Speed limits on deterministic chaos and dissipation

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Uncertainty in the initial conditions of dynamical systems can cause exponentially fast divergence of trajectories, a signature of deterministic chaos, or be suppressed by the dissipation of energy. Here, we derive a classical uncertainty relation that sets a speed limit on the rates of local observables underlying these behaviors. For systems with a time-invariant stability matrix, the speed limit we derive simplifies to a classical analogue of the Mandelstam-Tamm versions of the time-energy uncertainty relation. These classical bounds are set by fluctuations in the local stability of state space. To measure these fluctuations, we introduce a definition of the Fisher information in terms of Lyapunov vectors in tangent space, analogous to the quantum Fisher information defined in terms of wavevectors in Hilbert space. This information sets an upper bound on the speed at which classical, dynamical systems and their observables, instantaneous Lyapunov exponents and dissipation, evolve. This speed limit applies to systems that are open or closed, conservative or dissipative, actively driven or passively evolving, and directly connects the geometries of phase space and information.

Introduction. Quantum speed limits are fundamental constraints on the time evolution of quantum mechanical systems and their observables [1]. A milestone in their development is the Mandelstam-Tamm version of the time-energy uncertainty relation, which sets a speed limit on the observables of unitary quantum dynamics [2]. This and other bounds have been extended [3] to open quantum systems [4–7] and applied to many-body dynamics [8, 9]. They have also been connected to parameter estimation [10–13] and information theory [14–16] where they quantify the inherent limits on measurements of dynamical quantities [17, 18]. It was recently discovered that there are similar bounds on the evolution of classical systems, the earliest of which largely rely on the Hilbert space of the Liouville equation [19, 20], prompting a closer inspection of the classical nature of quantum speed limits [21]. For purely classical stochastic dynamics, there is now a growing number of thermodynamic speed limits [22–26] on the flux of energy and entropy between a system and external reservoirs. Included among them is a stochastic thermodynamic speed limit [26] that, when combined with the Mandelstam-Tamm bound, gives a more general speed limit on the observables of open quantum systems [27]. Despite this progress, all the currently known classical speed limits are on statistical dynamics. Still open is the question of whether there are speed limits on the underlying phase-space dynamics, dynamics that are just as important to statistical mechanics and often exhibit deterministic chaos. We address this question here.

Although many deterministic systems do not have stochastic fluctuations, they can be characterized by "uncertainty" associated with their evolution that originates from small disturbances in their initial conditions. This uncertainty is often analyzed through the linear stability of the, potentially nonlinear, dynamics where the local rates of convergence and divergence are intrinsic timescales for the evolution of perturbations. When perturbations corresponding to initially close phase space trajectories diverge [28], these intrinsic timescales are also a characteristic of deterministic chaos. These measures of instability have given insights into the physical mechanisms of the jamming transition in granular materials [29], self-organizing systems [30], evaporating collections of nuclei, equilibrium and nonequilibrium fluids [31–33], and critical phenomena [34]. The widespread analysis of the linear stability of dynamical systems, and the established connections between dynamical systems theory and nonequilibrium statistical mechanics [28, 35], suggest the possibility of classical speed limits on the intrinsic timescales of dynamical (in)stability that underlie deterministic chaos and the dissipation of energy.

In this Letter, we derive classical bounds on the observables and the state space of deterministic systems that parallel the Mandelstam-Tamm form of the time-energy uncertainty relation in quantum mechanics. Mandelstam and Tamm [2] considered isolated quantum systems evolving unitarily, proving that the rate of change of the expectation value $\langle \hat{O} \rangle$ of an arbitrary quantum observable $\hat{O}$ is bounded, $|d\langle \hat{O} \rangle/dt| \leq 2\Delta\hat{O}\Delta\hat{H}$, by the standard deviations of the observable and the Hamiltonian, $\hat{H}$. Perhaps more well known is their result that the minimum time $\tau^{\perp}$ for a system to evolve between two orthogonal states: $\tau^{\perp} \geq \pi/(2\Delta\hat{H})$. Here, we derive purely classical analogues of both of these bounds for dynamics that are not statistical. These bounds leverage a density matrix theory for the linearized dynamics of classical, deterministic systems. Within this theory, we can define observables, such as Lyapunov exponents and phase space contraction rates, and their intrinsic speeds. Defining a new classical Fisher information, we derive a speed limit on the evolution of these observables. We

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illustrate these bounds for local phase-space stretching rates and the energy dissipation rate in several model systems.

Linearized dynamics. To derive these speed limits, we start from the dynamical system, $\dot{\boldsymbol{x}} = \boldsymbol{F}[\boldsymbol{x}(t)]$, where $\boldsymbol{x}$ represents a point $\boldsymbol{x}(t) := [x^1(t), x^2(t), \ldots, x^n(t)]^\top$ in the n-dimensional state space, $\mathcal{M}$. Infinitesimal perturbation vectors $|\delta\boldsymbol{x}(t)\rangle := [\delta x^1(t), \delta x^2(t), \ldots, \delta x^n(t)]^\top \in T\mathcal{M}$ represent uncertainty about the initial condition. For a classical many-body system, these are perturbations of positions and momenta, $[\delta q^1(t), \ldots, \delta p_i(t), \ldots]$. Regardless of the state space, these vectors will generally stretch, contract, and rotate over time under the linearized dynamics,

$$d_t\,|\delta\boldsymbol{x}(t)\rangle = \boldsymbol{A}[\boldsymbol{x}(t)]\,|\delta\boldsymbol{x}(t)\rangle\,, \tag{1}$$

governed by the stability matrix, $\boldsymbol{A} := \boldsymbol{A}[\boldsymbol{x}(t)] = \boldsymbol{\nabla F}$ with elements $(\boldsymbol{A})^i_j = \partial\dot{x}^i(t)/\partial x^j(t)$. In our use of Dirac's notation here, the ket (bra) represents a finite-dimensional column (row) vector. While these linearized dynamics are an established approach to analyze the stability of nonlinear dynamical systems, there are not yet bounds on the time to transition between two classical *mechanical* states. However, this equation of motion has a formal similarity to the Schrödinger equation with the stability matrix occupying the position of the Hamiltonian operator, which suggests they may be a starting point for the derivation of classical speed limits.

One immediate challenge to setting limits on the dynamical evolution between two states is that these limits typically require expectation values. To overcome this challenge, we define a classical density matrix from the perturbations $|\delta\boldsymbol{x}(t)\rangle$. Once normalized, $|\delta\boldsymbol{u}(t)\rangle = |\delta\boldsymbol{x}(t)/\|\delta\boldsymbol{x}(t)\|\rangle$, these vectors have an outer product that defines a projection operator (a classical density matrix), $\boldsymbol{\varrho}(t) = |\delta\boldsymbol{u}(t)\rangle\langle\delta\boldsymbol{u}(t)|$, with the properties one expects of a quantum-mechanical pure state [36]. The equation of motion of this matrix,

$$d_t\boldsymbol{\varrho} = \bar{\boldsymbol{A}}\boldsymbol{\varrho} + \boldsymbol{\varrho}\bar{\boldsymbol{A}}^\top, \tag{2}$$

is akin to the von Neumann equation in quantum dynamics, with $\bar{\boldsymbol{A}} = \boldsymbol{A} - \mathrm{Tr}(\boldsymbol{A\varrho})$ playing the role of the Hamiltonian (Supplemental Material, SM I [37]. These norm-preserving dynamics hold regardless of whether the dynamical system is Hamiltonian or dissipative, and they enable a generalization of Liouville's theorem and Liouville's equation on phase space volumes [36]. In fact, the dynamics need not be mechanical; they could describe the evolution of any deterministic system from chemical reaction networks to power grids and biological populations [38].

Intrinsic speed of observables. Two well-known quantities in statistical physics and dynamical systems theory are averages over this density matrix. First, the instantaneous Lyapunov exponent or *local* stretching rate for the linearized dynamics is, $r := r(t) = r[\boldsymbol{x}(t)] = d_t \ln\|\delta\boldsymbol{x}(t)\| = \mathrm{Tr}(\boldsymbol{A}_+\boldsymbol{\varrho}) = \langle\boldsymbol{A}_+\rangle$, where $\boldsymbol{A}_+ = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^\top)$ is the symmetric part of $\boldsymbol{A}$. Time averaging this local rate gives the finite-time Lyapunov exponent [39]. Second, the dissipation is also an expectation value over this density matrix. The local phase space dissipation rate, Λ, is determined by the sum of stretching rates at a given phase point, $\Lambda = \sum_i^n r_i(t)$. For each perturbation vector, one can use a Gram-Schmidt [40] or covariant Lyapunov vector [41]. Here, we use the leading Lyapunov vector that gives the largest Lyapunov exponent in the asymptotic time limit [40].

These observables, and any other that is an expectation value over $\boldsymbol{\varrho}$, evolve in time. We can define the intrinsic speed from the time evolution $\langle\boldsymbol{O}\rangle$ of the observable $\boldsymbol{O}$,

$$d_t\langle\boldsymbol{O}\rangle = \mathrm{cov}(\boldsymbol{O}, 2\boldsymbol{A}^\top) + \langle d_t\boldsymbol{O}\rangle\,, \tag{3}$$

as we show in SM II [37]. The covariance, $\mathrm{cov}(\boldsymbol{X},\boldsymbol{Y}) = \langle\boldsymbol{X}\boldsymbol{Y}^\top\rangle - \langle\boldsymbol{X}\rangle\langle\boldsymbol{Y}^\top\rangle$, is composed of two pieces: the mean anticommutator, $\mathrm{cov}(\boldsymbol{O}, 2\boldsymbol{A}_+) = \langle\{\boldsymbol{O},\boldsymbol{A}_+\}\rangle - 2\langle\boldsymbol{A}_+\rangle\langle\boldsymbol{O}\rangle$, and the mean commutator, $\mathrm{cov}(\boldsymbol{O}, 2\boldsymbol{A}_-) = \langle[\boldsymbol{O},\boldsymbol{A}_-]\rangle$. We will express it more compactly as: $\dot{\mathcal{O}} := \mathrm{cov}(\boldsymbol{O}, 2\boldsymbol{A}^\top)$. This equation of motion is a deterministic analogue of the equation of motion for stochastic thermodynamic observables [26] (where the covariance could represent physical observables such as the heat flux), Price's equation in population biology [42], and Ehrenfest's theorem (Heisenberg's equation) for quantum mechanical observables [27, 43].

From this equation of motion, we can define the intrinsic speed of an observable and classical uncertainty relations that set limits on these speeds. One measure of the variation in $\boldsymbol{O}$ is the time it takes for the magnitude of this function $\mathcal{O} = \int\dot{\mathcal{O}}dt$ to have the value of one standard deviation $\Delta\boldsymbol{O}$. If $\dot{\mathcal{O}}$ is constant, this time $\tau_{\mathcal{O}}$ is approximately:

$$|\mathcal{O}| = \left|\int_{t_0}^{t_0+\tau_{\mathcal{O}}}\dot{\mathcal{O}}dt\right| \approx |\dot{\mathcal{O}}|\tau_{\mathcal{O}} \approx \Delta\boldsymbol{O}. \tag{4}$$

This condition motivates the definition of the time required for the observable to evolve to by one standard deviation. This observation motivates the definition of an intrinsic speed for $\boldsymbol{O}$,

$$\frac{1}{\tau_{\mathcal{O}}} := \frac{|\dot{\mathcal{O}}|}{\Delta\boldsymbol{O}} = \frac{|\,\mathrm{cov}(\boldsymbol{O}, 2\boldsymbol{A}^\top)|}{\Delta\boldsymbol{O}}, \tag{5}$$

similar to definitions in quantum mechanics [27, 43] and stochastic thermodynamics [26, 44]. Here, we will consider observables for the exponential growth of uncertainty in initial conditions, Lyapunov exponents, and observables for the energy dissipated.

Classical uncertainty relation and speed limit. Applying the Cauchy-Schwarz inequality to the covariance gives our main result:

$$\dot{\mathcal{O}}^2 = \mathrm{cov}(\boldsymbol{O}, 2\boldsymbol{A}^\top)^2 \leq 4\Delta\boldsymbol{O}^2\Delta\boldsymbol{A}^{\top 2}. \tag{6}$$

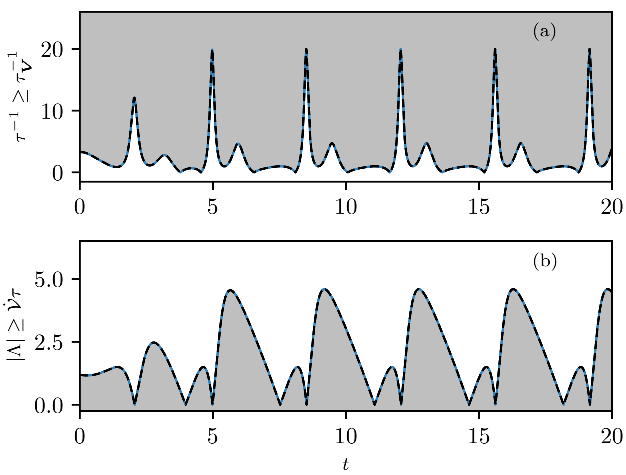

Figure 1. *Speed limit on the dissipation rate of the van der Pol oscillator.* (a) The Fisher information $\tau^{-1} = \sqrt{\mathcal{I}_F} = 2\Delta\boldsymbol{A}^\top$ upper bounds (dashed black curve) the intrinsic speed $\tau_{\boldsymbol{V}}^{-1}$ of energy dissipation $\Lambda = \langle\boldsymbol{V}\rangle$ with respect to a normalized perturbation $\boldsymbol{\varrho}$ (solid blue): $\tau^{-1} \geq \tau_{\boldsymbol{V}}^{-1}$. (b) The inequality also sets a lower bound on the magnitude of dissipation $|\Lambda| \geq \dot{\mathcal{V}}\tau$. Panel (b) shows $|\Lambda|$ (solid blue curve) is bounded from below by $\dot{\mathcal{V}}\tau$ (dashed black curve). The trajectory is on the stable limit cycle with damping parameter $\mu = 1.5$. Shaded regions mark regimes inaccessible to $\tau_{\boldsymbol{V}}^{-1}$ and $|\Lambda|$.

According to this classical uncertainty relation the uncertainty in $\boldsymbol{O}$ and the stability matrix $\boldsymbol{A}$ with respect to $\boldsymbol{\varrho}$ are a bound on $\dot{\mathcal{O}}$. With the intrinsic speed, $\tau_{\mathcal{O}}$, another form of this upper bound,

$$\tau_{\mathcal{O}}\,\Delta\boldsymbol{A}^\top \geq \tfrac{1}{2}, \tag{7}$$

is a *time-stability uncertainty relation.* From this relation, the fluctuations in local stability over the state space determines the speed at which observables, the instantaneous Lyapunov exponents and energy dissipation, evolve in time. For example, we the bound on energy dissipation for the van der Pol oscillator is shown in Fig. 1.

This classical uncertainty relation has a mathematical form that is strikingly similar to the Mandelstam-Tamm version of the time-energy uncertainty relation. Because of this similarity, this classical result can also be cast as a speed limit. The Mandelstam-Tamm bound is often expressed as a quantum speed limit $\tau_{\hat{O}}^{-1} \leq \tau_{\mathrm{QSL}}^{-1}$ on $\tau_{\hat{O}}^{-1} = \Delta\hat{O}^{-1} d\langle\hat{O}\rangle/dt$. For pure states, the limit is set by the fluctuations in energy $\tau_{\mathrm{QSL}} = \pi\hbar/(2\Delta\hat{H})$. Here, the intrinsic speed $\tau = 1/(2\Delta\boldsymbol{A}^\top)$ sets the limit on the speed $\tau_{\mathcal{O}}^{-1}$ of the observable, $\tau_{\mathcal{O}}^{-1} \leq \tau^{-1}$. That is, fluctuations in local stability, $\Delta\boldsymbol{A}$, related to the local curvature of state space dictate the maximum rate of any observable described by Eq. 4. This intrinsic feature of general dynamical systems bounds the local stretching rates that are used to measure chaos (deterministic and transient) and energy dissipation.

For certain observables, this classical speed limit can even more closely resemble the time-energy uncertainty relation [43] Recall that for an observable $\hat{O}$ of a closed quantum system, the mean commutator $\langle[\hat{O}, \hat{H}]\rangle$ takes the role of the covariance term in Eq. 5 and the term $\langle d_t\hat{O}\rangle$ in the Ehrenfest equation vanishes. Here, the stability matrix is the generator of the evolution and plays the role of the Hamiltonian in this analogy. In classical dynamical systems, $\langle d_t\boldsymbol{O}\rangle$ does not necessarily vanish because the observable $\boldsymbol{O}$ is usually time dependent. However, if $\boldsymbol{O}$ is time independent, the second term in Eq. 3 vanishes, $d_t\langle\boldsymbol{O}\rangle = \mathrm{cov}(\boldsymbol{O}, 2\boldsymbol{A}^\top)$. Applying the Cauchy-Schwarz inequality then leads to

$$d_t\langle\boldsymbol{O}\rangle \leq \Delta\boldsymbol{O}\Delta\boldsymbol{L} = 2\Delta\boldsymbol{O}\Delta\boldsymbol{A}^\top, \tag{8}$$

another classical analogue of the Mandelstam-Tamm uncertainty relation in quantum mechanics and the Cramér-Rao bound in classical statistics. Similar restrictions on the time-dependence of stochastic thermodynamic observables simplify a more general bound [26] for time-independent observables to bounds based on the Cramér-Rao inequality [23, 24]. In quantum mechanics [43], the analogue of Eq. 3 was only recently derived and shown to be a generalization of the Mandelstam-Tamm bound for open quantum systems [27].

Following Mandelstam-Tamm [2], this speed limit can put a bound on the time for a perturbation to evolve to an orthogonal state in the phase space. Choosing the observable to be the projection of the initial state $\boldsymbol{\varrho}(t_0) = |\delta\boldsymbol{u}(t_0)\rangle\langle\delta\boldsymbol{u}(t_0)|$, the time evolution of $\langle\boldsymbol{\varrho}(t_0)\rangle$ is lower bounded by $\langle\boldsymbol{\varrho}(t_0)\rangle \geq \cos^2(\Delta\boldsymbol{A}^\top t)$ in the time interval $0 \leq t \leq \pi/2\Delta\boldsymbol{A}^\top$. (A similar result holds for the quantum mechanical mean density operator $\langle\hat{\rho}(t_0)\rangle$[45].) This time interval also leads to a classical analogue of the time-energy uncertainty relation,

$$\tau^\perp\Delta\boldsymbol{A}^\top \geq \pi/2, \tag{9}$$

which bounds the time $\tau^\perp$ it takes for the initial state to evolve to an orthogonal state. Compared to our main result, Eq. 7, this bound holds for the comparatively few dynamical systems where $\boldsymbol{A}$ is time independent; two examples are the harmonic oscillator and the model for Chua's circuit [46]. While many nonlinear systems will violate this bound, they will satisfy more general bound, Eq. 7, which holds regardless of the time-independence of the observable.

Time-information uncertainty relation and tangent-space Fisher information. The Fisher information (parametrized by time) is often used to express thermodynamic and quantum speed limits because it is an intrinsic speed on the evolution of a system between neighboring states [47, 48]. It has a geometric representation through the Fisher information matrix, a Riemannian metric on statistical manifolds [13], and, through the Cramér-Rao inequality, it is a lower bound on the variance of unbiased estimators of parameters, making it a fundamental ingredient in optimal measurements of random variables. However, neither the classical nor quantum Fisher information are appropriate for the deterministic dynamics of the, potentially mechanical, systems we consider here. Though we are without classical probabilities, we can use the classical density matrix to define a new classical Fisher information—what we will call the tangent space Fisher information—to express the bounds above.

To define this information, consider the logarithmic derivative defined implicitly through $d_t\boldsymbol{\varrho} := \frac{1}{2}(\boldsymbol{\varrho}\boldsymbol{L} + \boldsymbol{L}^\top\boldsymbol{\varrho})$ [49, 50]. The form of $\boldsymbol{L}$,

$$\boldsymbol{L} = 2\bar{\boldsymbol{A}}^\top = 2(\boldsymbol{A}^\top - \langle\boldsymbol{A}\rangle), \tag{10}$$

comes from a comparison with Eq. 2. The tangent-space Fisher information is the variance of this classical logarithmic derivative (for "pure" states),

$$\mathcal{I}_F = \Delta\boldsymbol{L}^2 = \langle\boldsymbol{L}\boldsymbol{L}^\top\rangle = 4(\Delta\boldsymbol{A}^\top)^2, \tag{11}$$

and the expectation value of the Fisher information matrix $\boldsymbol{L}\boldsymbol{L}^\top$. As a point of comparison, the quantum Fisher information also derives from a logarithmic derivative in quantum information theory [51]; for pure quantum states evolving under a unitary dynamics, it is the variance in the energy $\hat{\mathcal{I}}_F = 4\Delta\hat{H}^2/\hbar^2$ [52].

With this classical Fisher information, $\mathcal{I}_F$, Eq. 7 becomes the time-information uncertainty relation,

$$\tau_{\mathcal{O}}\sqrt{\mathcal{I}_F} \geq 1. \tag{12}$$

That is, the Fisher information on the classical tangent space is the intrinsic speed $\sqrt{\mathcal{I}_F} = \tau^{-1}$ that sets the limit on the speed $\tau_{\mathcal{O}}^{-1}$ of any observable, $\tau_{\mathcal{O}}^{-1} \leq \tau^{-1}$, of the form in Eq. 7. When the observable is time independent, $\langle d_t\boldsymbol{O}\rangle = 0$, this uncertainty relation becomes, $\Delta\boldsymbol{O}/d_t\langle\boldsymbol{O}\rangle \geq 1/\sqrt{\mathcal{I}_F}$, reminiscent of the Cramér-Rao bound. These forms of the speed limit suggest the Fisher information on tangent space sets a fundamental limit on our ability to determine the state of a system given some *classical* uncertainty about its initial condition.

Model systems. The speed limit in Eq. 7 applies to any continuous-time, differentiable, classical dynamical system. Applying this bound, and those that follow from it, to model systems illustrate their implications for different physical observables. First, we consider the van-der Pol oscillator [53] where the speed limit here bounds the rate of energy dissipation. This oscillator is a 2D Liénard system [54] given by $\dot{x} = y$, $\dot{y} = -x - \mu(x^2 - 1)y$. It exhibits self sustained oscillations with nonlinear damping strength $\mu > 0$ and trajectories converge to a stable limit cycle. The local energy dissipation rate can be found exactly: $\Lambda = -\mu(x^2 - 1)$. To put a limit on the dissipation, we take a tangent vector $|\delta\boldsymbol{u}\rangle = (u, v)^\top$ and construct both an observable $\boldsymbol{V}$ and the density matrix,

$$\boldsymbol{V} = \Lambda\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad \boldsymbol{\varrho} = \begin{pmatrix} u^2 & uv \\ uv & v^2 \end{pmatrix}. \tag{13}$$

We choose this matrix representation of the observable because the dissipation rate Λ is the average (and standard deviation) of $\boldsymbol{V}$ over $\boldsymbol{\varrho}$. That is, $\langle\boldsymbol{V}\rangle = \mathrm{Tr}(\boldsymbol{V}\boldsymbol{\varrho})$ and

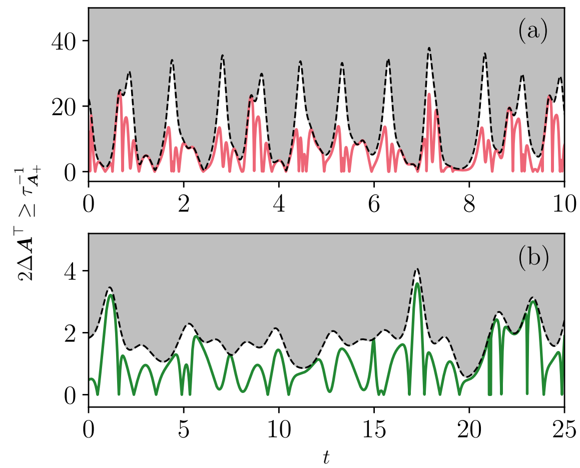

Figure 2. *Speed limit on chaos.* The instantaneous Lyapunov exponent, $r(t) = \langle\boldsymbol{A}_+\rangle$, for phase space orbits of (a) the Lorenz-Fetter model and (b) the Hénon-Heiles model. Square root of the tangent-space Fisher information (dashed black) $\sqrt{\mathcal{I}_F} = 2\Delta\boldsymbol{A}^\top$ upper bounds the speed $\tau_{\boldsymbol{A}_+}^{-1}$ (solid red and green). Shaded regions mark speeds not accessible to the observable. The parameters of the Lorenz-Fetter model are: $\sigma = 10$, $\beta = 8/3$, $\eta = 22$ (transient chaos). For the Hénon-Heiles system, the chaotic trajectory corresponds to the energy $E = 1/6$.

$\Delta\boldsymbol{V} = |\Lambda|$ because $\mathrm{Tr}\,\boldsymbol{\varrho} = 1$. The observable $\boldsymbol{V}$ is the matrix-representation of dissipation that can be generalized to higher dimensional systems.

For the van der Pol oscillator (SM III [37]), the intrinsic speed of dissipation is:

$$\tau_{\boldsymbol{V}}^{-1} = \frac{|\operatorname{cov}(\boldsymbol{V}, 2\boldsymbol{A}^\top)|}{|\Lambda|} = \left|\mathrm{Tr}\left[\begin{pmatrix} -2 - 4\mu xy & \Lambda \\ \Lambda & -2 \end{pmatrix}\boldsymbol{\varrho}\right]\right|.$$

Averages here are over $\boldsymbol{\varrho}$, so that $\tau_{\boldsymbol{V}}^{-1} = 2|\Lambda uv - 2\mu xyu^2 - 1|$. As shown in Fig. 1(a), the Fisher information $\mathcal{I}_F$ sets an intrinsic speed limit on the local energy dissipation rate of the van der Pol limit cycle: $\sqrt{\mathcal{I}_F} = \tau^{-1} \geq \tau_{\boldsymbol{V}}^{-1}$. The inequality nearly saturates. Equation 5 further suggests that the magnitude of $|\Lambda|$ is lower bounded by the product of the path observable $\dot{\mathcal{V}} = |\operatorname{cov}(\boldsymbol{V}, 2\boldsymbol{A}^\top)|$ and τ:

$$|\Lambda| \geq \dot{\mathcal{V}}\tau = \frac{\dot{\mathcal{V}}}{\sqrt{\mathcal{I}_F}}. \tag{14}$$

Figure 1(b) shows that because this bound nearly saturates, it sets a tight and finite lower bound on the energy dissipation rate. This bound suggests that, for a given $\mathcal{V}$, lowering the dissipation rate requires more (Fisher) information about the state relative to others infinitesimally nearby.

The bounds we report here apply to non-Hamiltonian, and even non-mechanical systems, such as chemical reaction networks and population dynamics. If the dynamics exhibit deterministic chaos, we can consider these bounds for the local rate of separation of chaotic trajectories, $\langle \boldsymbol{A}_+ \rangle$. The intrinsic speed of $\langle \boldsymbol{A}_+ \rangle$, $\tau^{-1} = |\operatorname{cov}(\boldsymbol{A}_+, 2\bar{\boldsymbol{A}}^\top)|/\Delta \boldsymbol{A}_+$ has an upper bound set by $\sqrt{\mathcal{I}_F} = 2\Delta \boldsymbol{A}^\top$. To illustrate this bound, we analyzed the Lorenz-Fetter model [55]. We chose parameters $(\sigma, \beta, \eta) = (10, 8/3, 22)$ for which this model exhibits transient chaos [56]; the dynamics are asymptotically non-chaotic but has a transient regime in which the oscillations are chaotic. As shown in Fig. 2(a), the intrinsic speed of $\langle \boldsymbol{A}_+ \rangle$ in the chaotic regime of one such trajectory is bounded by $\sqrt{\mathcal{I}_F}$ for $\boldsymbol{\varrho}$. The bound holds well through this transient regime and the decay regime, SM IV [37].

As an example of Hamiltonian dynamics, we analyze the Hénon-Heiles system [57]. Figure 2(b) shows the bound in Eq. 7 on local stretching rates on a chaotic trajectory with energy $E = 1/6$. For this energy, the Hénon-Heiles system [57] has a non-uniform phase space with algebraic decay of correlations and Poincaré recurrences [58]. This behavior is attributed to the existence of partial barriers to transport in the phase space, which leads to temporary trapping of orbits [59]. We note that the bound in Eq. 7 continues to hold for chaotic orbits visiting these so-called "sticky regimes" of a non-uniform phase space. The Fisher information therefore sets *local* upper bounds on the chaotic transport in Hamiltonian systems.

Chaotic systems are known to have the property of mixing characterized by relaxation or decay of correlations in the phase space [60]. The decay rate or the rate of mixing is determined by the leading Pollicott-Ruelle resonance [61–63]. In some simple systems, these resonances are related to the Lyapunov exponent [64]. For instance, in the Hamiltonian flow of the inverted harmonic potential with Hamiltonian $H = \lambda xy$, the resonances are integer multiples of the Lyapunov exponent λ [65]. Here, the intrinsic speed of the exponent λ, which gives the leading resonance for the inverted harmonic oscillator, saturates the bound in Eq. 7, SM V [37].

Conclusions. Uncertainty relations are one of the most prominent features of quantum mechanics. However, classical systems can also have uncertainty in their initial conditions, which can generate transient chaotic behavior or be suppressed by the dissipation of energy. Here, we show that for a broad class of dynamical systems, this uncertainty and the sensitivity to initial conditions also obey uncertainty relations. For deterministic, physical dynamics, we defined an intrinsic timescale, $\tau_{\mathcal{O}}$, for the mean of a given dynamical observable, $\boldsymbol{O}$, to change by the value of one initial standard deviation. Mirroring the Mandelstam-Tamm bounds, we derive speed limits on these observable timescales $\tau_{\mathcal{O}} \Delta \boldsymbol{A}^\top \geq \frac{1}{2}$ where the stability matrix, $\boldsymbol{A}$, plays the role of the Hamiltonian. As in quantum mechanics, this bounds also leads to $\tau^\perp \Delta \boldsymbol{A}^\top \geq \pi/2$, a classical analogue of the time-energy uncertainty relation for the time $\tau^\perp$ it takes for an initial perturbation to evolve to orthogonal state. We also recast these results by defining first the Fisher information, $\mathcal{I}_F$, for classical deterministic dynamics, $\tau_{\mathcal{O}}\sqrt{\mathcal{I}_F} \geq 1$. These speed limits are on the underlying dynamics of any classical system, be it open or closed, continuous or many-body, dissipative or conservative, passively evolving or actively driven. All of these speed limits are model independent and transform the longstanding statistical feature of uncertainty relations into a mechanical framework.

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