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A Klein TQFT: The local Real Gromov-Witten theory of curves

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In this paper we study the Real Gromov-Witten theory of local 3-folds over Real curves. We show that this gives rise to a 2-dimensional Klein TQFT defined on an extension of the category of unorientable surfaces. We use this structure to completely solve the theory by providing a closed formula for the local RGW invariants in terms of representation theoretic data, extending earlier results of Bryan and Pandharipande. As a consequence we obtain the local version of the real Gopakumar-Vafa formula that expresses the connected real Gromov-Witten invariants in terms of integer invariants. In the case of the resolved conifold the partition function of the RGW invariants agrees with that of the SO/Sp Chern-Simons theory.

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1. Introduction

A central problem in Gromov-Witten theory is understanding the structure of the Gromov-Witten invariants. Of special interest is the case when the target manifold is a Calabi-Yau threefold. Often, studying local versions of these theories (i.e. for non-compact targets) reveals much of the structure in the general case. The Gromov-Witten invariants come in several flavors: (a) closed (counting closed curves), (b) open (counting curves with boundary on a Lagrangian or SFT-type curves), and (c) real (counting closed curves invariant under an anti-symplectic involution).

In this paper we consider Real Gromov-Witten (RGW) invariants and we prove a structure result for the local RGW invariants of Real[1] 3-folds that are the total space of bundles over curves with an anti-symplectic involution (also referred to as a real structure). We show that the local RGW invariants give rise to a semi-simple 2d Klein TQFT which allows us to completely solve the theory. The motivation for considering 3-folds of this type comes from the virtual contribution to the real GW invariants of a Real elementary curve in a compact Real Calabi-Yau 3-fold, sometimes referred to as multiple-covers contribution, and the real Gopakumar-Vafa conjecture. The Gopakumar-Vafa conjecture [18] and its extension proved in [19] has an analogue in the setting of Real Calabi-Yau 3-folds, cf. [27]. The local version of the real GV conjecture is obtained in this paper as a consequence of the structure result. The case of compact 3-folds will be discussed in a subsequent paper.

A symmetric (or Real) Riemann surface is a Riemann surface Σ together with an anti-holomorphic involution $c : \Sigma \to \Sigma$. If $L \to \Sigma$ is a holomorphic line bundle, then the total space of

$$L \oplus c^*\overline{L} \to \Sigma \tag{1.1}$$

is a Real manifold with an anti-holomorphic involution

$$c_{tw}(z; u, v) = (c(z); v, u).$$

[1] We use Real with capital R for spaces with anti-J-invariant involutions, following Atiyah.

These are the Real 3-folds we consider in this paper, and we refer to them as local Real curves; note however that any rank 2 Real bundle $(V, \phi) \to (\Sigma, c)$ whose fixed locus V^ϕ is orientable is isomorphic to a Real bundle (1.1) for L a complex line bundle with $c_1(L) = \frac{1}{2}c_1(V)$, cf. [3, §4.1]. Moreover, an $U(1)$-action on the line bundle $L \to \Sigma$ induces an action on the 3-fold (1.1) compatible with the Real structure. In §2 we define local RGW invariants associated to the Real 3-fold (1.1) as pairings between the $U(1)$-equivariant Euler class of the index bundle Ind $\overline{\partial}_L$ (regarded as an element in K-theory) and the virtual fundamental class of the real moduli space $\overline{\mathcal{M}}^{c,\bullet}_{d,\chi}(\Sigma)$ of degree d real maps $f : C \to \Sigma$ from (possibly disconnected) domains of Euler characteristic χ. The real moduli space $\overline{\mathcal{M}}^{c,\bullet}_{d,\chi}(\Sigma)$ is orientable, but not a priori canonically oriented; the orientation depends on a choice of orientation data $\mathfrak{o}$ discussed in §2 and in the Appendix. The (shifted) generating functions for the local RGW invariants

$$RGW^{c,\mathfrak{o}}_d(\Sigma, L) \in \mathbb{Q}(t)((u))$$

take values in the localized equivariant cohomology ring of $U(1)$ generated by t; here u keeps track of the Euler characteristic of the domain.

We also consider a relative version of the RGW invariants for a branching divisor on (Σ, c) consisting of pairs of conjugate points. For the purposes of this paper, we can restrict attention to the case when none of the marked points or special points are real. The splitting formula of [14] then allows us to relate the local RGW invariants of Σ to the local RGW invariants of a decomposition of Σ along pairs of conjugate circles; see §4.

A priori, the local RGW invariants depend on the choice of an orientation data $\mathfrak{o}$ and the topological type of the real structure c on Σ. In §6, we show that there is a canonical choice of orientation for the local RGW invariants, and moreover these do not depend on the real structure c. We therefore use the notation

$$RGW_d(\Sigma, L) \in \mathbb{Q}(t)((u))$$

afterwards, when the canonical choice is assumed. Any other choice of a twisted orientation data changes RGW_d by $(\pm 1)^d$.

In §8 we show that the local RGW invariants determine an extension $\mathbf{RGW}_d$ of a 2-dimensional Klein TQFT. As we review in §7, a 2d Klein TQFT is a symmetric monoidal functor from the cobordism category 2**KCob** of unoriented surfaces to the category of R-modules for some ring R. Since 2**KCob** naturally contains the oriented cobordism category 2**Cob**, a Klein TQFT is an extension of a classical TQFT; it is equivalent to a Frobenius algebra with an involution Ω, which is the image of the orientation reversing tube, and a special element U, which is the image of the crosscap (Möbius band), cf. §7.

The connection with real Gromov-Witten theory is obtained by considering an equivalent category 2**SymCob** whose objects are pairs of closed oriented 1-dimensional manifolds and the cobordisms are symmetric (Real) surfaces. It is obtained from **2KCob** by passing to the orientation double cover. Then the involution Ω is the image of the symmetric cobordism swapping the components of an object, while U is the image of a symmetric sphere with a pair of (disjoint) conjugate disks removed. This perspective allows us to define in §7.2 an extension $2\mathbf{SymCob}^L$ which has the same objects but where the cobordisms also carry a complex vector bundle trivialized along the boundary. As we prove in §8, the local RGW invariants give rise to a symmetric monoidal functor $\mathbf{RGW}_d$ on $2\mathbf{SymCob}^L$; up to factors due to differing conventions, this extends the TQFT considered by Bryan and Pandharipande in [5] for the anti-diagonal action. In turn, the Bryan-Pandharipande construction similarly extends a classical construction studied by Dijkgraaf-Witten [7] and Freed-Quinn [12].

In §7.1 we discuss semi-simple KTQFTs, i.e. those for which the associated Frobenius algebra has an idempotent basis. Their restriction to the oriented cobordism category **2Cob** is determined by the eigenvalues $\{\lambda_\rho\}$ of the genus adding operator (which is diagonalized in the idempotent basis). To completely determine the KTQFT it then suffices to find the coefficients of Ω and U in the idempotent basis. We show that Ω restricts to an involution $v_\rho \mapsto v_{\rho^*}$ on the idempotent basis and that each coefficient U_ρ of U is 0 when $\rho \neq \rho^*$ and otherwise is equal to a squareroot $\pm\sqrt{\lambda_\rho}$ of the eigenvalue λ_ρ.

In §9 we prove that the KTQFT determined by the level 0 local RGW invariants is semisimple. It corresponds in fact to *signed* counts of degree d real Hurwitz covers. The idempotent basis is indexed by irreducible representations of the symmetric group S_d and $\Omega(v_\rho) = v_{\rho'}$ where ρ' is the conjugate representation. In order to calculate the coefficients of U in the idempotent basis, we introduce in §11 the signed Frobenius-Schur indicator (SFS). The SFS takes values $0, \pm 1$ on irreducible real representations, unlike the standard FS indicator which is $+1$ on them. The SFS is 0 if and only if the representation is *not* self-conjugate and the sign of a self-conjugate representation is given as a function of its characters. While these considerations are valid for real representations of any finite group, in the case of the symmetric group we find a simpler expression for the latter function using the Weyl formula. In particular, for an irreducible self-conjugate representation ρ of S_d,

$$SFS(\rho) = (-1)^{(d-r(\rho))/2},$$

where $r(\rho)$ is the rank of ρ, i.e. the length of the main diagonal of the Young diagram associated to ρ. This is precisely the sign that appears in the partition function of the SO/Sp Chern-Simons theory [4, (6.1)]; in the case of the resolved conifold, Theorems 1.1 and 1.2 below recover the partition function [4, (6.3)] and the free energy [4, (3.2)], respectively.

Combining these results we obtain in §9 a closed expression for the local RGW theory of the 3-fold (1.1) in terms of representation theoretic data, cf. Theorem 9.13. In the Calabi-Yau case it takes the following form:

Theorem 1.1 *(Local CY). Let Σ be a connected genus g symmetric surface and $L \to \Sigma$ a holomorphic line bundle with Chern number $g-1$. Then the generating function of the degree d local RGW invariants is equal to*

$$RGW_d(\Sigma, L) = \sum_{\rho=\rho'} \left((-1)^{\frac{d-r(\rho)}{2}} \prod_{\square \in \rho} 2\sinh \frac{h(\square)u}{2} \right)^{g-1}.$$

Here the sum is over all self-conjugate partitions ρ of d, the product is over all boxes $\square$ in the Young diagram of ρ, $h(\square)$ is the hooklength of $\square$, and $r(\rho)$ is the length of the main diagonal of the Young diagram of ρ.

The local RGW invariants correspond to possibly disconnected counts. As usual they can be expressed in terms of more basic invariants. In the real GW theory these basic counts come in two flavors, $CRGW_d(\Sigma, L)$ and $DRGW_d(\Sigma, L)$, corresponding to maps from connected Real domains and respectively from doublet domains i.e. domains consisting of two copies of a connected surface with opposite complex structures and the real structure exchanging the two copies. In fact

$$1 + \sum_{d=1}^{\infty} RGW_d(\Sigma, L) q^d = \exp\left(\sum_{d=1}^{\infty} CRGW_d(\Sigma, L) q^d + \sum_{d=1}^{\infty} DRGW_{2d}(\Sigma, L) q^{2d} \right).$$

Furthermore, the doublet invariants are related to half of the complex GW invariants whenever the target Σ is connected:

$$DRGW_{2d}(\Sigma, L)(u, t) = (-1)^{d(k+1-g)} \frac{1}{2} GW_d^{conn}(g|k, k)(iu, it),$$

where g is the genus of Σ, $k = c_1(L)[\Sigma]$ is the degree of L, and $GW_d^{conn}(g|k, k)$ are the connected invariants defined in [5] for the anti-diagonal action; see Corollary 3.9.

As a consequence of the structure result provided by Theorem 1.1, in §10 we obtain the local real Gopakumar-Vafa formula; for a complete statement, see Theorem 10.1.

Theorem 1.2 *(Local real GV formula). Let $L \oplus c^*\overline{L} \longrightarrow \Sigma$ be a local Real Calabi-Yau 3-fold with base a genus g symmetric surface (Σ, c). Then the generating function for the connected real Gromov-Witten invariants has the form:*

$$\sum_{d=1}^{\infty} CRGW_d(\Sigma|L)(u) q^d = \sum_{d=1}^{\infty} \sum_{h=0}^{\infty} n_{d,h}^{\mathbb{R}}(g) \sum_{k \text{ odd}} \tfrac{1}{k} (2\sinh(\tfrac{ku}{2}))^{h-1} q^{kd},$$

where the coefficients $n_{d,h}^{\mathbb{R}}(g)$, called the real BPS states, satisfy (i) (integrality) $n_{d,h}^{\mathbb{R}}(g) \in \mathbb{Z}$, (ii) (finiteness) for each d, $n_{d,h}^{\mathbb{R}}(g) = 0$ for large h, and (iii) (parity) $n_{d,h}^{\mathbb{R}}(g)$ has same parity as the complex BPS states $n_{d,h}^{\mathbb{C}}(g)$.

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2. Local Real Gromov-Witten invariants

2.1. Real GW invariants

We start with a brief overview of the real Gromov-Witten invariants. Let (X, ω) be a symplectic manifold and ϕ an anti-symplectic involution on X. A symmetric Riemann surface (C, σ) is a closed, oriented, possibly nodal, possibly disconnected Riemann surface Σ with an anti-holomorphic involution σ. A real map

$$f : (C, \sigma) \longrightarrow (X, \phi)$$

is a map $f : C \to X$ such that $u \circ \sigma = \phi \circ u$. Let $\mathcal{J}_\omega^\phi$ denote the space of ω-compatible almost complex structures J on X which satisfy $\phi^* J = -J$. For $\chi \in \mathbb{Z}$ and $B \in H_2(X, \mathbb{Z})$, denote by

$$\overline{\mathcal{M}}_{B,\chi}^{\phi,\bullet}(X)$$

the moduli space of equivalence classes (up to reparametrization of the domain) of stable degree B J-holomorphic real maps from symmetric Riemann surfaces of Euler characteristic χ, for $J \in \mathcal{J}_\omega^\phi$. We will consider only the case when the restriction of the maps to each connected component of the domain is *nontrivial.*

In this paper we restrict ourselves to target manifolds which are themselves symmetric Riemann surfaces. We will use (Σ, c) to denote the target curve and d for the degree of the map. The real moduli space is denoted

$$\overline{\mathcal{M}}_{d,\chi}^{c,\bullet}(\Sigma),$$

and consists of real maps $f : (C, \sigma) \to (\Sigma, c)$ whose domain may be disconnected. The involution on the domain decomposes the domain into real components and pairs of conjugate components. Following [16, (1.7)], an h-doublet is a real surface

$$(C, \sigma) = (C_1 \sqcup C_2, \sigma) = (S \sqcup \overline{S}, \sigma), \text{ where } \quad \sigma|_S = id : S \longrightarrow \overline{S}, \tag{2.1}$$

S is a genus h Riemann surface, and $\overline{S}$ denotes the curve S but with the opposite complex structure. Note that every real curve that has two components swapped by the involution is equivalent (up to reparametrization) to a doublet.

When Σ is connected, it is therefore convenient to consider the following two moduli spaces:

$$\overline{\mathcal{M}}^c_{d,h}(\Sigma) \quad \text{and} \quad \overline{D\mathcal{M}}^c_{d,h}(\Sigma), \tag{2.2}$$

where the first one consists of maps with connected domains of genus h and the second one consists of maps whose domains are h-doublets. Let

$$\overline{\mathcal{M}}^{\bullet}_{d,\chi}(\Sigma)$$

denote the classical moduli space of holomorphic maps from possibly nodal, possibly disconnected domains of Euler characteristic χ and degree d to Σ (whose restrictions to each connected component is nontrivial). Finally, denote by

$$\overline{\mathbb{R}\mathcal{M}}^{\bullet}_{\chi,\ell} \tag{2.3}$$

the real Deligne-Mumford moduli space parametrizing (possibly disconnected) symmetric surfaces (C, σ) of Euler characteristic χ with ℓ pairs of conjugate marked points

$$\{(y_1^+, y_1^-), \dots, (y_\ell^+, y_\ell^-)\}, \quad \text{where } y_i^- = \sigma(y_i^+). \tag{2.4}$$

The corresponding moduli spaces of connected real and doublet domains are denoted by $\overline{\mathbb{R}\mathcal{M}}_{g,\ell}$ and $\overline{D\mathcal{M}}_{g,\ell}$ respectively.

2.2. Twisted real orientations

The real moduli spaces are not in general orientable. In [15, Definition 1.2] a notion of real orientation was introduced whose existence ensures the orientability of the real moduli spaces when the target has odd complex dimension, cf. [15, Theorem 1.3]. This notion can be extended to a twisted orientation as in Definition 2.1 below when the target is a surface; see Definition A.1 for a general target. In the appendix we show that [15, Theorem 1.3] extends to this setting: a choice of twisted real orientation on an odd dimensional target determines a canonical orientation of the moduli spaces of real maps to that target. While a real orientation in the sense of [15] does not exist on a symmetric surface of even genus and fixed-point free involution, a twisted orientation exists on every symmetric surface.

As in (1.1), when $L \to (\Sigma, c)$ is a complex bundle then

$$(L \oplus c^*\overline{L}, c_{tw}) \longrightarrow (\Sigma, c), \quad \text{where} \quad c_{tw}(z; u, v) = (c(z); v, u), \tag{2.5}$$

is a Real bundle (i.e. a real bundle pair in the sense of [15, §1.1]). Note that the projection onto the first factor identifies the fixed locus of c_{tw} with

$$(L \oplus c^*\overline{L})^{c_{tw}} \cong L_{|\Sigma^c},$$

where Σ^c is the fixed locus of c.

Definition 2.1. Assume (Σ, c) is a symmetric surface. A twisted (real) orientation data

$$\mathfrak{o} = (\Theta, \psi, \mathfrak{s}) \tag{2.6}$$

for (Σ, c) consists of

(i) a complex line bundle Θ over Σ such that $c_1(\Theta \otimes c^*\overline{\Theta}) = -\chi(T\Sigma)$,
(ii) a homotopy class of isomorphisms

$$\Lambda^{\text{top}}(T\Sigma \oplus (\Theta \oplus c^*\overline{\Theta}), dc \oplus c_{tw}) \overset{\psi}{\cong} (\Sigma \times \mathbb{C}, c \times c_{std}) \tag{2.7}$$

where $c_{std} : \mathbb{C} \to \mathbb{C}$ is the standard complex conjugation.
(iii) a spin structure $\mathfrak{s}$ on the fixed locus $T\Sigma^c \oplus \Theta_{|\Sigma^c}$, compatible with the orientation induced by (2.7).

Up to deformation, rank r complex or holomorphic bundles on a Riemann surface are determined by their first Chern class. Similarly, rank r Real bundles $(V, \phi) \longrightarrow (\Sigma, c)$ are classified by $c_1(V)$ and $w_1(V^\phi)$, cf. [3]. In particular, condition (i) above ensures the existence of an isomorphism (2.7).

Example 2.2.

(a) When Σ^c is empty, there is no spin structure $\mathfrak{s}$ involved. Thus a choice of twisted orientation in this case corresponds only to a choice (Θ, ψ).
(b) When (Σ, c) is a g-doublet, Θ restricts to a line bundle on each component Σ_i; let $m_i = c_1(\Theta)[\Sigma_i]$ denote the degrees. Since $(c^*\overline{\Theta})_{|\Sigma_1} = c^*(\overline{\Theta}_{|\Sigma_2})$, condition (i) restricts the total degree

$$m = m_1 + m_2 = 2g - 2. \tag{2.8}$$

For any fixed complex line bundle Θ over the doublet satisfying (2.8), there is a unique isomorphism (2.7) up to homotopy (determined by the restriction to Σ_1). Moreover, Σ^c is empty for a doublet. Thus a twisted orientation for a doublet consists of a choice of the degrees $m_i \in \mathbb{Z}$ satisfying (2.8).
(c) When (Σ, c) is a connected genus g surface, the degrees of $c^*\overline{\Theta}$ and Θ are equal. In this case, condition (i) implies that the degree of Θ is $m = g - 1$; up to isomorphism, there is only one such complex line bundle.

A twisted orientation $\mathfrak{o}$ on (Σ, c) equips the real moduli spaces $\overline{\mathcal{M}}^{c,\bullet}_{d,\chi}(\Sigma)$ with a canonical orientation, cf. Appendix. In particular, it gives rise to a virtual fundamental class

$$[\overline{\mathcal{M}}^{c,\bullet}_{d,\chi}(\Sigma)]^{\mathrm{vir},\mathfrak{o}}$$

in dimension $b = d\chi(\Sigma) - \chi$.

2.3. Absolute RGW invariants

Consider a holomorphic bundle E over a complex curve Σ. Then the operator $\overline{\partial}_E$ determines a family of complex operators over moduli spaces of maps to Σ; the fiber at a stable map $f : C \to \Sigma$ is the pullback operator $\overline{\partial}_{f^*E}$. Denote by Ind $\overline{\partial}_E$ the index bundle associated to this family of operators, regarded as an element in K-theory.

Assume next L is a holomorphic line bundle over a symmetric surface (Σ, c), and let $E = L \oplus c^*\overline{L}$. It is a rank 2 holomorphic bundle over Σ which has a real structure c_{tw} given by (2.5). Let $\bar{\partial}_{(L\oplus c^*\bar{L}, c_{tw})}$ denote the restriction of $\bar{\partial}_{L\oplus c^*\bar{L}}$ to the invariant part of its domain and target, cf. [15, §4.3]. Via the projection onto the first factor, the kernel and cokernel of $\bar{\partial}_{(L\oplus c^*\bar{L}, c_{tw})}$ are canonically identified with the kernel and cokernel of $\bar{\partial}_L$.

Similarly $\bar{\partial}_{(L\oplus c^*\bar{L}, c_{tw})}$ determines a family of pullback operators over the real moduli space of maps to (Σ, c), and the projection onto the first factor identifies

$$\mathrm{Ind}\, \bar{\partial}_{(L\oplus c^*\bar{L}, c_{tw})} \overset{\pi_1}{\cong} \mathrm{Ind}\, \bar{\partial}_L. \tag{2.9}$$

The right hand side carries a natural complex structure, which pulls back to one on the left hand side. An $U(1)$-action on L induces one on $(L \oplus c^*\bar{L}, c_{tw})$, compatible with the real structure. In turn, these induce $U(1)$-actions on Ind $\overline{\partial}_L$ and Ind $\overline{\partial}_{(L\oplus c^*\bar{L}, c_{tw})}$ and the isomorphism (2.9) identifies their equivariant Euler classes.

Motivated by the Bryan-Pandharipande construction [5, §2.2], we consider the following real version, associated to a local Real 3-fold $(L \oplus c^*\overline{L}, c_{tw}) \to (\Sigma, c)$ defined by (2.5).

Definition 2.3. Assume (Σ, c) is a symmetric surface, L a holomorphic line bundle over Σ and $\mathfrak{o}$ a twisted orientation data (2.6) for (Σ, c). The local Real GW invariants are defined by the equivariant pairings:

$$RZ^{c,\mathfrak{o}}_{d,\chi}(\Sigma, L) = \int\limits_{[\overline{\mathcal{M}}^{c,\bullet}_{d,\chi}(\Sigma)]^{\mathrm{vir},\mathfrak{o}}} e_{U(1)}(-\mathrm{Ind}\, \overline{\partial}_{(L\oplus c^*\bar{L}, c_{tw})}) = \int\limits_{[\overline{\mathcal{M}}^{c,\bullet}_{d,\chi}(\Sigma)]^{\mathrm{vir},\mathfrak{o}}} e_{U(1)}(-\mathrm{Ind}\, \overline{\partial}_L). \tag{2.10}$$

Here $e_{U(1)}$ denotes the $U(1)$-equivariant Euler class.

As in [5, §2.2], we will primarily consider the shifted partition function:

$$RGW_d^{c,\mathfrak{o}}(\Sigma, L) = \sum_{\chi} u^{d(\frac{\chi(\Sigma)}{2}+c_1(L)[\Sigma])-\frac{\chi}{2}} RZ_{d,\chi}^{c,\mathfrak{o}}(\Sigma, L). \tag{2.11}$$

Intrinsically, (2.10) takes values in the equivariant cohomology of a point:

$$RZ_{d,\chi}^{c,\mathfrak{o}}(\Sigma, L) \in H^*_{U(1)}(pt) = H^*(\mathbb{CP}^\infty) = \mathbb{Q}[t].$$

Here t is the equivariant first Chern class of the standard representation of $U(1)$. Then the local invariant (2.10) can be expressed in terms of the equivariant parameter t and an *ordinary* integral:

$$RZ_{d,\chi}^{c,\mathfrak{o}}(\Sigma, L) = t^{\iota-b/2} \int_{[\overline{\mathcal{M}}_{d,\chi}^{c,\bullet}(\Sigma)]^{\mathrm{vir},\mathfrak{o}}} c_{b/2}(-\mathrm{Ind}\,\overline{\partial}_L). \tag{2.12}$$

Here b is the dimension of $\overline{\mathcal{M}}_{d,\chi}^{c,\bullet}(\Sigma)$ and ι the index (virtual complex rank) of $-\mathrm{Ind}\,\overline{\partial}_L$, given by:

$$\iota = \mathrm{rank}_{\mathbb{C}}(-\,\mathrm{Ind}\,\overline{\partial}_L) = -dc_1(L)[\Sigma] - \tfrac{1}{2}\chi. \tag{2.13}$$

Remark 2.4. The invariants $RGW_d^{c,\mathfrak{o}}(\Sigma, L)$ count maps from possibly disconnected real domains. The real structure σ acts on the components of the domain decomposing them into 'real components' (preserved by σ) and 'doublets' (pairs of conjugate components swapped by σ). When Σ is connected, we denote the connected domain invariants by

$$CRGW_d^{c,\mathfrak{o}}(\Sigma, L) = \sum_{h=0}^{\infty} u^{d(\frac{\chi(\Sigma)}{2}+c_1(L)[\Sigma])+h-1} \int_{[\overline{\mathcal{M}}_{d,h}^{c}(\Sigma)]^{\mathrm{vir},\mathfrak{o}}} e_{U(1)}(-\mathrm{Ind}\,\overline{\partial}_L) \tag{2.14}$$

and the doublet domain invariants (which appear only in even degree when Σ is connected) by

$$DRGW_d^{c,\mathfrak{o}}(\Sigma, L) = \sum_{h=0}^{\infty} u^{d(\frac{\chi(\Sigma)}{2}+c_1(L)[\Sigma])+2h-2} \int_{[\overline{D\mathcal{M}}_{d,h}^{c}(\Sigma)]^{\mathrm{vir},\mathfrak{o}}} e_{U(1)}(-\mathrm{Ind}\,\overline{\partial}_L). \tag{2.15}$$

Here $\overline{\mathcal{M}}_{d,h}^{c}(\Sigma)$ and $\overline{D\mathcal{M}}_{d,h}^{c}(\Sigma)$ are the moduli spaces (2.2) of degree d maps with connected genus h domain and h-doublet domain, respectively. Then

$$1 + \sum_{d=1}^{\infty} RGW_d^{c,\mathfrak{o}}(\Sigma, L) q^d = \exp\left(\sum_{d=1}^{\infty} CRGW_d^{c,\mathfrak{o}}(\Sigma, L) q^d + \sum_{d=1}^{\infty} DRGW_{2d}^{c,\mathfrak{o}}(\Sigma, L) q^{2d}\right). \tag{2.16}$$

2.4. Notation for partitions

A partition λ is a finite sequence of positive integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell)$. A partition of d, denoted $\lambda \vdash d$, is a partition such that the sum of its parts, denoted $|\lambda|$, is equal to d. Its length (number of parts ℓ) is denoted $\ell(\lambda)$. We can also write a partition in the form $\lambda = (1^{m_1} 2^{m_2} \dots)$ where m_k is the number of parts of λ equal to k. Then

$$d = |\lambda| = \sum_{i=1}^{\ell} \lambda_i = \sum_{k=1}^{\infty} k m_k \quad \text{and} \quad \ell(\lambda) = \ell = \sum_{k=1}^{\infty} m_k.$$

We will also consider the following combinatorial factor

$$\zeta(\lambda) = \prod m_k! k^{m_k}. \tag{2.17}$$

A partition λ is uniquely determined by its Young diagram and the conjugate partition λ' is obtained by reflecting λ across the main diagonal. The rank

$$r(\lambda) \tag{2.18}$$

of a partition is the length of the main diagonal of its Young diagram, cf. [13, §4.1].

2.5. Relative RGW invariants

Assume next that (Σ, c) is a *marked* symmetric surface, with r pairs of marked points

$$P_\Sigma = \{(x_1^+, x_1^-), \dots, (x_r^+, x_r^-)\}, \quad \text{where} \quad x_i^- = c(x_i^+), \tag{2.19}$$

cf. (2.4). So in particular we have a preferred marked point x_i^+ (the first element of a pair) in each pair of conjugate points.

We consider next the moduli spaces of real maps to (Σ, c) that have fixed ramification pattern over the marked points of Σ. This moduli space is a version of [5, Definition 3.1], adapted to the Real setting. The ramification pattern over each point is described by a partition λ.

Let $\vec{\lambda} = (\lambda^1, \dots, \lambda^r)$ be a collection of r partitions of d.

Definition 2.5. Denote by

$$\overline{\mathcal{M}}^{\bullet, c}_{d,\chi}(\Sigma)_{\lambda^1, \dots, \lambda^r} \tag{2.20}$$

the relative real moduli space of degree d stable real maps $f : (C, \sigma) \to (\Sigma, c)$ such that

- f has ramification pattern λ^i over x_i^+ (and thus also over $x_i^- = c(x_i^+)$), for all $i = 1, \dots, r$;

- the domain C is possibly disconnected and has total Euler characteristic χ;
- f is nontrivial on each connected component of C.

Here, as in [5, Definition 3.1], the inverse images of the marked points of the target are *not* ordered; in particular, an automorphism of f may permute domain components or points in the inverse image of the marked points of the target. It is straightforward to express these moduli spaces in terms of unions, products, and finite quotients of the relative moduli spaces where the points in the inverse images $f^{-1}(x_i^\pm) = \{y_{ij}^\pm\}_{j=1,\dots,\ell(\lambda^i)}$ are all marked, the points $y_{ij}^\pm$ are conjugate, $f(y_{ij}^+) = x_i^+$, and the ramification order of f at $y_{ij}^\pm$ is λ_j^i, for $j = 1, \dots, \ell(\lambda^i)$ and $i = 1, \dots, r$. The moduli space $\overline{\mathcal{M}}_{d,\chi}^{\bullet,c}(\Sigma)_{\lambda^1,\dots,\lambda^r}$ has virtual dimension b, where

$$b = d\chi(\Sigma) - \chi - 2\delta(\vec{\lambda}) \quad \text{and} \quad \delta(\vec{\lambda}) = \sum_{i=1}^{r} (d - \ell(\lambda^i)). \tag{2.21}$$

Here $\ell(\lambda^i)$ is the length of the partition λ^i, i.e. the cardinality of $f^{-1}(x_i^+)$.

The relative real moduli space is oriented using a twisted orientation $\mathfrak{o}$ as in Definition 2.1 but where $T\Sigma$ is the relative tangent space to the *marked* curve $\Sigma = (S, j, x_1^\pm, \dots x_r^\pm)$, i.e.

$$T\Sigma = TS \otimes \mathcal{O}\Big(- \sum_i x_i^+ - \sum_i x_i^- \Big); \tag{2.22}$$

see Appendix. Definition 2.3 then extends to the relative setting.

Definition 2.6. Assume (Σ, c) is a symmetric surface with r pairs of marked points. Let $L \to \Sigma$ be a holomorphic line bundle, $\mathfrak{o}$ a twisted orientation data for (Σ, c), and $\vec{\lambda} = (\lambda^1, \dots, \lambda^r)$ a collection of r partitions of d. The local real relative GW invariants associated with the Real 3-fold $(L \oplus c^*\overline{L}, c_{tw}) \to (\Sigma, c)$ and the orientation data $\mathfrak{o}$ are the equivariant pairings:

$$RZ_{d,\chi}^{c,\mathfrak{o}}(\Sigma, L)_{\vec{\lambda}} = \int_{[\overline{\mathcal{M}}_{d,\chi}^{c,\bullet}(\Sigma)_{\vec{\lambda}}]^{\mathrm{vir},\mathfrak{o}}} e_{U(1)}(-\mathrm{Ind}\, \overline{\partial}_{(L \oplus c^*\bar{L}, c_{tw})}) = \int_{[\overline{\mathcal{M}}_{d,\chi}^{c,\bullet}(\Sigma)_{\vec{\lambda}}]^{\mathrm{vir},\mathfrak{o}}} e_{U(1)}(-\mathrm{Ind}\, \overline{\partial}_L). \tag{2.23}$$

The shifted partition function (2.11) extends to the relative setting as

$$RGW_d^{c,\mathfrak{o}}(\Sigma, L)_{\vec{\lambda}} = \sum_{\chi} u^{d(\frac{\chi(\Sigma)}{2} + c_1(L)[\Sigma]) - \frac{\chi}{2} - \delta(\vec{\lambda})} RZ_{d,\chi}^{c,\mathfrak{o}}(\Sigma, L)_{\vec{\lambda}}, \tag{2.24}$$

where $\delta(\vec{\lambda})$ is as in (2.21). Note that the power of u is $b/2 + dk$, where b is the dimension (2.21) of $\overline{\mathcal{M}}_{d,\chi}^{c,\bullet}(\Sigma)_{\vec{\lambda}}$ and $k = c_1(L)[\Sigma]$.

The quantity (2.24) is invariant under (smooth) deformations, so it depends only on the topological type of $(\Sigma, c, \mathfrak{o})$, on $c_1(L)$, and on how the r partitions $\lambda^1, \dots, \lambda^r$ are distributed on the components of Σ. We use the notation

$$RGW_d^{c,\mathfrak{o}}(g|k)_{\vec{\lambda}} \tag{2.25}$$

for the case Σ is a connected genus g surface and $k = c_1(L)[\Sigma]$, and

$$RGW_d^{c,\mathfrak{o}}(g, g|k_1, k_2)_{\vec{\lambda}} \tag{2.26}$$

for the case Σ is a g-doublet, all the positive marked points are on the same component Σ_1 of Σ, and $k_i = c_1(L)[\Sigma_i]$ are the degrees of L on the two components.

As before, the local invariant (2.23) can be expressed in terms of the equivariant parameter t and an *ordinary* integral:

$$RZ_{d,\chi}^{c,\mathfrak{o}}(\Sigma, L)_{\lambda^1..\lambda^r} = t^{\iota - b/2} \int_{[\overline{\mathcal{M}}_{d,\chi}^{c,\bullet}(\Sigma)_{\lambda^1..\lambda^r}]^{\mathrm{vir},\mathfrak{o}}} c_{b/2}(-\mathrm{Ind}\, \overline{\partial}_L). \tag{2.27}$$

Here b is the dimension (2.21) of $\overline{\mathcal{M}}_{d,\chi}^{c,\bullet}(\Sigma)_{\lambda^1..\lambda^r}$ and ι the index (virtual complex rank) of $-\mathrm{Ind}\, \overline{\partial}_L$, given respectively by (2.21) and (2.13), so the power of t in (2.27) is

$$\iota - b/2 = -d(\chi(\Sigma)/2 + c_1(L)[\Sigma]) + \delta(\vec{\lambda}). \tag{2.28}$$

As in the absolute case, we use similar notions for the connected and doublet relative invariants and their moduli spaces, cf. Remark 2.4.

3. Doublet vs complex invariants

The doublet invariants (2.15) (and their extension to the relative setting) are real invariants associated with the moduli space of maps whose domain is a doublet (2.1). In this section we consider two situations: (a) when the target curve is a doublet and (b) when the target curve is connected. In both cases, we relate the doublet invariants to the residue invariants defined by Bryan and Pandharipande in [5] (for the anti-diagonal action). The latter are reviewed in §3.1.

Roughly speaking, the main idea is that a doublet can be identified with a complex curve by restricting to one of the components. This defines an identification $\mathcal{P}$ between the doublet moduli space and the usual (complex) moduli space, with matching deformation obstruction theories; moreover, a bundle on a doublet corresponds to two bundles, one for each component of the doublet.

The main results in this rather technical section are Corollaries 3.4 and 3.8, comparing the doublet invariants to the BP-invariants. They follow from the fact that in both cases

(i) the VFC of the doublet moduli space is equal up to a scalar multiple to that of the corresponding complex moduli space, cf. Lemmas 3.1 and 3.6 and (ii) the equivariant Euler classes of the index bundles are also equal up to sign, cf. Lemmas 3.3 and 3.7.

3.1. Complex GW invariants

We begin with a brief review of the complex moduli space and the residue GW-invariants defined by Bryan and Pandharipande in [5]. Assume Σ is a complex curve with r marked points $P = \{x_1, \dots x_r\}$, and let $\vec{\lambda} = (\lambda^1, \dots \lambda^r)$ be a collection of r partitions λ^i of d. Let

$$\overline{\mathcal{M}}^{\bullet}_{d,\chi}(\Sigma)_{\vec{\lambda}} \tag{3.1}$$

denote the usual (complex) relative moduli space [5, Definition 3.1] of degree d stable maps $f : C \to \Sigma$ from an Euler characteristic χ domain having ramification prescribed by $\vec{\lambda}$ over the points P (such that moreover the restriction of f to each connected component of the domain is nontrivial). Here the inverse images of the marked points of the target are *unordered*. The moduli space (3.1) is canonically oriented and carries a virtual fundamental class in dimension $2b$, where

$$b = d\chi(\Sigma) - \chi - \delta(\vec{\lambda})$$

and $\delta(\vec{\lambda})$ is as in (2.21).

If L_1, L_2 are two holomorphic bundles over Σ, the total space of

$$E = L_1 \oplus L_2 \to \Sigma \tag{3.2}$$

is a local holomorphic 3-fold with a $T = (\mathbb{C}^*)^2$ action. In [5, §3.2] Bryan-Pandharipande consider residue invariants by integrating a T-equivariant Euler class. When restricted to the anti-diagonal $U(1)$ action, the BP residue invariants are given by:

$$Z_{d,\chi}(\Sigma|L_1, L_2)_{\vec{\lambda}} = \int_{[\overline{\mathcal{M}}^{\bullet}_{d,\chi}(\Sigma)_{\vec{\lambda}}]^{\mathrm{vir}}} e_{U(1)}(-\mathrm{Ind}\, \overline{\partial}_{L_1 \oplus L_2}). \tag{3.3}$$

Their (shifted) generating function (cf. [5, §3.2]) is

$$GW_d(\Sigma|L_1, L_2)_{\vec{\lambda}} = \sum_{\chi} u^{d(\chi(\Sigma)+k_1+k_2)-\chi-\delta(\vec{\lambda})} Z_{d,\chi}(\Sigma|L_1, L_2)_{\vec{\lambda}}, \tag{3.4}$$

where $k_i = c_1(L_i)[\Sigma]$ and $\delta(\vec{\lambda})$ is as in (2.21). We denote by GW^{conn} the corresponding invariants associated to the moduli spaces of maps with *connected* domains.

3.2. Doublets and halves

For any doublet $(C = C_1 \sqcup C_2, \sigma)$, with r pairs of marked points P_C as in (2.19) the 'half' C_1 is a complex curve with r marked points and each of these marked points inherits a decoration of a $\pm$ sign. This process defines a map

$$(C = C_1 \sqcup C_2, \sigma) \mapsto C_1, \tag{3.5}$$

that takes a doublet to a connected complex curve with *signed* marked points. Formally, a complex curve with signed marked points is a complex curve Σ with marked points $P_\Sigma = \{x_1, \dots, x_r\}$ together with a choice $\varepsilon : P_\Sigma \to \{\pm 1\}$ of a sign associated to each point.

Conversely, to every complex curve C we can associate a doublet (2.1) via

$$(DC, \sigma)\,, \quad \text{where} \quad DC = C \sqcup \overline{C} = C_1 \sqcup C_2 \quad \text{and} \quad \sigma_{|C} = id : C \to \overline{C}. \tag{3.6}$$

Note that DC is the orientation double cover of C. When C has r signed marked points P_C, the double DC is marked: it has r pairs of conjugate points, and the sign ε of a marked point in P_C determines whether it is the first or second element of the corresponding pair in DC, with $+$ corresponding to first.

Therefore (3.5) is a correspondence.

3.3. Real maps to a doublet

Fix Σ a complex marked surface (with signed marked points) and let $D\Sigma = (\Sigma \sqcup \overline{\Sigma}, c) = (\Sigma_1 \sqcup \Sigma_2, c)$ denote its double (3.6). We next relate the local RGW invariants (2.10) of the double $D\Sigma$ to the BP-residue invariants (3.3) of Σ.

For any real map $f : (C, \sigma) \to D\Sigma = (\Sigma_1 \sqcup \Sigma_2, c)$, let

$$f_i : C_i \to \Sigma_i\,, \quad i = 1, 2, \tag{3.7}$$

denote its restriction to $C_i = f^{-1}(\Sigma_i)$, $i = 1, 2$. Conversely, any map $f : C \to \Sigma$ doubles to a real map

$$\tilde{f} : DC \to D\Sigma\,, \quad \text{with} \quad \tilde{f}_1 = f.$$

The signs of the marked points on Σ determine signs on the marked points of the domain C which are compatible under the doubling procedure (3.6). This defines a morphism

$$\mathcal{D} : \overline{\mathcal{M}}^{\bullet}_{d,\chi}(\Sigma)_{\lambda^1..\lambda^r} \longrightarrow \overline{\mathcal{M}}^{c,\bullet}_{d,2\chi}(D\Sigma)_{\lambda^1..\lambda^r}\,, \qquad f \mapsto \widetilde{f}, \tag{3.8}$$

between the moduli spaces, whose inverse is

$$\mathcal{P}(f) = f_1 \tag{3.9}$$

where f_1 is given by (3.7).

Lemma 3.1. *Fix an orientation data* $\mathfrak{o}$ *as in* (2.6) *for the doublet* $D\Sigma = \Sigma_1 \sqcup \Sigma_2$*. With the notation above, the identification* (3.9) *has degree* $(-1)^{dm_2+\ell_2}$*, i.e.*

$$[\overline{\mathcal{M}}^{c,\bullet}_{d,2\chi}(D\Sigma)_{\lambda^1..\lambda^r}]^{\mathrm{vir},\mathfrak{o}} = (-1)^{dm_2+\ell_2}\mathcal{D}_*[\overline{\mathcal{M}}^{\bullet}_{d,\chi}(\Sigma)_{\lambda^1..\lambda^r}]^{\mathrm{vir}}, \tag{3.10}$$

where m_2 *is the degree of* $\Theta_{|\Sigma_2}$ *and* ℓ_2 *is the sum of the lengths of the partitions associated to the positive points on* Σ_2*:*

$$m_2 = c_1(\Theta)[\Sigma_2] \quad \textit{and} \quad \ell_2 = \sum_{x_i^+\in\Sigma_2} \ell(\lambda^i). \tag{3.11}$$

Proof. The map (3.8) and its inverse (3.9) define an identification between the two moduli spaces, with matching deformation-obstruction theories. Thus it remains to compare the orientations. The argument is similar to that of [16, Theorem 1.3] taking into account the difference in the orientations induced by a twisted orientation data and a real orientation data in the sense of [15].

We first recall the procedures for orienting the complex and the real moduli spaces; the Appendix contains a more detailed discussion of the real case. The orientation sheaf of the real moduli space, (after stabilization of the domain if necessary), is canonically identified with

$$\det T\overline{\mathcal{M}}^{c,\bullet}_{d,\chi}(\Sigma)_{\lambda^1..\lambda^r} = \det \overline{\partial}_{(T\Sigma,dc)} \otimes \mathfrak{f}^* \det T\overline{\mathbb{R}\mathcal{M}}^{\bullet}_{\chi,\ell}. \tag{3.12}$$

Here $\mathfrak{f}$ is the map to the real Deligne-Mumford moduli space parametrizing real curves of Euler characteristic χ and ℓ pairs of conjugate marked points, and $\ell = \sum_{i=1}^{r} \ell(\lambda^i)$; see (A.13). Let

$$f : (C,\sigma) \longrightarrow (\Sigma, c)$$

be a point in the real moduli space. A choice of twisted orientation data $\mathfrak{o} = (\Theta, \psi, \mathfrak{s})$ determines a homotopy class of isomorphisms

$$f^*(T\Sigma \oplus \Theta \oplus c^*\overline{\Theta}, dc \oplus c_{tw}) \xrightarrow{\phi_{\mathfrak{o}}} (C \times \mathbb{C}^{\oplus 3}, \sigma \oplus c_{std}^{\oplus 3}). \tag{3.13}$$

Here $T\Sigma$ denotes the relative tangent bundle (2.22) of the marked curve. This induces an isomorphism

$$\det \bar{\partial}_{f^*(T\Sigma,dc)} = \det \bar{\partial}_{(\mathbb{C},c_{std})} \tag{3.14}$$

by using the canonical orientation on twice a bundle and the canonical complex orientation induced by the right-hand side of the identification

$$\det\bar\partial_{f^*(\Theta\oplus c^*\overline{\Theta},c_{tw})}\overset{\pi_1}{=}\det\bar\partial_{f^*\Theta}$$

as in (2.9). By [15, Theorem 1.3], there is also a canonical isomorphism

$$\det(T\overline{\mathbb{R}\mathcal{M}}_{h,\ell})=\det\bar\partial_{(\mathbb{C},c_{std})},\tag{3.15}$$

where the forgetful morphism of a pair of marked points is oriented via the first elements in the pairs. Then the orientation on the real moduli space is obtained by combining (3.14) and (3.15) within (3.12).

Similarly the complex moduli space at $f:C\to\Sigma$ is oriented via the complex orientation of $\det\bar\partial_{T\Sigma}$ and the complex orientation on the corresponding Deligne-Mumford moduli space as in (3.12).

Since the map $\mathcal{D}$ is compatible with the forgetful morphism to the corresponding DM spaces, its sign is determined by the comparison on the level of DM spaces and on the level of the index bundles.

When $(C,\sigma)=(C_1\sqcup C_2,\sigma)$ is a doublet and $(V,\phi)=(V_1\sqcup V_2,\phi)\to(C,\sigma)$ is a Real bundle, its index bundle $\mathrm{Ind}\,\bar\partial_{(V,\phi)}$ has a natural complex structure induced by the isomorphism:

$$\mathrm{Ind}\,\bar\partial_{(V,\phi)}\overset{p_1}{\cong}\mathrm{Ind}\,\bar\partial_{V_1}.\tag{3.16}$$

Here p_1 takes an invariant section $\xi=(\xi_1,\xi_2)$ of $(V_1\sqcup V_2,\phi)$ to its restriction ξ_1 to C_1. In particular, $\det\bar\partial_{(V,\phi)}$ has an induced orientation, which we refer to as the complex orientation, cf. [16, §3.1].

On the level of Deligne-Mumford spaces, the doubling map $\mathcal{D}$ from the complex moduli space (with signed marked points) to the real moduli induces an orientation on $\overline{D\mathcal{M}}_{h,\ell}$ which we call the complex orientation, cf. [16, §3.1]. By [16, Lemma 3.2], the comparison between the orientation on

$$\det(T\overline{D\mathcal{M}}_{h,\ell})\otimes\det\bar\partial_{(\mathbb{C},c_{std})},$$

induced by (3.15) and by the complex orientations on the two factors is $(-1)^{\chi/2+s}$, where χ is the Euler characteristic of C_1 and s is the number of negative marked points on the component C_1. Because C is a doublet, s is also equal to the number of positive marked points on the component C_2, i.e. the number ℓ_2 of points in the inverse image of marked points x_i^+ that lie on Σ_2.

We now turn to the comparison at the level of index bundles. The twisted orientation $\mathfrak{o}$ determines an orientation on

$$\det\bar\partial_{f^*(T\Sigma,dc)}\otimes\det\bar\partial_{(\mathbb{C},c_{std})}$$

via (3.13) and (3.14). In the case when the domain is a doublet, the two factors in this tensor product also have complex orientation as in (3.16). To understand the difference between the two orientations on the tensor product we consider the restriction of (3.13) to C_1. This restriction is a complex isomorphism of complex bundles and thus the induced isomorphism on the corresponding determinant bundles is orientation preserving. Therefore the difference between the two orientations on the tensor product corresponds to the difference between the complex orientation on $\det \bar\partial_{f^*(\Theta\oplus c^*\overline{\Theta}, c_{tw})}$ induced by (3.16) and the orientation (2.9) on

$$\det \bar\partial_{f^*(\Theta\oplus c^*\overline{\Theta}, c_{tw})} \overset{\pi_1}{=} \det \bar\partial_{f^*\Theta|_{\Sigma_1\sqcup\Sigma_2}}$$

used in the transition from (3.13) to (3.14).

By Lemma 3.2 below, the difference between these orientations is $(-1)^{\iota_2}$, where

$$\iota_2 = c_1(f^*\Theta_{|\Sigma_2}) + \chi/2 = dc_1(\Theta)[\Sigma_2] + \chi/2 = dm_2 + \chi/2$$

is the complex rank of the index bundle associated to $\Theta_{|\Sigma_2}$. Combined with the change $(-1)^{\chi/2+\ell_2}$ at the level of the DM moduli spaces this completes the proof. $\square$

Lemma 3.2. *The index bundle of* $(L \oplus c^*\overline{L}, c_{tw}) \longrightarrow (C_1 \sqcup C_2, c)$ *has two natural orientations:*

(i) *one induced by the isomorphism with* $\operatorname{Ind} \bar\partial_{L_{|C_1\sqcup C_2}}$ *via the projection* (2.9) *onto the first bundle.*

(ii) *another one induced by the isomorphism with* $\operatorname{Ind} \bar\partial_{(L\oplus c^*\overline{L})_{|C_1}}$ *via the restriction* (3.16) *to* C_1.

These orientations differ by a factor of $(-1)^{\iota_2}$*, where*

$$\iota_2 = \operatorname{rank}_{\mathbb{C}} \bar\partial_{L_{|C_2}} = c_1(L)[C_2] + \chi/2, \tag{3.17}$$

and χ *is the Euler characteristic of* C_2.

Proof. Holomorphic sections of $E = L \oplus c^*\overline{L} \longrightarrow C_1 \sqcup C_2$ invariant under the involutions c, c_{tw} have the form (ξ, η) where $\xi_i = \xi_{|C_i}$ are holomorphic sections of $L|_{C_i}$ while $\eta_i = \eta_{|C_i}$ are holomorphic sections of $(c^*\overline{L})|_{C_i}$, and

$$\eta_1 = c^*\xi_2 \quad \text{and} \quad \eta_2 = c^*\xi_1. \tag{3.18}$$

Note that η_1 is a section of $(c^*\overline{L})_{|C_1} = c^*(\overline{L}_{|C_2})$. In particular, there are two natural isomorphisms

$$\ker \bar\partial_{(E,c_{tw})} \longrightarrow \ker \bar\partial_L = \ker \bar\partial_{L_{|C_1}} \oplus \ker \bar\partial_{L_{|C_2}}, \quad (\xi,\eta) \mapsto \xi = (\xi_1, \xi_2)$$

and

$$\ker \bar{\partial}_{(E,c_{tw})} \longrightarrow \ker \bar{\partial}_{E_{|C_1}} = \ker \bar{\partial}_{L_{|C_1}} \oplus \ker \bar{\partial}_{(c^*\overline{L})_{|C_1}}, \quad (\xi,\eta) \mapsto (\xi,\eta)_{|C_1} = (\xi_1,\eta_1).$$

The same is true at the level of cokernels. Therefore we have two natural isomorphisms, the first one

$$\operatorname{Ind} \bar{\partial}_{(L\oplus c^*\overline{L},c_{tw})} \longrightarrow \operatorname{Ind} \bar{\partial}_{L_{|C_1 \sqcup C_2}} = \operatorname{Ind} \bar{\partial}_{L_{|C_1}} \oplus \operatorname{Ind} \bar{\partial}_{L_{|C_2}} \tag{3.19}$$

induced by the projection onto the first factor L of $E = L \oplus c^*\overline{L}$ and the second one

$$\operatorname{Ind} \bar{\partial}_{(L\oplus c^*\overline{L},c_{tw})} \longrightarrow \operatorname{Ind} \bar{\partial}_{(L\oplus c^*\overline{L})_{|C_1}} = \operatorname{Ind} \bar{\partial}_{L_{|C_1}} \oplus \operatorname{Ind} \bar{\partial}_{(c^*\overline{L})_{|C_1}} \tag{3.20}$$

induced by the restriction to the first component C_1 of the doublet $C = C_1 \sqcup C_2$. Both $\operatorname{Ind} \bar{\partial}_L$ and $\operatorname{Ind} \bar{\partial}_{(L\oplus c^*\overline{L})_{|C_1}}$ have natural complex structures and therefore induce two complex structures on $\operatorname{Ind} \bar{\partial}_{(L\oplus c^*\overline{L},c_{tw})}$ which we want to compare.

Moreover, there is a natural complex linear isomorphism

$$\operatorname{Ind} \bar{\partial}_{L_{|C_2}} \longrightarrow \overline{\operatorname{Ind} \bar{\partial}_{c^*(\overline{L}_{|C_2})}}, \qquad \text{induced by } \xi_2 \mapsto c^*\xi_2, \tag{3.21}$$

and using the fact that $\operatorname{ind} \bar{\partial}_{L_2 \to C_2}$ and $\operatorname{ind} \bar{\partial}_{\overline{L}_2 \to \overline{C}_2}$ have opposite complex structures. This combined with (3.18) implies that the orientations induced by (3.19) and (3.20) differ by $(-1)^{\iota_2}$, where ι_2 is the complex rank of the index of $L_{|C_2}$, given by (3.17). $\square$

Next, given two complex line bundles $L_1, L_2 \to \Sigma$ over a complex curve, we obtain a complex line bundle $L \to D\Sigma$ over the double $(D\Sigma, c) = (\Sigma \sqcup \overline{\Sigma}, c) = (\Sigma_1 \sqcup \Sigma_2, c)$ defined by

$$L_{|\Sigma_1} = L_1 \quad \text{and} \quad L_{|\Sigma_2} = c^*\overline{L}_2. \tag{3.22}$$

We denote such L by

$$D(L_1, L_2) \longrightarrow D\Sigma.$$

Note that if $L_1, L_2 \to \Sigma$ have degrees k_1, k_2, then $L_{|\Sigma_i}$ also has degree k_i, $i = 1, 2$.

Lemma 3.3. *With the notation above, the morphism* (3.9) *satisfies:*

$$e_{U(1)}(-\operatorname{Ind} \bar{\partial}_{D(L_1,L_2)}) = (-1)^{dc_1(L_2)[\Sigma]+\chi/2} \mathcal{P}^* e_{U(1)}(-\operatorname{Ind} \bar{\partial}_{L_1\oplus L_2}), \tag{3.23}$$

with the anti-diagonal action on $L_1 \oplus L_2$ used for the equivariant Euler class in the last expression.

Proof. When $L \to D\Sigma$ is a line bundle over a doublet $D\Sigma = \Sigma_1 \sqcup \Sigma_2$, the identification (3.19) induces an isomorphism

$$\mathrm{Ind}\,\overline{\partial}_L \longrightarrow \mathcal{P}_1^* \mathrm{Ind}\,\overline{\partial}_{L_{|\Sigma_1}} \oplus \mathcal{P}_2^* \mathrm{Ind}\,\overline{\partial}_{L_{|\Sigma_2}}$$

where $\mathcal{P}_i(f) = f_i$ are the restrictions (3.7) to the i-th component of the domain; in particular $\mathcal{P}_1 = \mathcal{P}$. Therefore

$$e_{U(1)}(-\mathrm{Ind}\,\overline{\partial}_L) = \sum_{m=0}^{\iota} t^m c_{\iota-m}(-\mathrm{Ind}\,\overline{\partial}_L) = \sum_{m+k+l=\iota} t^m \mathcal{P}_1^* c_k(-\mathrm{Ind}\,\overline{\partial}_{L_{|\Sigma_1}}) \mathcal{P}_2^* c_l(-\mathrm{Ind}\,\overline{\partial}_{L_{|\Sigma_2}}).$$

where $\iota = \mathrm{rank}_{\mathbb{C}}(-\mathrm{Ind}\,\overline{\partial}_L) = dc_1(L)[D\Sigma] - \frac{2\chi}{2}$ on $\overline{\mathcal{M}}^{c,\bullet}_{d,2\chi}(D\Sigma)_{\lambda^1..\lambda^r}$.

On the other hand, for the anti-diagonal action on $L_1 \oplus L_2$, we have

$$\begin{aligned} e_{U(1)}(-\mathrm{Ind}\,\overline{\partial}_{L_1\oplus L_2}) &= \Big(\sum_{k=0}^{\iota_1} c_k(-\mathrm{Ind}\,\overline{\partial}_{L_1}) t^{\iota_1-k}\Big)\Big(\sum_{l=0}^{\iota_2} c_l(-\mathrm{Ind}\,\overline{\partial}_{L_2})(-t)^{\iota_2-l}\Big) = \\ &= \sum_{k+l+m=\iota_1+\iota_2} t^m c_k(-\mathrm{Ind}\,\overline{\partial}_{L_1}) c_l(-\mathrm{Ind}\,\overline{\partial}_{L_2})(-1)^{\iota_2-l}. \end{aligned}$$

Here $\iota_i = \mathrm{rank}_{\mathbb{C}}(-\mathrm{Ind}\,\overline{\partial}_{L_i}) = -dc_1(L_i)[\Sigma] - \frac{\chi}{2}$ on $\overline{\mathcal{M}}^{\bullet}_{d,\chi}(\Sigma)_{\lambda^1..\lambda^r}$ for $i = 1, 2$. Note that $\iota = \iota_1 + \iota_2$.

Since (3.22) implies that $L_2 = c^*(\overline{L}_{|\Sigma_2}) \longrightarrow \Sigma_1$, then as in (3.21), we have

$$\mathcal{P}_2^*(-\mathrm{Ind}\,\overline{\partial}_{L_{|\Sigma_2}}) = \mathcal{P}_1^*(\overline{-\mathrm{Ind}\,\overline{\partial}_{L_2}}).$$

Thus

$$\mathcal{P}_2^* c_l(-\mathrm{Ind}\,\overline{\partial}_{L_{|\Sigma_2}}) = (-1)^l \mathcal{P}_1^* c_l(-\mathrm{Ind}\,\overline{\partial}_{L_2}),$$

and the claim follows. □

Since $\mathcal{P}$ and $\mathcal{D}$ are inverse morphisms, combining Lemmas 3.1 and 3.3 gives:

Corollary 3.4. *With the notation above, the local RGW invariants of a doublet and the BP invariants* (3.3) *of its half are related by:*

$$RZ^{\mathbf{o}}_{d,2\chi}(D\Sigma\,|D(L_1,L_2))_{\lambda^1..\lambda^r} = (-1)^{d(k_2+m_2)+\chi/2+\ell_2} Z_{d,\chi}(\Sigma\,|L_1,L_2)_{\lambda^1..\lambda^r}, \quad (3.24)$$

where m_2, ℓ_2 are as in (3.10), and $k_2 = c_1(L_2)[\Sigma]$ is the degree of L_2.

Remark 3.5. For a doublet target, the invariants (2.24) and the equality (3.24) are independent of the choice of first and second component of the target doublet. This can be seen as follows. The map $\mathcal{P}$ to the complex moduli space (3.9) is defined using the first component. Choosing the second component instead corresponds to switching the order of L_1, L_2 on the complex GW side. This switch results only in a change of the sign of the equivariant complex GW invariant by the parity of $\iota - b/2$, where ι is the complex rank of $-\mathrm{Ind}\,\overline{\partial}_{L_1\oplus L_2}$ and b is the dimension of the moduli space, cf. (2.27). The quantity $\iota - b/2 \mod 2$ is also the parity of the sum of the powers of (-1) in (3.24) for the two choices.

3.4. The doublet moduli space to a connected target

Assume next (Σ, c) is a genus g connected symmetric Riemann surface with r pairs of conjugate marked points, and $\vec{\lambda} = (\lambda^1, \dots, \lambda^r)$ is a collection of r partitions of $2d$. Recall that a reparametrization of a doublet domain C may swap its two components. So it is convenient to consider the two fold cover of the doublet moduli space

$$q : \widetilde{D\mathcal{M}}^c_{2d,h}(\Sigma)_\lambda \longrightarrow \overline{D\mathcal{M}}^c_{2d,h}(\Sigma)_\lambda \tag{3.25}$$

consisting of real maps whose domain is a doublet, up to reparametrizations *preserving* the order of its components. In particular,

$$[\overline{D\mathcal{M}}^c_{2d,h}(\Sigma)_{\vec{\lambda}}]^{\mathrm{vir},\mathfrak{o}} = \tfrac{1}{2} q_*[\widetilde{D\mathcal{M}}^c_{2d,h}(\Sigma)_{\vec{\lambda}}]^{\mathrm{vir},\mathfrak{o}}. \tag{3.26}$$

Every real map $f : (C_1 \sqcup C_2, \sigma) \to (\Sigma, c)$ from a doublet domain restricts to a pair of maps

$$f_i = f|_{C_i} : C_i \to \Sigma \quad \text{where} \quad f_2 = c \circ f_1 \circ \sigma_{|C_2}. \tag{3.27}$$

The ramification points of f get distributed on the two components of the domain: if f has ramification profile λ^i over x_i^+ (and therefore also over x_i^-), let $\lambda^i_\pm$ denote the ramification profile of its restriction f_1, cf. (3.27). Since $f_2 = c \circ f_1 \circ \sigma$ then f_2 has ramification λ^i_- over x_i^+ and ramification λ^i_+ over x_i^-.

This decomposes λ^i into

$$\lambda^i = \lambda^i_+ \sqcup \lambda^i_-, \qquad \text{where } \lambda^i_\pm \text{ are partitions of } d.$$

Note that if for example λ^i has parts $4, 3, 3, 2, 1$ then λ^i_+, λ^i_- could have parts $4, 2, 1$ and $3, 3, 1$ respectively. Denote such decompositions $\vec{\lambda} = \vec{\lambda}_+ \sqcup \vec{\lambda}_-$ where $\vec{\lambda}_\pm = (\lambda^1_\pm, \dots, \lambda^r_\pm)$ and let

$$\widetilde{D\mathcal{M}}^c_{2d,h}(\Sigma)_{\vec{\lambda}_+|\vec{\lambda}_-} \tag{3.28}$$

denote the corresponding relative moduli space of real maps from doublet domains, with *ordered* components, and so that the restriction to the first component has ramification λ^i_+ over x_i^+ and ramification λ^i_- over x_i^-, for all $i = 1, \dots r$. Therefore

$$\widetilde{D\mathcal{M}}^c_{2d,h}(\Sigma)_{\vec{\lambda}} = \bigsqcup_{\vec{\lambda}=\vec{\lambda}_+\sqcup\vec{\lambda}_-} \widetilde{D\mathcal{M}}^c_{2d,h}(\Sigma)_{\vec{\lambda}_+|\vec{\lambda}_-}. \tag{3.29}$$

Furthermore there is a morphism

$$\mathcal{P} : \widetilde{D\mathcal{M}}^c_{2d,h}(\Sigma)_{\vec{\lambda}_+|\vec{\lambda}_-} \longrightarrow \overline{\mathcal{M}}_{d,h}(\Sigma)_{\vec{\lambda}_+,\vec{\lambda}_-}, \qquad f \mapsto f_1 \tag{3.30}$$

cf. (3.27), where $\overline{\mathcal{M}}_{d,h}(\Sigma)_{\vec{\lambda}_+,\vec{\lambda}_-}$ denotes the classical moduli space of maps from a connected domain with ramification λ^i_+ over x^+_i and ramification λ^i_- over x^-_i, for $i = 1, \dots r$.

Conversely, every map $f : C \to \Sigma$ from a complex curve induces a real map

$$\widetilde{f} : (C \sqcup \overline{C}, \sigma) \to (\Sigma, c), \quad \text{where} \quad \widetilde{f}|_C = f,\ \widetilde{f}|_{\overline{C}} = c \circ f \circ \sigma|_{\overline{C}} \tag{3.31}$$

from the double of C to Σ. This defines the inverse $\mathcal{D}$ of (3.30).

Lemma 3.6. *Assume (Σ, c) is a connected symmetric marked curve with r pairs of conjugate points and let $\mathfrak{o} = (\Theta, \psi, \mathfrak{s})$ be twisted orientation data for it. Let $\lambda^i_\pm$, $i = 1, \dots, r$, be $2r$ partitions of d. Then, with the notation above,*

$$[\widetilde{D\mathcal{M}}^c_{2d,h}(\Sigma)_{\vec{\lambda}_+|\vec{\lambda}_-}]^{\mathrm{vir},\mathfrak{o}} = (-1)^{dm+\ell^-}\mathcal{D}_*[\overline{\mathcal{M}}_{d,h}(\Sigma)_{\vec{\lambda}_+,\vec{\lambda}_-}]^{\mathrm{vir}}, \tag{3.32}$$

where m is the degree of Θ and ℓ^- is the sum of the lengths of the partitions in $\vec{\lambda}_-$:

$$m = c_1(\Theta)[\Sigma] = g(\Sigma) - 1 + r \quad \textit{and} \quad \ell^- = \sum_{i=1}^r \ell(\lambda^i_-). \tag{3.33}$$

Proof. The proof is the same as that of Lemma 3.1. To compare orientations, it suffices to compare them on the level of DM spaces and on the level of index bundles. Let $f : (C_1 \sqcup C_2, \sigma) \to (\Sigma, c)$ denote an element of $\widetilde{D\mathcal{M}}^c_{2d,h}(\Sigma)_{\vec{\lambda}_+|\vec{\lambda}_-}$. Since $f \circ \sigma = c \circ f$ then

$$f^*(\Theta \oplus c^*\overline{\Theta}, c_{tw}) = (f^*\Theta \oplus \sigma^*(\overline{f^*\Theta}), \sigma_{tw})$$

where the involution σ_{tw} is given by (2.5) for the bundle $L = f^*\Theta \to (C_1 \sqcup C_2, \sigma)$. Moreover, C_1 has ℓ^- negative points and Euler characteristic $\chi = 2 - 2h$ thus the difference in orientations on the level of the DM spaces is $(-1)^{\chi/2+\ell^-}$ as before. On the level of index bundles, it similarly comes from the difference between the complex orientation on

$$\det \bar{\partial}_{f^*(\Theta\oplus c^*\overline{\Theta}, c_{tw})} = \det \bar{\partial}_{(f^*\Theta\oplus\sigma^* f^*\overline{\Theta}, \sigma_{tw})} \overset{p_1}{\cong} \det \bar{\partial}_{(f^*\Theta\oplus\sigma^* f^*\overline{\Theta})_{|C_1}}$$

induced by (3.16) and the orientation (2.9) on

$$\det \bar{\partial}_{f^*(\Theta\oplus c^*\overline{\Theta}, c_{tw})} \overset{\pi_1}{\cong} \det \bar{\partial}_{(f^*\Theta)_{|C_1\sqcup C_2}}.$$

By Lemma 3.2, this difference is $(-1)^\iota$, where $\iota = c_1(f^*\Theta)[C_2] + \chi/2 = dc_1(\Theta)[\Sigma] + \chi/2$. Finally, the fact that $m = c_1(\Theta)[\Sigma] = -\chi(T\Sigma) = g(\Sigma) - 1 + r$ is obtained as in Example 2.2(c), but for the relative tangent bundle $T\Sigma$, cf. (2.22). $\square$

Lemma 3.7. *Assume $L \to \Sigma$ is a holomorphic line bundle over a connected symmetric surface (Σ, c). Then the morphism (3.30) satisfies:*

$$e_{U(1)}(-\mathrm{Ind}\,\overline{\partial}_L \longrightarrow \widetilde{D\mathcal{M}}^{c}_{2d,h}(\Sigma)_{\vec{\lambda}_+|\vec{\lambda}_-}) = (-1)^\iota \mathcal{P}^* e_{U(1)}(-\mathrm{Ind}\,\overline{\partial}_{L\oplus L} \longrightarrow \overline{\mathcal{M}}_{d,h}(\Sigma)_{\vec{\lambda}_+,\vec{\lambda}_-}), \tag{3.34}$$

where $\iota = dc_1(L)[\Sigma] + 1 - h$ and the anti-diagonal action on $L \oplus L$ is used on the right hand side.

Proof. Denote $\widetilde{D\mathcal{M}} = \widetilde{D\mathcal{M}}^{c}_{2d,h}(\Sigma)_{\vec{\lambda}_+|\vec{\lambda}_-}$ and $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{d,h}(\Sigma)_{\vec{\lambda}_+,\vec{\lambda}_-}$. Then $\iota = -dc_1(L)[\Sigma] + h - 1$ is the complex rank of $-\mathrm{Ind}\,\overline{\partial}_L$ over $\overline{\mathcal{M}}$; the complex rank of $-\mathrm{Ind}\,\overline{\partial}_L$ over $\widetilde{D\mathcal{M}}$ is 2ι.

As in the proof of Lemma 3.3,

$$\begin{aligned} e_{U(1)}(-\mathrm{Ind}\,\overline{\partial}_L \to \widetilde{D\mathcal{M}}) &= \sum_{m=0}^{2\iota} t^m c_{2\iota-m}(-\mathrm{Ind}\,\overline{\partial}_L \to \widetilde{D\mathcal{M}}) = \\ &= \sum_{k+l+m=2\iota} t^m \mathcal{P}_1^* c_k(-\mathrm{Ind}\,\overline{\partial}_L \to \overline{\mathcal{M}}) \mathcal{P}_2^* c_l(-\mathrm{Ind}\,\overline{\partial}_L \to \overline{\mathcal{M}}) \end{aligned}$$

where $\mathcal{P}_i(f) = f_i$ is the restriction to the i-th component of the domain, cf. (3.27).

On the other hand, for the anti-diagonal action on $L \oplus L \to \Sigma$, for the index bundle over $\overline{\mathcal{M}}$,

$$\begin{aligned} e_{U(1)}(-\mathrm{Ind}\,\overline{\partial}_{L\oplus L}) &= \Big(\sum_{k=0}^{\iota} c_k(-\mathrm{Ind}\,\overline{\partial}_L) t^{\iota-k}\Big)\Big(\sum_{l=0}^{\iota} c_l(-\mathrm{Ind}\,\overline{\partial}_L)(-t)^{\iota-l}\Big) = \\ &= \sum_{k+l+m=2\iota} t^m c_k(-\mathrm{Ind}\,\overline{\partial}_L) c_l(-\mathrm{Ind}\,\overline{\partial}_L)(-1)^{\iota-l}. \end{aligned}$$

But as in (3.21), we have a canonical isomorphism $\mathrm{Ind}\,\overline{\partial}_{f_2^* L} \cong \overline{\mathrm{Ind}\,\overline{\partial}_{\sigma^* f_2^* \overline{L}}} = \overline{\mathrm{Ind}\,\overline{\partial}_{f_1^* c^* \overline{L}}}$ that varies continuously in f, and therefore

$$\mathcal{P}_2^*(-\mathrm{Ind}\,\overline{\partial}_L) \cong \mathcal{P}_1^*(-\overline{\mathrm{Ind}\,\overline{\partial}_{c^*\overline{L}}}) \cong \mathcal{P}_1^*(-\overline{\mathrm{Ind}\,\overline{\partial}_L}).$$

The last isomorphism follows because $c^*\overline{L}$ has the same degree as L on a connected surface, thus can be deformed to L, and the Euler class is deformation invariant. Therefore

$$\mathcal{P}_2^* c_l(-\mathrm{Ind}\,\overline{\partial}_L) = (-1)^l \mathcal{P}_1^* c_l(-\mathrm{Ind}\,\overline{\partial}_L).$$

Substituting into the first displayed equation and comparing it with the second one gives (3.34) (recall that $\mathcal{P} = \mathcal{P}_1$). □

Combining Lemmas 3.6 and 3.7 we obtain:

Corollary 3.8. *Assume* (Σ, c) *is a connected symmetric genus* g *surface with* r *pairs of conjugate marked points, and* $L \to \Sigma$ *a complex line bundle. With the notation above,*

$$\int_{[\widetilde{D\mathcal{M}}^{c}_{2d,h}(\Sigma)_{\vec{\lambda}_+|\vec{\lambda}_-}]^{\mathrm{vir},\mathfrak{o}}} e_{U(1)}(-\mathrm{Ind}\,\overline{\partial}_L) = (-1)^{s^-} \int_{[\overline{\mathcal{M}}_{d,h}(\Sigma)_{\vec{\lambda}_+,\vec{\lambda}_-}]^{\mathrm{vir}}} e_{U(1)}(-\mathrm{Ind}\,\overline{\partial}_{L\oplus L}), \tag{3.35}$$

where $s^- = dc_1(L)[\Sigma] + h - 1 + dm + \ell^-$ *and* m, ℓ^- *are as in* (3.33).

The right hand side of (3.35) corresponds to the connected GW invariants defined in [5], cf. §3.1. In particular, combining it with (3.26) we obtain the following corollary.

Corollary 3.9. *When* Σ *is a connected genus* g *symmetric surface with* r *pairs of conjugate marked points and* $L \to \Sigma$ *is a complex line bundle with* $c_1(L)[\Sigma] = k$*, the doublet invariants and the connected GW invariants of [5] are related via*

$$DRGW^{c,\mathfrak{o}}_{2d}(\Sigma, L)(u,t)_{\vec{\lambda}} = \tfrac{1}{2}(-1)^{d(k+g-1+r)} \sum_{\vec{\lambda}_+ \sqcup \vec{\lambda}_- = \vec{\lambda}} (-1)^{\ell^-} GW^{conn}_d(g|k,k)(iu,it)_{\vec{\lambda}_+,\vec{\lambda}_-},$$

with ℓ^- *as in* (3.33).

Proof. In this case $s^- = d(k+m) + h - 1 + \ell^-$, $m = g - 1 + r$, and the substitution $(u,t) \mapsto (iu, it)$ in (3.4) changes the coefficient $GW_{d,\chi}(g|k,k)$ by $(-1)^{\chi/2}$, where $\chi = 2 - 2h$. □

4. Splitting formulas

To every symmetric surface (Σ, c) with r pairs of conjugate marked points, every complex line bundle L over Σ, and every choice of twisted orientation data $\mathfrak{o}$ on (Σ, c), (2.24) associates a collection of invariants

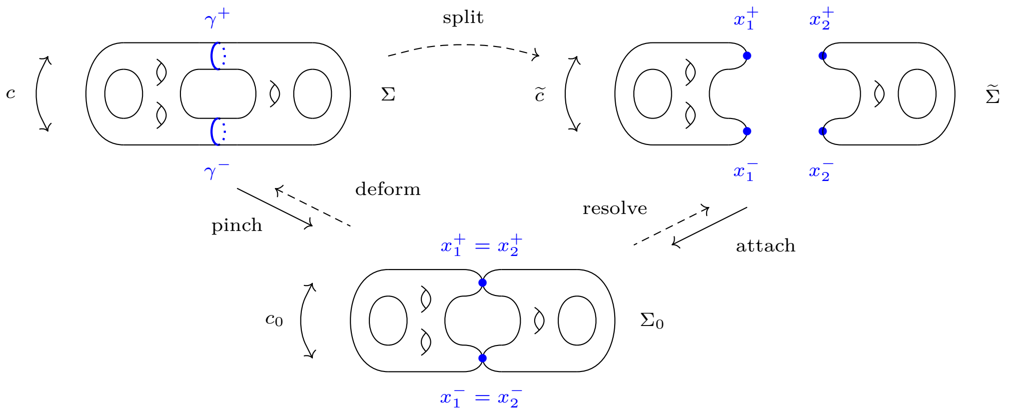
$$RGW^{c,\mathfrak{o}}_d(\Sigma, L)_{\mu^1\ldots\mu^r} = \sum_{\chi} u^{b/2+kd} \int_{[\overline{\mathcal{M}}^{c,\bullet}_{d,\chi}(\Sigma)_{\mu^1\ldots\mu^r}]^{\mathrm{vir},\mathfrak{o}}} e_{U(1)}(-\mathrm{Ind}\,\overline{\partial}_L),$$

where $\mu^1, \ldots, \mu^r$ are partitions of d. These are invariant not only under smooth deformations of the data $(\Sigma, c, L, \mathfrak{o})$, but also under deformations as the symmetric curve Σ pinches to acquire a pair of conjugate nodes as follows.

Recall that if Σ_0 is a nodal curve, then it has (a) a smooth resolution (normalization) $\widetilde{\Sigma}$ that replaces each node by a pair of marked points and (b) a family of deformations smoothing out the nodes.

This extends to symmetric surfaces as in [16, §4.2]. More precisely, assume (Σ_0, c_0) is a nodal symmetric surface with a pair of conjugate nodes and r pairs of conjugate marked points. It has a normalization $(\widetilde{\Sigma}, \widetilde{c})$ which has $r+2$ pairs of conjugate marked points. Similarly, (Σ_0, c_0) has a family of smooth deformations, simultaneously smoothing out the conjugate nodes using complex conjugate gluing parameters. The generic fiber (Σ, c) of the family is a symmetric surface with r pairs of conjugate marked points, and a pair of 'splitting circles' (disjoint vanishing cycles) swapped by the involution; as the gluing parameters converge to 0, these circles pinch to produce the two complex conjugate nodes of Σ_0.

A complex line bundle over the nodal curve extends to a line bundle over the family of deformations and lifts to a line bundle on the normalization. The relative tangent bundle to the family of marked curves restricts to the tangent bundle (2.22) of each fiber and gives rise to the tangent bundle of the normalization (regarded as a marked curve). Finally, a choice of orientation data as in Definition 2.1 on the nodal curve extends to orientation data over the family and lifts to orientation data on the normalization.



Furthermore, assume (Σ, c) is a marked symmetric surface with a pair of *conjugate splitting circles*, i.e. two embedded, disjoint circles $\gamma^{\pm}$ swapped by the involution and containing no marked points. Then (Σ, c) can be 'split' along these circles, i.e. it can be deformed to a nodal symmetric surface (Σ_0, c_0) which then has a smooth normalization $(\widetilde{\Sigma}, \widetilde{c})$. Any complex line bundle L over Σ and choice $\mathfrak{o}$ of twisted orientation data for (Σ, c) can be deformed to the nodal surface and then lifted to its normalization to give a line bundle $\widetilde{L}$ over $\widetilde{\Sigma}$ and a choice of orientation data $\widetilde{\mathfrak{o}}$ on the normalization $(\widetilde{\Sigma}, \widetilde{c})$, cf. [14, §7.1]. Lastly, every line bundle $\widetilde{L}$ and orientation data $\widetilde{\mathfrak{o}}$ on $\widetilde{\Sigma}$ descend to Σ_0 and can be deformed to a line bundle L and orientation data $\mathfrak{o}$ on Σ.

The splitting formula [5, Theorem 3.2] extends to the Real setting (cf. [14, Theorem 0.1 and Remark 2.2]) as follows:

Theorem 4.1 *([14, Thm 0.1]). Assume* (Σ, c) *is a marked symmetric surface with* r *pairs of conjugate points,* L *is a complex line bundle over* Σ*, and* $\mathfrak{o}$ *is an orientation data for* (Σ, c)*. Let* $(\widetilde{\Sigma}, \widetilde{c})$ *denote the symmetric surface obtained as described above from* (Σ, c) *by splitting it along two conjugate splitting circles, and let* $\widetilde{L}$ *and* $\widetilde{\mathfrak{o}}$ *be the corresponding line bundle and orientation data on* $\widetilde{\Sigma}$*.*

Then for any collection $\vec{\mu} = (\mu^1, \dots, \mu^r)$ *of* r *partitions of* d*, the RGW invariants* (2.24) *satisfy:*

$$RGW_d^{c,\mathfrak{o}}(\Sigma, L)_{\vec{\mu}} = \sum_{\lambda \vdash d} \zeta(\lambda) t^{2\ell(\lambda)} RGW_d^{\widetilde{c},\widetilde{\mathfrak{o}}}(\widetilde{\Sigma}, \widetilde{L})_{\vec{\mu},\lambda,\lambda}, \tag{4.1}$$

where $\zeta(\lambda)$ *is given by* (2.17)*,* t *is the equivariant parameter, and* $\ell(\lambda)$ *is the length of the partition* λ*.*

The basic idea of the proof comes from considering the family of moduli spaces of maps with values in Σ, as Σ deforms to become a nodal curve, cf. [6, Appendix A]. When regarded as maps into the total space of the family of deformations of Σ, maps with values in Σ limit to maps f_0 with values in Σ_0 that lift to maps with values in $\widetilde{\Sigma}$ having matching ramification pattern λ over the nodes of Σ_0. Since we are splitting along a pair of conjugate nodes, the local analysis of this deformation is the same as in the complex case, and the only difference is that the gluing at one of the nodes determines the gluing at the conjugate node. As in the proof of [5, Theorem 3.2], the multiplicity $\zeta(\lambda)$ comes from the number of ways such a map f_0 deforms to a map with values in Σ, and $t^{2\ell(\lambda)}$ comes from the difference in the Euler class of the index bundles (the index bundles differ by a trivial rank $2\ell(\lambda)$ bundle obtained by pulling back over the nodes of the domain the restriction of L to the nodes of the target). The comparison of the orientations is similar to that of [16, Theorem 1.2], except it uses the twisted orientation instead of the real orientation of [15].

Define the raising of the indices by the formula

$$RGW_d^{c,\mathfrak{o}}(\Sigma, L)_{\mu^1\dots\mu^r}^{\nu^1\dots\nu^s} = RGW_d^{c,\mathfrak{o}}(\Sigma, L)_{\mu^1\dots\mu^r,\nu^1\dots\nu^s} \left(\prod_{i=1}^{s} \zeta(\nu^i) t^{2\ell(\nu^i)} \right). \tag{4.2}$$

With this convention, (4.1) implies that for any splitting $(\widetilde{\Sigma}, \widetilde{c}, \widetilde{L}, \widetilde{\mathfrak{o}})$ of $(\Sigma, c, L, \mathfrak{o})$ along a pair of conjugate splitting circles,

$$RGW_d^{c,\mathfrak{o}}(\Sigma, L)_{\mu^1\dots\mu^r}^{\nu^1\dots\nu^s} = \sum_{\lambda \vdash d} RGW_d^{\widetilde{c},\widetilde{\mathfrak{o}}}(\widetilde{\Sigma}, \widetilde{L})_{\mu^1\dots\mu^r,\lambda}^{\nu^1\dots\nu^s,\lambda}. \tag{4.3}$$

In particular, for a splitting $(\widetilde{\Sigma}, \widetilde{c})$ of (Σ, c) along a pair of non-separating conjugated circles,

$$RGW_d^{c,\mathfrak{o}}(\Sigma, L)_{\mu^1\ldots\mu^r} = \sum_{\lambda\vdash d} RGW_d^{\widetilde{c},\widetilde{\mathfrak{o}}}(\widetilde{\Sigma}, L)^{\lambda}_{\mu^1\ldots\mu^r\lambda}, \tag{4.4}$$

while for a splitting along a pair of separating conjugated circles into (Σ', c') and (Σ'', c'') we have

$$RGW_d^{c,\mathfrak{o}}(\Sigma, L)^{\nu^1\ldots\nu^s}_{\mu^1\ldots\mu^r} = \sum_{\lambda\vdash d} RGW_d^{c',\mathfrak{o}'}(\Sigma', L')^{\lambda}_{\mu^1\ldots\mu^r} RGW_d^{c'',\mathfrak{o}''}(\Sigma'', L'')^{\nu^1\ldots\nu^s}_{\lambda} \tag{4.5}$$

where L', L'' and $\mathfrak{o}', \mathfrak{o}''$ denote the restrictions of $\widetilde{L}$ and $\widetilde{\mathfrak{o}}$ to Σ' and Σ'' respectively.

This will allow us to construct a Klein TQFT associated to these invariants in §8.

5. The level 0 theory

The main result in this section is a calculation of the level 0 theory for a symmetric sphere relative a pair of conjugate points, cf. Proposition 5.2. We start with the following preliminary result, for the level 0 theory, i.e. corresponding to the case when the line bundle L in (2.5) is trivial.

Lemma 5.1. *The level 0 RGW series (2.24) have no nonzero terms of positive degree in u.*

Proof. The level 0 RGW series are built from the following integrals:

$$\begin{aligned} RZ_{d,\chi}^{c,\mathfrak{o}}(\Sigma, \mathcal{O})_{\lambda^1\ldots\lambda^r} &= t^{\iota-b/2} \int\limits_{[\overline{\mathcal{M}}_{d,\chi}^{c,\bullet}(\Sigma)_{\lambda^1\ldots\lambda^r}]^{\mathrm{vir},\mathfrak{o}}} c_{b/2}(-\mathrm{Ind}\,\overline{\partial}_{\mathcal{O}}) \\ &= t^{\iota-b/2} \int\limits_{[\overline{\mathcal{M}}_{d,\chi}^{c,\bullet}(\Sigma)_{\lambda^1\ldots\lambda^r}]^{\mathrm{vir},\mathfrak{o}}} c_{b/2}(\mathbb{E}^\vee), \end{aligned}$$

where $\mathbb{E}^\vee$ denotes the dual of the Hodge bundle, and b is the dimension of the moduli space (2.21). Since the power of u in the level 0 RGW invariants (2.24) is $b/2$, it suffices to show that the only nonzero contribution to $RZ_{d,\chi}^{c,\mathfrak{o}}(\Sigma, \mathcal{O})_{\lambda^1..\lambda^r}$ comes from 0-dimensional moduli spaces. It suffices to show this is the case for the doublet and connected invariants of (Σ, c), when Σ is itself either a doublet or connected.

By Corollaries 3.4, 3.9, the doublet invariants for a connected or a doublet target are equal up to a scalar to the connected BP invariants. By the proof of [5, Lemma 7.5], for the antidiagonal action and level $(0, 0)$, the connected BP invariants vanish unless the dimension of the moduli space is 0.

So it remains to consider the case when both the domain and target are connected. Let $\overline{\mathbb{R}\mathcal{M}}_{g,\ell}$ and $\overline{\mathcal{M}}_{g,2\ell}$ denote the real and the complex Deligne-Mumford moduli spaces of connected genus g Riemann surfaces with ℓ pairs of conjugate and 2ℓ marked points, correspondingly. Consider the map

$$\overline{\mathbb{R}\mathcal{M}}_{g,\ell} \longrightarrow \overline{\mathcal{M}}_{g,2\ell} \tag{5.1}$$

forgetting the real structure on the curve. The image of this map falls into the fixed locus of the involution on $\overline{\mathcal{M}}_{g,2\ell}$ given by

$$[S, j, y_1, \ldots, y_{2\ell}] \mapsto [S, -j, y_2, y_1, \ldots y_{2\ell}, y_{2\ell-1}].$$

In general, the map (5.1) is neither injective nor surjective onto the fixed locus. However, the Hodge bundle $\mathbb{E}$ over the real Deligne-Mumford space is the pull-back via (5.1) of the Hodge bundle over the complex space. Over the real Deligne-Mumford space, the real structure σ on a Riemann surface representing a point in the space induces a complex conjugation on the fiber of the Hodge bundle over it. Therefore the Hodge bundle splits into invariant and anti-invariant parts of equal dimensions i.e.

$$\mathbb{E} \cong \mathbb{E}^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \overline{\mathbb{R}\mathcal{M}}_{g,\ell}.$$

This implies that

$$c_{2k+1}(\mathbb{E}) = 0 \in H^{4k+2}(\overline{\mathbb{R}\mathcal{M}}_{g,\ell}, \mathbb{Q}).$$

By Mumford's relations

$$0 = c_i(\mathbb{E} \otimes_{\mathbb{R}} \mathbb{C}) = \sum_{j=0}^{i} (-1)^j c_{i-j}(\mathbb{E}) c_j(\mathbb{E}).$$

In particular, for even index, $2c_{2k}(\mathbb{E}) + \sum_{j=1}^{2k-1} (-1)^j c_{2k-j}(\mathbb{E}) c_j(\mathbb{E}) = 0$. By induction on k, using the vanishing of the odd classes over the real moduli space we get

$$c_i(\mathbb{E}) = 0 \in H^{2i}(\overline{\mathbb{R}\mathcal{M}}_{g,\ell}, \mathbb{Q}) \quad \text{for all } i \neq 0.$$

Thus the only nonzero contributions to $RZ^{c,\mathbf{0}}_{d,\chi}(\Sigma, \mathcal{O})_{\lambda^1..\lambda^r}$ can come from integrating 1 over a 0-dimensional moduli space. □

5.1. Level 0 theory for a sphere relative two points

Consider next (Σ, c) a real sphere with a pair of conjugate points $x^{\pm}$. Up to reparametrization, there are only two real structures on $\Sigma = (\mathbb{P}^1, x^{\pm})$:

$$c_-(w) = -1/\overline{w} \quad \text{and} \quad c_+(w) = 1/\overline{w}.$$

The real locus Σ^c is empty for the first one and non-empty for the second one.

For the remainder of this section, we regard $\mathbb{P}^1$ as $\mathbb{C}\cup\infty$ with coordinate w, such that the preferred point x^+ corresponds to $w=0$ and x^- to $w=\infty$. Let S^1 be the unit circle $|w|=1$, which separates $\mathbb{P}^1$, and is oriented as the boundary of the component containing $x^+=0$. Then S^1 is the fixed locus when $c=c_+$ and is a cross-cap when $c=c_-$ (i.e. $c_-(w)=-w$ for all $w\in S^1$).

The relative tangent bundle $T\Sigma$, given by (2.22), is trivial for $\Sigma=(\mathbb{P}^1,x^\pm)$. Therefore a twisted orientation data $\mathfrak{o}=(\Theta,\psi,s)$ for $(\mathbb{P}^1,x^\pm,c)$ consists of a trivial complex line bundle $\Theta=\Sigma\times\mathbb{C}$ over $\Sigma=(\mathbb{P}^1,x^\pm)$, a choice of a homotopy class of Real isomorphisms (2.7), i.e.

$$\psi:\Lambda^{\mathrm{top}}(T\Sigma\oplus\Theta\oplus c^*\overline{\Theta},dc\oplus c_{tw})\xrightarrow{\cong}(\Sigma\times\mathbb{C},c\times c_{std}),\tag{5.2}$$

and a spin structure $\mathfrak{s}$ on $T\Sigma^c\oplus\mathbb{C}_{|\Sigma^c}$ over the real part of the target, compatible with the orientation induced by the isomorphism (5.2).

Note also that, up to homotopy, there is a unique trivialization

$$\phi:(T\Sigma,dc)\cong(\Sigma\times\mathbb{C},c\times c_{std})\quad\text{such that it restricts to}\tag{5.3}$$

$$(T\Sigma,dc)_{|S^1}=(TS^1\oplus JTS^1,dc)=(S^1\times(\mathbb{R}\oplus j\mathbb{R}),c\times c_{std}).\tag{5.4}$$

This is because there are two classes of trivializations (5.3) and they are distinguished by their restriction to S^1, cf. [11, Lemma 2.4]; we choose the one that satisfies (5.4).

Finally, to each partition $\lambda=(1^{m_1}2^{m_2}3^{m_3}\dots)$, associate the monomial

$$p_\lambda=\prod_{k=1}^{\infty}p_k^{m_k}.\tag{5.5}$$

With this notation, our main result in this section is:

Proposition 5.2. *Consider a Real sphere $\Sigma=(\mathbb{P}^1,x^\pm)$ with a pair of marked points and real structure c. Let $\mathfrak{o}$ be an orientation data for (Σ,c). Then for any partition λ of d,*

$$RGW_d^{c,\mathfrak{o}}(0|0)_\lambda=\exp\left(\varepsilon_{\mathfrak{o}}\sum_{k=0}^{\infty}\frac{p_{2k+1}}{(2k+1)t}-\sum_{m=1}^{\infty}\frac{p_m^2}{2mt^2}\right)_{[p_\lambda]},\tag{5.6}$$

where $\varepsilon_{\mathfrak{o}}=\pm1$ is independent of d. Here $(P)_{[p_\lambda]}$ denotes the coefficient of the monomial p_λ in the formal power series P.

Moreover, for each $\varepsilon=\pm1$ there exists a choice of a twisted orientation data $\mathfrak{o}$ such that $\varepsilon_{\mathfrak{o}}=\varepsilon$.

Proof. It suffices to calculate the connected and doublet invariants, since (2.16) extends to give

$$1+\sum_{\substack{d\geq 1\\ \lambda\vdash d}} RGW_d^{c,\mathfrak{o}}(\Sigma,L)_\lambda p_\lambda=\exp\left(\sum_{\substack{d\geq 1\\ \lambda\vdash d}} CRGW_d^{c,\mathfrak{o}}(\Sigma,L)_\lambda p_\lambda+\sum_{\substack{d\geq 1\\ \lambda\vdash d}} DRGW_{2d}^{c,\mathfrak{o}}(\Sigma,L)_{\lambda,\lambda} p_\lambda^2\right).$$

By Corollary 3.9, the doublet invariants $DRGW$ are related to the BP invariants GW^{conn} counting *connected* curves. The latter were computed in [5, Lemma 6.1] giving:

$$DRGW_{2d}^{c}(0|0)_\lambda(u,t) = -\tfrac{1}{2}GW_d^{conn}(0|0,0)_{\lambda_+,\lambda_-}(iu,it) = \tfrac{-1}{2d(-t)^2}\,, \qquad \text{for } \lambda_+ = \lambda_- = (d)$$

and vanish otherwise.

By Lemma 5.1, the only contribution to the connected real invariant $CRGW$ comes from 0 dimensional moduli spaces. The dimension of $\overline{\mathcal{M}}_{d,h}^{c}(\mathbb{P}^1)_\lambda$ is

$$b = 2d + 2h - 2 - 2d + 2\ell(\lambda) = 2h - 2 + 2\ell(\lambda).$$

It vanishes only when $h = 0$ and $\ell(\lambda) = 1$ i.e. $\lambda = (d)$. It suffices to show that in this case

$$\int_{[\overline{\mathcal{M}}_{d,0}^{c,\mathfrak{o}}(\mathbb{P}^1)_\lambda]^{\mathrm{vir}}} 1 = \begin{cases} \varepsilon_{\mathfrak{o}}\frac{1}{d}, & \text{if } d \text{ is odd,}\\ 0, & \text{if } d \text{ is even.}\end{cases} \tag{5.7}$$

Elements of $\overline{\mathcal{M}}_{d,0}^{c}(\mathbb{P}^1)_\lambda$ for $\lambda = (d)$ are real covers of a sphere by a sphere, fully ramified at the two points $x^\pm$, and equivariant with respect to a real structure σ on the domain and c on the target.

Case 1. Assume first that $c(w) = -1/\overline{w}$, so it has no fixed locus. Then σ cannot have fixed locus, and d must be odd (else the moduli space is empty). When d is odd, the moduli space consists of one solution $f(z) = z^d$, $\sigma(z) = -1/\overline{z}$, but which has d automorphisms $\phi(z) = \zeta z$ where $\zeta^d = 1$. It remains to calculate its sign and show it does not depend on d. We will first prove that there are two classes of twisted orientation data, giving rise to opposite invariants, and then we calculate the invariants for a canonical choice $\mathfrak{o} = \mathfrak{o}_{can}$ that corresponds to $\varepsilon_{\mathfrak{o}} = 1$.

A twisted orientation data $\mathfrak{o} = (\Theta, \psi, \mathfrak{s})$ in this case consists of a choice of an isomorphism (5.2) up to homotopy; the bundle $\Theta = \Sigma \times \mathbb{C}$ is trivial and the real locus of c is empty so the spin structure $\mathfrak{s}$ is irrelevant.

There are two real homotopy classes of isomorphisms (5.2) distinguished by the real homotopy class of ψ over the unit circle $|w| = 1$ in $\mathbb{P}^1 = \mathbb{C}\cup\infty$. One can switch between them by $\psi \mapsto -\psi$. The effect of this change on the orientation of the moduli space is via the change of the orientation on the bundle $\operatorname{Ind}\bar{\partial}_{(\mathbb{C},c_{std})}$, which is $(-1)^{\chi/2} = -1$ since the domains are spheres. In particular, if $\mathfrak{o}_1$ and $\mathfrak{o}_2$ denote the two choices of twisted orientation, then the level 0 connected invariants satisfy

$$CRGW_d^{c,\mathfrak{o}_1}(0|0)_\lambda = -CRGW_d^{c,\mathfrak{o}_2}(0|0)_\lambda. \tag{5.8}$$

We next determine the sign of the invariants in each degree by looking at the moduli space in more detail. The orientation on the moduli space is induced from the determinant bundle $\det \bar\partial_{(T\Sigma,dc)}$ and the Deligne-Mumford moduli space, cf. (A.13), after stabilization when necessary. So we add an extra pair of conjugate marked points $y_2^\pm$ on the domain. The moduli space is now 2 dimensional and it suffices to calculate the sign of the evaluation map at y_2^+, cf. [15, (2.2)]. For this we first exhibit an orientation on the moduli space for which the sign of the evaluation map is clear and then we compare it with orientation induced by the twisted orientation data.

The real DM moduli space $\mathbb{R}\mathcal{M}_{0,2}$ is 1-dimensional and consists of 3 intervals that compactify to a circle; one of the intervals corresponds to the case the involution on the domain is fixed-point free and the other two to the case the involution has fixed locus. We can assume that $\sigma(z) = \pm 1/\bar z$, $y_1^\pm$ are $z = 0, \infty$, and $y_2^+ = b \in \mathbb{R}_+$. The orientation on $\mathbb{R}\mathcal{M}_{0,2}$ agrees with the one induced by $b \in \mathbb{R}_+$ when σ has fixed locus, and is the opposite in the case σ is fixed-point free; see [16, §1.4].

When the domain is fixed, the moduli space of degree d real relative maps is

$$f_\tau : (\mathbb{P}^1, \sigma) \longrightarrow (\mathbb{P}^1, c) \quad z \mapsto e^{i\tau} z^d, \quad \tau \in \mathbb{R}/2\pi\mathbb{Z};$$

here $\sigma(z) = -1/\bar z$. Thus the relative moduli space with the extra pair $y_2^\pm$ of marked points is described by $(\tau, b) \in \mathbb{R} \times \mathbb{R}_+$, where b corresponds to the position of y_2^+ and τ gives the map f_τ. For the orientation induced by this identification, the evaluation map at y_2^+ is orientation reversing. The tangent space to the first factor corresponds naturally to $\operatorname{Ind} \bar\partial_{(T\Sigma,dc)}$ and the tangent space to the second factor to $T\mathbb{R}\mathcal{M}_{0,2}$. Recall that the canonical orientation on the latter is opposite that of $b \in \mathbb{R}_+$ when the domain involution is fixed-point free. Thus the evaluation map at y_2^+ would have positive sign if the orientation induced by a twisted orientation on $\operatorname{Ind} \bar\partial_{(T\Sigma,dc)}$ coincides with that induced by $\tau \in \mathbb{R}$. We next construct such twisted orientation data.

Let $\mathfrak{o}_{can}$ be the twisted orientation data for which (5.2) has the form $\psi = \phi \otimes \Lambda^{top}\theta_{tw}$, where ϕ is given by (5.3) and

$$\theta_{tw} : (\Sigma \times \mathbb{C} \oplus c^*(\Sigma \times \overline{\mathbb{C}}), c_{tw}) \cong (\Sigma \times \mathbb{C}^{\oplus 2}, c \times c_{std})$$

is orientation preserving at the level of index bundles when the first term has the complex orientation induced via (2.9) and the second term is oriented as twice a bundle. By Lemma 5.3 below, we can obtain such θ_{tw} as the composition of (5.9) and (5.10). For this choice, the twisted orientation data $\psi = \phi \otimes \Lambda^{top}\theta_{tw}$ induces precisely the isomorphism (5.4), as explained above (3.14). On the other hand, the isomorphism (5.4) induces an orientation on $\operatorname{Ind} \bar\partial_{(T\Sigma,dc)}$ that coincides with that of $\tau \in \mathbb{R}$. Therefore, for this choice of twisted orientation data, the evaluation map has positive degree for all odd d, completing the proof of (5.7).

Case 2. Assume $c(w) = 1/\bar w$, so the involution on the target has fixed locus. The argument in this case follows along the same lines. The fixed locus Σ^c is now the unit circle

S^1 and a twisted orientation data requires a choice $\mathfrak{s}$ of a spin structure on $T\Sigma^c \oplus \mathbb{C}_{|\Sigma^c}$ over the real part of the target, compatible with the orientation induced by the isomorphism (5.2). There is still one solution for d odd (with $\sigma(z) = 1/\overline{z}$ on the domain), but when d is even, there are now two solutions, with different real structures.

We next construct a twisted orientation data $\mathfrak{o} = \mathfrak{o}_{can}$ for which $\varepsilon_{\mathfrak{o}} = +1$. Let $\psi = \phi \otimes \Lambda^{top}\theta$, where ϕ is as in (5.3) and θ is the isomorphism (5.9). The isomorphism θ, along with the orientation of TS^1, induces a spin structure $\mathfrak{s}$, compatible with ψ. Denote these choices by $\mathfrak{o}_{can}$.

We repeat the same argument as in Case 1, taking into account that the orientation on the real DM moduli space is given by $b \in \mathbb{R}_+$ when σ has real locus, and by $-b$ when σ does not have real locus. Recall that in odd degree σ must have real locus, while in even degree there are two solutions, one with real locus and one without. By Lemma 5.3 below, at the level of index bundles, the isomorphism θ has sign $(-1)^{\mathrm{ind}\,\overline{\partial}_{\mathbb{C}}} = (-1)^{\chi/2} = -1$ and thus the orientation induced on $\mathrm{Ind}\,\bar{\partial}_{(T\Sigma,dc)}$ is opposite of that induced by $\tau \in \mathbb{R}$. So all maps whose domain involution has fixed locus contribute positively and all maps with fixed-point free domain involution contribute negatively. This implies that the maps in even degree cancel each other. In odd degree, the domains can only have real structure with fixed locus and thus contribute positively. This implies (5.7) for $\mathfrak{o} = \mathfrak{o}_{can}$ (with $\varepsilon_{\mathfrak{o}} = 1$).

It remains to calculate how the invariants depend on the orientation data $\mathfrak{o} = (\Sigma \otimes \mathbb{C}, \psi, \mathfrak{s})$. Up to homotopy, there are 4 choices, two for ψ and two for the spin structure $\mathfrak{s}$. As before, a change in the homotopy class of ψ changes the orientation on all maps thus giving (5.8). A change in the spin structure results in a change of $(-1)^d$ on the orientation of a degree d map as it changes the pullback spin structure on the domain only if the degree is odd, cf. [15, Corollary 5.7] and Lemma A.3. Since the even degree invariants vanish, changing the spin structure $\mathfrak{s}$ also gives (5.8), completing the proof of (5.7). □

When $(L, \phi) \to (\Sigma, c)$ is a Real bundle over a symmetric surface, then

$$\theta : (L \oplus c^*\overline{L}, c_{tw}) \cong (L \oplus L, \phi \oplus \phi), \qquad (z; x, y) \mapsto (z; x + \phi(y), -Jx + J\phi(y)) \quad (5.9)$$

is a Real isomorphism. The index of the LHS has a natural complex orientation while that of the RHS can be oriented as twice of a bundle. The next lemma compares these two orientations.

Lemma 5.3. *Assume $(L, \phi) \to (\Sigma, c)$ is a Real bundle. Then the index bundle $\mathrm{Ind}\,\overline{\partial}_{(L \oplus c^*\overline{L}, c_{tw})}$ has two natural orientations:*

(i) *one induced by the isomorphism (2.9) with $\mathrm{Ind}\,\overline{\partial}_L$ via the projection onto the first bundle.*
(ii) *the second one induced by (5.9) and the natural orientation on twice a bundle.*

The difference between these orientations is $(-1)^\iota$ *where* ι *is the complex rank of* $\operatorname{Ind}\overline{\partial}_L$*. Moreover,*

$$Id \oplus -Id : \operatorname{Ind}\overline{\partial}_{(L\oplus L,\phi\oplus\phi)} \to \operatorname{Ind}\overline{\partial}_{(L\oplus L,\phi\oplus\phi)}, \tag{5.10}$$

when both sides are oriented as twice a bundle, also has sign $(-1)^\iota$*.*

Proof. The isomorphism (2.9) with $\operatorname{Ind}\overline{\partial}_L$ induces a complex structure and therefore a complex orientation on the index bundle associated to the left hand side of (5.9). The isomorphism induced by θ at the level of index bundles would be orientation preserving if the complex orientation on twice of a bundle was used instead on the right hand side; the two choices differ by $(-1)^\iota$. The second statement is immediate. □

The proof of Proposition 5.2 constructs choices of orientation data $\mathfrak{o} = \mathfrak{o}_{can}$ that have the property that $\varepsilon_{\mathfrak{o}} = 1$; in particular, the sign of the degree 1 cover is +1. For such choices, (5.6) is equal to

$$RGW_d(0|0)_\lambda = \exp\left(\sum_{k=0} \frac{p_{2k+1}}{(2k+1)t} - \sum_{m=1} \frac{p_m^2}{2mt^2}\right)_{[p_\lambda]}, \tag{5.11}$$

while for any other choice of orientation data

$$RGW_d^{c,\mathfrak{o}}(0|0)_\lambda = (\varepsilon_{\mathfrak{o}})^d RGW_d(0|0)_\lambda$$

where $\varepsilon_{\mathfrak{o}} = \pm 1$ is the sign of the degree 1 cover. This follows because substituting $p_m \mapsto \varepsilon_{\mathfrak{o}} p_m$ for all $m = 1, 2, \ldots$ converts the sum in the exponential of (5.11) to the one in (5.6), but also changes $p_\lambda \mapsto (\varepsilon_{\mathfrak{o}})^d p_\lambda$ when λ is a partition of d.

6. Canonical orientation and independence of the target real structure

In this section we study how the RGW invariants depend on the choice of orientation data and on the real structure on the target. We show that a change in the orientation data or in the real structure results in a global change by a factor of $(\pm 1)^d$. We then use this information to define canonical RGW invariants which are compatible with the splitting formulas.

6.1. Dependence on the orientation data and real structure

Assume (Σ, c) is a symmetric Riemann surface with r pairs of conjugate marked points. We first describe how the RGW invariants depend on the choice of orientation data.

Lemma 6.1. *For any two orientation data $\mathfrak{o}_1$, $\mathfrak{o}_2$ for (Σ, c), there exists $m \in \mathbb{Z}$ such that*

$$RGW_d^{c,\mathfrak{o}_1}(\Sigma|L)_{\lambda^1\dots\lambda^r} = (-1)^{dm} RGW_d^{c,\mathfrak{o}_2}(\Sigma|L)_{\lambda^1\dots\lambda^r} \tag{6.1}$$

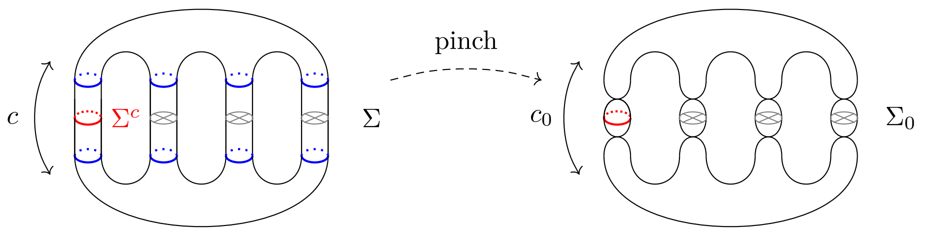
for all d and all collections of r partitions $\lambda^1, \dots, \lambda^r$ of d.

For every (Σ, c) there exist two orientation data for which the sign difference is $(-1)^d$.

Proof. It suffices to prove this when Σ is either connected or a doublet. Assume $\mathfrak{o}_i = (\Theta_i, \psi_i, \mathfrak{s}_i)$ are two orientation data for (Σ, c), cf. Definition 2.1.

Case 1. When Σ is a doublet, Lemma 3.1 implies that the *RGW* invariants for the two orientations differ by a factor of $(-1)^{dm_2}$, where m_2 is the difference between the degrees of the restrictions to the second component of Σ of the bundles Θ_i. Changing the degree of Θ_1 by 1 on the first component and by -1 on the second gives rise to a sign difference of $(-1)^d$.

Case 2. Assume next that Σ is connected. Choose a separating collection $\{\gamma_i\}$ of circles, each one of which is either fixed or a crosscap. Trivialize the complex line bundle L in a neighborhood of the γ_i, and split off a level 0-sphere containing no marked points, one for each γ_i. The complement of these spheres is then a doublet.



Any orientation data on Σ can similarly be split to induce an orientation data on the split surface Σ_0. For two different orientation data on the split surface, the invariants of the i'th sphere differ by a factor of ε_i^d, where $\varepsilon_i = \pm 1$ (by Proposition 5.2), and the invariants on the doublet by $(-1)^{dm}$ (as above). The splitting formula (4.5) then implies the same is true for the invariants of the original surface. $\square$

Lemma 6.2. *Assume Σ is a connected surface with $2r$ pairs of marked points, and c_1, c_2 are two real structures on Σ. Then for every orientation data $\mathfrak{o}_{c_1}$ on (Σ, c_1) there exists an orientation data $\mathfrak{o}_{c_2}$ on (Σ, c_2) so that*

$$RGW_d^{c_1,\mathfrak{o}_1}(\Sigma|L)_{\lambda^1\dots\lambda^r} = RGW_d^{c_2,\mathfrak{o}_2}(\Sigma|L)_{\lambda^1\dots\lambda^r}. \tag{6.2}$$

Proof. Real structures on Σ are classified topologically by the number of fixed circles of Σ and the orientability of Σ/c, see e.g. [21, §2.3]. We can transform (Σ, c_1) into (Σ, c_2) via a sequence of splittings of a sphere around either a crosscap or a fixed circle as above and replacing that sphere by a sphere with the other real structure. By Proposition 5.2 we can choose the orientation data on the new sphere so that its invariants match those of the old sphere. The claim follows from the splitting formula (4.3). □

Remark 6.3. When the target is connected, Lemma 3.6 implies that the orientation of the doublet moduli space depends neither on the choice of orientation data, nor on the real structure of the target.

When the target is a doublet, then up to deformation different choices of orientation data are distinguished by the degree of $\Theta|_{\Sigma_2}$, cf. Example 2.2(b). As in the proof above, the local RGW invariants then differ by a factor of $(-1)^{dm_2}$, where m_2 is the difference between these degrees.

6.2. Canonical RGW invariants

Assume (Σ, c) is a symmetric surface with r pairs of conjugate points. The discussion above partitions the choices of orientation data $\mathfrak{o}$ on (Σ, c) into two nonempty classes, distinguished by the sign $\varepsilon_{\mathfrak{o}} = \pm 1$ of the $d = 1$ cover of (Σ, c).

Definition 6.4. A canonical twisted orientation for (Σ, c) corresponds to a choice of twisted orientation data $\mathfrak{o}_{can} = \mathfrak{o}$ for which the degree 1 cover of Σ has sign $\varepsilon_{\mathfrak{o}} = +1$.

Corollary 6.5. *With the notation above,*

$$RGW_d(\Sigma|L)_{\lambda^1 \dots \lambda^r} = (\varepsilon_{\mathfrak{o}})^d RGW_d^{c,\mathfrak{o}}(\Sigma|L)_{\lambda^1 \dots \lambda^r} \tag{6.3}$$

is well defined, independent of the orientation data $\mathfrak{o}$ and of the real structure c on Σ; in particular,

$$RGW_d(\Sigma|L)_{\lambda^1 \dots \lambda^r} = RGW_d^{c,\mathfrak{o}_{can}}(\Sigma|L)_{\lambda^1 \dots \lambda^r}.$$

It is also compatible with the splitting formula (4.3), in the sense that

$$RGW_d(\Sigma, L)^{\nu^1 \dots \nu^s}_{\mu^1 \dots \mu^r} = \sum_{\lambda \vdash d} RGW_d(\widetilde{\Sigma}, \widetilde{L})^{\lambda, \nu^1 \dots \nu^s}_{\lambda, \mu^1 \dots \mu^r}. \tag{6.4}$$

Proof. The fact that (6.3) is independent of the orientation data $\mathfrak{o}$ on (Σ, c) follows from Lemma 6.1. Next, (6.3) is also independent of the real structure c on Σ by Lemma 6.2 since (6.2) for $d = 1$ implies that the sign of the degree 1 cover is the same for both $\mathfrak{o}_1$ and $\mathfrak{o}_2$. Finally, under the splitting (4.3) degree 1 covers split as degree 1 covers, giving (6.4). □

We end this section with a few consequences of this discussion.

Corollary 6.6. *The degree d, connected genus h real invariants of a connected genus g target vanish unless $d(g-1)+h-1 \equiv 0 \mod 2$.*

Proof. By Corollary 6.5 the RGW invariants (6.3) are independent of the choice of real structure and of orientation data on the target. For a connected target, the same is true for the doublet invariants $DRGW$ by Remark 6.3. Since the RGW invariants are equal to $\exp(CRGW + DRGW)$ as in (2.16) it follows that the connected RGW invariants of a connected target Σ are also independent of these choices. Finally, when the real structure on the connected genus g target has no fixed locus, there are no real degree d maps from a connected genus h surface to Σ unless $d(g-1)+h-1 \equiv 0 \mod 2$ cf. [17, Example 5.1]. Therefore the connected invariants vanish for any choice of real structure and orientation data unless this condition is satisfied. □

Lemma 6.7. *Exchanging the order within the i-th pair of conjugate marked points of Σ changes $RGW_d(\Sigma|L)_{\lambda^1\dots\lambda^r}$ by a factor of $(-1)^{d-\ell(\lambda^i)}$. Exchanging two pairs of conjugate points does not change the invariant.*

Proof. Exchanging $x_i^+ \leftrightarrow x_i^-$ in the target exchanges their $\ell(\lambda^i)$ preimages, contributing the $(-1)^{\ell(\lambda^i)}$ factor. For a degree 1 map, $\ell(\lambda^i) = 1$ and thus this changes the sign of the degree 1 cover by a factor of -1. This forces a change in the twisted orientation data to compensate for the $-$ sign on the degree 1 cover as in Lemma 6.1. The effect of this change on a degree d map is $(-1)^d$. Altogether, this implies the first claim. The second claim follows immediately since permuting pairs of conjugate points in the domain is relatively orientation preserving at the level of the DM moduli spaces. □

Corollary 6.8. *The degree d RGW invariants (6.3) of a connected target vanish unless $d-\ell(\lambda^i) \equiv 0$ mod 2 for all i.*

Proof. This follows by Lemma 6.7 since on a connected target we can find a path connecting x^+ to x^- and therefore continuously deform the pair of conjugate marked points (x^+, x^-) into (x^-, x^+). □

Corollary 6.9. *For a g-doublet target with all the positive marked points on the first component, we have*

$$RGW_d(g,g|k_1,k_2)_{\lambda^1\dots\lambda^r}(u,t) = (-1)^{dk_2} GW_d(g|k_1,k_2)_{\lambda^1\dots\lambda^r}(iu,it). \tag{6.5}$$

Proof. Since the degree $d = 1$ cover of a complex curve counts positively, Lemma 3.1 implies that, for a doublet target $\Sigma = \Sigma_1 \sqcup \Sigma_2$, $\mathfrak{o}_{can}$ corresponds to a choice $(\Theta, \psi, \mathfrak{s})$ such that

$$c_1(\Theta)[\Sigma_2] \equiv 0 \mod 2.$$

(Note that $\ell_2 = 0$ because by assumption all the + points are on the first component.) By Corollary 3.4, the real and complex invariants differ by a factor of $(-1)^{dk_2+\chi/2}$. Since the correspondence $(u,t) \mapsto (iu, it)$ changes the coefficient $GW_{d,\chi}(g|k_1,k_2)_{\lambda^1\ldots\lambda^r}$ by $(-1)^{\chi/2}$, we obtain (6.5). □

7. TQFT and Klein TQFT

We will use the local RGW invariants to define an extension of a semi-simple Klein TQFT in §8, which we completely solve in §9, obtaining explicit closed formulas for the local RGW invariants. This section contains a brief overview of TQFTs and Klein TQFTs, following [5, §4] and [2, §1] (up to some change in notation), and a discussion of semi-simple ones.

Let 2**Cob** be the usual (oriented, closed) 2-dimensional cobordism category. It is the symmetric monoidal category with objects given by compact oriented 1-manifolds (without boundary) and morphisms given by (diffeomorphism classes of) oriented cobordisms. A 2-dimensional topological quantum field theory (2d TQFT) with values in a commutative ring R is a symmetric monoidal functor

$$F : 2\mathbf{Cob} \to R\text{mod},$$

where Rmod is the category of R-modules. This is equivalent to a commutative Frobenius algebra over R; the product and co-product correspond to the pair of pants while the unit and co-unit to the cap and cup respectively, see Fig. 2.

In [5, §4.2], Bryan and Pandharipande enlarge the category 2**Cob** to a category $2\mathbf{Cob}^{L_1,L_2}$ with the same objects, but with extra morphisms. The morphisms are now equivalence classes of oriented cobordisms W decorated by a pair of complex line bundles $L_1, L_2 \to W$ trivialized over the boundary. The equivalence is up to bundle isomorphisms covering diffeomorphisms between the cobordisms (and compatible with the trivializations over the boundary). The composition is given by concatenation of the cobordisms and gluing the bundles using the trivializations over the boundary.

For example, a morphism in $2\mathbf{Cob}^{L_1,L_2}$ corresponding to a connected cobordisms W is completely determined by the genus g of W together with a pair of integers (k_1, k_2), called the level, recording the Euler classes $e(L_i) \in H^2(W, \partial W)$. Restricting the morphisms to $k_1 = k_2 = 0$ defines an embedding

$$2\mathbf{Cob} \subset 2\mathbf{Cob}^{L_1,L_2}.$$

In [5, §4.4] Bryan-Pandharipande use the local GW invariants to define a symmetric monoidal functor

$$\mathbf{GW} : 2\mathbf{Cob}^{L_1,L_2} \to R\text{mod}. \tag{7.1}$$

on this larger category. The functor (7.1) extends the classical 2d TQFT that appeared in the work of Dijkgraaf-Witten [7] and Freed-Quinn [12], and whose Frobenius algebra is the center $\mathbb{Q}[S_d]^{S_d}$ of the group algebra of the symmetric group S_d. It is used to completely solve the local Gromov-Witten theory.

A different extension of 2**Cob** is obtained by allowing unoriented and possibly unorientable surfaces as cobordisms; see [1,2]. We refer to this category as 2**KCob**, where **K** stands for Klein (surface). The objects are closed unoriented 1-manifolds and the morphisms are diffeomorphism classes of *unoriented* (and possibly unorientable) cobordisms. An equivalent point of view is to consider the orientation double covers of both the objects and the morphisms: (i) the objects are then closed oriented 1-manifolds with an orientation-reversing involution (deck transformation) exchanging the sheets of the cover and (ii) the morphisms are compact oriented 2-dimensional manifolds with a *fixed-point free* orientation-reversing involution extending the one on the boundary. Such 2-dimensional manifolds are called symmetric surfaces and we denote this category by 2**SymCob**. Moreover

$$2\mathbf{KCob} \equiv 2\mathbf{SymCob}$$

where the identification is obtained by passing to the orientation double cover in one direction and taking the quotient by the involution in the other direction. Working from the perspective of 2**SymCob** allows us to construct an extension of this category related to that of [5] and completely solve the local real Gromov-Witten theory. For this reason we describe 2**KCob** and 2**SymCob** in parallel below.

Remark 7.1. As mentioned after [2, Definition 1.7], it is convenient to identify 2**KCob** (and respectively 2**Cob**) with its skeleton, which is the full subcategory whose objects are disjoint unions of copies of a *fixed* oriented circle S^1. For 2**SymCob** we take the full subcategory whose objects are disjoint unions of two circles $\mathcal{S} = (S^1 \sqcup \overline{S^1}, \varepsilon)$, where $\overline{S^1}$ denotes the circle with opposite orientation and $\varepsilon|_{S^1} = id : S^1 \longrightarrow \overline{S^1}$. This way, 2**Cob** can be regarded as a subcategory of 2**KCob** with the same objects, but fewer morphisms:

$$2\mathbf{Cob} \subset 2\mathbf{KCob}.$$

Note that even if a cobordism in 2**KCob** is orientable, there may not be way to orient it in a way compatible with the boundary identifications. For example, Fig. 1 shows two different cobordisms, the first one being the tube (which induces the identity). The second one reverses the orientation of the S^1 and we refer to it as the involution Ω. It is a morphism in 2**KCob** but not in 2**Cob**. The difference is even more visible from the perspective of 2**SymCob**, cf. second cobordism in (7.3).

Fig. 1. The tube (identity) and the involution Ω in 2**KCob**.

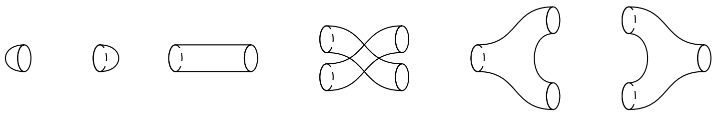


Fig. 2. The elementary cobordisms: cap, cup, tube, twist and the pairs of pants in 2**Cob** ⊂ 2**KCob**.

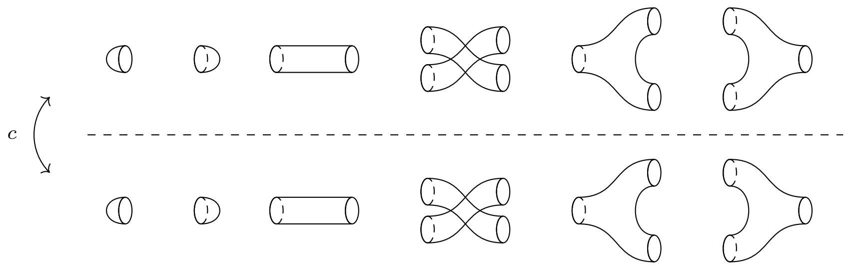
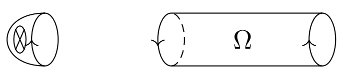


Fig. 3. The elementary cobordisms: cap, cup, tube, twist and the pairs of pants in 2**SymCob**.

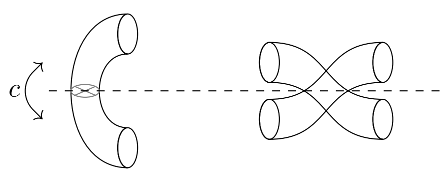
When 2**Cob** is regarded as a subcategory of 2**KCob** as described in Remark 7.1, its generators are given in Fig. 2 (cf. [2, Figure 1.1]). The corresponding elements of 2**SymCob** are their orientation double covers, cf. Fig. 3.

The category 2**KCob** has two extra generators, the cross-cap (a Möbius band) and the involution



(7.2)

respectively. In 2**SymCob** these correspond to their orientation double covers:



(7.3)

Note that in 2**SymCob** the involution swaps the two outgoing circles.

The extra generators satisfy certain relations in 2**KCob** (see p 1840-1841 of [2]). For example, moving a puncture once around the Möbius band changes the orientation of the puncture, cf. Fig. 4; equivalently, the involution acts trivially on the product of the cross-cap with another element, cf. (7.6).See Fig. 5

Another relation comes from decomposing the product of two cross-caps as in Fig. 6, cf. (7.7).

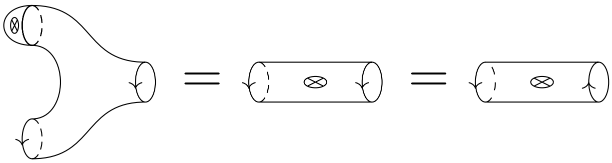


Fig. 4. The cobordism K and relations in 2**KCob**.

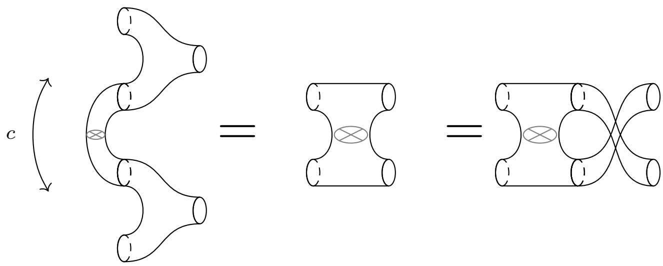


Fig. 5. The cobordism K and relations in 2**SymCob**.

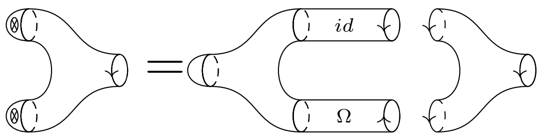


Fig. 6. Relation in 2**KCob**: decomposing the punctured Klein bottle.

7.1. Semi-simple Klein TQFT

Definition 7.2. A (closed) 2d Klein TQFT is a symmetric monoidal functor

$$F : 2\mathbf{KCob} \to R\mathrm{mod}. \tag{7.4}$$

In fact, cf. [2, Prop 1.11], a (closed) 2d Klein TQFT is equivalent to a commutative Frobenius algebra $H = F(S^1)$ together with two extra structures:

(a) an involutive (anti)-automorphism Ω of the Frobenius algebra H, denoted $x \mapsto x^*$. This means

$$(x^*)^* = x, \quad (xy)^* = y^*x^* \quad \text{and} \quad \langle x^*, y^*\rangle = \langle x, y\rangle \quad \text{for all } x, y \in H. \tag{7.5}$$

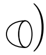
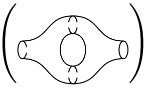
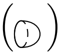
(b) an element $U \in H$ such that

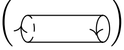
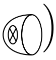
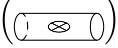
$$(aU)^* = aU \quad \text{for all } \ a \in H \quad \text{and} \tag{7.6}$$

$$U^2 = m(id \otimes \Omega)(\Delta(1)) = \sum \alpha_i\beta_i^* , \qquad \text{where the co-product} \quad \Delta(1) = \sum \alpha_i \otimes \beta_i. \tag{7.7}$$

The involution Ω and the element U correspond to the cobordisms (7.2). For an interpretation of the relations (b), see Fig. 4 and 6.

There are several elements of $2\mathbf{KCob}$ that play a special role; their images under (7.4) are denoted:

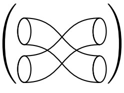
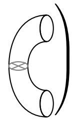
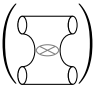
$F\Big($  $\Big) = 1$, $\quad F\Big($  $\Big) = G$, $\quad$ and $\quad F\Big($  $\Big) = C$, (7.8)

$F\Big($  $\Big) = \Omega$, $\quad F\Big($  $\Big) = U$, $\quad$ and $\quad F\Big($  $\Big) = K$. (7.9)

When (7.4) is regarded as a morphism on $2\mathbf{SymCob} \equiv 2\mathbf{KCob}$ via the orientation double cover construction, we denote it

$$\widetilde{F} : 2\mathbf{SymCob} \to R\text{mod}. \tag{7.10}$$

In particular,

$\widetilde{F}\Big($  $\Big) = \Omega$, $\quad \widetilde{F}\Big($  $\Big) = U$, $\quad$ and $\quad \widetilde{F}\Big($  $\Big) = K$. (7.11)

Definition 7.3. A semi-simple Klein TQFT is a Klein TQFT whose associated Frobenius algebra is semi-simple.

A semi-simple TQFT is determined by the structure constants $\{\lambda_\rho\}$, i.e. the coefficients of the co-multiplication $\Delta(v_\rho) = \lambda_\rho v_\rho \otimes v_\rho$ in the idempotent basis $\{v_\rho\}$. Moreover,

Proposition 7.4. *Assume (7.4) is a semisimple KTQFT with idempotent basis $\{v_\rho\}$, and assume that the ground ring R has no zero divisors. Then*

(i) $G(v_\rho) = \lambda_\rho v_\rho$ *and* $C(v_\rho) = \lambda_\rho^{-1}$.
(ii) Ω *defines an involution on the idempotent basis* $\Omega(v_\rho) = v_{\rho^*}$.
(iii) *If* $U = \sum_\rho U_\rho v_\rho$ *then* $U_\rho^2 = \lambda_\rho$ *if* $\rho = \rho^*$, *and* $U_\rho = 0$ *if* $\rho \neq \rho^*$.
(iv) $K(v_\rho) = U_\rho v_\rho$.

Proof. Property (i) holds for any semi-simple TQFT. To prove (ii), note that the second relation in (7.5) implies

$$\Omega(v_\rho)\Omega(v_\mu) = \Omega(v_\mu v_\rho)$$

for all ρ and μ. If $\Omega(v_\rho) = \sum_\nu \Omega_\rho^\nu v_\nu$, then since $\{v_\rho\}$ is an idempotent basis (i.e. $v_\rho v_\mu = \delta_{\rho\mu} v_\rho$ for all ρ, μ) this implies

$$\sum_\nu \Omega_\rho^\nu \Omega_\mu^\nu v_\nu = \delta_{\mu\rho} \sum_\nu \Omega_\rho^\nu v_\nu.$$

Therefore

$$\Omega_\rho^\nu \Omega_\mu^\nu = 0 \quad \text{for all } \rho \neq \mu \text{ and all } \nu, \quad \text{while} \tag{7.12}$$

$$(\Omega_\rho^\nu)^2 = \Omega_\rho^\nu \quad \text{so} \quad \Omega_\rho^\nu = 0 \text{ or } 1 \quad \text{for all } \rho,\ \nu \tag{7.13}$$

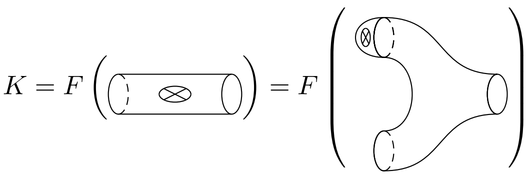
(because R has no zero divisors). Equation (7.12) implies that there is at most one non zero element in each row of the matrix associated to Ω in this basis. But since Ω is invertible, there must be exactly one non-zero element in each row, which by (7.13) must be equal to 1. The invertibility of Ω also implies that there is precisely one non-zero element in every column. This proves (ii).

Next, (7.7) (cf. Fig. 6) implies

$$U_\rho^2 = \Omega_\rho^\rho \lambda_\rho, \tag{7.14}$$

since $\Delta(1) = \sum_\rho \lambda_\rho v_\rho \otimes v_\rho$. This gives (iii).

Finally, property (iv) follows since $K(x) = U \cdot x$, i.e. K decomposes as



cf. Fig. 4. □

Assume Σ is a closed symmetric surface, considered as a morphism in 2**SymCob** from the ground ring to the ground ring.

Corollary 7.5. *With the notation of Proposition 7.4, the morphism (7.10) is given by:*

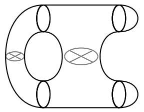
$$\widetilde{F}(\Sigma) = \sum_{\rho=\rho^*} U_\rho^{g-1}, \qquad \textit{when } \Sigma \textit{ is a connected genus } g \textit{ surface and}$$

$$\widetilde{F}(\Sigma \sqcup \overline{\Sigma}) = \sum_{\rho} \lambda_\rho^{g-1}, \qquad \textit{when } \Sigma \sqcup \overline{\Sigma} \textit{ is a } g\textit{-doublet.}$$

Proof. This follows from Proposition 7.4 by decomposing the surface Σ into elementary cobordisms, and the fact that connected symmetric surfaces (without real locus) are classified by their genus, cf. eg. [21, §2.3]. When $\Sigma = \mathbb{P}^1$,

$$\widetilde{F}(\mathbb{P}^1) = CU = \sum_\rho \lambda_\rho^{-1} U_\rho = \sum_{\rho=\rho^*} \lambda_\rho^{-1} U_\rho = \sum_{\rho=\rho^*} U_\rho^{-1}.$$

Similarly,

$$\widetilde{F}(T^2) = \widetilde{F}\left(\right.$$  $$\left.\right) = CKU \tag{7.15}$$

More generally, for a genus $g \geq 1$ symmetric surface Σ,

$$\widetilde{F}(\Sigma) = CK^g U = \sum_{\rho} \lambda_\rho^{-1} U_\rho^g U_\rho = \sum_{\rho=\rho^*} U_\rho^{g-1}.$$

Finally, on a g-doublet, the morphisms is

$$\widetilde{F}(\Sigma \sqcup \overline{\Sigma}) = CG^g(1) = \sum_{\rho} \lambda_\rho^{g-1},$$

recovering the classical theory. □

7.2. The category $2\mathbf{SymCob}^L$

We next construct a simultaneous extension of the categories $2\mathbf{Cob}^{L_1,L_2}$ and $2\mathbf{KCob} \equiv 2\mathbf{SymCob}$. Consider the category $2\mathbf{SymCob}^L$ whose

- objects are disjoint unions of copies of $\mathcal{S} = (S^1 \sqcup \overline{S^1}, \varepsilon)$, where ε swaps the two components, and
- morphisms correspond to isomorphism classes relative boundary of *decorated cobordisms*

$$W = (\Sigma, c, L),$$

where Σ is an oriented cobordism with a fixed-point free orientation-reversing involution c, extending ε, and L is a complex line bundle over Σ, trivialized along the boundary of Σ.

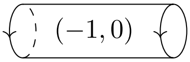
The level zero theory corresponds to a trivial bundle L, and defines an embedding:

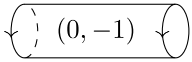
$$2\mathbf{Cob} \subset 2\mathbf{KCob} \equiv 2\mathbf{SymCob} \subset 2\mathbf{SymCob}^L. \tag{7.16}$$

The doubling procedure defines an embedding

$$2\mathbf{Cob}^{L_1,L_2} \subset 2\mathbf{SymCob}^L, \quad (\Sigma, L_1, L_2) \mapsto (\Sigma \sqcup \overline{\Sigma}, c_{|\Sigma} = id : \Sigma \to \overline{\Sigma}, L_1 \sqcup \overline{L}_2). \tag{7.17}$$

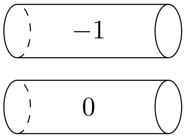
The category $2\mathbf{Cob}^{L_1,L_2}$ has 4 extra generators, the level $(\pm 1, 0), (0, \pm 1)$-caps, besides those of $2\mathbf{Cob}$, cf. [5, §4.3]. Similarly, the generators of the category $2\mathbf{SymCob}^L$ are those of $2\mathbf{SymCob}$ together with the images of the $(\pm 1, 0), (0, \pm 1)$-caps under (7.17). It is also useful to consider the tubes

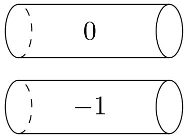


and


(7.18)

in $2\mathbf{Cob}^{L_1,L_2}$, and their images



and


(7.19)

in $2\mathbf{SymCob}^L$ under (7.17).

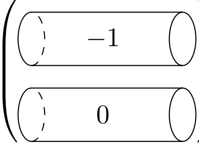
As in [5, Theorem 4.1], we obtain the following result.

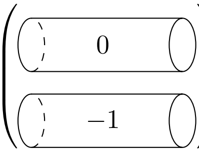
Proposition 7.6. *A symmetric monoidal functor*

$$F : 2\mathbf{SymCob}^L \longrightarrow Rmod \tag{7.20}$$

is uniquely determined by the level 0 *theory and the images* η *and* $\overline{\eta}$ *of the level* $(-1, 0)$ *and* $(0, -1)$*-caps.*

The images

$A = F$ (


) and $\bar{A} = F$ (


) (7.21)

of (7.19) are called the level-decreasing operators, and moreover

$$A(x) = \eta \cdot x, \quad \overline{A}(x) = \overline{\eta} \cdot x.$$

If the restriction of (7.20) to the level 0 theory defines a semi-simple KTQFT with idempotent basis $\{v_\rho\}$ then

$$A(v_\rho) = \eta_\rho v_\rho \text{ and } \bar{A}(v_\rho) = \overline{\eta}_\rho v_\rho, \quad \text{where} \quad \eta = \sum_\rho \eta_\rho v_\rho \text{ and } \bar{\eta} = \sum_\rho \overline{\eta}_\rho v_\rho. \tag{7.22}$$

As in Corollary 7.5, then the value of F on a closed connected genus g symmetric surface Σ at level $k = c_1(L)[\Sigma]$ is equal to

$$F(\Sigma|L) = CA^{-k}K^g(U) = \sum_{\rho=\rho^*} U_\rho^{g-1}\eta_\rho^{-k}. \tag{7.23}$$

The value of F on a g-doublet $\Sigma \sqcup \overline{\Sigma}$ with a line bundle $L_1 \sqcup L_2$ is similarly equal to

$$F(\Sigma \sqcup \overline{\Sigma}\,|L_1, L_2) = C(A^{-k_1}\bar{A}^{-k_2}G^g(1)) = \sum_\rho \lambda_\rho^{g-1}\eta_\rho^{-k_1}\bar{\eta}_\rho^{-k_2},$$

where $k_1 = c_1(L_1)[\Sigma]$ and $k_2 = c_1(L_2)[\overline{\Sigma}]$.

8. The Klein TQFT induced by the RGW invariants

In this section we use the local RGW invariants (6.3) to define an extension of a Klein TQFT, i.e. a functor **RGW** from the category $2\mathbf{SymCob}^L$ described in §7.2. This extends the Bryan-Pandharipande TQFT constructed from the *GW* theory for the anti-diagonal action; see §3.1.

Let $R = \mathbb{C}(t)((u))$ be the ring of Laurent series in u whose coefficients are rational functions of t and d be a positive integer. Denote by $\mathcal{S} = (S^1 \sqcup \overline{S^1}, \varepsilon)$ the disjoint union of two copies of a circle with opposite orientations and with the involution ε swapping them. To the object $\mathcal{S}$ we associate

$$\mathbf{RGW}_d(\mathcal{S}) = H = \bigoplus_{\alpha \vdash d} Re_\alpha, \tag{8.1}$$

the free module with basis $\{e_\alpha\}_{\alpha\vdash d}$ indexed by partitions α of d. Let

$$\mathbf{RGW}_d(\mathcal{S} \sqcup \cdots \sqcup \mathcal{S}) = H \otimes \cdots \otimes H.$$

To each cobordism $W = (\Sigma, c, L)$ in $2\mathbf{SymCob}^L$ from n copies of $\mathcal{S}$ to m copies of $\mathcal{S}$, associate the R-module homomorphism

$$\mathbf{RGW}_d(W) : H^{\otimes n} \to H^{\otimes m} \tag{8.2}$$

defined by

$$e_{\lambda^1} \otimes \cdots \otimes e_{\lambda^n} \mapsto \sum_{\mu^i \vdash d} RGW_d(\Sigma_W|L_W)^{\mu^1..\mu^m}_{\lambda^1..\lambda^n}\; e_{\mu^1} \otimes \cdots \otimes e_{\mu^m}.$$

Here (i) Σ_W is a closed marked symmetric Riemann surface whose topological type is that of Σ after removing small disks around the pairs of marked points, (ii) the first element in each pair of marked points of Σ_W corresponds to the first copy of S^1 in $\mathcal{S} = (S^1 \sqcup \overline{S^1})$, and (iii) $L_W \to \Sigma_W$ is a holomorphic line bundle whose first Chern class corresponds to the Euler class of $L \to \Sigma$. Finally, $RGW_d(\Sigma_W|L_W)_{\vec{\lambda}}$ are the local RGW invariants defined by (6.3) and (2.24), and the indices are raised by (4.2). The coefficients are invariant under smooth deformation, thus the assignment (8.2) is well-defined.

Theorem 8.1. *The assignment* (8.2) *defines a symmetric monoidal functor*

$$\mathbf{RGW}_d : 2\mathbf{SymCob}^L \to R\text{mod}. \tag{8.3}$$

Its restriction to $2\mathbf{KCob}$ *under* (7.16) *is a Klein TQFT, while its restriction to* $2\mathbf{Cob}^{L_1,L_2}$ *under* (7.17) *is*

$$\mathbf{RGW}_d(\Sigma \sqcup \overline{\Sigma}|L_1 \sqcup \overline{L}_2)(u,t) = (-1)^{dk_2}\mathbf{GW}_d(\Sigma|L_1, L_2)(iu, it). \tag{8.4}$$

Here k_i *is the total degree of* L_i *and* $\mathbf{GW}_d$ *is the TQFT* (7.1) *considered by Bryan-Pandharipande (for the anti-diagonal action).*

Proof. By Lemma 6.7, coefficients of the assignment (8.2) are invariant under permuting two *pairs* of conjugate points of Σ_W, thus (8.3) is symmetric. It is monoidal i.e.

$$\mathbf{RGW}_d(W_1 \sqcup W_2) = \mathbf{RGW}_d(W_1) \otimes \mathbf{RGW}_d(W_2)$$

because the real moduli space over a disjoint union of *Real* curves is the product of the real moduli spaces on each piece, and the index bundle naturally decomposes as the direct sum of the two index bundles. The composition law

$$\mathbf{RGW}_d(W_1 \circ W_2) = \mathbf{RGW}_d(W_1) \circ \mathbf{RGW}_d(W_2)$$

holds by (6.4), cf. (4.5) and (4.4).

When W is a doublet, Corollary 6.9 implies that the restriction of (7.1) to $2\mathbf{Cob}^{L_1,L_2}$ is the Bryan-Pandharipande TQFT (7.1) modified as stated. In particular (8.3) takes the identity in $2\mathbf{Cob}$ to the identity morphism, and therefore (8.3) is a functor. □

9. Solving the theory

In this section we show that the functor $\mathbf{RGW}_d$ defined by (8.3) restricts at level 0 to a semi-simple Klein TQFT. We also provide an explicit expression in terms of representation theoretic data of its value on a closed symmetric surface with a line bundle over it, thus solving the local RGW theory.

Conjugacy classes of the symmetric group S_d are indexed by partitions α of d. If ρ is an irreducible representation of S_d, let $\chi_\rho(\alpha)$ denote the trace of ρ on the conjugacy class α.

Recall that the level 0 part of $2\mathbf{SymCob}^L$ is naturally identified with $2\mathbf{SymCob} = 2\mathbf{KCob}$. Then

Lemma 9.1. *The restriction of the functor* $\mathbf{RGW}_d$ *to* $2\mathbf{KCob} \subset 2\mathbf{SymCob}^L$ *determines a semi-simple Klein TQFT with idempotent basis* (9.1).

Proof. This is a direct consequence of [5] with small modifications as follows. Let e_α be as in (8.1). Define a new basis

$$v_\rho = \frac{\dim \rho}{d!} \sum_\alpha (-t)^{\ell(\alpha)-d} \chi_\rho(\alpha) e_\alpha, \tag{9.1}$$

indexed by the irreducible representations ρ of S_d. Note that

$$e_\alpha = (-t)^{d-\ell(\alpha)} \sum_\rho \frac{d!}{\dim \rho} \frac{\chi_\rho(\alpha)}{\zeta(\alpha)} v_\rho. \tag{9.2}$$

The pair of pants product is determined by

$$RGW_d((0,0)|(0,0))^\gamma_{\alpha,\beta}$$

and by Corollary 6.9 and the last display on p. 113 of [5]

$$\begin{aligned} RGW_d((0,0)|(0,0))^\gamma_{\alpha,\beta}(t) &= GW_d(0|0,0)^\gamma_{\alpha,\beta}(it) \\ &= t^{d-\ell(\alpha)-\ell(\beta)+\ell(\gamma)} \sum_\rho \left(\frac{d!}{\dim \rho}\right) \frac{\chi_\rho(\alpha)\chi_\rho(\beta)}{\zeta(\alpha)\zeta(\beta)} \chi_\rho(\gamma). \end{aligned}$$

As in [5] this implies that $\{v_\rho\}$ is an idempotent basis and therefore $\mathbf{RGW}_d$ is semi-simple. □

Note that the relation between v_ρ, defined in (9.1), and v_ρ^{BP}, defined in [5, Equation (20)], is

$$v_\rho(t) = v_\rho^{BP}(it). \tag{9.3}$$

As discussed in §7, the theory is determined by the genus-adding operator G, the level-decreasing operators A, $\bar{A}$, the cross-cap U, and the involution Ω. Moreover,

Lemma 9.2. *In the idempotent basis $\{v_\rho\}$, the genus-adding operator G, the $(-1,0)$-tube A, and the $(0,-1)$-tube $\bar{A}$ have eigenvalues respectively*

$$\lambda_\rho = t^{2d}\left(\frac{d!}{\dim \rho}\right)^2, \quad \eta_\rho = t^d Q^{c_\rho/2}\left(\frac{\mathrm{dimh}_Q \rho}{\dim \rho}\right), \quad \overline{\eta}_\rho = t^d Q^{-c_\rho/2}\left(\frac{\mathrm{dimh}_Q \rho}{\dim \rho}\right). \tag{9.4}$$

Here $Q = e^u$, c_ρ is the total content of the Young diagram associated to ρ, and

$$\mathrm{dimh}_Q \rho = d! \prod_{\square \in \rho} \left(2 \sinh \tfrac{h(\square) u}{2}\right)^{-1} = d! \prod_{\square \in \rho} \left(Q^{\frac{h(\square)}{2}} - Q^{-\frac{h(\square)}{2}}\right)^{-1}, \tag{9.5}$$

where $h(\square)$ denotes the hooklength of the square $\square$ in the Young diagram associated to ρ.

Proof. By (8.4) and (9.3), the relation between the $\mathbf{RGW}_d$ and $\mathbf{GW}_d$ is obtained by the change of variables $(u,t) \mapsto (iu, it)$ and multiplication by $(-1)^{dc_1(L_2)}$ in both the standard and the idempotent bases. The result then follows from [5, §7.3]. In particular, (9.5) is related to the quantum dimension defined in [5] via

$$\mathrm{dimh}_Q\rho(u) = (-i)^d \dim_{Q_{BP}} \rho(iu), \qquad \text{where}$$
$$\frac{\dim_{Q_{BP}} \rho}{d!} = \prod_{\square\in\rho} \left(2\sin\frac{h(\square)u}{2}\right)^{-1} = \prod_{\square\in\rho} i\left(Q_{BP}{}^{\frac{h(\square)}{2}} - Q_{BP}{}^{-\frac{h(\square)}{2}}\right)^{-1}$$

and $Q_{BP} = e^{iu}$. □

It remains to determine U and Ω in the idempotent basis.

Proposition 9.3. *The involution Ω is given by*

$$\Omega(e_\alpha) = (-1)^{d-\ell(\alpha)} e_\alpha \quad \textit{and} \quad \Omega(v_\rho) = v_{\rho'} \tag{9.6}$$

in the standard basis $\{e_\alpha\}$ and in the idempotent basis $\{v_\rho\}$, respectively. Here ρ' denotes the conjugate representation.

Proof. Consider the moduli space of real maps into the doublet corresponding to Ω. It is the same as the moduli space of real maps into the doublet associated to the level 0 tube (the identity), except for the change $x_2^+ \leftrightarrow x_2^-$ of the order within the pair of marked points in the target corresponding to the outgoing boundary. Lemma 6.7 then implies the first equality. In the idempotent basis (9.1)

$$\Omega(v_\rho) = \frac{\dim\rho}{d!}\sum_\alpha (-t)^{\ell(\alpha)-d}\chi_\rho(\alpha)\Omega(e_\alpha) = \frac{\dim\rho}{d!}\sum_\alpha (-t)^{\ell(\alpha)-d}\chi_{\rho'}(\alpha)e_\alpha = v_{\rho'},$$

where the second equality holds since $\chi_{\rho'}(\alpha) = (-1)^{d-l(\alpha)}\chi_\rho(\alpha)$, cf. [22, page 42]. □

Note that (7.14) and (9.6) imply that the coefficients U_ρ vanish unless $\rho = \rho'$. If $\rho = \rho'$ then (7.14) and (9.4) imply that $U_\rho = \pm t^d \frac{d!}{\dim\rho}$ determining it up to a sign. Proposition 9.5 below calculates U directly, independent of these considerations, including the signs. The signed Frobenius-Schur indicator, defined in §11, plays a crucial role in this calculation.

9.1. The level 0 cross-cap U

Consider next the level 0 cross-cap U corresponding to (7.3). Its coefficients in the standard basis are obtained from the RGW invariants of a sphere with 2 marked points, real structure $c(z) = -1/\overline{z}$, and a trivial line bundle.

Before we proceed, it is convenient to make the following definition. For a partition λ of d, let

$$sq(\lambda) \tag{9.7}$$

denote the partition of d obtained from λ by splitting all of the even parts of λ into two equal parts e.g. $sq(4,3,3,2,1) = (2,2,3,3,1,1,1)$. This is motivated by the fact that if $g \in S_d$ has cycle type λ, then g^2 has cycle type $sq(\lambda)$. Recall that the sign morphism on S_d descends to the conjugacy class

$$\operatorname{sign}(g) = (-1)^{s(g)} = (-1)^{d-\ell(\lambda)} = \operatorname{sign}(\lambda), \tag{9.8}$$

where $s(g)$ is the parity of the permutation $g \in S_d$ and λ is its cycle type. This is also the parity of the number of even parts of λ and in particular, $\operatorname{sign}(sq(\lambda)) = +1$.

We start with the following combinatorial identity, which uses the notation (5.5); see also (5.6).

Lemma 9.4. *For any partition $\alpha \vdash d$, the coefficient r_α of the monomial p_α in the expansion below is given by*

$$r_\alpha = \left[\exp\left(\sum_{d \text{ odd}} \frac{1}{d} p_d - \frac{1}{2}\sum_{d=1}^{\infty} \frac{1}{d}(p_d)^2\right)\right]_{p_\alpha} = \sum_{\substack{\lambda \vdash d \\ sq(\lambda)=\alpha}} \frac{(-1)^{d-\ell(\lambda)}}{\zeta(\lambda)}. \tag{9.9}$$

In particular, r_α vanishes unless α has an even number of even parts.

Proof. The coefficient r_α on the LHS of (9.9) is the sum over all possible ways of decomposing α into a partition a containing only odd elements and 2 copies of a partition b:

$$\sum_\alpha r_\alpha p_\alpha = \sum_{a,b} \prod_{k \text{ odd}} \frac{p_k^{a_k}}{a_k! k^{a_k}} \prod_k \frac{(-1)^{b_k} p_k^{2b_k}}{2^{b_k} b_k! k^{b_k}}, \tag{9.10}$$

where a_k, b_k denote the multiplicities of k in the partitions a and b respectively. Every such decomposition $\alpha = a \sqcup b \sqcup b$ corresponds to a partition $\lambda = a \sqcup (2b)$, where $(2b)$ denotes the partition obtained from b by multiplying by 2 each of its parts; in particular, $sq(\lambda) = \alpha$ and $\sum_k b_k \equiv d - \ell(\lambda) \mod 2$. Therefore (9.10) becomes

$$\sum_\alpha r_\alpha p_\alpha = \sum_\alpha \Big(\sum_{\substack{\lambda \vdash d \\ sq(\lambda)=\alpha}} \frac{(-1)^{d-\ell(\lambda)}}{\zeta(\lambda)} \Big) p_\alpha. \quad \square$$

Combined with Proposition 5.2, Lemma 9.4 implies that the invariant $RGW(0|0)_\alpha$ is equal to

$$RGW(0|0)_\alpha = \frac{r_\alpha}{t^{\ell(\alpha)}} = \frac{1}{t^{\ell(\alpha)}} \sum_{\substack{\lambda \vdash d \\ sq(\lambda)=\alpha}} \frac{(-1)^{d-\ell(\lambda)}}{\zeta(\lambda)}. \tag{9.11}$$

The coefficients $RGW(0|0)^\alpha$ of U in the standard basis $\{e_\alpha\}$ are obtained by raising the indices in (9.11) via (4.2). Combinatorial considerations then allow us to determine the coefficients of

$$U = \sum_\alpha RGW(0|0)^\alpha e_\alpha = \sum_\rho U_\rho v_\rho \tag{9.12}$$

in the idempotent basis $\{v_\rho\}$.

Decompose the set of partitions λ of d into even and odd according to the parity of $d - \ell(\lambda)$, cf. (9.8).

Proposition 9.5. *The level 0 cross-cap (9.12) is equal to the sum over self-conjugate irreducible representations of S_d*

$$U = \sum_{\rho=\rho'} \varepsilon_\rho t^d \frac{d!}{\dim\rho} v_\rho, \quad \textit{where} \tag{9.13}$$

$$\varepsilon_\rho = (-1)^{o(\rho)} \quad \textit{and} \quad o(\rho) = \sum_{\substack{\beta\vdash d \\ \beta \text{ odd}}} \frac{\chi_\rho(sq(\beta))}{\zeta(\beta)}. \tag{9.14}$$

The expression $o(\rho)$ takes values 0 *or* 1 *on a self-conjugate irreducible representation ρ.*

Proof. By (9.12), (9.11) and (9.2)

$$U = \sum_\alpha RGW(0|0)^\alpha e_\alpha = \sum_\alpha \Big(\sum_{sq(\lambda)=\alpha} \frac{(-1)^{d-\ell(\lambda)}}{\zeta(\lambda) t^{\ell(\alpha)}} \Big) \zeta(\alpha) t^{2l(\alpha)} e_\alpha =$$

$$= \sum_\alpha \Big(\sum_{sq(\lambda)=\alpha} \frac{(-1)^{d-\ell(\lambda)}}{\zeta(\lambda)} \Big) \zeta(\alpha) t^{\ell(\alpha)} \Big((-t)^{d-\ell(\alpha)} \sum_\rho \frac{d!}{\dim\rho} \frac{\chi_\rho(\alpha)}{\zeta(\alpha)} v_\rho \Big) =$$

$$= \sum_\rho \Big(\sum_\alpha \sum_{sq(\lambda)=\alpha} (-1)^{d-\ell(\lambda)} \frac{\chi_\rho(\alpha)}{\zeta(\lambda)} \Big) \frac{d! t^d}{\dim\rho} v_\rho$$

In the last equality we used the fact that for $\alpha = sq(\lambda)$ the parity of d and $\ell(\alpha)$ is the same. It remains to show that the expression in the parenthesis is given by (9.14). For this we use the following combinatorial identity

$$\sum_{\substack{\lambda\vdash d \\ sq(\lambda)=\alpha}} (-1)^{d-\ell(\lambda)} \frac{\zeta(\alpha)}{\zeta(\lambda)} = \sum_{\rho=\rho'} \varepsilon_\rho \chi_\rho(\alpha) \tag{9.15}$$

cf. Lemma 11.3, which is of independent interest and whose proof is deferred to §11. Then

$$\sum_{\alpha} \sum_{sq(\lambda)=\alpha} (-1)^{d-\ell(\lambda)} \frac{\chi_\rho(\alpha)}{\zeta(\lambda)} = \sum_{\alpha} \sum_{\mu=\mu'} \varepsilon_\mu \chi_\mu(\alpha) \frac{\chi_\rho(\alpha)}{\zeta(\alpha)}$$

$$= \sum_{\mu=\mu'} \varepsilon_\mu \sum_{\alpha} \frac{\chi_\mu(\alpha)\chi_\rho(\alpha)}{\zeta(\alpha)} = \begin{cases} \varepsilon_\rho & \text{if } \rho = \rho', \\ 0 & \text{otherwise.} \end{cases}$$

The result follows. □

The next lemma provides a simpler expression for the sign ε_ρ appearing in (9.13).

Lemma 9.6. *Let ρ be an irreducible representation of S_d, $r(\rho)$ the length of the main diagonal of its Young diagram, and ε_ρ be as* (9.14)*. Then*

$$\varepsilon_\rho = (-1)^{\frac{d-r(\rho)}{2}}. \tag{9.16}$$

Proof. Let $x = (x_1, \ldots, x_n)$. The power sum functions $p_k(x)$ and the Schur functions $s_\rho(x)$ are related by

$$s_\rho(x) = \sum_{\lambda} \frac{\chi_\rho(\lambda)}{\zeta(\lambda)} p_\lambda(x) \qquad \text{and} \qquad p_\lambda(x) = \sum_{\rho} \chi_\rho(\lambda) s_\rho(x)$$

and form basis for the space of symmetric polynomials on n variables whenever $n > d$. By [22, 9c, p79], we have

$$\sum_{\rho=\rho'} (-1)^{(d+r(\rho))/2} s_\rho(x_1, \ldots, x_n) = \prod_i (1 - x_i) \prod_{i<j} (1 - x_i x_j),$$

where ρ' denotes the conjugate representation. This equality follows from Weyl's identity for B_n.

By Lemma 9.4,

$$\sum_{\alpha} \sum_{\substack{\lambda \\ sq(\lambda)=\alpha}} \frac{(-1)^{d-\ell(\lambda)}}{\zeta(\lambda)} p_\alpha = \exp\Big(\sum_{d \text{ odd}} \tfrac{1}{d} p_d - \tfrac{1}{2} \sum_{d=1}^{\infty} \tfrac{1}{d} (p_d)^2 \Big). \tag{9.17}$$

Substitute $p_d = p_d(x)$, and consider first the LHS of (9.17):

$$\sum_{\alpha} \sum_{\substack{\lambda \\ sq(\lambda)=\alpha}} \frac{(-1)^{d-\ell(\lambda)}}{\zeta(\lambda)} p_\alpha(x) = \sum_{\alpha} \sum_{\substack{\lambda \\ sq(\lambda)=\alpha}} \frac{(-1)^{d-\ell(\lambda)}}{\zeta(\lambda)} \sum_{\rho} \chi_\rho(\alpha) s_\rho(x) =$$

$$= \sum_{\rho} \Big(\sum_{\alpha} \sum_{\substack{\lambda \\ sq(\lambda)=\alpha}} \frac{(-1)^{d-\ell(\lambda)}}{\zeta(\lambda)} \chi_\rho(\alpha) \Big) s_\rho(x) = \sum_{\rho=\rho'} \varepsilon_\rho s_\rho(x).$$

For the last equality we used the displayed equation after (9.15). For the RHS of (9.17), we start with

$$\sum_{d \text{ odd}} \tfrac{1}{d} p_d(x) = \sum_{d \text{ odd}} \tfrac{1}{d} \sum_i x_i^d = \sum_i \log \left(\frac{1+x_i}{1-x_i} \right)^{1/2}$$

and

$$-\tfrac{1}{2} \sum_{d=1}^{\infty} \tfrac{1}{d} (p_d(x))^2 = -\tfrac{1}{2} \sum_{d=1}^{\infty} \tfrac{1}{d} \Big(\sum_i x_i^d \Big)^2 = -\tfrac{1}{2} \sum_{d=1}^{\infty} \tfrac{1}{d} \sum_{i,j} x_i^d x_j^d = \sum_{i,j} \log(1 - x_i x_j)^{1/2}.$$

Therefore,

$$\exp \left(\sum_{d \text{ odd}} \tfrac{1}{d} p_d(x) - \frac{1}{2} \sum_{d=1}^{\infty} \tfrac{1}{d} p_d(x)^2 \right) = \prod_i \left(\frac{1+x_i}{1-x_i} \right)^{1/2} \prod_{i,j} (1 - x_i x_j)^{1/2} =$$

$$\prod_i \left(\frac{1+x_i}{1-x_i} \right)^{1/2} \prod_{i<j} (1 - x_i x_j) \prod_{x_i = x_j} (1 - x_i x_j)^{1/2} = \prod_i (1 + x_i) \prod_{i<j} (1 - x_i x_j).$$

Since $s_\rho(-x) = (-1)^d s_\rho(x)$ then

$$\sum_{\rho=\rho'} (-1)^d \varepsilon_\rho s_\rho(x) = \sum_{\rho=\rho'} \varepsilon_\rho s_\rho(-x) = \prod_i (1 - x_i) \prod_{i<j} (1 - x_i x_j) = \sum_{\rho=\rho'} (-1)^{(d+r(\rho))/2} s_\rho(x).$$

But $s_\rho(x)$ is a basis so (9.16) holds. □

Lemma 9.6 and Proposition 9.5 then imply:

Corollary 9.7. *In the idempotent basis, the level 0 cross-cap U is given by*

$$U = \sum_{\substack{\rho \vdash d \\ \rho = \rho'}} (-1)^{(d-r(\rho))/2} \, t^d \frac{d!}{\dim \rho} v_\rho, \tag{9.18}$$

where $r(\rho)$ is the length of the main diagonal of the Young diagram of ρ.

9.2. Local Calabi-Yau over a sphere

Consider next the local RGW invariants associated to the Real Calabi-Yau 3-fold Y defined by (2.5) for $\Sigma = \mathbb{P}^1$ and $L = \mathcal{O}(-1)$. Note that Y is biholomorphic to the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$, thus contains no holomorphic curves besides multiple covers of the zero section. In particular, the only real curves in Y are the multiple covers of the zero section $\Sigma \subset Y$. Moreover, the discussion in the paragraph above (2.9) implies that the zero section in $L \oplus c^* \overline{L}$ with $L = \mathcal{O}(-1)$ is super-rigid and therefore (2.10) is precisely the contribution of its multiple covers to the real Gromov-Witten invariants of Y.

Theorem 9.8. *The generating function for the RGW invariants is*

$$1+\sum_{d=1}^{\infty} RGW_d(0|-1)q^d = 1+\sum_{\rho=\rho'}(-1)^{\frac{1}{2}(|\rho|-r(\rho))}\prod_{\square\in\rho}\left(2\sinh\tfrac{h(\square)u}{2}\right)^{-1}q^{|\rho|} \tag{9.19}$$

$$= \exp\left(\sum_{k\text{ odd}}\tfrac{1}{k}\left(2\sinh\tfrac{ku}{2}\right)^{-1}q^k - \tfrac{1}{2}\sum_{k}\tfrac{1}{k}\left(2\sinh\tfrac{ku}{2}\right)^{-2}q^{2k}\right). \tag{9.20}$$

In particular, the generating function for the connected real invariants is

$$\sum_{d=1}^{\infty} CRGW_d(0|-1)q^d = \sum_{k\text{ odd}}\tfrac{1}{k}\left(2\sinh\tfrac{ku}{2}\right)^{-1}q^k. \tag{9.21}$$

Proof. Recall that $RGW(0|-1) = CAU$ cf. (7.23) and the coefficients of C, A and U in the idempotent basis $\{v_\rho\}$ are given by (9.4) and (9.18). Since the content c_ρ of a self-conjugate partition vanishes, this gives (9.19).

Next, the invariants RGW are related to the connected and doublet invariants by

$$1+\sum_{d=1}^{\infty} RGW_d(0|-1)q^d = \exp\left(\sum_d CRGW_d(0|-1)q^d + \sum_d DRGW_{2d}(0|-1)q^{2d}\right)$$

cf. (2.16). Corollary 3.9 relates the doublet invariants to the connected GW invariants and along with the classical calculation of [10] we obtain

$$\sum_d DRGW_{2d}(0|-1)(u)q^{2d} = \tfrac{1}{2}\sum_d GW_d^{conn}(0|-1,-1)(iu)q^{2d} = -\tfrac{1}{2}\sum_k \tfrac{1}{k}\left(2\sinh\tfrac{ku}{2}\right)^{-2}q^{2k}.$$

It thus remains to prove (9.20). Substituting $p_d = (2\sinh\frac{du}{2})^{-1}q^d$ in (9.9) we obtain

$$\exp\left(\sum_{k\text{ odd}}\tfrac{1}{k}\left(2\sinh\tfrac{ku}{2}\right)^{-1}q^k - \tfrac{1}{2}\sum_k\tfrac{1}{k}\left(2\sinh\tfrac{ku}{2}\right)^{-2}q^{2k}\right)$$
$$= \sum_\alpha\Big(\sum_{\substack{\lambda\\ sq(\lambda)=\alpha}}\frac{(-1)^{d-l(\lambda)}}{\zeta(\lambda)}\Big)\frac{q^{|\alpha|}}{\prod_i 2\sinh\frac{\alpha_i u}{2}}$$

Using (9.15) the coefficient of q^d is equal to

$$\sum_{\alpha\vdash d}\sum_{\rho=\rho'}\varepsilon_\rho\frac{\chi_\rho(\alpha)}{\zeta(\alpha)}\frac{1}{\prod_i 2\sinh\frac{\alpha_i u}{2}} = \sum_{\rho=\rho'}\varepsilon_\rho Q^{d/2}\sum_\alpha\frac{\chi_\rho(\alpha)}{\zeta(\alpha)}\frac{(-1)^{\ell(\alpha)}}{\prod_i(1-Q^{\alpha_i})},$$

with $Q = e^u$. But $\varepsilon_\rho = (-1)^{\frac{d-r(\rho)}{2}}$ by Lemma 9.6, and the sum over α in the above expression equals the Schur function $s_{\rho'}$ for the conjugate representation ρ' times $(-1)^d$. Since

$$s_{\rho'} = Q^{c_{\rho'}-d/2}(-1)^d \frac{1}{\prod\limits_{\square\in\rho'} (Q^{h(\square)/2} - Q^{-h(\square)/2})}$$

cf. [22, page 45] and $\rho = \rho'$, $c_\rho = 0$, we obtain (9.20). $\square$

Remark 9.9. Note the similarity between (9.21) and the equivariant localization computation of the open GW invariants considered in [20, Theorem 7.2] for the weight $a = 0$ (see also [26, §6]). In this case the contributions of the graphs computing the invariants in the real and open case match in odd degree; for the real invariants in even degree, the graphs come in pairs depending on the type of the real structure, and there is a cancelation between open and crosscap contributions cf. [27, §3.3]. The sin vs sinh difference comes from the difference in orientation conventions, cf. [16, §3.1].

Remark 9.10. The right hand side of (9.20) has another expansion besides (9.19). By [25, (4.5)] with $t_1 = -t_2^{-1} = e^u$

$$\exp\Big(\sum_{k\text{ odd}} \tfrac{1}{k}\left(2\sinh\tfrac{ku}{2}\right)^{-1} q^k - \tfrac{1}{2}\sum_{k} \tfrac{1}{k}\left(2\sinh\tfrac{ku}{2}\right)^{-2} q^{2k}\Big) = \sum_\rho \frac{(-1)^{a(\rho)} q^{|\rho|}}{\prod\limits_{\square\in\rho} 2\sinh\frac{h(\square)u}{2}},$$

where $a(\rho)$ is the sum of the arm lengths of $\square \in \rho$. Note that this sum is over all representations, not only self-conjugate ones, and the signs $(-1)^{a(\rho)}$ and $\varepsilon_\rho = (-1)^{\frac{d-r(\rho)}{2}}$ are different in general. Nevertheless, the two sums are equal.

9.3. Local Calabi-Yau over a torus

Consider next the local RGW invariants associated to the Real CY 3-fold Y given by (2.5) for Σ a torus (elliptic curve) and L a degree 0 holomorphic line bundle. When L is not a torsion element in the Picard group, its total space contains no holomorphic curves other than the multiple covers of the zero section. Therefore as in §9.2, the zero section of Y is super-rigid and (2.10) is the contribution of its multiple covers to the real Gromov-Witten invariants of the 3-fold Y.

Theorem 9.11. *The generating function for the RGW invariants is*

$$\sum_d RGW_d(1|0)q^d = \sum_{\rho=\rho'} q^{|\rho|} = \exp\left(\sum_d (-1)^{d-1} \sum_{k\text{ odd}} \tfrac{1}{k} q^{dk} + \tfrac{1}{2}\sum_{d,k} \tfrac{1}{k} q^{2dk}\right). \quad (9.22)$$

Moreover, the generating function for the connected RGW invariants is

$$\sum_d CRGW_d(1|0)q^d = \sum_d (-1)^{d-1} \sum_{k \text{ odd}} \tfrac{1}{k} q^{dk}. \tag{9.23}$$

Proof. By (7.15),

$$RGW_d(1|0) = CKU = \sum_{\substack{\rho \vdash d \\ \rho = \rho'}} 1$$

giving the first equality in (9.22). Note that the generating function of the self-conjugate partitions is

$$\sum_{\rho=\rho'} q^{|\rho|} = \prod_d \frac{1}{1+(-q)^d}.$$

As in the proof of Theorem 9.8, relation (2.16) and Corollary 3.9 imply

$$\sum_d RGW_d(1|0)q^d = \exp\left(\sum_d CRGW_d(1|0)q^d + \sum_d DRGW_{2d}(1|0)q^{2d}\right) \quad \text{and}$$
$$\sum_d DRGW_{2d}(1|0)q^{2d} = \tfrac{1}{2}\sum_d GW_d^{conn}(1|0,0)q^{2d} = \tfrac{1}{2}\sum_{d,k} \tfrac{1}{k} q^{2dk}.$$

In the last equality we used the classical calculation

$$\exp\left(\sum_d GW_d^{conn}(1|0,0)q^d\right) = \sum_d GW_d(1|0,0)q^d = \sum_\rho q^{|\rho|} = \prod_d \frac{1}{1-q^d},$$

cf. [5, Corollary 7.3]. Since

$$\exp\left(\sum_d (-1)^{d-1} \sum_{k \text{ odd}} \tfrac{1}{k} q^{dk} + \tfrac{1}{2}\sum_{d,k} \tfrac{1}{k} q^{2dk}\right) = \prod_d \left(\frac{1-(-q)^d}{1+(-q)^d}\right)^{1/2} \left(\frac{1}{1-q^{2d}}\right)^{1/2},$$

we obtain the second equality in (9.22) and therefore (9.23). □

Remark 9.12. The connected invariants (9.23) can also be computed directly. By Lemma 5.1 and §6.1, it suffices to consider only real (unramified) covers of a torus without fixed locus by a torus; passing to the universal cover reduces this to a *signed* count of sub-lattices that are invariant under a lift of the complex conjugation. In fact, if we fix two separating crosscaps in the target, their inverse image consists of $d+d$ circles, each winding around the crosscap k times. One can show that exactly two of the circles are preserved by the involution in the domain (thus are crosscaps) and in particular k must be odd; d could be either even or odd. If d is odd, the two crosscaps in the domain map to the two crosscaps in the target; otherwise they map to a single crosscap in the target. Such a cover has degree dk, k automorphisms, and its sign is determined by whether or not the induced orientation on the crosscaps on the domain from the orientation on the crosscaps on the target coincides with the boundary orientation when we cut along the domain crosscaps (since the canonical orientation corresponds to having the crosscaps oriented in this manner). In particular, when d is odd we have +1 and when d is even -1, therefore contributing $(-1)^{d-1}\frac{1}{k}q^{dk}$ to (9.23).

9.4. The general case

Consider next a local Real 3-fold $(L \oplus c^*\overline{L}, c_{tw}) \to \Sigma$ over a connected surface.

Theorem 9.13. *Assume Σ is a connected genus g symmetric surface and $L \to \Sigma$ a holomorphic line bundle with $c_1(L) = k$. Then the degree d local RGW invariants are equal to*

$$RGW_d(g|k) = \sum_{\rho=\rho'} \left((-1)^{\frac{d-r(\rho)}{2}} t^d \frac{d!}{\dim \rho} \right)^{g-1} \left(t^d \, \frac{\mathrm{dimh}_Q \rho}{\dim \rho} \right)^{-k}. \tag{9.24}$$

Here the sum is over self-conjugate partitions ρ of d, $r(\rho)$ is the rank (2.18)*, and* $\mathrm{dimh}_Q\rho$ *is* (9.5)*.*

Proof. The result follows as before from $RGW(g|k) = CK^g A^{-k} U$, cf. (7.23). Note that when $g > 1$, $RGW(g|k)$ can also be obtained as the trace of the composition of the diagonal operators $K^{g-1}A^{-k}$. □

Corollary 9.14. *In the (Real) Calabi-Yau case, the contribution becomes*

$$RGW_d(g|g-1) = \sum_{\rho=\rho'} \left((-1)^{\frac{d-r(\rho)}{2}} \prod_{\square\in\rho} 2\sinh \tfrac{h(\square)u}{2} \right)^{g-1}. \tag{9.25}$$

In the equivariant Calabi-Yau case, the (complex) GW invariants defined in [5] are equal to

$$GW_d(g|g-1, g-1) = \sum_{\rho\vdash d} \Big(\prod_{\square\in\rho} 2\sin \tfrac{h(\square)u}{2} \Big)^{2g-2} \tag{9.26}$$

cf. [5, Corollary 7.3]. Note that here the sum is over all partitions of d, not just self-conjugate ones. Moreover, the generating function $GW^{conn}(g|g-1, g-1)$ of the connected GW invariants of [5] is

$$\sum_{d=1}^{\infty} \sum_{h} GW^{conn}_{d,h}(g|g-1, g-1) u^{2h-2} q^d = \log \left(1 + \sum_{d=1}^{\infty} \sum_{\rho\vdash d} \Big(\prod_{\square\in\rho} 2\sin \tfrac{h(\square)u}{2} \Big)^{2g-2} q^d \right). \tag{9.27}$$

On the other hand, by (2.16), the local RGW invariants (9.25) involve both the connected invariants $CRGW(g|g-1)$ and the doublet invariants $DRGW(g|g-1)$.

Remark 9.15. In the level 0 case, the proof of Lemma 5.1 implies that $RGW_d(g|0)$ is a *signed* count of degree d unramified real covers of a genus g Riemann surface (i.e. a real Hurwitz number), and (9.24) becomes:

$$RGW_d(g|0) = \sum_{\rho=\rho'} \left((-1)^{\frac{d-r(\rho)}{2}} t^d \frac{d!}{\dim \rho} \right)^{g-1}. \tag{9.28}$$

In contrast, the combinatorial count of real Hurwitz covers gives rise to a different KTQFT, cf. [1,24]; in this case, all covers count positively and the number of unramified real covers of a symmetric genus g surface (Σ, c) with *empty* real locus is equal to

$$H^{\mathbb{R}}_{(\Sigma,c)} = \sum_{\rho\vdash d} \left(\frac{d!}{\dim \rho} \right)^{g-1},$$

where the sum is over *all* partitions ρ of d. For this combinatorial KTQFT, the involution Ω is trivial and the coefficients U_ρ of the crosscap are equal to the *positive* square roots of the structure constants. However, unlike RGW, $H^{\mathbb{R}}_{(\Sigma,c)}$ depends on the real structure c.

10. The local real Gopakumar-Vafa formula

We are now ready to prove the real Gopakumar-Vafa formula (cf. [27, §5]) for the local RGW invariants defined in this paper. Let Σ be a connected genus g symmetric surface and $L \to \Sigma$ a holomorphic line bundle with $c_1(L) = g-1$. Consider the corresponding connected (complex) GW invariants $GW^{conn}_{d,h}(g|g-1, g-1)$ defined by [5], and the connected RGW invariants $CRGW_{d,h}(g|g-1)$ associated to (Σ, L), cf. (9.25)-(9.27).

The local GV conjecture in the classical setting, proved in [19, Proposition 3.4], states that the connected GW invariants of [5] have the following structure:

$$\sum_{d=1}^{\infty} \sum_{h} GW^{conn}_{d,h}(g|g-1, g-1) u^{2h-2} q^d = \sum_{d=1}^{\infty} \sum_{h} n^{\mathbb{C}}_{d,h}(g) \sum_{k=1}^{\infty} \tfrac{1}{k} (2\sin(\tfrac{ku}{2}))^{2h-2} q^{kd}, \tag{10.1}$$

where the coefficients $n^{\mathbb{C}}_{d,h}(g)$, called the local BPS states, satisfy (i) $n^{\mathbb{C}}_{d,h}(g) \in \mathbb{Z}$ and (ii) for each d, $n^{\mathbb{C}}_{d,h}(g) = 0$ for large h.

In the real setting, the local real GV formula takes the following form.

Theorem 10.1 *(Local real GV formula). Fix a genus g symmetric surface Σ and consider the local real Calabi-Yau 3-fold $(L \oplus c^*\overline{L}, c_{tw}) \to \Sigma$. Then the generating function for the connected local RGW invariants has the following structure:*

$$\sum_{d=1}^{\infty}\sum_{h=0}^{\infty} CRGW_{d,h}(g|g-1)u^{h-1}q^d = \sum_{d=1}^{\infty}\sum_{h=0}^{\infty} n^{\mathbb{R}}_{d,h}(g) \sum_{\substack{k \text{ odd} \\ k>0}} \tfrac{1}{k}(2\sinh(\tfrac{ku}{2}))^{h-1}q^{kd}, \quad (10.2)$$

where the coefficients $n^{\mathbb{R}}_{d,h}(g)$ *satisfy (i) (integrality)* $n^{\mathbb{R}}_{d,h}(g) \in \mathbb{Z}$*, (ii) (finiteness) for each* d*,* $n^{\mathbb{R}}_{d,h}(g) = 0$ *for large* h*, and (iii) (parity)* $n^{\mathbb{R}}_{d,h}(g) = n^{\mathbb{C}}_{d,h}(g) \mod 2$*. Moreover,*

(a) *for* $g = 0$*,* $n^{\mathbb{R}}_{d,h}(0) = 1$ *when* $d = 1$ *and* $h = 0$ *and vanish otherwise.*
(b) *for* $g = 1$*,* $n^{\mathbb{R}}_{d,h}(1) = (-1)^{d-1}$ *when* $h = 1$ *and vanish otherwise.*
(c) *for any* $g \geq 0$*,* $n^{\mathbb{R}}_{1,h}(g) = 1$ *when* $h = g$ *and vanish otherwise.*

Proof. The results for the genus $g \leq 1$ cases are obtained in (9.21) and (9.23). So it suffices to assume $g \geq 2$. For every integer $n \geq 0$, let

$$H_n(u,q) = \sum_{k \text{ odd}} \tfrac{1}{k}(2\sinh(\tfrac{ku}{2}))^n q^k = u^n q(1 + \dots). \quad (10.3)$$

Then $\{q^{-1}H_n(u,q^d)\}_{n\geq 0, d\geq 1}$ is a basis of the power series in u and q. In particular, for $g \geq 2$, there exists an expansion of the connected invariant in the form (10.2), for some coefficients $n^{\mathbb{R}}_{d,h}(g) \in \mathbb{Q}$, with $h \geq g$ (because there are no covers of a genus g curve by a lower genus curve).

Denote for simplicity by $Z_g = Z_g(u,q)$ the generating function of the RGW invariants (9.25) and by $C_g = C_g(u,q)$ and $D_g = D_g(u,q)$ the generating functions of the connected and the doublet invariants, respectively; then $Z_g = \exp(C_g + D_g)$ cf. (2.16).

Corollary 3.9, relating the doublet and the connected GW invariants of [5], and (10.1) imply

$$\begin{aligned} D_g &= \tfrac{1}{2}\sum_d GW^{conn}_d(iu)q^{2d} = \\ &= \tfrac{1}{2}\sum_{d=1}^{\infty}\sum_{h>0} n^{\mathbb{C}}_{d,h}(g)(-1)^{h-1}\sum_{k=1}^{\infty}\tfrac{1}{k}(2\sinh(\tfrac{ku}{2}))^{2h-2}q^{2kd}, \end{aligned}$$

where $n^{\mathbb{C}}_{d,h}(g)$ are integers and have the finiteness property. But $Z_g = \exp(C_g + D_g)$, so combined with (10.2) this gives:

$$Z_g = \exp\Big(\sum_{d,h>0} n^{\mathbb{R}}_{d,h}(g)\sum_{k \text{ odd}} \tfrac{1}{k} f(Q^k)^{h-1}q^{kd} + \tfrac{1}{2}\sum_{d,h>0} n^{\mathbb{C}}_{d,h}(g)\sum_{k>0}\tfrac{1}{k}F(Q^{2k})^{h-1}q^{2kd}\Big), \quad (10.4)$$

where

$$f(Q) = Q - Q^{-1}, \qquad F(Q) = 2 - Q - Q^{-1}, \qquad \text{and} \quad Q = e^{u/2}.$$

Note that for all $s \geq 0$,

$$f(Q)^s = \sum_{l=-s}^{s} \phi_l^s Q^l \quad \phi_l^s \in \mathbb{Z}, \qquad F(Q)^s = \sum_{l=-s}^{s} \psi_l^s Q^l \quad \psi_l^s \in \mathbb{Z} \tag{10.5}$$

are Laurent polynomials in Q with integer coefficients and with leading coefficients $\phi_{\pm s}^s$ and $\psi_{\pm s}^s$ equal to ± 1. Moreover,

$$f(Q) = F(Q) \mod 2 \quad \text{ and therefore } \phi_l^s = \psi_l^s \mod 2 \tag{10.6}$$

for all $-s \le l \le s$ and $s \ge 0$.

On the other hand, with this notation, (9.25) becomes

$$Z_g = 1 + \sum_{d=1}^{\infty} \sum_{\substack{\rho \vdash d \\ \rho=\rho'}} \Big(\epsilon_\rho \prod_{\square \in \rho} f(Q^{h(\square)})\Big)^{g-1} q^d \tag{10.7}$$

and therefore the coefficient of q^d is also a Laurent polynomial in Q with integer coefficients.

Comparing the coefficient of q^1 in (10.4) and (10.7) gives

$$[Z_g]_{q^1} = \sum_{h>0} n_{1,h}^{\mathbb{R}}(g) f(Q)^{h-1} = f(Q)^{g-1}.$$

As before, $\{f(Q)^n\}_{n\ge 0}$ are linearly independent, therefore $n_{1,h}^{\mathbb{R}}(g) = 1$ for $h = g$ and vanish otherwise, proving (c). In particular, $n_{1,h}^{\mathbb{R}}(g) \in \mathbb{Z}$. Recall also that $n_{1,h}^{\mathbb{C}}(g) = 1$ for $h = g$ and vanish otherwise, and therefore $n_{1,h}^{\mathbb{R}}(g) = n_{1,h}^{\mathbb{C}}(g)$ for all h.

We next proceed by induction on the degree, with initial step for $d = 1$ just proved. So we fix a degree $p \ge 2$ and assume by induction that

$$n_{d,h}^{\mathbb{R}}(g) \in \mathbb{Z} \quad \text{ and } \quad n_{d,h}^{\mathbb{R}} \equiv n_{d,h}^{\mathbb{C}}(g) \mod 2, \tag{10.8}$$

for all $d < p$. We also assume that for all $d < p$, $n_{d,h}^{\mathbb{R}}(g) = 0$ for h large.

Note that the coefficient of q^p in (10.4) has the form

$$\sum_{h>0} n_{p,h}^{\mathbb{R}}(g) f(Q)^{h-1} + \left[\exp\left(\sum_{\substack{d,h>0 \\ d\neq p}} n_{d,h}^{\mathbb{R}}(g) \sum_{\substack{k \text{ odd} \\ k>0}} \tfrac{1}{k} f(Q^k)^{h-1} q^{kd} + \tfrac{1}{2} \sum_{d,h>0} n_{d,h}^{\mathbb{C}}(g) \sum_{k=1}^{\infty} \tfrac{1}{k} F(Q^{2k})^{h-1} q^{2kd}\right)\right]_{q^p}$$

where the second term involves only $n_{d,h}^{\mathbb{R}}(g)$ with $d < p$.

Next, using the expansions (10.5), note that

$$\sum_{k \text{ odd}} \tfrac{1}{k} f(Q^k)^{h-1} q^{kd} = \sum_{k \text{ odd}} \tfrac{1}{k} \sum_{l=1-h}^{h-1} \phi_l^{h-1} Q^{kl} q^{kd} = \sum_{l=1-h}^{h-1} \phi_l^{h-1} \log\left(\tfrac{1+Q^l q^d}{1-Q^l q^d}\right)^{1/2}$$

and

$$\sum_{k=1}^{\infty} \tfrac{1}{k} F(Q^{2k})^{h-1} q^{2kd} = \sum_{k=1}^{\infty} \tfrac{1}{k} \sum_{l=1-h}^{h-1} \psi_l^{h-1} Q^{2kl} q^{2kd} = \sum_{l=1-h}^{h-1} \psi_l^{h-1} \log \tfrac{1}{1-Q^{2l}q^{2d}}.$$

Combining the last three displayed equations gives

$$[Z_g]_{q^p} = \sum_{h>0} n^{\mathbb{R}}_{p,h}(g) f(Q)^{h-1} + \left[\prod_{d\neq p} \prod_{h>0} \prod_{l=1-h}^{h-1} \frac{(1+Q^l q^d)^{\frac{1}{2}(n^{\mathbb{R}}_{d,h}(g)\phi_l^{h-1} - n^{\mathbb{C}}_{d,h}(g)\psi_l^{h-1})}}{(1-Q^l q^d)^{\frac{1}{2}(n^{\mathbb{R}}_{d,h}(g)\phi_l^{h-1} + n^{\mathbb{C}}_{d,h}(g)\psi_l^{h-1})}} \right]_{q^p}, \tag{10.9}$$

where the second summand is a Laurent polynomial in Q with integer coefficients by the induction hypothesis (10.8) and the fact that $\psi_l^s = \phi_l^s \mod 2$, cf. (10.6). Since $[Z_g]_{q^p}$ is a Laurent polynomial in Q with integer coefficients by (10.7), therefore so is

$$\sum_{h>0} n^{\mathbb{R}}_{p,h}(g) f(Q)^{h-1}.$$

Since the coefficients of its expansion and $n^{\mathbb{R}}_{p,h}(g)$ are related by an integral triangular transformation with 1's along the diagonal this implies $n^{\mathbb{R}}_{p,h}(g) \in \mathbb{Z}$ for all $h > 0$. This also shows the finiteness property of $n^{\mathbb{R}}_{p,h}(g)$ i.e. that for fixed g, p and large enough h these numbers vanish.

It remains to show that $n^{\mathbb{R}}_{p,h}(g) \equiv n^{\mathbb{C}}_{p,h}(g) \mod 2$ for all h. Similar considerations for the complex GW invariants

$$\exp(GW^{conn}(iu)) = GW(iu)$$

using (10.1) and (9.26) imply

$$\sum_{h>0} n^{\mathbb{C}}_{p,h}(g) F(Q)^{h-1} + \left[\prod_{d\neq p} \prod_{h>0} \prod_{l=1-h}^{h-1} \frac{1}{(1-Q^l q^d)^{n^{\mathbb{C}}_{d,h}(g)\psi_l^{h-1}}} \right]_{q^p} = \sum_{\rho} \left(\prod_{\square\in\rho} F(Q^{h(\square)}) \right)^{g-1} q^{|\rho|}.$$

Using again (10.6), we see that, mod 2, the Laurent series with integer coefficients

$$1 + \sum_{\rho=\rho'} \left(\epsilon(\rho) \prod_{\square\in\rho} f(Q^{h(\square)}) \right)^{g-1} q^{|\rho|} \equiv 1 + \sum_{\rho} \left(\prod_{\square\in\rho} F(Q^{h(\square)}) \right)^{g-1} q^{|\rho|} \mod 2$$

are equal, keeping in mind that the terms corresponding to ρ and ρ' on the RHS are equal, thus their contribution vanishes mod 2 unless ρ is self-conjugate.

The second inductive hypothesis (10.8) implies that, mod 2, the second summand in (10.9) equals

$$\left[\prod_{d \neq p} \prod_{h>0} \prod_{l=1-h}^{h-1} \frac{1}{(1-Q^l q^d)^{n^{\mathbb{C}}_{d,h}(g)\psi_l^{h-1}}} \right]_{q^p} \quad \mod 2.$$

Together these imply that

$$\sum_{h>0} n^{\mathbb{R}}_{p,h}(g) f(Q)^{h-1} \equiv \sum_{h>0} n^{\mathbb{C}}_{p,h}(g) F(Q)^{h-1} \mod 2,$$

which in turn implies $n^{\mathbb{R}}_{p,h}(g) \equiv n^{\mathbb{C}}_{p,h}(g) \mod 2$, completing the proof of the induction step. □

Corollary 10.2. *The coefficients* $n^{\mathbb{R}}_{d,h}(g)$ *vanish unless* $d(g-1)+h-1 \equiv 0 \mod 2$. *In particular,* $n^{\mathbb{C}}_{d,h}(g)$ *are even whenever* $d(g-1)+h-1 \not\equiv 0 \mod 2$.

Proof. By Corollary 6.6, the connected real invariants $CRGW_{d,h}(g|g-1)$ vanish unless $d(g-1)+h-1 \equiv 0 \mod 2$. Therefore the left hand side of (10.2) is invariant under the change of variables $(u,q) \to (-u, (-1)^{g-1}q)$. Making this change of variables on the right hand side of (10.2) and using the fact that the functions $\{H_n(u,q^d)\}$ are linearly independent (cf. (10.3)), implies that $n_{d,h}(g) = (-1)^{h-1+d(g-1)} n_{d,h}(g)$. The result follows. □

11. Signed Frobenius-Schur indicator

In this final section, which is of independent interest, we introduce the notion of a signed Frobenius-Schur indicator and show that it takes values $0, \pm 1$ on any irreducible real representation of a finite group (unlike the classical FS indicator, which is +1). This was used in §9 to determine the signs ε_ρ in the expression of the RGW-invariants.

The classical Frobenius-Schur indicator of a representation of a finite group is the character evaluated at the sum of squares of the group elements divided by the order of the group; its possible values for an irreducible representation are 1, 0, and -1, corresponding to the partition of the irreducible representations into real, complex, and quaternionic representations.

Below we construct a signed FS indicator for the symmetric group S_d, but these considerations remain valid for any real representation of a finite group G with a sign homomorphism $G \to \mathbb{Z}_2$.

Definition 11.1. The signed Frobenius-Schur indicator is defined by

$$SFS(\rho) \stackrel{\text{def}}{=} \frac{1}{d!} \sum_{g \in S_d} \chi_\rho(g^2)(-1)^{s(g)}, \tag{11.1}$$

where $s(g)$ is the parity of the permutation $g \in S_d$.

The sign morphism on S_d descends to conjugacy classes, which are indexed by partitions of d, decomposing them into even and odd ones. If α is a conjugacy class, let $sq(\alpha)$ denote the conjugacy class of g^2, where g is a representative of α, cf. (9.7).

Lemma 11.2. *On irreducible representations, the signed Frobenius-Schur indicator takes values* $0, \pm 1$. *Specifically,*

$$SFS(\rho) = \begin{cases} 0, & \text{if } \rho \text{ is not self-conjugate,} \\ \pm 1, & \text{if } \rho \text{ is self-conjugate.} \end{cases} \tag{11.2}$$

Furthermore, when ρ is self-conjugate, $SFS(\rho)$ is given by

$$\varepsilon_\rho = (-1)^{o(\rho)}, \quad \text{with} \quad o(\rho) = \frac{1}{d!} \sum_{\substack{g \in S_d \\ g \text{ odd}}} \chi_\rho(g^2) = \sum_{\substack{\alpha \vdash d \\ \alpha \text{ odd}}} \frac{\chi_\rho(sq(\alpha))}{\zeta(\alpha)}. \tag{11.3}$$

The expression $o(\rho)$ takes values 0 *or* 1 *on a self-conjugate representation ρ.*

Proof. The proof is similar to that for the standard Frobenius-Schur indicator; see for example [8, §3.2.3] or [9, §5.1]. The space $B = B(V)$ of bilinear forms on a vector space V can be identified with $\mathrm{Hom}(V, V^*)$ and $V^* \otimes V^*$ and the latter splits as a direct sum of symmetric and alternating forms

$$B \cong \mathrm{Hom}(V, V^*) \cong V^* \otimes V^* = Sym^2 \oplus Alt^2. \tag{11.4}$$

Let $\rho : S_d \to \mathrm{End}(V)$ be an irreducible representation. Recall that all representations of S_d are real, so in particular $\rho^* \cong \rho$, where $\rho^* : S_d \to \mathrm{End}(V^*)$ is the dual representation. Let $\rho' : S_d \to \mathrm{End}(V)$ be the conjugate representation, given by

$$\rho'(g) = (-1)^{s(g)} \rho(g),$$

where $s(g)$ is the parity of g, cf. (9.8); the Young diagram of ρ' is obtained from that of ρ by reflecting across the main diagonal.

On B, consider the following representation $T^{2'} = T^{2'}(\rho)$ given by

$$T^{2'} : S_d \longrightarrow \mathrm{End}(B), \quad T^{2'}(g) = (\rho' \otimes \rho)^*(g) = (-1)^{s(g)} \rho^*(g) \otimes \rho^*(g),$$

and denote by $Sym^{2'}$ and $Alt^{2'}$ the corresponding representations on symmetric and alternating forms. Note that with respect to the S_d action on B induced by $(\rho' \otimes \rho)^*$, its fixed loci can be identified with

$$B^{S_d} = (Sym^2)^{S_d} \oplus (Alt^2)^{S_d} \quad \text{and} \quad B^{S_d} \cong \mathrm{Hom}_{S_d}(V, V^*), \tag{11.5}$$

where $\mathrm{Hom}_{S_d}(V, V^*)$ denotes the space of (ρ', ρ^*)-equivariant morphisms. But ρ is irreducible, therefore the second part of (11.5) implies

$$\dim B^{S_d} \leq 1 \tag{11.6}$$

with equality iff $\rho' \cong \rho^*$ i.e. ρ is self-conjugate (since $\rho^* \cong \rho$). So in (11.5) the only possible pairs of dimensions

$$(\dim(Sym^2)^{S_d},\ \dim(Alt^2)^{S_d}) \tag{11.7}$$

are $(0,0)$, $(1,0)$, and $(0,1)$, with the latter two cases appearing only for self-conjugate representations.

Next, let

$$\pi = \frac{1}{d!}\sum_{g\in S_d} g \quad \text{thus} \quad h.\pi = \pi \quad \text{for all } h \in S_d.$$

Then for every $b \in B$, $T^{2'}(\pi)(b) \in B^{S_d}$ and for every $b \in B^{S_d}$, $T^{2'}(b) = b$. Taking trace we obtain

$$\frac{1}{d!}\sum_{g\in S_d} \chi_{T^{2'}}(g) = \dim B^{S_d} = \delta_{\rho\rho'}$$

cf. (11.6); note that $\chi_{T^{2'}}(g) = (-1)^{s(g)}\chi^2_\rho(g)$. Similarly, for the decomposition (11.5), we get

$$\frac{1}{d!}\sum_{g\in S_d} \chi_{Sym^{2'}}(g) = \dim(Sym^2)^{S_d} \quad \text{and} \quad \frac{1}{d!}\sum_{g\in S_d} \chi_{Alt^{2'}}(g) = \dim(Alt^2)^{S_d}. \tag{11.8}$$

On the other hand, for the standard $Sym^2\rho$ and $Alt^2\rho$ representations,

$$\chi_\rho(g^2) = \chi_{Sym^2\rho}(g) - \chi_{Alt^2\rho}(g) \tag{11.9}$$

cf. the proof of [9, Lemma 5.1.5] or [8, Lemma 3.2.17]. Moreover,

$$\chi_{Sym^{2'}}(g) = (-1)^{s(g)}\chi_{Sym^2\rho}(g) \quad \text{and} \quad \chi_{Alt^{2'}}(g) = (-1)^{s(g)}\chi_{Alt^2\rho}(g). \tag{11.10}$$

Therefore

$$\begin{aligned} SFS(\rho) &\underset{(11.1)}{=} \frac{1}{d!}\sum_{g\in S_d} \chi_\rho(g^2)(-1)^{s(g)} \underset{(11.9)}{=} \frac{1}{d!}\sum_{g\in S_d} (-1)^{s(g)}\chi_{Sym^2\rho}(g) - (-1)^{s(g)}\chi_{Alt^2\rho}(g) \\ &\underset{(11.10)}{=} \frac{1}{d!}\sum_{g\in S_d} \chi_{Sym^{2'}}(g) - \chi_{Alt^{2'}}(g) \underset{(11.8)}{=} \dim(Sym^2)^{S_d} - \dim(Alt^2)^{S_d}. \end{aligned}$$

Combined with (11.7), this gives (11.2).

It remains to determine when is the SFS indicator +1 and when -1. The standard FS indicator for S_d is always +1 therefore

$$1 = \frac{1}{d!}\sum_{g\in S_d}\chi_{Sym^2\rho}(g) = \frac{1}{2d!}\sum_{g\in S_d}\chi_\rho^2(g)+\chi_\rho(g^2) \tag{11.11}$$

$$0 = \frac{1}{d!}\sum_{g\in S_d}\chi_{Alt^2\rho}(g) = \frac{1}{2d!}\sum_{g\in S_d}\chi_\rho^2(g)-\chi_\rho(g^2). \tag{11.12}$$

Fix a self-conjugate representation ρ and let

$$e = \frac{1}{d!}\sum_{\substack{g\in S_d\\ g \text{ even}}}\chi_\rho(g^2) \qquad \text{and} \qquad o = \frac{1}{d!}\sum_{\substack{g\in S_d\\ g \text{ odd}}}\chi_\rho(g^2).$$

Subtracting the two equalities (11.12) and (11.11) gives $e+o=1$. Since $SFS(\rho) = e-o=\pm1$ for the self conjugate representation ρ, then $2o = 1-SFS(\rho)$ is either 0 or 2; in either case $SFS(\rho) = (-1)^o$ completing the proof. □

The following lemma played a key role in §9.

Lemma 11.3. *Let α be a partition of d. With the notations above*

$$\sum_{\substack{\beta\vdash d\\ sq(\beta)=\alpha}}(-1)^{s(\beta)}\frac{\zeta(\alpha)}{\zeta(\beta)} = \sum_\rho SFS(\rho)\chi_\rho(\alpha) = \sum_{\rho=\rho'}\varepsilon_\rho\chi_\rho(\alpha) \tag{11.13}$$

where ε_ρ is given by (11.3), and $s(\beta)$ denotes the parity of β.

Proof. The partition α corresponds to conjugacy class of S_d and let $a\in S_d$ denote a representative of it. Then

$$\sum_{\substack{\beta\vdash d\\ sq(\beta)=\alpha}}\frac{(-1)^{s(\beta)}\zeta(\alpha)}{\zeta(\beta)} = \sum_{\substack{g\in S_d\\ g^2=a}}(-1)^{s(g)}.$$

Consider the class function

$$\theta(a) = \sum_{\substack{g\in S_d\\ g^2=a}}(-1)^{s(g)}.$$

It has an expansion in the basis of irreducible characters $\{\chi_\rho\}$

$$\theta = \sum_\rho\langle\theta,\chi_\rho\rangle\chi_\rho,$$

where $\langle\,,\rangle$ is the inner product on the class functions. We have

$$\langle\theta,\chi_\rho\rangle = \frac{1}{d!}\sum_{h\in S_d}\theta(h)\chi_\rho(h) = \frac{1}{d!}\sum_{h\in S_d}\sum_{\substack{g\in S_d\\ g^2=h}}(-1)^{s(g)}\chi_\rho(h) = \frac{1}{d!}\sum_{g\in S_d}(-1)^{s(g)}\chi_\rho(g^2).$$

Therefore

$$\begin{aligned}\sum_{\substack{g\in S_d\\ g^2=a}}(-1)^{s(g)} = \theta(a) &= \sum_\rho \frac{1}{d!}\sum_{g\in S_d}(-1)^{s(g)}\chi_\rho(g^2)\chi_\rho(a)\\ &= \sum_\rho \chi_\rho(\alpha)\frac{1}{d!}\sum_{g\in S_d}(-1)^{s(g)}\chi_\rho(g^2). \quad\square\end{aligned}$$

Appendix A. Real orientation and twisted orientation data

In this appendix we review the key ideas of [15] that provided a criterion for orienting the real moduli space. We then describe a small variant that covers additional cases.

Let (X,ω,ϕ) be a Real symplectic manifold and consider the real moduli space $\overline{\mathcal{M}}^{\phi}_{B,g,\ell}(X)$ of pseudo-holomorphic real maps $f:(C,\sigma)\to(X,\phi)$ from a genus g surface with ℓ pairs of conjugate marked points and representing the class $B\in H_2(X)$.

We will use the setup of [15, §4.3]. For every real map $f:(C,\sigma)\to(X,\phi)$ and Real vector bundle $(V,\Phi)\longrightarrow(X,\phi)$ with a Φ-compatible connection ∇, let

$$D_{(V,\Phi),f}:\Gamma(f^*V)^\Phi\to\Lambda^{0,1}(f^*V)^\Phi$$

be the restriction of the real Cauchy-Riemann operator $D_{V,f}=\bar\partial^{f^*\nabla}$ to the space of invariant sections. Denote by $\det D_{(V,\Phi)}$ the determinant line bundle of the family of operators over the real moduli space; see [15, §4.3] for more details.

Unlike the classical line bundle $\det D_V$, which is always canonically oriented (cf. proof of [23, Theorem 3.1.5(i)]), the line bundle $\det D_{(V,\Phi)}$ is not always orientable.

The considerations of [15] imply that, after stabilization of the domain if necessary, the orientation sheaf of the real moduli space is canonically identified with

$$\det T\overline{\mathcal{M}}^{\phi}_{B,g,\ell}(X) = \det D_{(TX,d\phi)}\otimes\mathfrak{f}^*\det T\overline{\mathbb{R}\mathcal{M}}_{g,\ell} \tag{A.1}$$

cf. [15, (3.3)]. Here $\mathfrak{f}$ is the map to the real Deligne-Mumford moduli space $\overline{\mathbb{R}\mathcal{M}}_{g,\ell}$ parametrizing genus g real curves with ℓ pairs of conjugate marked points.

A notion of real orientation on a Real bundle $(V,\Phi)\to(X,\phi)$ was introduced in [15]. For the tangent bundle $(TX,d\phi)$, a real orientation consists of a Real line bundle $(L,\widetilde{\phi})\to(X,\phi)$ such that

$$\psi:(L,\widetilde{\phi})^{\otimes 2}\cong\Lambda^{\mathrm{top}}(TX,d\phi), \tag{A.2}$$

along with a choice of a homotopy class of such isomorphism and a spin structure on $TX^{\phi} \oplus 2(L^*)^{\widetilde{\phi}^*}$; see [15, Definition 1.2]. When the complex dimension of X is odd, such structure induces an orientation on all real moduli spaces $\overline{\mathcal{M}}^{\phi}_{B,g,l}(X)$, cf. [15, Theorem 1.3]. The main ingredients in the proof are as follows.

One of the key results of [15] is Proposition 5.2 which states that a real orientation on a rank n Real bundle $(V, \Phi) \to (\Sigma, \sigma)$ determines a canonical class of isomorphisms

$$\Psi : (V \oplus 2L^*, \Phi \oplus 2\widetilde{\phi}^*) \cong (\Sigma \times \mathbb{C}^{n+2}, \sigma \times c_{std}). \tag{A.3}$$

Here $(L^*, \widetilde{\phi}^*)$ denotes the dual of the Real bundle $(L, \widetilde{\phi})$. In turn, (A.3) induces a canonical isomorphism

$$\det D_{(V,\Phi)} \cong (\det \bar{\partial}_{(\mathbb{C},c_{std})})^{\otimes n}, \tag{A.4}$$

using the fact that

$$\det D_{(2L^*,2\widetilde{\phi}^*)} = (\det(D_{(L^*,\widetilde{\phi}^*)})^{\otimes 2}$$

is canonically oriented as twice a bundle. In [15, §5 and §6] family versions of this result are proved for families of (possibly nodal) surfaces. In particular, if $f : (C, \sigma) \to (X, \phi)$ is a point in the real moduli space, by [15, Proposition 5.2] the real orientation on the target determines by pullback a homotopy class of isomorphisms

$$f^*(TX \oplus 2L^*, d\phi \oplus 2\widetilde{\phi}^*) \cong (\Sigma \times \mathbb{C}^{n+2}, \sigma \times c_{std}) \tag{A.5}$$

which varies continuously with f. By the proof of [15, Theorem 1.3] this induces canonical isomorphisms

$$\det D_{(TX,d\phi)} \cong \det D_{(TX,d\phi)} \otimes (\det D_{(L^*,\widetilde{\phi}^*)})^{\otimes 2} \cong (\det \bar{\partial}_{(\mathbb{C},c_{std})})^{\otimes(n+2)}, \tag{A.6}$$

over the real moduli spaces $\overline{\mathcal{M}}^{\phi}_{B,g,l}(X)$. Here the first isomorphism is induced by the canonical orientation on twice a bundle, while the second one is induced by the isomorphism (A.5).

Moreover by [15, Theorem 1.3], there is also a canonical isomorphism

$$\det(T\overline{\mathbb{R}\mathcal{M}}_{h,\ell}) = \det \bar{\partial}_{(\mathbb{C},c_{std})}, \tag{A.7}$$

where the forgetful morphism of a pair of marked points is oriented via the first element in the pair. Therefore (A.1), (A.6) and (A.7) combine to give a canonical isomorphism

$$\det \overline{\mathcal{M}}^{\phi}_{d,g,l}(X) \cong (\det \bar{\partial}_{(\mathbb{C},c_{std})})^{\otimes(n+1)}, \tag{A.8}$$

cf. [15, Theorem 1.3], and therefore an orientation on all the real moduli spaces $\overline{\mathcal{M}}^{\phi}_{B,g,\ell}(X)$ when the complex dimension n of X is odd.

The canonical isomorphism (A.3) constructed in [15, Proposition 5.2] only requires

(a) a homotopy class of isomorphisms $\Lambda^{\mathrm{top}}(V \oplus 2L^*, \phi \oplus 2\widetilde{\phi}^*) \cong (\Sigma \times \mathbb{C}, \sigma \times c_{std})$ and
(b) a spin structure on $V^\phi \oplus 2(L^*)^{\widetilde{\phi}^*}$

and it does not depend on whether or not $E = 2L^*$ is twice of a bundle, cf. [15, Corollary 5.5]. The fact that E is twice of a bundle is only used in the proof of [15, Theorem 1.3] to argue that $\det \bar{\partial}_{(2L^*, 2\widetilde{\phi}^*)}$ is canonically oriented; cf. (A.6). Thus $(2L^*, 2\widetilde{\phi}^*)$ can be replaced by any real bundle pair $(E, \tilde{\phi})$ for which we know that the determinant line is canonically oriented.

Such a choice $(E, \tilde{\phi})$ can also be obtained as follows. Let $L \to X$ be a complex line bundle and let $E = L \oplus \phi^* \overline{L}$ with the real structure $\tilde{\phi} = \phi_{tw}$ defined as in (2.5). Then the projection onto the first factor induces a canonical isomorphism

$$\det D_{(E,\tilde{\phi})} = \det D_{(L \oplus \phi^* \overline{L}, \phi_{tw})} \overset{\pi_1}{\cong} \det D_L \tag{A.9}$$

over the real moduli space, as in (2.9). The right hand side is the determinant line of a real Cauchy-Riemann operator and is thus canonically oriented (cf. proof of [23, Theorem 3.1.5(i)]). Therefore it induces a canonical orientation on the left-hand side.

This motivates the following variant of [15, Definition 1.2].

Definition A.1. Assume (X, ϕ) is a Real symplectic manifold. A twisted orientation $\mathfrak{o} = (L, \psi, \mathfrak{s})$ for it consists of

(i) a complex line bundle $L \to X$ such that the bundle pair $(E, \tilde{\phi}) = (L \oplus \phi^* \overline{L}, \phi_{tw})$ satisfies:

$$w_2(TX^\phi) = w_2(E^{\tilde{\phi}}) \quad \text{and} \quad \Lambda^{\mathrm{top}}(TX^\phi, d\phi) \cong \Lambda^{\mathrm{top}}(E, \tilde{\phi}) \tag{A.10}$$

(ii) a homotopy class $[\psi]$ of isomorphisms satisfying (A.10).
(iii) a spin structure $\mathfrak{s}$ on the real vector bundle $TX^\phi \oplus (E^*)^{\tilde{\phi}^*}$ over the real locus, compatible with the orientation induced by ψ.

Then [15, Theorem 1.3] extends to give:

Lemma A.2. *A twisted orientation on (X, ϕ) induces a canonical orientation on the real moduli spaces $\overline{\mathcal{M}}^\phi_{B,g,\ell}(X)$ when the target X is odd complex dimensional.*

Proof. As in [15, Proposition 5.2], a twisted orientation determines by pullback a canonical homotopy class of isomorphisms

$$f^*(TX \oplus E^*, d\phi \oplus \widetilde{\phi}^*) \cong (\Sigma \times \mathbb{C}, \sigma \times c_{std})^{\oplus(n+2)} \tag{A.11}$$

varying continuously with $f \in \overline{\mathcal{M}}^{\phi}_{B,g,\ell}(X)$. The rest of the proof is the same as that of [15, Theorem 1.3] except now (A.6) is replaced by

$$\det D_{(TX,d\phi)} \cong \det D_{(TX,d\phi)} \otimes \det D_{(E^*,\tilde{\phi}^*)} \cong (\det \bar{\partial}_{(\mathbb{C},c_{std})})^{\otimes(n+2)} \tag{A.12}$$

over the real moduli space $\overline{\mathcal{M}}^{\phi}_{B,g,\ell}(X)$. Here the first isomorphism is induced by (A.9) and the complex orientation on $\det \bar{\partial}_{L^*}$ and the second isomorphism is induced by (A.11). Combined with (A.7) and (A.1) this determines a canonical homotopy class of isomorphisms (A.8) as in [15, Theorem 1.3]. □

These considerations similarly extend to the relative real moduli spaces considered in this paper, with the following modification. Assume the target (Σ, c) is a symmetric Riemann surface with r pairs of conjugate points, and consider the relative real moduli space

$$\overline{\mathcal{M}} = \overline{\mathcal{M}}^{\bullet,c}_{d,\chi}(\Sigma)_{\lambda^1,\dots,\lambda^r}$$

of Definition 2.5. The deformation-obstruction theory (with fixed domain) at a point $f \in \overline{\mathcal{M}}$ is determined by the linearization $\overline{\partial}_{f^*(T\Sigma,dc)}$ where $T\Sigma$ is the relative tangent bundle (2.22) to the *marked* curve $\Sigma = (S, j, \{x_i^{\pm}\}_{i=1,\dots,r})$. This is analogous with the situation for the complex moduli space and can be seen as follows. A point f in the moduli space is a real map $f : (C, \sigma) \to (\Sigma, c)$ which is ramified of order λ^i_j at the points $y^{\pm}_{ij}$, where $f^{-1}(x^{\pm}_i) = \{y^{\pm}_{ij}\}_{j=1,\dots,\ell(\lambda^i)}$, and $i = 1, \dots r$. Variations in f with fixed domain must vanish to order λ^i_j at $y^{\pm}_{ij}$, i.e. correspond to sections of

$$(f^*TS) \otimes \mathcal{O}\Big(-\sum_{i,j} \lambda^i_j y^+_{ij} - \sum_{i,j} \lambda^i_j y^-_{ij}\Big) = f^*\Big(TS \otimes \mathcal{O}\Big(-\sum_i x^+_i - \sum_i x^-_i\Big)\Big) = f^*T\Sigma$$

which are invariant under the involutions on the domain and target. Therefore the orientation sheaf of the relative real moduli space is canonically identified with

$$\det T\overline{\mathcal{M}}^{c,\bullet}_{d,\chi}(\Sigma)_{\vec{\lambda}} = \det \overline{\partial}_{(T\Sigma,dc)} \otimes \mathfrak{f}^* \det T\overline{\mathbb{R}\mathcal{M}}^{\bullet}_{\chi,\ell}, \tag{A.13}$$

where $\ell = \sum_i \ell(\lambda_i)$ is the total number of pairs of marked points on the domain, cf. (A.1). A twisted orientation on the *marked curve* (Σ, c) determines an orientation on these moduli spaces via the same procedure as in the absolute case. (Note that Definition A.1 and Definition 2.1 are equivalent, with $L^* = \Theta$.)

When (Σ, c) is a connected Real curve, a real orientation in the sense of [15] exists on it except in the case when c is fixed-point free and the genus of Σ is even. A twisted orientation exists on any Real curve, cf. [14, §7.1].

We end this appendix with the following observation. Assume (X, ϕ) is a Real symplectic manifold. For any Real line bundle $(L, \widetilde{\phi}) \to (X, \phi)$ there is an isomorphism

$$\theta : (L \oplus \phi^*\overline{L}, \phi_{tw}) \to (L \oplus L, \widetilde{\phi} \oplus \widetilde{\phi}), \tag{A.14}$$

as in (5.9); it induces an isomorphism

$$\theta^{\mathbb{R}} : L_{|X^\phi} \cong 2L^{\widetilde{\phi}}$$

along the fixed locus X^ϕ. In particular, if a real orientation $(L, \widetilde{\phi})$ exists, we can use either the real orientation or the twisted orientation $(L \oplus \phi^*\overline{L}, \phi_{tw})$ to induce an orientation on the moduli space. The difference between the two orientation procedures is determined by Lemma 5.3 as follows.

Lemma A.3. *Let $\mathfrak{o} = ((L, \widetilde{\phi}), \psi, \mathfrak{s})$ be a real orientation for (X, ϕ). Let $\mathfrak{o}' = (L \oplus c^*\overline{L}, \psi', \mathfrak{s}')$ denote the associated twisted orientation obtained from $\mathfrak{o}$ via the isomorphism (A.14), where $\mathfrak{s}' = (id \oplus \theta^{\mathbb{R}})^*\mathfrak{s}$ and $\psi' = \psi \circ \theta$. The difference between the orientation on the real moduli spaces $\overline{\mathcal{M}}^\phi(X)$ induced by $\mathfrak{o}$ and that induced by $\mathfrak{o}'$ is $(-1)^\iota$, where ι is the complex rank of* $\operatorname{Ind} \bar{\partial}_{L^*}$.

Proof. As explained at the beginning of the appendix, the two orienting procedures differ only in the way the auxiliary index bundle is oriented i.e. it is the difference in how the index bundles of $L^* \oplus \phi^*\overline{L}^*$ and $2L^*$ are oriented: in the first case via the projection onto the first factor and in the second case as twice a bundle. This difference is given by Lemma 5.3. □

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