $\kappa = \mathrm{kod}(X) = n$, X is said to be a minimal model of general type. In this case, $\Phi : X \to X$ is a birational map. Then $\mathcal{S}_X$ and $\mathcal{S}_{X_{\rm can}}$ are subvarieties of X and $X_{\rm can}$. Hence $X^\circ = X^\circ_{\rm can}$ is a Zariski open set of X or $X_{\rm can}$. The following lemma is a collection of results from [Tsu88, TZ06, Wan18].

LEMMA 2.1. *Let $g(t)$ be the long-time solution of the Kähler–Ricci flow (1.1). Then $g(t)$ converges smoothly to a unique Kähler-Einstein metric g_{KE} on X°_{can}. Futhermore, the scalar curvature and diameter of $g(t)$ are uniformly bounded and g_{KE} extends to a unique Kähler current ω_{KE} on X_{can} with bounded local potentials.*

Proof. The local smooth convergence is proved in [Tsu88]. The boundedness of local potentials of ω_{KE} is proved in [TZ06] built on the work of [Kol98, Zha06, EGZ09]. The scalar curvature bound is derived in [Zha09]. The diameter bound is recently proved in [Wan18] by developing and applying the local entropy for the Ricci flow with the Riemannian geometric structure of $X_{\rm can}$ from [Song2]. □

When $\kappa = \mathrm{kod}(X) = 0$, X is a Calabi-Yau manifold. The exponential convergence of the unnormalized Kähler–Ricci flow to a unique Ricci-flat Kähler metric is established in [Cao85]. In particular, the solution $g(t)$ of the normalized Kähelr-Ricci flow (1.1) will converge to a point with uniformly bounded Ricci curvature.

Therefore it suffices for us to study the case when $0 < \kappa = \mathrm{kod}(X) < n$. In this case, $\mathcal{S}_X$ is the set of all singular fibres of $\Phi : X \to X_{\rm can}$ and the Kähler–Ricci flow will collapse.

The following general testimates are established in [ST07, ST12, ST16].

LEMMA 2.2. *Let $g(t)$ be the long-time solution of the Kähler–Ricci flow (1.1). There exists $C > 0$ such that on $X \times [0, \infty)$,*

$$|\varphi| + tr_\omega(\chi) + \left|\frac{\partial \varphi}{\partial t}\right| + \left|\nabla \frac{\partial \varphi}{\partial t}\right| + \left|\Delta \frac{\partial \varphi}{\partial t}\right| + |R| \le C, \tag{2.9}$$

where R is the scalar curvature of $g(t)$, ∇ andΔ are the gradient and Laplace operators with respect to $g(t)$.

The following lemma is proved in [ST07, ST12] for convergence of the collapsing Kähler–Ricci flow and its limiting metric.

LEMMA 2.3. *Suppose $0 < \kappa = kod(X) < n$. Then $g(t)$ converges as a current to a unique smooth twisted Kähler-Einstein metric g_{can} on X°_{can} satisfying the following.*

(1) *g_{can} satisfies the twisted Kähler-Einstein equation*

$$Ric(g_{can}) = -g_{can} + g_{WP}$$

on X°_{can}, where g_{WP} is the Weil-Petersson metric for the variation of Calabi-Yau fibres of X° over X°_{can}.

(2) *The Kähler form associated to g_{can} extends to a unique Kähler current ω_{can} on X_{can} with bounded local potentials.*

A stronger convergence and fibrewise control are proved in [TWY18] in the following lemma.

LEMMA 2.4. *Suppose $0 \leq \kappa = kod(X) < n$. Then $g(t)$ converges to $\Phi^* g_{can}$ in C^0-topology on X° as $t \to \infty$. Furthermore, for any $\mathcal{K} \subset\subset X^\circ_{can}$, there exists $C > 0$ such that for any $x \in \mathcal{K}$ and $t \geq 0$,*

$$C^{-1} e^{-t} \omega|_{\Phi^{-1}(x)} \leq \omega(t)|_{\Phi^{-1}(x)} \leq C e^{-t} \omega_0|_{\Phi^{-1}(x)}.$$

3 The Relative Volume Comparison for the Ricci Flow

In this section, we will use the recent work in [Bam] to prove a relative volume comparison for the Ricci flow under suitable assumption on the scalar curvature bound. We will follow the notations from [Bam] for the following quantities on a Riemannian manifold or along the Ricci flow.

DEFINITION 3.1. Let (M, g) be a closed Riemannian manifold of dimension $n \geq 2$. The Nash entropy is defined by

$$\mathcal{N}(g, f, \tau) = (4\pi\tau)^{-n/2} \int_M f e^{-f} dV_g$$

for $f \in C^\infty(M)$ and $\tau > 0$.

Let $(M, g(t))$ be a Ricci flow, i.e., $g(t)$ satisfies the unnormalized Ricci flow equation

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}, \tag{3.10}$$

on an n-dimensional compact Riemannian manifold for $t \in I$, where I is an open or closed interval. The heat operator is defined by

$$\Box = \frac{\partial}{\partial t} - \Delta$$

and the conjugate heat operator is defined by

$$\Box^* = -\frac{\partial}{\partial t} - \Delta + \mathrm{R},$$

where Δ is the Laplace operator associated to $g(t)$ and R is the scalar curvature of $g(t)$. For fixed $(y, s) \in M \times I$, the heat kernel $K(\cdot, \cdot; y, s)$ based at (y, s) is given by

$$\Box K(\cdot, \cdot; y, s) = 0, \quad \lim_{t \to s^+} K(\cdot, t; y, s) = \delta_y,$$

where δ_y is the Dirac measure at y. By duality, for fixed $(x,t) \in M \times I$, the function $K(x,t;\cdot,\cdot)$ is the conjugate heat kernel at the base point (x,t) satisfying

$$\Box^* K(x,t;\cdot,\cdot) = 0, \quad \lim_{s \to t^-} K(x,t;\cdot,s) = \delta_x.$$

DEFINITION 3.2. Let $(M, g(t))$ be a solution of the Ricci flow for $t \in I$ and

$$d\nu_{x_0,t_0} = (4\pi\tau)^{-n/2} e^{-f} dV_g = K(x_0,t_0;\cdot,\cdot) dV_g$$

be a pointed conjugate heat kernel measure at the base point (x_0,t_0) with $\tau = t_0 - t$. Then the pointed Nash entropy at $(x_0,t_0) \in M \times I$ is defined by

$$\mathcal{N}_{x_0,t_0}(\tau) = \mathcal{N}(g(t_0-\tau), f(t_0-\tau), \tau) \tag{3.11}$$

with $\mathcal{N}_{x_0,t_0}(0) = 0$ and for $s < t_0$, the space-time function $\mathcal{N}^*_s$ is defined by

$$\mathcal{N}^*_s(x_0,t_0) = \mathcal{N}_{x_0,t_0}(t_0 - s).$$

DEFINITION 3.3. Let μ_1, μ_2 be two probability measures on a compact Riemannian manifold (X,g). The Wasserstein W_1-distance between μ_1 and μ_2 are defined by

$$d^g_{W_1}(\mu_1,\mu_2) = \sup_{f \in C^\infty(M), |\nabla f| \le 1} \left(\int_X f d\mu_1 - \int_X f d\mu_2 \right).$$

LEMMA 3.1. *For fixed base point (x_0,t_0), the pointed Nash entropy $\mathcal{N}_{x_0,t_0}(\tau)$ is non-increasing in τ.*

Equivalently, $\mathcal{N}^*_s(x_0,t_0)$ is non-decreasing in s. The following estimates are proved in [Bam] (Theorem 5.9 and Corollary 5.11).

LEMMA 3.2. *If $R(\cdot,s) \ge R_{min}$ for some $s \in I$ and $R_{min} \in \mathbb{R}$, then on $M \times (I \cap (s,\infty))$,*

$$|\nabla \mathcal{N}^*_s| \le \left(\frac{n}{2(t-s)} + |R_{min}| \right)^{\frac{1}{2}}.$$

Furthermore, if $s < t^* \le \min\{t_1,t_2\}$ and $s, t_1, t_2 \in I$, then for any $x_1, x_2 \in M$,

$$\mathcal{N}^*_s(x_1,t_1) - \mathcal{N}^*_s(x_2,t_2) \le \left(\frac{n}{2(t^*-s)} + |R_{min}| \right)^{\frac{1}{2}} d^{g(t^*)}_{W_1}(\nu_{x_1,t_1}(t^*), \nu_{x_2,t_2}(t^*)) + \frac{n}{2} \log \left(\frac{t_2 - s}{t^* - s} \right).$$

The pointed Nash entropy is roughly comparable to logarithmic of the volume ratio of suitable scale and scalar curvature assumption. The following volume non-inflation estimate is also proved in [Bam].

LEMMA 3.3. *Let $(M, g(t))$ be a solution of the Ricci flow for $t \in [-r^2, 0]$. If*

$$R \geq -nr^{-2}$$

on $M \times [-r^2, 0]$, then for any $A \geq 1$, there exists $C = C(n, A) > 0$ such that

$$Vol_{g(0)}(x, Ar) \leq Cr^n e^{\mathcal{N}^*_{-r^2}(x,0)}.$$

The following volume non-collapsing estimate is proved in [Bam] as a generalization of Perelman's κ-non-collapsing theorem.

LEMMA 3.4. *Let $(M, g(t))$ be a solution of the Ricci flow for $t \in [-r^2, 0]$. If*

$$R \leq r^{-2}, \text{ on } B_{g(0)}(x, r) \times [-r^2, 0],$$

then there exists $c = c(n) > 0$ such that

$$Vol_{g(0)}(B_{g(0)}(x, r)) \geq cr^n e^{\mathcal{N}^*_{-r^2}(x,0)}.$$

The following is the main result of this section by comparing volume of balls with same radius at the same time slice.

PROPOSITION 3.1. *Let $(M, g(t))$ be a solution to the Ricci flow on a compact n-dimensional Riemannian manifold M for $t \in [-r^2, 0]$, for some $r > 0$. If*

$$|R| \leq r^{-2}, \text{ on } M \times [-r^2, 0],$$

then for any $x_1, x_2 \in M$ with

$$d_{g(0)}(x_1, x_2) \leq A,$$

we have for some $c(n) > 0$

$$Vol_{g(0)}(B_{g(0)}(x_1, r)) \geq c(n) e^{-\sqrt{n} r^{-1} A} \; Vol_{g(0)}(B_{g(0)}(x_2, r)).$$

Proof. By Lemma 3.3 and Lemma 3.4, we have

$$\mathrm{Vol}_{g(0)}(B_{g(0)}(x_i, r)) \geq cr^n e^{\mathcal{N}^*_{-r^2}(x_i,0)}$$

and

$$\mathrm{Vol}_{g(0)}(B_{g(0)}(x_i, r)) \leq Cr^n e^{\mathcal{N}^*_{-r^2}(x_i,0)}.$$

By the gradient estimate for the pointed Nash entropy in Lemma 3.2, we have

$$\begin{aligned}
&\frac{\mathrm{Vol}_{g(0)}(B_{g(0)}(x_2, r))}{\mathrm{Vol}_{g(0)}(B_{g(0)}(x_1, r))} \\
&\quad \leq C' \exp\left(\mathcal{N}^*_{-r^2}(x_2, 0) - \mathcal{N}^*_{-r^2}(x_1, 0)\right) \\
&\quad \leq C' \exp\left(\sup_M |\nabla \mathcal{N}^*_{-r^2}(\cdot, 0)| d_{g(0)}(x_1, x_2)\right) \\
&\quad \leq C' \exp\left(\sqrt{n}\, \frac{d_{g(0)}(x_1, x_2)}{r}\right)
\end{aligned}$$

for some uniform $C' = C'(n) > 0$. □

We will also take the opportunity to present a different proof of the relative volume comparison of Tian-Zhang [TZ21] with an additional lower bound for the scalar curvature using the work of [Bam].

The following parabolic region is introduced in [Bam].

DEFINITION 3.4. Suppose $(x_0, t_0) \in M \times I$, r, T^-, $T^+ \geq 0$ and $t_0 - T^- \in I$. The P^*-parabolic neighborhood $P^*(x_0, t_0; r, T^-, T^+) \subset M \times I$ is defined by the set of $(x, t) \in M \times I$ satisfying

$$d_{W_1}^{g_{t_0-T^-}}\left(\nu_{x_0,t_0}(\cdot, t_0 - T^-), \nu_{x,t}(\cdot, t_0 - T^-)\right) < r, \quad t_0 - T^- \leq t \leq t_0 + T^+.$$

It is natural to compare the P^*-parabolic neighborhood $P^*(x_0, t_0; r, T^-, T^+)$ to the standard parabolic neighborhood

$$P(x_0, t_0; r, T^-, T^+) = B_{g(t_0)}(x_0, r) \times [t_0 - T^-, t_0 + T^+].$$

It is shown in [Bam] that along the Ricci flow, we have

$$d_{W_1}^{g(t)}(\nu_{x_1,t_0}(\cdot, t), \nu_{x_2,t_0}(\cdot, t)) \leq d_{g(t_0)}(x_1, x_2)$$

and the following comparison is proved in [Bam] (Corollary 9.6).

LEMMA 3.5. *Let $(M, g(t)_{t\in I})$ be a Ricci flow on an n-dimensional compact Riemannian manifold. For any $0 < \alpha \leq \mathcal{A} < \infty$, $K, \beta^-, \beta^+ \geq 0$, there exists $\underline{\mathcal{A}} = \underline{\mathcal{A}}(\alpha, \mathcal{A}, K, \beta^-, \beta^+) \geq 0$ such for any $\mathcal{B} \geq \underline{\mathcal{A}}$, if*

$$|Ric| \leq Kr^{-2}, \text{ in } P(x_0, t_0; \mathcal{A}r, -\beta^- r^2, \beta^+ r^2),$$

then

$$P(x_0, t_0; \mathcal{A}r, -\beta^- r^2, \beta^+ r^2) \subset P^*(x_0, t_0; \mathcal{B}r, -\beta^- r^2, \beta^+ r^2).$$

The following relative volume comparison theorem for the Ricci flow is a slightly weaker version of the one in [TZ21].

PROPOSITION 3.2. *For any $n \in \mathbb{Z}^+$ and $A \geq 1$, there exists $c(n, A) > 0$ such that the following holds. Let $(M, g(t))$ be a solution of the Ricci flow on a compact n-dimensional manifold M for $t \in [-r_0^2, r_0^2]$ such that*

$$|Ric| \leq r_0^{-2}, \ R \geq -nr_0^{-2}, \text{ in } B_{g(0)}(x_0, r_0) \times [0, r_0^2].$$

Then for any $B_{g(r_0^2)}(x, r) \subset B_{g(r_0^2)}(x_0, Ar_0)$ with $r \leq r_0$ satisfying

$$R|_{t=r_0^2} \leq r^{-2} \text{ in } B_{g(r_0^2)}(x, r),$$

we have

$$\frac{Vol_{g(r_0^2)}(B_{g(r_0^2)}(x, r))}{r^n} \geq c \, \frac{Vol_{g(0)}(B_{g(0)}(x_0, r_0))}{r_0^n}.$$

Proof. By parabolic scaling, we can assume that $r_0 = 1$. The pointed Nash entropy is comparable to volume of balls in principle.

First we will compare the pointed Nash entropy at the same time slice $t = 1$. By assumption, $x \in B_{g(1)}(x, r) \subset B_{g(1)}(x_0, A)$, hence

$$d_{g(1)}(x_0, x) \leq A.$$

By Lemma 3.2, after choosing $\mathrm{R}_{\min} = -n$, we have

$$|\nabla \mathcal{N}^*_{-1}(\cdot, 1)|_{g(1)} \leq 2n$$

on M. Therefore,

$$\left|\mathcal{N}^*_{-1}(x_0, 1) - \mathcal{N}^*_{-1}(x, 1)\right| \leq 2n \; d_{g(1)}(x_0, x) \leq 2nA. \tag{3.12}$$

Next, we will compare the pointed Nash entropy at different time slices. We apply Lemma 3.2 at two different base points $(x_0, 0)$ and $(x_0, 1)$ after choosing $s = -1$ and $t^* = 0$. Then

$$\begin{aligned}
&\left|\mathcal{N}^*_{-1}(x_0, 0) - \mathcal{N}^*_{-1}(x_0, 1)\right| \\
&\leq \left(\frac{n}{2(0-(-1))} + n\right)^{1/2} d^{g(0)}_{W_1}(\nu_{x_0,0}(0), \nu_{x_0,1}(0)) + \frac{n}{2} \log\left(\frac{1-(-1)}{0-(-1)}\right) \\
&\leq n \; d^{g(0)}_{W_1}(\nu_{x_0,0}(0), \nu_{x_0,1}(0)) + n.
\end{aligned}$$

By Lemma 3.5,

$$P(x_0, 0; 1, 0, 1) \subset P^*(x_0, 0; \gamma, 0, 1)$$

by the Ricci curvature assumption. Therefore, by the fact that $(x_0, 1) \in P^*(x_0, 0; \gamma, 0, 1)$ and definition, we have

$$d^{g(0)}_{W_1}\left(\nu_{x_0,0}(\cdot, 0), \nu_{x_0,1}(\cdot, 0)\right) = d^{g(0)}_{W_1}\left(\delta_{x_0}, \nu_{x_0,1}(\cdot, 0)\right) \leq \gamma.$$

Immediately, we have

$$\left|\mathcal{N}^*_{-1}(x_0, 0) - \mathcal{N}^*_{-1}(x_0, 1)\right| \leq n\gamma + n. \tag{3.13}$$

Finally, we are ready to proof the lemma. Combining estimates (3.12), (3.13) with Lemma 3.1, Lemma 3.3 and Lemma 3.4, we have

$$\begin{aligned}
&\frac{\mathrm{Vol}_{g(1)}(B_{g(1)}(x, r))}{\mathrm{Vol}_{g(0)}(B_{g(0)}(x_0, 1))} \\
&\quad \geq c r^n \exp\left(\mathcal{N}^*_{-r^2}(x, 1) - \mathcal{N}^*_{-1}(x_0, 0)\right) \\
&\quad \geq c r^n \exp\left(\mathcal{N}^*_{-1}(x, 1) - \mathcal{N}^*_{-1}(x_0, 0)\right) \\
&\quad \geq c r^n \exp\left(\mathcal{N}^*_{-1}(x, 1) - \mathcal{N}^*_{-1}(x_0, 1) + n\gamma + n\right) \\
&\quad \leq c r^n \exp\left(2nA + n\gamma + n\right).
\end{aligned}$$

This completes the proof of the theorem. □

4 Diameter Estimate

We now return to the Kähler–Ricci flow (1.1) discussed in §2. Let X be an n-dimensional Kähler manifold with semi-ample K_X. Recall that

$$\Phi : X \to X_{\mathrm{can}}$$

is the unique holomorphic map from X to its canonical model X_{can} induced by the pluricanonical system. We assume that

$$1 \leq \dim X_{\mathrm{can}} = \kappa \leq n-1.$$

The goal of this section is to establish a uniform diameter bound for long-time solutions of the Kähler–Ricci flow using the techniques developed in [FGS20, STZ].

We keep the same notations as before by letting $\mathcal{S}_{X_{\mathrm{can}}}$ be the set of critical values of Φ on X_{can} and $\mathcal{S}_X = \Phi^{-1}(\mathcal{S}_{\mathrm{can}})$, $X^\circ_{\mathrm{can}} = X_{\mathrm{can}} \setminus \mathcal{S}_{X_{\mathrm{can}}}$ and $X^\circ = X \setminus \mathcal{S}_X$. We can pick an effective $\mathbb{Q}$-Cartier divisor $\mathcal{D}$ on X_{can} satisfying the following.

(1) $\Phi^*\mathcal{D} = K_X$.
(2) $\mathcal{S}_{X_{\mathrm{can}}}$ is contained in the support of $\mathcal{D}$.

We let σ be the defining section of $\mathcal{D}$ and for conveniences we use σ for $\Phi^*\sigma$.

Now we consider a log resolution of X_{can} defined by

$$\Psi : W \to X_{\mathrm{can}}$$

such that

(1) W is smooth and the exceptional locus of Ψ is a union of smooth divisors of simple normal crossings.
(2) The pullback of $\mathcal{D}$ by Ψ, is a union of smooth divisors of simple normal crossings.

The Kähler form χ associated to the Fubini-Study metric χ restricted to X_{can} also lies in $[\mathcal{D}]$. For conveniences, we use σ for $\Psi^*\sigma$ on W. Let Z be the blow-up of X induced by $\Psi : W \to X_{\mathrm{can}}$. Then we can define the induced holomorphic maps $\Psi' : Z \to X$ and $\Phi' : Z \to W$ satisfying the following diagram.

$$\begin{array}{ccc} Z & \xrightarrow{\Psi'} & X \\ {\scriptstyle \Phi'}\downarrow & & \downarrow{\scriptstyle \Phi} \\ W & \xrightarrow{\Psi} & X_{\mathrm{can}} \end{array} \tag{4.14}$$

We also pick the hermitian metric h for the $\mathbb{Q}$-line bundle associated to $\mathcal{D}$ on X_{can} such that

$$\mathrm{Ric}(h) = \chi.$$

For conveniences, we still use h for Φ^*h on X, Ψ^*h on W and $(\Psi\circ\Phi')^*h$ on Z. Away from zeros of σ, Z can be identified as X by assuming the blow-ups take place at the support of σ.

For simplicity, we assume that

$$|\sigma|_h^2 \leq 1$$

on X_{can}. Let F be the standard increasing smooth cut-off function defined on $[0,\infty)$ satisfying

(1) $F(x)=0$, if $x\in[0,1/2]$,
(2) $F(x)=1$, if $x\in[1,\infty)$,

with uniformly bounded derivatives. Let

$$\eta_\varepsilon = \max\left\{\log|\sigma|_h^2, \log\varepsilon\right\}$$

for some sufficiently small $\varepsilon>0$ to be determined later. By the construction of σ and h, we have

$$\sqrt{-1}\partial\overline{\partial}\log|\sigma|_h^2+\chi\geq 0$$

as a current, therefore

$$\eta_\varepsilon\in \mathrm{PSH}(X,\chi)\cap C^0(X).$$

In particular, for sufficiently small $\varepsilon>0$, we have

$$\log\varepsilon\leq\eta_\varepsilon\leq 0.$$

We define ρ_ε by

$$\rho_\varepsilon = F\left(\frac{100\eta_\varepsilon}{\log\varepsilon}\right) \tag{4.15}$$

and

$$\mathcal{S}_\varepsilon = \left\{|\sigma|_h^{200}<\varepsilon\right\}. \tag{4.16}$$

For sufficiently large $k>0$, we have

$$\sup_X\left|\partial|\sigma|_h^{2k}\right|_\chi<\infty \tag{4.17}$$

due to Lemma 2.2 from the parabolic Schwarz lemma in [ST07, ST12]. Without loss of generality, we can assume $k=100$ for simplicity. The following lemma also shows that the open set $\mathcal{S}_\varepsilon$ has very small volume.

LEMMA 4.1. *Let $g(t)$ be the long-time solution of the Kähler–Ricci flow (1.1) on X and let $\omega(t)$ be the corresponding Kähler forms. Then for any $\delta > 0$, there exists $\varepsilon > 0$ such that for all $t \geq 0$,*

$$\int_{\mathcal{S}_\varepsilon} \omega(t)^n \leq \delta e^{-(n-\kappa)t},$$

or equivalently,

$$Vol_{g(t)}(\mathcal{S}_\varepsilon) \leq \delta e^{-(n-\kappa)t}.$$

Proof. It is straightforward to verify that

$$\lim_{\varepsilon \to 0} \int_{\{|\sigma|_h^{200} \leq \varepsilon\}} \Omega = 0.$$

From Lemma 2.2, we have

$$\omega(t)^n = e^{\frac{\partial \varphi}{\partial t} + \varphi - (n-\kappa)t} \Omega \leq C e^{-(n-\kappa)t} \Omega,$$

for some constant $C = C(X, g_0) < \infty$. Hence we have

$$\int_{\mathcal{S}_\varepsilon} \omega(t)^n \leq C e^{-(n-\kappa)t} \int_{\mathcal{S}_\varepsilon} \Omega \leq \delta e^{-(n-\kappa)t},$$

if we choose $\varepsilon > 0$ small enough. This completes the proof. □

The following lemma is proved by Song-Tian-Zhang (Lemma 2.7 [STZ]) based on the distance estimate in [FGS20] for the diameter bound and almost geodesic convexity for $(X^\circ_{\mathrm{can}}, g_{\mathrm{can}})$. We let

$$(\mathcal{Y}, d_{\mathcal{Y}}) = \overline{(X^\circ_{\mathrm{can}}, g_{\mathrm{can}})} \tag{4.18}$$

be the metric completion of $(X^\circ_{\mathrm{can}}, g_{\mathrm{can}})$.

LEMMA 4.2. *For any $\delta > 0$ and $\varepsilon > 0$, there exists $0 < \varepsilon' < \varepsilon$ such that for any two points $y_1, y_2 \in X_{can} \setminus \Phi(\mathcal{S}_\varepsilon)$, there exists a smooth path $\gamma \subset X_{can} \setminus \Phi(\mathcal{S}_{\varepsilon'})$ joining y_1 and y_2 satisfying*

$$\mathcal{L}_{g_{can}}(\gamma) \leq d_{\mathcal{Y}}(y_1, y_2) + \delta,$$

where $\mathcal{L}_{g_{can}}(\gamma)$ is the arc length of γ with respect to the metric g_{can}. In particular, $(\mathcal{Y}, d_{\mathcal{Y}})$ is a compact metric space and

$$diam_{g_{can}}(X^\circ_{can}) < \infty.$$

We remark that Lemma 4.2 is proved in [ZZ19, Zha19] for the special case of $\dim X_{can} = 1$.

Immediately, we can control the distance for the Kähler–Ricci flow away from singular fibres as shown in the following lemma.

COROLLARY 4.1. *For any $\delta > 0$ and $\varepsilon > 0$, there exists $T > 0$ such that for any two points $x_1, x_2 \in X \setminus \mathcal{S}_\epsilon > 0$ and $t \geq T$, we have*

$$d_{g(t)}(x_1, x_2) \leq diam_{d_{\mathcal{Y}}}(\mathcal{Y}) + \delta.$$

Proof. Since $g(t)$ converges to $\Phi^* g_{\mathrm{can}}$ uniformly in $C^0(X \setminus \mathcal{S}_\varepsilon)$, for any $x_1, x_2 \in X \setminus \mathcal{S}_\varepsilon$ and $t > T$, we have

$$\begin{aligned} d_{g(t)}(x_1, x_2) &\leq d_{g_{\mathrm{can}}}(\Phi(x_1), \Phi(x_2)) + \delta \\ &\leq \mathrm{diam}_{d_{\mathcal{Y}}}(\mathcal{Y}) + \delta, \end{aligned}$$

where the first inequality follows from Lemma 2.4 and the last inequality follows from Lemma 4.2. □

We now choose a fixed base point $x_0 \in X^\circ$ and $y_0 = \Phi(x_0) \in X^\circ_{\mathrm{can}}$. Since y_0 is a regular point of X_{can}, there exists $0 < \gamma < 1$ such that

$$B_{g_{\mathrm{can}}}(p_0, 2\gamma) \subset\subset X^\circ_{\mathrm{can}}.$$

LEMMA 4.3. *For any $0 < r_0 < \gamma$, there exists $T > 0$ such that for any $t > T$,*

$$\Phi^{-1}(B_{g_{can}}(p_0, 2^{-1} r_0)) \subset B_{g(t)}(x_0, r_0) \subset \Phi^{-1}(B_{g_{can}}(p_0, 2r_0)). \tag{4.19}$$

Furthermore, there exist $C = C(r_0, x_0) > 0$ such that for all $t > T$,

$$C^{-1} e^{-(n-\kappa)t} \leq Vol_{g(t)}(B_{g(t)}(x_0, r_0)) \leq C e^{-(n-\kappa)t}. \tag{4.20}$$

Proof. Since $g(t)$ converges $\Phi^* g_{\mathrm{can}}$ on X°, the containment (4.19) follows immediately for sufficiently large $t > 1$ due to the fact that $g(t)$ restricted to each fibre of Φ converges to 0 exponentially fast. The volume estimate (4.20) follows from the fact that

$$\frac{e^{(n-\kappa)t} \omega(t)^n}{\Omega}$$

is uniformly bounded for all $t \geq 0$ by Lemma 2.2. □

LEMMA 4.4. *For any $A, r_0 > 0$, there exists $c = (A, r_0, x_0) > 0$ such that for any $(x, t) \in M \times [0, \infty)$ with*

$$d_{g(t)}(x, x_0) \leq A,$$

we have

$$Vol_{g(t)}(B_{g(t)}(x, r_0)) \geq c e^{-(n-\kappa)t}.$$

Proof. By Lemma 2.2, the scalar curvature of $g(t)$ is uniformly bounded for all $t \geq 0$. Proposition 3.1 implies that there exists $c = c(A, r_0, x_0) > 0$ such that for any $t \geq 0$,

$$\mathrm{Vol}_{g(t)}(B_{g(t)}(x, r_0)) \geq c\mathrm{Vol}_{g(t)}(B_{g(t)}(x_0, r_0)).$$

The lemma then immediately follows by combining the above estimate and (4.20). □

PROPOSITION 4.1. *For any $\varepsilon > 0$, there exists $T > 0$ such that for all $t \geq T$ and $x \in X$,*

$$d_{g(t)}(x, x_0) < diam_{d_{\mathcal{Y}}}(\mathcal{Y}) + \varepsilon.$$

Proof. We will prove by contradiction. Let $\mathcal{D} = \mathrm{diam}_{d_{\mathcal{Y}}}(\mathcal{Y})$. Suppose there exist $\epsilon > 0$, $t_j \to \infty$ and $x_j \in X$ such that

$$\mathcal{D} + \epsilon \leq d_{g(t_j)}(x_j, x_0) \leq \mathcal{D} + 2\epsilon.$$

By Lemma 4.4, there exists $c > 0$ such that for all j,

$$\mathrm{Vol}_{g(t_j)}\left(B_{g(t_j)}\left(x_j, \frac{\varepsilon}{2}\right)\right) \geq 2ce^{-(n-\kappa)t_j},$$

and by Lemma 4.1, there exists an open $\mathcal{K} \subset\subset X^\circ$ such that for all $t \geq 0$,

$$\mathrm{Vol}_{g(t)}(X \setminus \mathcal{K}) \leq ce^{-(n-\kappa)t}.$$

By Corollary 4.1, there exist $\mathcal{K} \subset \mathcal{K}' \subset\subset X^\circ$ and $T > 0$ such that

$$\mathrm{diam}_{g(t)}(\mathcal{K}') < \mathcal{D} + \frac{\varepsilon}{2}$$

for $t \geq T$.

Therefore for sufficiently large j, we have

$$x_j \in X \setminus \mathcal{K}'$$

and

$$B_{g(t_j)}\left(x_j, \frac{\varepsilon}{2}\right) \subset X \setminus \mathcal{K}'.$$

This implies that

$$2ce^{-(n-\kappa)t_j} \leq \mathrm{Vol}_{g(t_j)}\left(B_{g(t_j)}\left(x_j, \frac{\varepsilon}{2}\right)\right) \leq \mathrm{Vol}_{g(t_j)}(X \setminus \mathcal{K}') \leq ce^{-(n-\kappa)t_j},$$

which gives contradiction. □

Theorem 1.1 immediately from Proposition 4.1. Corollary 1.1 is proved by combining Theorem 1.1, Proposition 3.1 and Lemma 4.3.

5 Global Convergence

In this section, we will study the convergence of the Kähler–Ricci flow (1.1) on an n-dimensional Kähler manifold X with semi-ample K_X.

LEMMA 5.1. *Let $g(t)$ be the solution of the Kähler–Ricci flow on the n-dimensional Kähler manifold X with semi-ample canonical bundle. Then for any $t_j \to \infty$, after possibly passing to a subsequence, $(X, g(t_j))$ converges to a compact metric space (X_∞, d_∞).*

Proof. For any $t \geq 0$ and any $\varepsilon > 0$, we let $B = \{B_{g(t)}(x, \varepsilon)\}_{x \in X}$ be an open covering of X with balls of radius $\varepsilon > 0$. By Vitali covering lemma, we can find a countable sub-collection $B_{\mathcal{J}} = \{B_{g(t)}(x_j(t), \varepsilon)\}_{j \in \mathcal{J}}$ of B such that

$$B_{g(t)}(x_{j_1}(t), \varepsilon) \cap B_{g(t)}(x_{j_2}(t), \varepsilon) = \phi$$

for any $j_1 \neq j_2 \in \mathcal{J}$ and

$$X \subset \cup_{j \in \mathcal{J}} B_{g(t)}(x_j(t), 5\varepsilon).$$

Let $|\mathcal{J}|$ be the cardinal number of the set $\mathcal{J}$. Also by Lemma 2.2, there exists $C > 0$ such that for any $t \geq 0$, By Corollary 1.1, there exists $c > 0$ such that for any $t \geq 0$ and any $x \in X$,

$$\mathrm{Vol}_{g(t)}(B_{g(t)}(x, \varepsilon)) \geq ce^{-(n-\kappa)t}.$$

This implies that

$$c|\mathcal{J}|e^{-(n-\kappa)t} \leq \sum_{j \in \mathcal{J}} \mathrm{Vol}_{g(t)}(B_{g(t)}(x_j(t), \varepsilon) \leq \mathrm{Vol}_{g(t)}(X) \leq Ce^{-(n-\kappa)t}$$

and so

$$|\mathcal{J}| \leq c^{-1}C.$$

Therefere for any $\varepsilon > 0$, there exists $C > 0$ such that for any $t \geq 0$, there exists a finite 5ε-net $\{x_j(t)\}_{j \in \mathcal{J}(\varepsilon,t)}$ of $(X, g(t))$ such that

$$|\mathcal{J}_{\varepsilon,t}| \leq C.$$

Then we can apply Gromov's precompactness theorem [Gro99] and any sequence $(X, g(t_j))$ with $t_j \to \infty$, $(X, g(t_j))$ converges to a compact metric space (X_∞, d_∞) after passing to subsequence. □

We let (Y, d_Y) be the metric completion of $(X^\circ_{\mathrm{can}}, g_{\mathrm{can}})$ and $(X_{\mathrm{can}}, g_{FS})$ be the canonical model equipped with the Fubini-Study metric. Since the solution of the Kähler–Ricci flow $g(t)$ converges in C^0 to the twisted Kähler-Einstein metric on X°_{can}. X°_{can} can be naturally embedded in X_∞ and Y, i.e., there exist identity maps from open sets of X_∞ and Y to X°_{can}. We then identify X°_{can} as $Y^\circ \subset Y$ and $X^\circ_\infty \subset X_\infty$. Naturally, one would ask if these identity maps extend to unique homeomorphisms. The following lemma gives the relation among (X_∞, d_∞), (Y, d_Y) and $(X_{\mathrm{can}}, g_{FS})$.

LEMMA 5.2. *The identity maps from Y° and X°_∞ to X°_{can} extend uniquely to the following Lipschitz maps*

$$\Psi : (Y, d_Y) \to (X_\infty, d_\infty), \ \ \Upsilon : (X_\infty, d_\infty) \to (X_{can}, g_{FS})$$

In particular, $\Upsilon|_{\Upsilon^{-1}(X^\circ_{can})}$ and $\Psi|_{(\Upsilon\circ\Psi)^{-1}(X^\circ_{can})}$ are identity maps from X°_∞ and Y° to X°_{can}.

Proof. Υ is well-defined because by Lemma 2.2, there exists $c > 0$ such that for any $t \geq 0$,

$$g(t) \geq c\ \Phi^* g_{FS}$$

on X. Therefore the extension of the identity map from X°_∞ to $X^\circ_{\rm can}$ coincides with the limit of Φ and it is Lipschitz from X_∞ to $X_{\rm can}$.

Since $g(t)$ converges to $g_{\rm can}$ on $X^\circ_{\rm can}$ in local C^0-topology, and by the result of [STZ], $(X^\circ_{\rm can}, g_{\rm can})$ is almost geodesic convex, then for any two points $y_1, y_2 \in X^\circ_{\rm can}$ and any $x_1 \in X_{y_1}$ and $x_2 \in X_{y_2}$,

$$\liminf_{t\to\infty} d_{g(t)}(x_1, x_2) \leq d_{g_{\rm can}|_{X^\circ_{\rm can}}}(y_1, y_2) = d_Y(y_1, y_2).$$

Now pick any two points $p, q \in X_\infty$. There exist $p_j, q_j \in X^\circ$ with $p_j \to p$ and $q_j \to q$ in Gromov–Hausdorff distance with respect to $g(t_j)$ as $t_j \to \infty$. On the other hand, $\Phi(p_j), \Phi(q_j) \in X^\circ_{\rm can}$ converge to some y_1 and $y_2 \in Y$ with respect to d_Y after passing to a subsequence. Then we have

$$d_\infty(p, q) = \lim_{j\to\infty} d_{g(t_j)}(p_j, q_j) \leq d_Y(y_1, y_2).$$

This implies that Ψ is well-defined and Lipschitz. □

LEMMA 5.3. *If $kod(X) = n$ or $kod(X) \leq 2$, then both Ψ and Υ are homeomorphic.*

Proof. If $\kod(X) = n$, by the result of [Song2], (Y, d_Y) is homeomorphic to the projective variety $X_{\rm can}$. This forces (X_∞, d_∞) to be homeomorphic to $X_{\rm can}$.

If $\kod(X) = 2$, then the canonical model $X_{\rm can}$ is an orbifold Kähler surfaces since $X_{\rm can}$ is KLT. By the result of Song-Tian-Zhang (Proposition 2.3 in [STZ]), (Y, d_Y) is homeomorphic to $X_{\rm can}$ and by Lemma 5.2 (X_∞, d_∞) must be homeomorphic to $X_{\rm can}$. The same argument extends to the case $\kod(X) \leq 1$ since $X_{\rm can}$ is either a point or a smooth Riemann surface. □

We now have completed the proof of Corollary 1.2 by combining Lemma 5.2 and Lemma 5.3.

Acknowledgments

The authors would like to thank Gang Tian, Zhenlei Zhang, Yalong Shi for inspiring discussions. The authors are grateful to Valentino Tosatti for pointing out a mistake in the Ricci curvature estimates in the earlier version. Part of this work was carried out during the first named author's visit at Beijing International Center for Mathematical Research (BICMR). He would like to thank BICMR and Kewei Zhang for the hospitality and support. W.Jian is supported by China Postdoctoral Science Foundation Grant No. 2019M660827 and the National Natural Science Foundation of China (No. 12288201). J. Song is supported by National Science Foundation Grants DMS-1711439.

References

[Aub78] T. Aubin. Équations du type Monge-Ampère sur les variétés kählériennes compactes,*Bull. Sci. Math. (2)* 102, 1 (1978), 63–95

[Bam18] R. Bamler. Convergence of Ricci flows with bounded scalar curvature, *Ann. Math. (2)* 188, 3 (2018), 753–831

[Bam] R. Bamler. Entropy and heat kernel bounds on a Ricci flow background, arXiv:2008.07093

[CHP16] F. Campana, A. Höring, and T. Peternell. Abundance for Kähler threefolds, *Annales de l'ENS,* 49, 3 (2016), 971–105

[CW17] X. Chen and B. Wang. Space of Ricci flows (II)—part A: moduli of singular Calabi-Yau spaces, *Forum Math. Sigma* 5 (2017), e32, 103 pp.

[Cao85] H.-D. Cao. Deformation of Kähler metrics to Kähler–Einstein metrics on compact Kähler manifolds, *Invent. Math.* 81, 2 (1985), 359–372

[DS10] R. Dervan and G. Szekelyhidi. The Kähler–Ricci flow and optimal degenerations, *J. Differ. Geom.* 116, 1 (2010), 187–203

[EGZ09] P. Eyssidieux, V. Guedj, and A. Zeriahi. Singular Kähler–Einstein metrics, *J. Am. Math. Soc.* 22, 3 (2009), 607–639

[FL] F.T.-H. Fong and M.C. Lee. Higher-order estimates of long-time solutions to the Kähler–Ricci flow, arXiv:2001.11555.

[FZ15] F.T.-H. Fong and Z. Zhang. The collapsing rate of the Kähler–Ricci flow with regular infinite time singularity, *J. Reine Angew. Math.* **703** (2015), 95–113

[FZ20] F.T.-H. Fong and Y. Zhang. Local curvature estimates of long-time solutions to the Kähler–Ricci flow, *Adv. Math.* 375 (2020) 107416. MR 4170232

[FGS20] X. Fu, B. Guo, and J. Song. Geometric estimates for complex Monge–Ampére equations, *J. Reine Angew. Math.* 765 (2020), 69–99

[Gro99] M. Gromov. Metric structures for Riemannian and non-Riemannian spaces, Progress in Mathematics, 152. *Birkhuser Boston, Inc.*, Boston, MA, (1999). xx+585 pp.

[GS] B. Guo and J. Song. Positivity of Weil–Petersson currents on canonical models and its applications, preprint

[GPS] B. Guo,D.H. Phong, and J. Sturm. On the Kähler–Ricci flow on Fano manifolds, arXiv:2001.06329

[GSW16] B. Guo, J. Song, and B. Weinkove. Geometric convergence of the Kähler–Ricci flow on complex surfaces of general type, *I.M.R.N.* 18 (2016) 5652–5669

[Ham82] R. S. Hamilton. Three-manifolds with positive Ricci curvature, *J. Differ. Geom.* 17, 2 (1982), 255–306

[HL] J. Han and C. Li. Algebraic uniqueness of Kähler–Ricci flow limits and optimal degenerations of Fano varieties, arXiv:2009.01010

[Jia20] W. Jian. Convergence of scalar curvature of Kähler–Ricci flow on manifolds of positive Kodaira dimension, *Adv. Math.* 371 (2020) 107253. MR 4108223

[JS] W. Jian and Y. Shi. A "boundedness implies convergence" principle and its applications to collapsing estimates in Kähler geometry, 1909.05521, accepted by Nonlinear Analysis

[Kaw85] Y. Kawamata. Pluricanonical systems on minimal algebraic varieties, *Invent. Math.* 79 (1985), 567–588

[Kaw92] Y. Kawamata. Abundance theorem for minimal threefolds, *Invent. Math.* 108 (1992), 229–246

[Kol98] S. Kolodziej. The complex Monge–Ampère equation, *Acta Math.* 180, 1 (1998), 69–117

[Miy98] Y. Miyaoka. Abundance conjecture for 3-folds: case $\nu = 1$, *Compos. Math.* 68 (1998), 203–332

[Pere1] G. Perelman. The entropy formula for the Ricci flow and its geometric applications, preprint, arXiv:math/0211159 [math.DG]

[Pere2] G. Perelman. Unpublished work on the Kähler–Ricci flow

[Ses08] N. Sesum and G. Tian. Bounding scalar curvature and diameter along the Kähler Ricci flow (after Perelman), *J. Inst. Math. Jussieu* 7, 3 (2008), 575–587

[Song1] J. Song. *Ricci flow and birational surgery*, arXiv:1304.2607

[Song2] J. Song. *Riemannian geometry of Kähler–Einstein currents*, arXiv:1404.0445

[ST07] J. Song, and G. Tian. The Kähler–Ricci flow on surfaces of positive Kodaira dimension, *Invent. Math.* 170, 3 (2007), 609–653

[ST12] J. Song and G. Tian. Canonical measures and Kähler–Ricci flow,*J. Am. Math. Soc.* 25, 2 (2012), 303–353

[ST17] J. Song and G. Tian. The Kähler–Ricci flow through singularities, *Invent. Math.* 207, 2 (2017), 519–595

[ST16] J. Song and G. Tian. Bounding scalar curvature for global solutions of the Kähler–Ricci flow, *Am. J. Math.* 138, 3 (2016), 683–695

[STZ] J. Song, G. Tian, and Z. Zhang. *Collapsing behavior of Ricci-flat Kähler metrics and long-time solutions of the Kähler–Ricci flow*, arXiv:1904.08345

[SW16] J. Song and B. Weinkove. Contracting exceptional divisors by the Kähler–Ricci flow, *Duke Math. J.* 162, 2 (2016), 367–415

[SW14] J. Song and B. Weinkove. Contracting divisors by the Kähler–Ricci flow II, *Proc. Lond. Math. Soc. (3)* 108, 6 (2014) 1529–1561

[SY12] J. Song and Y. Yuan. Metric flips with Calabi ansatz, *Geom. Funct. Anal.* 22, 1 (2012), 240–265
[TZ16] G. Tian and Z.L. Zhang. Convergence of Kähler–Ricci flow on lower-dimensional algebraic manifolds of general type, *Int. Math. Res. Not. IMRN* 21 (2016), 6493–6511
[TZ21] G. Tian and Z.L. Zhang. Relative volume comparison of Ricci Flow, *Sci. China Math.* (2021). https://doi.org/10.1007/s11425-021-1869-5
[TZ06] G. Tian and Z. Zhang. On the Kähler–Ricci flow on projective manifolds of general type, *Chin. Ann. Math. Ser. B* 27, 2 (2006), 179–192
[TZ07] G. Tian and X. Zhu. Convergence of Kähler–Ricci flow, *J. Am. Math. Soc.* 20, 3 (2007), 675–699
[TZZZ13] G. Tian, S. Zhang, Z. Zhang, and X.H. Zhu. Perelman's entropy and Kähler–Ricci flow on a Fano manifold, *Trans. Am. Math.* Soc. 365, 12 (2013), 6669–6695
[TWY18] V. Tosatti, B. Weinkove, and X. Yang. The Kähler–Ricci flow, Ricci-flat metrics and collapsing limits, *Am. J. Math.* 140, 3 (2018), 653–698
[Tsu88] H. Tsuji Existence and degeneration of Kähler–Einstein metrics on minimal algebraic varieties of general type, *Math. Ann.* 281 (1988), 123–133
[Wan18] B. Wang The local entropy along Ricci flow, Part A: the no-local-collapsing theorems, *Camb. J. Math.* 6, 3 (2018), 267-346.
[Yau78] S.-T. Yau On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I, *Commun. Pure Appl. Math.* 31 (1978), 339–411
[Zha19] Y. Zhang. Collapsing limits of the Kähler–Ricci flow and the continuity method, *Math. Ann.* 374, 1–2 (2019), 331–360
[ZZ19] Y. Zhang and Z. Zhang The continuity method on minimal elliptic Kähler surfaces, *Int. Math. Res. Not. IMRN* 10 (2019), 3186–3213
[Zha06] Z. Zhang. On degenerate Monge–Ampère equations over closed Kähler manifolds, *Int. Math. Res. Not.* 2006, Art. ID 63640, 18 pp
[Zha09] Z. Zhang. Scalar curvature bound for Kähler-Ricci flows over minimal manifolds of general type, *Int. Math. Res. Not.* 2009; https://doi.org/1093/imrn/rnp073

W. Jian
Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China. wangjian@amss.ac.cn

J. Song
Department of Mathematics, Rutgers University, Piscataway,
NJ 08854, USA. jiansong@math.rutgers.edu

Received: June 25, 2022
Revised: September 22, 2022
Accepted: September 29, 2022